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"A Two-Sided Model of Television Competition with Advertising Pricing and Endogenous Reinvestment"

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A Two-Sided Model of Television Competition with Advertising Pricing and Endogenous Reinvestment

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Abstract

This paper studies competition between television channels in a two-sided market with asymmetric firms. Motivated by a competition case in Colombia, we consider an oligopoly with three channels—two large and one small—that compete for viewers and advertisers. Advertising affects viewers both directly and indirectly through content quality, which is endogenously determined by the share of revenues that channels reinvest rather than distribute to shareholders. We first characterise the equilibrium of the subgame between viewers and advertisers and derive comparative statics linking audience levels to prices and payout policies. We then analyse the equilibrium of the game between channels, which jointly choose advertising prices and payout rates. While equilibrium prices are characterised implicitly, the model delivers closed-form solutions for payout decisions.

Our main result is that asymmetries in audience size translate into asymmetric competitive pressure on the advertising side, which weakens the smaller channel. This effect is amplified when advertisers are restricted to single-homing, as in the presence of exclusivity clauses. By concentrating advertising demand on dominant channels, exclusivity reduces the smaller channel's revenues and its incentives to invest in content quality, thereby limiting its ability to compete. These findings provide a novel mechanism through which exclusivity can generate exclusionary effects in two-sided media markets by affecting both demand allocation and endogenous investment decisions. We find that exclusivity reduces social welfare, mainly due to a decline in advertisers' surplus that is not offset by improvements on the viewers' side.

JEL codes: D43, L11, L13, L82, L86, M37

Keywords: Two-sided markets, free-TV, ad-financed business model, competitive bottleneck, exclusivity contracts.

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1 Introduction

This paper is motivated by a recent competition case in Colombia involving the television advertising market. In a market with three national channels, the two largest channels in terms of audience were found to impose exclusivity clauses on advertisers. Under these clauses, advertisers contracting with one of the dominant channels were prevented from placing advertisements on rival channels. In such an environment, the third, smaller channel faces significant difficulties in attracting advertising demand and, therefore, in effectively competing with the dominant incumbents.

The role of exclusivity in platform and media markets remains theoretically ambiguous and empirically unsettled. On the one hand, exclusive contracts raise significant anti-competitive concerns, as they may deter entry or foreclose rivals (Mathewson and Winter, 1987; Rasmusen *et al.*, 1991; Bernheim and Whinston, 1998). These concerns are further amplified in the presence of network externalities, where restricting access to one side of the market can disproportionately weaken competitors (Shapiro, 1999). In addition, exclusivity may limit consumer choice by preventing users on competing platforms from accessing content,products, or services, a concern that has motivated regulatory interventions promoting interoperability in two-sided markets. On the other hand, a substantial theoretical literature highlights potential pro-competitive effects of exclusivity. Exclusive arrangements may foster investment incentives and effort provision (Segal and Whinston, 2000), help platforms overcome coordination failures inherent to two-sided markets, and even facilitate entry by allowing challengers to build a critical mass of users. In this paper, we contribute to this debate by showing that, in an asymmetric oligopoly with endogenous content quality, exclusivity may have exclusionary effects by weakening smaller competitors through both demand reallocation and reduced investment incentives.

We study this setting through the lens of a two-sided market in which television channels compete simultaneously for viewers and advertisers. As is standard in the literature, there are cross-group network effects: advertisers value channels with larger audiences, while viewers are affected by advertising. However, in contrast to canonical models, advertising revenues finance content quality, which positively affects audience demand. This creates a feedback loop between advertising, content quality, and audience size.

We consider an asymmetric oligopoly with three firms: two large channels and one smaller competitor. Channels maximise the share of revenues distributed to their owners while endogenously choosing the fraction of revenues that is reinvested into content quality. Content quality enters positively into viewers' utility and is directly linked to the reinvestment decision of the channels. Our model is agnostic and considers that viewers may either dislike or enjoy exposure to advertising. We focus on free tv markets.[1]

Within this framework, we first characterise the subgame between advertisers and viewers. We derive the comparative statics of equilibrium audience levels with respect to advertising prices and to the fraction of revenues distributed to shareholders. These results highlight how pricing and reinvestment decisions affect audience outcomes through both direct and indirect channels.

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[1] See Peitz and Valletti (2008) for a model that compares Pay-Tv *versus* free-to-air channels,

Equipped with these results, we then characterise the equilibrium of the game between channels, which compete in advertising prices and payout policies. Although equilibrium prices are determined implicitly, the model delivers closed-form expressions for payout rates, allowing us to analyse how competitive asymmetries shape the trade-off between revenue extraction and investment in content quality.

Our main findings can be summarised as follows. First, asymmetries in audience size translate into asymmetries in competitive pressure on the advertising side, which disadvantage the smaller channel. Second, these asymmetries affect equilibrium investment decisions: large channels have stronger incentives to reinvest in content quality, while the small channel tends to distribute a larger share of its revenues to shareholders. Third, contractual features that increase the degree of exclusivity in the advertising market—captured in the model by an increase in the parameter governing substitution across channels—amplify these asymmetries. In particular, they weaken the competitive role of the small channel by reducing both its advertising revenues and its incentives to invest in quality. Taken together, these results provide a mechanism through which exclusivity clauses can generate exclusionary effects in two-sided media markets. The welfare analysis shows that increasing exclusivity systematically reduces social welfare. This decline is primarily driven by a loss in advertisers' surplus, which is only weakly offset by the reallocation of revenues toward dominant channels and the associated changes in content provision.

Related literature. This paper contributes to several strands of the literature. First, it relates to the theory of two-sided markets, as developed by Caillaud and Jullien (2003) and Rochet and Tirole (2003, 2006) where platforms internalise cross-group network effects between users and advertisers. Following Ferrando *et al.* (2008) and Anderson and Jullien (2020), our setting is also closely related to models of media markets in which advertising affects viewers' utility both directly and indirectly through content provision.[2]

Second, the paper contributes to the literature on competition and pricing in media industries. In contrast to much of this work, which typically assumes symmetric platforms or focuses on price competition alone, we consider an asymmetric oligopoly with endogenous investment in content quality financed by advertising revenues. In this respect, our approach is closer to models where platforms jointly determine prices and quality, but we introduce an additional layer by allowing firms to choose how much of their revenues to reinvest versus distribute to shareholders. Even though we investigate the effect of exclusivity contracts on competition, we assume entry away.[3]

Third, our analysis is related to the literature on multi-homing and single-homing in platform markets (see Athey *et al.*, 2018.). We depart from the standard binary treatment by introducing a reduced-form parameter that captures the intensity of substitution across platforms, allowing us to study a continuum between multi-homing and single-homing. This proves particularly useful for analysing the competitive effects of exclusivity clauses.

Finally, the paper speaks to the growing literature on exclusionary practices and vertical restraints in platform and media markets (Evans, 2025). By showing how exclusivity on

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[2] See also Anderson and Gabsewicz (2006).

[3] Crampes *et al.* (2009) focus their analysis on the effect of entry.

the advertising side can weaken smaller competitors through both demand reallocation and endogenous investment responses, our model provides a novel mechanism linking contractual restrictions to market outcomes in two-sided environments.

Roadmap. The remainder of the paper is organised as follows. Section 2 presents the model and introduces the behaviour of viewers and advertisers. Section 3 characterises the equilibrium of the subgame between these two sides of the market and derives the corresponding comparative statics. Section 4 analyses the competition between channels in advertising prices and payout policies, and provides a full characterisation of the equilibrium. Section 5 studies the comparative statics with respect to the degree of exclusivity in the advertising market. Section 6 discusses the implications of the model in light of the Colombian competition case that motivates the analysis. Section 7 concludes.

2 Set-up

Consider a television market with three private channels indexed by $i \in \{1, 2, 3\}$. Channels 1 and 2 are channels with large audiences, whereas channel 3 is a smaller one. Each channel acts as a two-sided platform connecting advertisers and viewers.

Each channel chooses an advertising price $p_i \geq 0$ and a payout rate $\alpha_i \in [0, 1]$. Advertising revenue is

$$R_i = p_i a_i,$$

where a_i is the advertising volume sold by channel i. A fraction α_i of advertising revenue is distributed to shareholders, while the remaining fraction $1 - \alpha_i$ is reinvested in content quality.

Content quality is given by

$$s_i = \bar{s}_i + \rho(1 - \alpha_i)R_i,$$

where $\bar{s}_i$ is baseline quality and $\rho > 0$ captures the productivity of reinvestment.

To introduce competition across channels on the advertisers' side, suppose that aggregate advertisers' demand is generated by the maximisation of the quadratic utility

$$U^A = \sum_{i=1}^{3} \Big(\mu + \beta(\nu q_i - p_i)\Big) a_i - \frac{1}{2} \sum_{i=1}^{3} a_i^2 - \tau \sum_{i<j} a_i a_j,$$

where $\mu > 0$ is the baseline attractiveness of television advertising, $\beta > 0$ measures the sensitivity of advertising demand to a channel's own attractiveness, $\nu > 0$ captures the positive effect of the audience on advertisers' willingness to advertise, and $\tau \in [0, 1)$ measures substitution across channels on the advertisers' side. An increase in τ makes advertising allocations across channels more substitutable, in the sense that higher investment in one channel reduces the marginal return of advertising in others. At the same time, it discourages multi-homing and leads advertisers to concentrate their spending on fewer platforms. Hence, higher τ simultaneously increases substitutability in demand and reduces the extent of multi-homing, which may weaken effective competition despite stronger substitution patterns.

The first-order conditions are:

$$\mu + \beta(\nu q_i^* - p_i) - a_i^* - \tau \sum_{j \neq i} a_j^* = 0, \qquad i = 1, 2, 3.$$

Modelling exclusivity. In the context we study, there are only three channels, and the two largest ones impose exclusivity clauses on advertisers. As a result, any advertiser contracting with one of these large channels is prevented from placing advertising on the other large channel as well as on the smaller one. Importantly, this also implies that advertisers choosing the small channel cannot simultaneously advertise on the large channels. Therefore, although the small channel does not impose exclusivity itself, the exclusivity clauses implemented by the two dominant channels effectively generate a single-homing environment.

Rather than modelling these contractual constraints explicitly, we adopt a reduced-form approach in which the parameter τ captures the intensity of substitution across channels. When τ is low, advertisers can spread their expenditure across multiple channels (multi-homing). As τ increases, advertising demand becomes more concentrated on a single channel, and in the limit, the model converges to a single-homing environment. This parsimonious formulation allows us to analyse the competitive effects of exclusivity while keeping the model tractable.

Viewers' utility function is given by:

$$U^V(q, a, s) = \sum_{i=1}^{i=3} (\delta_i + \eta a_i + \gamma s_i)\, q_i - \frac{1}{2} \sum_{i=1}^{i=3} q_i^2 - b \sum_{i<j} q_i q_j,$$

where δ_i is baseline attractiveness, η is the direct effect of advertising on viewers, $\gamma > 0$ measures the effect of quality on the audience, and $b > 0$ captures substitution across channels. The parameter η may be positive or negative. If $\eta < 0$, viewers dislike advertising on average; if $\eta > 0$, advertising has a positive direct effect on viewership. The first-order condition in q_i yields the following audience demand system:

$$q_i = \delta_i + \eta a_i + \gamma s_i - b \sum_{j \neq i} q_j. \qquad i = 1, 2, 3,$$

3 Equilibrium analysis

The timing of the game is as follows. In stage 1, each channel $i \in \{1, 2, 3\}$ simultaneously chooses an advertising price $p_i \geq 0$ and a payout rate $\alpha_i \in [0, 1]$. In stage 2, advertisers choose their advertising allocation across channels. In stage 3, viewers choose which channel to watch.

3.1 Subgame equilibrium

For given (p, α), a subgame equilibrium of stages 2 and 3 is a vector of advertising demands, qualities, and audiences $(a^*(p, \alpha),\ s^*(p, \alpha),\ q^*(p, \alpha))$ such that:

- advertising demand satisfies

$$\mu + \beta(\nu q_i^*(p,\alpha) - p_i) - a_i^* - \tau \sum_{j \neq i} a_j^* = 0. \qquad i = 1,2,3,$$

- quality is given by

$$s_i^*(p,\alpha) = \bar{s}_i + \rho(1-\alpha_i)R_i^*(p,\alpha), \qquad R_i^*(p,\alpha) = p_i a_i^*(p,\alpha).$$

- Audiences are given by:

$$q_i^*(p,\alpha) = \delta_i + \eta a_i^*(p,\alpha) + \gamma s_i^*(p,\alpha) - b\sum_{j \neq i} q_j^*(p,\alpha), \qquad i = 1,2,3.$$

3.2 Quality and audience

We now restrict attention to a symmetric equilibrium between channels 1 and 2. Denote

$$q_1 = q_2 =: q_H, \qquad p_1 = p_2 =: p_H, \qquad \alpha_1 = \alpha_2 =: \alpha_H, \qquad a_1 = a_2 =: a_H,$$

and

$$q_3 =: q_L, \qquad p_3 =: p_L, \qquad \alpha_3 =: \alpha_L, \qquad a_3 =: a_L.$$

Similarly, we have $\delta_1 = \delta_2 =: \delta_H$, $\bar{s}_1 = \bar{s}_2 =: \bar{s}_H$, $\delta_3 =: \delta_L$, and $\bar{s}_3 =: \bar{s}_L$. Under symmetry, the demand system yields:

$$\begin{cases} q_H = \dfrac{\delta_H + \eta a_H + \gamma s_H - b(\delta_L + \gamma s_L + \eta a_L)}{1+b-2b^2}; \\ q_L = \dfrac{(1+b)[\delta_L + \eta a_L + \gamma s_L] - 2b(\delta_H + \eta a_H + \gamma s_H)}{1+b-2b^2}. \end{cases}$$

Under symmetry, the advertisers' first-order conditions reduce to

$$\begin{cases} a_H = \dfrac{\mu(1-\tau) + \beta\left(\nu q_H - p_H - \tau(\nu q_L - p_L)\right)}{D}, \\ a_L = \dfrac{\mu(1-\tau) + \beta\left(\nu q_L - p_L)(1+\tau) - 2\tau(\nu q_H - p_H)\right)}{D}, \end{cases}$$

with

$$D := (1-\tau)(1+2\tau).$$

Introducing a_H and a_L in the expressions of q_H and q_L, we obtain a closed-form solution, the details of which are given in the appendix.

Lemma 1. *q_i is decreasing in α_i and increasing in α_j for $i = \{H, L\}$ and $j = \{H, L\}$ and $i \neq j$.*

Lemma 1 reveals that a higher payout rate reduces the audience of the channel that chooses it and increases the audience of the rival channels.

Proof. See appendix. ☐

Lemma 2. *q_i is increasing in p_i as long as τ is low enough and is ambiguous otherwise.*

Proof. See appendix. ☐

The sign of the equilibrium audience response to its own price depends on two components. The first one captures the tension between the direct effect of advertising on viewers and the indirect effect through quality. Second, there is a feedback effect through the rival side of the audience system.

Let us move to the channels' competition stage.

3.3 Channels' competition

In this reduced-form version, channel i directly maximises distributed profits:

$$\max_{p_i,\alpha_i} \alpha_i R_i.$$

For each channel i, assuming an interior solution, the first-order conditions with respect to p_i and α_i are given by

$$\begin{cases} a_i^* + p_i^* \dfrac{\partial a_i^*}{\partial p_i} = 0, \\ a_i^* + \alpha_i^* \dfrac{\partial a_i^*}{\partial \alpha_i} = 0. \end{cases}$$

Regarding payout rates, note that

$$\frac{\partial R_H}{\partial \alpha_H} = p_H \frac{\partial a_H}{\partial \alpha_H}, \qquad \frac{\partial R_L}{\partial \alpha_L} = p_L \frac{\partial a_L}{\partial \alpha_L}.$$

Since

$$\frac{\partial a_H}{\partial \alpha_H} = \frac{\nu \left(\frac{\partial q_H}{\partial \alpha_H} - \tau \frac{\partial q_L}{\partial \alpha_H} \right)}{D},$$

the large channel's payout condition becomes

$$R_H + \alpha_H p_H \frac{\nu \left(\frac{\partial q_H}{\partial \alpha_H} - \tau \frac{\partial q_L}{\partial \alpha_H} \right)}{D} = 0.$$

Likewise, since

$$\frac{\partial a_L}{\partial \alpha_L} = \frac{\nu \left((1+\tau) \frac{\partial q_L}{\partial \alpha_L} - 2\tau \frac{\partial q_H}{\partial \alpha_L} \right)}{D},$$

the small channel's payout condition becomes

$$R_L + \alpha_L p_L \frac{\nu \left((1+\tau) \frac{\partial q_L}{\partial \alpha_L} - 2\tau \frac{\partial q_H}{\partial \alpha_L} \right)}{D} = 0.$$

To close the first-order conditions, it remains to compute the derivatives of equilibrium audience with respect to prices and payout rates.

Substituting these expressions into the first-order conditions, the representative large channel's two optimality conditions become

$$a_H + p_H \frac{\nu\left(\frac{\partial q_H}{\partial p_H} - \tau \frac{\partial q_L}{\partial p_H}\right) - 1}{D} = 0,$$

and

$$R_H + \alpha_H p_H \frac{\nu\left(\frac{\partial q_H}{\partial \alpha_H} - \tau \frac{\partial q_L}{\partial \alpha_H}\right)}{D} = 0.$$

Similarly, the small channel's two optimality conditions become

$$a_L + p_L \frac{\nu\left((1+\tau)\frac{\partial q_L}{\partial p_L} - 2\tau \frac{\partial q_H}{\partial p_L}\right) - (1+\tau)}{D} = 0,$$

and

$$R_L + \alpha_L p_L \frac{\nu\left((1+\tau)\frac{\partial q_L}{\partial \alpha_L} - 2\tau \frac{\partial q_H}{\partial \alpha_L}\right)}{D} = 0.$$

This system captures two key feedback mechanisms:

- A *price effect*: increasing p_i reduces advertising demand directly but also affects audiences through reduced advertising and quality.
- An *investment effect*: increasing α_i reduces reinvestment, which lowers content quality, audience size, and thus advertising demand.

The balance between these effects determines the optimal pricing and payout strategies of the channels.

Proposition 1. *Conditional on prices (p_H, p_L), the equilibrium payout rates admit the closed-form expressions*

$$\alpha_H^* = 1 - \frac{1}{\gamma\rho p_H}\left[\frac{\beta\nu k_L(\eta + \gamma\rho p_L) - M_0}{\beta\nu\big(\beta\nu(\eta + \gamma\rho p_L) - k_H\big)} - \eta\right],$$

$$\alpha_L^* = 1 - \frac{1}{\gamma\rho p_L}\left[\frac{\beta\nu k_H(\eta + \gamma\rho p_H) - M_0}{\beta\nu\big(\beta\nu(\eta + \gamma\rho p_H) - k_L\big)} - \eta\right],$$

where

$$k_H = 1 + 2\tau b, \qquad k_L = 1 + \tau + b + 3\tau b, \qquad M_0 = D(1 + b - 2b^2), \qquad D = (1-\tau)(1+2\tau).$$

These expressions show that the payout rate a_i chosen by each type of channel depends not only on its own price but also on the rival side's effective quality-adjusted return through the terms $(\eta + \gamma\rho p_H)$ and $(\eta + \gamma\rho p_L)$. Each channel's payout decision trades off current profit extraction against the indirect effect of reinvestment on audience size and advertising demand. The expressions above show that the optimal payout rate depends not only on its own price but also on the rival side's effective return from advertising, captured by the terms $(\eta+\gamma\rho p_H)$ and $(\eta+\gamma\rho p_L)$. The asymmetry between k_H and k_L reflects the fact that the small channel faces stronger effective competitive pressure on the advertising side. As a result, its marginal return to reinvestment is lower, which tends to increase its equilibrium payout rate relative to the large channels.

Denote $X := \eta + \gamma\rho p$. We formulate the following assumption:

Assumption 1.

$$\beta^2\nu^2X^2 - \beta\nu(k_L + k_H)X + M_0 > 0.$$

This condition ensures that the marginal return to reinvestment in content quality is sufficiently large. Recall that $X = \eta + \gamma\rho p$ captures the total effect of advertising revenues on audience size, combining both the direct effect of advertising η, which can be positive or negative depending on whether consumers are ad-lovers or ad-averse, and the indirect effect through content quality. The condition, therefore, requires that advertising revenues translate sufficiently strongly into audience gains. This is a natural assumption in media markets, where higher advertising revenues typically finance better content and attract larger audiences. Under this condition, the reinvestment channel is strong enough for asymmetries across platforms to matter: larger channels benefit more from reinvestment, leading them to distribute less and invest more than smaller competitors.

Proposition 2. *Assume that equilibrium prices do not differ too strongly across channels, so that $p_H \approx p_L =: p$ and that Assumption 1 holds. Then, in equilibrium,*

$$\alpha_H^* < \alpha_L^*.$$

Proposition 2 reveals that when the prices charged by channels to firms are relatively close, large channels distribute a smaller fraction of their revenues to their shareholders and reinvest more in content quality than the smallest channel. The intuition is that reinvestment is more effective for large platforms: a marginal increase in quality translates into a larger increase in audience and advertising demand. By contrast, the small channel faces stronger effective competitive pressure on the advertising side, captured by $k_L > k_H$. As a result, the marginal return to reinvestment is lower, which induces the small channel to extract more revenue and reinvest less. In equilibrium, the stronger platforms invest, while the weaker platform extracts.

Proposition 3. *An increase in τ tends to amplify the asymmetry in payout policies between large and small channels. In particular, holding prices fixed, an increase in τ raises the effective competitive pressure faced by the small channel relative to the large channels, which reduces its marginal return to reinvestment. As a result, the equilibrium gap in payout rates,*

$$\alpha_L^* - \alpha_H^*,$$

tends to increase as τ rises.

Proposition 3 shows that a higher τ increases the relative competitive pressure faced by the small channel, as captured by the widening gap $k_L - k_H$. This reduces the effectiveness of reinvestment for the small channel, since any increase in quality translates into a smaller gain in audience and advertising demand. By contrast, the large channels benefit from the reallocation of advertising demand induced by a higher τ. This strengthens the return to reinvestment for large platforms. Therefore, moving from multi-homing to single-homing shifts the equilibrium trade-off between extraction and investment asymmetrically across channels: the large channels reinvest more, while the small channel extracts more.

3.4 Implicit characterisation of equilibrium prices

Although the payout rates admit closed-form solutions conditional on prices, the same is not true for equilibrium prices. The first-order conditions with respect to p_H and p_L nonetheless provide an implicit characterisation of the stage-1 equilibrium.

Proposition 4. *Consider an interior symmetric equilibrium between the two large channels. Then, the equilibrium prices p_H and p_L are characterised implicitly by the following two first-order conditions:*

$$a_H + p_H \left[-\frac{\beta}{D} + \frac{\beta\nu}{D\Delta} \Big(K_{11}(B_L + \tau A_L) - K_{21}(B_H + \tau A_H) \Big) \right] = 0, \tag{1}$$

and

$$a_L + p_L \left[-\frac{(1+\tau)\beta}{D} + \frac{\beta\nu}{D\Delta} \Big(K_{22}\big((1+\tau)A_H + 2\tau B_H\big) - K_{12}\big((1+\tau)A_L + 2\tau B_L\big) \Big) \right] = 0, \tag{2}$$

where

$$K_{12} = \frac{\tau\beta T_H}{D}, \qquad K_{21} = \frac{2\tau\beta T_L}{D},$$

and

$$K_{11} = -\frac{\beta\eta}{D} + \gamma\rho(1-\alpha_H)\left(a_H - \frac{\beta p_H}{D} \right),$$

$$K_{22} = -\frac{(1+\tau)\beta\eta}{D} + \gamma\rho(1-\alpha_L)\left(a_L - \frac{(1+\tau)\beta p_L}{D} \right).$$

The equilibrium prices balance two forces. First, a higher advertising price directly reduces advertising demand. Second, through the subgame equilibrium, it affects audience levels and therefore feeds back into advertising demand and revenues. The terms K_{11} and K_{22} summarise the trade-off between the direct effect of advertising on viewers and the indirect quality effect through reinvestment, while K_{12} and K_{21} capture cross-channel spillovers. Hence, even though closed-form solutions for prices are generally unavailable, the equilibrium prices are fully characterised in implicit form by (1) and (2).

3.5 Consequences of exclusivity contracts

This section studies how a change in τ affects equilibrium outcomes. Recall that τ measures the extent to which advertisers reallocate their demand across channels in response to differences in prices and audiences. A higher τ therefore captures a move from multi-homing toward single-homing or exclusivity.

Price asymmetry. Figure 1 plots the equilibrium price gap $p_H^*(\tau) - p_L^*(\tau)$ as a function of τ. The numerical simulations[4] show that the price differential between large and small channels increases with τ.

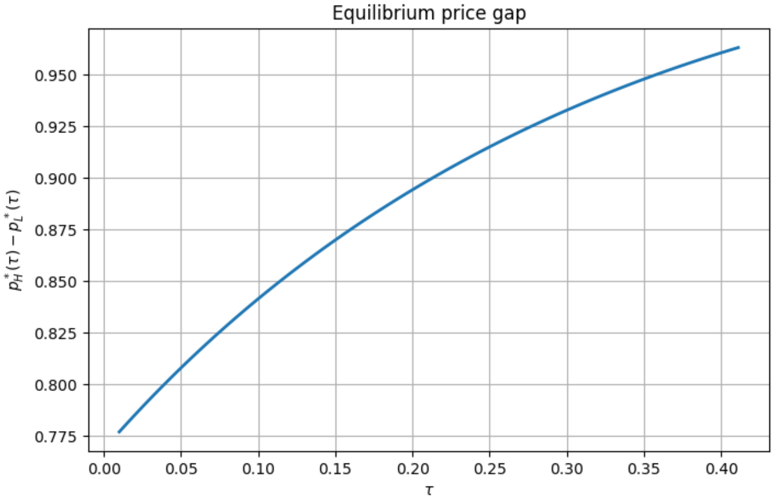

Figure 1: Price gap between p_H and p_L in function of τ

This pattern reflects the asymmetric reallocation of advertising demand induced by exclusivity. As τ rises, advertisers concentrate their expenditure on the large channels, which increases their effective market power relative to the fringe platform. In contrast, the small channel faces a shrinking residual demand, which limits its ability to sustain high prices. As a result, even in cases where both prices decline in levels, the dominant channels experience a relative advantage, leading to a widening price gap.[5] Hence, exclusivity primarily manifests itself through an increase in price asymmetry rather than a uniform increase in price levels.

Reallocation of advertising revenues. Figure 2 displays the share of total advertising revenues accruing to the large channels. Yet, we observe a monotonic increase in this share as τ rises.

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[4] The programme in Python is available in an online appendix.

[5] Advertising prices may decrease due to the fact that an increase of τ also contributes to decrease the size of the market.

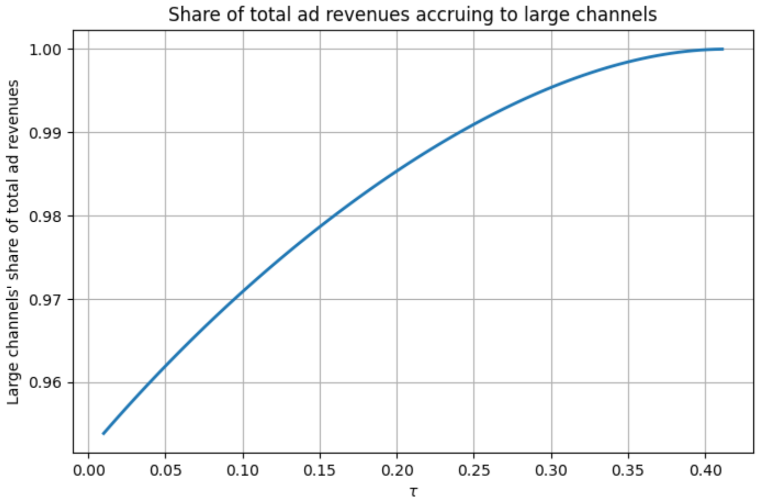

Figure 2: Share of total revenues in function of τ

This result highlights that exclusivity primarily operates through a reallocation mechanism. By discouraging multihoming, a higher τ induces advertisers to concentrate their spending on fewer platforms, which disproportionately benefits the dominant channels. Consequently, the large channels capture an increasing fraction of total advertising revenues, even when aggregate demand may decline. This mechanism provides a direct link between exclusivity and market dominance, independently of the level of prices.

Market concentration. Figure 3 reports the Herfindahl-Hirschman Index (HHI) computed from advertising revenues. The HHI increases with τ, indicating a higher level of market concentration as exclusivity intensifies. This finding is consistent with the reallocation of demand toward dominant platforms. As advertisers reduce multihoming, revenue becomes more unevenly distributed across channels, leading to a higher concentration index. Importantly, this increase in concentration arises even in environments where equilibrium prices do not increase. Hence, the numerical simulations of our model show that exclusivity can strengthen market power through a redistribution of demand rather than through higher price levels alone.

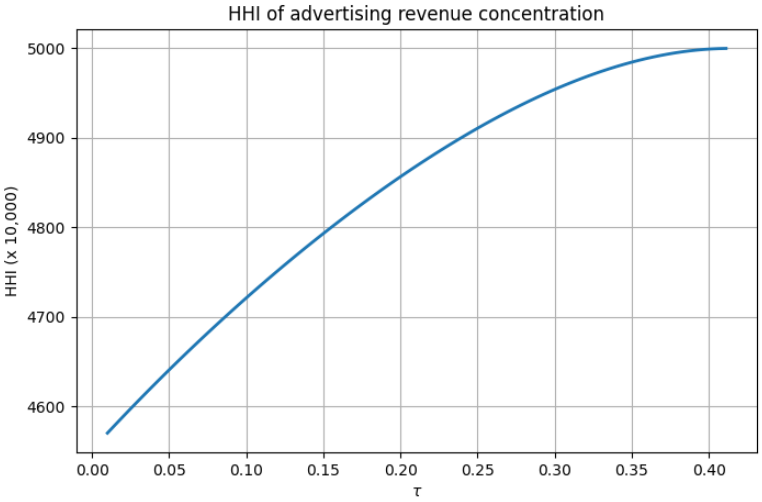

Figure 3: HHI of ad-revenues in function of τ

Foreclosure threshold. Finally, we identify a threshold value $\hat{\tau}$ such that the equilibrium advertising demand of the small channel becomes negligible, *i.e.* $a_L^*(\tau) \approx 0$. This threshold marks a transition from an interior allocation of advertising demand to a corner solution in which the fringe platform is effectively excluded from the advertising market. Economically, this reflects the cumulative effect of exclusivity. As τ increases, the small channel progressively loses advertising demand until it becomes non-viable. Beyond $\hat{\tau}$, advertisers concentrate all their expenditure on the large channels, leading to a *de facto* foreclosure of fringe competitors. This result highlights that exclusivity does not only distort competition at the margin but can ultimately eliminate smaller competitors from the market.

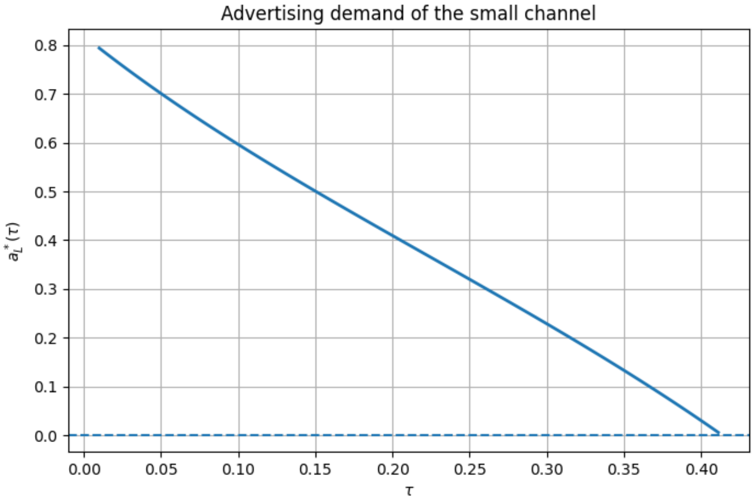

Figure 4: Enter Caption

Taken together, the simulations show that increasing exclusivity simultaneously (i) widens price asymmetries, (ii) reallocates advertising revenues toward dominant platforms, (iii) increases market concentration, and (iv) may ultimately lead to the exclusion of fringe competitors. These effects arise even in the absence of systematic increases in price levels, emphasising that exclusivity operates primarily through demand reallocation rather than through standard price competition mechanisms.

3.6 Social welfare

To evaluate the welfare implications of exclusivity, consider a utilitarian social welfare function that aggregates the utility of viewers, the utility of advertisers, and possibly a weighted component of channels' profits.

We then define social welfare as

$$W(\tau) = U^V\big(q^*(\tau), a^*(\tau), s^*(\tau)\big) + U^A\big(a^*(\tau), q^*(\tau), p^*(\tau), \tau\big) + \omega\Pi^*(\tau),$$

where $\omega \in [0,1]$ captures the weight assigned to channels' profits.

The effect of τ on welfare is, in general, ambiguous. Differentiating totally yields:

$$\frac{dW}{d\tau} = \frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial p_H}\frac{dp_H}{d\tau} + \frac{\partial W}{\partial p_L}\frac{dp_L}{d\tau} + \frac{\partial W}{\partial \alpha_H}\frac{d\alpha_H}{d\tau} + \frac{\partial W}{\partial \alpha_L}\frac{d\alpha_L}{d\tau}.$$

The direct effect of exclusivity is negative, since

$$\frac{\partial W}{\partial \tau} = -\sum_{i<j} a_i a_j,$$

which, under symmetry between the two large channels, becomes

$$\frac{\partial W}{\partial \tau} = -\big(a_H^2 + 2a_H a_L\big) < 0.$$

However, higher τ also reallocates advertising revenues toward dominant channels, which may stimulate their investment in content quality and increase viewers' utility. At the same time, it weakens the smaller channel, reduces competitive pressure, and may eventually lead to foreclosure.

Figure 5 reports the numerical welfare analysis as a function of τ, both under fixed payout rates and under endogenous payout decisions. Three main results emerge.

First, total welfare is monotonically decreasing in τ in both specifications. Hence, moving from a more competitive environment with substantial multi-homing to a more exclusive environment systematically reduces utilitarian social welfare. This result is consistent with the idea that exclusivity distorts the allocation of advertising demand, weakens the fringe platform, and reduces the gains from competition on both sides of the market.

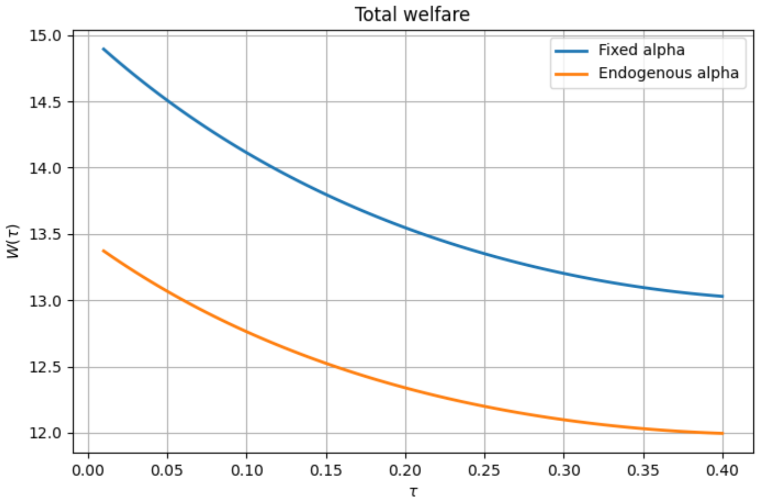

Figure 5: Welfare variation with respect to τ

Second, the welfare decomposition shows that the main source of this welfare loss comes from the advertisers' side. In both the fixed-α and endogenous-α cases, ΔW_A declines sharply with τ. This is intuitive in light of the direct effect of τ on the advertisers' utility function: a higher τ increases the effective penalty associated with spreading advertising expenditure across channels, thereby reducing advertisers' surplus. In other words, exclusivity directly lowers the value of multi-homing and makes advertising allocation less efficient.

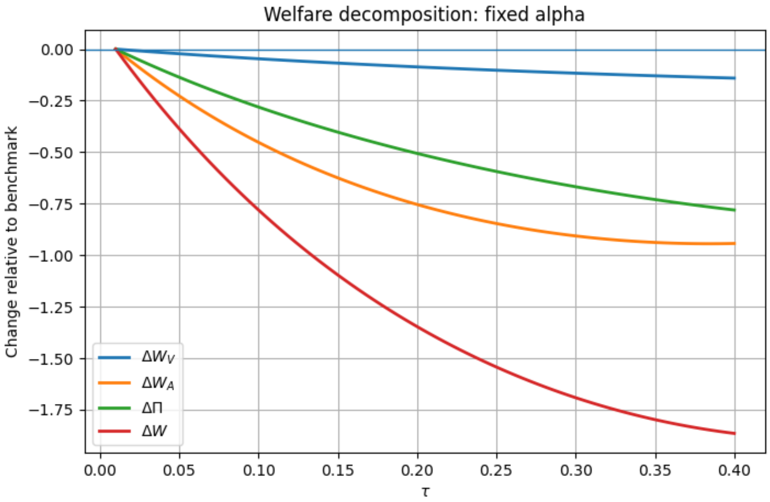

Figure 6: Welfare decomposition with fixed values of α

Third, viewers' welfare also declines with τ, although much more moderately than advertisers' welfare. This suggests that the increased concentration of advertising revenues in the large channels does not generate sufficiently strong quality improvements to compensate for the weakening of the fringe channel. Put differently, the reallocation of revenues toward dominant platforms is not welfare-enhancing enough on the viewers' side to offset the broader loss in competition and diversity.

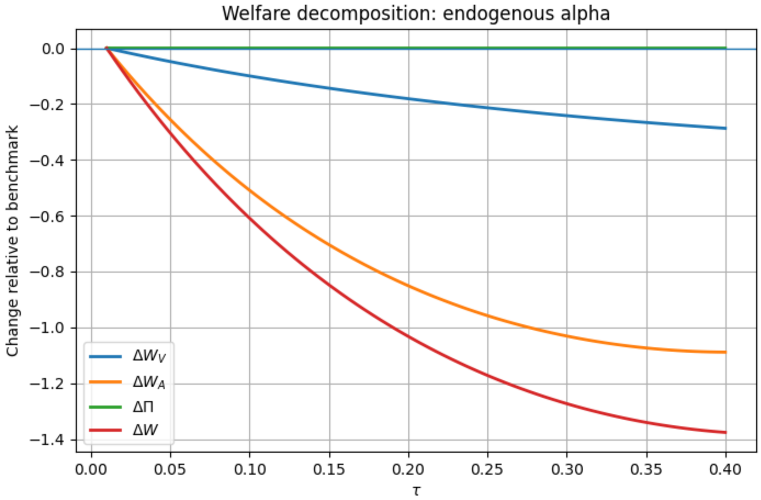

Figure 7: Welfare values with endogenous values of α

The comparison between the two specifications is also informative. Under fixed payout rates, channels' distributed profits decline with τ, and this further amplifies the fall in total welfare. Under endogenous payout rates, by contrast, distributed profits remain almost unchanged over the simulated range. This suggests that, once payout decisions are allowed to adjust optimally, firms partly absorb the effect of exclusivity through their internal allocation between reinvestment and profit distribution, without generating noticeable gains in aggregate profits. As a result, the negative welfare effect of exclusivity continues to be driven primarily by the deterioration in advertisers' surplus and, to a lesser extent, by the reduction in viewers' welfare.

Overall, these simulations indicate that exclusivity is welfare-reducing in this environment. Even though a higher τ reallocates advertising demand toward dominant channels, this does not translate into sufficiently large gains in quality or profits to offset the losses borne by advertisers and the weakening of the smaller channel. The quantitative exercises therefore support the view that exclusivity harms social welfare mainly by reducing the efficiency of advertising allocation and by reinforcing market concentration.

4 Conclusion

This paper develops a model of competition between television channels in a two-sided market, motivated by a competition case in Colombia. The key feature of the model is an asymmetric oligopoly with three channels—two large and one small—in which channels compete for both viewers and advertisers. A distinctive aspect of our framework is that content quality is endogenously determined by the fraction of advertising revenues that channels choose to reinvest, while the remaining share is distributed to shareholders. This setup allows us to capture the interaction between pricing, investment, and audience formation in a tractable way.

Within this environment, we characterise the equilibrium of the model and show how asymmetries across channels shape competitive outcomes. In particular, we find that the smaller channel faces stronger competitive pressure on the advertising side, which weakens its ability to attract advertising revenues and reduces its incentives to invest in content quality. By contrast, larger channels benefit from their audience advantage and have stronger incentives to reinvest. Importantly, we show that exclusivity—modelled as a move toward single-homing—amplifies these asymmetries. By making advertising demand more concentrated, exclusivity shifts revenues toward dominant channels, further weakens the smaller competitor, and reinforces the feedback loop between advertising revenues and content quality. In this context, exclusivity can have detrimental effects on competition by reducing the ability of smaller channels to exert competitive pressure.

The welfare analysis reinforces the central insight of the paper. Increasing exclusivity, captured by a higher value of τ, systematically reduces social welfare in our framework. This decline is primarily driven by a substantial loss on the advertisers' side, reflecting the reduced efficiency of allocating advertising expenditure when multi-homing becomes less attractive. While exclusivity reallocates revenues toward dominant channels, this does not generate sufficiently large gains in content quality or audience to offset the associated losses. As a result, even though exclusivity may strengthen the position of large platforms and increase market concentration, it does not improve overall efficiency. On the contrary, the analysis suggests that exclusivity distorts the functioning of the market and can ultimately be detrimental to welfare, especially when it weakens or excludes smaller competitors.

Our analysis could be extended in several directions. First, while we capture exclusivity through a reduced-form parameter governing the degree of single-homing, a natural extension would be to model exclusivity contracts explicitly and study their equilibrium formation. Second, the objective function of channels could be further decomposed by introducing a separation between ownership and management. In particular, one could consider a setting in which shareholders choose payout policies in an initial stage, while managers subsequently set advertising prices. This would allow the model to account for potential agency issues between owners and managers that are abstracted from in the current framework.

Overall, the paper highlights how contractual features on one side of a two-sided market can have far-reaching effects on competition, not only through demand allocation but also through endogenous investment decisions. These mechanisms are particularly relevant in media markets, where advertising revenues play a central role in financing content quality. These findings suggest that competition authorities should carefully assess exclusivity clauses in media markets, as their effects may go beyond short-run allocation and extend to long-run investment and market structure.

References

[1] Anderson, S. P., and Coate, S. (2005). Market provision of broadcasting: A welfare analysis. *Review of Economic Studies*, 72(4), 947–972.

[2] Athey, S., E. Calvano, and J. Gans. 2018. The Impact of Consumer Multi-homing on Advertising Markets and Media Competition. *Management Science*, 64, 1574-1590.

[3] Anderson, S. P., and Jullien, B. (2020). The advertising-financed business model in two-sided media markets, Working paper.

[4] Caillaud, B., and Jullien, B. (2003). Chicken & egg: Competition among intermediation service providers. *RAND Journal of Economics*, 34(2), 309–328.

[5] Crampes, C., C. Haritchabalet, and B. Jullien (2009). Advertising, Competition and Entry in Media Industries. *Journal of Industrial Economics*, 57(1), 7-31.

[6] Evans, D.S., (2025). Economics of Vertical Restraints for Multi-Sided Platforms. University of Chicago Institute for Law Economics Olin Research Paper No. 626.

[7] Ferrando, J., J.J. Gabszewicz, D. Laussel, and N. Sonnac (2008). Intermarket network externalities and competition: An application to the media industry. *International Journal of Economic Theory*, 4(3), 357-379.

[8] Gabszewicz, J.J., D. Laussel, and N. Sonnac (2004). Programming and Advertising Competition in the Broadcasting Industry. *Journal of Economics & Management Strategy*, 13(4), 657-669.

[9] Mathewson, G. Frank, and R.A. Winter. (1987). The Competitive Effects of Vertical Agreements: Comment. *American Economic Review* 77 (5): 1057–62.

[10] Peitz, M., and Valletti, T. (2008). Content and advertising in the media: Pay-tv versus free-to-air. *International Journal of Industrial Organization*, 26(4), 949–965.

[11] Rasmusen, E.B., Ramseyer, J.M. and J.S. Wiley, 1991. Naked Exclusion. *American Economic Review* 81 (5): 1137–45.

[12] Rochet, J.-C., and Tirole, J. (2003). Platform competition in two-sided markets. *Journal of the European Economic Association*, 1(4), 990–1029.

[13] Rochet, J.-C., and Tirole, J. (2006). Two-sided markets: A progress report. *RAND Journal of Economics*, 37(3), 645–667.

[14] Segal, I.R., and M.D. Whinston. 2000. Exclusive Contracts and Protection of Investments. *RAND Journal of Economics* 31 (4): 603–33.

[15] Shapiro, C. 1999. Exclusivity in Network Industries. *George Mason Law Review* 7 (3): 673–83.

5 Appendix

5.1 Comparative static within the subgame equilibrium

First, we reorganise the system that defines the audiences at the subgame equilibrium, and next we differentiate this system with respect to the channels' strategic variables to provide comparative static results.

5.1.1 Characterisation of the audiences at the subgame equilibrium

Substituting for quality, the audience variables become:

$$\begin{aligned} q_H &= \delta_H + \gamma \bar{s}_H + \big[\eta + \gamma\rho(1-\alpha_H)p_H\big]a_H - b(q_H + q_L), \\ q_L &= \delta_L + \gamma \bar{s}_L + \big[\eta + \gamma\rho(1-\alpha_L)p_L\big]a_L - 2bq_H. \end{aligned}$$

Define

$$T_H := \eta + \gamma\rho(1-\alpha_H)p_H, \qquad T_L := \eta + \gamma\rho(1-\alpha_L)p_L.$$

Substituting the expressions for a_H and a_L yields

$$\begin{cases} q_H = \delta_H + \gamma \bar{s}_H + \dfrac{T_H}{D}\Big[\mu(1-\tau) + \beta\nu q_H - \tau\beta\nu q_L - \beta p_H + \tau\beta p_L\Big] - b(q_H + q_L), \\ q_L = \delta_L + \gamma \bar{s}_L + \dfrac{T_L}{D}\Big[\mu(1-\tau) + (1+\tau)\beta\nu q_L - 2\tau\beta\nu q_H + 2\tau\beta p_H - (1+\tau)\beta p_L\Big] - 2bq_H. \end{cases}$$

Rearranging, we obtain

$$\begin{cases} \Big[1 + b - \dfrac{\beta\nu T_H}{D}\Big]q_H + \Big[b + \dfrac{\tau\beta\nu T_H}{D}\Big]q_L = \delta_H + \gamma \bar{s}_H + \dfrac{T_H}{D}\Big[\mu(1-\tau) - \beta p_H + \tau\beta p_L\Big], \\ \Big[2b + \dfrac{2\tau\beta\nu T_L}{D}\Big]q_H + \Big[1 - \dfrac{(1+\tau)\beta\nu T_L}{D}\Big]q_L = \delta_L + \gamma \bar{s}_L + \dfrac{T_L}{D}\Big[\mu(1-\tau) + 2\tau\beta p_H - (1+\tau)\beta p_L\Big]. \end{cases}$$

Define

$$\begin{aligned} A_H &:= 1 + b - \frac{\beta\nu T_H}{D}, \qquad B_H := b + \frac{\tau\beta\nu T_H}{D}, \\ A_L &:= 2b + \frac{2\tau\beta\nu T_L}{D}, \qquad B_L := 1 - \frac{(1+\tau)\beta\nu T_L}{D}, \\ C_H &:= \delta_H + \gamma \bar{s}_H + \frac{T_H}{D}\Big[\mu(1-\tau) - \beta p_H + \tau\beta p_L\Big], \end{aligned}$$

and

$$C_L := \delta_L + \gamma \bar{s}_L + \frac{T_L}{D}\Big[\mu(1-\tau) + 2\tau\beta p_H - (1+\tau)\beta p_L\Big].$$

Then, the subgame equilibrium is characterised by

$$\begin{cases} A_H q_H + B_H q_L = C_H, \\ A_L q_H + B_L q_L = C_L. \end{cases}$$

Its determinant is given by

$$\Delta := A_H B_L - A_L B_H.$$

Whenever $\Delta \neq 0$, the unique subgame equilibrium is

$$q_H = \frac{C_H B_L - B_H C_L}{\Delta}, \qquad q_L = \frac{A_H C_L - A_L C_H}{\Delta}.$$

5.1.2 Comparative static of q_H and q_L with respect to α_H and α_L

Derivatives with respect to α_H. Differentiating the audience system with respect to α_H gives

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial \alpha_H} + B_H \dfrac{\partial q_L}{\partial \alpha_H} = \dfrac{\partial C_H}{\partial \alpha_H} - \dfrac{\partial A_H}{\partial \alpha_H} q_H - \dfrac{\partial B_H}{\partial \alpha_H} q_L, \\ A_L \dfrac{\partial q_H}{\partial \alpha_H} + B_L \dfrac{\partial q_L}{\partial \alpha_H} = 0. \end{cases}$$

Since

$$\frac{\partial T_H}{\partial \alpha_H} = -\gamma \rho p_H,$$

we obtain

$$\frac{\partial A_H}{\partial \alpha_H} = \frac{\beta \nu \gamma \rho p_H}{D}, \qquad \frac{\partial B_H}{\partial \alpha_H} = -\frac{\tau \beta \nu \gamma \rho p_H}{D}$$

and

$$\frac{\partial C_H}{\partial \alpha_H} = -\frac{\gamma \rho p_H}{D} \Big[\mu(1-\tau) - \beta p_H + \tau \beta p_L \Big].$$

Define

$$K_{31} := \frac{\partial C_H}{\partial \alpha_H} - \frac{\partial A_H}{\partial \alpha_H} q_H - \frac{\partial B_H}{\partial \alpha_H} q_L.$$

Then, the previous system becomes:

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial \alpha_H} + B_H \dfrac{\partial q_L}{\partial \alpha_H} = K_{31}, \\ A_L \dfrac{\partial q_H}{\partial \alpha_H} + B_L \dfrac{\partial q_L}{\partial \alpha_H} = 0. \end{cases}$$

Applying Cramer's rule, we obtain:

$$\frac{\partial q_H}{\partial \alpha_H} = \frac{K_{31} B_L}{\Delta}, \qquad \frac{\partial q_L}{\partial \alpha_H} = -\frac{A_L K_{31}}{\Delta}.$$

Derivatives with respect to α_L. Differentiating the audience system with respect to α_L gives

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial \alpha_L} + B_H \dfrac{\partial q_L}{\partial \alpha_L} = 0, \\ A_L \dfrac{\partial q_H}{\partial \alpha_L} + B_L \dfrac{\partial q_L}{\partial \alpha_L} = \dfrac{\partial C_L}{\partial \alpha_L} - \dfrac{\partial A_L}{\partial \alpha_L} q_H - \dfrac{\partial B_L}{\partial \alpha_L} q_L. \end{cases}$$

Since

$$\frac{\partial T_L}{\partial \alpha_L} = -\gamma \rho p_L,$$

we obtain

$$\frac{\partial A_L}{\partial \alpha_L} = -\frac{2\tau \beta \nu \gamma \rho p_L}{D},$$

$$\frac{\partial B_L}{\partial \alpha_L} = \frac{(1+\tau) \beta \nu \gamma \rho p_L}{D},$$

and

$$\frac{\partial C_L}{\partial \alpha_L} = -\frac{\gamma \rho p_L}{D}\Big[\mu(1-\tau) + 2\tau\beta p_H - (1+\tau)\beta p_L\Big].$$

Define

$$K_{32} := \frac{\partial C_L}{\partial \alpha_L} - \frac{\partial A_L}{\partial \alpha_L} q_H - \frac{\partial B_L}{\partial \alpha_L} q_L.$$

Then, the previous system becomes:

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial \alpha_L} + B_H \dfrac{\partial q_L}{\partial \alpha_L} = 0, \\ A_L \dfrac{\partial q_H}{\partial \alpha_L} + B_L \dfrac{\partial q_L}{\partial \alpha_L} = K_{32}. \end{cases}$$

Applying Cramer's rule yields:

$$\frac{\partial q_H}{\partial \alpha_L} = -\frac{B_H K_{32}}{\Delta}, \qquad \frac{\partial q_L}{\partial \alpha_L} = \frac{A_H K_{32}}{\Delta}.$$

Variation of the audiences with respect to α_H and α_L The terms K_{31} and K_{32} admit a particularly simple expression. Recall that

$$K_{31} = \frac{\partial C_H}{\partial \alpha_H} - \frac{\partial A_H}{\partial \alpha_H} q_H - \frac{\partial B_H}{\partial \alpha_H} q_L.$$

Using the previous comparative static on $(A_H, B_H, C_H, A_L, B_L, C_L)$ with respect to α_H and α_L we obtain:

$$K_{31} = -\frac{\gamma \rho p_H}{D}\Big[\mu(1-\tau) - \beta p_H + \tau\beta p_L + \beta\nu(q_H - \tau q_L)\Big].$$

The term in brackets is precisely $D a_H$. Hence,

$$K_{31} = -\gamma \rho p_H a_H < 0.$$

Exactly the same reasoning applies to K_{32}. Recall that

$$K_{32} = \frac{\partial C_L}{\partial \alpha_L} - \frac{\partial A_L}{\partial \alpha_L} q_H - \frac{\partial B_L}{\partial \alpha_L} q_L.$$

we obtain:

$$K_{32} = -\gamma \rho p_L a_L < 0.$$

Substituting into the previous expressions gives

$$\frac{\partial q_H}{\partial \alpha_H} = -\frac{\gamma \rho p_H a_H B_L}{\Delta} < 0, \qquad \frac{\partial q_L}{\partial \alpha_H} = \frac{\gamma \rho p_H a_H A_L}{\Delta} > 0,$$

and

$$\frac{\partial q_H}{\partial \alpha_L} = \frac{\gamma \rho p_L a_L B_H}{\Delta} > 0, \qquad \frac{\partial q_L}{\partial \alpha_L} = -\frac{\gamma \rho p_L a_L A_H}{\Delta} < 0.$$

5.1.3 Comparative static of q_H and q_L with respect to p_H and p_L

Derivatives with respect to p_H. Differentiating the audience system with respect to p_H gives

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial p_H} + B_H \dfrac{\partial q_L}{\partial p_H} = \dfrac{\partial C_H}{\partial p_H} - \dfrac{\partial A_H}{\partial p_H} q_H - \dfrac{\partial B_H}{\partial p_H} q_L, \\ A_L \dfrac{\partial q_H}{\partial p_H} + B_L \dfrac{\partial q_L}{\partial p_H} = \dfrac{\partial C_L}{\partial p_H} - \dfrac{\partial A_L}{\partial p_H} q_H - \dfrac{\partial B_L}{\partial p_H} q_L. \end{cases}$$

Since T_L does not depend on p_H, one has:

$$\frac{\partial A_L}{\partial p_H} = 0, \qquad \frac{\partial B_L}{\partial p_H} = 0, \qquad \frac{\partial C_L}{\partial p_H} = \frac{2\sigma T_L}{D}.$$

Differentiating the different terms with respect to p_H gives:

$$\frac{\partial T_H}{\partial p_H} = \gamma\rho(1-\alpha_H),$$

so we have:

$$\frac{\partial A_H}{\partial p_H} = -\frac{\beta\nu\gamma\rho(1-\alpha_H)}{D}, \qquad \frac{\partial B_H}{\partial p_H} = \frac{\tau\beta\nu\gamma\rho(1-\alpha_H)}{D},$$

$$\frac{\partial C_H}{\partial p_H} = \frac{1}{D}\left[\gamma\rho(1-\alpha_H)\Big(\mu(1-\tau) - \beta p_H + \tau\beta p_L\Big) - \beta T_H\right],$$

and

$$\frac{\partial C_L}{\partial p_H} = \frac{2\tau\beta T_L}{D}.$$

Define

$$K_{11} := \frac{\partial C_H}{\partial p_H} - \frac{\partial A_H}{\partial p_H} q_H - \frac{\partial B_H}{\partial p_H} q_L, \qquad K_{21} := \frac{\partial C_L}{\partial p_H} = \frac{2\tau\beta T_L}{D}.$$

Then, the previous system becomes:

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial p_H} + B_H \dfrac{\partial q_L}{\partial p_H} = K_{11}, \\ A_L \dfrac{\partial q_H}{\partial p_H} + B_L \dfrac{\partial q_L}{\partial p_H} = K_{21}. \end{cases}$$

Applying Cramer's rule, we obtain:

$$\frac{\partial q_H}{\partial p_H} = \frac{K_{11}B_L - B_H K_{21}}{\Delta}, \qquad \frac{\partial q_L}{\partial p_H} = \frac{A_H K_{21} - A_L K_{11}}{\Delta}.$$

Derivatives with respect to p_L. Differentiating the audience system with respect to p_L gives

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial p_L} + B_H \dfrac{\partial q_L}{\partial p_L} = \dfrac{\partial C_H}{\partial p_L}, \\ A_L \dfrac{\partial q_H}{\partial p_L} + B_L \dfrac{\partial q_L}{\partial p_L} = \dfrac{\partial C_L}{\partial p_L} - \dfrac{\partial A_L}{\partial p_L} q_H - \dfrac{\partial B_L}{\partial p_L} q_L, \end{cases}$$

because T_H does not depend on p_L. Differentiating various terms with respect to p_L give:

$$\frac{\partial C_H}{\partial p_L} = \frac{\tau\beta T_H}{D}, \qquad \frac{\partial C_L}{\partial p_L} = \frac{1}{D}\left[\gamma\rho(1-\alpha_L)\Big(\mu(1-\tau) + 2\tau\beta p_H - (1+\tau)\beta p_L\Big) - (1+\tau)\beta T_L\right],$$

$$\frac{\partial A_L}{\partial p_L} = \frac{2\tau\beta\nu\gamma\rho(1-\alpha_L)}{D}, \qquad \frac{\partial B_L}{\partial p_L} = -\frac{(1+\tau)\beta\nu\gamma\rho(1-\alpha_L)}{D}.$$

Let us define

$$K_{12} := \frac{\tau\beta T_H}{D}, \qquad K_{22} := \frac{\partial C_L}{\partial p_L} - \frac{\partial A_L}{\partial p_L}q_H - \frac{\partial B_L}{\partial p_L}q_L.$$

Then, the previous system can be rewritten:

$$\begin{cases} A_H \dfrac{\partial q_H}{\partial p_L} + B_H \dfrac{\partial q_L}{\partial p_L} = K_{12}, \\ A_L \dfrac{\partial q_H}{\partial p_L} + B_L \dfrac{\partial q_L}{\partial p_L} = K_{22}. \end{cases}$$

Applying Cramer's rule yields:

$$\frac{\partial q_H}{\partial p_L} = \frac{K_{12}B_L - B_H K_{22}}{\Delta}, \qquad \frac{\partial q_L}{\partial p_L} = \frac{A_H K_{22} - A_L K_{12}}{\Delta}.$$

Sign of the equilibrium audience responses to direct prices The previous expressions imply

$$\frac{\partial q_H}{\partial p_H} = \frac{K_{11}B_L - B_H K_{21}}{\Delta}, \qquad \frac{\partial q_L}{\partial p_L} = \frac{A_H K_{22} - A_L K_{12}}{\Delta}.$$

Hence, we have:

$$\frac{\partial q_H}{\partial p_H} = \frac{B_L K_{11} - B_H \frac{2\sigma T_L}{D}}{\Delta},$$

and

$$\frac{\partial q_L}{\partial p_L} = \frac{A_H K_{22} - A_L \frac{\sigma T_H}{D}}{\Delta}.$$

Sign of the cross-price audience responses The previous derivations imply

$$\frac{\partial q_H}{\partial p_L} = \frac{K_{12}B_L - B_H K_{22}}{\Delta}, \qquad \frac{\partial q_L}{\partial p_H} = \frac{A_H K_{21} - A_L K_{11}}{\Delta}.$$

or

$$\frac{\partial q_H}{\partial p_L} = \frac{\frac{\tau T_H}{D} B_L - B_H K_{22}}{\Delta},$$

and

$$\frac{\partial q_L}{\partial p_H} = \frac{\frac{2\tau T_L}{D} A_H - A_L K_{11}}{\Delta}.$$

5.2 Proof of Proposition 1

Substituting the comparative static expressions of q_H and q_L with respect to α_H and α_L into the first-order condition with respect to these payout rates gives:

$$R_H - \alpha_H p_H \frac{\nu\gamma\rho p_H a_H (B_L + \tau A_L)}{D\Delta} = 0,$$

that becomes

$$\alpha_H = \frac{D\Delta}{\nu\gamma\rho p_H (B_L + \tau A_L)}.$$

Similarly, substituting these comparative static expressions into the first-order condition in α_L for the small channel gives:

$$R_L - \alpha_L p_L \frac{\nu\gamma\rho p_L a_L \big((1+\tau)A_H + 2\tau B_H\big)}{D\Delta} = 0,$$

that is,

$$\alpha_L = \frac{D\Delta}{\nu\gamma\rho p_L \big((1+\tau)A_H + 2\tau B_H\big)}.$$

Simplifying these expressions, we obtain:

$$B_L + \tau A_L = 1 + 2\tau b - \beta\nu T_L,$$

and

$$(1+\tau)A_H + 2\tau B_H = 1 + \tau + b + 3\tau b - \beta\nu T_H.$$

Define

$$k_H := 1 + 2\tau b, \qquad k_L := 1 + \tau + b + 3\tau b.$$

Then

$$\alpha_H = \frac{D\Delta}{\nu\gamma\rho p_H (k_H - \beta\nu T_L)}, \qquad \alpha_L = \frac{D\Delta}{\nu\gamma\rho p_L (k_L - \beta\nu T_H)}.$$

Next, using

$$\Delta = A_H B_L - A_L B_H,$$

one obtains after simplification

$$D\Delta = M_0 - \beta\nu k_H T_H - \beta\nu k_L T_L + \beta^2\nu^2 T_H T_L,$$

where

$$M_0 := D(1 + b - 2b^2).$$

Substituting this expression into the formula for α_H gives

$$\alpha_H \nu\gamma\rho p_H (k_H - \beta\nu T_L) = M_0 - \beta\nu k_H T_H - \beta\nu k_L T_L + \beta^2\nu^2 T_H T_L.$$

Using

$$T_H = \eta + \gamma\rho(1 - \alpha_H)p_H = \eta + \gamma\rho p_H - \gamma\rho p_H \alpha_H,$$

we can rewrite this equality as

$$\alpha_H \nu\gamma\rho p_H(k_H - \beta\nu T_L) = M_0 - \beta\nu k_H(\eta + \gamma\rho p_H - \gamma\rho p_H\alpha_H) - \beta\nu k_L T_L + \beta^2\nu^2(\eta + \gamma\rho p_H - \gamma\rho p_H\alpha_H)T_L,$$

that yields to

$$T_L = \frac{\beta\nu k_H(\eta + \gamma\rho p_H) - M_0}{\beta\nu\big(\beta\nu(\eta + \gamma\rho p_H) - k_L\big)}.$$

Using the definition of T_L, we obtain:

$$\alpha_L = 1 - \frac{T_L - \eta}{\gamma\rho p_L},$$

which yields

$$\alpha_L^* = 1 - \frac{1}{\gamma\rho p_L}\left[\frac{\beta\nu k_H(\eta + \gamma\rho p_H) - M_0}{\beta\nu\big(\beta\nu(\eta + \gamma\rho p_H) - k_L\big)} - \eta\right].$$

Proceeding symmetrically for α_L, we start from

$$\alpha_L \nu\gamma\rho p_L(k_L - \beta\nu T_H) = M_0 - \beta\nu k_H T_H - \beta\nu k_L T_L + \beta^2\nu^2 T_H T_L,$$

and substitute

$$T_L = \eta + \gamma\rho(1 - \alpha_L)p_L = \eta + \gamma\rho p_L - \gamma\rho p_L\alpha_L.$$

Again, the terms in α_L cancel out, leaving

$$M_0 - \beta\nu k_H T_H - \beta\nu k_L(\eta + \gamma\rho p_L) + \beta^2\nu^2 T_H(\eta + \gamma\rho p_L) = 0.$$

Hence, we obtain:

$$T_H = \frac{\beta\nu k_L(\eta + \gamma\rho p_L) - M_0}{\beta\nu\big(\beta\nu(\eta + \gamma\rho p_L) - k_H\big)}.$$

Yet, using the definition of T_H, we have:

$$\alpha_H = 1 - \frac{T_H - \eta}{\gamma\rho p_H},$$

which yields

$$\alpha_H^* = 1 - \frac{1}{\gamma\rho p_H}\left[\frac{\beta\nu k_L(\eta + \gamma\rho p_L) - M_0}{\beta\nu\big(\beta\nu(\eta + \gamma\rho p_L) - k_H\big)} - \eta\right].$$

Therefore, conditional on prices (p_H, p_L), the equilibrium payout rates admit the closed-form expressions given in Proposition 1. This concludes the proof. ■

5.3 Proof of Proposition 4

Large channel. Using the expression of a_H and differentiating with respect to p_H gives:

$$\frac{\partial a_H}{\partial p_H} = -\frac{\beta}{D} + \frac{\beta\nu}{D}\frac{\partial q_H}{\partial p_H} - \frac{\tau\beta\nu}{D}\frac{\partial q_L}{\partial p_H}.$$

Therefore, the first-order condition becomes

$$a_H + p_H \left[-\frac{\beta}{D} + \frac{\beta\nu}{D}\frac{\partial q_H}{\partial p_H} - \frac{\tau\beta\nu}{D}\frac{\partial q_L}{\partial p_H} \right] = 0.$$

Using the comparative statics obtained in the subgame, we obtain

$$a_H + p_H \left[-\frac{\beta}{D} + \frac{\beta\nu}{D}\frac{K_{11}B_L - B_H K_{21}}{\Delta} - \frac{\tau\beta\nu}{D}\frac{A_H K_{21} - A_L K_{11}}{\Delta} \right] = 0.$$

Collecting terms in K_{11} and K_{21} gives

$$a_H + p_H \left[-\frac{\beta}{D} + \frac{\beta\nu}{D\Delta}\Big(K_{11}(B_L + \tau A_L) - K_{21}(B_H + \tau A_H) \Big) \right] = 0.$$

This is equation (1).

Small channel. Recall that Similarly, differentiating a_L with respect to p_L yields

$$\frac{\partial a_L}{\partial p_L} = -\frac{(1+\tau)\beta}{D} + \frac{(1+\tau)\beta\nu}{D}\frac{\partial q_L}{\partial p_L} - \frac{2\tau\beta\nu}{D}\frac{\partial q_H}{\partial p_L}.$$

Hence, the first-order condition becomes

$$a_L + p_L \left[-\frac{(1+\tau)\beta}{D} + \frac{(1+\tau)\beta\nu}{D}\frac{\partial q_L}{\partial p_L} - \frac{2\tau\beta\nu}{D}\frac{\partial q_H}{\partial p_L} \right] = 0.$$

Using the comparative static of audiences variables, we obtain:

$$a_L + p_L \left[-\frac{(1+\tau)\beta}{D} + \frac{(1+\tau)\beta\nu}{D}\frac{A_H K_{22} - A_L K_{12}}{\Delta} - \frac{2\tau\beta\nu}{D}\frac{K_{12}B_L - B_H K_{22}}{\Delta} \right] = 0.$$

Collecting terms in K_{22} and K_{12} gives

$$a_L + p_L \left[-\frac{(1+\tau)\beta}{D} + \frac{\beta\nu}{D\Delta}\Big(K_{22}\big((1+\tau)A_H + 2\tau B_H\big) - K_{12}\big((1+\tau)A_L + 2\tau B_L\big) \Big) \right] = 0.$$

This is equation (2). ■

1. Numerical simulation of the equilibrium

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import least_squares

# ====================================================================
# 1. PARAMETERS
# ====================================================================

mu = 1.4065982215119535
beta = 2.330775345945348
nu = 0.7069620186197285

eta = -0.01472736692983806
gamma = 0.5839506641806507
rho = 0.13420306070568444
b = 0.19240990930474677

delta_H = 2.0806931662975083
delta_L = 0.8389340442014385
sbar_H = 0.8712669740748864
sbar_L = 0.32640562471907925

tau_min = 0.01
tau_max = 0.43
n_tau = 90

h_fd = 1e-6
aL_tol = 1e-4
```

```
# price bounds
p_lower = 1e-4
p_upper = 10.0

# alpha bounds for clipping
alpha_eps = 1e-4



# ============================================================
# 2. AUXILIARY FUNCTIONS
# ============================================================

def D_of_tau(tau):
    return (1.0 - tau) * (1.0 + 2.0 * tau)

def M0_of_tau(tau):
    return D_of_tau(tau) * (1.0 + b - 2.0 * b**2)

def kH_of_tau(tau):
    return 1.0 + 2.0 * tau * b

def kL_of_tau(tau):
    return 1.0 + tau + b + 3.0 * tau * b



# ============================================================
# 3. UNCONSTRAINED ALPHAS FROM PROPOSITION 1
#    THEN CLIP TO [alpha_eps, 1-alpha_eps]
# ============================================================

def alpha_unconstrained_from_prices(pH, pL, tau):
```

```
    if pH <= 0 or pL <= 0:
        return None

    kH = kH_of_tau(tau)
    kL = kL_of_tau(tau)
    M0 = M0_of_tau(tau)

    XH = eta + gamma * rho * pH
    XL = eta + gamma * rho * pL

    den_H = beta * nu * XL - kH
    den_L = beta * nu * XH - kL

    if abs(den_H) < 1e-10 or abs(den_L) < 1e-10:
        return None

    term_H = (beta * nu * kL * XL - M0) / den_H
    term_L = (beta * nu * kH * XH - M0) / den_L

    alpha_H_u = 1.0 - (term_H - eta) / (gamma * rho * pH)
    alpha_L_u = 1.0 - (term_L - eta) / (gamma * rho * pL)

    if not np.isfinite(alpha_H_u) or not np.isfinite(alpha_L_u):
        return None

    return alpha_H_u, alpha_L_u


def alpha_bounded_from_prices(pH, pL, tau):
    vals = alpha_unconstrained_from_prices(pH, pL, tau)
    if vals is None:
```

```
        return None

    alpha_H_u, alpha_L_u = vals

    alpha_H = np.clip(alpha_H_u, alpha_eps, 1.0 - alpha_eps)
    alpha_L = np.clip(alpha_L_u, alpha_eps, 1.0 - alpha_eps)

    return alpha_H, alpha_L, alpha_H_u, alpha_L_u



# ============================================================
# 4. SUBGAME WITH BOUNDED ALPHAS
# ============================================================

def subgame_quantities(pH, pL, tau):
    if pH <= 0 or pL <= 0 or tau <= 0 or tau >= 0.5:
        return None

    alpha_vals = alpha_bounded_from_prices(pH, pL, tau)
    if alpha_vals is None:
        return None

    alpha_H, alpha_L, alpha_H_u, alpha_L_u = alpha_vals

    D = D_of_tau(tau)
    if D <= 0:
        return None

    TH = eta + gamma * rho * (1.0 - alpha_H) * pH
    TL = eta + gamma * rho * (1.0 - alpha_L) * pL
```

```
    AH = 1.0 + b - (beta * nu * TH) / D
    BH = b + (tau * beta * nu * TH) / D

    AL = 2.0 * b + (2.0 * tau * beta * nu * TL) / D
    BL = 1.0 - ((1.0 + tau) * beta * nu * TL) / D

    CH = delta_H + gamma * sbar_H + (TH / D) * (mu * (1.0 - tau) - beta * pH + tau * beta * pL)
    CL = delta_L + gamma * sbar_L + (TL / D) * (mu * (1.0 - tau) + 2.0 * tau * beta * pH - (1.0 + tau) * beta * pL)

    M = np.array([[AH, BH],
                  [AL, BL]], dtype=float)
    rhs = np.array([CH, CL], dtype=float)

    detM = np.linalg.det(M)
    if abs(detM) < 1e-10:
        return None

    try:
        qH, qL = np.linalg.solve(M, rhs)
    except np.linalg.LinAlgError:
        return None

    aH = (mu * (1.0 - tau) + beta * (nu * qH - pH - tau * (nu * qL - pL))) / D
    aL = (mu * (1.0 - tau) + beta * (1.0 + tau) * (nu * qL - pL) - 2.0 * tau * beta * (nu * qH - pH)) / D

    return {
        "alpha_H": alpha_H,
        "alpha_L": alpha_L,
        "alpha_H_u": alpha_H_u,
        "alpha_L_u": alpha_L_u,
```

```
        "qH": qH,
        "qL": qL,
        "aH": aH,
        "aL": aL,
    }


# ============================================================
# 5. NUMERICAL q-DERIVATIVES
# ============================================================

def numerical_q_derivatives(pH, pL, tau, h=h_fd):
    plus = subgame_quantities(pH + h, pL, tau)
    minus = subgame_quantities(pH - h, pL, tau)
    if plus is None or minus is None:
        return None

    dqH_dpH = (plus["qH"] - minus["qH"]) / (2.0 * h)
    dqL_dpH = (plus["qL"] - minus["qL"]) / (2.0 * h)

    plus = subgame_quantities(pH, pL + h, tau)
    minus = subgame_quantities(pH, pL - h, tau)
    if plus is None or minus is None:
        return None

    dqH_dpL = (plus["qH"] - minus["qH"]) / (2.0 * h)
    dqL_dpL = (plus["qL"] - minus["qL"]) / (2.0 * h)

    return dqH_dpH, dqL_dpH, dqH_dpL, dqL_dpL
```

```
# ============================================================
# 6. PRICE RESIDUALS
# ============================================================

def price_residuals(x, tau):
    pH, pL = x

    sg = subgame_quantities(pH, pL, tau)
    if sg is None:
        return np.array([1e3, 1e3], dtype=float)

    qH, qL = sg["qH"], sg["qL"]
    aH, aL = sg["aH"], sg["aL"]

    if (qH <= 0) or (qL <= 0) or (aH <= 0) or (aL <= 0):
        return np.array([1e3, 1e3], dtype=float)

    derivs = numerical_q_derivatives(pH, pL, tau)
    if derivs is None:
        return np.array([1e3, 1e3], dtype=float)

    dqH_dpH, dqL_dpH, dqH_dpL, dqL_dpL = derivs
    D = D_of_tau(tau)

    daH_dpH = -beta / D + (beta * nu / D) * dqH_dpH - (tau * beta * nu / D) * dqL_dpH
    daL_dpL = -(1.0 + tau) * beta / D + ((1.0 + tau) * beta * nu / D) * dqL_dpL - (2.0 * tau * beta * nu / D) * dqH_dpL

    FH = aH + pH * daH_dpH
    FL = aL + pL * daL_dpL
```

```
    return np.array([FH, FL], dtype=float)



# ============================================================
# 7. SOLVER AT A GIVEN tau
# ============================================================

def solve_equilibrium_at_tau(tau, guesses):
    best = None
    best_cost = np.inf

    for guess in guesses:
        try:
            sol = least_squares(
                lambda x: price_residuals(x, tau),
                x0=np.array(guess, dtype=float),
                bounds=([p_lower, p_lower], [p_upper, p_upper]),
                xtol=1e-10,
                ftol=1e-10,
                gtol=1e-10,
                max_nfev=600
            )
        except Exception:
            continue

        pH, pL = sol.x
        res = price_residuals(sol.x, tau)
        cost = np.linalg.norm(res)

        sg = subgame_quantities(pH, pL, tau)
        if sg is None:
```

```
    continue

alpha_H = sg["alpha_H"]
alpha_L = sg["alpha_L"]
qH, qL = sg["qH"], sg["qL"]
aH, aL = sg["aH"], sg["aL"]

if (qH <= 0) or (qL <= 0) or (aH <= 0) or (aL <= 0):
    continue
if qH <= qL:
    continue

RH = pH * aH
RL = pL * aL
total_R = 2.0 * RH + RL
if total_R <= 0:
    continue

s1 = RH / total_R
s2 = RH / total_R
s3 = RL / total_R

share_large = 2.0 * RH / total_R
HHI_rev = s1**2 + s2**2 + s3**2

if cost < best_cost and cost < 1e-5:
    best_cost = cost
    best = {
        "success": True,
        "tau": tau,
        "pH": pH,
```

```
            "pL": pL,
            "alpha_H": alpha_H,
            "alpha_L": alpha_L,
            "alpha_H_u": sg["alpha_H_u"],
            "alpha_L_u": sg["alpha_L_u"],
            "qH": qH,
            "qL": qL,
            "aH": aH,
            "aL": aL,
            "RH": RH,
            "RL": RL,
            "price_gap": pH - pL,
            "share_large": share_large,
            "HHI_rev": HHI_rev,
            "residual_norm": cost,
        }

    if best is None:
        return {
            "success": False,
            "tau": tau,
            "pH": np.nan,
            "pL": np.nan,
            "alpha_H": np.nan,
            "alpha_L": np.nan,
            "alpha_H_u": np.nan,
            "alpha_L_u": np.nan,
            "qH": np.nan,
            "qL": np.nan,
            "aH": np.nan,
            "aL": np.nan,
```

```
            "RH": np.nan,
            "RL": np.nan,
            "price_gap": np.nan,
            "share_large": np.nan,
            "HHI_rev": np.nan,
            "residual_norm": np.nan,
        }

    return best



# =============================================================
# 8. SOLVE OVER tau GRID
# =============================================================

def solve_over_tau(tau_grid):
    results = []

    guesses = [
        [1.10, 0.36],
        [1.00, 0.30],
        [1.30, 0.40],
        [0.90, 0.25],
        [1.50, 0.50],
    ]

    for tau in tau_grid:
        res = solve_equilibrium_at_tau(tau, guesses)
        results.append(res)

        if res["success"]:
```

```
            pH_star, pL_star = res["pH"], res["pL"]
            guesses = [
                [pH_star, pL_star],
                [1.02 * pH_star, 0.98 * pL_star],
                [0.98 * pH_star, 1.02 * pL_star],
                [min(pH_star + 0.03, p_upper), max(pL_star - 0.01, p_lower)],
                [max(pH_star - 0.03, p_lower), min(pL_star + 0.01, p_upper)],
            ]
        else:
            guesses = [
                [1.10, 0.36],
                [1.00, 0.30],
                [1.30, 0.40],
                [0.90, 0.25],
                [1.50, 0.50],
            ]

    return results


# ============================================================
# 9. RUN
# ============================================================

tau_grid = np.linspace(tau_min, tau_max, n_tau)
results = solve_over_tau(tau_grid)

tau_vals = np.array([r["tau"] for r in results], dtype=float)
price_gap_vals = np.array([r["price_gap"] for r in results], dtype=float)
share_large_vals = np.array([r["share_large"] for r in results], dtype=float)
HHI_rev_vals = np.array([r["HHI_rev"] for r in results], dtype=float)
```

```
aL_vals = np.array([r["aL"] for r in results], dtype=float)

alpha_H_vals = np.array([r["alpha_H"] for r in results], dtype=float)

alpha_L_vals = np.array([r["alpha_L"] for r in results], dtype=float)

alpha_H_u_vals = np.array([r["alpha_H_u"] for r in results], dtype=float)

alpha_L_u_vals = np.array([r["alpha_L_u"] for r in results], dtype=float)

residual_norm_vals = np.array([r["residual_norm"] for r in results], dtype=float)


mask = ~np.isnan(price_gap_vals)


tau_ok = tau_vals[mask]

price_gap_ok = price_gap_vals[mask]

share_large_ok = share_large_vals[mask]

HHI_rev_ok = HHI_rev_vals[mask]

aL_ok = aL_vals[mask]

alpha_H_ok = alpha_H_vals[mask]

alpha_L_ok = alpha_L_vals[mask]

alpha_H_u_ok = alpha_H_u_vals[mask]

alpha_L_u_ok = alpha_L_u_vals[mask]

residual_ok = residual_norm_vals[mask]


tau_hat = np.nan

for t, aL in zip(tau_ok, aL_ok):

    if aL <= aL_tol:

        tau_hat = t

        break


print("Successful equilibria:", np.sum(mask), "/", len(tau_vals))

if len(residual_ok) > 0:

    print("Max residual norm among retained equilibria:", np.max(residual_ok))

if np.isnan(tau_hat):

    print(f"No tau_hat found up to tau = {tau_max:.3f} using tolerance {aL_tol}.")
```

```
else:
    print(f"Estimated tau_hat (first tau with a_L <= {aL_tol}): {tau_hat:.4f}")

# ============================================================
# 10. PLOTS
# ============================================================

plt.figure(figsize=(8, 5))
plt.plot(tau_ok, price_gap_ok, linewidth=2)
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
plt.xlabel(r"$\tau$")
plt.ylabel(r"$p_H^*(\tau)-p_L^*(\tau)$")
plt.title("Equilibrium price gap")
plt.grid(True)
plt.show()

plt.figure(figsize=(8, 5))
plt.plot(tau_ok, share_large_ok, linewidth=2)
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
plt.xlabel(r"$\tau$")
plt.ylabel("Large channels' share of total ad revenues")
plt.title("Share of total ad revenues accruing to large channels")
plt.grid(True)
plt.show()

plt.figure(figsize=(8, 5))
plt.plot(tau_ok, 10000.0 * HHI_rev_ok, linewidth=2)
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
```

```
plt.xlabel(r"$\tau$")
plt.ylabel("HHI (x 10,000)")
plt.title("HHI of advertising revenue concentration")
plt.grid(True)
plt.show()

plt.figure(figsize=(8, 5))
plt.plot(tau_ok, aL_ok, linewidth=2)
plt.axhline(aL_tol, linestyle="--")
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
plt.xlabel(r"$\tau$")
plt.ylabel(r"$a_L^*(\tau)$")
plt.title("Advertising demand of the small channel")
plt.grid(True)
plt.show()

# Optional diagnostic plot
plt.figure(figsize=(8, 5))
plt.plot(tau_ok, alpha_H_ok, label=r"$\alpha_H$ bounded", linewidth=2)
plt.plot(tau_ok, alpha_L_ok, label=r"$\alpha_L$ bounded", linewidth=2)
plt.xlabel(r"$\tau$")
plt.ylabel(r"$\alpha$")
plt.title("Bounded payout rates")
plt.legend()
plt.grid(True)
plt.show()
```

2. Numerical simulations of welfare

```
# ======================================================================
# 1. PARAMETERS
# ======================================================================

mu = 1.4065982215119535
beta = 2.330775345945348
nu = 0.7069620186197285

eta = -0.01472736692983806
gamma = 0.5839506641806507
rho = 0.13420306070568444
b = 0.19240990930474677

delta_H = 2.0806931662975083
delta_L = 0.8389340442014385
sbar_H = 0.8712669740748864
sbar_L = 0.32640562471907925

tau_min = 0.01
tau_max = 0.43
n_tau = 90

h_fd = 1e-6
aL_tol = 1e-4

# price bounds
p_lower = 1e-4
p_upper = 10.0
```

```
# alpha bounds for clipping

alpha_eps = 1e-4



# ============================================================

# 2. AUXILIARY FUNCTIONS

# ============================================================


def D_of_tau(tau):

    return (1.0 - tau) * (1.0 + 2.0 * tau)


def M0_of_tau(tau):

    return D_of_tau(tau) * (1.0 + b - 2.0 * b**2)


def kH_of_tau(tau):

    return 1.0 + 2.0 * tau * b


def kL_of_tau(tau):

    return 1.0 + tau + b + 3.0 * tau * b



# ============================================================

# 3. UNCONSTRAINED ALPHAS FROM PROPOSITION 1

#    THEN CLIP TO [alpha_eps, 1-alpha_eps]

# ============================================================


def alpha_unconstrained_from_prices(pH, pL, tau):

    if pH <= 0 or pL <= 0:

        return None
```

```
    kH = kH_of_tau(tau)
    kL = kL_of_tau(tau)
    M0 = M0_of_tau(tau)

    XH = eta + gamma * rho * pH
    XL = eta + gamma * rho * pL

    den_H = beta * nu * XL - kH
    den_L = beta * nu * XH - kL

    if abs(den_H) < 1e-10 or abs(den_L) < 1e-10:
        return None

    term_H = (beta * nu * kL * XL - M0) / den_H
    term_L = (beta * nu * kH * XH - M0) / den_L

    alpha_H_u = 1.0 - (term_H - eta) / (gamma * rho * pH)
    alpha_L_u = 1.0 - (term_L - eta) / (gamma * rho * pL)

    if not np.isfinite(alpha_H_u) or not np.isfinite(alpha_L_u):
        return None

    return alpha_H_u, alpha_L_u


def alpha_bounded_from_prices(pH, pL, tau):
    vals = alpha_unconstrained_from_prices(pH, pL, tau)
    if vals is None:
        return None

    alpha_H_u, alpha_L_u = vals
```

```
    alpha_H = np.clip(alpha_H_u, alpha_eps, 1.0 - alpha_eps)
    alpha_L = np.clip(alpha_L_u, alpha_eps, 1.0 - alpha_eps)

    return alpha_H, alpha_L, alpha_H_u, alpha_L_u


# ============================================================
# 4. SUBGAME WITH BOUNDED ALPHAS
# ============================================================

def subgame_quantities(pH, pL, tau):
    if pH <= 0 or pL <= 0 or tau <= 0 or tau >= 0.5:
        return None

    alpha_vals = alpha_bounded_from_prices(pH, pL, tau)
    if alpha_vals is None:
        return None

    alpha_H, alpha_L, alpha_H_u, alpha_L_u = alpha_vals

    D = D_of_tau(tau)
    if D <= 0:
        return None

    TH = eta + gamma * rho * (1.0 - alpha_H) * pH
    TL = eta + gamma * rho * (1.0 - alpha_L) * pL

    AH = 1.0 + b - (beta * nu * TH) / D
    BH = b + (tau * beta * nu * TH) / D
```

```
    AL = 2.0 * b + (2.0 * tau * beta * nu * TL) / D
    BL = 1.0 - ((1.0 + tau) * beta * nu * TL) / D

    CH = delta_H + gamma * sbar_H + (TH / D) * (mu * (1.0 - tau) - beta * pH + tau * beta * pL)
    CL = delta_L + gamma * sbar_L + (TL / D) * (mu * (1.0 - tau) + 2.0 * tau * beta * pH - (1.0 + tau) * beta * pL)

    M = np.array([[AH, BH],
                  [AL, BL]], dtype=float)
    rhs = np.array([CH, CL], dtype=float)

    detM = np.linalg.det(M)
    if abs(detM) < 1e-10:
        return None

    try:
        qH, qL = np.linalg.solve(M, rhs)
    except np.linalg.LinAlgError:
        return None

    aH = (mu * (1.0 - tau) + beta * (nu * qH - pH - tau * (nu * qL - pL))) / D
    aL = (mu * (1.0 - tau) + beta * (1.0 + tau) * (nu * qL - pL) - 2.0 * tau * beta * (nu * qH - pH)) / D

    return {
        "alpha_H": alpha_H,
        "alpha_L": alpha_L,
        "alpha_H_u": alpha_H_u,
        "alpha_L_u": alpha_L_u,
        "qH": qH,
        "qL": qL,
        "aH": aH,
```

```
        "aL": aL,
    }



# =============================================================
# 5. NUMERICAL q-DERIVATIVES
# =============================================================

def numerical_q_derivatives(pH, pL, tau, h=h_fd):
    plus = subgame_quantities(pH + h, pL, tau)
    minus = subgame_quantities(pH - h, pL, tau)
    if plus is None or minus is None:
        return None

    dqH_dpH = (plus["qH"] - minus["qH"]) / (2.0 * h)
    dqL_dpH = (plus["qL"] - minus["qL"]) / (2.0 * h)

    plus = subgame_quantities(pH, pL + h, tau)
    minus = subgame_quantities(pH, pL - h, tau)
    if plus is None or minus is None:
        return None

    dqH_dpL = (plus["qH"] - minus["qH"]) / (2.0 * h)
    dqL_dpL = (plus["qL"] - minus["qL"]) / (2.0 * h)

    return dqH_dpH, dqL_dpH, dqH_dpL, dqL_dpL



# =============================================================
# 6. PRICE RESIDUALS
# =============================================================
```

```
def price_residuals(x, tau):
    pH, pL = x

    sg = subgame_quantities(pH, pL, tau)
    if sg is None:
        return np.array([1e3, 1e3], dtype=float)

    qH, qL = sg["qH"], sg["qL"]
    aH, aL = sg["aH"], sg["aL"]

    if (qH <= 0) or (qL <= 0) or (aH <= 0) or (aL <= 0):
        return np.array([1e3, 1e3], dtype=float)

    derivs = numerical_q_derivatives(pH, pL, tau)
    if derivs is None:
        return np.array([1e3, 1e3], dtype=float)

    dqH_dpH, dqL_dpH, dqH_dpL, dqL_dpL = derivs
    D = D_of_tau(tau)

    daH_dpH = -beta / D + (beta * nu / D) * dqH_dpH - (tau * beta * nu / D) * dqL_dpH
    daL_dpL = -(1.0 + tau) * beta / D + ((1.0 + tau) * beta * nu / D) * dqL_dpL - (2.0 * tau * beta * nu / D) * dqH_dpL

    FH = aH + pH * daH_dpH
    FL = aL + pL * daL_dpL

    return np.array([FH, FL], dtype=float)
```

```
# ============================================================
# 7. SOLVER AT A GIVEN tau
# ============================================================

def solve_equilibrium_at_tau(tau, guesses):
    best = None
    best_cost = np.inf

    for guess in guesses:
        try:
            sol = least_squares(
                lambda x: price_residuals(x, tau),
                x0=np.array(guess, dtype=float),
                bounds=([p_lower, p_lower], [p_upper, p_upper]),
                xtol=1e-10,
                ftol=1e-10,
                gtol=1e-10,
                max_nfev=600
            )
        except Exception:
            continue

        pH, pL = sol.x
        res = price_residuals(sol.x, tau)
        cost = np.linalg.norm(res)

        sg = subgame_quantities(pH, pL, tau)
        if sg is None:
            continue

        alpha_H = sg["alpha_H"]
```

```
alpha_L = sg["alpha_L"]
qH, qL = sg["qH"], sg["qL"]
aH, aL = sg["aH"], sg["aL"]

if (qH <= 0) or (qL <= 0) or (aH <= 0) or (aL <= 0):
    continue
if qH <= qL:
    continue

RH = pH * aH
RL = pL * aL
total_R = 2.0 * RH + RL
if total_R <= 0:
    continue

s1 = RH / total_R
s2 = RH / total_R
s3 = RL / total_R

share_large = 2.0 * RH / total_R
HHI_rev = s1**2 + s2**2 + s3**2

if cost < best_cost and cost < 1e-5:
    best_cost = cost
    best = {
        "success": True,
        "tau": tau,
        "pH": pH,
        "pL": pL,
        "alpha_H": alpha_H,
        "alpha_L": alpha_L,
```

```
            "alpha_H_u": sg["alpha_H_u"],
            "alpha_L_u": sg["alpha_L_u"],
            "qH": qH,
            "qL": qL,
            "aH": aH,
            "aL": aL,
            "RH": RH,
            "RL": RL,
            "price_gap": pH - pL,
            "share_large": share_large,
            "HHI_rev": HHI_rev,
            "residual_norm": cost,
        }

    if best is None:
        return {
            "success": False,
            "tau": tau,
            "pH": np.nan,
            "pL": np.nan,
            "alpha_H": np.nan,
            "alpha_L": np.nan,
            "alpha_H_u": np.nan,
            "alpha_L_u": np.nan,
            "qH": np.nan,
            "qL": np.nan,
            "aH": np.nan,
            "aL": np.nan,
            "RH": np.nan,
            "RL": np.nan,
            "price_gap": np.nan,
```

```
            "share_large": np.nan,
            "HHI_rev": np.nan,
            "residual_norm": np.nan,
        }

    return best



# ============================================================
# 8. SOLVE OVER tau GRID
# ============================================================

def solve_over_tau(tau_grid):
    results = []

    guesses = [
        [1.10, 0.36],
        [1.00, 0.30],
        [1.30, 0.40],
        [0.90, 0.25],
        [1.50, 0.50],
    ]

    for tau in tau_grid:
        res = solve_equilibrium_at_tau(tau, guesses)
        results.append(res)

        if res["success"]:
            pH_star, pL_star = res["pH"], res["pL"]
            guesses = [
                [pH_star, pL_star],
```

```
            [1.02 * pH_star, 0.98 * pL_star],
            [0.98 * pH_star, 1.02 * pL_star],
            [min(pH_star + 0.03, p_upper), max(pL_star - 0.01, p_lower)],
            [max(pH_star - 0.03, p_lower), min(pL_star + 0.01, p_upper)],
          ]
        else:
          guesses = [
            [1.10, 0.36],
            [1.00, 0.30],
            [1.30, 0.40],
            [0.90, 0.25],
            [1.50, 0.50],
          ]

  return results



# ============================================================
# 9. RUN
# ============================================================

tau_grid = np.linspace(tau_min, tau_max, n_tau)
results = solve_over_tau(tau_grid)

tau_vals = np.array([r["tau"] for r in results], dtype=float)
price_gap_vals = np.array([r["price_gap"] for r in results], dtype=float)
share_large_vals = np.array([r["share_large"] for r in results], dtype=float)
HHI_rev_vals = np.array([r["HHI_rev"] for r in results], dtype=float)
aL_vals = np.array([r["aL"] for r in results], dtype=float)
alpha_H_vals = np.array([r["alpha_H"] for r in results], dtype=float)
alpha_L_vals = np.array([r["alpha_L"] for r in results], dtype=float)
```

```
alpha_H_u_vals = np.array([r["alpha_H_u"] for r in results], dtype=float)
alpha_L_u_vals = np.array([r["alpha_L_u"] for r in results], dtype=float)
residual_norm_vals = np.array([r["residual_norm"] for r in results], dtype=float)

mask = ~np.isnan(price_gap_vals)

tau_ok = tau_vals[mask]
price_gap_ok = price_gap_vals[mask]
share_large_ok = share_large_vals[mask]
HHI_rev_ok = HHI_rev_vals[mask]
aL_ok = aL_vals[mask]
alpha_H_ok = alpha_H_vals[mask]
alpha_L_ok = alpha_L_vals[mask]
alpha_H_u_ok = alpha_H_u_vals[mask]
alpha_L_u_ok = alpha_L_u_vals[mask]
residual_ok = residual_norm_vals[mask]

tau_hat = np.nan
for t, aL in zip(tau_ok, aL_ok):
  if aL <= aL_tol:
    tau_hat = t
    break

print("Successful equilibria:", np.sum(mask), "/", len(tau_vals))
if len(residual_ok) > 0:
  print("Max residual norm among retained equilibria:", np.max(residual_ok))
if np.isnan(tau_hat):
  print(f"No tau_hat found up to tau = {tau_max:.3f} using tolerance {aL_tol}.")
else:
  print(f"Estimated tau_hat (first tau with a_L <= {aL_tol}): {tau_hat:.4f}")
```

```
# ==============================================================
# 10. PLOTS
# ==============================================================

plt.figure(figsize=(8, 5))
plt.plot(tau_ok, price_gap_ok, linewidth=2)
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
plt.xlabel(r"$\tau$")
plt.ylabel(r"$p_H^*(\tau)-p_L^*(\tau)$")
plt.title("Equilibrium price gap")
plt.grid(True)
plt.show()

plt.figure(figsize=(8, 5))
plt.plot(tau_ok, share_large_ok, linewidth=2)
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
plt.xlabel(r"$\tau$")
plt.ylabel("Large channels' share of total ad revenues")
plt.title("Share of total ad revenues accruing to large channels")
plt.grid(True)
plt.show()

plt.figure(figsize=(8, 5))
plt.plot(tau_ok, 10000.0 * HHI_rev_ok, linewidth=2)
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
plt.xlabel(r"$\tau$")
plt.ylabel("HHI (x 10,000)")
plt.title("HHI of advertising revenue concentration")
```

```
plt.grid(True)
plt.show()


plt.figure(figsize=(8, 5))
plt.plot(tau_ok, aL_ok, linewidth=2)
plt.axhline(aL_tol, linestyle="--")
if not np.isnan(tau_hat):
    plt.axvline(tau_hat, linestyle="--")
plt.xlabel(r"$\tau$")
plt.ylabel(r"$a_L^*(\tau)$")
plt.title("Advertising demand of the small channel")
plt.grid(True)
plt.show()


# Optional diagnostic plot
plt.figure(figsize=(8, 5))
plt.plot(tau_ok, alpha_H_ok, label=r"$\alpha_H$ bounded", linewidth=2)
plt.plot(tau_ok, alpha_L_ok, label=r"$\alpha_L$ bounded", linewidth=2)
plt.xlabel(r"$\tau$")
plt.ylabel(r"$\alpha$")
plt.title("Bounded payout rates")
plt.legend()
plt.grid(True)
plt.show()
```