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Stochastic Burgers Equation Driven by a Hermite Sheet with Additive Noise: Existence, Uniqueness, and Regularity

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Abstract

We study the stochastic Burgers equation driven by a Hermite sheet of order $q \geq 1$ with **additive noise**, establishing the well-posedness of mild solutions via a fixed-point argument in suitable Banach spaces. Under appropriate conditions on the Hurst parameters $\mathbf{H} = (H_0, H_1, \ldots, H_d) \in (1/2, 1)^{d+1}$, we prove existence and uniqueness of solutions through a Picard iteration scheme. The solution exhibits spatial and temporal Hölder regularity, with exponents determined by the Hurst parameters of the driving noise. Furthermore, we demonstrate that the solution inherits the self-similarity property from the Hermite sheet, providing explicit scaling exponents. Uniform moment estimates in space and time are derived, forming the foundation for the regularity analysis. The additive noise formulation allows us to use the standard Wiener integral construction for Hermite processes, thereby avoiding the technical complications of Malliavin calculus required for multiplicative noise. This restriction is mathematically justified as it circumvents the need for Malliavin derivative bounds essential for random integrands with Hermite processes of order $q \geq 2$, a key difficulty highlighted in recent literature. The work develops the stochastic integration theory with respect to Hermite sheets for deterministic integrands and establishes a complete framework for analyzing nonlinear SPDEs with non-Gaussian noise, contributing to the understanding of stochastic systems with long-range dependence and non-Gaussian fluctuations.

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1 Introduction

The stochastic Burgers equation is a fundamental model in mathematical physics, lying at the crossroads of fluid dynamics, turbulence theory, and stochastic growth phenomena. While the deterministic Burgers equation describes viscous fluid motion with nonlinear transport, its stochastic counterpart incorporates random forcing and provides insight into phenomena such as turbulent energy transfer and interface growth. As a prototypical nonlinear stochastic partial differential equation (SPDE), the Burgers equation offers a mathematically tractable yet physically meaningful framework for studying the interaction between nonlinearity and randomness.

The theory of SPDEs driven by Gaussian noise has undergone significant development over the past decades. Foundational results were established by Da Prato and Zabczyk [1], while recent breakthroughs in the analysis of singular SPDEs were achieved through the theory of regularity structures introduced by Hairer [2, 3]. Within this context, the stochastic Burgers equation occupies a prominent position, notably through its connection to the Kardar–Parisi–Zhang (KPZ) universality class [4], and as a testing ground for novel analytical techniques.

In contrast, SPDEs driven by non-Gaussian noise remain far less explored. Hermite processes provide a natural framework for such investigations, extending fractional Brownian motion to non-Gaussian regimes of order $q \geq 2$ while preserving key properties such as self-similarity, stationary increments, and long-range dependence [6, 7]. These processes are defined via multiple Wiener–Itô integrals and exhibit probabilistic features that markedly differ from the Gaussian case. While linear SPDEs driven by Hermite noise have been studied in recent years [8, 9], the nonlinear setting—and in particular equations with quadratic nonlinearities such as the Burgers equation—poses substantial analytical challenges due to the combined effects of nonlinearity and non-Gaussianity.

The present work contributes to this line of research by studying the stochastic Burgers equation driven by a $(d+1)$-parameter Hermite sheet $Z^{q,\mathbf{H}}$ of order $q \geq 1$, under the assumption of *additive noise*. More precisely, we consider

$$\partial_t u(t,\mathbf{x}) = \nu\Delta u(t,\mathbf{x}) - \frac{1}{2}\nabla\cdot\left(u(t,\mathbf{x})^2\right) + \sigma(t,\mathbf{x})\,\frac{\partial^{d+1} Z^{q,\mathbf{H}}}{\partial t\,\partial x_1\cdots\partial x_d}, \tag{1}$$

where $\mathbf{H} = (H_0, H_1, \dots, H_d) \in (1/2, 1)^{d+1}$ denotes the vector of Hurst parameters governing the temporal and spatial regularity of the noise, and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a deterministic coefficient function. The nonlinear term is written in conservative form $\nabla \cdot (u^2) = \sum_{i=1}^d \partial_{x_i}(u^2)$, which naturally extends the one-dimensional Burgers nonlinearity $\partial_x(u^2)$ to higher spatial dimensions and is compatible with the mild formulation used throughout the paper.

1.1 Novelty and Challenges

The main novelty of this work lies in extending the well-posedness and regularity theory for the stochastic Burgers equation to the case of non-Gaussian Hermite sheet noise of arbitrary order $q \geq 1$. While the additive noise case might appear as a simplification, it presents several non-trivial challenges distinct from the Gaussian setting ($q = 1$):

1. **Stochastic integration:** Stochastic integration with respect to Hermite sheets for deterministic integrands requires a careful analysis of the covariance structure and the associated Hilbert spaces.

2. **Condition on Hurst parameters:** The condition for the well-posedness of the stochastic convolution,
$$2H_0 + \sum_{i=1}^d H_i > d + 1 - \frac{1}{q},$$
becomes more restrictive as q increases, reflecting the increasing singularity of the kernel representation of Hermite processes. This condition is *sufficient* for our approach and is not claimed to be optimal in the nonlinear setting. Improving this condition would likely require different techniques such as renormalization or Malliavin-type arguments, which are beyond the scope of the present work.

3. **Hypercontractivity:** The hypercontractivity properties of Wiener chaos, essential for moment estimates, differ from the Gaussian case and depend on the order q.

4. **Regularity exponents:** The regularity exponents in the Hölder estimates are limited by the Hurst parameters of the Hermite sheet, and the non-Gaussian nature of the noise influences the scaling properties of the solution.

To our knowledge, this is the first work that establishes a complete well-posedness and regularity theory for the stochastic Burgers equation with additive Hermite sheet noise of order $q \geq 2$ in multiple spatial dimensions.

1.2 Technical Justification for Additive Noise

The restriction to additive noise is not merely a simplification but a mathematical necessity when considering Hermite processes of order $q \geq 2$. For multiplicative noise $\sigma(t, \mathbf{x}, u(t, \mathbf{x}))$, the stochastic integral involves an adapted random integrand, and its construction requires Malliavin calculus with bounds involving Malliavin derivatives, as shown in [10] for Rosenblatt processes ($q = 2$) and in recent work on general Hermite processes [11].

The recent work by [11] explicitly demonstrates that for Hermite processes of order $q \geq 2$, the divergence-type integral with random integrands requires control of both the $\mathcal{H}$-norm of the integrand *and* the $\mathcal{H} \otimes \mathcal{H}$-norm of its Malliavin derivative. Specifically, they establish bounds of the form:
$$\mathbb{E}\left[\left|\int_0^t \Phi(s) dZ_s^{q,H}\right|^2\right] \leq C\left(\mathbb{E}[\|\Phi\|_{\mathcal{H}}^2] + \mathbb{E}[\|D\Phi\|_{\mathcal{H}\otimes\mathcal{H}}^2]\right),$$
where $D\Phi$ denotes the Malliavin derivative. By focusing on the additive case with deterministic $\sigma(t, \mathbf{x})$, we avoid these technical complications while still obtaining a mathematically rigorous

theory that captures essential features of the equation. This approach allows us to use the standard Wiener integral construction for deterministic integrands with respect to Hermite sheets.

The restriction to additive noise is thus essential in the non-Gaussian Hermite setting, as Malliavin-type tools are currently unavailable for handling multiplicative interactions in this context. This strategic choice allows us to establish a solid foundation for future investigations of more complex noise structures.

1.3 Advantages of the Additive Noise Framework

While the additive noise case represents a mathematical simplification, it offers significant advantages that justify its independent study:

1. **Foundation for future work:** The additive case provides essential insights into the interaction between Hermite noise and nonlinear dynamics, serving as a necessary stepping stone toward the multiplicative case.

2. **Rich mathematical structure:** Even with additive noise, the solution inherits complex properties from the Hermite sheet, including non-Gaussianity, long-range dependence, and anisotropic self-similarity.

3. **Physical relevance:** Additive noise models external random forcing independent of the system state, which arises naturally in many physical contexts such as turbulent environments or thermal fluctuations.

4. **Analytical tractability:** The deterministic integrand allows for explicit computations and sharp regularity estimates that reveal how the Hurst parameters and Hermite order influence solution behavior.

1.4 Main Results

Our main contributions can be summarized as follows:

1. **Existence and uniqueness:** Under the condition $2H_0 + \sum_{i=1}^{d} H_i > d + 1 - \frac{1}{q}$, we prove local-in-time existence and uniqueness of mild solutions in suitable Banach spaces via a fixed-point argument (Theorem 4.3). In spatial dimension $d \geq 2$, the result is local in time, while in dimension one the Burgers equation is known to enjoy better well-posedness properties.

2. **Moment estimates:** We derive uniform moment estimates for the solution, which grow at most polynomially in the order of the moment (Proposition 4.5).

3. **Hölder regularity:** We establish spatial and temporal Hölder regularity of the solution, with exponents explicitly given by the Hurst parameters (Theorem 5.2). The exponents are shown to be sharp in the sense that they cannot be improved without additional assumptions on the noise. We also address the appearance of logarithmic terms in spatial regularity estimates, noting that these are standard in Burgers-type equations and do not affect the final Hölder continuity results.

4. **Self-similarity:** Under appropriate scaling of the initial condition and coefficient σ, we prove that the solution inherits the self-similarity property from the driving Hermite sheet (Theorem 5.4). This property is motivated by its importance in the study of scaling limits and universality classes in stochastic PDEs.

1.5 Organization of the Paper

The paper is organized as follows. Section 2 recalls essential background on Hermite processes, multiple Wiener–Itô integrals, and the associated covariance spaces. In Section 3, we introduce the mild formulation of (1) and state the assumptions underlying our analysis. Section 4 is devoted to the proof of local well-posedness via a fixed-point argument and to the derivation of key moment estimates. In Section 5, we investigate the regularity and scaling properties of the solution, including Hölder continuity and conditional self-similarity. Finally, Section 6 provides a summary and discusses perspectives and open problems.

This work advances the theoretical understanding of nonlinear SPDEs driven by non-Gaussian noise and provides analytical tools that may be applicable to a broader class of stochastic systems exhibiting long-range dependence. The additive noise framework studied here constitutes a natural first step toward the more challenging analysis of multiplicative Hermite-driven equations.

2 Preliminaries

This section recalls the main probabilistic tools used throughout the paper, including multiple Wiener–Itô integrals, Wiener chaos, and Hermite sheets. These objects form the foundation of the stochastic integration framework employed in the analysis of the stochastic Burgers equation with additive Hermite noise.

2.1 Multiple Wiener–Itô Integrals and Wiener Chaos

Let

$$W = \{W(A) : A \in \mathcal{B}_b(\mathbb{R}^{d+1})\}$$

be a Gaussian white noise on $\mathbb{R}^{d+1}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{B}_b$ denotes the collection of bounded Borel sets. By definition, W is a centered Gaussian family with covariance

$$\mathbb{E}[W(A)W(B)] = \lambda(A \cap B),$$

where λ denotes the Lebesgue measure.

For an integer $q \geq 1$ and a symmetric function $f \in L^2((\mathbb{R}^{d+1})^q)$, the *multiple Wiener–Itô integral of order* q, denoted by $I_q(f)$, is defined as follows. For elementary symmetric functions of the form

$$f = \sum_{i_1,\dots,i_q} a_{i_1,\dots,i_q} \mathbf{1}_{A_{i_1} \times \cdots \times A_{i_q}},$$

where the sets A_j are disjoint and $a_{i_1,\dots,i_q} = 0$ whenever two indices coincide, one sets

$$I_q(f) = \sum_{i_1,\dots,i_q} a_{i_1,\dots,i_q} W(A_{i_1}) \cdots W(A_{i_q}).$$

This mapping extends uniquely by density and isometry to all symmetric functions in $L^2((\mathbb{R}^{d+1})^q)$.

The following properties are fundamental:

- **Isometry:**
$$\mathbb{E}[I_q(f)I_p(g)] = \delta_{p,q}\, q!\, \langle \tilde{f}, \tilde{g} \rangle_{L^2((\mathbb{R}^{d+1})^q)},$$
where $\tilde{f}$ denotes the symmetrization of f.

- **Orthogonality:** Wiener integrals of different orders are orthogonal in $L^2(\Omega)$.

- **Hypercontractivity:** For any $p \geq 2$, there exists a constant $C_{p,q} > 0$ such that

$$\mathbb{E}[|I_q(f)|^p] \leq C_{p,q}(\mathbb{E}[|I_q(f)|^2])^{p/2}$$

  [12].

The space $L^2(\Omega, \sigma(W), \mathbb{P})$ admits the orthogonal *Wiener chaos decomposition*

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q,$$

where $\mathcal{H}_0 = \mathbb{R}$ and, for $q \geq 1$, $\mathcal{H}_q$ is the closed linear subspace generated by random variables of the form $I_q(f)$ with $f \in L^2((\mathbb{R}^{d+1})^q)$ symmetric. We refer to [12, 13, 14] for further details.

2.2 Hermite Processes and Hermite Sheets

Hermite processes constitute a class of self-similar stochastic processes with stationary increments that generalize fractional Brownian motion beyond the Gaussian framework. They are defined through multiple Wiener–Itô integrals.

Definition 2.1 (Hermite process)**.** *Let $q \geq 1$ and $H \in (1/2, 1)$. The Hermite process $(Z_t^{q,H})_{t\geq 0}$ of order q and Hurst parameter H is defined by*

$$Z_t^{q,H} = c(H,q) \int_{\mathbb{R}^q} \left(\int_0^t \prod_{j=1}^{q} (s - y_j)_+^{-(1/2+(1-H)/q)} \, ds \right) dW(y_1) \cdots dW(y_q),$$

where $c(H,q) > 0$ is a normalizing constant ensuring $\mathbb{E}[(Z_1^{q,H})^2] = 1$.

For $q = 1$, the Hermite process coincides with fractional Brownian motion, while for $q \geq 2$ it is non-Gaussian [5, 7].

This construction extends naturally to multiple parameters.

Definition 2.2 (Hermite sheet)**.** *Let $q \geq 1$, $d \geq 1$, and $\mathbf{H} = (H_0, \ldots, H_d) \in (1/2, 1)^{d+1}$. The $(d+1)$-parameter Hermite sheet $(Z_{\mathbf{t}}^{q,\mathbf{H}})_{\mathbf{t}\in\mathbb{R}_+^{d+1}}$ is defined by*

$$Z_{\mathbf{t}}^{q,\mathbf{H}} = c(\mathbf{H},q) \int_{(\mathbb{R}^{d+1})^q} \left[\int_{[0,\mathbf{t}]} \prod_{j=1}^{q} \prod_{i=0}^{d} (s_i - y_{j,i})_+^{-(1/2+(1-H_i)/q)} \, d\mathbf{s} \right] dW(\mathbf{y}_1) \cdots dW(\mathbf{y}_q), \qquad (2)$$

where $\mathbf{s} = (s_0, \ldots, s_d)$ and $\mathbf{y}_j = (y_{j,0}, \ldots, y_{j,d})$.

The Hermite sheet is a centered random field belonging to the qth Wiener chaos $\mathcal{H}_q$ as a random variable in $L^2(\Omega)$. It is non-Gaussian for $q \geq 2$ and possesses self-similarity and long-range dependence. Its covariance function is given by

$$\mathbb{E}[Z_{\mathbf{t}}^{q,\mathbf{H}} Z_{\mathbf{s}}^{q,\mathbf{H}}] = \prod_{i=0}^{d} \frac{1}{2} \left(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i} \right).$$

2.3 Stochastic Integration with Respect to the Hermite Sheet

We restrict attention to deterministic integrands, which allows the construction of stochastic integrals with respect to the Hermite sheet via multiple Wiener–Itô integrals, without invoking Malliavin calculus.

Definition 2.3 (Covariance Hilbert space)**.** *Let $\mathcal{H}$ denote the Hilbert space obtained as the closure of smooth compactly supported functions on $[0,T]\times\mathbb{R}^d$ with respect to the inner product*

$$\langle \varphi,\psi\rangle_{\mathcal{H}} = \alpha_{\mathbf{H}} \int_{[0,T]^2}\int_{\mathbb{R}^{2d}} \varphi(s,\mathbf{y})\psi(r,\mathbf{z})|s-r|^{2H_0-2}\prod_{i=1}^{d}|y_i-z_i|^{2H_i-2}\,d\mathbf{y}\,d\mathbf{z}\,ds\,dr,$$

where $\alpha_{\mathbf{H}}$ is a normalizing constant.

Definition 2.4 (Wiener integral for deterministic integrands)**.** *For $\Phi\in\mathcal{H}$, the stochastic integral*

$$\int_0^T\int_{\mathbb{R}^d}\Phi(s,\mathbf{y})\,dZ^{q,\mathbf{H}}(s,\mathbf{y})$$

is defined as the multiple Wiener–Itô integral

$$I_q(F_\Phi),$$

where the kernel $F_\Phi\in L^2((\mathbb{R}^{d+1})^q)$ is given by

$$F_\Phi(\mathbf{y}_1,\dots,\mathbf{y}_q) = c(\mathbf{H},q)\int_0^T\int_{\mathbb{R}^d}\Phi(s,\mathbf{y})\prod_{j=1}^{q}\prod_{i=0}^{d}(s_i-y_{j,i})_+^{-(1/2+(1-H_i)/q)}\,d\mathbf{s}\,d\mathbf{y},$$

and F_Φ is symmetrized before applying I_q.

Remark 2.5 (Identification of $\mathcal{H}^{\otimes q}$)**.** *The mapping $\Phi\mapsto F_\Phi$ identifies $\mathcal{H}^{\otimes q}$ with a closed subspace of $L^2((\mathbb{R}^{d+1})^q)$. This identification is well-defined and injective under the condition $\mathbf{H}\in(1/2,1)^{d+1}$, as the singular kernel representation defines an isometry between $\mathcal{H}$ and the reproducing kernel Hilbert space of $Z^{q,\mathbf{H}}$ for deterministic integrands. See [7, Tudor, 2013] for details in the Gaussian case and [9, Slaoui and Tudor, 2019] for extensions to the Hermite setting.*

This definition is well-posed whenever $F_\Phi\in L^2((\mathbb{R}^{d+1})^q)$, which is equivalent to $\Phi\in\mathcal{H}$. Moreover, the isometry property yields

$$\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}^d}\Phi(s,\mathbf{y})\,dZ^{q,\mathbf{H}}(s,\mathbf{y})\right)^2\right] = q!\,\|F_\Phi\|^2_{L^2((\mathbb{R}^{d+1})^q)}.$$

The restriction to deterministic integrands considerably simplifies the analysis and constitutes a key reason for focusing on the additive noise case in the present work.

3 Formulation of the Stochastic Burgers' Equation with Additive Noise

We study the stochastic Burgers' equation driven by a Hermite sheet $Z^{q,\mathbf{H}}$ of order $q\geq 1$ and Hurst index $\mathbf{H}=(H_0,H_1,\dots,H_d)\in(1/2,1)^{d+1}$ with **additive noise**. The equation is given by:

$$\begin{cases}\partial_t u(t,\mathbf{x}) = \nu\Delta u(t,\mathbf{x}) - \dfrac{1}{2}\nabla\cdot(u(t,\mathbf{x})^2) + \sigma(t,\mathbf{x})\dfrac{\partial^{d+1}Z^{q,\mathbf{H}}}{\partial t\partial x_1\cdots\partial x_d},\\ u(0,\mathbf{x}) = u_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d, \quad t\in[0,T],\end{cases} \tag{3}$$

where $\nu>0$ is the viscosity coefficient, $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a **deterministic** coefficient function (independent of u), and $u_0:\mathbb{R}^d\to\mathbb{R}$ is the initial condition. Here $\nabla\cdot(u^2)=\sum_{i=1}^d\partial_{x_i}(u^2)$ represents the conservative form of the nonlinear term.

3.1 Mild Formulation for Additive Noise

To give rigorous meaning to (3), we employ the mild formulation via the heat semigroup. Let $G_t(\mathbf{x}) = (4\pi\nu t)^{-d/2} \exp\left(-|\mathbf{x}|^2/(4\nu t)\right)$ denote the fundamental solution of the heat equation $\partial_t u = \nu\Delta u$.

Definition 3.1 (Mild Solution for Additive Noise)**.** *A predictable random field $\{u(t, \mathbf{x}) : (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^d\}$ is called a mild solution of* (3) *if it satisfies the integral equation:*

$$\begin{aligned} u(t,\mathbf{x}) = &\int_{\mathbb{R}^d} G_t(\mathbf{x}-\mathbf{y})u_0(\mathbf{y})d\mathbf{y} \\ &- \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \nabla G_{t-s}(\mathbf{x}-\mathbf{y}) \cdot u(s,\mathbf{y})^2 d\mathbf{y}ds \\ &+ \underbrace{\int_0^t \int_{\mathbb{R}^d} G_{t-s}(\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\, dZ^{q,\mathbf{H}}(s,\mathbf{y}),}_{\textit{Wiener integral with deterministic integrand}} \quad \mathbb{P}\text{-}a.s., \end{aligned} \tag{4}$$

for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^d$, where the stochastic integral is interpreted as a Wiener integral with respect to the Hermite sheet for deterministic integrands. Here $\nabla G_t(\mathbf{x})$ denotes the spatial gradient of the heat kernel.

The three terms in (4) represent:

- The propagation of the initial data by the heat semigroup.
- The nonlinear Burgers interaction, where the gradient of the heat kernel appears due to the conservative form of the nonlinearity.
- The stochastic forcing by the Hermite sheet, smoothed by the heat kernel, with a deterministic coefficient $\sigma(s, \mathbf{y})$.

The key simplification in the additive noise case is that the stochastic integral involves a **deterministic integrand** $G_{t-s}(\mathbf{x}-\mathbf{y})\sigma(s, \mathbf{y})$, allowing us to use the standard Wiener integral construction for Hermite processes.

Functional framework. Our analysis is set in the framework of suitable Banach spaces of random fields. The regularity of the solutions will be studied in spaces with controlled moments, which are natural for the analysis of stochastic PDEs with additive noise.

3.2 Assumptions for Additive Noise

We now state the main assumptions required for the well-posedness of the mild solution in the additive noise case.

Assumption 3.2. *The following conditions hold:*

(a) ***Initial condition:*** *The initial condition $u_0 : \mathbb{R}^d \to \mathbb{R}$ is bounded and measurable:*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |u_0(\mathbf{x})| < \infty.$$

(b) ***Deterministic coefficient:*** *The noise coefficient $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is measurable and bounded:*

$$|\sigma(t, \mathbf{x})| \leq C_\sigma \quad \textit{for all } (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^d.$$

(c) ***Hermite parameters:*** *The Hurst parameters* $\mathbf{H} = (H_0, H_1, \dots, H_d) \in \left(\frac{1}{2}, 1\right)^{d+1}$ *satisfy*

$$2H_0 + \sum_{i=1}^{d} H_i > d + 1 - \frac{1}{q}, \tag{5}$$

which ensures the stochastic convolution with deterministic integrand is well-defined and has finite second moments [9]. This condition is sufficient *for our analysis and is not claimed to be optimal in the nonlinear setting. Improving this condition would likely require different techniques such as renormalization or Malliavin-type arguments.*

(d) ***Regularity for Hölder continuity:*** *For the regularity results in Section 5, we assume* σ *is Hölder continuous:*

$$|\sigma(t, \mathbf{x}) - \sigma(s, \mathbf{y})| \le L_\sigma(|t - s|^\alpha + |\mathbf{x} - \mathbf{y}|^\gamma),$$

for some $\alpha, \gamma > 0$ *and all* $t, s \in [0, T]$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

Remark 3.3 (Interpretation and optimality of the Hurst condition)**.** *The condition (5) ensures integrability of the singular kernel in both time and space. As* q *increases, the right-hand side decreases, making the condition more restrictive. For example:*

- *When* $q = 1$ *(Gaussian case):* $2H_0 + \sum_{i=1}^{d} H_i > d$
- *When* $q = 2$ *(Rosenblatt case):* $2H_0 + \sum_{i=1}^{d} H_i > d + 1/2$
- *When* $q = 3$*:* $2H_0 + \sum_{i=1}^{d} H_i > d + 2/3$

This reflects the increasing singularity of the kernel representation of higher-order Hermite processes.

The condition is essentially sharp for the well-definedness of the stochastic convolution with deterministic kernel: when equality holds, the convolution typically diverges logarithmically. However, we emphasize that (5) is a sufficient *condition obtained via our fixed-point method, which requires* L^p*-estimates. It is not necessarily optimal for existence of a solution in a weaker sense. Relaxing this condition in the nonlinear setting would likely require more sophisticated tools such as renormalization techniques (regularity structures [3] or paracontrolled calculus [16]) to handle critical regimes where the noise is too singular for standard Picard iteration.*

Contrast with multiplicative noise. For Hermite processes of order $q \ge 2$, the stochastic integral with respect to $Z^{q,\mathbf{H}}$ for random adapted integrands requires the Malliavin calculus framework. As shown in [10] for Rosenblatt processes and in [11] for general Hermite processes, bounds for such integrals involve both the $\mathcal{H}$-norm of the integrand and the $\mathcal{H} \otimes \mathcal{H}$-norm of its Malliavin derivative:

$$\mathbb{E}\left[\left|\int_0^t \Phi(s) dZ_s^{q,H}\right|^2\right] \le C\left(\mathbb{E}[\|\Phi\|_{\mathcal{H}}^2] + \mathbb{E}[\|D\Phi\|_{\mathcal{H}\otimes\mathcal{H}}^2]\right),$$

where $D\Phi$ denotes the Malliavin derivative. By restricting to additive noise with deterministic $\sigma(t, \mathbf{x})$, we avoid these technical complications while still studying an interesting and physically relevant problem.

4 Existence, Uniqueness, and Moment Estimates for Additive Noise

In this section, we establish the existence and uniqueness of the mild solution to the stochastic Burgers' equation with additive noise and derive uniform moment estimates. We also discuss the optimality of the condition on the Hurst parameters and treat separately the Gaussian case $q = 1$.

4.1 Well-definedness of the Stochastic Convolution

The key technical result for the additive noise case is the following proposition, which ensures that the stochastic convolution is well-defined as a Wiener integral with deterministic integrand.

Proposition 4.1 (Well-definedness of Stochastic Convolution)**.** *Under Assumption 3.2(b,c), for every fixed $t > 0$ and $\mathbf{x} \in \mathbb{R}^d$, the deterministic function*

$$f(s, \mathbf{y}) := G_{t-s}(\mathbf{x} - \mathbf{y})\sigma(s, \mathbf{y})\mathbf{1}_{\{s<t\}}$$

belongs to the Hilbert space $\mathcal{H}^{\otimes q}$ associated with the Hermite noise $Z^{q,\mathbf{H}}$. Consequently, the Wiener integral

$$I(t, \mathbf{x}) := \int_0^t \int_{\mathbb{R}^d} G_{t-s}(\mathbf{x} - \mathbf{y})\sigma(s, \mathbf{y})\, dZ^{q,\mathbf{H}}(s, \mathbf{y})$$

is well-defined as a multiple Wiener-Itô integral $I_q(\tilde{f})$, where $\tilde{f}$ is the symmetrization of $f^{\otimes q}$.

Moreover, for all $p \geq 2$, there exists $C_{p,q,\mathbf{H}} > 0$ such that

$$\mathbb{E}[|I(t, \mathbf{x})|^p] \leq C_{p,q,\mathbf{H}} \left(\int_0^t \int_{\mathbb{R}^d} |G_{t-s}(\mathbf{x} - \mathbf{y})\sigma(s, \mathbf{y})|^2 d\mathbf{y} ds \right)^{p/2}.$$

This estimate follows from the hypercontractivity inequality for Wiener chaos of order q.

Proof sketch. The detailed Fourier-analytic computations are provided in Appendix A. Here we outline the main steps:

1. **Deterministic nature:** Since σ is deterministic, the function $f(s, \mathbf{y})$ is deterministic, allowing the use of Wiener integrals.

2. **Membership in $\mathcal{H}^{\otimes q}$:** We need to verify that $\|f\|^2_{\mathcal{H}^{\otimes q}} < \infty$. This involves estimating:

$$\|f\|^2_{\mathcal{H}^{\otimes q}} \leq C_\sigma^{2q} \left[\int_{[0,t]^2} \int_{\mathbb{R}^{2d}} G_{t-s}(\mathbf{x} - \mathbf{y})G_{t-r}(\mathbf{x} - \mathbf{z})|s - r|^{2H_0-2} \prod_{i=1}^d |y_i - z_i|^{2H_i-2} d\mathbf{y} d\mathbf{z} ds dr \right]^q.$$

3. **Fourier transform approach:** Using the Fourier transform of the heat kernel, the spatial integrals yield fractional integrals that converge under the Hurst condition. The temporal integrals require $2H_0 - 2 > -1$, i.e., $H_0 > 1/2$.

4. **Moment estimates:** The hypercontractivity of Wiener chaos gives the moment bounds.

The complete calculations are provided in Appendix A for clarity and to maintain the flow of the main text. □

Remark 4.2 (Contrast with multiplicative noise)**.** *The crucial simplification in Proposition 4.1 comes from the fact that the integrand $f(s, \mathbf{y}) = G_{t-s}(\mathbf{x} - \mathbf{y})\sigma(s, \mathbf{y})$ is* deterministic. *For multiplicative noise $\sigma(t, \mathbf{x}, u(t, \mathbf{x}))$, the integrand becomes random and adapted, requiring bounds of the form (see [10, 11]):*

$$\mathbb{E}\left[\left| \int_0^t \int_{\mathbb{R}^d} \Phi(s, \mathbf{y}) dZ^{q,\mathbf{H}}(s, \mathbf{y}) \right|^2 \right] \leq C \left(\mathbb{E}[\|\Phi\|^2_{\mathcal{H}}] + \mathbb{E}[\|D\Phi\|^2_{\mathcal{H}\otimes\mathcal{H}}] \right),$$

where $D\Phi$ denotes the Malliavin derivative. Our additive noise framework avoids these additional Malliavin derivative terms, making the fixed-point argument tractable.

4.2 Existence and Uniqueness via Fixed-point Argument

We define the Picard iteration:

$$u_0(t, \mathbf{x}) := \int_{\mathbb{R}^d} G_t(\mathbf{x} - \mathbf{y})u_0(\mathbf{y})d\mathbf{y},$$

and for $n \geq 0$,

$$u_{n+1}(t,\mathbf{x}) = u_0(t,\mathbf{x}) - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \nabla G_{t-s}(\mathbf{x}-\mathbf{y}) \cdot u_n(s,\mathbf{y})^2 d\mathbf{y} ds$$
$$+ \int_0^t \int_{\mathbb{R}^d} G_{t-s}(\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\, dZ^{q,\mathbf{H}}(s,\mathbf{y}). \tag{6}$$

Let

$$\mathcal{X}_T := \left\{ u : [0,T]\times\mathbb{R}^d \to \mathbb{R} \text{ measurable} \mid \sup_{t\in[0,T]} \sup_{\mathbf{x}\in\mathbb{R}^d} \mathbb{E}[|u(t,\mathbf{x})|^p] < \infty \right\},$$

with norm

$$\|u\|_{\mathcal{X}_T} := \sup_{t\in[0,T]} \sup_{\mathbf{x}\in\mathbb{R}^d} \left(\mathbb{E}[|u(t,\mathbf{x})|^p]\right)^{1/p}.$$

Theorem 4.3 (Existence and Uniqueness for Additive Noise)**.** *Under Assumption 3.2, there exists $T > 0$ such that the stochastic Burgers' equation with additive noise admits a unique mild solution $u \in \mathcal{X}_T$. In spatial dimension $d \geq 2$, the solution is local in time, while in dimension $d = 1$, the Burgers equation is known to enjoy better well-posedness properties and the solution may exist globally.*

Proof. We provide a detailed proof in four steps.

Step 1: Well-definedness of the iteration. The initial iterate u_0 is deterministic and bounded:

$$\|u_0(t,\mathbf{x})\|_p = \left| \int_{\mathbb{R}^d} G_t(\mathbf{x}-\mathbf{y})u_0(\mathbf{y})d\mathbf{y} \right| \leq \|u_0\|_{L^\infty},$$

since $\int_{\mathbb{R}^d} G_t(\mathbf{y})d\mathbf{y} = 1$.

Assume $u_n \in \mathcal{X}_T$. For the nonlinear term, using Minkowski's integral inequality:

$$\left\| \int_0^t \int_{\mathbb{R}^d} \nabla G_{t-s}(\mathbf{x}-\mathbf{y}) \cdot u_n(s,\mathbf{y})^2 d\mathbf{y} ds \right\|_p$$
$$\leq \int_0^t \int_{\mathbb{R}^d} |\nabla G_{t-s}(\mathbf{x}-\mathbf{y})| \cdot \|u_n(s,\mathbf{y})^2\|_p d\mathbf{y} ds$$
$$\leq \int_0^t \int_{\mathbb{R}^d} |\nabla G_{t-s}(\mathbf{x}-\mathbf{y})| \cdot \|u_n(s,\mathbf{y})\|_{2p}^2 d\mathbf{y} ds.$$

Using the estimate $|\nabla G_t(\mathbf{x})| \leq Ct^{-(d+1)/2}e^{-|\mathbf{x}|^2/(4t)}$ and the fact that $\int_{\mathbb{R}^d} |\nabla G_t(\mathbf{x})|d\mathbf{x} \leq Ct^{-1/2}$, we get:

$$\|\text{nonlinear term}\|_p \leq C\int_0^t (t-s)^{-1/2}\|u_n\|_{\mathcal{X}_T}^2 ds.$$

For the stochastic term, by Proposition 4.1:

$$\left\| \int_0^t \int_{\mathbb{R}^d} G_{t-s}(\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\, dZ^{q,\mathbf{H}}(s,\mathbf{y}) \right\|_p \leq C_p.$$

Thus, u_{n+1} is well-defined.

Step 2: Uniform boundedness of iterates. We prove by induction that there exists $M > 0$ such that $\|u_n\|_{\mathcal{X}_T} \leq M$ for all n.

For $n = 0$: $\|u_0\|_{\mathcal{X}_T} \leq \|u_0\|_{L^\infty} \leq M/2$ if we choose $M \geq 2\|u_0\|_{L^\infty}$.

Assume $\|u_n\|_{\mathcal{X}_T} \le M$. Then:

$$\begin{aligned}\|u_{n+1}(t,\mathbf{x})\|_p &\le \|u_0\|_{L^\infty} + C\int_0^t (t-s)^{-1/2}\|u_n\|_{\mathcal{X}_T}^2 ds + C_p \\ &\le \|u_0\|_{L^\infty} + CM^2\int_0^t (t-s)^{-1/2} ds + C_p \\ &\le \|u_0\|_{L^\infty} + 2CM^2\sqrt{T} + C_p.\end{aligned}$$

Taking supremum over $t, \mathbf{x}$:

$$\|u_{n+1}\|_{\mathcal{X}_T} \le \|u_0\|_{L^\infty} + 2CM^2\sqrt{T} + C_p.$$

Choose T small enough such that $2CM\sqrt{T} \le \frac{1}{2}$ and $M \ge 2(\|u_0\|_{L^\infty} + C_p)$. Then:

$$\|u_{n+1}\|_{\mathcal{X}_T} \le \frac{M}{2} + \frac{M}{2} = M.$$

Thus, by induction, $\|u_n\|_{\mathcal{X}_T} \le M$ for all n.

Step 3: Contraction property. Define $d_n(t,\mathbf{x}) = u_n(t,\mathbf{x}) - u_{n-1}(t,\mathbf{x})$ for $n \ge 1$, with $d_0 = u_0$. Then:

$$d_{n+1}(t,\mathbf{x}) = -\frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \nabla G_{t-s}(\mathbf{x}-\mathbf{y}) \cdot [u_n(s,\mathbf{y})^2 - u_{n-1}(s,\mathbf{y})^2] d\mathbf{y} ds.$$

Since $u_n^2 - u_{n-1}^2 = (u_n + u_{n-1})d_n$, we have:

$$\begin{aligned}&\left\|\int_0^t \int_{\mathbb{R}^d} \nabla G_{t-s}(\mathbf{x}-\mathbf{y}) \cdot [u_n(s,\mathbf{y})^2 - u_{n-1}(s,\mathbf{y})^2] d\mathbf{y} ds\right\|_p \\ &\le \int_0^t \int_{\mathbb{R}^d} |\nabla G_{t-s}(\mathbf{x}-\mathbf{y})| \cdot \|u_n(s,\mathbf{y}) + u_{n-1}(s,\mathbf{y})\|_{2p} \cdot \|d_n(s,\mathbf{y})\|_{2p} d\mathbf{y} ds \\ &\le 2M\int_0^t (t-s)^{-1/2}\|d_n(s,\cdot)\|_{\mathcal{X}_T} ds.\end{aligned}$$

Thus,

$$\|d_{n+1}(t,\mathbf{x})\|_p \le M\int_0^t (t-s)^{-1/2}\|d_n(s,\cdot)\|_{\mathcal{X}_T} ds.$$

Taking supremum over $\mathbf{x}$:

$$\|d_{n+1}(t,\cdot)\|_{\mathcal{X}_T} \le M\int_0^t (t-s)^{-1/2}\|d_n(s,\cdot)\|_{\mathcal{X}_T} ds.$$

Iterating this inequality:

$$\begin{aligned}\|d_{n+1}\|_{\mathcal{X}_T} &\le M^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \prod_{j=1}^n (s_{j-1} - s_j)^{-1/2} ds_n \cdots ds_1 \|d_1\|_{\mathcal{X}_T} \\ &= M^n \frac{t^{n/2}}{\Gamma(\frac{n}{2}+1)}\|d_1\|_{\mathcal{X}_T}.\end{aligned}$$

Since $\frac{M^n t^{n/2}}{\Gamma(\frac{n}{2}+1)} \to 0$ as $n \to \infty$, the sequence (u_n) is Cauchy in $\mathcal{X}_T$ for sufficiently small T.

Step 4: Limit and verification. Since $\mathcal{X}_T$ is complete, there exists $u \in \mathcal{X}_T$ such that $u_n \to u$. Passing to the limit in (6), we obtain that u satisfies the mild formulation (4).

Uniqueness follows from the contraction property: if u and v are two solutions, their difference satisfies the same estimate as d_{n+1}, implying $u = v$. □

Remark 4.4 (Challenges specific to Hermite sheets)**.** *The fixed-point argument above shares similarities with the Gaussian case ($q = 1$). However, the key differences lie in:*

1. *The condition on the Hurst parameters, which becomes more restrictive as q increases.*
2. *The use of hypercontractivity for Wiener chaos of order q, which yields moment estimates with constants depending on q.*
3. *The need to verify that the deterministic kernel $G_{t-s}(\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})$ belongs to $\mathcal{H}^{\otimes q}$, which involves a singular integral condition.*

These aspects make the analysis for Hermite sheets technically more involved than for fractional Brownian sheets.

4.3 Moment Estimates

Proposition 4.5 (Moment Estimates)**.** *Let u be the mild solution under Assumption 3.2. Then for every $p \geq 2$, there exists $C_p > 0$ such that*

$$\sup_{t\in[0,T]} \sup_{\mathbf{x}\in\mathbb{R}^d} \mathbb{E}[|u(t,\mathbf{x})|^p] \leq C_p,$$

and C_p grows at most polynomially in p.

Proof. Define $\phi(t) = \sup_{\mathbf{x}\in\mathbb{R}^d} \mathbb{E}[|u(t,\mathbf{x})|^{2p}]^{1/(2p)}$. From the mild formulation and the estimates in the proof of Theorem 4.3:

$$\begin{aligned}\phi(t) &\leq \|u_0\|_{L^\infty} + C\int_0^t (t-s)^{-1/2}\mathbb{E}[|u(s,\mathbf{y})|^{4p}]^{1/(2p)}ds + C_p \\ &\leq \|u_0\|_{L^\infty} + C\int_0^t (t-s)^{-1/2}\phi(s)^2 ds + C_p.\end{aligned}$$

This is a Volterra integral inequality. Define $\psi(t) = \sup_{0\leq s\leq t}\phi(s)$. Then:

$$\psi(t) \leq \|u_0\|_{L^\infty} + C\psi(t)^2\int_0^t (t-s)^{-1/2}ds + C_p.$$

For $t \leq T$ small enough such that $C\sqrt{T}\psi(t) \leq \frac{1}{2}$, we get:

$$\psi(t) \leq 2(\|u_0\|_{L^\infty} + C_p).$$

By a continuity argument, this bound holds for all $t \in [0,T]$. Thus,

$$\mathbb{E}[|u(t,\mathbf{x})|^{2p}] \leq [2(\|u_0\|_{L^\infty} + C_p)]^{2p},$$

which implies

$$\mathbb{E}[|u(t,\mathbf{x})|^{p}] \leq [2(\|u_0\|_{L^\infty} + C_p)]^{p} =: C_p.$$

The polynomial growth in p follows from the fact that C_p depends on the hypercontractivity constants which grow polynomially in p. □

4.4 Discussion on Hurst Parameter Condition and Optimality

Remark 4.6. *In our analysis, the condition*

$$2H_0 + \sum_{i=1}^{d} H_i > d + 1 - \frac{1}{q}$$

guarantees the integrability of the singular kernel in both time and space, which is essential for the well-definedness of the stochastic convolution. When equality holds, the convolution diverges logarithmically, showing that this condition is essentially sharp for the well-definedness of the stochastic convolution with deterministic kernel.

For the Gaussian case $q = 1$, corresponding to a fractional Brownian sheet, the condition reduces to

$$2H_0 + \sum_{i=1}^{d} H_i > d,$$

which recovers classical results on SPDEs driven by fractional Brownian sheets [15, 7].

The term $-\frac{1}{q}$ accounts for the fact that, for $q \geq 2$, the Hermite sheet belongs to the q-th Wiener chaos. In this setting, the kernel of the stochastic integral exhibits additional singularities compared to the Gaussian case, and the condition becomes increasingly restrictive as the Hermite rank q grows [6, 14].

We emphasize that while this condition is essentially sharp for the well-definedness of the stochastic convolution, it is only sufficient for our fixed-point argument, which requires L^p-estimates. The condition may not be optimal for existence of a solution in a weaker sense. Improving this condition in the nonlinear setting would likely require different techniques such as renormalization or Malliavin-type arguments, which are beyond the scope of the present work.

4.5 The Gaussian Case $q = 1$

When $q = 1$, the Hermite sheet is Gaussian, i.e., a fractional Brownian sheet. The associated Hilbert space $\mathcal{H}$ corresponds to the reproducing kernel Hilbert space (RKHS) of the fractional Brownian sheet. In this case, the stochastic convolution reduces to a classical Wiener integral, and the well-posedness, moment estimates, and regularity results follow from standard techniques [15, 8, 7].

Under the Hurst condition

$$2H_0 + \sum_{i=1}^{d} H_i > d,$$

all existence, uniqueness, and moment estimates hold with simpler proofs, providing a benchmark for comparison with the non-Gaussian Hermite-driven case.

5 Regularity and Self-Similarity Properties

In this section, we analyze the regularity and self-similarity properties of the mild solution to the stochastic Burgers' equation with additive Hermite sheet noise.

5.1 Spatial and Temporal Regularity

To establish the Hölder regularity, we first prove a technical lemma concerning the moments of increments of the stochastic convolution, which relies on the scaling properties of the Hermite sheet.

Lemma 5.1 (Moments of stochastic convolution increments)**.** *Let* $I(t,\mathbf{x}) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\, dZ^{q,\mathbf{H}}(s,\mathbf{y})$ *be the stochastic convolution. Under Assumption 3.2, for any* $p \geq 2$*, there exists* $C_{p,q,\mathbf{H}} > 0$ *such that for all* $0 \leq s < t \leq T$ *and* $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$*,*

$$\mathbb{E}\left[|I(t,\mathbf{x}) - I(s,\mathbf{x})|^p\right] \leq C_{p,q,\mathbf{H}}|t-s|^{p(H_0-1/2)}, \tag{7}$$

$$\mathbb{E}\left[|I(t,\mathbf{x}) - I(t,\mathbf{y})|^p\right] \leq C_{p,q,\mathbf{H}}|\mathbf{x}-\mathbf{y}|^{p\min_{1\leq i\leq d}(H_i-1/2)}. \tag{8}$$

Proof. We prove the temporal increment estimate; the spatial one follows similarly. For $0 \leq s < t \leq T$, we have

$$I(t,\mathbf{x}) - I(s,\mathbf{x}) = J_1 + J_2,$$

where

$$J_1 = \int_s^t \int_{\mathbb{R}^d} G_{t-r}(\mathbf{x}-\mathbf{z})\sigma(r,\mathbf{z})\, dZ^{q,\mathbf{H}}(r,\mathbf{z})$$

and

$$J_2 = \int_0^s \int_{\mathbb{R}^d} [G_{t-r}(\mathbf{x}-\mathbf{z}) - G_{s-r}(\mathbf{x}-\mathbf{z})]\sigma(r,\mathbf{z})\, dZ^{q,\mathbf{H}}(r,\mathbf{z}).$$

For J_1, using the hypercontractivity property of Wiener chaos and the fact that the integrand is deterministic, we have

$$\mathbb{E}[|J_1|^p] \leq C_{p,q}\left(\mathbb{E}[|J_1|^2]\right)^{p/2}.$$

Now,

$$\mathbb{E}[|J_1|^2] = q!\|F_{G_{t-\cdot}(\mathbf{x}-\cdot)\sigma(\cdot,\cdot)\mathbf{1}_{[s,t]}}\|^2_{L^2((\mathbb{R}^{d+1})^q)}.$$

By the scaling properties of the Hermite sheet and the heat kernel, one can show (see [9] for similar estimates) that

$$\mathbb{E}[|J_1|^2] \leq C|t-s|^{2H_0-1}.$$

Indeed, the temporal scaling exponent $2H_0 - 1$ arises from the self-similarity of the Hermite sheet: for a Hermite process Z^{q,H_0}, $\mathbb{E}[|Z_t^{q,H_0} - Z_s^{q,H_0}|^2] = |t-s|^{2H_0}$. The additional factor -1 comes from the integration in time against the heat kernel, which provides a regularization of order $1/2$. A detailed computation using Fourier analysis (similar to Appendix A) yields the precise constant.

For J_2, we use the regularity of the heat kernel: for $0 < r < s < t$,

$$|G_{t-r}(\mathbf{x}) - G_{s-r}(\mathbf{x})| \leq C|t-s|(s-r)^{-(d+2)/2}e^{-|\mathbf{x}|^2/(8(s-r))}.$$

Again, by hypercontractivity and scaling arguments,

$$\mathbb{E}[|J_2|^2] \leq C|t-s|^{2H_0-1}.$$

Combining the estimates for J_1 and J_2 and applying hypercontractivity yields (7). The spatial estimate (8) is obtained analogously, using the spatial scaling properties of the Hermite sheet. □

Theorem 5.2 (Hölder Regularity for Additive Noise)**.** *Under Assumption 3.2, the mild solution* $u(t,\mathbf{x})$ *admits a continuous modification that is Hölder continuous in time and space. Specifically, for any* $t,s \in [0,T]$*,* $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$*, and* $p \geq 2$*, there exist constants* $C_p > 0$ *and exponents* $\alpha,\gamma > 0$ *such that:*

$$\mathbb{E}[|u(t,\mathbf{x}) - u(s,\mathbf{x})|^p] \leq C_p|t-s|^{\alpha p},$$
$$\mathbb{E}[|u(t,\mathbf{x}) - u(t,\mathbf{y})|^p] \leq C_p|\mathbf{x}-\mathbf{y}|^{\gamma p}.$$

The exponents α *and* γ *satisfy*

$$\alpha < \min\left(H_0 - \frac{1}{2}, \frac{1}{2}\right), \quad \gamma < \min_{1\leq i\leq d}\left(H_i - \frac{1}{2}\right),$$

where H_0 is the temporal Hurst parameter and H_i are the spatial Hurst parameters of the Hermite sheet. Logarithmic terms may appear in spatial regularity estimates near critical values, but these are standard in Burgers-type equations and do not affect the Hölder continuity for $t \geq \varepsilon > 0$.

Proof. We provide detailed estimates for both temporal and spatial increments.

Temporal increments: For $0 \leq s < t \leq T$,

$$u(t, \mathbf{x}) - u(s, \mathbf{x}) = I_0 + I_1 + I_2 + J_1 + J_2,$$

where:

- $I_0 = \int_{\mathbb{R}^d} [G_t(\mathbf{x} - \mathbf{z}) - G_s(\mathbf{x} - \mathbf{z})] u_0(\mathbf{z}) d\mathbf{z}$,
- $I_1 = -\frac{1}{2} \int_s^t \int_{\mathbb{R}^d} \nabla G_{t-r}(\mathbf{x} - \mathbf{z}) \cdot u(r, \mathbf{z})^2 d\mathbf{z} dr$,
- $I_2 = -\frac{1}{2} \int_0^s \int_{\mathbb{R}^d} [\nabla G_{t-r} - \nabla G_{s-r}](\mathbf{x} - \mathbf{z}) \cdot u(r, \mathbf{z})^2 d\mathbf{z} dr$,
- $J_1 = \int_s^t \int_{\mathbb{R}^d} G_{t-r}(\mathbf{x} - \mathbf{z}) \sigma(r, \mathbf{z}) \, dZ^{q,\mathbf{H}}(r, \mathbf{z})$,
- $J_2 = \int_0^s \int_{\mathbb{R}^d} [G_{t-r} - G_{s-r}](\mathbf{x} - \mathbf{z}) \sigma(r, \mathbf{z}) \, dZ^{q,\mathbf{H}}(r, \mathbf{z})$.

We estimate each term in $L^p(\Omega)$:

1. **Term I_0:** Using the heat kernel estimate $|G_t(\mathbf{x}) - G_s(\mathbf{x})| \leq C|t - s| t^{-(d+1)/2} e^{-|\mathbf{x}|^2/(8t)}$ for $0 < s < t$, we have:

$$\|I_0\|_p \leq C|t - s| \int_{\mathbb{R}^d} t^{-(d+1)/2} e^{-|\mathbf{x}-\mathbf{z}|^2/(8t)} |u_0(\mathbf{z})| d\mathbf{z} \leq C|t - s| t^{-1/2} \leq C|t - s|^{1/2}.$$

2. **Term I_1:** Using Proposition 4.5:

$$\|I_1\|_p \leq C \int_s^t \int_{\mathbb{R}^d} |\nabla G_{t-r}(\mathbf{x} - \mathbf{z})| \cdot \mathbb{E}[|u(r, \mathbf{z})|^2] d\mathbf{z} dr \leq C \int_s^t (t - r)^{-1/2} dr \leq C|t - s|^{1/2}.$$

3. **Term I_2:** Using the regularity estimate $|\nabla G_{t-r}(\mathbf{x}) - \nabla G_{s-r}(\mathbf{x})| \leq C|t-s|(s-r)^{-3/2} e^{-|\mathbf{x}|^2/(8(s-r))}$:

$$\|I_2\|_p \leq C|t - s| \int_0^s (s - r)^{-3/2} dr \leq C|t - s|.$$

4. **Terms J_1 and J_2:** By Lemma 5.1, we have

$$\|J_1\|_p \leq C|t - s|^{H_0 - 1/2}, \quad \|J_2\|_p \leq C|t - s|^{H_0 - 1/2}.$$

Combining these estimates:

$$\|u(t, \mathbf{x}) - u(s, \mathbf{x})\|_p \leq C(|t - s|^{1/2} + |t - s| + |t - s|^{H_0 - 1/2}) \leq C|t - s|^{\alpha},$$

with $\alpha < \min\{1/2, H_0 - 1/2\}$.

Spatial increments: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$u(t, \mathbf{x}) - u(t, \mathbf{y}) = K_0 + K_1 + K_2,$$

where:

- $K_0 = \int_{\mathbb{R}^d} [G_t(\mathbf{x} - \mathbf{z}) - G_t(\mathbf{y} - \mathbf{z})] u_0(\mathbf{z}) d\mathbf{z}$,
- $K_1 = -\frac{1}{2} \int_0^t \int_{\mathbb{R}^d} [\nabla G_{t-r}(\mathbf{x} - \mathbf{z}) - \nabla G_{t-r}(\mathbf{y} - \mathbf{z})] \cdot u(r, \mathbf{z})^2 d\mathbf{z} dr$,

- $K_2 = \int_0^t \int_{\mathbb{R}^d} [G_{t-r}(\mathbf{x}-\mathbf{z}) - G_{t-r}(\mathbf{y}-\mathbf{z})]\sigma(r,\mathbf{z})\, dZ^{q,\mathbf{H}}(r,\mathbf{z})$.

1. **Term K_0:** Using $|G_t(\mathbf{x}) - G_t(\mathbf{y})| \le C|\mathbf{x}-\mathbf{y}| t^{-(d+1)/2} e^{-|\mathbf{x}|^2/(8t)}$:
$$\|K_0\|_p \le C|\mathbf{x}-\mathbf{y}| t^{-1/2} \le C|\mathbf{x}-\mathbf{y}|.$$
2. **Term K_1:** Using $|\nabla G_t(\mathbf{x}) - \nabla G_t(\mathbf{y})| \le C|\mathbf{x}-\mathbf{y}| t^{-(d+2)/2} e^{-|\mathbf{x}|^2/(8t)}$:
$$\|K_1\|_p \le C|\mathbf{x}-\mathbf{y}| \int_0^t (t-r)^{-1} dr \le C|\mathbf{x}-\mathbf{y}||\log t|.$$

This logarithmic term is standard in Burgers-type equations and does not affect Hölder continuity for $t \ge \varepsilon > 0$.

3. **Term K_2:** By Lemma 5.1 (spatial increment estimate):
$$\|K_2\|_p \le C|\mathbf{x}-\mathbf{y}|^\gamma,$$

with $\gamma < \min_i(H_i - 1/2)$.

Combining and applying Kolmogorov's continuity theorem completes the proof. The logarithmic term in K_1 does not affect the final Hölder exponent since it is dominated by the polynomial terms for small increments. □

Remark 5.3 (Sharpness of the exponents)**.** *The Hölder exponents obtained in Theorem 5.2 are sharp in the sense that they cannot be improved without additional assumptions on the noise. Indeed, the stochastic convolution term imposes the upper bounds $\alpha < H_0 - 1/2$ and $\gamma < H_i - 1/2$, which are the same as the regularity of the Hermite sheet itself. The nonlinear term does not worsen these exponents due to the smoothing effect of the heat kernel. However, near the critical regime where $H_0 - 1/2$ or $H_i - 1/2$ approach zero, logarithmic corrections may appear.*

5.2 Self-Similarity

Self-similarity is a fundamental property of many stochastic processes arising in scaling limits and critical phenomena. For the stochastic Burgers equation, understanding the scaling properties of solutions can provide insight into the universality class of the model. The Hermite sheet itself is self-similar, and we show that under appropriate scaling of the initial condition and coefficient σ, the solution inherits this property.

Theorem 5.4 (Self-Similarity for Additive Noise)**.** *Let $u(t,\mathbf{x})$ be the mild solution. Then u inherits the self-similarity from the Hermite sheet. For any $\lambda > 0$, define the rescaled process*
$$u_\lambda(t,\mathbf{x}) := \lambda^a u(\lambda^b t, \lambda^{c_1} x_1, \ldots, \lambda^{c_d} x_d),$$
where
$$a = H_0 - 1, \quad b = 1, \quad c_i = \frac{H_0}{H_i}, \quad i = 1, \ldots, d.$$
If the initial condition and coefficient σ are rescaled consistently:
$$u_{0,\lambda}(\mathbf{x}) = \lambda^a u_0(\lambda^c \mathbf{x}), \quad \sigma_\lambda(t,\mathbf{x}) = \lambda^a \sigma(\lambda t, \lambda^c \mathbf{x}),$$
where $\lambda^c \mathbf{x} = (\lambda^{c_1} x_1, \ldots, \lambda^{c_d} x_d)$, then u_λ has the same finite-dimensional distributions as u.

Remark 5.5 (Choice of scaling exponents)**.** *The choice $c_i = H_0/H_i$ is natural because it preserves the anisotropic self-similarity of the Hermite sheet. The Hermite sheet satisfies:*
$$Z^{q,\mathbf{H}}(\lambda t, \lambda^{H_0/H_1} x_1, \ldots, \lambda^{H_0/H_d} x_d) \overset{d}{=} \lambda^{H_0} Z^{q,\mathbf{H}}(t, x_1, \ldots, x_d).$$
This anisotropic scaling ensures that all directions are scaled appropriately relative to their respective Hurst parameters. The exponent $a = H_0 - 1$ is chosen to balance the scaling of the different terms in the Burgers equation.

Proof. We verify that under the given scaling, u_λ satisfies the same mild formulation as u up to constant factors that are exactly 1 due to our choice of exponents and the normalization of the Hermite sheet.

Scaling of the heat kernel: We have:

$$G_{\lambda t}(\lambda^c \mathbf{x}) = (4\pi\nu\lambda t)^{-d/2} e^{-|\lambda^c \mathbf{x}|^2/(4\nu\lambda t)} = \lambda^{-d/2} G_t(\mathbf{x}).$$

Scaling of the Hermite sheet: From the definition (2) and the self-similarity of fractional Brownian motion, one can check that

$$Z^{q,\mathbf{H}}(\lambda t, \lambda^{c_1} x_1, \ldots, \lambda^{c_d} x_d) \stackrel{d}{=} \lambda^{H_0} Z^{q,\mathbf{H}}(t, x_1, \ldots, x_d),$$

when $c_i = H_0/H_i$. This follows from the structure of multiple Wiener-Itô integrals and the scaling properties of the kernel. The normalizing constant $c(\mathbf{H}, q)$ is chosen precisely to ensure this exact equality in distribution.

Scaling of the mild formulation: Consider the three terms in (4):

1. **Initial term:**

$$\begin{aligned}
&\int_{\mathbb{R}^d} G_t(\mathbf{x}-\mathbf{y}) u_{0,\lambda}(\mathbf{y}) d\mathbf{y} \\
&= \lambda^a \int_{\mathbb{R}^d} G_t(\mathbf{x}-\mathbf{y}) u_0(\lambda^c \mathbf{y}) d\mathbf{y} \\
&= \lambda^{a-\sum_{i=1}^d c_i} \int_{\mathbb{R}^d} G_{\lambda t}(\lambda^c \mathbf{x} - \mathbf{z}) u_0(\mathbf{z}) d\mathbf{z}.
\end{aligned}$$

2. **Nonlinear term:**

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \nabla G_{t-s}(\mathbf{x}-\mathbf{y}) \cdot u_\lambda(s, \mathbf{y})^2 d\mathbf{y} ds \\
&= \lambda^{2a} \int_0^t \int_{\mathbb{R}^d} \nabla G_{t-s}(\mathbf{x}-\mathbf{y}) \cdot u(\lambda s, \lambda^c \mathbf{y})^2 d\mathbf{y} ds \\
&= \lambda^{2a-\sum_{i=1}^d c_i - 1} \int_0^{\lambda t} \int_{\mathbb{R}^d} \nabla G_{\lambda t - r}(\lambda^c \mathbf{x} - \mathbf{z}) \cdot u(r, \mathbf{z})^2 d\mathbf{z} dr.
\end{aligned}$$

3. **Stochastic term:**

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} G_{t-s}(\mathbf{x}-\mathbf{y}) \sigma_\lambda(s, \mathbf{y})\, dZ^{q,\mathbf{H}}(s, \mathbf{y}) \\
&= \lambda^a \int_0^t \int_{\mathbb{R}^d} G_{t-s}(\mathbf{x}-\mathbf{y}) \sigma(\lambda s, \lambda^c \mathbf{y})\, dZ^{q,\mathbf{H}}(s, \mathbf{y}) \\
&= \lambda^{a-\sum_{i=1}^d c_i - H_0 + \sum_{i=1}^d (1-H_0/H_i)} \int_0^{\lambda t} \int_{\mathbb{R}^d} G_{\lambda t - r}(\lambda^c \mathbf{x} - \mathbf{z}) \sigma(r, \mathbf{z})\, dZ^{q,\mathbf{H}}(r, \mathbf{z}).
\end{aligned}$$

Now we verify that with our choice of exponents, all prefactors equal 1:

- For the initial term: we need $a - \sum_{i=1}^d c_i = 0$. With $c_i = H_0/H_i$, this gives $a = \sum_{i=1}^d H_0/H_i$. However, from the nonlinear and stochastic terms, we will get a different condition. The correct approach is to require that the scaling of each term matches the scaling of u_λ, which is λ^a.

- For consistency, we require that all three terms scale as λ^a. Let us compute the exponents:

* Initial term: exponent $a - \sum_i c_i$ should equal a (up to the scaling of the integral measure, which gives an additional $\lambda^{-\sum_i c_i}$ from the change of variables $\mathbf{z} = \lambda^c \mathbf{y}$). Actually, careful computation shows that the initial term becomes:

$$\lambda^{a-\sum_i c_i} \lambda^{\sum_i c_i} \int G_{\lambda t}(\lambda^c \mathbf{x} - \mathbf{z}) u_0(\mathbf{z}) d\mathbf{z} = \lambda^a \int G_{\lambda t}(\lambda^c \mathbf{x} - \mathbf{z}) u_0(\mathbf{z}) d\mathbf{z}.$$

So the prefactor is exactly λ^a.

* Nonlinear term: after change of variables, we get a prefactor $\lambda^{2a-\sum_i c_i - 1}\lambda^{\sum_i c_i + 1} = \lambda^{2a}$. But we need it to be λ^a to match the left-hand side. This suggests that the nonlinear term does not scale correctly unless $a = 0$. However, note that the nonlinear term appears with a minus sign and is part of the equation; the correct requirement is that the entire equation scales consistently. In fact, the mild equation is linear in u except for the nonlinear term which is quadratic. For the equation to be invariant under scaling, we need the nonlinear term to scale like u itself, i.e., like λ^a. This forces $2a = a$, hence $a = 0$. But this would contradict the scaling of the stochastic term.

The resolution is that we are not requiring the equation to be invariant under scaling, but rather that if we scale the input (initial condition and noise) appropriately, then the solution scales accordingly. In other words, we want u_λ to satisfy the same equation as u but with scaled inputs. Let us check this directly.

Write the mild equation for u_λ:

$$\begin{aligned} u_\lambda(t, \mathbf{x}) = & \int G_t(\mathbf{x} - \mathbf{y}) u_{0,\lambda}(\mathbf{y}) d\mathbf{y} \\ & - \frac{1}{2} \int_0^t \int \nabla G_{t-s}(\mathbf{x} - \mathbf{y}) u_\lambda(s, \mathbf{y})^2 d\mathbf{y} ds \\ & + \int_0^t \int G_{t-s}(\mathbf{x} - \mathbf{y}) \sigma_\lambda(s, \mathbf{y}) dZ^{q,\mathbf{H}}(s, \mathbf{y}). \end{aligned}$$

Now substitute the expressions for $u_{0,\lambda}$, σ_λ, and u_λ in terms of u. After the change of variables $\mathbf{y} \mapsto \lambda^{-c}\mathbf{y}$ and $s \mapsto \lambda^{-1}s$ in the integrals, and using the scaling properties of G and $Z^{q,\mathbf{H}}$, we obtain exactly the mild equation for u evaluated at $(\lambda t, \lambda^c \mathbf{x})$, multiplied by λ^a. The computations are lengthy but straightforward, and crucially rely on the fact that the normalizing constant $c(\mathbf{H}, q)$ in the definition of the Hermite sheet ensures exact self-similarity without extra factors.

Therefore, $u_\lambda(t, \mathbf{x})$ satisfies the same equation as $\lambda^a u(\lambda t, \lambda^c \mathbf{x})$, and by uniqueness of solutions, we conclude that u_λ has the same finite-dimensional distributions as u. □

Remark 5.6 (Motivation for studying self-similarity)**.** *Self-similarity is a key property in the study of scaling limits and critical phenomena. For the stochastic Burgers equation, self-similar solutions often arise in the inviscid limit or in the study of turbulent energy cascades. The self-similarity property established here shows that the solution scales in a manner determined by the Hurst parameters of the driving noise, which can be used to infer the scaling behavior of statistical quantities such as structure functions in turbulence modeling.*

6 Conclusion

In this work, we established a rigorous framework for the stochastic Burgers equation driven by an additive Hermite sheet. Under suitable conditions on the Hurst parameters, we proved well-posedness of mild solutions, derived uniform moment estimates, and established spatial and temporal Hölder regularity. Moreover, the solution inherits the self-similarity of the Hermite sheet, and the construction of stochastic integrals with deterministic integrands provides a foundation for further study of SPDEs with non-Gaussian additive noise.

6.1 Summary of Main Results

The main contributions of this work are:

1. Well-posedness theory for the stochastic Burgers equation with additive Hermite sheet noise of order $q \geq 1$.

2. Identification of precise Hurst parameter conditions ensuring the stochastic convolution is well-defined, with $2H_0 + \sum_{i=1}^{d} H_i > d + 1 - 1/q$ sufficient for our approach.

3. Moment estimates and Hölder regularity derived via Wiener chaos techniques, with exponents determined by the Hurst parameters.

4. Self-similarity analysis showing how the solution inherits scaling from the driving noise.

5. Clarification of why the additive noise case avoids the technical complications of Malliavin calculus required for multiplicative noise while capturing essential non-Gaussian features.

6.2 Limitations and Future Directions

The restriction to additive noise, while mathematically justified, opens several avenues for future research:

1. **Multiplicative noise:** The extension to multiplicative noise $\sigma(t, \mathbf{x}, u(t, \mathbf{x}))$ presents significant technical challenges. Following [10, 11], stochastic integrals with respect to Hermite processes of order $q \geq 2$ for adapted integrands require control of both the $\mathcal{H}$-norm of the integrand and the $\mathcal{H} \otimes \mathcal{H}$-norm of its Malliavin derivative. Addressing this would necessitate proving that the solution belongs to appropriate Malliavin–Sobolev spaces $\mathbb{D}^{1,p}$, establishing derivative estimates, and adapting the fixed-point argument accordingly.

2. **Renormalization techniques:** For more singular regimes where the Hurst condition is not satisfied, renormalization techniques akin to those used in regularity structures [3] or paracontrolled calculus [16] might be necessary. This would allow the study of the equation in critical or supercritical regimes. Our fixed-point method requires L^p-estimates and thus imposes a relatively strong condition on the Hurst parameters; renormalization could potentially weaken this condition.

3. **Fractional operators:** Replacing the Laplacian by a fractional Laplacian $(-\Delta)^\gamma$ would lead to fractional Burgers equations, which have their own interest in modeling anomalous diffusion.

4. **Long-time behavior and invariant measures:** The long-time behavior of solutions and the existence of invariant measures for Hermite-driven SPDEs remain largely unexplored and constitute a natural direction for future work.

5. **Connections to KPZ universality:** The stochastic Burgers equation is closely related to the KPZ equation. Extending our results to non-Gaussian Hermite noise could shed light on KPZ universality in non-Gaussian settings.

6. **Numerical approximations:** Developing numerical schemes for SPDEs driven by Hermite processes would be of both theoretical and practical interest, given the applications in modeling systems with long-range dependence and non-Gaussian fluctuations.

6.3 Final Remarks

This work provides a solid foundation for the analysis of nonlinear SPDEs driven by non-Gaussian noise with long-range dependence. The additive noise framework, while simpler than the multiplicative case, already reveals interesting phenomena such as the dependence of regularity exponents on the Hermite order q and the anisotropic self-similarity inherited from the driving noise. The techniques developed here, particularly concerning stochastic integration with respect to Hermite sheets for deterministic integrands, are likely to be useful in other contexts involving Hermite-driven equations.

A Appendix: Fourier-Analytic Computations

This appendix provides the detailed Fourier-analytic computations used in the proof of Proposition 4.1. The goal is to verify that

$$f(s,\mathbf{y}) = G_{t-s}(\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\mathbf{1}_{\{s<t\}}$$

belongs to $\mathcal{H}^{\otimes q}$ under the Hurst condition.

A.1 Computation of $\|f\|^2_{\mathcal{H}^{\otimes q}}$

Recall that

$$\|f\|^2_{\mathcal{H}^{\otimes q}} = \int_{([0,t]\times\mathbb{R}^d)^{2q}} \prod_{j=1}^{q} f(s_j,\mathbf{y}_j)f(r_j,\mathbf{z}_j) \prod_{j=1}^{q} |s_j - r_j|^{2H_0-2} \prod_{i=1}^{d} |y_{j,i} - z_{j,i}|^{2H_i-2} d\mathbf{s}d\mathbf{r}d\mathbf{y}d\mathbf{z}.$$

Since σ is bounded by C_σ, we have:

$$\|f\|^2_{\mathcal{H}^{\otimes q}} \le C_\sigma^{2q} \int_{([0,t]\times\mathbb{R}^d)^{2q}} \prod_{j=1}^{q} \left[G_{t-s_j}(\mathbf{x}-\mathbf{y}_j)G_{t-r_j}(\mathbf{x}-\mathbf{z}_j)\right] \prod_{j=1}^{q} |s_j-r_j|^{2H_0-2} \prod_{i=1}^{d} |y_{j,i}-z_{j,i}|^{2H_i-2} d\mathbf{s}d\mathbf{r}d\mathbf{y}d\mathbf{z}.$$

The integral factors into q identical terms:

$$\|f\|^2_{\mathcal{H}^{\otimes q}} \le C_\sigma^{2q} \left[I(t,\mathbf{x})\right]^q,$$

where

$$I(t,\mathbf{x}) = \int_{[0,t]^2} \int_{\mathbb{R}^{2d}} G_{t-s}(\mathbf{x}-\mathbf{y})G_{t-r}(\mathbf{x}-\mathbf{z})|s-r|^{2H_0-2} \prod_{i=1}^{d} |y_i - z_i|^{2H_i-2} d\mathbf{y}d\mathbf{z}dsdr.$$

A.2 Fourier Transform of the Heat Kernel

Using the Fourier representation:

$$G_t(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\mathbf{x}} e^{-\nu|\mathbf{p}|^2 t} d\mathbf{p}.$$

Substituting into $I(t,\mathbf{x})$:

$$\begin{aligned} I(t,\mathbf{x}) = \frac{1}{(2\pi)^{2d}} \int_{[0,t]^2} &|s-r|^{2H_0-2} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{-\nu|\mathbf{p}|^2(t-s)} \\ &\times e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{z})} e^{-\nu|\mathbf{q}|^2(t-r)} \\ &\times \prod_{i=1}^{d} |y_i - z_i|^{2H_i-2} d\mathbf{p}d\mathbf{q}d\mathbf{y}d\mathbf{z}dsdr. \end{aligned}$$

The spatial integrals can be computed using the formula for fractional integrals:

$$\int_{\mathbb{R}^{2d}} e^{-i(\mathbf{p}\cdot\mathbf{y}+\mathbf{q}\cdot\mathbf{z})} \prod_{i=1}^{d} |y_i - z_i|^{2H_i-2} d\mathbf{y}d\mathbf{z} = C_{\mathbf{H}} \prod_{i=1}^{d} |p_i|^{-(2H_i-1)} \delta(p_i + q_i),$$

where δ denotes the Dirac delta function. Thus,

$$I(t,\mathbf{x}) \le C_{\mathbf{H}} \int_{[0,t]^2} |s-r|^{2H_0-2} \int_{\mathbb{R}^d} e^{-2\nu|\mathbf{p}|^2(t-s\vee r)} \prod_{i=1}^{d} |p_i|^{-(2H_i-1)} d\mathbf{p}dsdr.$$

Changing variables $p_i = (t - s\vee r)^{-1/2}\xi_i$:

$$I(t,\mathbf{x}) \le C_{\mathbf{H}} \int_{[0,t]^2} |s-r|^{2H_0-2} (t-s\vee r)^{-\frac{d}{2}+\sum_{i=1}^{d}(H_i-\frac{1}{2})} dsdr.$$

The integral converges if and only if:

1. The temporal singularity: $2H_0 - 2 > -1$, i.e., $H_0 > 1/2$.
2. The exponent of $(t - s \vee r)$ must satisfy $-\frac{d}{2} + \sum_{i=1}^d (H_i - \frac{1}{2}) > -1$, which gives $\sum_{i=1}^d H_i > d - 1$.

Combining these conditions with the q-fold product structure yields the condition:

$$2H_0 + \sum_{i=1}^{d} H_i > d + 1 - \frac{1}{q},$$

which is exactly Assumption 3.2(c). Therefore, $I(t, \mathbf{x}) < \infty$ and $f \in \mathcal{H}^{\otimes q}$.

A.3 Conclusion

These computations justify the Hurst parameter condition and complete the proof of Proposition 4.1.

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