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**“Dynamic Efficiency With Private
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Dynamic Efficiency With Private Information

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Abstract

This paper studies the efficient allocation of capital and consumption in a production economy with many agents, private information, and aggregate risk. It extends the influential work of Andrew Atkeson and Robert E. Lucas Jr. (1992), who analyzed a related problem in an exchange economy. In a dynamic production setting, the planner faces a fundamental trade-off between providing some insurance against privately observed idiosyncratic risk and sustaining productive investment and economic growth. Using mean-field control techniques, we derive the infinite dimensional Hamilton–Jacobi–Bellman equation that characterizes constrained-efficient allocations. Under constant relative risk aversion preferences, the solution admits a simple characterization. We show that constrained-efficient allocations can be decentralized through a competitive market in which goods trade against a single safe asset supplied by fiscal or monetary authorities. Dynamic efficiency requires setting the growth rate of the safe asset to balance the demand of agents for insurance with the investment needed to maintain optimal growth.

Keywords: Economies with private information, Mean Field Control, Monetary Policy.

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1 Introduction

In an influential paper, Atkeson and Lucas (1992) characterized efficient distribution of consumption in a dynamic exchange economy with a large number of agents subject to privately observable idiosyncratic shocks. This paper extends their results to a production economy.

The study of dynamic efficiency in Economics has a long history, going back at least to Ramsey (1928) and Malinvaud (1953). In models with an infinitely lived representative agent such as Cass (1965) and Koopmans (1965), the first welfare theorem applies, and competitive equilibrium allocations are dynamically efficient. Following Samuelson (1958) and Diamond (1965), a vast literature has shown that this result does not apply in OLG models. Whenever the steady-state interest rate lies below the growth rate of the economy ($r < g$), the equilibrium allocation is dynamically inefficient. Tirole (1985) showed that, in such situations, there exists a "bubbly" steady state in which both the capital stock and the output are lower because bubbles divert resources away from capital accumulation¹ But welfare is higher because bubbles reduce the over-accumulation of capital. However, Abel et al. (1989) suggest that real economies are often characterized by the opposite inequality ($r > g$) and are thus dynamically efficient² This appears to limit the relevance of the theory of rational bubbles.

However, in the presence of financial frictions, the interest rate need not accurately reflect the return to capital, and bubbles may exist even if the economy is efficient. Financial frictions not only relax the conditions for the existence of bubbles, but also enable bubbles to expand the capital stock and output. Martin and Ventura (2012) stress the wealth effect of bubble creation: new bubbles reallocate resources because they are sold by productive to unproductive agents. Kocherlakota (2009) showed that bubbles also act as providers of liquidity by allowing agents to transfer resources over time.

In our model, the possibility of bubbles is completely disconnected from dynamic efficiency. We show that there is a unique interest rate r that guarantees dynamic efficiency, and it can be either $>$ or $<$ g . Similar results are obtained by Farhi and Tirole (2012), who study an OLG model with financial friction due to moral hazard. Similarly, Reis(2021) shows that bubbles can be sustainable when $r < g < \mu$, where μ denotes the marginal productivity of capital.

We consider a continuous-time, infinite-horizon economy in the spirit of Angeletos (2007). There is a single good that can be consumed or invested as productive capital. Agents operate their own production technologies, subject to privately observed idiosyncratic shocks and a publicly observed common aggregate shock. Since idiosyncratic shocks are privately observable, allocations of capital and consumption have to satisfy incentive compatibility constraints. Using the powerful techniques of mean field control, we characterize constrained optimal allocations by a functional Hamilton Jacobi Bellman (HJB) equation. When utilities have Constant Relative Risk Aversion (CRRA), this equation can be solved almost explicitly. Moreover, constrained optimal allocations can be decentralized by a market where the good is traded against a (quasi)safe asset that can be interpreted as fiat money or as a real bond³. The key to dynamic efficiency is for the principal to set an appropriate growth rate of this quasi-safe asset.

Our paper connects three main strands of macroeconomic theory. The first is the dynamic contracting literature following Lucas and Atkeson (1992), which studies optimal insurance in environments with private information and incentive constraints. The second is the heterogeneous-agent macro literature initiated by Aiyagari (1994), where incomplete markets and idiosyncratic risk generate wealth heterogeneity and affect

¹See the survey by Martin and Ventura (2018).

²See, however the critiques of Geerolf (2013).

³We show that when there are aggregate shocks, it is optimal to index this asset on these shocks, which implies that this asset is not completely safe.

aggregate outcomes. The third is the literature on public debt and safe assets initiated Barro (1974). We combine these approaches in a production economy in the spirit of Angeletos (2007) with private information. In our framework, the planner’s problem is a dynamic contracting environment with heterogeneous agents, while the implementation relies on a single safe asset—interpreted as fiat money or government bonds—whose supply determines the equilibrium allocation.

Our work also relates to the literature on economic growth initiated by Barro (1990). In Barro’s model like in ours, government policy affects the long-run growth rate through its impact on the accumulation of productive capital. However, rather than focusing on productive public spending, we consider how the provision of safe assets affects capital accumulation when agents face private information and incomplete insurance. In this setting, monetary or fiscal policy indirectly influences the growth rate by shaping incentives and risk-sharing opportunities available to agents.

Our paper is also related to the literature on the liquidity and insurance services provided by a safe asset (such as money or government bonds). This literature was initiated by Barro (1974). Recent papers along these lines are Guerrieri and Lorenzoni (2017) and Di Tella (2020). Brunnermeier Merkel Sannikov (2024) decompose the price of safe assets into the discounted sums of future cash flows and future service flows: re-trading allows partial insurance of idiosyncratic risk in an incomplete markets setting.

On the technical side, our analysis is related to the mean-field literature, see, e.g., Lasry and Lions (2007), Cardaliaguet (2012), and Carmona and Delarue (2018). More precisely, we study a Mean Field Control problem. This is related to, but different from, Mean Field Games. Mean Field Games have been introduced in macroeconomics by Achdou, Buera, Lasry, Lions, and Moll (2022). In their model, each agent solves an optimal control problem associated with a Bellman equation, while the distribution of wealth among agents is determined by a Fokker-Planck equation. In our analysis, there is a single control problem, that of the planner who controls the distribution of individual continuation utilities. That control problem, whose state variables involve the infinite-dimensional distribution of agents’ utilities, is a mean-field control problem.

Our paper is also related to the macro-theory literature on dynamic heterogeneity in constrained economies, see, e.g., Aiyagari (1994), Angeletos (2007), Achdou et al (2022), Bewley (1977), Huguet (1997), and Krusell and Smith (1998). These models and ours share the property that idiosyncratic risks cannot be fully insured, implying that the dynamics of capital and consumption depend on the whole distribution of wealth across agents, an infinite-dimensional variable. This generates technical difficulties, which Krusell and Smith (1998) have approached by *assuming* that agents’ expectations only depend on a one-dimensional statistics, the average of the wealth distribution. They calibrate this “approximated equilibrium” and show that it fits the US consumption and investment data. In contrast, we *prove* that in our model, when utilities have constant relative risk aversion (CRRA), constrained optimal allocations only depend on a one-dimensional statistics.

Aiyagari (1994), Angeletos (2007), Achdou et al (2022), Bewley (1977), Huguet (1997), and Krusell and Smith (1998) consider exogenous financial constraints. In these papers, the only available financial instrument is a risk-free bond, and agents’ wealth cannot fall below an exogenous threshold. This constraint binds only at the boundary, which generates a Dirac mass in the distribution of wealth. In contrast, our paper provides a microfoundation of the impossibility of fully insuring idiosyncratic risks, namely, the incentive compatibility constraint. We show that this constraint binds everywhere, so that the distributions are smooth. Moreover, while these papers focus on stationary distributions, we show that the optimal mechanism does not lead to a stationary distribution. This is similar to the approach in Atkeson and Lucas (1992), who also characterize the optimal mechanism. However, Atkeson and Lucas (1992) considered a stationary endowment economy, while we consider a production economy, in which the total amount of resources

grows endogenously. Correspondingly, while in Atkeson and Lucas (1992) an agent’s utility goes to minus infinity with probability one, this is not the case in our model when average productivity is large enough. Another difference is that in Atkeson and Lucas (1992) there is only idiosyncratic risk, while we also consider aggregate risk, which has been a challenge in the literature. The treatment of aggregate risk is an important contribution of our article. It necessitates employing Ito calculus in infinite-dimensional spaces, a technique developed only recently by mathematicians and explained in Villani (2009) or Carmona and Delarue (2018). Using this technique, we can characterize how aggregate risks are shared between the principal and agents. The remainder of the paper is organized as follows:

- Section 2 presents the model for a finite number N of agents and the planning problem that characterizes dynamically efficient allocations.
- Section 3 studies the limit of the model and the planning problem when $N \rightarrow \infty$.
- Section 4 shows how constrained optimal allocations can be decentralized by a single market where the good is exchanged for a safe asset issued by the government.
- Section 5 extends the analysis to aggregate risk.
- Section 6 concludes.

2 Model

We study a standard growth model with many agents subject to privately observed idiosyncratic output shocks in the spirit of Angeletos (2007). Since idiosyncratic shocks are privately observable, incentive compatibility conditions are needed to prevent output diversion by agents. In the limit case $N \rightarrow \infty$, we characterize optimal allocations of consumption and capital under these incentive compatibility constraints by a functional HJB equation. When utilities are CRRA, the solutions to this equation are quasi explicit. Then we show how constrained efficient allocations can be decentralized by a market where the good is traded against fiat money. The key to dynamic efficiency is to set the appropriate growth rate of money supply.

2.1 Preferences and Technology

Consider a continuous-time economy, $t \in [0, \infty)$, populated by N agents. Each agent $i \in \{1, \dots, N\}$ has a mass $1/N$, such that the total population is normalized to unity. We normalize capital allocation and consumption to facilitate the analysis of the limit case of a continuum of agents as $N \rightarrow \infty$. Thus, agent i manages at date t an amount of capital $\frac{k_t^i}{N}$ and consumes $\frac{c_t^i}{N}$ per unit of time. Agents are infinitely lived and maximize their intertemporal expected utility:

$$\mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} u(c_t^i) dt \right], \tag{1}$$

where $\rho > 0$ is the discount rate and $u(\cdot)$ is an increasing and concave utility function. Following a standard convention in the literature, we scale the flow utility by ρ so that a constant consumption stream c produces a lifetime utility of $u(c)$.⁴

There is a single tradeable good that can be consumed or invested in N stochastic technologies with constant return to-scale. Technology i is operated by agent i . If agent i invests an amount k_t^i/N in their

⁴Specifically, $\int_0^\infty \rho e^{-\rho t} u(c) dt = u(c)$.

own technology, it generates a cumulative output process Y_t^i evolving according to:

$$dY_t^i = \frac{k_t^i}{N} [\mu dt + \sigma dZ_t^i + \sigma_A dZ_t^A], \quad (2)$$

where μ is the expected rate of return of the technology and the production risk is decomposed into two components:

- *Idiosyncratic Shocks:* $(Z_t^i)_{i=1}^N$ are project-specific productivity shocks, independent between agents.
- *Aggregate Shocks:* Z_t^A is a common productivity shock to which all technologies are equally exposed.

The processes $(Z_t^A, Z_t^1, \dots, Z_t^N)$ are independent standard Brownian motions. We denote by $(\mathcal{F}_t)_{t \geq 0}$ the augmented filtration generated by this Wiener process of dimension $(N + 1)$. All decision and payoff processes are assumed to be square-integrable and progressively measurable with respect to (\mathcal{F}_t) .

2.2 Capital

The aggregate capital stock at time t is defined as the average capital across all producers:

$$K_t := \frac{1}{N} \sum_{i=1}^N k_t^i. \quad (3)$$

Let c_t^i/N denote the consumption flow of agent i at time t . The law of motion for aggregate capital is determined by the net investment, where aggregate output is adjusted for the total consumption outflows:

$$dK_t = \sum_{i=1}^N dY_t^i - \left(\frac{1}{N} \sum_{i=1}^N c_t^i \right) dt. \quad (4)$$

Substituting the technology specification from (2), the evolution of the aggregate capital stock satisfies the following:

$$dK_t = (\mu K_t - \bar{c}_t) dt + \sigma_A K_t dZ_t^A + \frac{\sigma}{N} \sum_{i=1}^N k_t^i dZ_t^i, \quad (5)$$

where $\bar{c}_t = \frac{1}{N} \sum c_t^i$ is the average consumption. Equation (5) represents the resource constraint. A key feature of this environment is the diversification of risk: since idiosyncratic shocks $\{Z_t^i\}$ are independent, the final term in (5) vanishes as $N \rightarrow \infty$ by a Law of Large Numbers.⁵ Consequently, in the limit, only the aggregate productivity shock σ_A drives fluctuations in the aggregate capital stock.

A consumption path for an agent is a progressively measurable nonnegative process $(c_t)_t$ such that

$$\mathbb{E} \int_0^\infty \rho e^{-\rho s} |u(c_s)| ds < +\infty.$$

For a given consumption path (c_t^i) , the expected continuation utility of agent i at time t is

$$\omega_t^i := \mathbb{E}_t \int_t^\infty \rho e^{-\rho(s-t)} u(c_s^i) ds. \quad (6)$$

⁵We provide a formal treatment of the continuum limit and the resulting deterministic cross-sectional properties in Section 3.

2.3 The Promise-Keeping Condition

To characterize the dynamics of the agent's welfare, let M_t^i denote the expectation of agent i 's lifetime utility conditional on the information available at time t :

$$M_t^i \equiv \mathbb{E}_t \left[\int_0^\infty \rho e^{-\rho s} u(c_s^i) ds \right] = \int_0^t \rho e^{-\rho s} u(c_s^i) ds + e^{-\rho t} \omega_t^i. \quad (7)$$

Under assumption (??), M_t^i is the conditional expectation of an integrable random variable and is therefore a \mathcal{F}_t -martingale. Applying the Martingale Representation Theorem, innovations in M_t^i can be expressed as a stochastic integral with respect to the underlying Brownian motions $(Z_t^A, Z_t^1, \dots, Z_t^N)$. Differentiating (7), the law of motion for the continuation utility ω_t^i is given by:

$$d\omega_t^i = \rho [\omega_t^i - u(c_t^i)] dt + \frac{\sigma}{N} \sum_{j=1}^N \beta_t^{ij} dZ_t^j + \sigma_A \beta_t^{A,i} dZ_t^A, \quad (8)$$

where the diffusion coefficients β_t^{ij} and $\beta_t^{A,i}$ are progressively measurable processes. In the context of the optimal mechanism, these coefficients represent the incentive intensities or exposures:

- $\beta_t^{A,i}$ reflects the exposure of the agent i to the aggregate productivity shock.
- β_t^{ij} reflects the exposure of the agent i to idiosyncratic project-specific shocks (where $j = i$ is the agent's own shock and $j \neq i$ are cross-exposures).

Equation (8) is the *promise-keeping constraint*. It describes how the principal must adjust the agent's future utility to account for current consumption and realized productivity shocks, ensuring that the initial utility promised to the agent is delivered in expectation.

2.4 Incentive Compatibility

The presence of private information regarding individual output realizations introduces a fundamental agency friction. We assume that agents can secretly divert and consume a portion of the output they generate, a feature standard in the dynamic contracting literature (see, e.g., Bolton and Scharfstein, 1990; DeMarzo and Sannikov, 2006). By the Revelation Principle, we restrict our attention to direct, truthful revelation mechanisms.

The second-best allocation is determined by a mechanism that maps the history of realized aggregate and idiosyncratic shocks into allocations, subject to the resource constraint and the incentive compatibility (IC) constraint. The timing of events within a representative interval $[t, t + dt)$ is as follows:

1. *Production*: Each agent i operates the technology with capital k_t^i/N .
2. *Reporting*: Shocks dZ_t^i and dZ_t^A are realized. Agent i reports a realization $d\hat{Z}_t^i$ to the planner and transfers the corresponding output $d\hat{Y}_t^i$.
3. *Allocation*: Based on the reported history, the planner assigns the consumption flow c_t^i , the capital stock for the subsequent period k_{t+dt}^i , and a promised continuation utility ω_{t+dt}^i .

Under truthful reporting ($d\hat{Z}_t^i = dZ_t^i$), and assuming that all other agents report truthfully, the agent's instantaneous utility and change in continuation value, normalized by ρdt , are:

$$\rho u(c_t^i) dt + d\omega_t^i = \rho u(c_t^i) dt + \frac{\sigma}{N} \sum_{j=1}^N \beta_t^{ij} dZ_t^j + \sigma_A \beta_t^{A,i} dZ_t^A. \quad (9)$$

Conversely, if an agent chooses to under-report output by an amount proportional to δdt (i.e., $d\hat{Z}_t^i = dZ_t^i - \delta dt$), they divert the hidden output $\sigma\delta(k_t^i/N)$ to private consumption. The agent's total payoff under this deviation is:

$$\rho u\left(c_t^i + \sigma\delta\frac{k_t^i}{N}\right) dt + \frac{\sigma}{N}\beta_t^{ii}(dZ_t^i - \delta dt) + \frac{\sigma}{N}\sum_{j\neq i}\beta_t^{ij}dZ_t^j + \sigma_A\beta_t^{A,i}dZ_t^A. \quad (10)$$

The IC constraint requires that the payoff in (9) weakly exceeds that in (10) for any $\delta > 0$. Taking the first-order condition with respect to δ at $\delta = 0$ and exploiting the concavity of $u(\cdot)$, we obtain the following necessary and sufficient condition:

$$\beta_t^{ii} \geq \rho k_t^i u'(c_t^i). \quad (11)$$

Because $k_t^i u'(c_t^i) > 0$, condition (11) implies that agents continuation utilities must be sensitive to their idiosyncratic productivity shocks to prevent diversion. This *minimum risk exposure* is increasing in the level of capital managed by the agent and decreasing in their current consumption level, reflecting the standard trade-off between insurance and incentives.

2.5 The Aggregate Planning Problem

The planner's objective is to determine an allocation that minimizes the aggregate capital $K(\mathbb{W})$ required to deliver a target vector of continuation utilities $\mathbb{W} = (\omega^1, \dots, \omega^N)$ under feasibility and incentive compatibility conditions. In the finite- N case, this problem is highly complex, due to interactions between idiosyncratic incentive compatibility constraints and the optimal sharing of individual risks.

However, as the economy approaches the continuum limit ($N \rightarrow \infty$), the idiosyncratic productivity shocks diversify at the aggregate level. In this limit, the resource constraint is driven solely by aggregate shocks, which are publicly observable, while the idiosyncratic risk is managed through the distribution of continuation utilities. This simplification allows us to shift our analytical focus to the limit case of a continuum of agents. This is what we do in the remainder of the paper.

3 The Continuum Limit and the Functional Planning Problem

In the continuum limit, the state of the economy is no longer a finite vector of utilities but rather a continuous probability measure \mathbb{P} on \mathbb{R} representing the cross-sectional distribution of continuation utilities. The resulting planning problem is a stochastic control problem in an infinite-dimensional space. Consequently, the value function and the associated optimal controls are defined as *functionals* of the measure \mathbb{P} . To analyze this problem, we use the tools of stochastic calculus on Wasserstein spaces, as developed in the recent Mean Field Games literature (see, e.g., Villani, 2009; Cardaliaguet, 2012; Carmona and Delarue, 2018). In what follows, we characterize the functional Hamilton-Jacobi-Bellman (HJB) equation that governs the optimal evolution of the economy.

3.1 Statement of the Control Problem

We assume that as $N \rightarrow \infty$, the empirical distribution of the continuation utility vector \mathbb{W} converges weakly to a probability measure \mathbb{P} in $\mathcal{P}_2(\mathbb{R})$, the set of probability measures on \mathbb{R} that have a second moment⁶. The

⁶We endow $\mathcal{P}_2(\mathbb{R})$ with the Wasserstein distance. For $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}_2(\mathbb{R})$, $\Pi(\mathbb{P}_1, \mathbb{P}_2)$ is the set of transport plans, that is, probability measures on $\mathbb{R} \times \mathbb{R}$ with respective marginals \mathbb{P}_1 and \mathbb{P}_2 . The Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$ is defined as the square root of $\min_{\gamma \in \Pi(\mathbb{P}_1, \mathbb{P}_2)} \int_{\mathbb{R}^2} |y - x|^2 d\gamma(x, y)$.

state variable of the limit problem at time t is therefore \mathbb{P}_t , representing the conditional distribution at date t of the agents' continuation utilities given the filtration generated by the aggregate shocks.

From the theory of *propagation of chaos* (Sznitman, 1991), the idiosyncratic shocks are diversified in the aggregate, and the aggregate consumption flow converges to the cross-sectional mean:

$$\bar{C}(\mathbb{P}_t) = \int_{\mathbb{R}} c(\mathbb{P}_t, \omega) d\mathbb{P}_t(\omega). \quad (12)$$

In this limit, the resource constraint (5) converges to:

$$dK_t = \left(\mu K_t - \int_{\mathbb{R}} c(\mathbb{P}_t, \omega) d\mathbb{P}_t(\omega) \right) dt + \sigma_A K_t dZ_t^A, \quad (13)$$

implying that the only stochastic driver of the aggregate capital stock is the aggregate productivity shock dZ_t^A . Simultaneously, the dynamics of an individual agent's continuation utility are reduced to two stochastic components: the aggregate shock and the agent's own idiosyncratic shock. By the *sufficient statistic principle* (Holmström, 1979), it is suboptimal to index an agent's welfare to the idiosyncratic shocks of others. Thus, by the martingale representation theorem, we can write:

$$d\omega_t = \rho(\omega_t - u(c_t)) dt + \sigma \beta_t dZ_t + \sigma_A \beta_t^A dZ_t^A. \quad (14)$$

For the remainder of this section, we focus on the deterministic benchmark where $\sigma_A = 0$. In this case, integrating (13) and imposing the transversality condition $\lim_{t \rightarrow \infty} e^{-\mu t} K_t = 0$ yields a single resource constraint:

$$K = \int_0^\infty e^{-\mu t} \left[\int_{\mathbb{R}} c(\mathbb{P}_t, \omega) d\mathbb{P}_t(\omega) \right] dt. \quad (15)$$

The Planning Problem

The constrained-efficient allocation minimizes the total capital $K(\mathbb{P})$ required to support the distribution \mathbb{P} under promise keeping, feasibility and incentive compatibility constraints. We state the associated recursive control problem as follows:

$$K(\mathbb{P}) = \min_{\{k, c, \beta\}} \int_0^\infty e^{-\mu t} \left[\int_{\mathbb{R}} c(\mathbb{P}_t, \omega) d\mathbb{P}_t(\omega) \right] dt \quad (16)$$

subject to the following constraints for all t :

$$d\omega_t = \rho[\omega_t - u(c_t)] dt + \sigma \beta_t dZ_t \quad (\text{Promise Keeping}) \quad (17)$$

$$\beta_t \geq \rho k_t u'(c_t) \quad (\text{Incentive Compatibility}) \quad (18)$$

$$K(\mathbb{P}_t) = \int_{\mathbb{R}} k(\mathbb{P}_t, \omega) d\mathbb{P}_t(\omega) \quad (\text{Capital Allocation}) \quad (19)$$

The mapping $K(\cdot)$ is a functional defined from the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ to \mathbb{R} . We will characterize the solution of this infinite dimensional control problem⁷ by showing that it satisfies a functional Hamilton-Jacobi-Bellman equation in $\mathcal{P}_2(\mathbb{R})$. We first need to introduce the necessary tools of stochastic calculus in measure spaces.

⁷Note that this problem has an unusual feature: the value function appears in the capital allocation constraint.

3.2 Differential calculus in the space of measures

The space is infinite-dimensional and does not have a vector space structure. To characterize the sensitivity of the value function K with respect to variations in the probability measure \mathbb{P} , we need to define the derivative of a functional $F : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$. We start by recalling the notions of first and second variations and L-differentiability for functions of measures relying on the convexity of $\mathcal{P}_2(\mathbb{R})$ (see Carmona and Delarue Vol 2, Chapter 4, Assumptions (A1) and (A2)).

Definition 1. • *The functional F admits a first variation in $\mathbb{P} \in \mathcal{P}_2(\mathbb{R})$ if there exists a continuous real-valued function $\nabla F(\mathbb{P}) : \mathbb{R} \rightarrow \mathbb{R}$, such that for all \mathbb{Q} in $\mathcal{P}_2(\mathbb{R})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F((1 - \varepsilon)\mathbb{P} + \varepsilon\mathbb{Q}) - F(\mathbb{P})) = \int_{\mathbb{R}} \nabla F[\mathbb{P}](\omega) d(\mathbb{Q} - \mathbb{P})(\omega).$$

- *The functional F admits a second variation at $\mathbb{P} \in \mathcal{P}_2(\mathbb{R})$ if for every ω , the functional $\nabla F(\cdot)(\omega)$ admits a first variation at \mathbb{P} .*
- *The functional F is L-differentiable in \mathbb{P} if it admits a first variation in $\mathbb{P} \in \mathcal{P}_2(\mathbb{R})$ and the functional $\nabla F(\mathbb{P})$ is twice differentiable in \mathbb{R} . We will denote by $\partial_{\omega} \nabla F(\mathbb{P})$ and $\partial_{\omega\omega} \nabla F[\mathbb{P}]$ its first and second derivatives.*

3.3 The Social Planner's problem with a continuum of agents

To characterize the social planner's problem, we rely on the recursive representation of the value functional. Let $K(\mathbb{P})$ denote the minimum cost of providing the required utilities given the current distribution \mathbb{P} . Formally, the value functional satisfies the Dynamic Programming Principle (DPP):

$$K(\mathbb{P}_t) = \inf_{c, k, \beta} \left[\int_t^{t+dt} \left(\int_{\mathbb{R}} c(\mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) ds + e^{-\mu dt} K(\mathbb{P}_{t+dt}) \right]. \quad (20)$$

To determine the law of motion for K , we will now pass to a more compact infinitesimal version of the dynamic programming principle by considering the first variation $\nabla K(\mathbb{P})(\cdot)$, which will take the form of an ordinary differential equation. It relies on expansions of $e^{-\mu dt} K(\mathbb{P}_{t+dt})$ on the right-hand side of (20) based on the following the Itô formula for a real-valued function F defined on $\mathcal{P}_2(\mathbb{R})$ which takes the following form (see Carmona and Delarue (2018), Vol 1, Chapter 5, Th. 5.92) under the underlying state dynamics:

$$F(\mathbb{P}_t) = F(\mathbb{P}_0) + \int_0^t \rho \mathbb{E} [\partial_{\omega} \nabla F(\mathbb{P}_s)(\omega_s)(\omega_s - u(c(\mathbb{P}_s, \omega_s)))] ds \quad (21)$$

$$+ \frac{1}{2} \int_0^t \mathbb{E} [\partial_{\omega\omega}^2 \nabla F(\mathbb{P}_s)(\omega_s) \sigma^2 \beta^2(\mathbb{P}_s, \omega_s)] ds. \quad (22)$$

Applying Itô's Lemma and suppressing the arguments (\mathbb{P}, ω) for brevity, we obtain the following functional HJB equation:

$$\mu K(\mathbb{P}) = \inf_{c, k, \beta} \int_{\mathbb{R}} \left[c + \rho(\omega - u(c)) \partial_{\omega} \nabla K + \frac{\sigma^2 \beta^2}{2} \partial_{\omega\omega}^2 \nabla K \right] d\mathbb{P}(\omega), \quad (23)$$

subject to the incentive compatibility (IC) and aggregate capital allocation constraints (18)-(19). As is standard in this class of problems, we assume (and later verify) that $\partial_{\omega\omega}^2 \nabla K > 0$. This implies that the IC constraint is always binding: $\beta = \rho k u'(c)$. Notably, in the absence of agency frictions, the allocation of capital k would be indeterminate due to the ex-ante homogeneity of agents. Here, however, capital serves as

a critical incentive-provision device. Let $\lambda(\mathbb{P})$ denote the Lagrange multiplier associated with the aggregate resource constraint $\int k(\mathbb{P}, \omega) d\mathbb{P} = K(\mathbb{P})$. This leads to the following constrained functional Hamilton-Jacobi-Bellman (HJB) equation.

$$(\mu - \lambda(\mathbb{P}))K(\mathbb{P}) = \inf_{c(\cdot), k(\cdot)} \int_{\Omega} \left[c + \rho(\omega - u(c))\partial_{\omega} \nabla K + \frac{(\rho\sigma k u'(c))^2}{2} \partial_{\omega\omega}^2 \nabla K - \lambda(\mathbb{P})k \right] d\mathbb{P}. \quad (24)$$

Equation (24) establishes that the value functional K (if it exists) is a solution to a second-order differential equation in the space of measures. While such functional equations may admit multiple solutions, the following verification theorem identifies the unique solution to the planner's problem.

Proposition 1. (*Verification Theorem*) *Let $\lambda(\cdot)$ be a given Lagrange multiplier, and $K^{\lambda}(\mathbb{P})$ be a L -differentiable function as defined in Definition 1. If*

i) K^{λ} is a solution to (24) with the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\mu t} K^{\lambda}(\mathbb{P}_t) = 0,$$

ii) there exists a control $(c_{\lambda}^, k_{\lambda}^*)$ that reaches the minimum in (24),*

iii) and there is a Lagrange multiplier λ^ such that for every $t \geq 0$,*

$$\int k_{\lambda^*}^*(\omega) d\mathbb{P}_t(\omega) = K^{\lambda^*}(\mathbb{P}_t),$$

then K^{λ^} coincides with the optimal value functional K of the planner's problem.*

Proof. The proof is in the appendix. □

While solving the HJB equation for arbitrary utility specifications is generally intractable, the problem becomes manageable under the assumption of Constant Relative Risk Aversion (CRRA). In the following subsection, we employ a "guess-and-verify" approach to derive a closed-form characterization of the optimal allocation.

3.4 Constrained Optimal Allocations when u is CRRA

In this section, we assume $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$, with $\alpha > 1$.

Proposition 2. *The optimal controls (k^*, c^*) in the HJB equation do not depend on \mathbb{P} .*

They can be parameterized by two constants a^ and y^* such that*

$$k(\omega) = a^* u^{-1}(\omega), \quad c(\omega) = \frac{\rho k(\omega)}{y^*},$$

where a^ is the unique solution of*

$$\frac{\mu}{\rho} = \inf_y \left[\frac{1}{y} + \frac{(\frac{\rho a^*}{y})^{1-\alpha} - 1}{\alpha - 1} + \frac{\alpha \sigma^2}{2\rho} (\rho a^*)^{2-2\alpha} y^{2\alpha} \right], \quad (25)$$

and y^ is the value of y at which the infimum is obtained.*

Proof. The proof is in the appendix. □

The next proposition characterizes dynamically efficient allocations when utilities are CRRA.

Proposition 3. *Consider a dynamic allocation (k_t, c_t) of capital and consumption as a function of individual performance. This allocation is constrained efficient if and only if*

- *individual consumption is proportional to individual capital*

$$\forall t, c_t = \frac{\rho}{y^*} k_t.$$

- *The sensitivity of individual capital to performance is constant:*

$$\frac{dk_t}{k_t} = g^* dt + \sigma(y^*)^\alpha (\rho a^*)^{1-\alpha} dZ_t,$$

where $g^* = \mu - \frac{\rho}{y^*}$ is the constant growth rate of aggregate capital.

Note that $g^* < \mu$, which implies that the transversality condition is satisfied:

$$\lim_{t \rightarrow \infty} K_t e^{-\mu t} = \lim_{t \rightarrow \infty} K_0 e^{(g^* - \mu)t} = 0.$$

Under CRRA preferences, the optimal contract exhibits a fundamental separability between the cross-sectional distribution of utility and the dynamic provision of incentives, echoing the structural properties identified by Atkeson and Lucas (1992). This decoupling allows the social planner's problem to be decomposed into two distinct components:

- **The Distribution Problem:** The initial distribution of welfare is addressed via the assignment of starting capital and consumption levels. For an agent with initial promised utility ω , the allocations are pinned down by:

$$k_0(\omega) = a^* u^{-1}(\omega), \quad c_0(\omega) = \frac{\rho k_0(\omega)}{y^*}$$

Here, the initial capital stock k_0 serves as the state variable that scales the agent's lifetime claims, while consumption is proportional to this initial endowment.

- **The Incentive Problem:** To maintain incentive compatibility, the planner dynamically reallocates k_t and c_t according to the realization of idiosyncratic performance shocks, dZ_t . The evolution of the agent's capital and the corresponding consumption rule are characterized by the following stochastic differential equation:

$$\frac{dk_t}{k_t} = g^* dt + \sigma(y^*)^\alpha (\rho a^*)^{1-\alpha} dZ_t, \quad c_t = \frac{\rho k_t}{y^*}$$

3.5 A Closed Form Solution when Utility is Logarithmic

Under logarithmic preferences, the decoupling of the distribution and incentive problems results in a closed-form solution. This allows us to pin down the equilibrium parameters through a single-variable optimization and a cubic equation, facilitating a direct inspection of how volatility governs the agent's marginal propensity to consume and their subsequent risk exposure.

Proposition 4. *When $u(c) = \log c$ the solution is explicit. a^* is uniquely defined by*

$$\frac{\mu}{\rho} + \log \rho a^* = \inf_y \left[\frac{1}{y} + \log y + \frac{\sigma^2 y^2}{2\rho} \right],$$

while y^* is the unique real root of a cubic equation:

$$\frac{\sigma^2 y^3}{\rho} + y = 1.$$

Proof. If we take the limit of equation (36) for $\alpha \rightarrow 1$, we obtain the following relation:

$$\frac{\mu}{\rho} = \inf_y \left[\frac{1}{y} - \log \frac{\rho a^*}{y} + \frac{\sigma^2 y^2}{2\rho} \right]. \quad (26)$$

This establishes the first part of Proposition 4. The second part is a simple consequence of the first order condition of this minimization problem. \square

Note that one can also write

$$y^* = \operatorname{argmax}_y \left[\log \frac{\rho}{y} + \frac{\mu - \frac{\rho}{y} - \frac{\sigma^2 y^2}{2}}{\rho} \right]. \quad (27)$$

The interpretation of this condition is that y^* trades-off optimally between the utility of consumption, the growth rate, and individual risk exposure.

4 Decentralized Implementation

4.1 The Institutional Set-up

Having characterized the constrained efficient allocations, we now demonstrate how these can be decentralized in a market environment where fiat money serves as the medium of exchange.⁸ We assume the existence of a central bank that issues fiat money—represented as either physical currency or a Central Bank Digital Currency (CBDC)—directly to private agents. The achievement of the social optimum depends on the implementation of an appropriate monetary policy. Given our stylized framework, which abstracts from labor supply, multi-sector production, nominal rigidities, and financial intermediation, the model does not aim to capture the multifaceted transmission channels of monetary policy explored in, for instance, Kaplan et al. (2009). In our setting, the central bank maintains direct access to agents and exercises perfect control over the nominal money supply, M_t . Reflecting the stationarity of the optimal allocations in an economy growing at a constant rate, we restrict our attention to stationary monetary policies. Specifically, we assume the central bank commits to a constant money growth rate g_m , such that:

$$M_t = M_0 \exp(g_m t).$$

Since the principal does not consume, the seigniorage revenue is distributed entirely to the agents. Our analysis focuses on stationary rational expectations equilibria characterized by a constant anticipated inflation rate π . We now proceed to examine the dynamics of individual wealth within this equilibrium framework.

Consider a typical agent with capital k_t and money m_t at date t . His real wealth at time t is

$$w_t = k_t + \frac{m_t}{p_t}. \quad (28)$$

⁸An alternative implementation is possible through a market for bonds with an endogenously determined interest rate.

It evolves according to the following dynamics:

$$dw_t = k_t(\mu dt + \sigma dZ_t) + [\tau w_t - \pi(w_t - k_t) - c_t]dt. \quad (29)$$

The increase in real wealth is thus decomposed as output plus transfers⁹ τw_t , minus the inflation tax πm_t minus consumption c_t . The budget constraint of the central bank is (recall that there are no public expenditures or remittances to the government):

$$\frac{dM_t}{dt} \equiv g_m M_t = \tau p_t W_t,$$

where W_t denotes aggregate real wealth. Seignorage revenue is therefore entirely transferred to the agents.

4.2 Individual behavior

The representative agent faces a continuous-time stochastic control problem, choosing consumption c_t and capital investment k_t to maximize the expected discounted lifetime utility:

$$v(w) = \sup_{\{k_t, c_t\}} \mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} \log c_t dt \right]$$

subject to the following wealth dynamics:

$$dw_t = k_t(\mu dt + \sigma dZ_t) + [\tau w_t - \pi(w_t - k_t) - c_t] dt.$$

The value function $v(w)$ is characterized by the Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho v(w) = \sup_{k,c} \left\{ \rho \log c + v'(w)[k(\mu + \pi) + w(\tau - \pi) - c] + \frac{\sigma^2 k^2}{2} v''(w) \right\}$$

Given the logarithmic utility specification¹⁰ and the linear structure of the budget constraint, the value function is additive-logarithmic in wealth: $v(w) = \log w + v(1)$. This homogeneity implies the following derivatives:

$$v'(w) = \frac{1}{w}, \quad v''(w) = -\frac{1}{w^2}$$

The first-order conditions for consumption and capital investment yield:

$$c = \rho w, \quad k = \frac{\mu + \pi}{\sigma^2} w$$

Under this optimal policy, agents maintain a constant portfolio allocation, investing a fixed fraction $y = \frac{\mu + \pi}{\sigma^2}$ of their wealth in productive capital, with the remainder held in money. Along the balanced growth path, the economy grows at a constant rate:

$$g = \mu - \frac{\rho}{y}$$

⁹To maintain the homogeneity property of the individual optimization problem, we assume that transfers are proportional to individual wealth.

¹⁰In appendix, we show that implementation is slightly more complex but qualitatively similar when utilities are power functions.

Consistent with rational expectations, the equilibrium inflation rate is endogenously determined by:

$$\pi = \sigma^2 y - \mu$$

4.3 Equilibrium

Individual demand for money is

$$m_t = (1 - y)p_t w_t = \frac{1 - y}{y} p_t k_t.$$

Aggregate money demand is thus $M_t = \frac{1-y}{y} p_t K_t$. The market clearing condition can be expressed by equating the money supply and the aggregate money demand:

$$M_0 \exp g_m t = \frac{(1 - y)}{y} p_t K_t = \frac{(1 - y)}{y} (p_0 \exp \pi t) (K_0 \exp g t)$$

It is equivalent to two conditions:

$$M_0 = \frac{(1 - y)}{y} p_0 K_0,$$

showing that the initial price is proportional to the initial aggregate quantity of money M_0 (neutrality), and

$$g_m = \pi + g,$$

which expresses that the growth rate of money is the sum of inflation and real growth rates. Now we have

$$g = \mu - \frac{\rho}{y}, \quad \pi = \sigma^2 y - \mu.$$

Thus,

$$g_m = g + \pi = \sigma^2 y - \frac{\rho}{y},$$

which uniquely determines y .

Proposition 5. *For any constant money growth rate $g_m < \sigma^2 - \rho$ set by the principal, there is a unique stationary rational expectation equilibrium where money has a positive value.*

It is such that

- *Agents invest a constant fraction $y < 1$ of their wealth in capital (and the rest in money), where y is the unique solution of*

$$g_m = \sigma^2 y - \frac{\rho}{y}.$$

- *Agents consume a constant fraction ρ of their wealth.*
- *Aggregate capital K_t grows at a constant rate $g = \mu - \frac{\rho}{y}$.*
- *Price p_t increases at a constant rate $\pi = \sigma^2 y - \mu$.*

4.4 The Welfare Theorems

The following table shows the link between equilibrium allocations and constrained optimal allocations.

	Constrained Optima	Equilibria
capital allocations	$k(\omega_t) = a^* \exp \omega_t$	$k_t = y w_t$
consumption	$c_t = \frac{\rho k_t}{y^*}$	$c_t = \rho w_t$
growth rate	$g^* = \mu - \frac{\rho}{y^*}$	$g = \mu - \frac{\rho}{y}$
capital dynamics	$\frac{dk_t}{k_t} = g^* dt + \sigma y^* dZ_t$	$\frac{dk_t}{k_t} = g dt + \sigma y dZ_t$

When y equals y^* and $w_t = \frac{a^*}{y^*} \exp \omega_t$, we see that $k_t = y^* w_t = a^* \exp \omega_t$, and the two allocations coincide.

Thus, the two welfare theorems are valid in our set-up:

Proposition 6. *If the Central Bank chooses the money growth rate $g_m^* = \sigma^2 y^* - \frac{\rho}{y^*}$, then*

1. *The competitive equilibrium allocation is constrained efficient for any initial distribution of wealth.*
2. *In contrast, any constrained optimal allocation can be implemented as an equilibrium allocation after an initial reallocation of wealth.*

In our model where financial markets are endogenously incomplete, the two welfare theorems are valid if and only if the growth rate of money supply is well chosen. This condition completely characterizes dynamic efficiency in our set-up.

4.5 Optimal Monetary Policy without Aggregate Risk

Proposition 6 allows us to derive the optimal monetary policy in our model. In our stationary environment, the central bank simply has to set the growth rate of money at a constant (and appropriate) level.

Proposition 7. *When there is no aggregate risk and utilities are logarithmic¹¹, optimal monetary policy is characterized by a constant growth rate of the money supply:*

$$g_m^* = \sigma^2 y^* - \frac{\rho}{y^*},$$

where y^* is the unique real root of the cubic equation

$$\frac{\sigma^2 y^3}{\rho} + y = 1.$$

This condition ensures that the economy grows at the optimal real rate $g^* = \mu - \frac{\rho}{y^*}$.

Fiat money is not necessary for implementing constrained optimal allocations. Government bonds could also do the job. Suppose indeed that, rather than money, the government issues real bonds that can be traded one for one with the good. The model becomes entirely real (money disappears). The government commits to pay a constant interest rate r on bonds and constantly issues the quantity of bonds demanded by the agents at this particular interest rate. By a simple adaptation of Proposition 6, we can show that for any interest rate r there is a unique equilibrium where

- agents invest a constant fraction $y = \frac{\mu - r}{\sigma^2}$ of their wealth in capital and the rest in bonds.

¹¹See the appendix for the extension to power utilities.

- agents consume a constant fraction ρ of their wealth
- the economy grows at rate $g = \mu - \frac{\rho}{y}$.

Constrained dynamic efficiency is obtained when $y = y^*$, which corresponds to an optimal interest rate

$$r^* = \mu - \sigma^2 y^*.$$

The budget constraint of the government becomes:

$$\tau = (g^* - r^*)(1 - y^*).$$

The optimal growth rate g^* of the economy can be larger or smaller than the interest rate r^* . Thus dynamic efficiency is unrelated with the existence of bubbles. To see this, it is convenient to express all policy variables as functions of y^* rather than σ . Indeed

$$\frac{\sigma^2}{\rho} = \frac{1 - y^*}{(y^*)^3}.$$

Thus

$$g^* - r^* = \sigma^2 y^* - \frac{\rho}{y^*} = \rho \left[\frac{1 - y^*}{(y^*)^2} - \frac{1}{y^*} \right] = \rho \left[\frac{1 - 2y^*}{(y^*)^2} \right].$$

This quantity can be positive or negative. Thus, there are two possible regimes:

- $\sigma^2 > 4\rho$: $y^* < \frac{1}{2}, g^* > r^*$: agents receive positive transfers. Bonds are not backed by future taxes.
- $\sigma^2 < 4\rho$: $y^* > \frac{1}{2}, g^* < r^*$: in this case, the fiscal theory of the price level (Cochrane 2023) applies, as the value of government bonds equals the present value of future taxes.

5 Extension to Aggregate Risk

We now assume that $\sigma_A > 0$: there is an aggregate productivity shock $\sigma_A dZ_t^A$, the same for all agents, where Z_t^A is also a Brownian motion, independent of Z_t .

5.1 Statement of the Problem

Aggregate capital K_t becomes stochastic:

$$dK_t = (\mu K_t - \int c_t(\omega) d\mathbb{P}) dt + \sigma_A K_t dZ_t^A.$$

By the martingale representation theorem, the dynamics of ω_t also has a new component:

$$d\omega_t = \rho(\omega_t - u(c_t)) dt + \sigma \beta_t dZ_t + \sigma_A \beta_t^A dZ_t^A.$$

We will assume that the optimal mechanism satisfies the Markov property: $K_t = K(\mathbb{P}_t)$, where K is a deterministic functional. Recall the standard Ito formula: when $k(\cdot)$ is a \mathbb{C}^2 function and ω_t a diffusion process, $k_t \equiv k(\omega_t)$ is also a diffusion process that satisfies

$$dk_t = k'(\omega_t) d\omega_t + \frac{1}{2} k''(\omega_t) (d\omega_t)^2.$$

There is a **functional** equivalent of the Ito formula: when $K(\cdot)$ is a smooth **functional**, and \mathbb{P}_t a diffusion process in the space of measures, then $K_t \equiv K(\mathbb{P}_t)$ is also a diffusion process such that

$$dK_t = \int f(\omega)[d\mathbb{P}_{t+dt} - d\mathbb{P}_t](\omega) + \frac{1}{2} \int \int h(\omega, \omega')[d\mathbb{P}_{t+dt} - d\mathbb{P}_t](\omega)[d\mathbb{P}_{t+dt} - d\mathbb{P}_t](\omega'),$$

where f denotes the first variation and h the second variation of K . By using the dynamics of ω_t :

$$d\omega_t = \rho(\omega_t - u(c_t)) dt + \sigma\beta dZ_t + \sigma_A\beta_A dZ_t^A,$$

we can write the stochastic differential of K_t :

$$\begin{aligned} dK_t = & \int \left[\rho[\omega - u(c)]f'(\omega) + \frac{\sigma^2\beta^2 + \sigma_A^2\beta_A^2}{2} f''(\omega) \right] d\mathbb{P}(\omega) dt + \sigma_A \int \beta_A(\omega) f'(\omega) d\mathbb{P}(\omega) dZ_t^A \\ & + \frac{\sigma_A^2}{2} \int \int h(\omega, \omega') \beta_A(\omega) \beta_A(\omega') d\mathbb{P}(\omega) d\mathbb{P}(\omega') dt. \end{aligned}$$

By the feasibility constraint we also have another expression of dK_t :

$$dK_t = (\mu K_t - \int c_t(\omega) d\mathbb{P}) dt + \sigma_A K_t dZ_t^A.$$

By identifying the stochastic components of these two formulas, we obtain an additional constraint for the allocation of systemic risk:

$$\int \beta_A(\omega) f'(\omega) d\mathbb{P} = K(\mathbb{P}). \quad (30)$$

The value functional $K(\mathbb{P})$ solves a stochastic control problem in the space of measures:

$$K(\mathbb{P}) = \min_{(k, c, \beta, \beta_A)} \mathbb{E} \left[\int_0^\infty e^{-\mu t} (\int c_t(\mathbb{P}, \omega) d\mathbb{P}) dt \right], \quad (31)$$

under promise keeping

$$d\omega_t = \rho[\omega_t - u(c_t)] dt + \sigma\beta_t dZ_t + \sigma_A\beta_A dZ_t^A, \quad (32)$$

plus **three** constraints: incentive compatibility, capital allocation, and risk allocation (30). Like before, we will assume (and guess ex-post) that $f''(\omega) > 0$, which implies that the incentive compatibility constraint is binding everywhere:

$$\beta(\omega) \equiv \rho k(\omega) u'(c(\omega)).$$

5.2 The functional HJB equation in the stochastic case

Proposition 8. *If the value functional K has a first variation f and a second variation h that are both twice continuously differentiable, for all \mathbb{P} there exist two multipliers λ and λ' , associated respectively with capital allocation and risk allocation constraints, such that*

$$\begin{aligned} (\mu - \lambda - \lambda') K(\mathbb{P}) = & \inf_{c, k, \beta_A} \int \left[c + \rho[\omega - u(c)]f'(\omega) + \frac{(\sigma\rho k(\omega)u'(c(\omega)))^2 + \sigma_A^2\beta_A^2}{2} f''(\omega) \right] d\mathbb{P}(\omega) \\ & + \frac{\sigma_A^2}{2} \int \int h(\omega, \omega') \beta_A(\omega) \beta_A(\omega') d\mathbb{P}(\omega) d\mathbb{P}(\omega') - \lambda \int k(\omega) d\mathbb{P}(\omega) - \lambda' \int \beta_A(\omega) f'(\omega) d\mathbb{P}(\omega). \end{aligned}$$

This is the **functional form** of the stochastic Hamilton Jacobi equation.

5.3 The case of log utilities

When $u(c) = \log c$, aggregate risk does not fundamentally change the properties of constrained optimal allocations:

- Agents invest a constant fraction y^* (the same as in the absence of aggregate risk) of their wealth in productive capital and the rest in money.
- The economy grows at a random rate:

$$\frac{dK_t}{K_t} = \left(\mu - \frac{\rho}{y^*}\right)dt + \sigma_A dZ_t^A.$$

In order for the portfolio choice of agents to remain unchanged by aggregate risk, we need the real return on money to be

$$-\pi^* dt + \sigma_A dZ_t^A,$$

which is stochastic. The intuition is that aggregate risk must be borne equally by all agents. This can be implemented if the central bank sets a constant growth rate of the money supply g_m , as in the case without aggregate risk. The inflation rate fluctuates around the target π^* . This can be interpreted as flexible inflation targeting:

$$\frac{dp_t}{p_t} = \pi^* dt - \sigma_A dZ_t^A.$$

The dynamics of individual wealth is

$$dw_t = k_t(\mu dt + \sigma dZ_t) - [c_t + \pi(w_t - k_t) - \tau w_t]dt + \sigma_A dZ_t^A,$$

so that the portfolio choice of agents is characterized by the same program as before. The proportion y^* of wealth that agents invest in productive capital is given by

$$\sup_y \left[\mu y - \pi^*(1 - y) - \frac{\sigma^2 y^2 + \sigma_A^2}{2} \right]$$

Proposition 9. *Like before, optimal allocations can be implemented if the central bank sets a constant money growth rate $g_m^* = g^* + \pi^*$. The only difference is that inflation is no more constant but fluctuates around the target π^* :*

$$\frac{dp_t}{p_t} = \pi^* dt - \sigma_A dZ_t^A.$$

The real return on money is stochastic, in order for aggregate risk to be borne equally by all agents..

This is exactly the recommendation of Friedman (1960), p.106: "The simpler rule is that the stock of money be increased at a fixed rate year-in and year-out without any variation in the rate of increase to meet cyclical needs." The only difference is that Friedman recommended $g_m = g$, so that $\pi = 0$ (no inflation), whereas our simple model instead suggests $g_m = g + \pi^*$ with an endogenously determined optimal level of inflation π^* . This conclusion is in sharp contrast with the New Keynesian literature on monetary policy.

6 Conclusion

This paper makes three contributions.

First, we develop mean field control techniques that can be used for characterizing Pareto optima in dynamic economies with a large number of heterogeneous agents. They correspond to the solutions of a functional HJB equation that satisfy a transversality condition. We use these techniques to determine constrained optimal allocations in an AK growth model with a continuum of agents subject to unobservable idiosyncratic shocks. They can be used as well for other purposes, such as determining the impact of competition between private and public monies, as in our companion paper Biais et al. (2026), or characterizing optimal contracts between a large investor and a continuum of fund managers, as in Biais et al (2025).

Second, we find closed form solutions to this equation when utilities have constant relative risk aversion. Like in Atkeson and Lucas (1992) the planning problem has simple solutions when utilities are CRRA. The incentive compatibility problem and the distribution problem are independent of each other. Our functional HJB equation in the space of measure is equivalent to a scalar HJB equation that has closed form solutions.

Third, we show that these allocations can be implemented by a central bank that provides agents with fiat money that can be used to trade the good at an endogenous equilibrium price, or a government that issues real bonds that can be exchanged one to one for the good.

Our results differ from those generated by other frictions, such as price rigidities or OLG models. In our set-up, the possibility of bubbles is disconnected from dynamic efficiency: at the optimum policy rule, we can have $r > or < g$ according to the model parameters.

The relevance of our implementation by controlling the growth rate of the money supply seems to be at odds with reality because modern central banks no longer conduct monetary policy by targeting monetary aggregates. Instead, policy is implemented primarily through the setting of short-term interest rates and through the management of central bank liabilities such as reserves. Nevertheless, the core insight of the model can be translated into the institutional framework of contemporary monetary policy.

The main result of the paper is that dynamic efficiency requires the planner to choose the appropriate growth rate of the safe asset. In the decentralized implementation characterized in Section 4, this policy determines the return on the safe asset and, therefore, the equilibrium portfolio allocation between productive capital and the safe asset. Agents allocate a constant fraction of their wealth to productive capital and the remainder to the safe asset, and this portfolio share determines the growth rate of capital and the dynamics of consumption. What matters in this environment is therefore not the nominal quantity of money per se but the return on the safe asset relative to the return on productive capital since this relative return governs agents' portfolio choices.

In modern monetary systems, households and firms typically do not hold central bank liabilities directly. Instead, their holdings of safe assets essentially take the form of bank deposits, which are liabilities of commercial banks. These deposits are widely perceived as safe because they are backed by a combination of central bank reserves, prudential regulation, and public guaranties such as deposit insurance. Banks hold reserves in the central bank and create deposit liabilities held by households, effectively intermediating access to the safe asset supplied by the central bank. Through its control of short-term interest rates and the remuneration and quantity of reserves, the central bank influences the return on these safe assets relative to the return on productive investment. Although the model describes policy in terms of the growth rate of money, modern central banks instead implement policy by choosing the short-term interest rate, which determines the return on safe assets relative to productive investment. In both cases, policy works through the same channel: it determines the relative return on safe assets, and therefore the equilibrium allocation of wealth between safe assets and productive capital.

Looking ahead, the insights of the model may become even more directly relevant in a future monetary system that includes central bank digital currencies (CBDCs). In the model, fiat money is a safe asset issued directly by the central bank and held by agents. A widely accessible retail CBDC would bring the institutional structure of the monetary system closer to this theoretical environment than the current system, in which households primarily hold bank deposits rather than central bank liabilities. By influencing both the quantity and the remuneration of digital central bank money held by the public, central banks could more directly affect the supply and return of the safe asset available to households, making the link between the theoretical mechanism and the implementation of monetary policy even more transparent.

More broadly, the analysis suggests that the key policy variable is the supply and return of safe assets in the economy: by shaping the relative return between safe assets and productive capital, monetary policy determines the portfolio allocation of agents and, therefore, the long-term allocation of resources, growth, and welfare.

Appendix: proofs

Proof of Proposition 1 Fix $\mathbb{P} \in \mathbb{P}_2(\mathbb{R})$ and a Lagrange multiplier λ . Let \mathbb{P}_t be the probability distribution of the random variable ω_t when the initial probability distribution of ω_0 is \mathbb{P} . We apply Itô's formula (21) to $K^\lambda(\mathbb{P}_t)$ between $s = 0$ and $s = t$ for $t > 0$.

$$\begin{aligned} e^{-\mu t} K^\lambda(\mathbb{P}_t) &= K^\lambda(\mathbb{P}) \\ &\quad - \int_0^t \mu e^{-\mu s} K^\lambda(\mathbb{P}_s) ds \\ &\quad + \int_0^t e^{-\rho s} \int \partial_\omega \nabla K^\lambda[\mathbb{P}_s](\omega) (\rho\omega - u(c(\mathbb{P}_s, \omega))) d\mathbb{P}_s(\omega) ds \\ &\quad + \int_0^t e^{-\rho s} \int \partial_{\omega\omega}^2 \nabla K^\lambda[\mathbb{P}_s](\omega) \frac{\sigma^2}{2} y^2(\mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) ds. \end{aligned}$$

The HJB equation satisfied by K^λ implies that

$$K^\lambda(\mathbb{P}) \leq e^{-\mu t} K^\lambda(\mathbb{P}_t) + \int_0^t e^{-\mu s} \left(c(\mathbb{P}_s, \omega) + \lambda(K_s, \mathbb{P}_s) \left(K_s - \int k(\mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds.$$

Letting t tend to $+\infty$ and using the transversality condition, we obtain

$$K^\lambda(\mathbb{P}) \leq \int_0^\infty e^{-\mu s} \left(\bar{C}(\mathbb{P}_s) + \lambda(\mathbb{P}_s) \left(K_s - \int k(\mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds.$$

Since the control pair is arbitrary, we have

$$K^\lambda(\mathbb{P}) \leq \inf_{c, k} \int_0^\infty e^{-\mu s} \left(\bar{c}(\mathbb{P}_s) + \lambda(\mathbb{P}_s) \left(K_s - \int k(\mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds.$$

On the other hand, one can apply the same Itô argument with the optimal control pair satisfying ii) to obtain

$$K^\lambda(\mathbb{P}) = \inf_{c, k} \int_0^\infty e^{-\mu s} \left(\bar{C}(\mathbb{P}_s) + \lambda(\mathbb{P}_s) \left(K_s - \int k(\mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds.$$

Choosing λ^* allows us to conclude the proof. QED

Proof of Proposition 2

The proof follows a constructive verification procedure. We postulate that the solution is independent of the continuation utility distribution \mathbb{P} and that optimal controls are proportional to the inverse utilities of the agents. We then compute the associated value function, and show that it satisfies the HJB equation and the transversality condition.

If the solution is independent of \mathbb{P} , then $k(\mathbb{P}, \omega) = k(\omega)$ and the first variation of

$$K(\mathbb{P}) = \int k(\omega) d\mathbb{P}(\omega)$$

coincides with $k(\cdot)$. The HJB equation (24) becomes:

$$(\mu - \lambda) \int k(\omega) d\mathbb{P}(\omega) = \inf_{c(\cdot), k(\cdot)} \int \left[c + \rho(\omega - u(c))k'(\omega) + \frac{(\rho\sigma k u'(c))^2}{2} k''(\omega) - \lambda k \right] d\mathbb{P}, \quad (33)$$

where the multiplier λ is also independent of \mathbb{P} . This equation must be true for all \mathbb{P} . Since the term between brackets does not depend on \mathbb{P} , the HJB equation must be valid pointwise:

$$(\mu - \lambda)k(\omega) = \inf_{c,k} [c + \rho(\omega - u(c))k'(\omega) + \frac{(\rho\sigma k u'(c))^2}{2} k''(\omega) - \lambda k], \quad (34)$$

for some constant multiplier λ . We postulate that the optimal controls are

$$k(\omega) = a^* u^{-1}(\omega), \quad c(\omega) = \frac{\rho k(\omega)}{y^*},$$

where a^* and y^* are constants to be determined. For proving optimality, it is convenient to redefine our controls as

$$k = a u^{-1}(\omega), \quad c = \gamma u^{-1}(\omega)$$

and show that it is optimal to take $a = a^*$ and $\gamma = \frac{\rho a^*}{y^*}$ for all ω . This reparameterization leads to the following formulas:

$$u(c) = \gamma^{1-\alpha} \omega, \quad u'(c) = [\gamma u^{-1}(\omega)]^{-\alpha}, \quad k'(\omega) = a^* [u^{-1}(\omega)]^\alpha, \quad k''(\omega) = \alpha a^* [u^{-1}(\omega)]^{2\alpha-1}.$$

Plugging these formulas into equation (34), we see that all terms are proportional to $u^{-1}(\omega)$, which implies that optimal controls do not depend on ω . The simplified condition is equivalent to

$$(\mu - \lambda)a^* = \inf_{a,\gamma} [\gamma + \rho a^* \frac{1 - \gamma^{1-\alpha}}{1 - \alpha} + \alpha a^* \frac{(\rho\sigma a \gamma^{-\alpha})^2}{2} - \lambda a], \quad (35)$$

for some constant λ . Optimization with respect to a is obtained for $a = a^*$ when $\lambda = \alpha(a^* \rho \sigma \gamma^{-\alpha})^2$. Now equation (34) becomes, after adding λa^* to both sides:

$$\mu a^* = \inf_{\gamma} [\gamma + \rho a^* \frac{1 - \gamma^{1-\alpha}}{1 - \alpha} + \frac{\alpha (a^*)^3 (\rho \sigma \gamma^{-\alpha})^2}{2}].$$

If we take $\gamma = \frac{\rho a^*}{y}$ and divide by ρa^* we obtain

$$\frac{\mu}{\rho} = \inf_y \left[\frac{1}{y} + \frac{(\frac{\rho a^*}{y})^{1-\alpha} - 1}{\alpha - 1} + \frac{\alpha \sigma^2}{2\rho} (\rho a^*)^{2-2\alpha} y^{2\alpha} \right]. \quad (36)$$

Since $\alpha > 1$, the function between brackets

$$H(y, a^*) \equiv \frac{1}{y} + \frac{(\frac{\rho a^*}{y})^{1-\alpha} - 1}{\alpha - 1} + \frac{\alpha \sigma^2}{2\rho} (\rho a^*)^{2-2\alpha} y^{2\alpha}$$

is decreasing in a^* and U-shaped in y . Moreover,

$$H(y, 0) = \infty, \quad H(y, \infty) = \frac{1}{y}.$$

Let us define

$$I(a^*) \equiv \min_y H(y, a^*).$$

This function is well defined for all $a^* > 0$, and decreasing in a^* as a minimum of decreasing functions of a^* .

Moreover

$$I(0) = \infty, \quad I(\infty) = \min_y \frac{1}{y} = 0.$$

Thus there is exactly one value a^* such that $I(a^*) = \frac{\mu}{\rho}$. Finally, y^* is the unique value of y for which the inf is reached for this particular value of a^* . QED

Note that $g^* = \mu - \frac{\rho}{y^*} < \mu$, which implies that the transversality condition is satisfied:

$$\lim_{t \rightarrow \infty} K_t e^{-\mu t} = \lim_{t \rightarrow \infty} K_0 e^{(g^* - \mu)t} = 0.$$

Equilibria and Optima with Power Utility

We assume that $u(c) = \frac{c^{1-\alpha}}{1-\alpha}$, with $\alpha > 1$. The government issues real bonds and commits to pay a constant interest rate r on these bonds. Interest payments are financed by issuing more bonds and taxing agents at a constant rate τ .

Individual behavior

Each agent solves an individual control problem:

$$v(w) = \sup_{(k,c)_t} \mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} u(c_t) dt \right],$$

under the state equation

$$dw_t = k_t(\mu dt + \sigma dZ_t) + [r(w_t - k_t) - \tau w_t - c_t] dt.$$

The individual value function $v(w)$ satisfies the scalar HJB equation

$$\rho v(w) = \sup_{k,c} \left[\rho u(c) + v'(w)[\mu k - c - \tau w + r(w - k)] + \frac{\sigma^2 k^2 v''(w)}{2} \right].$$

By homogeneity, we know that $v(w) = v u(w)$, for some constant v to be determined. This implies

$$v'(w) = v w^{-\alpha}, \quad v''(w) = -\alpha v w^{-\alpha-1}$$

The first order conditions with respect to consumption and capital holdings imply:

$$c = h w, \quad k = x w,$$

where

$$h = \left[\frac{\rho}{v} \right]^{\frac{1}{\alpha}}, \quad x = \frac{\mu - r}{\alpha \sigma^2}.$$

Thus, agents invest a constant fraction $x = \frac{\mu - r}{\alpha \sigma^2}$ of their wealth in productive capital and the rest in money. The economy grows at a constant rate

$$g = \mu - \frac{h}{x}.$$

Finally, the HJB equation gives, after simplifications

$$\rho = \alpha h + (1 - \alpha) \left[\mu - \tau + \alpha \sigma^2 \left(\frac{x^2}{2} - x \right) \right] \tag{37}$$

Equilibrium

The economy grows at rate $g = \mu - \frac{h}{x}$. The budget of the government is balanced when

$$\tau = (r - g)(1 - x) = \left(\alpha\sigma^2x - \frac{h}{x}\right)(1 - x). \quad (38)$$

We want to implement constrained optimal allocations. The following table, similar to the one presented in the main text (for log utilities) shows the link between equilibrium allocations and constrained optimal allocations for power utilities.

	Constrained Optima	Equilibria
capital allocations	$k(\omega_t) = a^*u^{-1}(\omega_t)$	$k_t = xw_t$
consumption	$c_t = \frac{\rho k_t}{y^*}$	$c_t = \frac{h}{x}k_t$
growth rate	$g^* = \mu - \frac{\rho}{y^*}$	$g = \mu - \frac{h}{x}$
capital dynamics	$\frac{dk_t}{k_t} = g^*dt + \sigma(\rho a^*)^{1-\alpha}(y^*)^\alpha dZ_t$	$\frac{dk_t}{k_t} = gdt + \sigma x dZ_t$

Thus constrained optimal allocations coincide with equilibria under two conditions:

$$x = (\rho a^*)^{1-\alpha}(y^*)^\alpha,$$

and

$$\frac{h}{x} = \frac{\rho}{y^*}.$$

These two conditions determine the interest rate $r = \mu - \alpha\sigma^2x$ and the tax rate τ . It is easy to see that the budget constraint of the government is satisfied.

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