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“Lorentz Tranformation is necessarily mathematically linear”

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Lorentz Tranformation is necessarily mathematically linear

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Abstract

We show that the Lorentz Transformation, that appears in the Einstein's Special Relativity Theory, is necessarily linear.

keywords: special relativity, Minkowski metric

1 Introduction

The Lorentz Transformation, at the heart of the special theory of relativity invented by Albert Einstein [2], is assumed to be linear from physical assumptions. In this short note, we prove that they are also mathematically linear, under smooth regularity assumptions.

Consider that we have two Cartesian coordinate systems, also called inertial reference frames, $R(x, y, z, t)$ (or simply R) and $R'(x', y', z', t')$ (or simply R'), where t and t' denote the time coordinates, (x, y, z) and (x', y', z') denote the spatial coordinates. The reference frame R' moves relative to R with the velocity v in along the $x-$axis. We know that the coordinates y and z perpendicular to the velocity are the same in both reference frames: $y = y'$ and $z = z'$. So, it is sufficient to consider only a transformation of the coordinates (t, x) from the reference frame R to the coordinates (t', x') with

$$t' = F(t, x) \tag{1}$$

$$x' = G(t, x) \tag{2}$$

in the reference frame R'.

From the translation symmetry of space and time, Physicists conclude that the functions F and G must be linear functions. And it is now well

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known that, when the transformation is linear, we have the following relations [3] (with the additional constraint that $x' = 0$ if and only if $x = vt$)

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} f(ct,x) \\ g(ct,x) \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{3}$$

where c is a positive constant (which turns out to be the light speed), $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The (linear) Lorentz Transformation is equivalently given by (1) and (2) or by (3) with

$$t' = F(t,x) = \gamma(t - x\frac{v}{c^2}) = \frac{1}{c} f(ct,x) \tag{4}$$

$$x' = G(t,x) = \gamma(-vt + x) = g(ct,x). \tag{5}$$

One way to get relations (3) is to assume the conservation of the pseudo-norm of Minkowski, derived from the pseudo-scalar product of Minkowski, through a linear application from R to R'.

The pseudo-scalar product of Minkowski is the non-degenerate symmetric bilinear form η given by

$$\eta(\vec{P_1}, \vec{P_2}) = \tau_1\tau_2 - x_1x_2,$$

where $\vec{P_1} = (\tau_1, x_1) \in \mathbb{R}^2$ and $\vec{P_2} = (\tau_2, x_2) \in \mathbb{R}^2$.

2 Main result: the Lorentz Transformation must be linear

The Lorentz Transfomation (f, g), given by (3), is the unique linear transformation from $\mathbb{R}^2$ onto $\mathbb{R}^2$ that preserves the pseudo-norm of Minkowski, that is

$$c^2t'^2 - x'^2 = c^2t^2 - x^2,$$

or

$$\tau'^2 - x'^2 = \tau^2 - x^2,$$

for all $(t, x) \in \mathbb{R}^2$, with $\tau = ct$, $\tau' = ct' = f(ct, x)$ and $x' = g(ct, x)$.

In this short note, we will show that, under smooth regularity assumptions, a transformation $L = (f, g)$, from $\mathbb{R}^2$ to $\mathbb{R}^2$, such that, for any two points (τ_1, x_1) and (τ_2, x_2) in $\mathbb{R}^2 \times \mathbb{R}^2$

$$f(\tau_1, x_1)f(\tau_2, x_2) - g(\tau_1, x_1)g(\tau_2, x_2) = \tau_1\tau_2 - x_1x_2 \tag{6}$$

is linear. In other words, we will show that, under smooth regularity assumptions, if a transformation preserves the pseudo-scalar product of Minkowski, it is necessarily linear. Note that we do not need to suppose that L is one to one.

Benz [1] gives a more general result since he considers Lorentz Transformation defined as follows: let X be a real inner product space of arbitrary finite or infinite dimension ≥ 2. The mapping $\lambda : X \to X$ is called a Lorentz Transformation if and only if it preserves the pseudo-distance of Minkowski, derived from the eponym pseudo-norm. Then, the Lorentz Transformation is explicitly written by means of Lorentz boosts and orthogonal transformations.

Our result is more restrictive since we only consider the space $\mathbb{R}^2$. Our ambition and intention are just to give a simple pedagogical proof, with differentiability arguments unlike simple geometric ones used by Benz.

Let us now give our result.

Theorem 2.1 *Let $L = (f, g)$ be a application defined on $\mathbb{R}^2$ with values in $\mathbb{R}^2$. We assume that the second partial derivatives $f^{(2,0)}$, $f^{(0,2)}$, $g^{(2,0)}$ and $g^{(0,2)}$ exist everywhere. The application L preserves the pseudo-scalar product of Minkowski if and only if $L = (f, g)$ is defined by*

$$\begin{pmatrix} \tau' \\ x' \end{pmatrix} = \begin{pmatrix} f(\tau, x) \\ g(\tau, x) \end{pmatrix} = \begin{pmatrix} u & k_0 l \\ k_0 u & l \end{pmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix}, \tag{7}$$

for all $(\tau, x) \in \mathbb{R}^2$, where u and l are constant such that $u^2 = l^2 = \frac{1}{1-k_0^2}$, with $k_0^2 < 1$.

Proof. It is easy to check that if $L = (f, g)$ is defined by (7) then (6) is satisfied.

It takes more "time" (relatively not absolutly!) to prove that if (6) is satisfied by L then the latter is necessarily given by (7).

$L = (f, g)$ preserves the pseudo-scalar product of Minkowski means that, for any two points (τ_1, x_1) and (τ_2, x_2) in $\mathbb{R}^2 \times \mathbb{R}^2$, we have the following equation

$$\tau_1 \tau_2 - x_1 x_2 = f(\tau_1, x_1) f(\tau_2, x_2) - g(\tau_1, x_1) g(\tau_2, x_2). \tag{8}$$

We differentiate equation (8) twice with respect to τ_1 to get

$$f^{(2,0)}(\tau_1, x_1) f(\tau_2, x_2) - g^{(2,0)}(\tau_1, x_1) g(\tau_2, x_2) = 0. \tag{9}$$

If there exists a point (τ^0, x^0) such that $f^{(2,0)}(\tau^0, x^0) \neq 0$, then we get from (9), for all $(\tau, x) \in \mathbb{R}^2$ and for $(\tau_1, x_1) = (\tau^0, x^0)$

$$f(\tau, x) = k g(\tau, x), \tag{10}$$

with $k = \dfrac{g^{(2,0)}(\tau^0, x^0)}{f^{(2,0)}(\tau^0, x^0)}$, so that we must get from (8) for $(\tau_1, x_1) = (\tau_2, x_2) = (\tau, x)$

$$\tau^2 - x^2 = (k^2 - 1) g^2(\tau, x). \tag{11}$$

which is not true for all (τ, x) in $\mathbb{R}^2$, in contradiction with (8). Thus, for all (τ, x) in $\mathbb{R}^2$

$$f^{(2,0)}(\tau, x) = 0,$$

so that $f^{(1,0)}(\tau, x) = u(x)$, where u is a function defined on $\mathbb{R}$. Similarly, we can show that $g^{(2,0)}(\tau, x) = 0$, so that $g^{(1,0)}(\tau, x) = v(x)$, for all (τ, x) in $\mathbb{R}^2$, where v is a function defined on $\mathbb{R}$.

Now, we differentiate equation (8) twice with respect to x_1 to get

$$f^{(0,2)}(\tau_1, x_1) f(\tau_2, x_2) - g^{(0,2)}(\tau_1, x_1) g(\tau_2, x_2) = 0. \tag{12}$$

Like previously, we get in a similar way

$$f^{(0,1)}(\tau, x) = w(\tau)$$
$$g^{(0,1)}(\tau, x) = l(\tau)$$

for all (τ, x) in $\mathbb{R}^2$, where w and l are two functions defined on $\mathbb{R}$.

At this step, we have the following summary: for all (τ, x) in $\mathbb{R}^2$, the Jacobian matrix J of the transformation (f, g) has the following form

$$J(\tau, x) = \begin{pmatrix} u(x) & w(\tau) \\ v(x) & l(\tau) \end{pmatrix}.$$

Now, let's take the derivative of equation (8) with respect to τ_1 to get

$$\tau_2 = f^{(1,0)}(\tau_1, x_1) f(\tau_2, x_2) - g^{(1,0)}(\tau_1, x_1) g(\tau_2, x_2). \tag{13}$$

And we now take the derivative of equation (13) with respect to x_2 to get

$$f^{(1,0)}(\tau_1, x_1) f^{(0,1)}(\tau_2, x_2) - g^{(1,0)}(\tau_1, x_1) g^{(0,1)}(\tau_2, x_2) = 0. \tag{14}$$

Thus, we get for all (τ, x) in $\mathbb{R}^2$

$$u(x) w(\tau) = v(x) l(\tau). \tag{15}$$

Then assume that $l(\tau) = 0 = g^{(0,1)}(\tau, x)$ for all $\tau \in \mathbb{R}$, thus g is of the form $g(\tau, x) = L(\tau)$, where L is a function defined on $\mathbb{R}$. Thus, we get from (8), for all (τ, x) in $\mathbb{R}^2$

$$\tau^2 - x^2 = f^2(\tau, x) - L^2(\tau). \tag{16}$$

Differentiating twice this last equation with respect to x gives

$$w^2(\tau) = -1$$

which is absurd. Thus, there exists τ_0 such that $l(\tau_0) \neq 0$ for which equation (15) becomes

$$v(x) = \frac{w(\tau_0)}{l(\tau_0)} u(x) = k_0 u(x),$$

for all x in $\mathbb{R}$.Then, from (15), we get

$$u(x)w(\tau) = k_0 u(x) l(\tau). \tag{17}$$

We can similarly show that there exists x_0 such that $u(x_0) \neq 0$ so that

$$u(x_0)w(\tau) = k_0 u(x_0) l(\tau), \tag{18}$$

leads to

$$w(\tau) = k_0 l(\tau),$$

for all τ in $\mathbb{R}$.

Now, from (8), with $(\tau_1, x_1) = (\tau_2, x_2) = (\tau, x)$, we have

$$\tau^2 - x^2 = f^2(\tau, x) - g^2(\tau, x). \tag{19}$$

Taking twice the second derivative of this last equation with respect to τ, we obtain

$$(1 - k_0^2) u^2(x) = 1,$$

and taking twice the second derivative of the same equation (19) with respect to x, we get

$$(1 - k_0^2) l^2(\tau) = 1.$$

Finally, if L preserves the pseudo-scalar product of Minkowski, the functions u, l and so v and w are necessarily piecewise constant functions, and the constant k_0 is necessarily such that $k_0^2 < 1$. The Jacobian matrix J of the transformation (f, g) can be re-written as follows, for all (τ, x) in $\mathbb{R}^2$

$$J(\tau, x) = \begin{pmatrix} u(x) & k_0 l(\tau) \\ k_0 u(x) & l(\tau) \end{pmatrix},$$

with $u^2(x) = l^2(\tau) = \frac{1}{1-k_0^2}$.

Suppose now that for $x_1 \neq x_2$ we have $u(x_1)u(x_2) = \frac{-1}{1-k_0^2}$, which means that $u(x_1)$ and $u(x_2)$ are of opposite signs. Then, from equation (8), we get the following contradiction $1 = -1$. We get the same contradiction by considering two different τ_1 and τ_2 for which $l(\tau_1)$ and $l(\tau_2)$ are of opposite signs.

Finally, the Jacobian matrix J of the transformation (f, g) takes necessarily the following form, for all (τ, x) in $\mathbb{R}^2$

$$J(\tau, x) = \begin{pmatrix} u & k_0 l \\ k_0 u & l \end{pmatrix},$$

where u and l are constant such that $u^2 = l^2 = \frac{1}{1-k_0^2}$, with k_0 satisfying $k_0^2 < 1$. $\square$

3 Conclusion

At this stage, we now know that $L = (f, g)$ is of the form

$$\begin{pmatrix} \tau' \\ x' \end{pmatrix} = \begin{pmatrix} f(\tau, x) \\ g(\tau, x) \end{pmatrix} = \begin{pmatrix} u & k_0 l \\ k_0 u & l \end{pmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix}, \tag{20}$$

where u and l are constant such that $u^2 = l^2 = \frac{1}{1-k_0^2}$, with $k_0^2 < 1$.

From physical considerations, in particular the symmetric role played by the reference frames R and R' and the same sign for $\tau = ct$ and $\tau' = ct'$, we get that $u = l = \frac{+1}{\sqrt{1-k_0^2}}$. In conclusion, we have

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} f(ct, x) \\ g(ct, x) \end{pmatrix} = \begin{pmatrix} u & k_0 u \\ k_0 u & u \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \tag{21}$$

where $u = l = \frac{1}{\sqrt{1-k_0^2}}$, with k_0 such that $k_0^2 < 1$.

With the additional constraint that $x' = 0$ if and only if $x = vt$, we obtain that $k_0 = -\frac{v}{c} = -\beta$, and the well known (linear) Lorentz Transformation (3).

References

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