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Bruno Biais, Hans Gersbach,
Jean-Charles Rochet,
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and Stéphane Villeneuve

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Bruno Biais*, Hans Gersbach[†], Jean-Charles Rochet[‡]

Ernst Ludwig von Thadden[§] and Stéphane Villeneuve[¶]

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*HEC Paris. Email: biaisb@hec.fr

[†]ETH Zürich and KOF Swiss Economic Institute, Email: hgersbach@ethz.ch.

[‡]Toulouse School of Economics. Email: jeancharles.rochet@gmail.com.

[§]Universität Mannheim, CEPR, and ECGI. Email: vthadden@uni-mannheim.de

[¶]Toulouse School of Economics. Email: Stephane.Villeneuve@TSE-fr.eu

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Abstract

We analyze dynamic capital allocation and risk sharing between a principal and many agents, who privately observe their output. The state variables of the mechanism design problem are aggregate capital and the distribution of continuation utilities across agents. This gives rise to a Bellman equation in an infinite dimensional space, which we solve with mean-field techniques. We fully characterize the optimal mechanism and show that the level of risk agents must be exposed to for incentive reasons is decreasing in their initial outside utility. We extend classical welfare theorems by showing that any incentive-constrained optimal allocation can be implemented as an equilibrium allocation, with appropriate money issuance and wealth taxation by the principal.

1 Introduction

How should capital be allocated and risks shared in a dynamic production economy without aggregate risk? In the absence of informational frictions, the answer is clear: capital should be allocated according to individual productivities and risks should be eliminated by diversification. However, when information about individual outputs is private, incentive compatibility constraints must be taken into account. This paper studies how these constraints affect capital accumulation and risk sharing.

To address these issues, we consider an infinite horizon economy with a large number of risk averse agents and a single good that can be consumed or invested as capital. Each agent operates a project whose output is proportional to the amount of capital under his/her management and subject to idiosyncratic shocks. Individual unit outputs are i.i.d. so that a version of the law of large numbers applies, implying that aggregate output is deterministic.

We assume agents privately observe their individual output and can secretly consume some of it, as in Bolton and Scharfstein [11].¹ In contrast to output and consumption, capital is observable. Applying the revelation principle, we study revelation mechanisms, in which agents truthfully report their output to the principal, who then allocates consumption and capital according to the reports. The optimal dynamic mechanism allocates capital and consumption to maximize the principal's utility, subject to the participation and incentive constraints of the agents and the aggregate resource constraint.²

To provide agents with incentives not to divert output, the optimal contract specifies an increase (resp. decrease) of consumption **and** capital for agents whose output is larger (resp. smaller) than

¹See also Rampini and Viswanathan [45].

²The principal in our model signs a dynamic contract with each agent. The set of contracts between the principal and the population of agents is the mechanism.

expected. Lucky agents (those that perform better in a given period) obtain more capital to manage in the next period, not because they are more skilled (performance is i.i.d. across agents and across periods) but because this provides incentives to report good performance instead of diverting output. In contrast with the symmetric information case, insurance is imperfect, because full insurance is not incentive compatible. So, the optimal mechanism exposes agents to a fraction of their idiosyncratic risk.

From a mathematical viewpoint, finding the optimal mechanism is challenging, as we need to extend to an infinite number of agents the martingale techniques introduced by Sannikov [47] in the one agent case. With only one agent, the Bellman equation that characterizes the optimal mechanism involves the partial derivatives of the value function with respect to two state variables: capital and the continuation utility promised to the (single) agent by the principal. In contrast, in our model with a continuous approximation of an economy with a large number of agents, the state variables are aggregate capital and the **entire distribution** of continuation utilities across agents, which belongs to the space of probability distributions over \mathbb{R} . Thus, we need to use mean-field games techniques,³ because the value function of the principal solves a Bellman equation in an infinite dimensional space. We first determine this Bellman equation, which involves a generalized notion of derivative (the L-derivative⁴) of the value function with respect to the probability distribution of continuation utilities. Then, thanks to our log utility specification, we show that the dimension of state variables can be reduced to two: aggregate capital and the expectation of (a function of) agents' continuation utilities. These are sufficient statistics for the characterization of the optimal mechanism. Thanks to the reduction in the dimension of the state space from infinity to two, we can fully characterize the

³Mean-field games techniques, introduced by Lasry and Lions [42] allow to approximate the solutions of games with a large number N of agents by the solution of the limit game associated with $N = \infty$. They were first applied to economics by Achdou et al. [1]. For a more recent application of mean-field games techniques to economics, see Achdou et al. [2].

⁴This notion of derivative is defined in our Appendix B and developed in Carmona and Delarue [14].

dynamics of capital and consumption allocations as well as the distribution of continuation utilities across agents.

The optimal direct mechanism is remarkably simple: consumption and capital are allocated among agents proportionally to each agent's equivalent permanent consumption, defined as the constant lifetime stream of consumption giving the agent the same continuation utility as the mechanism. The equivalent permanent consumption of each agent grows at a constant rate in expectation, but is impacted by the agent's performance. The innovation in the growth rate of an agent's consumption or capital is proportional, by a positive constant y , to the agent's idiosyncratic output shock. The proportionality constant y measures the extent to which the agent is exposed to the risk of his/her idiosyncratic output shock. Raising y reduces allocative efficiency by reducing insurance, but it also relaxes the incentive compatibility condition, enabling the principal to extract more rents. Thus there is a rent-efficiency trade-off. We characterize the set of information-constrained Pareto optimal allocations, which can be parametrized by y . The larger the agents' initial outside utility, the larger the fraction of the surplus they must be given, the lower their risk exposure y . Because agents are exposed to their idiosyncratic shocks, inequality increases over time and agents become more and more heterogenous.⁵ Moreover, while aggregate capital and output grow over time, growth is lower than under symmetric information. This is because incentive compatibility constrains how much new capital can be delegated to agents.

As an application, we study if the information-constrained optimal allocation obtaining in our framework can arise as an equilibrium allocation. To do so, we study equilibrium in a market in which agents exchange goods against money issued by the principal, and are subject to wealth taxes levied by the principal. When trading in the market, agents face a dynamic portfolio problem à la

⁵More precisely, the coefficient of variation (standard deviation divided by the mean) of continuation utilities across agents increases over time.

Merton [43]. They choose how much to invest in capital and in money, bearing in mind that the latter is exposed to inflation but the former is risky. The principal influences this portfolio choice by controlling money supply and thus the inflation rate. An appropriate monetary policy gives rise to an inflation rate such that, in equilibrium, the agents choose the same risk exposure in their portfolio as in the optimal mechanism.

Our results can be contrasted with the classical welfare theorems. These theorems state that, in a convex economy with complete markets and without frictions, all competitive equilibria are efficient (first welfare theorem) and, conversely, all efficient allocations can be decentralized by a competitive equilibrium after appropriate transfers between agents (second welfare theorem). The classical welfare theorems do not apply in our economy with asymmetric information and endogenously incomplete markets. Thus, in contrast with the first theorem of welfare, laissez faire competitive equilibria in our framework are generically constrained inefficient. However, all constrained optimal allocations can be implemented as market equilibria, provided there are appropriate taxes and an appropriate supply of money, which agents exchange against capital in order to buffer their productivity shocks. Our implementation result can thus be viewed as an extension of the two welfare theorems to an economy with endogenously incomplete markets.

Literature: Our paper complements several strands of literature.

First, our analysis of dynamic contracting between one principal and many agents is related to the literature analyzing dynamic contracting between one principal and one agent, in particular the seminal work of DeMarzo and Fishman [15], [16] and Sannikov [47], and the following analyses of Biais, Mariotti, Plantin, and Rochet [9], DeMarzo and Sannikov [17], Feng and Westerfield [24], and Di Tella and Sannikov [22]. As in Biais, Mariotti, Rochet, and Villeneuve [10] and DeMarzo, Fishman, He, and Wang [18], firm size is determined by the optimal contract and is useful to provide incentives.

However, in contrast to these last two papers, in the present paper there are no capital adjustment costs. This enhances tractability, and gives rise to continuous reallocation of capital. He (2009) offers an interesting alternative approach in which firm size is affected by unobservable agent's effort. This differs from our model in which firm size is directly controlled by the principal, and what is unobservable is output.

The major contribution of the present paper relative to that literature is to embed the contracting problem into a general equilibrium context, with a population of agents and aggregate resource constraints. Thus we shed light on the impact of incentive constraints on the allocation of capital and consumption between agents. In particular, we show that incentive constraints imply that inequality between agents increases over time. Moreover, we show how constrained optimal allocations can be implemented if the principal issues money which the agents can store or trade against goods and sets appropriate tax rates. We thus extend the two welfare theorems to an economy with frictions.

Second, our analysis is related to the dynamic macrofinance literature analyzing risk with exogenously incomplete markets (see Bewley [7], Aiyagari [3], Huggett [35] and [36], Krusell and Smith [40], Angeletos [5], He and Krishnamurthy [32] and [33], Brunnermeier and Sannikov [12], Di Tella [21], and Achdou et al [2]). In particular, our focus on the distribution across agents and our reliance on mean-field techniques are in line with Achdou et al [2]. See also Achdou et al. [1]

The distinguishing characteristic of the present paper relative to that literature is to provide microfoundations for market incompleteness.⁶ Thus, the institutions and constraints we consider are endogenous features of the optimal dynamic mechanism. This helps clarify the consequences of informational frictions. For example, we reconcile two effects which, as explained by Angeletos [5], had so far been viewed as distinct. While the literature in line with Bernanke and Gertler

⁶Another difference is that, while most of that literature studies labor income risk, our paper, like Angeletos [5] considers capital return risk.

(1989) emphasizes how wealth affects the *ability* to invest in capital, Angeletos [5] emphasizes how wealth affects the *willingness* to hold risky capital. Our mechanism design approach clarifies the common origin of these two forces: incentive compatibility constrains both how much capital agents are allocated and how much of the corresponding idiosyncratic risk they must bear. Consequently, in contrast with Angeletos [5], in our analysis frictions unambiguously lower capital accumulation.

In line with Angeletos [5], we prove that it is optimal to use the average of (a function of) agent's continuation utility as state variable, instead of the whole distribution of agents' continuation utilities.⁷ This complements the analysis of Krusell and Smith [40], who show numerically that in their dynamic stochastic equilibrium the average wealth of the agents is almost perfectly in line with the behaviour of macroeconomic aggregates.

In line with Achdou et al. [1] and Achdou et al. [2] our paper applies mean-field games techniques to economics. An important difference between our paper and [1] and Achdou et al. [2] is that we take an optimal contracting approach, in which market incompleteness arises endogenously. This offers a new and interesting application of mean-field techniques because in our contracting approach the distribution of continuation utilities across the continuum of agents is a state variable for the principal.

Third, our focus on money in the implementation of the optimal mechanism links our paper to the new monetarist literature initiated by the seminal papers of Kiyotaki and Wright ([37], [38]) and reviewed in Williamson and Wright [51]. A common theme with that literature is that money arises endogenously, as a useful instrument, instead of being a constraint as in cash in advance models or exogenous as in money-in-the-utility-function models. Money in our implementation encodes the memory of past performance in line with Kocherlakota [39] and provides consumption insurance in

⁷A major difference is that in Angeletos [5] institutions and market incompleteness constraints are exogenous while in our paper, they are features of the endogenous optimal mechanism.

line with Berentsen and Rocheteau [6].

There are important differences, however, between our analysis and the new monetarist literature. First, instead of starting from a characterization of optimal allocations in a setting with money, we characterize the optimal mechanism in a real economy with only goods and no money, and then we introduce money as a tool to implement the optimal mechanism. Second, while the new monetarist literature assumes large households (Shi [48]) or the alternation of decentralized and centralized markets (Lagos and Wright, [41]) so that at the beginning of each period, all agents start with the same amount of money, in our framework, agents have endogenously heterogeneous money holdings, and we characterize the dynamics of this heterogeneity.⁸ The third difference is a consequence of the second one: In the new monetarist literature, agents are homogeneous at the beginning of each period, so the optimal allocation is pinned down by a static mechanism. In contrast, in our setting agents' continuation utilities vary stochastically over time, so the optimal allocation is set by a dynamic mechanism.⁹

Fourth, we complement the mechanism design approach to optimal taxation pioneered by Mirrlees [44], Diamond and Mirrlees [20], and Diamond [19], and further developed by the new dynamic public finance literature, e.g., Golosov, Kocherlakota and Tsyvinski [27], Golosov and Tsyvinski [28], and Farhi and Werning [23]. A major difference is that, in these papers, risk and information asymmetry are about wage earners' skills, while, in our paper, risk and information asymmetry are

⁸Rocheteau, Weill, and Wong [46] offer an interesting extension of the Lagos and Wright [41] approach in which they characterize the equilibrium distribution of money holdings. But, in line with previous new monetarist analyses, their approach relies on money from the start, in contrast with our approach, which starts with a direct mechanism without money and then implements it with money.

⁹Another related paper is Aiyagari and Williamson [4]. In our paper like in theirs a continuum of agents have random outputs and want to share risk, but this is difficult because individual outputs are privately observed. A major difference between their work and ours is that in Aiyagari and Williamson [4] agents use labor but not capital, while in our analysis agents' outputs are increasing in the capital they are allocated. Thus, in our paper, unlike in Aiyagari and Williamson [4] capital allocation is key in the provision of incentives, and information asymmetry reduces capital accumulation relative to the first best. Another difference is that Aiyagari and Williamson [4] study the consequences of transportation costs, which are absent in our framework.

about managers' capital returns. Correspondingly, unlike in these papers, the dynamic of capital allocation plays a key role in our analysis. Another major difference is that the optimal taxation literature focuses on one policy tool, namely the tax system, while in our set-up, the implementation of the optimal mechanism relies on money issuance as well as on taxation.

Structure of the paper: The rest of the paper is organized as follows. Section 2 introduces the continuous time model, and solves the symmetric information case, which provides a useful benchmark for the analysis of the asymmetric information case. In Section 3 we determine the Bellman equation that characterizes the principal's value function under asymmetric information. Then we make a guess on the form of the optimal policy, qualitatively close to that obtained under symmetric information, and finally we show that this candidate policy is indeed the full solution of our problem. Section 4 shows that the optimal direct mechanism can be implemented with money and taxes. Section 5 concludes. Proofs that are not in the main text are in Appendix A. Appendix B provides a brief introduction to generalized differential calculus in Wasserstein spaces.

2 The model

We model an economy with an infinite horizon and continuous time. Idiosyncratic shocks are captured by independent Brownian motions, which are easy to define when there is a finite number N of agents, but more difficult with a continuum. We therefore start by describing the model with N agents and then take the limit as N tends to infinity.

2.1 The economy with finitely many agents

Agents, indexed by $i = 1, \dots, N$, and principal are infinitely lived with discount rate ρ and logarithmic utility. There is a single good, which can be used for consumption or as capital input in a stochastic constant returns to scale technology operated by the agents. If agent i invests $k_t^{(N),i}/N$ units¹⁰ of the good in his production process, his instantaneous output (net of depreciation) is

$$dY_t^{(N),i} = \frac{k_t^{(N),i}}{N} [\mu dt + \sigma dZ_t^i], \quad (1)$$

where μ is the expected rate of return (net of depreciation) of the technology and (Z_t^i) , $i = 1, \dots, N$, are independent and identically distributed Brownian motions, whose increments can be interpreted as idiosyncratic non persistent productivity shocks. Let $(\mathcal{F}_t^{(N)})_{t \geq 0}$ be the filtration generated by the N -dimensional Brownian motion $(Z_t^1, \dots, Z_t^N)_{t \geq 0}$. All processes introduced in this section are assumed to be square-integrable and adapted to $(\mathcal{F}_t^{(N)})$. The total amount of capital in the economy at time t is¹¹

$$K_t^{(N)} := \frac{1}{N} \sum_{i=1}^N k_t^{(N),i}. \quad (2)$$

If agent i 's instantaneous consumption at time t is $c_t^{(N),i}/N$ and that of the principal $c_t^{(N),P}$, the law of motion of total capital is

$$dK_t^{(N)} = \left(\mu K_t^{(N)} - \frac{1}{N} \sum_{i=1}^N c_t^{(N),i} - c_t^{(N),P} \right) dt + \frac{\sigma}{N} \sum_{i=1}^N k_t^{(N),i} dZ_t^i. \quad (3)$$

¹⁰We denote capital and consumption allocations of each agent in this way (divided by N) to simplify the comparison with the limit case $N = \infty$. In some sense, each agent has "mass" $1/N$, so as to keep the total mass of agents to 1. In this way, aggregate capital and consumption converge to the integral of individual capital and consumption allocations when N goes to ∞ .

¹¹Throughout the paper, individual and aggregate capital are strictly positive at every point in time. It is never optimal to run down the capital to zero in every case we are considering.

Equation (3) is an intertemporal resource constraint stating that net aggregate investment is equal to total output (net of depreciation) minus total consumption.

We seek to characterize the Pareto frontier of the economy by computing the value function V of the principal, defined as the maximum expected utility she can obtain with a total volume of capital K when agent i is guaranteed a total expected utility ω^i for $i = 1, \dots, N$. This value function is obtained by finding capital and consumption paths $(k_t^{(N),i})$, $(c_t^{(N),i})$, and $(c_t^{(N),P})$ that maximize the principal's expected utility

$$\mathbb{E} \int_0^\infty e^{-\rho t} \log c_t^{(N),P} dt, \quad (4)$$

where $\rho > 0$ is the discount rate and maximization is subject to the law of motion of aggregate capital (3), the initial condition $K_0^{(N)} = K$, the agents' participation constraints:

$$\mathbb{E} \int_0^\infty e^{-\rho t} \log c_t^{(N),i} dt = \omega^i, \quad (5)$$

for $i = 1, \dots, N$, and the capital allocation constraint:

$$K_t^{(N)} = \frac{1}{N} \sum_{i=1}^N k_t^{(N),i}.$$

In the above contracting problem we assume that agents and principal commit to the contract concluded at date 0.¹² Depending on the informational environment, there will be further constraints beyond those specified above. We first consider the case of symmetric information, in which idiosyncratic shocks and thus individual outputs are publicly observable. This case (the first best) will later serve as a benchmark for the case in which agents privately observe shocks and can secretly divert output.

¹²As a consequence, we do not specify outside options for the agents beyond time 0.

2.2 Allocations under symmetric information with finitely many agents

In the first best, capital allocation has no impact on the dynamics of the continuation payoffs, and only matters for the volatility of aggregate capital. Risk aversion implies that this volatility should be minimized, and therefore that $k_t^{(N),i} \equiv K_t^{(N)}$. We can therefore drop the $(k_t^{(N),i})$ from the list of controls. Since production and capital accumulation are Markovian and preferences are stationary, there is no need to condition on previous realizations. Hence, we can assume that the controls $(c_t^{(N),i})$ and $(c_t^{(N),P})$ are feedback controls and thus only depend on the current state $K_t^{(N)}$, the current time t , and the initial values ω^i , $1 \leq i \leq N$. More precisely, we assume that there are functions \bar{c}^A and c^P such that

$$c_t^{(N),i} = \bar{c}^A(t, K_t^{(N)}, \omega^i, \nu^{(N)}) \text{ and } c_t^{(N),P} = c^P(t, K_t^{(N)}, \nu^{(N)}),$$

where $\nu^{(N)}$ denotes the empirical measure of the N expected utilities at time 0. Since all agents are allocated the same amount of capital, the law of motion of aggregate capital (3) becomes

$$dK_t^{(N)} = \left(\mu K_t^{(N)} - \frac{1}{N} \sum_{i=1}^N \bar{c}^A(t, K_t^{(N)}, \omega^i, \nu^{(N)}) - c^P(t, K_t^{(N)}, \nu^{(N)}) \right) dt + \sigma K_t^{(N)} dA_t^{(N)}, \quad (6)$$

where

$$A_t^{(N)} \equiv \frac{1}{N} \sum_{i=1}^N Z_t^i \quad (7)$$

are normal random variables with mean 0 and variance $\frac{t}{N}$. The principal's problem therefore is to determine feedback controls \bar{c}^A, c^P that maximize (4) subject to the law of motion (6), the initial condition $K_0^{(N)} = K$, and the agents' participation constraints (5). We now simplify the problem by looking at the limit case where $N \rightarrow \infty$.

2.3 Optimal allocations under symmetric information in the limit economy

We assume that $\nu^{(N)}$ weakly converges to some probability \mathbb{P} which represents the initial distribution of utilities. When $N \rightarrow \infty$, the propagation of chaos theory (Sznitman, [49]) shows that the agents' average consumption

$$\frac{1}{N} \sum_{i=1}^N \bar{c}^A(t, K_t^{(N)}, \omega^i, \nu^{(N)}) = \int \bar{c}^A(t, K_t^{(N)}, \omega, \nu^{(N)}) d\nu^{(N)}(\omega)$$

converges to

$$c^A(t, K_t, \mathbb{P}) = \int_{\mathbb{R}} \bar{c}^A(t, K_t, \omega, \mathbb{P}) d\mathbb{P}(\omega) \quad (8)$$

and the stochastic differential equation (6) converges to the ordinary differential equation

$$\dot{K}_t = \mu K_t - c^A(t, K_t, \mathbb{P}) - c^P(t, K_t, \mathbb{P}). \quad (9)$$

When N goes to infinity, there is no risk any more. Intuitively, this is because, by the law of large numbers, $A_t^{(N)}$ goes to 0 as N goes to infinity. Thus capital accumulation and consumption are deterministic.¹³ These remarks drastically simplify the control problem, which can now be written as

$$\max_{\bar{c}^A, c^P} \int_0^\infty e^{-\rho t} \log c^P(t, K_t, \mathbb{P}) dt \quad (10)$$

¹³To facilitate the exposition, the argument is only made heuristically at this stage. In the technical online Appendix we develop this theory in full rigour, assuming Lipschitz continuity of the controls with respect to all variables.

subject to the law of motion (9), the initial condition $K_0 = K$, the agents' participation constraints

$$\int_0^\infty e^{-\rho t} \log \bar{c}^A(t, K_t, \omega, \mathbb{P}) dt = \omega \quad (11)$$

for all $\omega \in \mathbb{R}$, and the definition of agents' aggregate consumption (8). Since optimality implies the transversality condition $\lim_{t \rightarrow \infty} e^{-\mu t} K_t = 0$, equation (9) and the initial condition $K_0 = K$ together can be integrated into:

$$K = \int_0^\infty e^{-\mu t} (c^A(t, K_t, \mathbb{P}) + c^P(t, K_t, \mathbb{P})) dt. \quad (12)$$

Taking first-order conditions of the above defined problem, we obtain our first lemma (whose proof is in Appendix A):

Lemma 1. *The solution of (10) subject to (11) and (12) (if it exists), is such that aggregate consumption is a constant fraction of capital, equal to the discount rate ρ :*

$$c^P(t, K_t, \mathbb{P}) + c^A(t, K_t, \mathbb{P}) = \rho K_t, \quad (13)$$

and the consumption of the principal is a constant fraction of capital:

$$c^P(t, K_t, \mathbb{P}) = \gamma^P K_t = (\rho - a e^{-\frac{\mu - \rho}{\rho}}) K_t, \quad (14)$$

where

$$a \equiv \frac{1}{K} \int_{\mathbb{R}} e^{\rho \omega} d\mathbb{P}(\omega). \quad (15)$$

The parameter a relates the initial values of the two state variables K and \mathbb{P} of the principal's problem to each other. For each outside option ω , $e^{\rho \omega}$ can be interpreted as an agent's "equivalent

permanent consumption” for ω , namely the constant lifetime stream of consumption that would give the agent total utility ω . Integrating over all agents with respect to distribution \mathbb{P} , this gives the total expected equivalent permanent consumption demanded by all agents. This is set in relation to K , the total capital stock at the disposal of the principal to satisfy these demands over time.

Since the principal’s consumption must be positive, (14) implies that the problem has a solution only if

$$a < \rho e^{\frac{\mu-\rho}{\rho}}$$

i.e., if the following condition on the initial data K, \mathbb{P} of the principal’s problem holds:

$$\rho K e^{\frac{\mu-\rho}{\rho}} > \int_{\mathbb{R}} e^{\rho\omega} d\mathbb{P}(\omega). \quad (16)$$

Intuitively, the condition means that the initial amount of aggregate capital (K) is sufficiently large to give his/her reservation utility (ω) to each agent, while leaving enough resources available for the principal to have non-negative consumption.

Equation (13) states that aggregate consumption is a constant fraction ρ of aggregate capital K . This stems from our assumption that the principal and the agents have logarithmic utility. For the same reason, the consumption of the principal is a constant fraction of aggregate capital, as stated in Equation (14). In turn, this implies that the aggregate consumption of the agents is a constant fraction of aggregate capital. Our next proposition spells out how this aggregate consumption is allocated across individual agents.

To conclude the characterization of the optimal allocation under symmetric information, we define

the expected utility at any time t of an agent who starts out with reservation utility ω as

$$\omega_t = \int_t^{+\infty} e^{-\rho(s-t)} \log \bar{c}^A(s, K_s, \omega, \mathbb{P}) ds. \quad (17)$$

Introducing the new variable ω_t makes it possible to replace the integral constraints (11) by the initial conditions $\omega_0 = \omega$ and work recursively. Differentiating (17) yields the ODE

$$\dot{\omega}_t = \rho\omega_t - \log \bar{c}^A(t, K_t, \omega, \mathbb{P}), \quad \text{with } \omega_0 = \omega. \quad (18)$$

The next proposition summarizes the properties of the optimal allocation under symmetric information.

Proposition 1. *The first-best contracting problem has a solution if and only if $a < \rho e^{\frac{\mu-\rho}{\rho}}$. In this case the value function of the principal is*

$$V(K, \mathbb{P}) = \frac{1}{\rho} \left(\log K + \log(\rho e^{\frac{\mu-\rho}{\rho}} - a) \right). \quad (19)$$

Moreover the optimal allocation has the following properties:

1. Capital grows at the constant rate $\mu - \rho$:

$$K_t = K e^{(\mu-\rho)t}. \quad (20)$$

2. Agents' continuation utilities grow linearly:

$$\omega_t = \omega + \frac{\mu - \rho}{\rho} t. \quad (21)$$

3. At each date t , an agent with continuation utility ω_t consumes $\gamma^A e^{\rho\omega_t}$, where $\gamma^A = e^{-\frac{\mu-\rho}{\rho}}$.

4. At each date t , the principal consumes a constant fraction $\gamma^P = \rho - \gamma^A a$ of aggregate capital K_t .

Using (19) and the definition of a , we have that

$$e^{\rho V} + \int e^{\rho\omega} d\mathbb{P} = \rho K e^{\frac{\mu-\rho}{\rho}}, \quad (22)$$

The left-hand side of (22) is the sum of the principal's equivalent permanent consumption and the aggregate agents' permanent consumption, while the right-hand side is the total amount of equivalent permanent consumption to be allocated among the principal and the agents. Thus, (22) describes the Pareto frontier in the space of equivalent permanent consumptions, pinning down how total surplus is divided among agents and principal.

Equation (13) stated that aggregate consumption was a constant fraction ρ of aggregate capital K . Since capital yields output at rate μ , the investment rate is $\mu - \rho$, which gives rise to the dynamics of the capital stock stated in (20).

Equation (14) stated that the consumption of the principal was a constant fraction γ^P . Jointly with (13) and the dynamics of agent's utility (21) this yields the value of γ^A in Proposition 1.

3 Optimal allocations under asymmetric information

We now turn to the case in which agents privately observe their individual output. By the revelation principle, we consider direct revelation mechanisms that are incentive compatible. A mechanism is a mapping from the output \hat{Y}_t^i , reported and delivered by agent i to the principal, into consumption $(c_t^i)_{t \geq 0}$ and capital allocations $(k_t^i)_{t \geq 0}$ for the agent. The processes $(c_t^i)_{t \geq 0}$ and $(k_t^i)_{t \geq 0}$ are positive and

adapted to the filtration (\mathcal{F}_t) generated by the N reported outputs \hat{Y}_t^i . Since agents privately observe their own output, they can be tempted to divert a part of it and secretly consume it. To avoid this, the mechanism must induce truthful revelation, i.e., it must be incentive compatible.

3.1 Incentive compatibility

Consider an agent with contract $(c_t^i, k_t^i)_{t \geq 0}$ who considers under-reporting his Brownian increments by some amount $\Delta_t dt$. Defining

$$d\hat{Z}_t^i = dZ_t^i - \Delta_t dt, \quad (23)$$

the dynamics of reported output writes as

$$d\hat{Y}_t^i = \mu k_t^i dt + \sigma k_t^i d\hat{Z}_t^i.$$

Since the agent cannot store, diversion cannot be negative: $\Delta_t \geq 0$ for every t . The time t expected utility of agent i who adopts a diversion strategy $(\Delta_s)_{s \geq t}$ is

$$\omega_t^{i,\Delta} = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} \log(c_s^i + \sigma k_s^i \Delta_s) ds \right].$$

The martingale representation theorem implies that the dynamics of the agent's continuation utility is

$$d\omega_t^{i,\Delta} = (\rho \omega_t^{i,\Delta} - \log(c_t^i)) dt + \sigma y_t^i d\hat{Z}_t^i, \quad (24)$$

where y_t^i is a \mathcal{F}_t -adapted process. The drift term on the right-hand side of (24) is similar to that prevailing in the first best, see (18). The second term on the right-hand side of (24) is the product of the agent's reported productivity shock $(d\hat{Z}_t^i)$ by σy_t^i . Intuitively, y_t^i is the sensitivity of the agent's

continuation utility with respect to his/her report.

To provide incentives for truthful revelation, the principal must propose a contract generating an appropriate sensitivity. We now offer an intuitive examination of the corresponding incentive compatibility condition. Equation (24) implies the local incentive compatibility condition is that the agent i be better off revealing dZ_t^i truthfully, and getting

$$\log(c_t^i)dt + \sigma y_t^i dZ_t^i$$

than under-reporting $d\hat{Z}_t^i = dZ_t^i - \Delta dt$ and getting

$$\log(c_t^i + \sigma k_t^i \Delta)dt + \sigma y_t^i d\hat{Z}_t^i = \log(c_t^i + \sigma k_t^i \Delta)dt + \sigma y_t^i (dZ_t^i - \Delta dt).$$

Therefore, the local incentive compatibility condition is

$$\sigma y_t^i \geq \sup_{\Delta \geq 0} \frac{\log((c_t^i + \sigma k_t^i \Delta) - \log(c_t^i))}{\Delta} = \frac{\sigma k_t^i}{c_t^i}.$$

This means that, for incentive compatibility, the sensitivity of continuation utility to performance must be larger than the product of the capital k_t^i managed by the agent by his/her marginal utility of consumption, i.e.,

$$y_t^i \geq \frac{k_t^i}{c_t^i}.$$

The following lemma, whose proof is in Appendix A, states that the condition above is also sufficient.

Lemma 2. *The incentive compatibility condition is equivalent to the inequality*

$$\forall t, \quad y_t^i \geq \frac{k_t^i}{c_t^i}. \tag{25}$$

The incentive compatibility condition (25) is in the spirit of Sannikov [47]. It implies that, in contrast with the symmetric information case, agents cannot fully share the risk of their idiosyncratic shocks. Condition (25) also shows there is a tradeoff between risk-sharing and investment: providing more insurance to the agent, by reducing the sensitivity of his/her continuation value to output shocks is possible only at the cost of reducing capital relative to consumption. This is because increasing capital, and therefore output, increases the amount of resources the agent can divert, which tightens the incentive constraint. This tradeoff is similar to that arising in Biais, Mariotti, Rochet and Villeneuve [10], where the size of operation (similar to capital in the present context) was limited by incentive compatibility.

3.2 Mean-field economy

As in the symmetric information, to simplify the problem we consider the limiting case in which N tends to infinity. As in the first best, the value function of the principal will depend on the capital stock K and the distribution \mathbb{P} of agents' continuation payoffs. We introduce the space $\mathcal{P}_2(\mathbb{R})$ of probability measures on \mathbb{R} with a finite second moment, which we endow with the Wasserstein distance W_2 (see for example Sznitman [49], Carmona and Delarue [14], Cardaliaguet [13] and Villani [50] and Appendix B). The main difference with the first best case is that the principal has to leave a part of the idiosyncratic risk to each agent, for incentive reasons. The agents being risk averse, it is never optimal for the principal to expose them to more risk than required by the incentive compatibility condition. That is, in any optimal allocation, the incentive constraint (25) is always binding and we can eliminate the capital allocation variable by writing $k_t^i = y_t^i c_t^i$. As a consequence,

the individual continuation utility of agent i will evolve in the N agents economy as

$$d\omega_t^{i,(N)} = (\rho\omega_t^{i,(N)} - \log(c_t^i)) dt + \sigma \frac{k_t^i}{c_t^i} dZ_t^i.$$

Moreover, we denote by $\mathbb{P}_t^{(N)} = \frac{1}{N} \sum_{i=1}^N \delta_{\omega_t^{i,(N)}}$ the empirical measure, where δ_x denotes the Dirac measure in x . We now assume the following feedback nature of the controls:

$$c_t^i = c(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)}) \quad (26)$$

$$k_t^i = k(K_t^{(N)}, \mathbb{P}_t^{(N)}, \omega_t^{i,(N)}) \quad (27)$$

$$c_t^P = c^P(K_t^{(N)}, \mathbb{P}_t^{(N)}) \quad (28)$$

for each $i = 1, \dots, N$, where c , c^P , and k are continuous¹⁴ functions $\mathbb{R}^+ \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$. The theory of propagation of chaos (see Sznitman[49] section 1 for details) shows that, when $N \rightarrow \infty$, for any fixed integer m , the joint distribution of the $m + 1$ -dimensional process $(K_t^{(N)}, \omega_t^{1,(N)}, \dots, \omega_t^{m,(N)})_{t \geq 0}$ weakly converges to the product distribution $\delta_{K_t} \otimes \mathbb{P}_t^{\otimes m}$ where

$$dK_t = \left(\mu \int k(K_t, \mathbb{P}_t, \cdot) d\mathbb{P}_t - \int c(K_t, \mathbb{P}_t, \cdot) d\mathbb{P}_t - c^P(K_t, \mathbb{P}_t) \right) dt, \quad K_0 = K, \quad (29)$$

and \mathbb{P}_t is the marginal distribution of the mean-field stochastic differential equation (SDE)

$$d\omega_t = [\rho\omega_t - \log c(K_t, \mathbb{P}_t, \omega_t)]dt + \sigma y(K_t, \mathbb{P}_t, \omega_t) dZ_t, \quad (30)$$

¹⁴We will assume a Lipschitz condition for the functions defining the drift in equations (29) and (30) to guarantee the well-posedness of the mean-field limit McKean Vlasov equation and check the Lipschitz condition holds at the optimum. A function f is Lipschitz if there exists a constant $L > 0$, such that for, $|f(K_1, \mathbb{P}_1, \omega_1) - f(K_0, \mathbb{P}_0, \omega_0)| \leq L(|K_1 - K_0| + |\omega_1 - \omega_0| + W_2(\mathbb{P}_1, \mathbb{P}_0))$.

where $(Z_t)_t$ is a Brownian motion and y denotes the ratio $\frac{k}{c}$.

As can be seen in equation (29), when $N \rightarrow \infty$, the dynamics of aggregate capital is deterministic. Intuitively, as in the symmetric information case, the idiosyncratic shocks average out according to a form of law of large numbers.

Under an appropriate Lipschitz condition, Proposition 9 in the online Appendix establishes that for any initial condition there exists a unique solution $(\omega_t, K_t, \mathbb{P}_t)$ to the above mean-field Stochastic Differential Equation. Hereafter, we focus on the limit case in which N goes to infinity. The pair (K_t, \mathbb{P}_t) plays the role of the state variable of the principal problem in the mean-field economy on which we apply the recursive formulation.

3.3 Characterization of second best allocations.

The main difficulty for exploiting the dynamic programming principle is to differentiate functionals defined on the Wasserstein space. Various notions of derivatives with respect to measures have been developed, in connection with the theory of optimal transport¹⁵. We use here the Wasserstein metric on the space of probability measures and the notion of L-differentiability that is presented in Appendix B. Following the traditional approach for control problems, we first determine the shape of the Hamilton-Jacobi-Bellman (HJB) equation that the value function of the principal must satisfy (necessary condition) and then establish a verification theorem showing that regular solutions of this HJB equation solve our control problem (sufficient condition). To do so, consider the control problem of the principal

$$V(K, \mathbb{P}) = \sup_{(c, c^P, y) \in \mathcal{K}} \int_0^\infty e^{-\rho t} \log c_t^P dt, \quad (31)$$

¹⁵See the books by Carmona and Delarue [14] and Villani [50] and our more formal presentation in Appendix B of the present paper.

where the state equations are given by the capital allocation constraint (29) and the stochastic differential equation (30). The supremum is taken over the set \mathcal{K} of admissible feedback controls (c, c^P, y) , such that for all $t \geq 0$,

$$\int y(., \omega) c(., \omega) d\mathbb{P}_t(\omega) = K_t. \quad (32)$$

Since the process K_t is deterministic, in any optimal solution c_t^P has to be deterministic, which simplifies the formulation of the problem.

A second difficulty is that this control problem involves a constraint (32) that mixes control variables and state variables. To deal with this constraint, we introduce a related, unconstrained, problem as follows: for each function λ (defined on the product space $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$), which we call the *Lagrange multiplier*, we consider the control problem

$$V_\lambda = \sup_{(c, c^P, y)} \int_0^\infty e^{-\rho t} \left(\log c^P(K_t, \mathbb{P}_t) + \lambda(K_t, \mathbb{P}_t) \left(K_t - \int y(., \omega) c(., \omega) d\mathbb{P}_t(\omega) \right) \right) dt.$$

We first state a result that establishes a link between the principal's value V and V_λ .

Proposition 2. *If i) for every Lagrange multiplier process one can find an optimal control $\alpha_\lambda = (c_\lambda, c_\lambda^P, y_\lambda)$ such that*

$$V_\lambda = \int_0^\infty e^{-\rho t} \left(\log c_\lambda^P + \lambda(K_t, \mathbb{P}_t) \left(K_t - \int y_\lambda(., \omega) c_\lambda(., \omega) d\mathbb{P}_t(\omega) \right) \right) dt$$

and ii) there exists a Lagrange multiplier λ_0 such that for all $t \geq 0$,

$$K_t = \int y_{\lambda_0}(., \omega) c_{\lambda_0}(., \omega) d\mathbb{P}_t(\omega),$$

i.e. $\alpha_{\lambda_0} \in \mathcal{K}$, then $V = V_{\lambda_0}$ and α_{λ_0} solves the principal problem.

We are now in a position to derive the HJB equation associated with the unconstrained problem.

Proposition 3. *If the value function of the principal V is sufficiently regular,¹⁶ it satisfies the following HJB equation:*

$$\begin{aligned} \rho V(K, \mathbb{P}) = \sup_{c, c^P, y} & \left\{ \log c^P(K, \mathbb{P}) + \lambda_0(K, \mathbb{P}) \left(K - \int y(K, \mathbb{P}, \omega) c(K, \mathbb{P}, \omega) d\mathbb{P}(\omega) \right) \right. \\ & + \frac{\partial V}{\partial K}(K, \mathbb{P}) \left(\mu K - c^P(K, \mathbb{P}) - \int c(K, \mathbb{P}, \omega) d\mathbb{P}(\omega) \right) \\ & \left. + \int \partial_\omega \nabla V[K, \mathbb{P}](\omega) (\rho \omega - \log c(K, \mathbb{P}, \omega)) d\mathbb{P}(\omega) + \int \partial_{\omega\omega} \nabla V[K, \mathbb{P}](\omega) \frac{\sigma^2}{2} y^2(K, \mathbb{P}, \omega) d\mathbb{P}(\omega) \right\}, \end{aligned} \quad (33)$$

where ∇V denotes the L -gradient of V with respect to the measure \mathbb{P} and ∂_ω (respectively $\partial_{\omega\omega}$) denote its first (respectively second) partial derivative in ω , while λ_0 denotes the Lagrange multiplier associated with the capital allocation constraint given in Proposition 2 ii).

The first two lines of equation (33) are similar to their counterparts in the one-agent problem. The first term on the right-hand side represents the instantaneous utility from consumption, the second term reflects the capital allocation constraint, and the third term is the derivative of the value function with respect to capital multiplied by the growth rate of capital. The last two terms of equation (33) are similar to those that would arise in a one-agent problem in that they involve the first derivative of the value function of the principal with respect to the continuation utility of the agent multiplied by the drift of this continuation utility, and the second derivative multiplied a quadratic term arising because of Ito's lemma. They, however, differ from their one-agent counterpart because they reflect that the principal faces a population of agents, with different ω , so that they involve integrals and L -gradients with respect to the measure $\mathbb{P}(\omega)$.

¹⁶By this we mean that it is differentiable in K and L -differentiable in \mathbb{P} (see Appendix B).

Inspired by classical verification theorems for stochastic control of diffusion processes, we prove in Appendix A the following result, which is a consequence of the Itô formula given in Appendix B for functions defined on the Wasserstein space.

Proposition 4. (*Verification Theorem*) *Let $\lambda(\cdot)$ be a Lagrange multiplier, and $v^\lambda(K, \mathbb{P})$ be a regular function. If there exist i) a solution v^λ to (33) with the transversality condition $\lim_{t \rightarrow +\infty} e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t) = 0$, and ii) a control α_λ^* attaining the maximum in (33), then $v^\lambda = V_\lambda$. Moreover, if there is a Lagrange multiplier λ_0 such that $\alpha_{\lambda_0}^* \in \mathcal{K}$ then $v^{\lambda_0} = V$.*

3.4 A guess-and-verify approach

We now make a conjecture, a “guess”, about the form of the solution to the optimal control problem and show that the corresponding value function satisfies the Hamilton-Jacobi-Bellman equation (33), so that the guess is the actual solution of the problem, by the verification theorem (Proposition 4).

3.4.1 The guess

In line with the first best case, we make the following conjecture

$$c^P(K, \mathbb{P}) = \gamma^P K, \quad c(K, \mathbb{P}; \omega_t) = \gamma^A \exp(\rho \omega_t),$$

where γ^P and γ^A are positive constants. We also guess that $y(\cdot)$ is a positive constant, which we denote by y . The binding incentive compatibility constraint then implies that an agent’s consumption is proportional to this agent’s capital:

$$c(K, \mathbb{P}; \omega_t) = \frac{k(K, \mathbb{P}; \omega_t)}{y}.$$

Our conjecture implies that total consumption at date t equals $(\gamma^P + \frac{1}{y})K_t$ and thus that K_t grows at a constant rate

$$\frac{\dot{K}_t}{K_t} \equiv g = \mu - \gamma^P - \frac{1}{y}. \quad (34)$$

We now characterize the solution of the problem under our conjecture. Then we will show that this is also the solution to the full problem.

The principal's utility $V(K, \mathbb{P})$ under our conjecture satisfies

$$\rho V(K, \mathbb{P}) = \int_0^\infty \rho e^{-\rho t} \log(\gamma^P K \exp(gt)) dt = \log(\gamma^P K) + \frac{g}{\rho}.$$

Similarly, by the participation constraint, an agent's initial utility ω satisfies

$$\rho \omega = \mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} \log\left(\frac{k(K, \mathbb{P}; \omega_t)}{y}\right) dt \right].$$

Integrating by parts, this becomes

$$\rho \omega = \mathbb{E} \left[\log \frac{k(K, \mathbb{P}, \omega)}{y} + \int_0^\infty \rho e^{-\rho t} d[\log k(K, \mathbb{P}; \omega_t)] \right].$$

It remains to determine the dynamics of $\log k(K, \mathbb{P}; \omega_t)$. We know that $k_t \equiv k(K, \mathbb{P}; \omega_t)$ is proportional to $\exp(\rho \omega_t)$ and grows on average at rate g . Moreover by definition of y , the volatility of ω_t is σy .

Therefore, by Ito's lemma:

$$\frac{dk_t}{k_t} = \frac{d(\exp(\rho \omega_t))}{\exp(\rho \omega_t)} = g dt + \rho \sigma y dZ_t.$$

and:

$$d(\log k_t) = \left(g - \frac{\rho^2 \sigma^2 y^2}{2}\right) dt + \rho \sigma y dZ_t.$$

Consequently:

$$\rho\omega_t = \log \frac{k(K, \mathbb{P}; \omega_t)}{y} + \frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

Hence the capital managed by an agent is proportional to this agent's equivalent permanent consumption $\exp(\rho\omega_t)$:

$$\exp(\rho\omega_t) = \frac{k(K, \mathbb{P}; \omega_t)}{y} \exp \left[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2} \right]. \quad (35)$$

Integrating (35) with respect to the measure \mathbb{P} , the capital allocation constraint rewrites:

$$\int \exp(\rho\omega_t) d\mathbb{P} = \frac{K_t}{y} \exp \left[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2} \right]. \quad (36)$$

Equation (36) shows that, under our conjecture, the ratio of aggregate capital to expected equivalent permanent consumption is constant over time. More precisely, we have

$$a = \frac{\int \exp(\rho\omega_t) d\mathbb{P}}{K_t} = \frac{1}{y} \exp \left[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2} \right], \quad (37)$$

where the constant a is as defined in the analysis of the symmetric information case in equation (15).

The value function of the principal is thus $V(K, \mathbb{P}) = \frac{\log K}{\rho} + v(a)$ where

$$\rho v(a) = \max \log \gamma^P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} \quad (38)$$

under constraint (36). A priori, the value function V depends on the aggregate capital K and the entire distribution \mathbb{P} of continuation utilities. Inspecting (36) and (38), however, one can note that the value function and the constraint depend on the individual continuation utilities only through their expectation: $\int \exp(\rho\omega) d\mathbb{P}$, which is therefore a sufficient statistics for \mathbb{P} . This implies that V

only depends on two scalar variables, K and $a = \frac{\int \exp(\rho\omega)d\mathbb{P}}{K}$. Expressing the constraint (36) in logs, we obtain

$$\rho V(K, \mathbb{P}) = \log K + \rho v(a) = \log K + \sup_{\gamma^P, y} \log \gamma^P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho},$$

subject to the capital allocation constraint

$$\log(ya) = \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

Inserting the value of the growth rate from (34) into the capital allocation constraint (36) expressed as an inequality yields

$$a \leq \frac{1}{y} \exp \left[\frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2} \right].$$

This is consistent with $\gamma^P > 0$ only if $a < \frac{1}{y} \exp[\frac{\mu - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}]$. Thus, like in the symmetric information case, the principal's problem only has a solution when a is less than some value defined here as

$$a_{max} = \max_y \frac{1}{y} \exp \left(\frac{\mu - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2} \right).$$

Building on the above analysis, we obtain the next proposition, whose proof is in Appendix A:

Proposition 5. *Under our conjecture, the value function of the principal's problem is $\frac{\log K}{\rho} + v(a)$, where $v(a)$ is defined for $0 < a < a_{max}$ by*

$$\rho v(a) = \sup_{\gamma^P, y} \left(\log \gamma^P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} \right), \quad (39)$$

$$s.t. \log ya = \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

The solution of this problem is such that $\gamma^P = \rho - \frac{1}{y + \rho\sigma^2 y^3}$, where $y = y(a)$ is defined implicitly by

$$a = \frac{1}{y} \exp\left[\frac{\mu - \rho}{\rho} - \frac{\rho\sigma^2 y}{1 + \rho\sigma^2 y^2} - \frac{\rho\sigma^2 y^2}{2}\right]. \quad (40)$$

The allocation of surplus between the principal and the agents is parametrized by y . As can be seen in (39) the utility of the principal increases with y . a is the initial aggregate utility of the agents (per unit of initial capital), which they must be offered because of the participation constraint. The right-hand side of (40) is decreasing in y . So y decreases with a . Thus, as a goes up, the participation constraint of the agents gets more demanding, resulting in a lower value of y , corresponding to a lower utility for the principal.

3.4.2 Verifying our guess

We now show that our guess is correct: the value function $V(K, \mathbb{P}) = \frac{\log K}{\rho} + v(a)$ obtained under our guess satisfies the Bellman equation (33). To do so, we first compute the derivatives of V , using the definition $a = \frac{\int \exp(\rho\omega) d\mathbb{P}}{K}$:

$$V_K = \frac{1}{\rho K} - \frac{av'(a)}{K}, \quad \nabla V = \exp(\rho\omega) \frac{v'(a)}{K}.$$

Then

$$\partial_\omega \nabla V = \rho \nabla V, \quad \partial_{\omega\omega} \nabla V = \rho^2 \nabla V.$$

Substituting these derivatives, and allowing γ^A and y to depend on ω unlike in our simplifying guess (and correspondingly expressing an agent's consumption as $\gamma^A(\omega) \exp(\rho\omega)$), the Bellman equation

writes as

$$\begin{aligned} \rho v(K, a) = & \sup_{y(\cdot), \gamma^P, \gamma^A(\cdot)} \left[\log \gamma^P K + \lambda \left(K - \int y(\omega) \gamma^A(\omega) \exp(\rho \omega) d\mathbb{P} \right) \right. \\ & \left. + \left(\frac{1}{\rho K} - \frac{av'(a)}{K} \right) \left[\mu K - \gamma^P K - \int \gamma^A(\omega) \exp(\rho \omega) d\mathbb{P} \right] + \int \rho \exp(\rho \omega) \frac{v'(a)}{K} \left[-\log \gamma^A(\omega) + \frac{\rho \sigma^2}{2} y^2(\omega) \right] d\mathbb{P}(\omega) \right]. \end{aligned} \quad (41)$$

Pointwise maximization inside the integral shows that the optimal controls $(y(\cdot), \gamma^A(\cdot))$ are constant since ω only appears through the common multiplicative factor $\exp(\rho \omega) d\mathbb{P}(\omega)$. With constant controls y and γ^A , the HJB equation takes a simpler expression

$$\begin{aligned} \rho v(K, a) = & \sup_{y, \gamma^P, \gamma^A} \left(\log \gamma^P K + \lambda (1 - y \gamma^A a) K \right. \\ & \left. + \left(\frac{1}{\rho} - av'(a) \right) [\mu - \gamma^P - \gamma^A a] + \rho av'(a) \left[-\log \gamma^A + \frac{\rho \sigma^2}{2} y^2 \right] \right). \end{aligned}$$

Replacing γ^A by $\frac{1}{ya}$, the second term cancels, reflecting that the capital allocation constraint holds as an equality. The Bellman equation becomes:

$$\rho v(K, a) = \log K + \sup_{y, \gamma^P} \left[\log \gamma^P + (1 - \rho av'(a)) \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} + \rho av'(a) \left[\log(ya) + \frac{\rho \sigma^2 y^2}{2} \right] \right].$$

This coincides with the definition of v given in proposition (5), once we have noted that the Lagrange multiplier of the capital allocation constraint equals $\rho av'(a)$. Thus we can state the main result of our paper:

Proposition 6. *The value function of the principal's problem under asymmetric information is*

$$V(K, \mathbb{P}) = \frac{\log K}{\rho} + v(a),$$

where the function v is defined in Proposition (5).

3.5 Properties of second best allocations

Taking stock of the analysis above, the next proposition summarizes the properties of optimal information-constrained allocations. These properties are drastically simplified by the fact that date t allocations only depend on two scalars, namely the capital stock K_t and the expected equivalent permanent consumption of agents $\int \exp(\rho\omega_t)d\mathbb{P}$, and that the ratio of the latter to the former is equal to the constant denoted by a , so that optimal controls can all be expressed as functions of $y = y(a)$ defined in Proposition 5.

Proposition 7. *Second best optimal allocations can be parameterized by y . They are such that, at each date t :*

1. *Aggregate capital grows at a constant rate $g = \mu - \rho - \frac{\rho\sigma^2 y}{1+\rho\sigma^2 y^2}$.*

2. *Agents' continuation utilities follow a drifted Brownian motion:*

$$\omega_t = \omega + \left(\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2} \right) t + \sigma y Z_t. \quad (42)$$

3. *The capital managed by an agent is $k_t = \frac{\exp(\rho\omega_t)}{a}$ and his consumption $c_t = \frac{k_t}{y}$.*

4. *The principal consumes a constant fraction of aggregate capital: $c_t^P = \gamma^P K_t$, where*

$$\gamma^P = \rho - \frac{1}{y + \rho\sigma^2 y^3}. \quad (43)$$

5. *The parameter y can take any value in (y_{\min}, ∞) , where y_{\min} is the unique positive root of the*

equation $\rho\sigma^2y^3 + y = \frac{1}{\rho}$. This guarantees that the consumption of the principal is positive, i.e.,

$$\gamma^P > 0.$$

Property 1 shows that frictions reduce growth, since $g = \mu - \rho - \frac{\rho\sigma^2y}{1+\rho\sigma^2y^2}$ is lower than the first best growth rate $\mu - \rho$. This reflects incentive constraints, which restrict investment. When $\sigma = 0$, there is no incentive problem and the growth rate is equal to its first best level.

Property 2 implies that the cross section of agents' continuation payoffs gets more dispersed as time goes by. Even if all agents are ex-ante identical, inequality necessarily increases over time, due to incentive compatibility constraints. Moreover, there is a simple relation between the continuation utility of an agent at date t and its performance over $(0, t)$. Indeed, the average productivity of the agent over $(0, t)$ is just $\mu + \sigma \frac{Z_t}{t}$. Optimal compensation implies a simple, affine, relation between the continuation utility ω_t and this performance measure, similarly to Holmstrom and Milgrom [34].

Properties 3 and 4 are similar to the first best case, in that consumptions are equal to capital multiplied by a constant. This simplicity is due to our assumption that utilities are logarithmic.

Property 5 states that information-constrained allocations are parameterized by the sensitivity of agent's continuation utility to performance, y which can take any value such that $\sigma^2\rho^2y^3 + \rho y > 1$, to ensure that $\gamma^P > 0$. This provides a simple description of the information constrained Pareto frontier which we present next.

3.6 Information constrained Pareto frontier

The above analysis yields a characterization of the information constrained Pareto frontier in the space of equivalent permanent consumptions. Substituting the expression of γ^P from (43) into the equivalent permanent consumptions of agents, in equation (35), and the principal, from equation (38),

we obtain

$$\exp(\rho\omega) = \frac{k(\omega)}{y} \exp \left[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2} \right], \quad (44)$$

where g is the growth rate given in Proposition 7, and

$$\exp \rho V(K, \mathbb{P}) = \left(\rho - \frac{1}{y + \rho\sigma^2 y^3} \right) \text{Exp} \left[\frac{g}{\rho} \right]. \quad (45)$$

Equation (44) reflects that each agent consumes a fraction $\frac{1}{y}$ of its capital under management, which grows at average rate g , with volatility σy generating a risk premium, and is discounted at rate ρ . Similarly (45) reflects that the principal consumes a fraction $(\rho - \frac{1}{y + \rho\sigma^2 y^3})$ of the capital stock, growing at rate g , but is not impacted by any risk, so that unlike in (44), there is no risk premium. As mentioned above, when $\sigma = 0$, there is no incentive problem. Correspondingly (44) and (45) boil down to the first best Pareto frontier in (22).

4 Implementation

In the first best, idiosyncratic risk is fully mutualized and agents are exposed to aggregate risk only. This can be implemented with insurance markets, in which agents fully hedge their idiosyncratic risk, so that, net of hedging trades, agents all hold the same portfolio, linked to aggregate output, i.e., GDP.

In the second best, incentive compatibility precludes full mutualization of idiosyncratic risks. Correspondingly, agents are exposed to their idiosyncratic risk (with sensitivity y) and the aggregate risk (with sensitivity y^A). In this section we show that this can be implemented with a market in which agents hold a portfolio of risky capital, reflecting their own idiosyncratic risk, and GDP-linked

bonds, reflecting the aggregate risk. To implement the optimal mechanism, the government must set interest rates and debt-issuance such that the weights in the agent's portfolio match the optimal sensitivities, y and y^A .

4.1 Government policy

Consider the following policy: At time 0 the government issues B_0 perpetual GDP-indexed bonds, paying rate

$$r dt + \sigma_A dZ_t^A.$$

In this context, at time t an agent's wealth, a_t , is the sum of his/her bond holdings, b_t , and capital holdings, k_t , i.e.,

$$a_t = b_t + k_t.$$

Aggregate bond holdings are denoted B_t and agents' aggregate wealth is A_t .

At time t , the government issues new bonds dB_t , and redistributes proceeds net of interest payments, with transfers T , proportional to agents' wealth. Thus, the government budget balance constraint at time t is

$$dB_t - B_t(r dt + \sigma_A dZ_t^A) = T A_t dt,$$

where the left-hand side is bond issuance proceeds net of interest payments, while the right-hand side is transfers to agents. Thus, once the issuance policy $\frac{dB_t}{B_t}$ and the interest rate r are set, this determines T :

$$T \frac{A_t}{B_t} dt = \frac{dB_t}{B_t} - (r dt + \sigma_A dZ_t^A). \quad (46)$$

(46) implies that when the growth rate of government debt is larger (resp. lower) than interest rate

payments, transfers to agents are positive (resp. negative). We show below that, for well chosen issuance, interest rate and tax policies, market equilibrium leads to allocations that are identical to the second best allocation.

4.2 Agent's optimal policy

The problem faced by an agent is similar to Merton (1969). The dynamics of an agent's wealth is

$$da_t = k_t(\mu dt + \sigma dZ_t + \sigma_A dZ_t^A) + b_t(r dt + \sigma_A dZ_t^A) - c dt + T a_t dt.$$

Substituting $a_t = b_t + k_t$, this is

$$da_t = k_t(\mu dt + \sigma dZ_t) + b_t r dt + a_t \sigma_A dZ_t^A - c dt + T a_t dt. \quad (47)$$

That is, the agent's exposure to aggregate risk does not depend on the agent's portfolio choice between k_t and a_t .

Taking into account the budget constraint constraint of the agent, the value function v of the agent satisfies the following Bellman equation (for brevity subscripts t are hereafter omitted):

$$\rho v(a) = \max_{k,c} u(c) + v'(a) [(\mu - r)k + ar - c + Ta] + \frac{\sigma^2 k^2 + \sigma_A^2 a^2}{2} v''(a).$$

The first order condition with respect to c is

$$u'(c) = v'(a).$$

The first order condition with respect to k is

$$v'(a)(\mu - r) = -\sigma^2 v''(a)k.$$

As in Merton (1969), for logarithmic utility, one can rely on a homogeneity argument and the value function is of the form

$$v(a) = \frac{\log(a)}{\rho} + v(1),$$

which implies $v'(a) = \frac{1}{\rho a}$ and $v''(a) = -\frac{1}{\rho a^2}$. So the first order condition with respect to c writes

$$c = \rho a. \tag{48}$$

That is, an agent's consumption is a constant fraction ρ of the agent's wealth. The more impatient the agent, the larger the discount rate ρ , the larger the fraction of his/her wealth that the agent consumes.

The first order condition with respect to k writes

$$k = \frac{\mu - r}{\sigma^2} a. \tag{49}$$

Denoting

$$x \equiv \frac{\mu - r}{\sigma^2}, \tag{50}$$

an agent's optimal portfolio is to allocate a fraction x of his/her wealth to investment in the production technology and $1 - x$ to the GDP indexed bond. The fraction of the agent's wealth that is invested in the risky production technology increases with its productivity but decreases with the expected

interest rate on bonds. Thus, by setting r the government can influence the portfolio choice of the agent. As we will see below, this will be key in the implementation.

4.3 Equilibrium

At time t , an agent holds capital

$$k_t = xa_t,$$

and bonds

$$b_t = (1 - x)a_t = \frac{1 - x}{x}k_t.$$

This implies that aggregate bond supply by B_t , aggregate agents' wealth A_t , and aggregate capital K_t are proportional, with

$$\frac{A_t}{B_t} = \frac{1}{1 - x}. \quad (51)$$

Aggregate capital growth is equal to aggregate output minus aggregate consumption, that is

$$dK_t = K_t(\mu dt + \sigma_A dZ_A) - \frac{\rho}{x}K_t dt.$$

Since aggregate capital, aggregate wealth and aggregate bonds are proportional, they have the same growth rate. Thus, by 51, the government budget balance equation 46) rewrites

$$\frac{T}{1 - x}dt = (\mu dt + \sigma_A dZ_A) - \frac{\rho}{x}dt - (r dt + \sigma_A dZ_t^A).$$

The Brownian term simplifies, so the government budget balance condition is

$$T = (1 - x) \left[\left(\mu - \frac{\rho}{x} \right) - r \right]. \quad (52)$$

The term in the brackets on the right-hand side is the expected growth rate net of the interest rate. So, when expected growth is larger than interest rates, debt issuance is larger than debt service and the government transfers the difference to the agents.

The bond market clearing condition is

$$B_t = \frac{1-x}{x} K_t, \quad (53)$$

where the left-hand side is the aggregate supply of bonds, while the right-hand side is the aggregate demand.

4.4 Implementation and welfare theorem

Substituting the value of T , c and k from (52), (48), and (49), into (47), agent's wealth dynamics write

$$\frac{da_t}{a_t} = \left(\mu - \frac{\rho}{x} \right) dt + \sigma x dZ_t + \sigma_A dZ_t^A.$$

Since agent's wealth and capital are proportional, this is also the dynamics of agent's capital. Table 3 offers a summary comparison of our results on consumption, capital and growth in the second best characterized in the previous section and in the market equilibrium presented just above.

Table 3: Summary comparison between second best & equilibrium

	SB	Equilibrium
k	$k_{SB} \exp(\rho\omega)$	xa
$c(\omega)$	$\frac{k}{y}$	$\rho a = \frac{\rho}{x} k$
$E(g)$	$\mu - \frac{1}{y}$	$\mu - \frac{\rho}{x}$
$\frac{dk}{k}$	$\left(\mu - \frac{1}{y}\right) dt$ $+ \rho y \sigma dZ + \sigma^A dZ^A$	$\left(\mu - \frac{\rho}{x}\right) dt$ $+ \sigma x dZ + \sigma_A dZ^A$

As can be seen in Table 3, setting $x = \rho y$ identifies the dynamics of agent's capital prevailing in the second best and in equilibrium. Consequently, if agents start with the same capital allocation at time 0 in equilibrium and in the second best, they hold exactly the same amount of capital from that point on in equilibrium and in the second best. Because in both case consumption is proportional to capital, agents also have the same consumption in equilibrium and in the second best. Consequently, as can be seen in the table, for $x = \rho y$, expected aggregate growth is the same in equilibrium and in the second best. By (50), to set $x = \rho y$, the government must set r such that

$$\frac{\mu - r}{\sigma^2} = \rho y.$$

That is

$$r = \mu - \rho y \sigma^2. \tag{54}$$

We can therefore state our next proposition:

Proposition 8. *If agents start with the same initial capital allocation as in the second best, and T is as given in (52) while r is as given in (54), then the equilibrium allocation is identical to the second best allocation.*

In the optimal mechanism, the state variable for an agent is continuation utility, or, equivalently, equivalent permanent consumption $\exp(\rho\omega)$ and capital is proportional to this variable. In the implementation, the state variable for an agent is wealth, and capital is proportional to this variable. By setting the interest rate appropriately, the government can induce portfolio choices such that the dynamics of the agent's state variable is the same in the market equilibrium as in the optimal mechanism.¹⁷

Greenwald and Stiglitz [30] have shown that the classical welfare theorems are not valid in an economy with incomplete markets and asymmetric information like ours. In particular, competitive equilibrium allocations are generically constrained inefficient. This is the case in our model. However, our results imply that simple policy interventions can restore the equivalence between information constrained optimal allocations and equilibrium allocations. As inflation increases, holding money becomes less attractive relative to holding physical capital. Thus, the fraction of wealth invested in the risky asset x is increasing in the inflation rate π .

5 Conclusion

This paper characterizes optimal capital allocation and risk sharing between a principal and many agents, who privately observe their individual output and can secretly consume some of it, as in Bolton and Scharfstein [11]. To provide agents with incentives for truthful revelation, the optimal dynamic

¹⁷This is in line with the implementation of the optimal contract in Biais-Mariotti-Plantin-Rochet [9], the cash reserves of the firm were informationally equivalent to continuation utility of the agent.

mechanism increases agents' continuation utility after good reported performance and reduces it after bad reported performance. In this context, agents' continuation utilities are random variables and their distribution across agents is a state variable of the problem faced by the principal. This gives rise to a Bellman equation in an infinite dimensional space. To solve this difficult problem we rely on mean-field techniques. Under the assumption of logarithmic we fully characterize the optimal dynamics of capital and consumption as well as the distribution of continuation utilities across agents.

Moreover, we show that the optimal dynamic mechanism can be implemented in equilibrium with a market in which agents trade goods for money, issued by the principal, and wealth taxation. An appropriately set money supply growth rate gives rise to an optimal inflation rate, inducing agents to allocate their wealth among capital and money in a way that gives rise to the same risk exposure as in the optimal mechanism. Wealth taxation complements money to engineer in equilibrium the same level of risk-sharing as in the optimal mechanism.

This implementation result is akin to the second welfare theorem: For any Pareto optimal allocation, there exists a combination of fiscal and monetary policies implementing that allocation in equilibrium. However, while in the classical welfare theorem, markets are perfect and complete, in our analysis markets are endogenously incomplete because of information asymmetry. Moreover, while in the classical second welfare theorem, any proportional tax is distortive, in our analysis, proportional wealth taxes optimally affect agents' behaviour. Finally note that the first theorem of welfare also has to be modified in our context. Only a subset of the equilibria arising in our setting are information-constrained Pareto optima. In particular, the laissez-faire equilibrium, obtained with no taxation and no public expenditure, is not constrained Pareto optimal.

6 Appendix A: Proofs

Proof of Lemma 1 The problem can be solved by recursive methods, which we develop in Section 3. Here, we provide a heuristic derivation of the solution in terms of Lagrange multipliers. Denoting by $\lambda(\cdot)$ and η the Lagrange multipliers associated with the infinite-dimensional constraints (11) and (12), respectively, the *Lagrangian* is

$$L = \int_0^\infty e^{-\rho t} \left[\log c^P(t, K_t, \mathbb{P}) - \int_{\mathbb{R}} \lambda(\omega) \log \bar{c}^A(t, K_t, \omega, \mathbb{P}) d\mathbb{P}(\omega) - \eta e^{-(\mu-\rho)t} (c^P(t, K_t, \mathbb{P}) + c^A(t, K_t, \mathbb{P})) \right] dt + \eta K.$$

By assuming sufficient integrability of the functions $c^P(t, K_t, \mathbb{P})$ and $\bar{c}^A(t, K_t, \omega, \mathbb{P})$, the first order conditions of this problem are for every $t \geq 0$ and every $\omega \in \mathbb{R}$,

$$c^P(t, K_t, \mathbb{P}) = \frac{1}{\eta} e^{(\mu-\rho)t} \quad \text{and} \quad \bar{c}^A(t, K_t, \omega, \mathbb{P}) = -\frac{\lambda(\omega)}{\eta} e^{(\mu-\rho)t}.$$

Since $K_0 = K$, we have for every $t \geq 0$ and every $\omega \in \mathbb{R}$,

$$c^P(t, K_t, \mathbb{P}) = e^{(\mu-\rho)t} c^P(0, K, \mathbb{P}), \tag{55}$$

$$\bar{c}^A(t, K_t, \omega, \mathbb{P}) = e^{(\mu-\rho)t} \bar{c}^A(0, K, \omega, \mathbb{P}). \tag{56}$$

Integrating (56) by means of (8) yields

$$c^A(t, K_t, \mathbb{P}) = e^{(\mu-\rho)t} c^A(0, K, \mathbb{P}) \tag{57}$$

Inserting (55) and (57) into (12) and using (55) and (57) again, we get

$$c^P(t, K_t, \mathbb{P}) + c^A(t, K_t, \mathbb{P}) = \rho K e^{(\mu-\rho)t}. \quad (58)$$

Inserting (58) into the ODE (9) now yields

$$\dot{K}_t = \mu K_t - \rho K e^{(\mu-\rho)t},$$

which has the solution

$$K_t = K e^{(\mu-\rho)t}. \quad (59)$$

(58) then can be written in the recursive formulation

$$c^P(t, K_t, \mathbb{P}) + c^A(t, K_t, \mathbb{P}) = \rho K_t, \quad (60)$$

which means that aggregate consumption is a constant fraction ρ of aggregate capital K . This property reflects our assumptions that the agents and the principal have logarithmic utility.

Next, combining (57) and (11), integrating partially, and re-arranging yields the consumption of each agent

$$\bar{c}^A(t, K_t, \omega, \mathbb{P}) = e^{(\mu-\rho)t} e^{-\frac{\mu-\rho}{\rho}} e^{\rho\omega} \quad (61)$$

Using (59), (61), and (60), the lemma obtains. QED

Proof of Proposition 1 Inserting \bar{c}^A from (61) into (18) and verifying yields

$$\omega_t = \omega + \frac{\mu - \rho}{\rho} t.$$

QED

Proof of Lemma 2 We omit the index i to alleviate notations. We will apply the martingale optimality principle to the process family indexed by Δ defined as

$$R_t^\Delta = e^{-\rho t} \omega_t + \int_0^t e^{-\rho s} \log(c_s + \sigma k_s \Delta_s) ds,$$

where

$$d\omega_t = (\rho\omega_t - \log(c_t)) dt + \sigma y_t d\hat{Z}_t.$$

We have

$$\begin{aligned} dR_t^\Delta &= e^{-\rho t} (y_t d\hat{Z}_t + (\log(c_t + \sigma k_t \Delta_t) - \log(c_t)) dt) \\ &= e^{-\rho t} y_t dZ_t + (\log(c_t + \sigma k_t \Delta_t) - \log(c_t) - y_t \Delta_t) dt \end{aligned}$$

Because \log is concave, we observe

$$\log(c_t + \sigma k_t \Delta_t) - \log(c_t) - y_t \Delta_t \leq \Delta \left(\frac{\sigma k_t}{c_t} - y_t \right) \leq 0.$$

Therefore, the process $(R_t^\Delta)_t$ is a \mathbb{P} supermartingale for every nonnegative process Δ and a martingale for the truthful report case $\Delta = 0$. Thus, we have

$$R_0 = \omega_0 = \mathbb{E} \left[\int_0^\infty e^{-\rho s} \log(c_s) ds \right] \geq \mathbb{E} \left[\int_0^\infty e^{-\rho s} \log(c_s + \sigma k_s \Delta_s) ds \right],$$

which implies the optimality of a truthful report. QED

Proof of Proposition 2: Let $\alpha = (c, c^P, y)$ be an admissible feedback control. We denote

$$J_\lambda^\alpha = \int_0^\infty e^{-\rho t} \left(\log c_t^P + \lambda(K_t, \mathbb{P}_t) \left(K_t - \int y_t(\omega) c_t(w) d\mathbb{P}_t(\omega) \right) \right) dt,$$

and

$$J^\alpha = \int_0^\infty e^{-\rho t} \log(c_t^P) dt.$$

For every Lagrange multiplier λ , we have by assumption i),

$$V_\lambda = J_\lambda^{\alpha_\lambda} \geq J_\lambda^\alpha.$$

In particular, for $\lambda = \lambda_0$,

$$V_{\lambda_0} = J_{\lambda_0}^{\alpha_{\lambda_0}} = J^{\alpha_{\lambda_0}}$$

On the other hand, for $\alpha \in \mathcal{K}$, we have $V_{\lambda_0} = J_{\lambda_0}^{\alpha_{\lambda_0}} \geq J_{\lambda_0}^\alpha = J^\alpha$ yielding $V_{\lambda_0} \geq \sup_{\alpha \in \mathcal{K}} J^\alpha$.

Because $\alpha_{\lambda_0} \in \mathcal{K}$, $V = V_{\lambda_0}$ and the proof is complete.

QED

Proof of Proposition 3: If the value function of the principal V is regular, the result follows from a direct application of the Itô's formula (73) given in Appendix B.

Proof of Proposition 4: Fix $\mathbb{P} \in \mathbb{P}_2(\mathbb{R})$ and a Lagrange multiplier λ . Let \mathbb{P}_t be the probability distribution of the random variable ω_t when the initial probability distribution of ω_0 is \mathbb{P} . Let us consider some arbitrary feedback control $\alpha(K_t, \mathbb{P}_t, \omega_t)$. We apply Itô's formula (73) to $v^\lambda(K_t, \mathbb{P}_t)$ between $s = 0$ and $s = t$ for $t > 0$.

$$\begin{aligned}
e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t) &= v^\lambda(K, \mathbb{P}) \\
&+ \int_0^t e^{-\rho s} \left(-\rho v^\lambda(K_s, \mathbb{P}_s) + v_K^\lambda(K_s, \mathbb{P}_s) \left(\mu K_s - c^P(K_s, \mathbb{P}_s) - \int c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds \\
&+ \int_0^t e^{-\rho s} \int \partial_\omega \nabla v^\lambda[K_s, \mathbb{P}_s](\omega) (\rho \omega - \log c(K_s, \mathbb{P}_s, \omega)) d\mathbb{P}_s(\omega) ds \\
&+ \int_0^t e^{-\rho s} \int \partial_{\omega\omega} \nabla v^\lambda[K_s, \mathbb{P}_s](\omega) \frac{\sigma^2}{2} y^2(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) ds.
\end{aligned}$$

We deduce from the Bellman equation (33) satisfied by v^λ that

$$\begin{aligned}
v^\lambda(K, \mathbb{P}) &\geq e^{-\rho t} v^\lambda(K_t, \mathbb{P}_t) \\
&+ \int_0^t e^{-\rho s} \left(\log(c^P(K_s, \mathbb{P}_s)) + \lambda(K_s, \mathbb{P}_s) \left(K_s - \int y(K_s, \mathbb{P}_s, \omega) c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds.
\end{aligned}$$

Letting t tend to $+\infty$ and using the transversality condition, we obtain

$$v^\lambda(K, \mathbb{P}) \geq \int_0^\infty e^{-\rho s} \left(\log(c^P(K_s, \mathbb{P}_s)) + \lambda(K_s, \mathbb{P}_s) \left(K_s - \int y(K_s, \mathbb{P}_s, \omega) c(K_s, \mathbb{P}_s, \omega) d\mathbb{P}_s(\omega) \right) \right) ds = J_\lambda^\alpha.$$

Since the control α is arbitrary, we obtain $v^\lambda(K, \mathbb{P}) \geq V_\lambda$. On the other hand, let us apply the same

Itô argument with the control α_λ^* attaining the maximum in (33). We obtain

$$v^\lambda(K, \mathbb{P}) = J_\lambda^{\alpha_\lambda^*} \leq V_\lambda,$$

which yields that $v^\lambda = V_\lambda$. We conclude the proof by applying Proposition 2.

QED

Proof of Proposition 5: Denote $A_t \equiv \int \exp(\rho\omega) d\mathbb{P}$. To obtain the dynamics of A_t , we substitute $\gamma^A = 1/(ya)$ in $c(\omega) = \gamma^A \exp(\rho\omega)$, and then substitute the resulting expression into (30), which yields

$$d\omega_t = \log(ya)dt + \sigma y dZ_t. \quad (62)$$

(62) and $A_t = \mathbb{E}[\exp(\rho\omega_t)]$ yield

$$A_t = A_0 \mathbb{E} [\exp(\rho (\log(ya)t + \sigma y Z_t))] = A_0 \exp \left(\left(\rho \log(ya) + \frac{\rho^2 \sigma^2 y^2}{2} \right) t \right), \quad (63)$$

which gives

$$\frac{dA_t}{A_t} = \left(\rho \log(ya) + \frac{\rho^2 \sigma^2 y^2}{2} \right) dt. \quad (64)$$

Equality of the growth rates of K_t and A_t means that

$$\mu - \gamma^P - \frac{1}{y} = \rho \log(ya) + \frac{\rho^2 \sigma^2 y^2}{2}. \quad (65)$$

The restricted principal's problem is thus characterized by the following maximization problem:

$$V(K, \mathbb{P}) = \sup_{\gamma^P, y} \int_0^{+\infty} e^{-\rho t} \log(\gamma^P K_t) dt, \quad (66)$$

under the constraint (65) and the dynamics of capital

$$K_t = K \exp((\mu - \gamma^P - \frac{1}{y})t). \quad (67)$$

Substituting K_t from (67) into (66), the latter writes

$$V(K, \mathbb{P}) = \sup_{\gamma_P, y} \int_0^{+\infty} [e^{-\rho t} \left(\log(\gamma_P K) + (\mu - \gamma^P - \frac{1}{y})t \right)] dt, \text{ s.t., (65)}. \quad (68)$$

Easy computations then show that (68) can be rewritten as

$$\rho V(K, \mathbb{P}) = \log K + \sup_{\gamma_P, y} \left(\log \gamma_P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} \right), \text{ s.t., (65)}. \quad (69)$$

Using (65) we can express γ^P as a function of y and a

$$\gamma^P = \mu - \frac{1}{y} - \rho \left(\log(ya) + \frac{\rho \sigma^2 y^2}{2} \right) = \psi(y) - \rho \log(a).$$

Substituting the value of γ^P into (69), the latter writes as

$$\rho V(K, \mathbb{P}) = \log K + \sup_y \left(\log \left(\mu - \frac{1}{y} - \rho \left(\log(ya) + \frac{\rho \sigma^2 y^2}{2} \right) \right) + \log(ya) + \frac{\rho \sigma^2 y^2}{2} \right). \quad (70)$$

There exists a solution to (70) when the feasible set is non empty, i.e. when it is possible to find values of y for which the argument of the first log is positive. This is equivalent to $a \leq a_{max}$. Taking the first order condition in (70) and denoting

$$\rho v^*(a) := \sup_y \left(\log \left(\mu - \frac{1}{y} - \rho \left(\log(ya) + \frac{\rho \sigma^2 y^2}{2} \right) \right) + \log(ya) + \frac{\rho \sigma^2 y^2}{2} \right), \quad (71)$$

we obtain that $\rho V(K, \mathbb{P}) = \log K + \rho v^*(a)$. QED

Proof of Proposition 6: The proof consists in two steps. First, we have to prove the existence of a Lagrangian multiplier λ_0 , such that the feasibility constraint (32) is satisfied. Second, we have to prove that v^* satisfies the transversality condition for the class of admissible controls defined below to apply Theorem 4.

Step 1: The first order conditions in (41) give

$$-\lambda Ky - \left(\frac{1}{\rho} - av'(a) \right) - \rho \frac{v'(a)}{\gamma^A} = 0$$

and

$$-\lambda K \gamma_A + \rho^2 \sigma^2 y v'(a) = 0.$$

Solving the two equations above gives the optimal controls $y^*(\lambda)$ and $\gamma^A(\lambda)$,

$$y^*(\lambda) = \frac{-\rho \sigma^2 \left(\frac{1}{\rho} - av'(a) \right) + \sqrt{\rho^2 \sigma^2 \left(\frac{1}{\rho} - av'(a) \right)^2 - 4\rho \sigma^2 \lambda^2 K^2}}{2\lambda K \rho \sigma^2}$$

and

$$\gamma^{*,A}(\lambda) = \frac{\rho^2 \sigma^2 v'(a)}{\lambda K} y^*(\lambda).$$

A tedious computation shows that the feasibility constraint $\gamma^{*,A}(\lambda) y^*(\lambda) a = 1$ gives a cubic equation for λ that admits a solution.

Step 2: For $\varepsilon > 0$, let us define the set

$$\mathcal{A}_\varepsilon = \{ \alpha \text{ admissible feedback controls s.t. } \int \exp(\rho \omega) d\mathbb{P}_t^{(\alpha)}(\omega) \leq (a_{\max} - \varepsilon) K_t^{(\alpha)} \text{ for all } t \geq 0 \}$$

and define the set \mathcal{A} as the union of \mathcal{A}_ε . We will prove that for every control $\alpha \in \mathcal{A}$, we have the transversality condition $\lim_{t \rightarrow +\infty} e^{-\rho t} \left(\frac{\log K_t^\alpha}{\rho} + v^*(A_t^\alpha) \right) = 0$ where

$$A_t^\alpha = \frac{\int \exp(\rho\omega) d\mathbb{P}_t^{(\alpha)}(\omega)}{K_t^{(\alpha)}}.$$

Take $\alpha \in \mathcal{A}$. There is $\varepsilon > 0$, such that $\alpha \in \mathcal{A}_\varepsilon$. Because v^* is continuous, v^* is bounded by a constant C_ε on the interval $[0, a_{max} - \varepsilon]$, and we have

$$e^{-\rho t} \left(\frac{\log K_t^\alpha}{\rho} + v^*(A_t^\alpha) \right) \leq e^{-\rho t} \frac{\mu}{\rho} t + e^{-\rho t} C_\varepsilon,$$

which proves the transversality condition. QED

Proof of Proposition 7: To prove Point 1 in Proposition 7 we start by observing that (29) states that the growth rate of capital is

$$g = \mu - \frac{\int c(\omega) d\mathbb{P}(\omega)}{K} - \frac{c^P}{K}$$

and that (3.4.1) states that

$$c^P = \gamma^P K, c(\omega) = \gamma^A \exp(\rho\omega).$$

Substituting the latter in the former, we have

$$g = \mu - \frac{\gamma^A \int \exp(\rho\omega) d\mathbb{P}(\omega)}{K} - \gamma^P.$$

By the definition of A_t , this is

$$g = \mu - \gamma^A \frac{A_0}{K} - \gamma^P. \quad (72)$$

As explained in the analysis of the restricted problem, (3.4.1) implies $\frac{A_t}{K_t}$ is a constant, denoted by a , and $\gamma^A = \frac{1}{ya}$. Substituting in (72) yields

$$g = \mu - \frac{1}{y} - \gamma^P.$$

Substituting γ^P , we obtain Point 1 in Proposition 7.

To prove Point 2 in Proposition 7, we start by recalling that (62) states

$$d\omega = \log(ya)dt + \sigma y dZ_t$$

and that (65) implies

$$\log(ya) = \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

Noting that the first term on the right-hand side is $\frac{g}{\rho}$, we obtain Point 2 in Proposition 7.

Point 3 in Proposition 7 is just a restatement of γ^P , while Points 4 and 5 are established at the beginning of the analysis of the restricted problem.

QED

Proof of Proposition ?? Proposition 7 implies that the value of y corresponding to an information-constrained Pareto optimum in which $\gamma^P = 0$ is the root of

$$\rho\sigma^2 y^3 + y = \frac{1}{\rho}.$$

Substituting $x = \rho y$ in this equation, we obtain

$$\frac{\sigma^2}{\rho}x^3 + x = 1,$$

whose unique positive root is x_{min} , as stated in Proposition ?? . By Proposition ?? this is the portfolio structure chosen by agents in equilibrium when the tax rate is that stated in Proposition ??

$$\tau^* = (1 - x_{min})\left(\frac{\rho}{x_{min}} - \sigma^2 x_{min}\right).$$

This establishes the second part of proposition ?? . To establish the first part, distribute initial capital proportionally to $\exp(\rho\omega)$ and set $\tau = \tau^*$.

QED

Appendix B: Differential calculus in the Wasserstein space

Consider a real-valued function F defined on $\mathcal{P}_2(\mathbb{R})$, which is the set of probability measures on \mathbb{R} with finite second moment. We endow $\mathcal{P}_2(\mathbb{R})$ with the Wasserstein distance. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $\Pi(\mu, \nu)$ is the set of transport plans, that is, probability measures on $\mathbb{R} \times \mathbb{R}$ with respective marginals μ and ν . The Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$ is defined as the square root of

$$\min_{\gamma \in \Pi(\mu, \nu)} \int_{\mathbb{R}^2} |y - x|^2 d\gamma(x, y).$$

To apply a verification argument for the principal problem, we are interested in Itô's formula for F to describe the dynamic $t \rightarrow F(\mathbb{P}_t)$, where \mathbb{P}_t is the marginal probability distribution of the process $(w_t)_t$ given by Equation (30). Itô's formula for F naturally requires differential calculus on the space of measures. We start by introducing the two notions of first variation and L-differentiability

for functions of measures relying on the convexity of $\mathcal{P}_2(\mathbb{R})$ (for a more rigorous treatment, see for instance the standard references ([14] Definition 5.43 and Proposition 5.48), [13] and [50]).

Definition 1. *We will say that*

- *A function F admits a first variation at $\mu \in \mathcal{P}_2(\mathbb{R})$ if there exists a real-valued and continuous function $\nabla F[\mu] : \mathbb{R} \rightarrow \mathbb{R}$, such that for all ν in $\mathcal{P}_2(\mathbb{R})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F((1 - \varepsilon)\mu + \varepsilon\nu) - F(\mu)) = \int_{\mathbb{R}} \nabla F[\mu](\omega) d(\nu - \mu)(\omega).$$

- *A function F that admits a first variation at $\mu \in \mathcal{P}_2(\mathbb{R})$ is L -differentiable at μ if the function $\nabla F[\mu]$ is twice differentiable on \mathbb{R} . We will denote by $\partial_{\omega} \nabla F[\mathbb{P}]$ and $\partial_{\omega\omega} \nabla F[\mathbb{P}]$ its first and second L -derivatives.*

For a function F that is L -differentiable, Itô's formula, associated to the dynamics $t \rightarrow F(\mathbb{P}_t)$ with the dynamics of ω_t given in (30), takes the following form (see [14], Chapter 5, Th. 5.99):

$$\begin{aligned} F(\mathbb{P}_t) &= F(\mathbb{P}_0) + \int_0^t \mathbb{E} [\partial_{\omega} \nabla F[\mathbb{P}_s](\omega_s)(\rho\omega_s - \log c^A(K_s, \mathbb{P}_s, \omega_s))] ds \\ &\quad + \frac{1}{2} \int_0^t \mathbb{E} [\partial_{\omega\omega} \nabla F[\mathbb{P}_s](\omega_s)\sigma^2 y^2(K_s, \mathbb{P}_s, \omega_s)] ds. \end{aligned} \tag{73}$$

A general class of L -differentiable functions can be described as follows. Let ϕ be a twice continuously differentiable function on \mathbb{R} with quadratic growth and v a continuously differentiable function on \mathbb{R} . We consider the function F defined on $\mathcal{P}_2(\mathbb{R})$ by

$$F(\mu) = v \left(\int_{\mathbb{R}} \phi(\omega) \mu(d\omega) \right).$$

Then, F is L-differentiable with

$$\nabla F[\mu] = v' \left(\int_{\mathbb{R}} \phi(\omega) \mu(d\omega) \right) \phi, \partial_{\omega} \nabla F[\mu] = v' \left(\int_{\mathbb{R}} \phi(\omega) \mu(d\omega) \right) \phi' \text{ and } \partial_{\omega\omega} \nabla F[\mu] = v' \left(\int_{\mathbb{R}} \phi(\omega) \mu(d\omega) \right) \phi''.$$

Online Appendix.

The objective of this technical note is to clarify the existence of the limit equations (29) and (30) by adapting for sake of completeness the techniques developed in [49].

To simplify, we consider that the function y is constant and we normalize it to 1. The proof below can be easily extended to the case where the function y is Lipschitz by applying Burkholder, Davis and Gundy inequality but we choose not for simplicity of exposition.

Equations (30) and (31) can be summarized by the McKean Vlasov SDE

$$\begin{cases} d\omega_t = b_1(\omega_t, K_t, \mu_t) dt + dZ_t \\ dK_t = b_2(K_t, \mu_t) dt, \end{cases} \quad (74)$$

where μ_t is the probability distribution of ω_t for all t and the two functions b_i are Lipschitz on $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R})$ respectively.

Proposition 9. *There exists a unique strong solution of the McKean Vlasov equation (74).*

Proof: Fix $T \geq 0$. Define the truncated supremum norm $\|x\| = \sup_{s \in [0, t]} |x_s|$ for $x \in \mathcal{C} = \mathcal{C}([t, T], \mathbb{R})$. We define the following truncated distance on $\mathcal{P}_2(\mathcal{C})$,

$$d_t^2(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathcal{C} \times \mathcal{C}} \|x - y\|_t \pi(dx, dy).$$

Fix $\mu \in \mathcal{P}_2(\mathcal{C})$ and consider the SDE

$$\begin{cases} d\omega_t^\mu = b_1(\omega_t^\mu, K_t^\mu, \mu_t) dt + dZ_t \\ dK_t^\mu = b_2(K_t^\mu, \mu_t) dt, \end{cases} \quad (75)$$

Because the function b_i are Lipschitz, there exists a unique square-integrable solution (ω^μ, K^μ) to (75). Define the map $\Phi : \mathcal{P}_2(\mathcal{C}) \times \mathcal{P}_2(\mathcal{C})$ that assigns to μ the probability distribution of ω^μ . Observe that fixed points of Φ are solutions of the McKean Vlasov equation (74).

Let $\mu, \nu \in \mathcal{P}_2(\mathcal{C})$, for $s \leq t \leq T$, we have using Cauchy-Schwarz inequality

$$|K_s^\mu - K_s^\nu| \leq s \int_0^s |b_2(K_u^\mu, \mu_u) - b_2(K_u^\nu, \nu_u)|^2 du.$$

Using the Lipschitz assumption, the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and taking the supremum over $[0, t]$, we obtain

$$\mathbb{E} [||K^\mu - K^\nu||_t^2] \leq 2tL^2 \mathbb{E} \left(\int_0^t (||K^\mu - K^\nu||_s^2 + W_2^2(\mu_s, \nu_s)) ds \right).$$

Using Fubini's theorem and Gronwall's inequality, we obtain

$$\mathbb{E} [||K^\mu - K^\nu||_t^2] \leq C \int_0^t W_2^2(\mu_s, \nu_s) ds \leq C \int_0^t d_s^2(\mu, \nu) ds. \quad (76)$$

Repeating the same argument, we have

$$\begin{aligned} \mathbb{E} [||\omega^\mu - \omega^\nu||_t^2] &\leq 2tL^2 \mathbb{E} \left(\int_0^t (||\omega^\mu - \omega^\nu||_s^2 + ||K^\mu - K^\nu||_s^2 + W_2^2(\mu_s, \nu_s)) ds \right) \\ &\leq 2tL^2 \mathbb{E} \left(\int_0^t (||\omega^\mu - \omega^\nu||_s^2 + W_2^2(\mu_s, \nu_s)) ds \right) + 2t^2L^2 \mathbb{E} [||K^\mu - K^\nu||_t^2] \\ &\leq C \mathbb{E} \left(\int_0^t (||\omega^\mu - \omega^\nu||_s^2 + W_2^2(\mu_s, \nu_s)) ds \right), \end{aligned}$$

where the last inequality uses (76). Using Fubini's theorem and Gronwall's inequality again, we obtain

$$\mathbb{E} [||\omega^\mu - \omega^\nu||_t^2] \leq C \int_0^t d_s^2(\mu, \nu) ds.$$

By definition of d_t^2 and observing that the joint distribution of (ω^μ, ω^ν) is a coupling, we finally obtain

$$d_t^2(\Phi(\mu), \Phi(\nu)) \leq \mathbb{E} [||\omega^\mu - \omega^\nu||_t^2] \leq C \int_0^t d_s^2(\mu, \nu) ds.$$

The proof of existence and uniqueness now follows from the usual Picard iteration procedure.

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