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Sovereign Debt Sustainability with Involuntary Default

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Abstract

We study the sustainability of sovereign debt under the assumption of involuntary and costly default: governments do their utmost to avoid default, which reduces the resources available for debt service. We show that costly default tightens Blanchard's $g > r$ condition. We derive a formula for a government's maximum sustainable debt (MSD), which depends on the mean and the volatility of the country's growth rate, the government's maximum primary surplus, the risk-free rate, and the fraction of resources available to the government in default. We compute MSD for 12 Eurozone countries and examine the role of the European Stability Mechanism in increasing MSD.

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1 Introduction

The sustainability of sovereign debt in industrialized countries has received much attention in recent years. This is as it should be, in view of the explosion in sovereign debt to levels previously reached only in time of war. Thus, at the end of 2023, sovereign debt levels were 255% of GDP in Japan, 168% in Greece, 144% in Italy, 123% in the United States, 110% in France, and 104% in the United Kingdom. It is natural to ask whether such high debt levels are sustainable.[1]

In an influential paper, Blanchard (2019) has provided some reassurance regarding the sustainability of sovereign debt in the USA. Blanchard reports evidence that, in the USA, the average growth rate $\overline{g}$ has typically exceeded the average risk free rate r, a condition that implies that economic growth can be counted upon to resorb large temporary increases in debt, or to make permanent small deficits feasible.[2] An implicit assumption underlying the $\overline{g} > r$ condition is that there is no cost to default: a government's failure to service its debt affects neither the government's ability to roll over existing debt nor its ability to raise new debt.

If default is costly—if, for instance, default disrupts the process of new debt issuance—then the $\overline{g} > r$, condition must be modified to account for such cost.[3] This is the purpose of the present paper. New debt issuance is central to debt service. If lenders provide less new debt in default, they constrain the extent to which future primary surpluses can be brought forward for debt service. We show that this constraint effectively amounts to a reduction in the growth rate to be compared to the interest rate in the $\overline{g} > r$ condition. Default tightens that condition by replacing the growth rate $\overline{g}$ by a rate γ made lower by constrained reliance on future primary surpluses: $\gamma < \overline{g}$. We examine the validity of the two conditions for twelve Eurozone countries, namely Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, the Netherlands, Portugal, and Spain. We find that while the $\overline{g} > r$ condition is satisfied for all twelve Eurozone countries over the period 1999-2023, the $\gamma > r$ condition is satisfied for none. Accounting for the cost of default makes for a less sanguine assessment of sovereign debt sustainability.

Does the condition $\overline{g} > r$ not make the cost of default irrelevant by ruling out default? The answer is no. Debt sustainability precludes reliance on ever increasing debt-to-GDP ratios for the purpose of debt service, that is, it precludes reliance on a Ponzi scheme. Under a condition that we show to be $\gamma < r$ when default is costly, sustainability implies the existence of a finite upper bound on the debt-to-GDP ratio. Thus, if a government's debt should be close to the upper bound and a large negative growth shock were to occur, driving the debt-to-GDP ratio above the upper bound, default would occur and the cost of default would be incurred; this would be true regardless of whether $\overline{g} > r$ or $\overline{g} < r$, as long as $\gamma < r$. Whilst there would be no default if $\gamma > r$, this does not imply that default costs are irrelevant as γ itself depends on these

[1] We assume an independent central bank that prevents the government from resorting to inflation for servicing its debt

[2] We use the notation $\overline{g}$ to denote average growth rate and g_t for the growth rate at date t.

[3] There is extensive evidence of countries' difficulty to issue debt in the aftermath of default; see for example Borensztein and Panizza (2009), Gelos, Sahay, and Sandleris (2011), and Cruces and Trebesch (2013).

costs.

Our generalization of the $\overline{g} > r$ condition proceeds from our characterization of a country's maximum sustainable debt. We assume that the country's government does its utmost to service its debt. This assumption is consistent with our focus on *maximum* debt: lenders naturally lend more to those they expect to strive to avoid default than to those they fear may strategically default on the debt.[4] We characterize maximum sustainable debt as the fixed point that equates *(i)* debt beyond which default occurs and *(ii)* the maximum level of resources available for debt service. These resources in turn are the sum of *(ii–a)* the maximum primary surplus that the government can achieve, that which maximizes revenues and minimizes expenses to those strictly necessary to the functioning of the State, and *(ii–b)* the maximum proceeds that can be had from new debt issuance.

New debt proceeds naturally depend on the probability of default associated with the level of newly issued debt, that is, they depend on the level of new debt beyond which default will occur. There is therefore a recursive relationship between present and future debt beyond which default occurs. That debt is *maximum* as it equals the maximum level of resources available for debt service. It is *sustainable* only if future debt remains finite: debt that relies for repayment on the issuance of infinite debt, even if only far into the future, cannot be considered sustainable. When both the risk-free interest rate and the rate of growth in GDP are stationary, maximum sustainable debt is the fixed point of the equation that equates the maximum debt that can be owed to the maximum resources that are available to service such debt, where these resources in turn depend on the maximum debt that can be issued, and consequently owed. The necessary and sufficient condition for a fixed point to exist is $\gamma < r$; that fixed point is the maximum sustainable debt (MSD): no debt level greater than MSD can be sustained. Conversely, when $\gamma > r$, no fixed point exists; growth can be counted upon both to resorb large temporary increases in debt and to make permanent small deficits feasible. The $\gamma > r$ condition reduces to the $\overline{g} > r$ condition when there is no decrease in resource availability in default, that is, when there is no cost of default. This is the sense in which Blanchard's $\overline{g} > r$ condition is a special case of our $\gamma > r$ condition.

We derive the expressions for maximum sustainable debt as well as its associated proceeds, debt face value, default probability, interest rate, and primary surplus. The average primary surplus of a government that continuously issues maximum debt is well below the maximum primary surplus which, together with maximum borrowing proceeds, determines maximum sustainable debt; the continuous issuance of maximum debt may in fact be consistent with an average primary deficit rather than surplus. We consider the role of subsidized financing such as provided by the European Stability Mechanism in increasing MSD. We show that, while subsidized financing is effective at increasing MSD, it generally crowds out private sector financing and results in a lower increase in MSD than the present value of the subsidy it provides.

Default in our model is involuntary: governments do their utmost to avoid default. How

[4]We further justify this assumption below.

realistic is that assumption? Very! Levy Yeyati and Panizza (2011) report numerous instances of government reluctance to default: governments appear to default only as a last resort, after they have tried every possible way of staving default off. Defaults delayed well past the point at which throwing in the towel would have been optimal have prompted the IMF (2013, p.1) to comment that "debt restructurings have often been too little and too late." While debt service is costly, default can be even costlier, especially from the point of view of a government that can expect to lose power in the aftermath of default (Borensztein and Panizza (2009); Malone (2011)), and whose members' prospects for alternative employment would be jeopardized were they deemed too prone to default. Even a less than fully self-interested government may do its utmost to avoid default: Grossman and van Huyck (1988) and Tomz (2007) have argued that creditors are much more lenient towards borrowers for whom default was clearly unavoidable than those who are perceived to have been too quick to default; Bolton and Jeanne (2011) and Gennaioli, Martin, and Rossi (2014, 2018) have noted the potential of sovereign default to jeopardize the proper functioning of an entire banking system, in view of government bonds' importance as collateral for bank loans.

The paper proceeds as follows. Section 2 reviews the literature. Section 3 presents the model and derives the condition for sustainability and the expressions for maximum sustainable debt, borrowing proceeds, interest rate, default probability, and primary surplus. Section 4 considers the role of the European Stability Mechanism. Section 5 presents the data, parametrization, and results. A last section concludes. The Appendix contains selected derivations and the proofs.

2 Literature review

There is an immense literature on sovereign debt sustainability, to which the present literature review cannot but fail to do justice. Recent surveys are those of D'Erasmo, Mendoza, and Zhang (2016), Debrun, Ostry, Willems, and Wyplosz (2019), and Mitchener and Trebesch (2021). Here, we limit ourselves to discussing those papers directly related to our own: our purpose is to place our work in the context of previous work.

Central to work on sustainable government debt has been the intertemporal government budget constraint (IGBC), which relates present and future debt, the growth and interest rates, and a country's primary balance. Sustainable debt has generally been viewed as the fixed point of the IGBC, and sustainability conditions correspondingly have been viewed as those that ensure the existence of such fixed point.[5] Although Bohn (2007) has shown that debt diverging to infinity may nonetheless satisfy the IGBC, he implicitly assumes that the primary balance may grow unbounded if necessary, a rather implausible assumption.

Our derivation of maximum sustainable debt follows that previous work. The equation we consider, however, relates not actual but maximum debt: we derive a recursive equation which, as noted in the Introduction, equates present maximum debt to maximum resources available, which depend on the primary balance as well as the proceeds from the issuance of

[5]See for example the discussion in Section 2.1 of D'Erasmo, Mendoza, and Zhang (2016).

new debt, themselves a function of future maximum debt. As does the IGBC, our recursive equation includes the interest and growth rates, the former used to discount future maximum debt and the latter reflecting the expression of that debt as a fraction of future GDP. As do Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013, GKMOQ) and Collard, Habib, and Rochet (2015, CHR), and unlike Tanner (2013) or Blanchard (2019) for example, we explicitly account for the possibility of default that arises from the nature of government liabilities, specifically debt. We differ from GKMOQ in allowing for refinancing in default and in assuming a constant maximum primary balance; in contrast, GKMOQ's primary balance is estimated along the lines of Bohn's (1998) fiscal reaction function.[6]

Our work extends CHR in a number of directions. In addition to allowing for refinancing in default, thereby making possible a generalization of Blanchard (2019), we derive an expression for the primary surplus at maximum debt issuance. Interestingly, and somewhat surprisingly, we find that average primary surplus is well below maximum primary surplus; average primary surplus can in fact be negative, i.e., a primary deficit. We also extend CHR to examine the effect of the provision of subsidized financing on maximum sustainable debt.

We have noted the relation of our work to Blanchard (2019). Recent work has argued that the proper discount rate to be compared to the growth rate is not the risk free rate considered by Blanchard (2019), but a rate that includes a risk premium as compensation for output risk. This is because government debt is the present value of future primary surpluses, which are correlated with output, and are in fact made riskier than output by the stronger procyclicality of taxes than of government spending. De Vette, Olijslagers, and Van Wijnbergen (2020) find that the discounted value of Dutch primary surpluses is roughly half the market value of Dutch government debt. They examine, and dismiss, a number of potential explanations for the gap between present and market values. The one explanation they do not dismiss is that of future fiscal adjustments that would increase the primary surplus; these range from 1.20% of GDP under partial equilibrium, that is, neglecting the effect of the adjustment in decreasing the risk premium, to 0.28% of GDP under general equilibrium. Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) find that the PV of future surpluses in the US is *negative* 155% of GDP, in contrast to government debt that has market value equal to 37% of GDP. The surprising result that the PV of future surpluses is negative can be explained as follows. Surpluses are the difference between taxes and government spending. Both are procyclical at medium to long horizons, but the former more so than the latter; indeed, government spending is countercyclical at short horizons. This implies that the risk premium for taxes is higher than the risk premium for government spending. As taxes and spending are more or less equal, the PV of taxes is lower than that of spending, making the PV of primary surpluses negative. As do De Vette, Olijslagers, and Van Wijnbergen (2020), Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) consider a number of potential explanations for the gap between present and market values;

[6]A fiscal reaction function relates a country's primary balance to its past debt. Bohn (1998) shows that a sufficient condition for debt sustainability is that the coefficient of past debt in the reaction function be strictly positive.

none appears to be able to close the gap between values if considered in isolation. For example, Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) estimate that a spending cut of 7.7% of GDP would have to occur with a probability of 42% in order to reconcile the present value of future surpluses with the market value of the debt; they deem such probability unrealistic.

In contrast to such work, ours uses the risk-free rate to discount future primary surpluses: although not all Eurozone countries can be considered small open economies, most can; investors can diversify the risk presented by fluctuations in these countries' primary surpluses to an extent they cannot that presented by the US primary surplus. We share the use of the risk-free rate with Mehrotra and Sergeyev (2021), with whom we further share the assumption of a maximum primary surplus and an explicit allowance for the possibility of default. We differ from Mehrotra and Sergeyev (2021) in that default in their paper is due to extraordinary growth disasters, whereas it is due to ordinary output fluctuations in ours. Our focus on ordinary output fluctuations is consistent with our desire to estimate maximum sustainable debt. In a setting without refinancing in default, Collard, Habib, and Rochet (2015) have shown that maximum sustainable debt is lower when extraordinary growth disasters can occur.

Reis (2020) decomposes government debt into the sum of (i) the present value of future primary surpluses, discounted at the marginal product of capital (MPK) and (ii) a bubble term, made possible by the fact that investors are willing to hold government bonds despite such bonds paying an interest rate lower than the MPK; the interest rate on government bonds is lowered by the convenience yield received for holding these uniquely safe assets which constitute highly desirable collateral.[7] Reis (2020) shows that a government can run a perpetual deficit if the MPK exceeds the growth rate, but that this deficit can be no larger than a fraction $\overline{g} - r$ of the assets in the economy. Cochrane (2019) notes that forecasted deficits in the United States are well above the $\overline{g} - r$ fraction of GDP that is consistent with a stable debt-to-GDP ratio.

Brunnermeier, Merkel, and Sannikov (2020) consider the role of government bonds in partially insuring otherwise uninsurable idiosyncratic risk: investors trade safe government bonds to smooth consumption when productive capital is subject to uninsurable idiosyncratic shocks; investors therefore are willing to pay a premium for what Brunnermeier, Merkel, and Sannikov (2020) refer to as the service flow of government bonds. Brunnermeier, Merkel, and Sannikov (2020) argue that the presence of such flows can serve to reconcile the pattern of quasi continual primary deficits that characterizes countries such as Japan with the positive market value of these countries' government bonds: service flows give rise to a bubble term that offsets the stream of primary deficits.[8] A necessary condition for such bubble term to exist is that $r < \overline{g}$. In a setting such as ours in which default costs transform the preceding inequality into the inequality

[7] See for example Krishnamurthy and Vissing-Jorgensen (2012) .

[8] Chien, Cole, and Lustig (2023) consolidate the asset and liability positions of the Japanese government and central bank. They attribute the sustainability of Japan's debt to the discrepancy between long duration assets and short duration liabilities. Combined with Japan's low interest rate policy and with financial repression that has deterred investment into foreign securities, this discrepancy has generated an annual 3% of GDP that has contributed to the servicing and sustainability of Japanese debt. We do not believe the rationale of Chien, Cole, and Lustig (2023) applies to the Eurozone countries we consider, as free movement of capital within the Eurozone precludes financial repression.

$r < \gamma$, the failure of that latter inequality to hold—it is false for all the Eurozone countries we consider—suggests that service flows cannot reconcile actual and maximum sustainable debt levels where the former are larger than the latter.

Our work assumes involuntary default; this assumption distinguishes our work from the work on strategic default in the mold of Eaton and Gersovitz (1981).[9] The latter type of default has governments repeatedly compare the payoffs from debt service and default to choose that course of action which maximizes the present and future welfare of their population. Both types of default define a level of maximum sustainable debt: it is the level beyond which the government *cannot* pay under involuntary default, the level beyond which the government *will not* pay under strategic default. We believe our assumption of involuntary default is more appropriate given our focus on maximum debt: lenders naturally lend more to those they expect will do their utmost to service the debt that those they suspect continually will weigh the costs and benefits of servicing the debt.[10] Thus, in line with Gelpern and Panizza (2022), we turn our attention from sovereign authority's discretion to renege on debt to authority's power to mobilize resources to service debt.

3 Sovereign Debt under Involuntary Default

3.1 The Model

We assume one-period debt.[11] We consider a government that has issued in period $t-1$ debt of face value D_t to be paid in period t. That the government does it utmost to stave off default implies that default occurs only when the maximum level of resources available to the government for debt service in period t are smaller than D_t. These resources are the maximum primary surplus, αY_t, and the maximum borrowing capacity, $b_{M,t} Y_t$. Expressed as a fraction of GDP Y_t, maximum primary surplus (MPS) α and maximum borrowing proceeds $b_{M,t}$ together define the critical debt to GDP ratio ω_t in period t

$$\omega_t = \alpha + b_{M,t}. \tag{1}$$

Default occurs when $D_t > \omega_t Y_t$. Since debt D_t was issued in period $t-1$, we express it as a fraction d_t of GDP in period $t-1$: $D_t = d_t Y_{t-1}$. Default therefore occurs when $d_t Y_{t-1} > \omega_t Y_t$ or, rearranging, when

$$1 + g_t \equiv \frac{Y_t}{Y_{t-1}} < \frac{d_t}{\omega_t}. \tag{2}$$

This formulation makes clear the central role of the GDP growth rate g_t. We assume that the growth factor $G_t \equiv 1 + g_t$ is independently and identically distributed and denote F its c.d.f. and f its p.d.f.

[9] See the surveys by Aguiar and Amador (2014) and Aguiar, Chatterjee, Cole, and Stangebye (2016).

[10] We nonetheless acknowledge the limited effectiveness of reputation at deterring strategic default: quantitative models of sovereign debt show that reputation has almost no impact on willingness to pay (Schmitt-Groheé and Uribe (2017)).

[11] We assume one-period debt for tractability. Short-term debt unlike long-term debt is not vulnerable to dilution (see for example Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012)), but it is vulnerable to coordination failures (see for example Cole and Kehoe (2000)). We abstract from both considerations.

It remains to determine maximum borrowing proceeds $b_{M,t}$. For that purpose, we denote r the risk-free interest rate which we assume is constant for simplicity. We further assume that, if default should occur in period $t+1$, only a fraction $\kappa \in [0,1]$ of resources available for debt service absent default, $\alpha Y_{t+1} + b_{M,t+1} Y_{t+1}$, would in fact be available for debt service: default limits both the availability of the primary surplus and the ability to issue new debt.[12] Using condition (1) and $Y_{t+1} = (1+g_{t+1})Y_t$ to write

$$\alpha Y_{t+1} + b_{M,t+1} Y_{t+1} = \omega_{t+1} Y_{t+1} = \omega_{t+1}(1+g_{t+1})Y_t,$$

we have

$$b_{M,t}Y_t = \frac{1}{1+_f} \max_{d_{t+1}} \left[d_{t+1}Y_t \left(1 - F\left(\frac{d_{t+1}}{\omega_{t+1}}\right)\right) + \kappa \int_0^{\frac{d_{t+1}}{\omega_{t+1}}} \omega_{t+1} Y_t G \mathrm{d}F(G) \right], \tag{3}$$

where we have used the i.i.d. property of growth to drop the time subscript. With risk-neutral lenders, period t proceeds equal period $t+1$ expected payment discounted at the risk-free rate r.[13] Payment absent default is $D_{t+1} = d_{t+1}Y_t$; from (2), such payment is received with probability $1 - F(d_{t+1}/\omega_{t+1})$. Payment in case of default is $\kappa\omega_{t+1}(1+g_{t+1})Y_t$; it has expected value $\kappa \int_0^{d_{t+1}/\omega_{t+1}} \omega_{t+1} Y_t G \mathrm{d}F(G)$ over the range of growth factors ($G \equiv 1+g$) for which default occurs $[0, d_{t+1}/\omega_{t+1})$. There is maximization over d_{t+1} because $b_{M,t}$ represents *maximum* borrowing proceeds.

Dividing (3) by Y_t and using (1), we obtain the recursive equation for the evolution of the debt-to-GDP ratio

$$\omega_t = \alpha + \frac{1}{1+r} \max_{d_{t+1}} \left[d_{t+1} \left(1 - F\left(\frac{d_{t+1}}{\omega_{t+1}}\right)\right) + \kappa \int_0^{\frac{d_{t+1}}{\omega_{t+1}}} \omega_{t+1} G \mathrm{d}F(G) \right]. \tag{4}$$

The RHS of Equation (4) recalls Merton's 1974 formula for the pricing of risky corporate debt. The recursive nature of the equation nonetheless reveals an important difference: absent the ability of a government's creditors to seize the country's assets in case of default, the value of government debt necessarily depends on future government debt in a way that need not be true of corporate debt.

3.2 A Condition for Sustainability

Equation (4) characterized the evolution of debt; we now ask to what extent and under which condition a given level of debt in sustainable. We start by defining sustainability in our framework.

Definition 1 *A given debt-to-GDP ratio ω is sustainable if the sequence ω_t defined by $\omega_0 = \omega$ and (4) for all $t \geqslant 1$ is bounded above.*

[12] We assume for tractability that the same resource availability rate κ applies to both sources of funds.

[13] The assumption of of risk-neutral lenders could be relaxed under the alternative assumption of complete markets. The analysis would proceed along very similar lines, with risk-adjusted probabilities replacing actual probabilities.

We are now in a position to derive a condition for sustainability and discuss the similarity and the difference between this condition and that of Blanchard (2019). Let us rewrite (4) as

$$\omega_t = \alpha + \phi\left(\omega_{t+1}\right), \tag{5}$$

where

$$\phi\left(\omega\right) \equiv \frac{1}{1+r} \max_d d\left[1 - \ell\left(\frac{d}{\omega}\right)\right] \tag{6}$$

and

$$\ell\left(\rho\right) \equiv F\left(\rho\right) - \frac{\kappa}{\rho}\int_0^\rho G \mathrm{d}F\left(G\right). \tag{7}$$

The function ϕ represents maximum proceeds from new debt issuance; these depend on expected loss from default, represented by the function ℓ. Thus, in line with our previous discussion, (5) expresses maximum debt as the sum of maximum primary surplus and maximum new debt issuance proceeds.

Note that ϕ is linear. To see this, make the change of variable $G \equiv d/\omega$ and rewrite (6) as

$$\phi\left(\omega\right) = \frac{1}{1+r}\max_{G\geqslant 0} \omega\, G\left[1 - \ell\left(G\right)\right] = \frac{1+\gamma}{1+r}\omega, \tag{8}$$

where

$$1+\gamma \equiv \max_G G\left[1-\ell\left(G\right)\right] = G_M\left[1-\ell\left(G_M\right)\right] = \left(1+g_M\right)\left[1-\ell\left(1+g_M\right)\right], \tag{9}$$

where G_M is the growth factor that solves the maximization problem and $g_M \equiv G_M - 1$ is the corresponding growth rate. Note that, as will become clear in Section 3.3, g_M represents the growth rate below which the government defaults. Using (8), Equation (5) rewrites as the linear difference equation

$$\omega_t = \alpha + \frac{1+\gamma}{1+r}\omega_{t+1}. \tag{10}$$

Equation (10) has solution

$$\omega_t = \left(\frac{1+r}{1+\gamma}\right)^t\left(\omega - \omega_M\right) + \omega_M, \tag{11}$$

where[14]

$$\omega_M = \frac{\alpha(1+r)}{r-\gamma}. \tag{13}$$

Recall from Section 3.1 that $\omega = \omega_0$ is the debt-to-GDP ratio whose sustainability is to be determined. Define $\overline{G} \equiv 1 + \overline{g} \equiv \mathbb{E}\left[G\right]$. We establish the following proposition.[15]

[14]Proposition 2 will show ω_M to be the maximum sustainable debt. It is the fixed point of (10), thereby satisfying

$$\omega_M = \alpha + \frac{1+\gamma}{1+r}\omega_M. \tag{12}$$

[15]All proofs are reported in the Appendix.

Proposition 2 (Debt Sustainability)

1. *When $r < \gamma$, ω_t decreases over time for all ω; any debt-to-GDP ratio ω is sustainable.*

2. *When $r > \gamma$, ω_t decreases over time if $\omega \leq \omega_M$ but tends to ∞ if $\omega > \omega_M$; the debt-to-GDP ratio ω is sustainable if and only if $\omega \leq \omega_M$.*

3. *When there is no decrease in resource availability in default, $\kappa = 1$, then $\gamma = \overline{g}$: our condition $r < \gamma$ coincides with the Blanchard condition $r < \overline{g}$.*

4. *When there is decreased resource availability in default, $\kappa < 1$, then the condition for any debt-to-GDP ratio to be sustainable is more restrictive than the Blanchard condition: $\gamma < \overline{g}$.*

When there is no decrease in resource availability in default, borrowing proceeds are maximized by issuing debt of infinite face value, thereby ensuring that there is default in every period, in turn ensuring that debtholders receive the entirety of the primary surplus and of new issuance proceeds in every period. Debtholders effectively become shareholders. This explains the replacement of γ by $\overline{g}$.

When there is decreased resource availability in default, the effectiveness of growth at providing the additional resources necessary for debt resorption diminishes; this makes the attainment of the environment analyzed by Blanchard (2019) more difficult: when $\gamma < \overline{g}$, $r < \gamma$ implies $r < \overline{g}$ but the converse is not true.

In order to test the condition $\gamma < r$, assume that the growth factor $1 + g$ is lognormally distributed, $\log(1+g) \sim N(\mu, \sigma^2)$ with $1 + \overline{g} = \exp[\mu + \sigma^2/2]$; (9) becomes[16]

$$1 + \gamma = (1 + \overline{g}) \left\{ \max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right] [1 - N(x)] + \kappa N(x - \sigma) \right\} \tag{14}$$

The condition $\gamma < r$ in turn becomes[17]

$$\overline{g} < \frac{1 + r}{\max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right] [1 - N(x)] + \kappa N(x - \sigma)} - 1 \equiv g_c. \tag{15}$$

Inequality (15) is the condition we shall test empirically. It is similar to the Blanchard Condition in that it compares a growth rate, $\overline{g}$, to an interest rate, the risk-free rate grossed up to account for the possibility of default; this interest rate defines the cutoff growth rate, g_c.

What if $\overline{g} > g_c$ or, equivalently, $\gamma > r$? Then, as just noted, there is no maximum debt-to-GDP ratio: high growth can be counted upon to resorb large temporary increases in debt, or to make permanent small deficits feasible. This naturally recalls the environment analyzed by Blanchard (2019) under the condition $\overline{g} > r$.

With the growth factor $1 + g$ lognormally distributed, MSD is a function of five parameters: the mean μ and the volatility σ of $\log(1 + g)$, the maximum primary surplus α, the fraction of resources available in default κ, and the risk-free interest rate r. We show

[16] The derivation is in the Appendix.
[17] The derivation is in the Appendix.

Proposition 3 (Comparative Statics)

1. *Maximum sustainable debt* ω_M *is increasing in the maximum primary surplus* α *and decreasing in the risk-free rate* r.

2. *MSD* ω_M *is increasing through* γ *in the resource availability rate* κ *and in the mean* μ *of* $\log(1+g)$.

3. *MSD* ω_M *is decreasing through* γ *in the volatility* σ *of* $\log(1+g)$ *for constant expected growth factor* $1+\overline{g}$*; MSD's variation in* σ *when* μ *is kept constant is indeterminate.*

The results are intuitive. The maximum level of debt ω_M a government can sustain is higher, the higher is the present value of the resources available to the government. This value in turn is higher, the higher is the maximum primary surplus the government can achieve (α), the higher is the mean of the log growth factor (μ), the more resources are available to the government in case the government should be forced into default (κ), and the lower is the interest rate at which future primary surpluses are discounted, that rate in turn being lower the lower is the risk-free interest rate (r). For a given expected growth rate $\overline{g}$, the present value of the resources available to the government is lower, the higher is the volatility of the log growth factor (σ), because more volatile growth implies a greater likelihood of low growth rates at which default occurs; default decreases the resources available to the government. Recognizing that the expected growth rate $\overline{g}$ increases in volatility under the assumption of lognormality introduces a counteracting effect to volatility, making volatility's overall effect indeterminate.

3.3 The Equilibrium at Maximum Sustainable Debt

We now characterize the equilibrium that prevails when debt is at MSD. We consider the proceeds from debt issuance, the face value of the debt, the probability of default, and implicit interest rate. We leave the discussion of the primary surplus to Section 3.4.

From (1) evaluated at maximum sustainable debt ω_M, we obtain maximum borrowing proceeds associated with MSD; we refer to these as maximum sustainable borrowing (MSB) $b_M \equiv \omega_M - \alpha$. From (13), b_M equals

$$b_M = \frac{\alpha(1+\gamma)}{r-\gamma}. \tag{16}$$

Recalling the definition of d_t in Section 3.1, we define d_M to be the face value of the debt that corresponds to MSD ω_M and MSB b_M: a government that has issued in period $t-1$ zero-coupon debt $D_t = d_M Y_{t-1}$ due in period t defaults in period t if GDP Y_t is such that $d_M Y_{t-1} > \omega_M Y_t$; $b_M Y_{t-1}$ are the proceeds from debt issuance. From the change of variable $G \equiv d/\omega$ made to solve the maximization problem in (6), which is recalled to have solution $G_M \equiv 1+g_M$, ω_M is such that

$$d_M = (1+g_M)\omega_M. \tag{17}$$

Default at MSD occurs when $1+g_t = Y_t/Y_{t-1} < d_M/\omega_M = 1+g_M$; the corresponding probability of default is $P_M \equiv F\left(1+g_M\right)$.

We now compute the implicit interest rate r_M on the zero-coupon debt. We have[18]

$$1 + r_M = \frac{d_M}{b_M} = \frac{1 + r}{1 - \ell\left(1 + g_M\right)}. \tag{18}$$

The function $\ell(\cdot)$ in (7) is a resource availability-adjusted probability of default;[19] the implicit interest rate r_M compensates risk-neutral investors for expected default loss.

Government debt often is expressed as a present value of future primary surpluses. We provide two such characterizations of b_M. The first characterization is[20]

$$b_M = \frac{\alpha(1 + g_M)}{r_M - g_M}. \tag{19}$$

Maximum sustainable borrowing MSB therefore equals the present value of future MPS α, growing at the rate g_M below which default occurs, discounted at the default risk-adjusted interest rate r_M.

Alternatively, MSB b_M can be decomposed into the difference of two terms, the present value of future MPS α and the present value of expected default costs, both growing at rate g_M and both discounted at the risk-free rate r. Formally[21]

$$b_M = \alpha \sum_{n=1}^{\infty} \left(\frac{1 + g_M}{1 + r}\right)^n - \alpha \sum_{n=1}^{\infty} \left(\frac{1 + g_M}{1 + r}\right)^n \left(1 - \left[1 - \ell\left(1 + g_M\right)\right]^n\right). \tag{20}$$

The first term on the RHS is the present value of future MPS, the second is the present value of expected default costs. Recalling that ℓ is a (resource availability-adjusted) probability of default, $\left[1 - \ell\left(1 + g_M\right)\right]^n$ can be interpreted as the probability that no default occurs up to and including period $t + n$ and $1 - \left[1 - \ell\left(1 + g_M\right)\right]^n$ the probability that default has occurred in any period up to and including that period. Default in any period up to and including $t + n$ makes the MPS in that period no longer available for debt service. The expected cost in period $t + n$ of default in any period up to and including that period is therefore $\left(1 - \left[1 - \ell\left(1 + g_M\right)\right]^n\right) \alpha(1 + g_M)^n$, expressed as a fraction of current period GDP Y_t: it is the MPS in period $t + n$, foregone because of default in any period up to and including that period. The present value of these costs is the second term on the RHS of (20).

[18]The derivation is in the Appendix.

[19]To see that ℓ is a probability function, note that

$$l'\left(\rho\right) = \left(1 - \kappa\right) f\left(\rho\right) + \frac{\kappa}{\rho^2} \int_0^{\rho} G \mathrm{d}F\left(G\right) > 0,$$

$$\lim_{\rho \to 0} \ell\left(\rho\right) = F\left(0\right) = 0,$$

and

$$\lim_{\rho \to \infty} \ell\left(\rho\right) \equiv F\left(\infty\right) = 1,$$

where we have used l'Hospital rule to obtain the first limit:

$$\lim_{\rho \to 0} \frac{\kappa}{\rho} \int_0^{\rho} G \mathrm{d}F\left(G\right) = \lim_{\rho \to 0} \kappa \rho f\left(\rho\right) = 0.$$

[20]The derivation is in the Appendix.

[21]The derivation is in the Appendix.

3.4 The Primary Surplus

The results in sections 3.2 and 3.3 make clear the importance of Maximum Primary Surplus α, which appears in the expressions for ω_M, b_M, and, indirectly, d_M.[22] However, the effective primary surplus necessary to service maximum debt d_M can be much lower. Consider for example a government that issues maximum debt of face value $d_M Y_t$ to obtain proceeds $b_M Y_t$. In period $t+1$ in which the debt is due, the primary surplus $\eta_{t+1} Y_{t+1}$ necessary to service the debt is such that[23]

$$\begin{aligned} d_M Y_t &= \eta_{t+1} Y_{t+1} + b_M Y_{t+1} \\ \Leftrightarrow \eta_{t+1} &= \frac{d_M}{1+g_{t+1}} - b_M, \end{aligned} \tag{21}$$

which can be rewritten as[24]

$$\eta_{t+1} = \left(\frac{1+r_M}{1+g_{t+1}} - 1 \right) b_M. \tag{22}$$

The average primary surplus therefore equals

$$\overline{\eta} \equiv \mathbb{E}\left[\eta\right] = \left((1+r_M)\mathbb{E}\left[\frac{1}{1+g}\right] - 1 \right) b_M, \tag{23}$$

where we have used the stationarity of the GDP growth rate. We have

Proposition 4 *An average primary deficit, $\overline{\eta} < 0$, is sustainable if and only if*

$$\mathbb{E}\left[\frac{1}{1+g}\right] < \frac{1}{1+r_M};$$

.

The sustainability of an average primary deficit reflects either a high growth rate, that is a low $\mathbb{E}\left[1/(1+g)\right]$, or a low cost of debt, that is a high $1/(1+r_M)$. In such case, a country can consistently run primary deficits $\overline{\eta} = ((1+r_M)\mathbb{E}\left[1/(1+g)\right] - 1)\, b_M$.[25]

Summarizing our results so far, we have derived a condition for sustainability, which we have related to the Blanchard Condition; we have also derived the expressions for MSD ω_M, MSB b_M, for the corresponding face value of the debt d_M, the probability of default P_M, the interest rate r_M, and the average primary surplus $\overline{\eta}$. We shall compute these values for the twelve Eurozone countries we consider below.

An additional result is that (4) implies the Blanchard Condition $\overline{g} > r$.[26] The stricter nature of (4) as compared to the Blanchard Condition, both making continual small permanent deficits feasible, confirms the more restrictive nature of a setting such as ours in which default can occur.

[22]Interestingly, MPS α does not appear in the expressions for P_M and r_M. This is because α does not affect g_M: the maximization problem (9) which g_M solves does not depend on α.

[23]Note that $\eta_{t+1} = \alpha$ at $g_{t+1} = g_M$; to see this, use (17) and $b_M = \omega_M - \alpha$.

[24]The derivation is in the Appendix.

[25]We note that the lognormality of the growth factor assumed in Section 3.2 implies $\mathbb{E}\left[1/(1+g)\right] = \exp\left[-\mu + \sigma^2/2\right]$.

[26]To see this, note that $(1+\overline{g})^{-1} < \mathbb{E}\left[(1+g)^{-1}\right]$ by Jensen's inequality and $(1+r_M)^{-1} < (1+r)^{-1}$ as $r_M > r$ from (18).

4 European Stability Mechanism

We now extend our analysis to consider the role of the European Stability Mechanism (ESM).

4.1 Modeling the Effect of the ESM

We assume the ESM lends a Eurozone government an amount that makes up a constant fraction m of that country's GDP. What distinguishes the ESM as lender from other lenders is that it charges the government the subsidized interest rate r_S, as opposed to the market interest rate r_M in (18). Concessionary financing naturally involves the charging of a lower interest rate, $r_S < r_M$. We wish to determine to extent to which ESM lending on concessionary terms increases a country maximum sustainable debt ω_M, at what cost to the ESM.

Default in period t occurs when the sum of zero coupon debt $D_t = d_t Y_{t-1}$ due to market lenders and subsidized interest bearing debt $m(1+r_S)Y_{t-1}$ due to the ESM, both raised in the previous period $t-1$ and expressed as a fraction of that period's GDP Y_{t-1}, is larger than the combined resources available to the government in the current period t. These are the sum of the maximum primary surplus αY_t, new ESM lending mY_t and maximum proceeds from new market debt issuance $b_{M,t}Y_t$. Together, these define maximum debt $\omega_t Y_t = \alpha Y_t + mY_t + b_{M,t}Y_t$. The analogue to (1) is therefore

$$\omega_t = \alpha + m + b_{M,t} \tag{24}$$

and default occurs if and only if

$$1 + g_t < \frac{d_t + m(1+r_S)}{\omega_t}.$$

The analogue to (3), divided by Y_t, in turn is

$$\begin{aligned} b_{M,t} =& \frac{1}{1+r} \max_{d_{t+1}} \left[d_{t+1} \left[1 - F\left(\frac{d_{t+1} + m(1+r_S)}{\omega_{t+1}} \right) \right] + \kappa \frac{d_{t+1}}{d_{t+1} + m(1+r_S)} \int_0^{\frac{d_{t+1}+m(1+r_S)}{\omega_{t+1}}} \omega_{t+1} G \mathrm{d}F(G) \right] \\ =& \frac{1}{1+r} \max_{d_{t+1}} d_{t+1} \left[1 - F\left(\frac{d_{t+1} + m(1+r_S)}{\omega_{t+1}} \right) + \kappa \frac{\omega_{t+1}}{d_{t+1} + m(1+r_S)} \int_0^{\frac{d_{t+1}+m(1+r_S)}{\omega_{t+1}}} G \mathrm{d}F(G) \right] \\ =& \frac{1}{1+r} \max_{d_{t+1}} d_{t+1} \left[1 - \ell\left(\frac{d_{t+1} + m(1+r_S)}{\omega_{t+1}} \right) \right], \end{aligned}$$

where we have assumed (i) that resources available in default ($\kappa\omega_{t+1}$) are shared in proportion to the amounts due private creditors (d_{t+1}) and the ESM ($m(1+r_S)$) and (ii) that default decreases the availability of ESM financing (from m to κm) to the same extent as it does other financing sources (from $\alpha + b_{M,t}$ to $\kappa(\alpha + b_{M,t})$).[27] This makes the analogues to (5) and (6)

$$\omega_t = \alpha + m + \phi(\omega_{t+1}) \tag{25}$$

and

$$\phi(\omega) \equiv \frac{1}{1+r} \max_d d \left[1 - \ell\left(\frac{d + m(1+r_S)}{\omega} \right) \right]. \tag{26}$$

[27] These two assumptions imply that the single difference between ESM and private sector financing is the ESM's subsidized interest rate, $r_S < r_M$. ESM loans are thus *pari passu* with their private sector counterparts.

Similarly to Section 3, we make the change of variable $G \equiv [d + m(1 + r_S)] / \omega$ to rewrite (26) as

$$\phi(\omega) = \frac{1}{1+r} \max_G [\omega G - m(1 + r_S)] [1 - \ell(G)] . \tag{27}$$

Unlike in Section 3, the function ϕ is no longer linear; this is because $\arg\max_G [\omega G - m(1 + r_S)] [1 - \ell(G)]$ is a function of ω. There is therefore no analogue to the explicit solution (11).

When $\phi'(\cdot) = G_M [1 - \ell(G_M)] / (1 + r) < 1$, ϕ is a contraction, (25) has a unique fixed point ω_M with corresponding $G_M = \arg\max_G [\omega_M G - m(1 + r_S)] [1 - \ell(G)]$ and $g_M \equiv G_M - 1$;[28] (25) can be rewritten as

$$\omega_t - \omega_M = \phi(\omega_{t+1}) - \phi(\omega_M). \tag{28}$$

As ϕ is increasing, the sign of $\omega_t - \omega_M$ is the same for all t and is equal to the sign of $\omega - \omega_M$. Moreover, as ϕ is a contraction, its inverse ϕ^{-1} is an expansion and the forward looking dynamics in (28) imply that $|\omega_t - \omega_M| \to \infty$.[29] Thus, similarly to Section 3, ω is sustainable if and only if it is less than the fixed point ω_M.

4.2 Perspectives on ESM support

We first confirm that ESM financing does indeed increase MSD ω_M. We also show that $\partial\omega_M/\partial\kappa > 0$: the greater availability of resources in default increases MSD.

Proposition 5 *$\partial\omega_M/\partial m > 0$ and $\partial\omega_M/\partial\kappa > 0$.*

What is true of MSD ω_M does not necessarily extend to b_M. From (24) evaluated at MSD ω_M, we obtain

$$b_M = \omega_M - \alpha - m.$$

Differentiating with respect to m and using (34), we obtain

$$\frac{\partial b_M}{\partial m} = \frac{\partial \omega_M}{\partial m} - 1 = \frac{1 + r - (1 + r_S) [1 - \ell(1 + g_M)]}{1 + r - (1 + g_M) [1 - \ell(1 + g_M)]} - 1.$$

We have

$$\frac{\partial b_M}{\partial m} > 0 \Leftrightarrow g_M > r_S.$$

The intuition for this result is as follows: g_M is the minimum rate of growth necessary to avoid default when debt owed equals MSD ω_M, r_S is the interest rate charged by the ESM; the former measures the rate of growth in the resources available for debt service, the latter the rate at which these resources are claimed by the ESM (recall that all variables are expressed as fractions of GDP). If the latter is larger than the former, ESM financing decreases resources available to private creditors, who consequently decrease the amount they are willing to lend. In contrast, it is easy to see that $\partial b_M/\partial\kappa = \partial\omega_M/\partial\kappa > 0$.

[28] Note that g_M again denotes the rate of growth below which the government defaults.

[29] To see this, use (25) to write $\omega_{t+1} = \phi^{-1}(\omega_t - \alpha - m)$; likewise write $\omega_M = \phi^{-1}(\omega_M - \alpha - m)$ and subtract to obtain

$$\omega_{t+1} - \omega_M = \phi^{-1}(\omega_t - \alpha - m) - \phi^{-1}(\omega_M - \alpha - m).$$

We now derive the analogue to (20). Maximum borrowing proceeds are now the sum of proceeds from the private sector b_M and proceeds from the ESM m. Using $b_M = \phi(\omega_M)$ and (27) we can write[30]

$$b_M + m = \frac{\left[\alpha + b_M + m + \frac{m}{1+g_M}(r_M - r_S)\right](1+g_M)\left[1 - \ell(1+g_M)\right]}{1+r}; \tag{29}$$

the term $[m/(1+g_M)](r_M - r_s)$ represents the value of the annual subsidy provided by the ESM. The division by $1+g_M$ reflects the fact that the subsidy is a fraction of GDP in the period that precedes the period in which debt is serviced. Iterating, we obtain

$$\begin{aligned} b_M + m &= \left[\alpha + \frac{m(r_M - r_S)}{1+_M}\right]\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r}\right)^n\left[1 - \ell(1+g_M)\right]^n \\ &= \left[\alpha + \frac{m(r_M - r_S)}{1+g_M}\right]\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r}\right)^n \\ &\quad - \left[\alpha + \frac{m(r_M - r_S)}{1+g_M}\right]\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r}\right)^n\left(1 - \left[1 - \ell(1+g_M)\right]^n\right); \end{aligned}$$

the first term in the last expression is the present value of future primary surpluses and subsidies, the second is the present value of expected default costs, which include foregone subsidies: recall our assumption that default decreases the availability of both public and private financing. The PV of the subsidy provided by the ESM, expressed as a fraction of current period GDP, equals[31]

$$PVS = \frac{m(r_M - r_S)}{r_M - g_M}. \tag{30}$$

The PV of the subsidy equals the annual value of the subsidy expressed as a fraction of GDP, growing at g_M and discounted at r_M.

We now compare the increase in borrowing proceeds made possible by ESM financing to the value of the subsidy PVS. The former is $b_M + m - b_M^0$, where

$$b_M^0 = \alpha\sum_{n=1}^{\infty}\left(\frac{1+g_M^0}{1+r}\right)^n\left[1 - \ell\left(1+g_M^0\right)\right]^n,$$

with $1 + g_M^0 \equiv G_M^0 = \arg\max_G G\left[1 - \ell(G)\right]$.[32] We show[33]

$$b_M + m - b_M^0 = \alpha\left[\frac{(1+g_M)\left[1-\ell(1+g_M)\right]}{1+r-(1+g_M)\left[1-\ell(1+g_M)\right]} - \frac{(1+g_M^0)\left[1-\ell(1+g_M^0)\right]}{1+r-(1+g_M^0)\left[1-\ell(1+g_M^0)\right]}\right] + PVS. \tag{31}$$

Intuitively, any net benefit of ESM financing, that is, any increase in borrowing proceeds over and above the subsidy, depends on the extent to which ESM financing makes MPS α more frequently available for debt service.

[30] The derivation is in the Appendix.
[31] The derivation is in the Appendix.
[32] The values b_M^0 and g_M^0 are those derived in Section 3; the superscript is needed to distinguish these values from those derived in the present section.
[33] The derivation is in the Appendix.

Now, recalling that $1 + g_M \equiv G_M = \text{argmax}_G \, (\omega_M G - mR_S) \, [1 - \ell \, (G)]$ with $m > 0$, it is clear that $(1 + g_M^0) \, [1 - \ell \, (1 + g_M^0)] > (1 + g_M) \, [1 - \ell \, (1 + g_M)]$: MPS α is made less frequently available as a result of ESM financing.

The preceding appears to suggest that ESM financing is actually detrimental to total payoff, in the sense that its cost, the present value of the subsidy provided by the ESM, is larger than its benefit, the increase in maximum borrowing proceeds that ESM financing makes possible. Still, as shown in Proposition 5 and as will be confirmed by our simulations below, ESM increases MSD ω_M.

5 Quantitative results

5.1 Data and Parametrization

This section presents summary statistics for the data used in the paper. All data are sourced from the World Economic Outlook Database published by the IMF.

We start with real GDP growth, computed as the percentage change in real GDP in Euros.[34] Table 1 shows that, over the period 1999-2023 under analysis, the simple average of the twelve countries' average annual growth rates was nearly 1.9%. There is, however, substantial cross-country variation, with the average growth rate ranging from about 0.5% in Italy to more than 5.5% in Ireland. There is also substantial within-country growth volatility, with the within-country standard deviation surpassing average growth for all countries but Luxembourg.

Table 1: Real GDP Growth, Primary Surplus

	$\mathbb{E}[\Delta \log(Y)]$	$\sigma[\Delta \log(Y)]$	$\mathbb{E}[PS/Y]$	$\max(PS/Y)$
Austria	1.56	2.42	−0.60	2.04
Belgium	1.73	2.15	0.86	6.24
France	1.38	2.48	−1.86	1.25
Germany	1.19	2.25	0.27	2.67
Italy	0.49	3.01	0.65	4.33
Luxembourg	3.01	2.96	0.74	4.44
Netherlands	1.74	2.34	−0.17	3.67
Finland	1.58	2.86	0.32	7.62
Greece	0.71	4.81	−1.15	4.19
Ireland	5.71	6.03	−1.45	6.46
Portugal	1.13	3.12	−1.44	2.90
Spain	1.80	3.69	−2.01	3.38
Average	1.84	3.18	−0.49	4.09

All quantities are expressed in percents. PS/Y: Primary surplus/Gross Domestic Product

The second key variable of interest for our analysis is the primary balance over GDP. Table 1 shows that, among the twelve countries we consider, seven ran primary deficits on average and

[34]Recall that $1 + g_t = Y_t/Y_{t-1}$ is assumed to be log-normally distributed; this implies that $\log(1 + g_t) = \Delta \log(Y_t)$ is normally distributed with mean $\mu = \mathbb{E}[\Delta \log(Y_t)]$ and standard deviation $\sigma = \sigma[\Delta \log(Y_t)] \equiv \sqrt{\mathbb{E}\,[\Delta \log(Y_t)^2] - \mathbb{E}[\Delta \log(Y_t)]^2}$.

the remaining five ran primary surpluses; the average deficit across all twelve countries was 0.49%. Small primary deficits do in fact appear to be the norm. Of particular interest to our analysis is the maximum primary surplus (MPS), ranging from 1.25% in France to 7.62% in Finland. We set MPS α equal to 7.5%. It is important to bear in mind that 7.5% is the maximum not the average primary surplus: recall from Footnote 23 that $\eta_{t+1} = \alpha$ only at $g_{t+1} = g_M$; we show below that $\overline{\eta}$ ranges from a low of -2.95% to a high of 1.83%; the negative number indicates an average deficit rather than surplus, made possible by very high growth. We also set $\kappa = 75\%$. We motivate this choice by the 63% recovery rate estimated by Cruces and Trebesch (2013);[35] in line with the findings of Graf von Luckner, Meyer, Reinhart, and Trebesh (2024), we assume that the level of resources available in default is higher for advanced economies such as those of the Eurozone than for the mainly emerging market economies considered by Cruces and Trebesch (2013).[36] The comparative statics analysis that follows shows that MSD is relatively little sensitive to the fraction of resources available in default for $\kappa < 90\%$. Finally, we set the nominal risk-free rate—used to compute the real risk-free rates—equal to 2.4%, Germany's average over the period 1999-2023.[37]

5.2 Main Results

Figure 1 plots the real risk-free rates r, the cutoff growth rates g_c, recalled from Section 3.2 to be the risk-free rate grossed up to account for the possibility of default, and the mean growth rates $\overline{g} = \exp\left[\mu + \sigma^2/2\right] - 1$ for the twelve Eurozone countries.[38] We let the real risk-free rates differ across countries; they are obtained by subtracting each country's inflation rate from Germany's nominal rate.

The mean growth rates of all twelve countries are higher than their respective risk-free rates, $\overline{g} > r$. All twelve countries thus satisfy the Blanchard Condition. Yet no country satisfies the $\overline{g} > g_c$ condition; equivalently, no country satisfies the $\gamma > r$ condition. From Proposition 2, this implies the existence of an upper bound on debt, maximum sustainable debt ω_M in (13).

The fifth column of Table 2 shows MSD.[39] Belgium, Luxembourg, and the Netherlands have the highest MSD. That Belgium and the Netherlands should have higher MSD than Ireland, 227% and 207% vs. 154%, despite Ireland's much higher average growth rate, 5.71% vs. 1.73% and 1.74% (see Table 1) underlines the greater importance of volatility in determining MSD: the standard deviations of Belgium and the Netherland's growth rates are markedly lower than Ireland's, 2.15% and 2.34% vs. 6.03%; it is Ireland's very high volatility that explains the very high value of its cutoff growth rate g_c, which somewhat along with Greece's towers above those of the remaining ten countries in Figure 1. That Greece has the lowest MSD, 93%, reflects both

[35] Cruces and Trebesch (2013), p. 86, find that "the average sovereign haircut is 37 percent,"

[36] Graf von Luckner, Meyer, Reinhart, and Trebesh (2024) find a negative relation between haircut and per capital income.

[37] Our parametrization uses yearly as opposed to quarterly data in order to facilitate comparisons with actual debt-to-GDP ratios.

[38] Again recall that $1 + g_t$ is assumed to be log-normally distributed.

[39] We now use the same real risk-free rate for all twelve countries; we do so for consistency. That rate is obtained by subtracting the twelve countries' average inflation rate from Germany's nominal risk-free rate.

Figure 1: Sustainability in the Eurozone

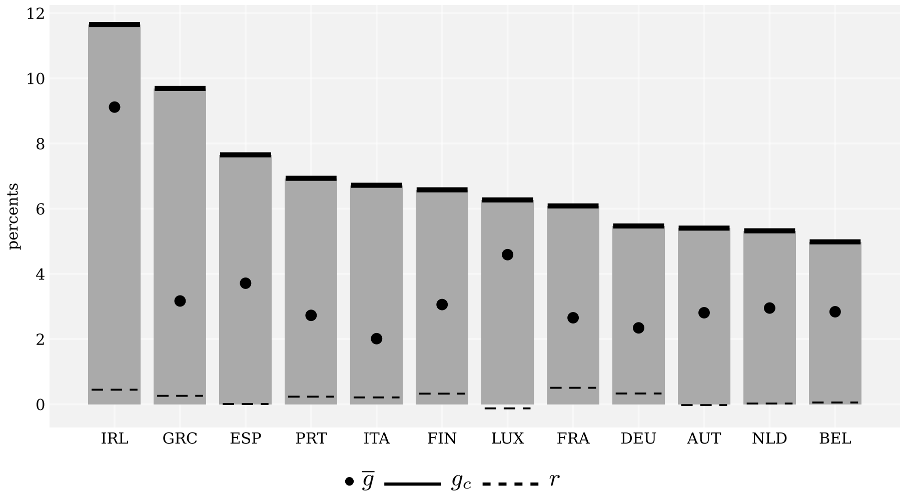


Table 2: Interest Rate r_M , Default Probability P_M, Average Primary Surplus $\overline{\eta}$, MSD ω_M

Country	r_M	P_M	$\overline{\eta}$	ω_M	$(GD/Y)_{2023}$	$\max(GD/Y)$
Austria	1.44	4.36	−0.30	192	75	84
Belgium	1.28	3.79	−1.04	227	106	115
France	1.47	4.50	0.09	178	110	115
Germany	1.34	4.01	0.21	188	66	82
Italy	1.79	5.65	1.50	128	144	155
Luxembourg	1.76	5.53	−2.78	221	28	28
Netherlands	1.40	4.21	−0.75	207	50	68
Finland	1.70	5.32	0.10	163	74	75
Greece	3.00	9.85	1.83	93	168	212
Ireland	3.94	12.88	−2.95	154	43	120
Portugal	1.86	5.89	0.87	138	108	135
Spain	2.22	7.19	0.42	133	107	120

All quantities are expressed in percents. $(GD/Y)_{2023}$ and $\max(GD/Y)$ denote respectively the gross debt-to-GDP ratio in 2023 and its maximal value over the sample period.

its low average growth rate, 0.71%, and its high growth rate volatility, 4.81%. Italy has the second lowest MSD, 128%; it also has the lowest growth rate, 0.49%, and relatively high growth rate volatility, 3.01%.

The sixth and sevenths columns of Table 2 report the gross debt-to-GDP ratio in 2023 and its maximal value over the sample period.[40] A comparison of Greece's 93% MSD with levels of debt reported in columns 6 and 7 explains why Greece's government was forced to default on its debt owed to private creditors once lenders were put on notice that other Eurozone governments, Germany's in particular, would not bail them out. The end 2023 debt level, which at 168% is nearly twice as large as Greece's MSD, overwhelmingly is owed not to private creditors but to public lenders, the European Stability Mechanism in particular. Italy, which as noted in the Introduction had debt equal to 144% of GDP at end 2023, 16 percentage points above its 128% MSD, might well have had to default had it not been for the European Central Bank Governor Mario Draghi's famous July 2012 'whatever it takes' speech, which effectively committed the ECB to act as Eurozone government debt 'purchaser of last resort.'[41] That policy surely must have been beneficial to Portugal (135% peak debt at end 2020, 138% MSD) and to Spain (120% end 2020, 133%) as well. The case of Ireland (120% end 2013, 154%) is less clear-cut. The rapid decrease in Irish debt post 2013 suggests that Ireland never faced a debt sustainability problem; instead, it was hit by a very large temporary negative shock, which could have spiralled into a bad equilibrium had the European Central Bank not played a coordinating role and supplied liquidity when needed. The remaining Eurozone countries appear to be comfortably below MSD, at end 2023 at least.

The second and third columns of Table 2 show the interest rate r_M and the probability of default P_M at MSD ω_M. Both are decreasing in the average growth rate and increasing in growth rate volatility. This is despite the fact that, as we have just seen, MSD itself shares these two properties: a country with lower and more volatile growth that seeks to remain at MSD is constrained to issue less debt yet is exposed to a higher probability of default that requires the country to pay a higher interest rate. For r_M and P_M as for ω_M, volatility appears paramount: Ireland's has the highest probability of default at MSD, 12.88%, despite having the highest average growth rate, 5.71%; this is because, as noted above, it also has the most volatile growth rate, 6.03%.

The fourth column of Table 2 shows the average primary surplus $\overline{\eta}$ at maximum debt issuance; note that a negative number indicates an average deficit rather than surplus. It is noteworthy that the average primary surplus is well below the maximum primary surplus α, recalled to equal 7.5%. In contrast to MSD ω_M, interest rate r_M, and default probability P_M for which growth rate volatility was paramount, it is the average growth rate than now appears to dominate: the

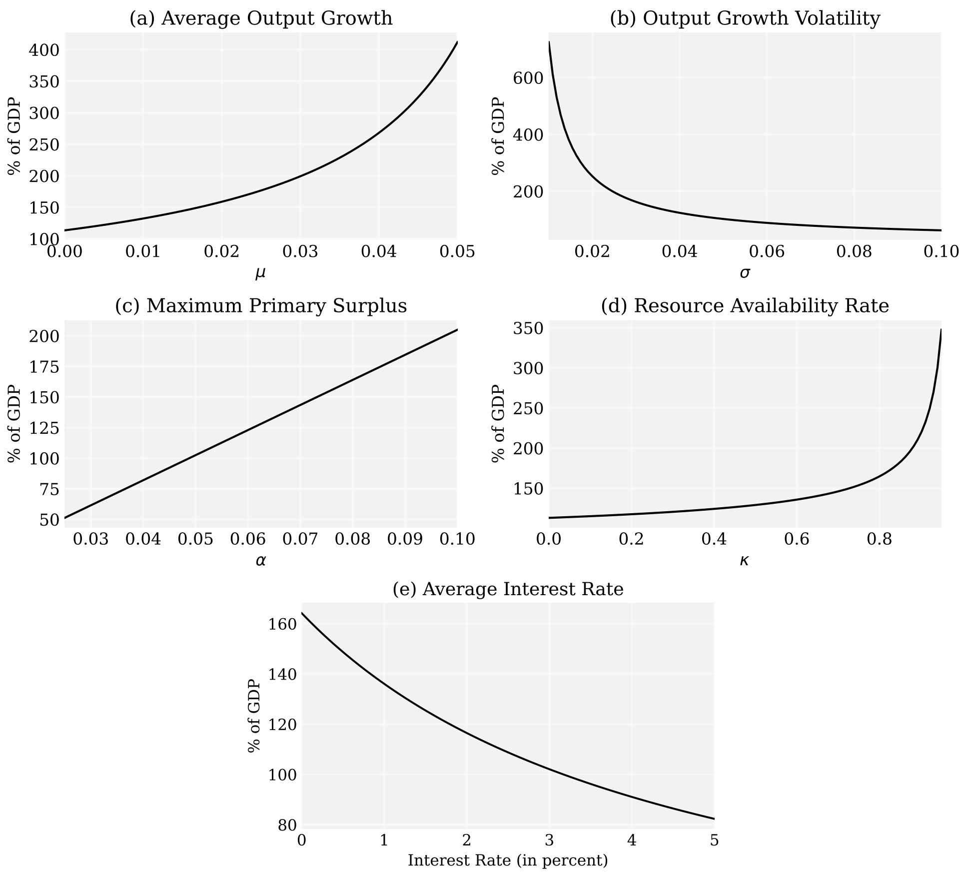
[40]The data is sourced from the IMF World Economic Outlook database; it refers to general government debt. We use gross rather than net debt because the latter is unavailable for some countries in some years. We do not subtract central bank holdings of government debt because we do not want to assume that all central bank financing is granted on concessionary terms; as the discussion that follows makes clear, some surely is.

[41]The share of Italian government debt held by the Bank of Italy and backed by the Eurosystem increased from less than 5% in July 2012 to about 25% at the end of 2023.

countries with the lowest average growth rates are those that must run the highest average primary surpluses (Greece, Italy); conversely, the countries with the highest average growth rates can afford to run average deficits (Ireland, Luxembourg). Albeit less important than the average growth rate, growth rate volatility matters too: Austria, Belgium, and the Netherlands can run average deficits, despite their relatively low average growth rates (1.56%, 1.73%, and 1.74%, respectively), because of their lower growth rate volatilities (2.42%, 2.15%, and 2.34%, respectively). Regardless of whether it runs an average primary surplus or a deficit, a country's debt cannot exceed that country's MSD if that debt is to be sustainable, in the sense of not relying on ever increasing debt ratios.

We now examine the comparative statics of MSD by way of simulations. The simulations in Panels (a) and (b) of Figure 2 confirm that MSD is higher for higher average growth rate and lower for more volatile growth rate; the decrease in MSD is quite dramatic over the range of volatilities observed, specifically 2.15% (Belgium) to 6.03% (Ireland).

Figure 2: Maximal Sustainable Debt: Comparative Statics

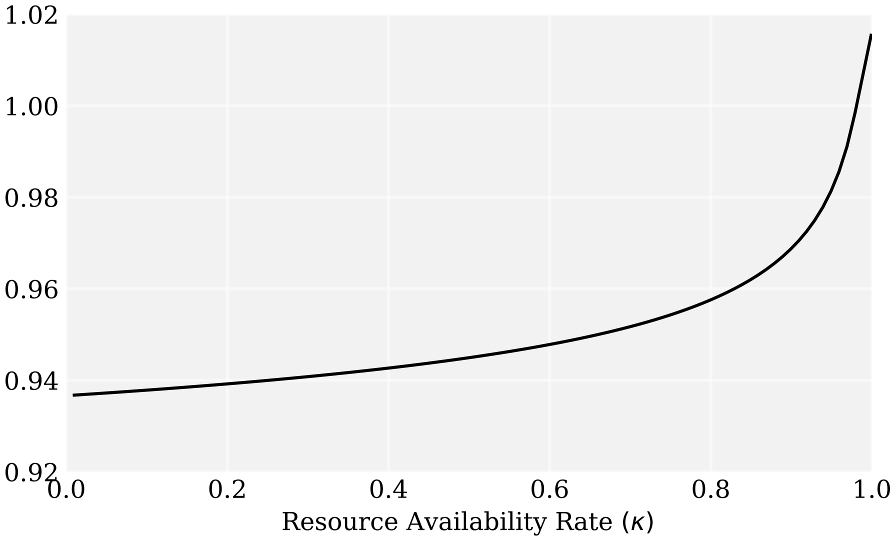


Panel (c) illustrates the importance of the maximum primary surplus α. MSD increases linearly in α, an immediate implication of (13). Our simulation of an average Eurozone country, one whose growth rate is the GDP-weighted average of the growth rates of the twelve countries we consider, shows that an increase of the MPS from 7.5% to 10% increases MSD from 154% to 205%. Symmetrically, a decrease in MPS to 5% decreases MSD to 103%. We recall that maximum primary surplus is not the same as average primary surplus: Eichengreen and Panizza (2016) show that large and persistent primary surpluses are rare.

Panel (d) of Figure 2 shows that an increase in the resource availability rate κ increases MSD at an accelerating rate. Consistently with the discussion following Proposition 2, MSD asymptotically tends to infinity as the resource availability rate tends to one. Note the slow increase in MSD for $\kappa < 60\%$: MSD equals 113% at $\kappa = 0$ and 136% at $\kappa = 60\%$. Finally, Panel (e) of Figure 2 shows how the MSD of the average Eurozone country varies with the risk-free interest rate.

We present two further results. Figure 3 confirms the result in Proposition 3 that γ increases in κ to reach $\bar{g}$ at $\kappa = 1$: $1 + \gamma = 1.019 = \exp\left[\mu + \sigma^2/2\right] = 1 + \bar{g}$ at $\kappa = 1$, with $\mu = 1.84\%$ and $\sigma = 3.18\%$ from Table 1. Figure 4 shows the MPS α and resource availability rate κ combinations for MSD isoquants; α and κ clearly are substitutes: both increase the resources available for debt service, the former directly and the latter indirectly through borrowing proceeds.[42]

Figure 3: $1 + \gamma(\kappa)$ Function



5.3 European Stability Mechanism

Despite the controversies that have accompanied its birth (Tooze (2019)), the ESM appears to be moderately effective at increasing MSD. Panel (a) of Figure 5 shows that having the ESM hold government debt equal to 50% of GDP increases MSD from 154% to 177% of GDP.

This is a far from negligible increase, yet it amounts to less than one-half of the debt subscribed by the ESM. This is because ESM financing displaces private sector financing: Panel (b) of Figure 5 shows that maximum sustainable borrowing from the private sector, b_M in (16), decreases in ESM financing, from 147% to 120% of GDP. This suggests that $g_M < r_S$ in the present case: recall from Section 4.2 that $g_M < r_S$ implies that fewer resources are available to

[42]Note that $\alpha = 0$ when κ has increased to the value that equates γ to r. This result is not to be taken literally, as the equality (13) that it seeks to satisfy holds only if $r > \gamma$.

Figure 4: α vs κ: 2 tales for a MSD

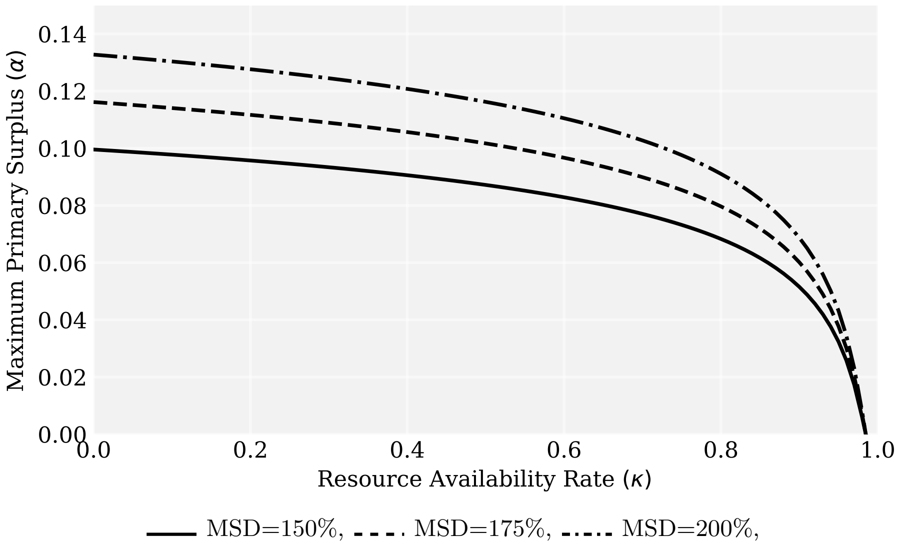
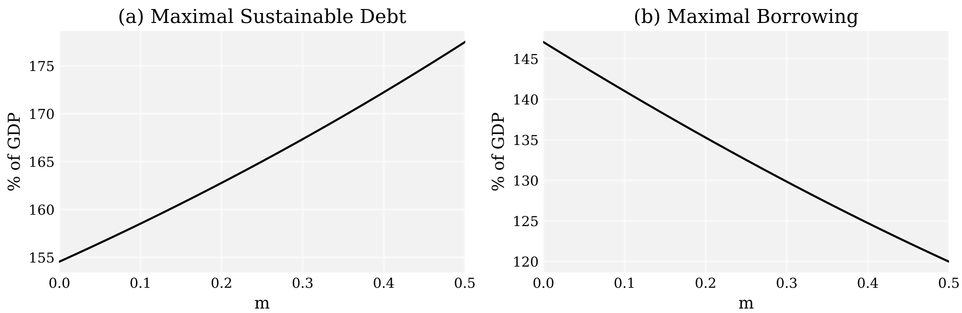


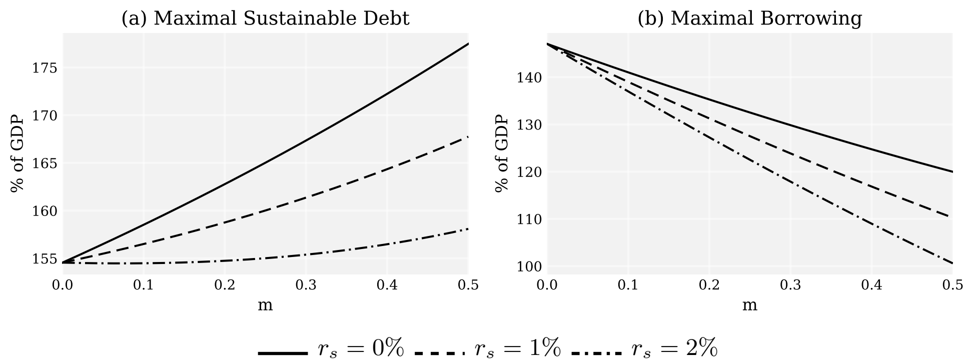
Figure 5: European Stability Mechanism



Note: This graph was generated assuming an interest rate $r_S = 0\%$.

private creditors, who consequently decrease the amount they are willing to lend. When the subsidized interest r_S charged by the ESM increases from zero to 2%, the increase in MSD made possible by ESM financing is only to 158%, a very modest 4 percentage point increase. This is shown in Panel (a) of Figure 6; Panel (b) shows the corresponding change in MSB from the private sector.

Figure 6: European Stability Mechanism: Sensitivity to r_S



6 Conclusion

We study public debt sustainability under the assumption that countries always try to repay but might be pushed into default because they lack the resources necessary to service existing debt. In our set up, default is always involuntary but it nevertheless limits both the availability of the primary surplus and the ability to issue new debt. These limits determine the cost of default. We derive a simple formula for maximum sustainable debt; its main determinants are the mean and the volatility of the GDP growth rate, the real risk-free interest rate, the maximum primary surplus, and the resource availability rate in the aftermath of default. Our model encompasses and clarifies Blanchard's well-known result that, as long as $r < \bar{g}$, countries can permanently run small deficits. We show that this result only holds if the cost of default is zero.

In models rooted in the Eaton and Gersovitz (1981) tradition, a higher cost of default increases willingness to pay and leads to a higher debt limit. In our model, willingness to pay is always there and a higher cost of default leads to lower maximum sustainable debt. The result that a higher cost of default increases maximum sustainable debt has important policy implications. Reforms of the international financial architecture aimed at reducing the costs of default have often been criticized on the grounds that they would be inefficient ex-ante because they reduce willingness to pay and thus limit countries' ability to borrow (Dooley (2000)).[43] We show that when default is involuntary (an assumption in line with the new consensus that countries tend to default too little and too late, IMF (2013)), such reforms might be efficient both ex-post and ex-ante.

However, we also show that such reforms need to be carefully calibrated. We study the

[43] Aguiar (2023) argues that such limitation may in fact be beneficial to a country's population, as governments often borrow more than is in their populations' interest.

case of an institution that provides loans at a subsidized rate (as we focus on the Eurozone, we call this institution the ESM, in a global setting our analysis could be generalized to the IMF). While the presence of such an institution can increase a country's debt limit, we show that there is no free lunch. In fact, in our model, this is an expensive lunch, with the present value of the subsidy provided by the ESM being larger than the increase in maximum sustainable debt brought about by ESM intervention.[44]

We focus on advanced economies and assume a truly independent central bank that is unwilling to monetize the debt. Our model could thus be immediately applied to an emerging market country that only borrows abroad or that has a credible fixed exchange rate.[45] Extending the model to an emerging market country that borrows in both foreign and domestic currency and that does not have a truly independent central bank would require modeling both debt composition (i.e., the choice between domestic and foreign currency debt) and the temptation to inflate away domestic currency debt. Such a model could yield new avenues for studying debt sustainability in emerging market countries and for jointly analyzing external and domestic debt sustainability.

[44] Note that we have assumed that (i) ESM loans are *pari passu* with those of private creditors and (ii) there are no coordination failures in lending. Abandoning the first assumption would probably strengthen our results, abandoning the second would very likely weaken them.

[45] In the first case, we would need to replace real GDP growth and its volatility with the growth and the volatility of GDP measured in foreign currency.

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—APPENDIX—

Proof of Proposition 2:
Results 1 and 2: The results are immediate from equation (11).
Result 3: (6) and (7) become when $\kappa = 1$

$$\phi(\omega) = \frac{1}{1+r} \max_d d \left[1 - F\left(\frac{d}{\omega}\right) + \frac{\omega}{d} \int_0^{d/\omega} G \mathrm{d}F(G) \right],$$

which has FOC

$$1 - F\left(\frac{d}{\omega}\right) + \frac{\omega}{d} \int_0^{d/\omega} G \mathrm{d}F(G) + d \left[-f\left(\frac{d}{\omega}\right) \frac{1}{\omega} - \frac{\omega}{d^2} \int_0^{d/\omega} G \mathrm{d}F(G) + \frac{\omega}{d} \frac{1}{\omega} \frac{d}{\omega} f\left(\frac{d}{\omega}\right) \right] = 0$$

$$\Leftrightarrow 1 - F\left(\frac{d}{\omega}\right) = 0,$$

which in turn has solution $d = \infty$.

Using (1) and (4), we can write

$$b_{M,t} = \frac{1}{1+r} \int_0^{\infty} (\alpha + b_{M,t+1}) G \mathrm{d}F(G) = \frac{1+\overline{g}}{1+r} (\alpha + b_{M,t+1}),$$

where $1+\overline{g} = \mathbb{E}[G] = \int_0^{\infty} G \mathrm{d}F(G)$ is recalled. The preceding equation recalls (10); it has solution similar to (11), with $(1+r)/(1+\overline{g})$ replacing $(1+r)/(1+\gamma)$. The result is immediate: there is no maximum debt when $r < \overline{g}$; this is the Blanchard Condition.
Result 4: From (7) and (9), we can write

$$\begin{aligned}
1+\gamma &= \max_G G[1-\ell(G)] \\
&= \max_G \left\{ G[1-F(G)] + \kappa \int_0^G s \mathrm{d}F(s) \right\} \\
&< \max_G \left\{ G[1-F(G)] + \int_0^G s \mathrm{d}F(s) \right\} \\
&= \int_0^{\infty} s \mathrm{d}F(s) \\
&= \overline{G} \\
&= 1+\overline{g}.
\end{aligned}$$

■

Derivation of (14): We have

$$\begin{aligned}
1+\gamma =& \max_G G[1-\ell(G)] \\
=& \max_x \exp[\mu+\sigma x][1-F(\exp[\mu+\sigma x])] + \kappa \int_{-\infty}^{x} \exp[\mu+\sigma s] \mathrm{d}N(s) \\
=& \exp\left[\mu + \frac{\sigma^2}{2}\right] \left\{ \max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right] [1-N(x)] + \kappa N(x-\sigma) \right\} \\
=& (1+\overline{g}) \left\{ \max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right] [1-N(x)] + \kappa N(x-\sigma) \right\}.
\end{aligned}$$

■

Derivation of (15): Use (14) to rewrite $1+\gamma < 1+r$ as

$$1+\overline{g} < \frac{1+r}{\max_x \exp\left[\sigma x - \frac{\sigma^2}{2}\right] [1-N(x)] + \kappa N(x-\sigma)}; \tag{32}$$

■

Proof of Proposition 3:
Result 1: The results are immediate from equation (13).

Result 2: That ω_M is increasing in γ is immediate from (13); that γ is increasing in κ and μ is immediate from (14) and $1+\overline{g} = \exp\left[\mu + \sigma^2/2\right]$.
Result 3: From (14), keeping $\overline{G} \equiv 1+\overline{g}$ constant and using the Envelope Theorem, we have

$$\left.\frac{\partial\gamma}{\partial\sigma}\right|_{\overline{G}} = \overline{G}\left\{(x-\sigma)\exp\left[\sigma x - \frac{\sigma^2}{2}\right][1-N(x)] - \kappa N'(x-\sigma)\right\}.$$

Substituting into the preceding expression the First-Order Condition of (14), specifically

$$\sigma\exp\left[\sigma x - \frac{\sigma^2}{2}\right][1-N(x)] - \exp\left[\sigma x - \frac{\sigma^2}{2}\right]N'(x) + \kappa N'(x-\sigma) = 0,$$

we obtain

$$\left.\frac{\partial\gamma}{\partial\sigma}\right|_{\overline{G}} = \overline{G}\exp\left[\sigma x - \frac{\sigma^2}{2}\right]\{x[1-N(x)] - N'(x)\} = \overline{G}\exp\left[\sigma x - \frac{\sigma^2}{2}\right]\Psi(x),$$

where $\Psi(x) = x[1-N(x)] - N'(x)$. Noting that $\Psi'(x) = 1 - N(x) > 0$ as $N''(x) = -xN'(x)$ for the standard normal distribution and that $lim_{x\to\infty}\Psi(x) = 0$, we conclude that the function Ψ is everywhere negative and that $\partial\gamma/\partial\sigma|_{\overline{G}} < 0$.

When $\overline{G} = \exp\left[\mu + \sigma^2/2\right]$ is not kept constant, the fact that $\partial\overline{G}/\partial\sigma > 0$ makes the sign of $\partial\gamma/\partial\sigma$ indeterminate. ■

Derivation of (18): Using (13) and (16) to write $b_M = ((1+\gamma)/(1+r))\,\omega_M$ and using (9) and (17), we have

$$\begin{aligned} 1+r_M &= \frac{d_M}{b_M} \\ &= \frac{1+g_M}{\frac{1+\gamma}{1+r}} \\ &= \frac{1+r}{1-\ell(1+g_M)}. \end{aligned} \tag{33}$$

Note from (33) that the condition $r > \gamma$ for the existence of MSD ω_M is equivalent to $r_M > g_M$. ■

Derivation of (19): Use (9), (16), and (18) to write

$$\begin{aligned} b_M &= \frac{\alpha(1+\gamma)}{r-\gamma} \\ &= \frac{\alpha(1+g_M)[1-\ell(1+g_M)]}{1+r-(1+g_M)[1-\ell(1+g_M)]} \\ &= \frac{\alpha(1+g_M)}{r_M-g_M}. \end{aligned}$$

■

Derivation of (20): Using (9), (12), and $b_M \equiv \omega_M - \alpha$, we can write

$$b_M = \frac{\omega_M(1+g_M)[1-\ell(1+g_M)]}{1+r} = \frac{(\alpha+b_M)(1+g_M)[1-\ell(1+g_M)]}{1+r}.$$

Iterating forward, we have[46]

[46]Note that the inequality $1+\gamma = (1+g_M)[1-\ell(1+g_M)] < 1+r$ ensures convergence.

$$b_M = \alpha \sum_{n=1}^{\infty} \left(\frac{1+g_M}{1+r} \right)^n \left[1 - \ell\left(1+g_M\right)\right]^n.$$

Replacing $\left[1 - \ell\left(1+g_M\right)\right]^n$ by $1 - \left(1 - \left[1 - \ell\left(1+g_M\right)\right]^n\right)$ we obtain

$$b_M = \alpha \sum_{n=1}^{\infty} \left(\frac{1+g_M}{1+r} \right)^n - \alpha \sum_{n=1}^{\infty} \left(\frac{1+g_M}{1+r} \right)^n \left(1 - \left[1 - \ell\left(1+g_M\right)\right]^n\right).$$

■

Derivation of (22): We have

$$\begin{aligned}
\eta_{t+1} &= \frac{d_M}{1+g_{t+1}} - b_M \\
&= \frac{\omega_M(1+g_M)}{1+g_{t+1}} - b_M \\
&= \left(\frac{(1+r)(1+g_M)}{(1+\gamma)(1+g_{t+1})} - 1 \right) b_M \\
&= \left(\frac{1+r}{\left[1 - l\left(1+g_M\right)\right]\left(1+g_{t+1}\right)} - 1 \right) b_M \\
&= \left(\frac{1+r_M}{1+g_{t+1}} - 1 \right) b_M,
\end{aligned}$$

where the second equality is obtained from (17), the third from $b_M = \left((1+\gamma)/(1+r)\right)\omega_M$, the fourth from (9), and the last from (18). ■

Proof of Proposition 5: Differentiate

$$\omega_M = \alpha + m + \phi\left(\omega_M\right),$$

with ϕ in (27) with respect to m to obtain

$$\frac{\partial \omega_M}{\partial m} = 1 + \left(\frac{1+g_M}{1+r} \frac{\partial \omega_M}{\partial m} - \frac{1+r_S}{1+r} \right) \left[1 - \ell\left(1+g_M\right)\right]$$

$$\begin{aligned}
\Leftrightarrow \quad \frac{\partial \omega_M}{\partial m} &= \frac{1 - \frac{1+r_S}{1+r}\left[1 - \ell\left(1+g_M\right)\right]}{1 - \frac{1+g_M}{1+r}\left[1 - \ell\left(1+g_M\right)\right]} \\
&= \frac{1+r-\left(1+r_S\right)\left[1 - \ell\left(1+g_M\right)\right]}{1+r-\left(1+g_M\right)\left[1 - \ell\left(1+g_M\right)\right]} \qquad (34) \\
&> 0.
\end{aligned}$$

In order to determine the sign of $\partial\omega_M/\partial m$, we have used the two inequalities $\left(1+g_M\right)\left[1 - \ell\left(1+g_M\right)\right] < 1 + r$ and $r_S < r_M$. The first inequality is the condition for ϕ to be a contraction. The second inequality reflects the provision of ESM financing on concessionary terms.[47]

Now differentiate ω_M with respect to κ to obtain

$$\frac{\partial \omega_M}{\partial \kappa} = \frac{1+g_M}{1+r} \frac{\partial \omega_M}{\partial \kappa} \left[1 - \ell\left(1+g_M\right)\right] - \left(\frac{\omega_M(1+g_M) - m(1+r_S)}{1+r} \right) \frac{\partial \ell\left(1+g_M\right)}{\partial \kappa}$$

$$\Leftrightarrow \left(1 - \frac{1+g_M}{1+r}\left[1 - \ell\left(1+g_M\right)\right]\right) \frac{\partial \omega_M}{\partial \kappa} = -\left(\frac{\omega_M(1+g_M) - m(1+r_S)}{1+r} \right) \frac{\partial \ell\left(1+g_M\right)}{\partial \kappa}$$

$$\Leftrightarrow \frac{\partial \omega_M}{\partial \kappa} = -\frac{\omega_M(1+g_M) - m(1+r_S)}{1+r-\left(1+g_M\right)\left[1 - \ell\left(1+g_M\right)\right]} \frac{\partial \ell\left(1+g_M\right)}{\partial \kappa} > 0,$$

[47] Note that there would be no increase in MSD if there were no subsidy: $\partial\omega_M/\partial m = 0$ for $r_S = r_M$.

where we have used $\partial\ell/\partial\kappa < 0$, $\omega_M(1+g_M) > m(1+r_S)$, and $(1+g_M)[1-\ell(1+g_M)] < 1+r$: the first inequality is immediate from (7), the second from the First-Order Condition for G_M at ω_M and $\ell'(\cdot) > 0$ from the observation in Footnote 19, and the third from the condition for ϕ to be a contraction. ■

Derivation of (29): We have

$$\begin{aligned} b_M + m &= \frac{[\omega_M(1+g_M) - m(1+r_S)][1-\ell(1+g_M)]}{1+r} + m \\ &= \frac{\left[(\alpha + b_M + m)(1+g_M) - m(1+r_S) + \frac{m(1+r)}{1-\ell(1+g_M)}\right][1-\ell(1+g_M)]}{1+r} \\ &= \frac{\left[\alpha + b_M + m + \frac{m}{1+g_M}[r_M - r_S]\right](1+g_M)[1-\ell(1+g_M)]}{1+r}. \end{aligned}$$

■

Derivation of (30): We have

$$\begin{aligned} PVS &= \frac{m(r_M - r_S)}{1+g_M}\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r}\right)^n [1-\ell(1+g_M)]^n \\ &= \frac{m(r_M - r_S)}{1+g_M}\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r_M}\right)^n \\ &= \frac{m(r_M - r_S)}{1+r_M}\frac{1}{1-\frac{1+g_M}{1+r_M}} \\ &= \frac{m(r_M - r_S)}{r_M - g_M}. \end{aligned}$$

■

Derivation of (31): We have

$$\begin{aligned} b_M + m - b_M^0 &= \left[\alpha + \frac{m(r_M - r_S)}{1+g_M}\right]\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r}\right)^n [1-\ell(1+g_M)]^n - \alpha\sum_{n=1}^{\infty}\left(\frac{1+g_M^0}{1+t_f}\right)^n \left[1-\ell\left(1+g_M^0\right)\right]^n \\ &= \alpha\left[\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r_M}\right)^n - \sum_{n=1}^{\infty}\left(\frac{1+g_M^0}{1+r_M^0}\right)^n\right] + \underbrace{\frac{m(r_M - r_S)}{1+g_M}\sum_{n=1}^{\infty}\left(\frac{1+g_M}{1+r_M}\right)^n}_{PVS} \\ &= \alpha\left[\frac{1+g_M}{r_M - g_M} - \frac{1+g_M^0}{r_M^0 - g_M^0}\right] + PVS \\ &= \alpha\left[\frac{(1+g_M)[1-\ell(1+g_M)]}{1+r-(1+g_M)[1-\ell(1+g_M)]} - \frac{(1+g_M^0)\left[1-\ell\left(1+g_M^0\right)\right]}{1+r-(1+g_M^0)[1-\ell(1+g_M^0)]}\right] + PVS. \end{aligned}$$

■