“A simple calculus of sums of powers”

Olivier Perrin
A simple calculus of sums of powers

Olivier Perrin

June 17, 2024

Abstract

We introduce a new and simple method for computing the sums

\[ S_p(n) = \sum_{i=1}^{n} i^p, \]

without induction nor a priori knowledge of the Bernoulli numbers.

**keywords:** Bernoulli polynomials, Bernoulli numbers, sums of powers

1 Introduction and notations

Let \( p \) and \( n \) be two natural integers, and we note

\[ S_p(n) = \sum_{i=1}^{n} i^p, \]

that we call the Bernoulli sum of order \( p \), with the convention \( S_p(0) = 0 \).

We recall

\[
S_0(n) = n, \\
S_1(n) = \frac{n^2}{2} + \frac{n}{2}, \\
S_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}, \\
S_3(n) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.
\]

There exists a general formula, proved for instance in [1]

\[ S_p(n) = \frac{1}{p+1} \sum_{k=0}^{p} C_{p+1}^k B_k n^{p+1-k}, \]

\[ \text{[1]} \text{Toulouse School of Economics, France, E-mail: olivier.perrin@tse-fr.eu} \]

\[ \text{[2]} \text{Université Toulouse Capitole} \]
where the $C_{p+1}^k$, $0 \leq k \leq p$, are the binomial coefficients and the $B_k$’s, $k \in \mathbb{N}$, are the (modified) Bernoulli numbers; $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = \frac{1}{30}, \ldots$. These modified Bernoulli numbers can be obtained by induction as follows

$$B_0 = 1; \quad \text{and for } k \geq 1: \quad B_k = \frac{1}{k + 1} \sum_{j=0}^{k-1} C_{k+1}^j B_j (-1)^{k+1-j}.$$ 

We recall that the classical Bernoulli numbers are obtained with the following induction

$$B_0 = 1; \quad \text{and for } k \geq 1: \quad B_k = -\frac{1}{k + 1} \sum_{j=0}^{k-1} C_{k+1}^j B_j.$$ 

The only difference is that in our (modified) case $B_1 = \frac{1}{2}$ instead of the classical $B_1 = -1/2$. If we set $B_k = B^k (B \text{ power } k)$, $k \in \mathbb{N}$, we can use the Newton’s binomial formula in a formal way for writing equality (2) like this

$$S_p(n) = \frac{1}{p + 1} \left( (B + n)^{p+1} - B^{p+1} \right).$$ 

(3)

2 Main result: a simple iterative calculus of $S_p(n)$

We extend the formula (2) to any real number $x$, so that we get the polynomials $S_p : x \mapsto S_p(x)$ with

$$S_p(x) = \frac{1}{p + 1} \sum_{k=0}^{p} C_{p+1}^k B_k x^{p+1-k},$$

keeping in mind that the meaning of $S_p(x) = \sum_{i=1}^{x} i^p$ is valid only for $x \in \mathbb{N}$.

For instance, the polynomials $S_p$, for $p = 0, 1, 2, 3$, are defined by

$$S_0(x) = x, \quad S_1(x) = \frac{x^2}{2} + \frac{x}{2}, \quad S_2(x) = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}, \quad S_3(x) = \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4},$$

Then, let us consider a modified version of the Bernoulli polynomials as follows

2
**Definition 2.1** The (modified) Bernoulli polynomials are the unique sequence of polynomials $B_p$, $p \in \mathbb{N}$, such that $B_0 = 1$ and for all $p$

\[ B'_{p+1} = (p + 1)B_p, \]
\[ \int_0^1 B_p(x)dx = 1. \]

The first four Bernoulli polynomials are defined for any real $x$ by

\[ B_0(x) = 1, \]
\[ B_1(x) = x + \frac{1}{2}, \]
\[ B_2(x) = x^2 + x + \frac{1}{6}, \]
\[ B_3(x) = x^3 + \frac{3x^2}{2} + \frac{x}{2}. \]

More generally, for all $p \in \mathbb{N}$

\[ B_p(x) = \sum_{k=0}^{p} C_k^p B_k x^{p-k}. \]

(4)

Compared to the classical Bernoulli polynomials, the only difference here is that the coefficient of the second highest degree term, $x^{p-1}$, is positive instead of being negative. The corresponding modified Bernoulli numbers satisfy

\[ B_p = B_p(0). \]

(5)

Equalities (4) and (5) are proved like in [1] for the classical Bernoulli polynomials and numbers. Finally, it is easy to verify that for all real $x$

\[ S_p(x) = \frac{B_{p+1}(x) - B_{p+1}(0)}{p+1}. \]

(6)

Consequently, for all natural integer $n \geq 1$

\[ S_p(n) = \frac{B_{p+1}(n) - B_{p+1}(0)}{p+1}. \]

(7)

Note that this last equality is formally equivalent to (3), more precisely

\[ B_{p+1}(n) = (B + n)^{p+1}. \]

Thus, from Definition 2.1, for all $p \geq 1$

\[ S'_{p}(n) = \frac{B'_{p+1}(n)}{p+1} = B_p(n) = pS_{p-1}(n) + B_p(0), \]

from which we deduce our main result
Proposition 2.1 For all $p \geq 1$, and all $n \in \mathbb{N} - \{0\}$

$$S_p(n) = \int_0^n pS_{p-1}(x)dx + B_p(0)n,$$

For instance, $\int_0^n 2S_1(x)dx = \int_0^n (x^2 + x)dx = \frac{n^3}{3} + \frac{n^2}{2}$. We thus find again that

$$S_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Remark that the factor $B_p$ can be obtained by identification because $S_p(1) = 1$. For the above example

$$B_2 = \frac{1}{6} = S_2(1) - \frac{1^3}{3} - \frac{1^2}{2}.$$

In conclusion, we have introduced an easy method for computing all the Bernoulli sums $S_p(n)$ without induction, nor a priori knowledge of the Bernoulli numbers. We just need to know how to compute iteratively the integrals, from 0 to $n$, of functions $x \mapsto x^i$, $i > 0$.

References