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"A simple calculus of sums of powers"

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A simple calculus of sums of powers

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Abstract

We introduce a new and simple method for computing the sums $S_p(n) = \sum_{i=1}^{n} i^p$, without induction nor a priori knowledge of the Bernoulli numbers.

keywords: Bernoulli polynomials, Bernoulli numbers, sums of powers

1 Introduction and notations

Let p and n be two natural integers, and we note

$$S_p(n) = \sum_{i=1}^{n} i^p, \tag{1}$$

that we call the Bernoulli sum of order p, with the convention $S_p(0) = 0$. We recall

$$\begin{aligned} S_0(n) &= n, \\ S_1(n) &= \frac{n^2}{2} + \frac{n}{2}, \\ S_2(n) &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}, \\ S_3(n) &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}. \end{aligned}$$

There exists a general formula, proved for instance in [1]

$$S_p(n) = \frac{1}{p+1} \sum_{k=0}^{p} C_{p+1}^{k} B_k n^{p+1-k}, \tag{2}$$

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where the C_{p+1}^k's, $0 \leq k \leq p$, are the binomial coefficients and the B_k's, $k \in \mathbb{N}$, are the (modified) Bernoulli numbers; $B_0 = 1, B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = \frac{-1}{30}, \ldots$ These modified Bernoulli numbers can be obtained by induction as follows

$$B_0 = 1; \quad \text{and for } k \geq 1: \quad B_k = \frac{1}{k+1} \sum_{j=0}^{k-1} C_{k+1}^j B_j (-1)^{k+1-j}.$$

We recall that the classical Bernoulli numbers are obtained with the following induction

$$B_0 = 1; \quad \text{and for } k \geq 1: \quad B_k = \frac{-1}{k+1} \sum_{j=0}^{k-1} C_{k+1}^j B_j.$$

The only difference is that in our (modified) case $B_1 = 1/2$ instead of the classical $B_1 = -1/2$. If we set $B_k = B^k$ (B power k), $k \in \mathbb{N}$, we can use the Newton's binomial formula in a formal way for writing equality (2) like this

$$S_p(n) = \frac{1}{p+1} \left((B+n)^{p+1} - B^{p+1} \right). \tag{3}$$

2 Main result: a simple iterative calculus of $S_p(n)$

We extend the formula (2) to any real number x, so that we get the polynomials $S_p : x \mapsto S_p(x)$ with

$$S_p(x) = \frac{1}{p+1} \sum_{k=0}^{p} C_{p+1}^k B_k x^{p+1-k},$$

keeping in mind that the meaning of $S_p(x) = \sum_{i=1}^{x} i^p$ is valid only for $x \in \mathbb{N}$.
For instance, the polynomials S_p, for $p = 0, 1, 2, 3$, are defined by

$$\begin{aligned} S_0(x) &= x, \\ S_1(x) &= \frac{x^2}{2} + \frac{x}{2}, \\ S_2(x) &= \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}, \\ S_3(x) &= \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4}, \end{aligned}$$

Then, let us consider a modified version of the Bernoulli polynomials as follows

Definition 2.1 *The (modified) Bernoulli polynomials are the unique sequence of polynomials B_p, $p \in \mathbb{N}$, such that $B_0 = 1$ and for all p*

$$\begin{aligned} B'_{p+1} &= (p+1)B_p, \\ \int_0^1 B_p(x)dx &= 1. \end{aligned}$$

The first four Bernoulli polynomials are defined for any real x by

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x + \frac{1}{2}, \\ B_2(x) &= x^2 + x + \frac{1}{6}, \\ B_3(x) &= x^3 + \frac{3x^2}{2} + \frac{x}{2}. \end{aligned}$$

More generally, for all $p \in \mathbb{N}$

$$B_p(x) = \sum_{k=0}^{p} C_p^k B_k x^{p-k}. \tag{4}$$

Compared to the classical Bernoulli polynomials, the only difference here is that the coefficient of the second highest degree term, x^{p-1}, is positive instead of being negative. The corresponding modified Bernoulli numbers satisfy

$$B_p = B_p(0). \tag{5}$$

Equalities (4) and (5) are proved like in [1] for the classical Bernoulli polynomials and numbers. Finally, it is easy to verify that for all real x

$$S_p(x) = \frac{B_{p+1}(x) - B_{p+1}(0)}{p+1}. \tag{6}$$

Consequetly, for all natural integer $n \geq 1$

$$S_p(n) = \frac{B_{p+1}(n) - B_{p+1}(0)}{p+1}. \tag{7}$$

Note that this last equality is formally equivalent to (3), more precisely

$$B_{p+1}(n) = (B+n)^{p+1}.$$

Thus, from Definition 2.1, for all $p \geq 1$

$$S'_p(n) = \frac{B'_{p+1}(n)}{p+1} = B_p(n) = pS_{p-1}(n) + B_p(0),$$

from which we deduce our main result

Proposition 2.1 *For all $p \geq 1$, and all $n \in \mathbb{N} - \{0\}$*

$$S_p(n) = \int_0^n pS_{p-1}(x)dx + B_p(0)n,$$

For instance, $\int_0^n 2S_1(x)dx = \int_0^n (x^2 + x)dx = \frac{n^3}{3} + \frac{n^2}{2}$. We thus find again that

$$S_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Remark that the factor B_p can be obtained by idenfication because $S_p(1) = 1$. For the above example

$$B_2 = \frac{1}{6} = S_2(1) - \frac{1^3}{3} - \frac{1^2}{2}.$$

In conclusion, we have introduced an easy method for computing all the Bernoulli sums $S_p(n)$ without induction, nor a priori knoweledge of the Bernoulli numbers. We just need to know how to compute iteratively the integrals, from 0 to n, of functions $x \mapsto x^i$, $i > 0$.

References

[1] Gay, L. & Lemonnier, F. (2013). *Fascinants nombres de Bernoulli.* Lecture dirigée par Bernard Le Stum, Université de Rennes 1 - ENS Ker Lann. https://perso.eleves.ens-rennes.fr/people/florian.lemonnier/documents/rapportBer.pdf