WORKING PAPERS

N° 1536

MAY 2024

"Multidimensional Screening After 37 years"

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Revised Version

May 16, 2024

ABSTRACT

This expository article surveys the literature that has followed my paper "A Necessary and Sufficient Condition for Rationalizability in a Quasi-linear Context" that was published in the Journal of Mathematical Economics in 1987.

Keywords: multidimensional screening, rationalizability, bunching, mechanism design.

1 Introduction

When my paper "A Necessary and Sufficient Condition for Rationalizability in a Quasi-linear Context" was published in the Journal of Mathematical Economics in 1987, I did not anticipate that it would be cited almost 600 times in the next 37 years. It was a short (10 pages) methodological note characterizing the implementability of allocation mechanisms in a context of privately informed agents and quasi-linear utilities. Mechanism design theory was initiated by Mirrlees (1971) and further developed by Myerson (1981), and Laffont and Maskin (1981), among many others. This theory was very successful because it could be applied to a large spectrum of economic problems: Regulation of firms, non-linear pricing, auction design, optimal taxation,... But it was confined to situations with one good and one dimension of un-observable heterogeneity, under the assumption, called the single crossing property, that agents' marginal utilities for the good could be ranked independently of the level of consumption: higher types are always ready to pay more

[*]I gratefully acknowledge the encouragements of the Editor, Andres Carvajal, the useful comments of an anonymous referee, and financial support of the European Research Council (ERC grant n° 101055239, DIPAMUTA). Views and opinions expressed are however only mine and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

than lower types for the next marginal unit. This single crossing property was popularized by Spence (1974) in signalling contexts and Mirrlees (1971) in mechanism design contexts.

In the uni-dimensional case, when the Spence Mirrlees condition applies, implementability is equivalent to monotonicity of the allocation of the good: higher types never get a lower quantity or quality than lower types. Moreover, local downwards incentive compatibility constraints are always binding and informational rents are easily computed. Second best optimality is associated with the maximization of a "virtual surplus", equal to the difference between social surplus and informational rents. This difference is straightforward to compute, except when there is "bunching"[1]. Finally, when the distribution of types is bounded, there is no distortion at the top (because the allocation of the highest type never attracts lower types) and the lowest types get no informational rent: the participation constraint binds at the bottom.

However, the single crossing property is not satisfied in the majority of applications, in particular when heterogeneity is multidimensional and types cannot be ranked independently of the allocation. My 1987 note was providing a simple condition (called cyclical monotonicity[2]) for characterizing implementability for general quasi linear preferences. To my surprise, this condition proved very useful in many different contexts[3]. Moreover, it revealed a particular tree structure in the type space that illustrates the incredible richness of multidimensional screening.

The remainder of this note is organized as follows. Section 2 presents the main result of my 1987 paper: an allocation is implementable in dominant strategies if and only if it is cyclically monotone. It also discusses the subsequent implementability literature that has simplified this condition by adding more structure. Section 3 surveys the main applications of this condition that have been developed in the last 37 years. Section 4 illustrates the implicit tree structure that emerges in multidimensional screening problems, by using one of the most popular applications: multiple goods monopoly pricing. Section 5 concludes.

2 The implementability condition (Rochet 1987)

Consider a screening model where a principal has to assign different goods or bundles of goods $y_i \in Y$ (where Y is a subset of $\mathbb{R}^m$), to a population of agents of different types

[1]Bunching occurs when different types get the same allocation. In the uni-dimensional case, bunching can be excluded by reasonable assumptions.

[2]It is also called cyclic or cycle monotonicity.

[3]I will not review some of the applications which are too far away from the core mechanism design literature. An example is the rational inattention literature initiated by Christopher Sims. It turns out that the main result in Rochet (1987) can be useful in this literature: see for example Caplin and Dean (2015) and Caplin et al. (2017). This result is also used a lot in computer science.

$i = 1, ...N$. Agents have quasi-linear preferences. The utility of agents of type i is

$$U_i = b_i(y_i) - p_i, \tag{1}$$

where $b_i(.)$ is a utility function from Y to $\mathbb{R}$ and p_i is the price paid by agent i for the bundle y_i. The number $b_i(y)$ can be seen as the maximum amount of money that agent i is ready to pay for purchasing bundle y, in which case the participation constraint of agent i writes $U_i \geq 0$. An allocation $y = (y_1, ...y_N)$ is implementable if and only if there exists a vector of prices $(p_1, ..., p_N)$ such that

$$\forall(i,j), b_i(y_i) - p_i \geq b_i(y_j) - p_j. \tag{2}$$

This condition can also be written as

$$\forall(i,j), U_i - U_j \geq b_i(y_j) - b_j(y_j). \tag{3}$$

If the different types are viewed as different agents, these inequalities can be interpreted as no-envy conditions. Our characterization is then a characterization of envy free allocations. Note that the only assumption about preferences is quasi-linearity: the model is completely non-parametric.

2.1 Cyclical Monotonicity

The main result in Rochet (1987) is

Proposition 1 *An allocation $i \to y_i$ for $i = 1, ...N$ is implementable if and only if, for any finite cycle $i_0, i_1, ..., i_{c+1} = i_0$ among types*

$$\sum_{j=0}^{j=c}(b_{i_{j+1}}(y_{i_j}) - b_{i_j}(y_{i_j})) \leq 0. \tag{4}$$

To provide an intuition for condition (4), one can define the incremental utility between type i_{j+1} and type i_j as the difference between the utility of type i_{j+1} and the utility of type i_j when consuming the bundle assigned to type i_j. Condition (4) means that, for any cycle in the set of types, the sum of incremental utilities along the cycle is nonpositive. The necessity of this condition is easy to show by adding up incentive compatibility constraints along a cycle. The sufficiency part is much less obvious. In fact, this result is the non parametric version of a famous convex analysis result due to Rockafellar:

Proposition 2 *A mapping $\theta \to y(\theta)$ from a convex subset Θ of $\mathbb{R}^m$ into $\mathbb{R}^m$ is a selection of the subdifferential of a convex function U (meaning that $\forall\theta \in \Theta, y(\theta) \in \partial U(\theta)$) if and* *only if for all finite cycle $\theta_0, \theta_1, ..., \theta_{c+1} = \theta_0$,*

$$\sum_{j=0}^{j=c} (\theta_{j+1} - \theta_j).y(\theta_j) \leq 0.$$

This property is called cyclical (or cycle) monotonicity. When types are parameterized by a vector θ of $\mathbb{R}^m$ instead of an index i in $(1, ..., N)$, and utilities are bilinear in θ and y, meaning that $b(\theta, y) = \theta.y$, implementability of an allocation $\theta \rightarrow y(\theta)$ is equivalent to finding prices $p(\theta)$ such that

$$\forall \theta, U(\theta) = \sup_{\theta'} [\theta.y(\theta') - p(\theta')],$$

which is in turn equivalent to two properties: U is convex and

$$\forall \theta, y(\theta) \in \partial U(\theta).$$

Applying Rockafellar's condition thus shows that implementability in the bilinear case is equivalent to cyclical monotonicity. My result was just an extension of Rockafellar's result to non parametric preferences. Interestingly, Propositions 1 and 2 are also valid when the number of types is infinite. Under regularity conditions, cyclical monotonicity is equivalent to a set of differential equations, analogous to Slutsky's equations, that express the symmetry of the derivative of the mapping y (which is the Hessian of U), together with positivity conditions, expressing the convexity of U.

2.2 Two Simple Cases

Proposition 1 has two simple corollaries:

Proposition 3 *In the unidimensional case, when the Spence Mirrlees condition is satisfied, cyclical monotonicity is equivalent to classical monotonicity: higher types get more.*

Proposition 4 *In the multidimensional case, when utilities are linear w.r.t. types:*

$$u(\theta, y) = \theta.v(y),$$

an allocation $y(\theta)$ is implementable if and only if there exists a convex function of types, denoted $U(\theta)$, such that, for all θ, $v(y(\theta))$ belongs to $\partial U(\theta)$, the subdifferential of U .

The equivalence between this property and cyclical monotonicity had already been established by Rockafellar. Thus the main result of my note was based on the remark that Rockafellar's result is valid for arbitrary utility functions.[4] As we show below, Proposition

[4]I was lucky to meet with Tyrell Rockafellar, who was visiting Paris at the time were I was working on the paper. He kindly encouraged me to generalize his equivalence result to the non parametric case.

4 is particularly useful when the distribution of types is continuous.

2.3 The Implementability Literature

This literature aims at simplifying the characterization in Rochet (1987) by adding more structure. For example, Saks and Yu (2005) show that any monotone function is cyclically monotone on convex domains if the set of outcomes is finite. Bikhchandani et al. (2006) obtain a spectacular result by restricting attention to deterministic mechanisms: they show that, for such mechanisms, implementability is equivalent to weak monotonicity, which is just cyclical monotonicity for cycles of order 2. It coincides with the natural extension of monotonicity to multidimensional set-ups. Of course optimal multidimensional incentive compatible mechanisms often involve stochastic allocations, as we show below. Berger at al. (2017) extend the results of Saks and Yu (2005) by showing that allocation rules are implementable in a multidimensional context if and only if they are implementable on any two-dimensional convex subset of the type set. These results are achieved by using directed graphs methods that I comment in Section 4. Mishra et al. (2014) show the equivalence of monotone and cyclically monotone allocations for single peaked preferences. Using algebraic topology methods, Kushnir and Lokutsievskiy (2021) show that, under a gross substitutes condition on preferences, any monotone allocation is also cyclically monotone. Finally, Rahman (2023) adopts an alternative approach to both the proof and interpretation of Proposition 1, based on linear duality. This duality reveals a formal relationship between incentives and detection, much like that between prices and quantities. Rahman generalizes Proposition 1 and obtains a subdifferential characterization of implementing payments, revenue equivalence as differentiability of a value function, as well as budget-balanced implementation.

We now explore the mechanism design literature, that has examined several economic questions to which these results could be applied.

3 The Mechanism Design Literature

3.1 Multiproduct Price Discrimination

A monopolist wants to sell m indivisible goods to a population of heterogeneous buyers characterized by their unobservable "type" $\theta \in \mathbb{R}^m$ interpreted as the vector of their valuations for the m different goods and distributed on a convex subset Θ of $\mathbb{R}^m$ according to a probability distribution[5] F on $\mathbb{R}^m$. This distribution is typically assumed to be absolutely continuous with respect to the Lebesgue measure, with a density f. An alternative interpretation of this set-up is a multi-item auction with a single bidder whose valuations

[5]The total size of the population is normalized to 1 without loss of generality.

are unknown to the auctioneer. It can be generalized to several bidders.

Allowing stochastic mechanisms, where y_i is interpreted as the probability that the agent obtains good i, the feasible consumption set is the hypercube $[0,1]^m$. Production costs are normalized to zero. When $m = 1$, Riley and Zeckhauser (1983) have shown that the optimal mechanism is always to post a price p^* so that consumers purchase the good if and only if their valuation is above p^*. The optimal mechanism is thus deterministic in dimension 1, which Riley and Zeckhauser interpret as the seller making a take-it or leave it offer to the buyers. Haggling is never optimal for a single good. With multiple goods, things are more complex. First, the seller typically benefits from a "bundling" strategy, namely offering bundles of goods at prices that are lower than the sum of the prices of the different goods composing the bundle. These bundling strategies were explored by Adams and Yellen (1976) and Mc Afee et al. (1989). But stochastic mechanisms can sometimes do even better.

To see this, we have to use Proposition 3: a mechanism is implementable if and only if the associated indirect utility function $U(\theta) = \theta.y(\theta) - p(\theta)$ is convex and the allocation $y(\theta)$ belongs to the subdifferential ∂U. Thus we can characterize the optimal mechanism by the indirect utility function U that maximizes the monopolist's revenue (remember that costs are normalized to zero):

$$B(U) = \int_\Theta p(\theta)dF(\theta) = \int_\Theta [\theta.\nabla U(\theta) - U(\theta)]dF(\theta),$$

over the set of functions U such that $U \geq 0$ (participation constraint) and U is convex (implementability). When the distribution of types is absolutely continuous, the set of types where U is not differentiable has zero measure and we can write ∇U instead of ∂U.

This problem has been studied by many authors. A striking result is that the monopolist can sometimes benefit from using a stochastic mechanism (Thanassoulis 2004, Manelli and Vincent 2006, 2007, 2012). By contrast, Haghpanah and Hartline (2021) find conditions under which pure bundling (i.e. only selling the full bundle of goods) is optimal. Similarly, Bikhchandani and Mishra (2022) obtain conditions under which deterministic mechanisms are optimal for selling two identical, indivisible objects to a single buyer. Pavlov (2011) obtains a complete characterization of the solution when $m = 2$ and the distribution of types is uniform[6]on $[c, c+1]^2$. The optimal mechanism is stochastic when $0 < c < 0.77$. However the menu of stochastic bundles is finite: the allocation $y(\theta)$ belongs to the set $[(0,0),(a,1),(1,a),(1,1)]$, where $0 < a < 1$. Using duality methods, Daskalakis et al (2017) show that when the distribution of types has a beta distribution, the optimal mechanism may require an infinity of options. When the distribution of types is uniform, but $m > 2$, Giannakopoulos and Koutsoupias (2018) inquire whether the optimal mech-

[6]Pavlov (2011) and Thanassoulis (2004) also study the case where the goods are substitutable. In this case the feasible set is $[y \geq 0, \sum_i y_i \leq 1.]$

anism involves deterministic bundling (what they call Straight Jacket Auctions). They use LP duality methods to show that the optimal mechanism is deterministic for $m \leq 6$ and conjecture that this is true for any m. Using deep learning methods, Dütting et al (2022) design a neural network architecture (which they call RochetNet) to investigate numerically whether the Giannakopoulos and Koutsoupias conjecture is correct. They obtain a numerical confirmation of this conjecture for $m \leq 10$. They also obtain new analytical results for alternative distributions of types. However, Hart and Reny (2015)[7] obtain very surprising results about revenue maximizing mechanisms for selling multiple goods: They show in particular that, unlike the case of one good, when the buyer's values for the goods increase, the seller's maximal revenue may well decrease! They also clarify how randomization can increase the seller's revenue in the multiple-good case.

Rochet and Thanassoulis (2019) study the two-product monopoly profit maximization problem for a seller who can commit to a dynamic pricing strategy. They show that if consumers' valuations are not strongly ordered, then optimality for the seller can require intertemporal price discrimination: the seller offers a choice between supplying a complete bundle now, or delaying the supply of a component of that bundle until a later date.

3.2 Multiproduct Non Linear Pricing

The context of multidimensional non linear pricing is very similar to the previous case, but y is now interpreted as a vector of attributes for a durable good (say a car). This is the multidimensional extension of the hedonic model of Mussa and Rosen (1978). Potential buyers buy at most one unit of the good, and choose the vector of attributes y at price $p(y)$ that maximizes their net utility $U = \theta.y - p(y)$ among the product line Y offered by the seller. The unit cost $C(y)$ is a convex function of attributes y. The profit of the seller is thus the expectation of the difference between economic surplus $S(\theta, y) = \theta.y - C(y)$ and U, the part of the surplus left to the buyers:

$$B(U) = \int_\Theta [p(\theta) - C(y(\theta))]dF(\theta) = \int_\Theta [\theta.\nabla U(\theta) - C(\nabla U(\theta)) - U(\theta)]f(\theta)d\theta,$$

which is to be maximized over the set of convex non negative functions. If we forget the convexity constraint, we obtain a variations calculus problem with an inequality constraint. This is called an obstacle problem and has been well studied in Physics. A very good introduction to obstacle problems can be found in Rodrigues (1981). When the inequality constraint is not binding, the solution satisfies an Euler equation:

$$div([(\theta - \nabla C(\nabla U))f(\theta)] = -f(\theta),$$

[7] See also Hart and Nisan (2014a) and (2014b).

where *div* denotes the divergence operator. In the unidimensional case, this Euler equation can be integrated, leading to the familiar condition

$$(\theta - C'(U'))f(\theta) = 1 - F(\theta).$$

If the distribution of θ is well behaved, the optimal allocation is given implicitly by the equation

$$C'(y(\theta)) = \theta - \frac{1 - F(\theta)}{f(\theta)},$$

and different types obtain different allocations:

$$\theta_1 < \theta_2 \implies y(\theta_1) < y(\theta_2).$$

In this case the convexity constraint on U is not binding. "Bunching", i.e. identical treatment of consumers with different characteristics, only occurs when $\theta - \frac{1-F(\theta)}{f(\theta)}$ is not an increasing function of θ, in which case an "ironing" technique, introduced by Myerson (1981), has to be used. In that case U is not strictly convex: it is linear on the intervals where the convexity constraint binds.

Armstrong (1996) was one of the first to study this problem in the multidimensional case. He was able to find an analytical solution for a class of distributions that possess a radial symmetry. More importantly, he showed that, contrarily to the unidimensional case, the optimal mechanism typically involved the exclusion of a positive measure of potential consumers. It is easy to see that the monopolist always chooses a mechanism such that at least one consumer has a zero surplus (the participation constraint is binding)[8]. But if only one consumer has a zero surplus, then the multidimensional monopolist typically gains by further increasing its prices by ϵ. Indeed, under reasonable assumptions on the distribution of types, the gain will be of order ϵ on all active buyers, while the loss will be of order ϵ^m on the lost consumers. When $m > 1$ this cannot be optimal. Thus Armstrong (1996) shows that a special form of bunching (namely exclusion of some consumers) is very general in multidimensional problems.

Rochet and Choné (1998) go further and develop a characterization of the optimal mechanism by using "sweeping operators" that generalize ironing techniques to multiple dimensions. They also find that a more sophisticated form of bunching is typical of multidimensional screening problems. When a whole set of types (a "bunch") get the same allocation, the Euler equation that characterizes solutions of variations calculus problems is only satisfied on average over the "bunch". Rochet and Choné show on an example that it may be optimal for the monopolist to offer less variety of products for the consumers that are not ready to pay high prices for the good. Only the consumers that have "high types" get specific products fitting their heterogeneous preferences, while

[8] Otherwise all prices could be increased by a small constant, yielding a higher profit

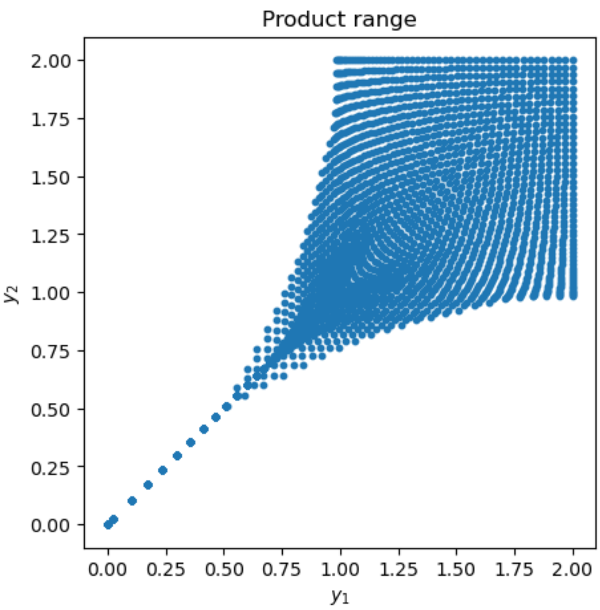

Figure 1: The Product Range in a discrete approximation of the Rochet Choné problem

"low types" get a more standard product. I conjecture that this second type of bunching is quite general for multidimensional screening problems. In other words, the solution to the obstacle problem (the relaxed problem without the convexity constraint) is almost never convex.

Ekeland and Moreno (2010) solve a discretized version of the same model where the distribution of types is uniform on a square. They point out a mistake in the solution conjectured by Rochet and Choné for the continuous case. Rochet and Choné had postulated that the optimal product line was the union of a straight line and a square. Ekeland and Moreno (2010) show that it is more like a "stingray", in the sense that the junction of the line and the square is obtained by a smooth curve. But the fundamental property remains: high types have access to more product variety than low types. The above figure, drawn from Carlier et al. (2023) represents this "stingray" in a discrete approximation of the Rochet-Choné problem for a uniform distribution of types on the square $[1, 2]^2$. Mirebeau (2014) has developed a very efficient algorithm for finding solutions of discretized variations calculus problems with global convexity constraints on a regular grid.

3.3 Related Forms of the Implementability Condition

Heydenreich et al.(2009) give a characterization of the cases where the famous revenue equivalence property discovered by Myerson (1981) can be extended to multidimensional frameworks. Their idea is based on a graph theoretic interpretation of the incentive compatibility constraints. Carbajal and Ely (2013) extend the standard implementability result to environments where the envelope theorem and revenue equivalence principle fail due to non-convex and non-differentiable valuations. They obtain a characterization of incentive compatibility based on the Mirrlees representation of the indirect utility and a monotonicity condition on the allocation rule, which pin down the range of possible payoffs as a function of the allocation rule. They illustrate their approach by deriving the optimal selling mechanism in a buyer–seller situation where the buyer is loss-averse.

Pavan et al. (2014) consider mechanism design problems in dynamic quasi-linear environments where private information arrives over time and decisions are made over multiple periods. They provide a necessary condition for incentive compatibility, called the integral monotonicity condition. This condition takes the form of an envelope formula for the derivative of an agent's equilibrium expected payoff with respect to his current type. Even if an agent's current type is uni-dimensional, his report can affect allocations in multiple periods.

3.4 Robust Mechanism Design

Carroll (2017) studies robust implementation, based on ideas from operations research and robust optimization. Instead of assuming full knowledge of the distribution of valuations for bundles of goods, Carroll assumes knowledge only of the marginal distributions of values of individual goods. Robust mechanism design has some relations with approximately optimal mechanism design (see Roughgarden and Talgam-Cohen 2019), but it aims at max–min optimality rather than approximation. The optimal mechanism is the mechanism that maximizes the max–min objective for every partial instance. An excellent survey of this literature is Carroll (2019).

3.5 Relaxing the Quasi-linearity Assumption

Optimal mechanisms are much more difficult to characterize if the quasi-linearity assumption is relaxed. However, some progress has recently been made. Nöldeke and Samuelson (2018) show that a duality relationship known as Galois connection can be used to gain new insights into these difficult mechanism design problems. On the same vein, Kazumura et al (2020) characterize incentive compatible mechanisms in a non quasi-linear context and obtain a revenue equivalence result that can be applied to several contexts.

4 The Underlying Tree Structure of Discrete Problems

In his beautiful book, Vohra (2011) provides a very elegant and unified approach to mechanism design by using linear programming methods. This section follows an alternative, graph theoretical approach that illustrates well the richness and diversity of multidimensional screening problems. Let us go back to the discrete set-up, where there are N distinct consumer types $\theta_i \in \mathbb{R}^m$, $i = 1, ...N$. For the sake of a simple exposition we keep the linear parameterization $U_i = \theta_i.y - p$ but the reasoning is valid for more general quasi-linear utilities. We simplify the notation by denoting $y_i = y(\theta_i)$. The expected profit of the monopolist is

$$B(y, U) = \sum_i f_i[\theta_i.y_i - C(y_i) - U_i],$$

which must be maximized under the discrete incentive compatibility constraints:

$$\forall (i, j), U_i - U_j \geq (\theta_i - \theta_j).y_j, \tag{5}$$

and individual rationality constraints:

$$\forall i, U_i \geq 0. \tag{6}$$

Note the similarity with the continuous version of the problem. However, contrarily to this continuous problem, where y could be replaced by ∇U leading to a variation calculus problem involving only U and ∇U, here we must keep track of the allocation y because there are several possible choices that are compatible with the incentive compatibility constraints. The allocation must belong to the sub-differential of U but U is typically non differentiable at some of the θ_i, which have a positive mass. This why y cannot be determined from U. Following Spence (1980), it is natural to decompose the firm's problem in two sub-problems:

1. For a fixed allocation $y = (y_1, ..., y_N)$, compute the minimum of expected utilities $\sum_i f_i U_i$ under incentive compatibility and individual rationality constraints, which can be called the expected informational rent.

2. Choose y to maximize the difference between expected surplus and expected informational rent.

It is remarkable that the first sub-problem has a general solution that can be found by a relatively simple algorithm. Let us indeed denote by $\mathcal{U}(y)$ the set of utility vectors that implement the allocation y. For convenience, we can add a fictitious type $\theta_0 = 0$ which has a zero utility, in order to capture the individual rationality constraint. In what follows, it will be useful to consider arbitrary paths in the set $\Theta = (\theta_0, \theta_1, ..., \theta_N)$. We will denote such a path from type θ_i to θ_j by a function γ . The "length" l of γ is the number of segments used to connect $\theta_i = \gamma(0)$ to $\theta_j = \gamma(l)$. Hence, γ maps $(0, 1, ..., l)$ into Θ. Finally, a path of length l is "closed" if $\gamma(0) = \gamma(l)$. With this notation for discrete paths, Proposition 1 can be formulated as

Proposition 5 *$\mathcal{U}(y)$ is nonempty if and only if for every closed path γ*

$$\sum_k y_{\gamma(k)}.(\theta_{\gamma(k+1)} - \theta_{\gamma(k)}) \leq 0. \tag{7}$$

Then we have:

Proposition 6 *When condition (7) is satisfied, $\mathcal{U}(y)$ has a unique minimal element $u^{min}(y)$ characterized for $i = 1, ..., N$ by*

$$u_i^{min}(y) \equiv sup_\gamma \sum_k y_{\gamma(k)}(\theta_{\gamma(k+1)} - \theta_{\gamma(k)}), \tag{8}$$

where the supremum is taken over all open paths from 0 to i , and $u_0^{min} = 0$.

Condition (8) means that agent i's guaranteed utility level u_i^{min} is equal to the maximum of all sums of incremental utilities along any path connecting 0 to θ_i. We will refer to $u_i^{min}(y)$ as the **informational rent** of agent i corresponding to allocation y. Note that this rent does not depend on the frequencies $(f_1, ..., f_N)$ of the distribution of types, but only on its support $\Theta = (\theta_1, ..., \theta_N)$. Formula (8) shows that the informational rent of each agent can be computed by a recursive algorithm. Intuitively, it is as if each type i would choose the path from 0 to θ_i that maximizes the sum of incremental utilities. Denote by u_i^l the maximum of formula (8) over all paths of length less than or equal to l from 0 to θ_i. Then u_i^l can be computed recursively by a Bellman-type formula:

$$u_i^{l+1} = \sup_j [u_j^l + (\theta_i - \theta_j).y_j].$$

Condition (7) implies that this algorithm has no cycles. The set of types being finite, u_i^l converges to the rent of agent i in a finite number of steps. For any allocation y, the dynamic programming principle implies that if j belongs to the optimal path γ from 0 to i , the truncation of γ to the path between 0 and j defines the optimal path from 0 to j. This allows us to define a partial ordering $\prec$ on Θ: $j \prec i$ if and only if j belongs to one of the optimal paths from 0 to i . For generic [9] allocations, there is a unique optimal path γ from 0 to i , and the rent of i is easily computed:

$$u_i^{min}(y, \prec) = \sum_{k=1}^{k=l-1} y_{\gamma(k)}[\theta_{\gamma(k+1)} - \theta_{\gamma(k)}].$$

Graphically, the collection of optimal paths comprises a "tree", i.e. a connected graph without cycles such that, from the "root" vertex 0, there is a unique path to any other point in the graph) denoted Γ. The binding incentive constraints are the branches of the

[9]However, the optimal allocation may be such that there are several optimal paths, in which case the expected rent is not differentiable.

tree emanating from type θ_0, where the individual rationality constraint binds. One can define for all i, j such that $i \prec j$, the "immediate successor" $s(i,j)$ of i in the direction of j by the formula

$$s(i,j) = min[k|i \prec k \prec j, k \neq i].$$

Then, it is easy to see that the expected rent can be written as

$$\mathcal{R}(y, \prec) = \sum_i \sum_{i \prec j} f_j y_i [(\theta_{s(i,j)} - \theta_i]$$

In the unidimensional case, and when the single-crossing holds, condition (7) reduces to the well-known monotonicity condition $y_1 \leq y_2 \leq ... \leq y_N$ and $\prec$ always coincides with the complete ordering: $\theta_1 < \theta_2 < \ < \theta_N$. The associated tree has a single branch. In this case

$$\mathcal{R}(y, \prec) = \sum_i [1 - F_i] y_i . [\theta_{i+1} - \theta_i]$$

In the general case, the binding incentive compatibility constraints (corresponding to the agent's optimal paths defining the tree Γ) depend on the allocation y. The expected rent is not differentiable everytime there are several possible paths for connecting one type and the fictitious type θ_0. In general, the virtual surplus does not have a simple expression. In fact, the virtual surplus approach works only when one can anticipate a priori the optimal paths i.e., which incentive compatibility constraints will be binding, like in the example with radial symmetry solved by Armstrong (1996).

When i is a maximal element for $\prec$, the set $(j|i \prec j)$ is empty and $\mathcal{R}$ does not depend on y_i. Thus there is no reason to distort y_i, which coincides with the first best allocation at θ_i. This is a generalization of the "no distortion at the top" result for uni-dimensional problems. In the multidimensional case, the order $\prec$ depends on the allocation, which implies that a fixed point problem has to be solved. There may be several maximal types for which the allocation is not distorted, a property that can never happen in dimension one.

The following figure, drawn from Carlier et al. (2023) shows an example of the underlying tree structure of the set of types for a discrete approximation of the Rochet-Choné problem with a uniform distribution on the square $[1, 2]^2$. The approximation grid has 5 points on each dimension. The green dots represent the points on the grid where the individuality constraint binds (the multidimensional version of the "no rent at the bottom" result). The arrows represent the binding incentive compatibility constraints. On the "north-east" part of the square, only local downward (and "westward") incentive compatibility constraints bind (a discrete version of the so-called "relaxed problem" where only local incentive compatibility constraints are binding). As a consequence, the distortions of the allocations are orthogonal to the boundary (a multidimensional version

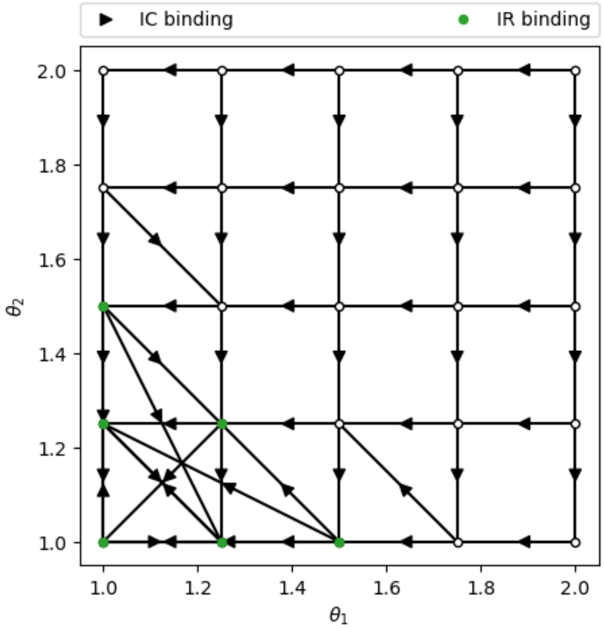

Figure 2: The Underlying Tree in a discrete version of the Rochet Choné problem

of the "no-distortion at the top" result). However as we move to the south west part of the square, we see that "transverse" incentive compatibility constraints start binding, for example between types $(1, 1.5)$ and $(1.25, 1)$. Ultimately, as we move closer to the south west corner of the square, global incentive compatibility constraints start binding. This corresponds to the "bunching at the bottom" result of Rochet and Choné (1998) where "low types" obtain the same allocations on the lines $\theta_1 + \theta_2 = a$. This is the case on the figure for $a = 2.5$ and $a = 3$.

5 Conclusion

Multidimensional screening models are difficult because they give rise to an endogenous ordering of types. In other words, the set of binding incentive compatibility constraints is endogenous to the choice of the allocation y. Therefore the expected informational rent does not have a uniform expression, which precludes the use of the virtual surplus technique, so efficient in dimension one. The incentive compatibility conditions are frequently binding not only among local types but also more globally, and the discrete analog of the first-order approach is not generally valid. However we have seen that there is a simple algorithm that allows to compute numerically the informational rent, which is typically a non differentiable of the allocation. Instead of using non smooth optimization techniques, it is more efficient to transform the optimization problem into a max-min problem as in Carlier et al (2023) and use primal-dual algorithms. This approach is starting to unveil the surprising rich features of multidimensional screening problems and has potentially many new applications.

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