WORKING PAPERS

N° 1534

May 2024

"Optimal Fiscal Rules and Macroprudential Policies with Sovereign Default Risk"

Gerard Maideu-Morera

Toulouse School of Economics

Optimal Fiscal Rules and Macroprudential Policies with Sovereign Default Risk*

Gerard Maideu-Morera†

April 2024

Abstract

The European Sovereign debt crises (2010-2012) showcased how excessive private leverage can threaten sovereign debt sustainability, making the existing fiscal rules targeting only public debt insufficient. In this paper, I study the optimal joint design of fiscal rules and macroprudential policies with sovereign default risk. I first consider a stylized two-period model of a small open economy where both the local government and a representative household borrow internationally. A central authority internalizes externalities from sovereign default by the local government and designs fiscal rules and macroprudential policies. The model yields two insights: (i) it provides a novel rationale for macroprudential policies, and (ii) sovereign debt limits that are a function of the quantity of private debt (private-debt-dependent fiscal rules) can implement the optimal allocation. Then, I generalize these results to a multiperiod model with heterogeneous households, aggregate risk, and a rich asset structure. Finally, I calibrate a quantitative version of the model to compute the private-debt-dependent fiscal rules and the size of the macroprudential wedges.

JEL codes: F34, F41, F45, E44, G28

Keywords: Fiscal rules, macroprudential policy, sovereign default, endogenous borrowing constraints, economic unions

*I am grateful to Eugenia Gonzalez-Aguado, Christian Hellwig and Nicolas Werquin for their advice and guidance. I would also like to thank Fernando Arce, Charles Brendon, Fabrice Collard, Andreas Schaab, Jean Tirole and participants at the EEA (2023) and TSE workshops.

†Toulouse School of Economics. Email: gerard.maideumorera@tse-fr.eu.

1 Introduction

Ireland and Spain experienced large increases in sovereign bond spreads during the European sovereign debt crisis years (2010-2012). Yet, both countries had previously complied with the European Union sovereign debt limits with a ratio of public debt to GDP below 40% in 2007. Thus, the fiscal rules in place targeting the quantity of public debt were insufficient to prevent the crises. Instead, excessive private leverage was highlighted as one of the leading causes of the crises for these economies (Lane (2012)), suggesting that macroprudential policies targeting private debt –lacking at the time– would have been more effective than tighter fiscal discipline (Martin and Philippon (2017)). These events illustrated the need for an integrated policy design that accounts for how financial instability can threaten sovereign debt sustainability.

This paper studies the optimal joint design of fiscal rules (i.e., restrictions to sovereign debt) and macroprudential policies (i.e., restrictions to private debt) with sovereign default risk. I consider a small open economy where the local government and households can borrow internationally. The local government finances expenditures through distortionary taxes and external debt and can default strategically. A central authority designs fiscal rules and macroprudential policies, taking into account extra costs from sovereign default that the local government does not internalize. Consider, for example, a central authority in a monetary union taking into account externalities from sovereign default on the other countries of the union (Tirole (2015)).

In this environment, I proceed in three steps. First, I study a stylized two-period model where I present two main results: (i) the optimality of macroprudential policies and (ii) the implementation of the optimal allocation through sovereign debt limits that are a function of private debt (private-debt-dependent fiscal rules). Then, I generalize and extend these results in a richer model. Finally, I calibrate a quantitative model to compute and quantify the optimal fiscal rules and macroprudential policies.

Stylized two-period model. In the two-period model (Section 2), I assume a representative household that borrows with uncontingent and non-defaultable bonds. Starting with the (laissez-faire) local government's problem, I first show that although private debt is not defaultable, higher household leverage increases the probability of sovereign default because the gain of not taxing and defaulting increases with the household's marginal utility (i.e., its leverage). Moreover, because taxes are distortionary, the local government borrows to smooth tax distortions. As a result, the Frisch elasticity of the household's income – a measure of the government's tax capacity – is a key parameter determining the quantity of sovereign and private borrowing.

The central authority maximizes the household's utility but also takes into account externalities from sovereign default. It can impose restrictions on sovereign and private debt but cannot enforce the repayment by the local government. For exposition, I first assume the default externalities are high enough so that default is never optimal. Therefore, its problem reduces to finding the combination of sovereign and private debt that maximizes the household's value while preventing default.

The first result of the paper shows that we can decentralize the solution of the central authority's problem by imposing a limit on sovereign debt that is a function of the quantity of private debt and then letting the local government choose the allocation freely. This is a *solvency constraint that is not too tight* as in Alvarez and Jermann (2000) because it binds only when the government would (weakly) prefer defaulting, but it is a function of private debt. We can think of this type of solvency constraint as a private-debt-dependent fiscal rule. In practice, fiscal rules typically impose limits on deficits and sovereign debt invariant to other macroeconomic factors, such as private debt. For example, the Maastricht Treaty and the Stability and Growth Pact intended to limit public deficits to 3% and sovereign debt to 60% of GDP.[1] The model indicates that better arrangements can be obtained when these limits are a (decreasing) function of private debt.

The second result characterizes the optimal macroprudential policy as a function of the restrictions on sovereign debt. That is, I derive a formula linking the wedges on sovereign and private debt, which are defined as distortions relative to the (laissez-faire) allocation of local government's problem. The wedge on private debt inherits the sign of the wedge on sovereign debt, implying that it is optimal to restrict private debt whenever sovereign debt is restricted. More concretely, the wedge on private debt equals the wedge on sovereign debt times two adjustment factors. The first one is the derivative of the sovereign debt limit with respect to private debt. Because the default probability increases with private debt, reducing private borrowing allows for an increase in sovereign debt while holding the default probability constant. The second one measures the benefits of an increase in sovereign debt in the first period by adjusting for the deadweight loss of taxation. Intuitively, when sovereign debt is restricted, taxes are inefficiently high in period one. So, increasing sovereign debt enables a decrease in taxes and a smaller tax distortion.

The formula provides a rationale for macroprudential policies based on distortionary tax-

[1] This type of fiscal rules are prevalent, see Bova *et al.* (2015).

ation and sovereign default externalities. This motive is different from the pecuniary (Lorenzoni (2008), Bianchi (2011) and Dávila and Korinek (2018)) and aggregate demand (Farhi and Werning (2016)) externalities identified in the literature. Arce (2023) studies the interactions between private borrowing with pecuniary externalities and sovereign default in a quantitative model and finds that it implies larger optimal macroprudential taxes. However, I show that macroprudential policies can be optimal even without pecuniary externalities and zero sovereign debt spreads.

This simple model admits two other formulas relating the wedges in terms that can be more closely linked to the data. In the first one, this relation only depends on the ratio of marginal utilities in default and repayment. To first order, this is equivalent to the coefficient of relative risk aversion times the percentage increase in consumption after default – which is increasing in the public and private leverage and the output gains from eliminating tax distortions. The second one is expressed as the ratio of the derivatives of the default probability with respect to every type of debt times the deadweight loss from taxes.

Then, I consider the case where the default externalities are small enough that allowing for a positive default probability can be optimal. In this case, the local government may conduct macroprudential policies by itself (i.e. restrictions relative to the decentralized borrowing of the representative household) to lower the cost of sovereign borrowing. I show that the previous results on the private-debt-dependent fiscal rules and the optimal macroprudential wedges easily generalize.

Finally, I also study the effect of suboptimal policies that only restrict either private or sovereign debt. These policies are less effective if the economy substitutes and borrows more in the unrestricted debt (see Martin and Philippon (2017)). The Frisch elasticity determines this substitution because, from the local government's perspective, it determines how (price-) substitutable the two types of debt are. For a country with high tax capacity – i.e. low Frisch – the cost of not smoothing taxes is small, so it can easily substitute between the two types of debt after a change in relative prices. Conversely, substitution is more costly for a country with low tax capacity. Thus, these suboptimal policies may be less effective if the country has a high tax capacity.

General model. In the second part of the paper (Section 3), I generalize the previous results on private-debt-dependent fiscal rules and optimal macroprudential wedges formulas in a dynamic model with heterogeneous households, aggregate risk, and a rich asset structure. The definition of the solvency constraints is now more involved because, with state-contingent assets, the states where the government may default can be a non-monotone function of the households' portfolios. However, I show how the solvency constraints can be appropriately defined to derive similar formulas for the optimal macroprudential wedges. These wedges are state, household, and asset-specific. So, it allows us to study how the optimal macroprudential interventions depend on the payoff structure of each asset or the household's characteristics. Finally, I also allow for domestic holdings of sovereign debt and household default. I assume the households automatically default on some assets after a sovereign default. In this case, the wedge formulas are similar to those of the domestic sovereign debt and imply larger distortions than when the households cannot default.[2]

Quantitative model. Finally, I solve and calibrate a quantitative version of the model to compute the private-debt-dependent fiscal rules and the size of the macroprudential wedges (Section 4). I extend the model by introducing long-term sovereign debt to better match the data. I calibrate an exogenous borrowing limit on private debt to match the level of private external debt of Spain in 2007. As in the two-period model, I find that higher private debt increases the default probability. The optimal private-debt-dependent fiscal rule can be computed directly from the solution of the local government's problem, which, in turn, allows us to infer the magnitudes of the optimal macroprudential wedges. Hence, this approach allows us to quantify the optimal policies without solving the full Ramsey problem of the central authority.

The quantitative results show that the private-debt-dependent fiscal rule is rather flat. That is, how much the government is allowed to borrow is not very sensitive to the level of private debt. In particular, going from zero private debt to the (exogenous) borrowing limit allows the government to increase sovereign debt by 7.8%. In turn, this implies that the optimal macroprudential wedges will be small. However, in the model, private debt only increases the default probability mechanically by increasing the household's marginal utility. These estimates should be interpreted as a lower bound. The slope of the fiscal rule and the size of the wedges could be much larger, with, for example, pecuniary externalities (Arce (2023)), bank-sovereign doom loops (Farhi and Tirole (2018)), or private default following a sovereign default.

[2] In Appendix D, I also explore an extension where I allow the (representative) household to default strategically without sovereign default. The – appropriately defined – private-debt-dependent fiscal rule still implements the optimal allocation. Moreover, when only the no private default constraint binds, the macroprudential wedges are larger than the wedge on sovereign debt.

Related literature. This paper contributes to the literature studying the interaction between sovereign default and private debt by studying the joint design of fiscal rules and macroprudential policies. Arce (2023) studies a quantitative sovereign default model with private debt and pecuniary externalities as in Bianchi (2011). It shows that the interaction between sovereign default and private debt increases the frequency of financial crises and the size of the optimal macroprudential taxes. I contribute by studying the design of fiscal rules and macroprudential policies jointly from the perspective of a central authority that takes into account default externalities. Moreover, I show that in this context, macroprudential interventions are optimal even without pecuniary externalities or sovereign debt spreads and derive analytical formulas for the macroprudential wedges in a model with heterogeneous agents and multiple state-contingent securities.

A related literature studies the welfare effects of centralized international borrowing (public debt) versus decentralized borrowing (private debt) (see Wright (2006), Jeske (2006) and Kim and Zhang (2012)). In the benchmark model, I study an economy with two assets (public and private debt), but where only public debt is defaultable. Moreover, since I solve the Ramsey problem of the local government, I am focusing on centralized borrowing. Other papers studying the interaction between sovereign default risk and the private sector include Mendoza and Yue (2012), Arellano *et al.* (2017), and Kaas *et al.* (2020). They study the productivity losses from a sovereign debt crisis due to domestic firms' inability to finance imports or reduced access to financing due to depressed banks' balance sheets.

The EU sovereign debt crisis (2010-12) has motivated a large body of positive and normative.[3] Martin and Philippon (2017) use a DSGE model to identify the drivers of the European crisis across different countries. For Ireland and Spain, they find that the leading cause of the crisis was the large build-up of private debt, so macroprudential policies would have been preferable to tighter fiscal discipline. Building on their work, I study the optimal fiscal rules and macroprudential policies in a model where the government can default strategically. Moreover, they find that a strategic government may respond to macroprudential policies by increasing public debt. I identify the Frisch elasticity as the key parameter that determines this substitution. A large literature has emerged that studies the feedback loops between sovereign and bank insolvency –the so-called doom loops– and derives implications for macroprudential policy and banking supervision and

[3]For positive work see, for example, Lane (2012), Gourinchas *et al.* (2017), Chodorow-Reich *et al.* (2019) or Brunnermeier and Reis (2019) for a broad overview. For normative work see Tirole (2015) and Gourinchas *et al.* (2020). Abrahám *et al.* (2018), Ferrari *et al.* (2020), and Liu *et al.* (2022) use dynamic contracting methods to study optimal risk-sharing arrangements in the European context. Relatedly, on optimal risk sharing and fiscal policy in monetary unions, see Auclert and Rognlie (2014), Farhi and Werning (2017), Aguiar *et al.* (2015), or Eijffinger *et al.* (2018)

regulation. Some important contributions include Acharya *et al.* (2014), Bocola (2016), Brunnermeier *et al.* (2016), and Farhi and Tirole (2018). Instead, I focus on the leverage of the household sector and study the implications for the design of fiscal rules.

Finally, I also contribute to the literatures studying fiscal rules with sovereign default (Hatchondo *et al.* (2022)) and fiscal rules in economic unions (Chari and Kehoe (2007)), Dovis and Kirpalani (2020), Broner *et al.* (2021), Sublet (2022), Berriel *et al.* (2023a), and Berriel *et al.* (2023b)).[4] I study the optimal design of fiscal rules with private borrowing and show that private-debt-dependent fiscal rules can implement the optimal allocation.

2 Two-period model

There are two periods $t \in \{1,2\}$. I consider a small open economy composed of two agents: a representative household and a benevolent local government. Both can borrow internationally in the first period, but only the local government can default. The third agent is a central authority. As the local government, it wants to maximize the utility of representative household, but it also internalizes some extra costs arising from sovereign default. A natural interpretation is that the small open economy (e.g., Spain or Ireland) belongs to an economic union, and the central authority (e.g., European Commission) takes into account that sovereign default imposes externalities on other members in the union (Tirole (2015)).[5] I first describe and characterize the problems of the representative household and the local government in Sections 2.1 and 2.2, respectively. Then, I introduce the central authority in Section 2.3.

2.1 Households

The representative household's per-period utility is

$$u(c_t - v(y_t)),$$

[4] Hatchondo *et al.* (2022) argue that fiscal rules based on sovereign debt spreads are preferable to rules based on the quantity of sovereign debt. As I discuss in the paper, if the spreads price the default probability correctly, this type of rule is equivalent to the private-debt-dependent fiscal rule and can also implement the optimal allocation.

[5] The model also fits the European context in that I assume that the local government does not have access to independent monetary policy and that the exchange rate is fixed.

where c_t denotes consumption and y_t the household's income.[6] The function u is increasing and concave, and the disutility of producing income v is increasing, convex, and isoelatic so that the Frisch elasticity $\eta \equiv \frac{v'(y_1)}{v''(y_1)y_1} > 0$ is constant. The representative household's discount factor is $\beta \in (0,1)$. At $t = 1$, the households can borrow internationally with one-period uncontingent bonds a_2, so $-a_2$ is the quantity of private debt. I assume that the households cannot default and that they borrow at the international risk-free rate $\frac{1}{q} - 1$. I relax this assumption in Section 3, where I consider an extension in which the households also default internationally after the government defaults. At every period, the households' income is taxed at a linear rate $\tau_t \in [0,1]$. From the households' perspective, the income tax in the second period is stochastic as it depends on the government's default decisions. More precisely, $\tau_2(\theta)$ is a function of a random variable $\theta \in [\underline{\theta}, \overline{\theta}]$ realized at $t = 2$ and that affects the government's default decisions, as I specify in the next section.

Household's problem. Given a tax system $\{\tau_1, (\tau_2(\theta))_{\theta \in [\underline{\theta}, \overline{\theta}]}\}$, the representative household maximizes

$$V_1^{HH} = \max_{\substack{c_1, y_1, a_1, \\ c_2(\theta), y_2(\theta)}} u(c_1 - v(y_1)) + \beta \mathbb{E}_\theta \left[u(c_2(\theta) - v(y_2(\theta))) \right]$$

subject to the budget constraints at $t = 1$

$$c_1 + qa_2 = (1 - \tau_1) y_1, \tag{1}$$

and at $t = 2$

$$c_2(\theta_2) = a_2 + (1 - \tau_2(\theta)) y_2(\theta), \tag{2}$$

for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

The solution to the household's problem is characterized by the following optimality condition for income

$$1 - \tau_t = v'(y_t), \tag{3}$$

for all τ_t, and the Euler equation

$$u_1' = \beta q^{-1} \mathbb{E}[u_2'(\theta)], \tag{4}$$

[6] I write income in the utility function directly. But, in the background, we can think of an economy where households supply labor l with disutility $v(l)$ and a competitive representative firm produces with the constant returns production $f(l) = l$. The price of the final good is normalized to one, so the equilibrium wage is also one, and $y = l$.

where u_1' and $u_2'(\theta)$ denote the marginal utility of consumption in the first period and in the second period in state θ, respectively.

2.2 Local government

At $t = 1$, the local government taxes income linearly and borrows externally with one-period uncontingent bonds to finance some (exogenous) government expenditures G_1.[7] Let B_2 denote the bonds issued by the government, so $-B_2$ is the quantity of sovereign debt. At $t = 2$, the local government may default, so the foreign lenders demand a price $Q^B(B_2,a_2)$, which may depend on the default probability.[8] However, I assume that $Q^B(B_2,a_2) = q$ when the default probability is zero. Therefore, the budget constraint of the government at $t = 1$ writes

$$G_1 + Q^B(B_2,a_2)B_2 = \tau_1 y_1. \tag{5}$$

At the beginning of period $t = 2$, the local government decides whether to repay B_2 or default. I introduce taste shocks to the value of defaulting by assuming that default carries a stochastic utility penalty $\theta \in [\underline{\theta},\overline{\theta}]$ realized at the beginning of period 2, with $\underline{\theta} \geq 0$ and continuous density $\pi(\theta)$. Therefore, from the government's perspective, the utility at period $t = 2$ after default is $u(c_2 - v(y_2)) - \theta$. I assume that there are no government expenditures to be financed at t = 2. As a result, if the government repays, we have the following budget constraints for the government and the household

$$-B_2 = \tau_2 y_2, \tag{6}$$

$$c_2 = a_2 + (1 - \tau_2)y_2, \tag{7}$$

where the quantity of sovereign debt directly pins down the required tax rate. After default, because there are no government expenditures in the second period, the government does not tax ($\tau_2 = 0$), so the budget constraint of the household is

$$c_2 = a_2 + y_2. \tag{8}$$

Thus, we can think of the zero tax on the household after a default essentially as a bailout. It is important to understand why public and private debt are non-fungible in the model

[7] I assume G_1 is low enough such that it can be financed with income taxes.

[8] The setting allows for the case where foreign lenders expect to always be repaid by the central authority after a default, so $Q^B(B_2,a_2)$ is constant.

so we cannot collapse them and consider only total external debt $B_2 + a_2$. First, sovereign debt is defaultable, but private debt is not. Although private debt affects the incentives to default, it does not do it in the same way as sovereign debt. Second, transferring resources between the budget constraints of the government and the household is costly because taxation is distortionary. Thus, even without sovereign default, we need to keep track of B_2 and a_2 separately. Hence, fungibility would emerge if taxes were lump-sum (i.e., $\eta \to 0$) and both or none of the two types of debt were defaultable.

Government's problem. The local government chooses government expenditures, sets taxes, borrows, and makes default decisions to maximize the representative household's utility. That is, it solves a Ramsey problem. Hence, we use the optimality condition (3) of the household as an implementability condition for the output tax rate. Because we solve the primal problem, the government directly chooses how indebted the household is, i.e., c_1 and a_2, and there is no associated implementability condition because the price q is fixed. The implicit assumption is that the government has sufficient instruments to control private debt perfectly.[9]

At period 1, the local government maximizes

$$V_1 = \max_{c_1, y_1, B_2, a_2} u(c_1 - v(y_1)) + \beta V_2(B_2, a_2) \tag{9}$$

subject to the budget constraints (1) and (5), the implementability condition (3), and the bond price function $Q^B(B_2, a_2)$. The expected value at period $t = 2$ is defined as

$$V_2(B_2, a_2) = \mathbb{E}_\theta \left[\max\{V_2^R(B_2, a_2), V_2^D(a_2, \theta)\} \right], \tag{10}$$

where $V_2^R(B_2, a_2)$ is the value of defaulting and $V_2^D(a_2, \theta)$ the value of repaying. The value of repaying writes:

$$V_2^R(B_2, a_2) = \max_{c_2, y_2} u\left(c_2 - v\left(y_2\right)\right)$$

subject to the budget constraints (6), (7) and the implementability condition (3). The value defaulting writes:

$$V_2^D(a_2, \theta) = \max_{c_2, y_2} u\left(c_2 - v\left(y_2\right)\right) - \theta$$

subject to the budget constraint (8).

[9]More broadly, although full control may be a strong assumption, it is reasonable to assume that it has some influence on private debt through, for example, tax instruments, the strictness of the banking supervision (Farhi and Tirole (2018)), etc.

Properties of the government's problem. Because taxation is distortionary (i.e. the Frisch elasticity is positive $\eta > 0$), the government wants to borrow to smooth tax distortions across the two periods. Hence, it is generally not optimal to set $B_2 = 0$ and only have private borrowing –even if it is cheaper ex-ante– as this would require $\tau_2 = 0$, so all G_1 would have to be financed with τ_1. To see this more formally, let λ^G and λ^{HH} be the multipliers on the government and the household budget constraints, respectively. Combining the first-order conditions for c_1 and y_1, we can derive

$$\lambda^G = \left(1 - \eta \frac{\tau_1}{1 - \tau_1}\right) \lambda^{HH}, \tag{11}$$

where $\lambda^{HH} = u_1'$. All the derivations can be found in Appendix A.1. The term $\left(1 - \eta \frac{\tau_1}{1-\tau_1}\right)$ links the multiplier on the household and the government's budget constraints. First, notice $\eta \frac{\tau_1}{1-\tau_1} = \frac{\tau_1}{y_1} \frac{dy_1}{d\tau_1}$ is the elasticity of income with respect to the tax rate, and so it measures the deadweight loss of taxation. Hence, it captures how distortionary taxes are, i.e., the tax capacity of the government. More generally, we can think of it as the cost of transferring resources from the household's budget constraint to that of the government.[10] Second, the multipliers λ^G and λ^{HH} also pin down the marginal costs of borrowing in sovereign and private debt. Notice that the optimality conditions for B_2 and a_2 write:

$$\lambda^G (Q^B + B_2 \frac{\partial Q^B}{\partial B_2}) = \beta \frac{\partial V_2}{\partial B_2} \tag{12}$$

$$\lambda^{HH} q = \beta \frac{\partial V_2}{\partial q_2} \tag{13}$$

Therefore, the government will choose the combination of public and private debt that minimizes borrowing costs while also accounting for the costs associated with the tax distortions. In Section 2.6, I show that η determines how much the government substitutes between sovereign and private debt and, therefore, how it responds to changes in debt prices.

Finally, before studying the central authority's problem, it is useful to understand how each type of debt affects the default probability. Let $\delta(B_2, a_2) = \mathbb{P}_\theta(V^R(B_2, a_2) \leq V^D(a_2, \theta))$ denote the default probability as a function of the quantity of sovereign and private debt. The changes in default probability from an increase in sovereign debt ($-B_2$) and private

[10] If $\eta \to 0$, taxes become lump-sum, and this term is equal to one, so there is no deadweight loss from taxation. By contrast, at the top of the Laffer curve $\frac{\tau_1}{1-\tau_1} = \frac{1}{\eta}$, and so the term goes to zero.

debt ($-a_2$) are proportional to (see Appendix A.1)

$$\frac{d\delta(B_2,a_2)}{d(-B_2)} \propto -\frac{dV_2^R}{d(-B_2)} = \frac{u_2'(R)}{1-\eta\frac{\tau_2}{1-\tau_2}} > 0 \tag{14}$$

$$\frac{d\delta(B_2,a_2)}{d(-a_2)} \propto -\left(\frac{dV_2^R}{d(-a_2)} - \frac{dV_2^D}{d(-a_2)}\right) = u_2'(R) - u_2'(D) > 0, \tag{15}$$

where $u_2'(R)$ and $u_2'(D)$ denote the marginal utility of consumption in the second period if the government repays and if it defaults, respectively. Equation (14) shows that higher sovereign debt increases the default probability because it decreases the value of repaying. An increase in sovereign debt requires an increase in taxes in the second period to repay it, which brings a deadweight loss of $\eta\frac{\tau_2}{1-\tau_2}$, and so lowers the households' utility by $\frac{u_2'(R)}{1-\eta\frac{\tau_2}{1-\tau_2}}$. Equation (15) shows that private debt also has an (indirect) effect on the default probability. Higher private debt lowers the household's consumption at $t=2$ due to the larger repayment. Because the marginal utility is higher after repayment, an increase in private debt lowers the value of repaying relatively more than the value of defaulting, thereby increasing the default probability. Intuitively, when private leverage is high, the consumption gain from defaulting brings a larger increase in utility for the household. However, because $1-\eta\frac{\tau_2}{1-\tau_2} < 1$, sovereign debt always has a larger effect on the default probability.

2.3 Central authority's problem

The objective of the central authority is to maximize the representative households's utility, but taking into account that sovereign default has an extra cost $S > 0$. We may think of this cost as externalities on the rest of the countries of the union (Tirole (2015)). These may include the contagion to other countries through financial markets or the real economy, costs from a bailout, etc. The objective of the central authority is

$$W_1 = V_1 - S\delta(B_2,a_2). \tag{16}$$

The central authority solves a similar primal problem as the local government: It sets taxes and chooses private and sovereign debt, but it cannot force the local government not to default. More concretely, the central authority is constrained by the fact that there is default whenever $V_2^R(B_2,a_2) < V_2^D(a_2,\theta)$. I assume that S is high enough such that having a positive default probability would never be optimal. Thus, the problem of the central authority is the same as that of the local government (problem (9)), but with the extra constraints

$$V_2^R(B_2, a_2) \geq V_2^D(a_2, \theta) \ \forall \theta \in [\underline{\theta}, \overline{\theta}]. \tag{17}$$

This assumption simplifies the exposition because it allows us to keep the price of sovereign debt fixed. In Section 2.5, I show how the main results on optimal fiscal rules and macro-prudential policies easily generalize to the case where S is small and the central authority allows for a positive default probability. Throughout, I will define a fiscal rule (macro-prudential policy) as any intervention to the choice of sovereign (private) debt relative to the solution of the local government's problem (9).

General solvency constraints and private-debt-dependent fiscal rules. Instead of solving the central authority's problem with constraint (17), it is more illustrative to use an equivalent constraint where we impose a limit on sovereign debt that is a function of the quantitative of private debt. Moreover, this can be directly interpreted as a decentralization of the central authority's solution where the local government can choose public and private borrowing freely as long as it satisfies the private-debt-dependent limit on sovereign debt.

To this end, notice first that because $V_2^D(a_2, \theta)$ is decreasing in θ, we only need to consider the constraint $V_2^R(B_2, a_2) \geq V_2^D(a_2, \underline{\theta})$.[11] I define the function $B^{max}(a_2)$ as the maximal level of sovereign debt that guarantees there is no default given a quantity of private debt a_2. That is, $B^{max}(a_2)$ is implicitly defined as

$$V_2^R(B^{max}(a_2), a_2) = V_2^D(a_2, \underline{\theta}). \tag{18}$$

Then we can substitute constraint (17) by the following limit on sovereign debt

$$B_2 \geq B^{max}(a_2). \tag{19}$$

Notice that this is a slight generalization of the *solvency constraints that are not too tight* defined in Alvarez and Jermann (2000) because it binds only when the agent would (weakly) prefer defaulting, but it is a function of private debt. We can think of this constraint as a private-debt-dependent fiscal rule. That is, how much the government is allowed to borrow depends on the level of private debt in the economy. In practice, fiscal rules are usually independent of other macroeconomic and financial variables such as private debt.

[11] This may not be the case with more general asset structures and shocks, as the state where default is more attractive can depend on the household's portfolio. I will show how this can be handled in Section 3.

For example, the Maastricht treaty and the Stability and Growth Pact intended to impose a limit of 60% of public debt to GDP and budget deficits at 3%.[12] As I show below, as long as more private debt increases the incentives to default, it is optimal that the sovereign debt limit is a function of private debt.

In Section 3, I generalize this constraint in a richer model. The sovereign debt limit B^{max} will be a function of the financial wealth positions and sovereign debt holdings of all household types in all $t+1$ aggregate states.

2.4 Optimal wedges

I define the wedges on sovereign and private debt as the distortions in the allocation of the central authority relative to the problem of the local government (i.e., problem (9)). More concretely, they are defined as deviations from the optimality conditions for sovereign debt (equation (12)) and private debt (equation (13)) of the local government's problem with zero default probability. Therefore, the wedge on sovereign debt τ^B is defined as:

$$\tau^B = 1 - \beta q^{-1} \left(\frac{u_1'}{1-\eta\frac{\tau_1}{1-\tau_1}} \right)^{-1} \frac{u_2'(R)}{1-\eta\frac{\tau_2}{1-\tau_2}}. \tag{20}$$

Similarly, I define the wedge on private debt τ^a as:

$$\tau^a = 1 - \beta q^{-1} \left(u_1' \right)^{-1} u_2'(R). \tag{21}$$

If the local government would never default, i.e., constraint (19) (or equivalently constraint (17)) does not bind, we have $\tau^B = 0$. Otherwise, sovereign debt is restricted and $\tau^B > 0$. Then we ask, if $\tau^B > 0$, is it optimal also to restrict the choice of private debt? And by how much? The following proposition derives a relation between the two wedges.

Proposition 1. *The optimal wedge on private debt satisfies the following equation:*

$$\tau^a = \left(-\frac{\partial B^{max}(a_2)}{\partial a_2} \right) \frac{\tau^B}{1-\eta\frac{\tau_1}{1-\tau_1}}. \tag{22}$$

Moreover, whenever $\tau^B > 0$, we have $\tau^B > \tau^a > 0$.

All results are derived in the Appendix. The proposition shows that it is optimal to restrict private debt (i.e., $\tau^a > 0$) whenever sovereign debt is restricted (i.e., $\tau^B > 0$). This provides

[12] This type of rules are also common outside the EU, see Bova *et al.* (2015)

a rationale for macroprudential policies based on externalities from sovereign default. Pecuniary (Lorenzoni (2008), Bianchi (2011) and Dávila and Korinek (2018)) or aggregate demand (Farhi and Werning (2016)) externalities are not required for macroprudential policies to be optimal.

Equation (22) also helps us understand the rationale for restricting private debt in this model. We can always prevent default by only reducing sovereign debt (the local government never defaults if $B_2 = 0$). But this is not optimal, and we can do better by restricting private debt. The government wants to borrow to smooth the tax distortions of financing G_1. Without default, it is easy to show that the local government would set $\tau_1 = \tau_2$. However, when $\tau^B > 0$, taxes are inefficiently high at $t = 1$. By reducing private debt, sovereign debt can be increased by $-\frac{\partial B^{max}(a_2)}{\partial a_2} > 0$ while keeping the default probability at zero. This allows the government to cut taxes at $t = 1$, which induces a resource gain proportional to $\frac{1}{1-\eta\frac{\tau_1}{1-\tau_1}}$. If private debt did not affect the default incentives, we would have $\frac{\partial B^{max}(a_2)}{\partial a_2} = 0$ and $\tau^a = 0$. Finally, the proposition shows that whenever sovereign debt is restricted, the wedge on private debt is always smaller than the wedge on sovereign debt. Intuitively, because private debt always has a smaller effect on the default probability, it is optimal that the distortion is smaller.

Arce (2023) studies the interactions between sovereign default and pecuniary externalities in private debt and shows that sovereign default increases the size of the optimal macroprudential wedges. However, even without pecuniary externalities, the incentives to default are higher when the household is more leveraged. So, combined with the sovereign default externalities and the distortionary taxation, it is sufficient to rationalize macroprudential policies in a sovereign default context.[13]

In Section 3, I will show how this formula generalizes to a fully dynamic model with heterogeneous households, aggregate shocks, and a more general asset structure. However, we can also write the wedge formula in terms that can be mapped more directly to the data.

Proposition 2. *The wedge on private debt can be rewritten as*

$$\frac{\tau^a}{1-\tau^a} = \left(1 - \frac{u_2'(D)}{u_2'(R)}\right) \frac{\tau^B}{1-\tau^B} \tag{23}$$

[13]In Arce (2023), because the government has access to lump-sum taxes, the choice of sovereign and private debt balances the costs of sovereign debt spreads and hitting the borrowing constraint for private debt. By contrast, when taxation is distortionary, the tax smoothing motive pins down sovereign borrowing. Therefore, the macroprudential policy is optimal – even when sovereign debt spreads are zero – because it allows for better tax smoothing.

and

$$\tau^a = \left(\frac{\frac{\partial \delta(B_2,a_2)}{\partial a_2}}{\frac{\partial \delta(B_2,a_2)}{\partial B_2}} |_{\delta(B_2,a_2)=0} \right) \frac{\tau^B}{1-\eta\frac{\tau_1}{1-\tau_1}}. \tag{24}$$

The first equation shows that only the difference in marginal utilities after repayment and default is required to pin down the wedge on private debt.[14] The simplification comes from that, in this simple model, the only deviation from full tax smoothing comes from the wedges.[15] The second equation uses the fact that $\frac{\partial -B^{max}(a_2)}{\partial a_2}$ is also the ratio of the change in the default probability from an increase in private and sovereign debt evaluated at the region with 0 default probability.

Assuming constant relative risk aversion $\sigma \equiv -\frac{u''(c)c}{u'(c)}$, to first order the first formula can be rewritten as

$$\frac{\tau^a}{1-\tau^a} \approx \sigma \frac{\Delta c}{c_2(R)} \frac{\tau^B}{1-\tau^B}. \tag{25}$$

Hence, the relation between the wedges depends on the coefficient of relative risk aversion and the percentage increase in consumption from defaulting.[16] In particular, the macroprudential wedge is increasing in both the risk aversion and the increase in consumption. Using the household's budget constraints (7) and (8), we get

$$\frac{\Delta c}{c_2(R)} = \frac{1}{1+\frac{a_2}{y_2(R)}+\frac{B_2}{y_2(R)}} \left(\underbrace{(-\frac{B_2}{y_2(R)})}_{\text{Direct gain default}} + \underbrace{\frac{y_2(R)-y_2(D)}{y_2(R)}}_{\text{Gain lower tax distortion}} \right).$$

So, the percentage change in consumption is mechanically increasing in the public leverage $(-\frac{B_2}{y_2(R)})$ and in the relative output gains from eliminating tax distortions after default. Moreover, because higher private debt lowers consumption after repayment but does not affect the change in consumption from default, the macroprudential wedges are also increasing in the quantity of private debt.

Discussion of the decentralization and spread-based rules. The decentralization with a private-debt-dependent fiscal rule works because, conditional on not defaulting, the

[14] In the general model, if we assume no income effects, we will also obtain an extended version of this formula.

[15] Recall that to minimize tax distortions, the local government borrows to perfectly smooth tax (i.e $\tau_1 = \tau_2$). With the wedges instead we have $\frac{1-\eta\frac{\tau_1}{1-\tau_1}}{1-\eta\frac{\tau_2}{1-\tau_2}} = \frac{1-\tau^B}{1-\tau^a}$.

[16] Interestingly, this expression bears some resemblance with the Baily-Chetty formula for optimal unemployment insurance (Baily (1978), Chetty (2006)).

preferences of the local government and the central authority are perfectly aligned. That is, they both agree on the pair of sovereign and private debt that minimizes distortions. Therefore, imposing this private-debt-dependent fiscal rule guarantees that the local government implements the optimal macroprudential policy by itself. Hatchondo *et al.* (2022) have shown that spread-break rules, that is, limits on fiscal deficits when spreads are above a certain threshold, can be preferable to limits on sovereign debt. In this model, if spreads ($\frac{q}{Q^B(B_2,a_2)} - 1$) price the default probability correctly, the private-debt-dependent limit (19), and a rule that imposes zero spreads are clearly equivalent. So a government faced with this spread-based rule would optimally choose the same combination of sovereign and private debt and implement the optimal macroprudential policy.

2.5 Optimal policies with positive default probability

I now show that the above results on the optimal macroprudential wedges and the implementation through private-debt-dependent fiscal rules extend when S is smaller so that allowing for a positive default probability may be optimal.

First, it is interesting to study the local government's problem when the default probabilities are positive. In this case, the local government wants to restrict private debt even without any intervention by the central authority. The Euler equation of the household is: $u_1' = \beta q^{-1} \frac{\partial V_2}{\partial a_2}$. However, the Euler equation for private debt of the local government has a wedge $\tau^{a,LG}$ defined as $\tau^{a,LG} = 1 - \beta q^{-1}(u_1')^{-1} \frac{\partial V_2}{\partial a_2}$, which is equal to

$$\tau^{a,LG} = \frac{1}{1 - \eta \frac{\tau_1}{1-\tau_1}} q^{-1} \frac{\partial Q^B}{\partial a_2}(-B_2).$$

Because higher private debt increases the default probability, the local government can lower the spread on sovereign debt by restricting private debt. A lower spread carries a resource gain for the government proportional to $(-B_2)$, which allows for a tax cut with a further resource gain of $\frac{1}{1-\eta\frac{\tau_1}{1-\tau_1}}$ due to reduced tax distortion. Hence, when the default probability is positive, the local government implements macroprudential policies to lower the spreads on sovereign debt.[17]

Private-debt-dependent fiscal rules with positive default probability. The previous results can be generalized by imposing a private-debt-dependent fiscal rule that guaran-

[17]This rationale is also discussed in Arce (2023); with distortionary taxes, the tax smoothing motive implies larger gains from lower sovereign debt spreads and so the size of the macroprudential wedge.

tees that the default probability never exceeds a certain threshold. Notice that the default probability can be expressed as $\delta(B_2, a_2) = \mathbb{P}(\theta \leq \theta^*(B_2, a_2))$, where $V_2^R(B_2, a_2) = V_2^D(a_2, \theta^*(B_2, a_2))$. Given the one-to-one relationship between the default cutoff θ^* and the default probability, it is convenient to directly define the debt limit as a function of θ^* instead of the default probability. That is, we define the function $B^{max}(a_2; \theta^*)$ as

$$V_2^R(B^{max}(a_2; \theta^*), a_2) = V_2^D(a_2, \theta^*), \tag{26}$$

and impose the solvency constraint

$$B_2 \geq B^{max}(a_2; \theta^*). \tag{27}$$

Then, the central authority chooses the cutoff θ^* that maximizes its value. More concretely, letting $V_1(\theta^*)$ denote the local government's value under constraint (27), the optimal default cutoff for the central authority is

$$\theta^* = \arg\max_{\tilde{\theta}} V_1(\tilde{\theta}) + \Pi(\tilde{\theta})(-S).$$

I now define the wedge on sovereign debt τ^B as

$$\tau^B = 1 - \beta(Q^B + \frac{\partial Q^B}{\partial B_2} B_2)^{-1} \left(\frac{u_1'}{1 - \eta \frac{\tau_1}{1-\tau_1}} \right)^{-1} \frac{\partial V_2(B_2, a_2)}{\partial B_2},$$

and on private debt τ^a as

$$\tau^a = 1 - \tau^{a,LG} - \beta q^{-1} (u_1')^{-1} \frac{\partial V_2(B_2, a_2)}{\partial a_2}.$$

Notice that because the wedges are defined relative to the solution of the local government's problem, the wedge $\tau^{a,LG}$ is not included in τ^a. The following proposition shows that this fiscal rule implements the optimal allocation and derives formulas relating the wedges on private and sovereign debt akin to those derived in the previous section.

Proposition 3. *The optimal allocation of the central authority can be decentralized by imposing the solvency constraint (27) with the optimal default cutoff θ^*. Moreover, the wedges on private* *and sovereign debt are linked through:*

$$\tau^a = \frac{Q^B + \frac{\partial Q^B}{\partial B_2} B_2}{q} \left(- \frac{\partial B^{max}(a_2; \theta^*)}{\partial a_2} \right) \frac{1}{1 - \eta \frac{\tau_1}{1-\tau_1}} \tau^B$$
$$= \frac{Q^B + \frac{\partial Q^B}{\partial B_2} B_2}{q} \left(1 - \frac{u_2'(D)}{u_2'(R)} \right) \frac{1 - \eta \frac{\tau_2}{1-\tau_2}}{1 - \eta \frac{\tau_1}{1-\tau_1}} \tau^B$$

The macroprudential wedge formulas are the same as before, except for the term $\frac{Q^B + \frac{\partial Q^B}{\partial B_2} B_2}{q}$. This is the ratio of the increases in resources from a marginal increase in sovereign and private debt. Hence, this term takes into account that with positive default probability, an increase in sovereign debt lowers its price, so the resources for the government only increase by $Q^B + \frac{\partial Q^B}{\partial B_2} B_2$ instead of q.

2.6 Tax capacity, substitution and the effects of third best policies

It is also useful to understand the effects of suboptimal policies that only restrict one type of debt. Martin and Philippon (2017) found that in their model, a biased government responds by borrowing more after private debt is restricted through macroprudential policies. This is also the case here. After one type of debt is restricted or its price increases, the local government can substitute and borrow more with the other type of debt.[18] So, these policies will be less effective when the local government can substitute more easily. I show in this section that the Frisch elasticity η is the parameter that determines the magnitude of this substitution.

For exposition, assume that the local goverment cannot default and consider the following policies: (i) an increase in the cost of sovereign debt by $\varepsilon^B > 0$, i.e., $q(1 - \varepsilon^B)B_2$; and (ii) an increase the cost of private debt by $\varepsilon^a > 0$, i.e., $q(1 - \varepsilon^a)a_2$. The following proposition derives a tax smoothing formula that only depends on the difference in debt costs and the Frisch elasticity.

Proposition 4. *The difference in optimal taxes across periods can be approximated by the following formula:*

$$\frac{\tau_2}{1 - \tau_2} - \frac{\tau_1}{1 - \tau_1} \approx \frac{1}{\eta} \left[\varepsilon^a - \varepsilon^B \right]. \tag{28}$$

[18] Anecdotically, before the EU sovereign debt crises (2010-2012), Ireland and Spain were complying with the sovereign debt limits dictated by the Maastricht Treaty but experienced large increases in private debt.

Because G_1 is fixed, the left-hand side of the formula uniquely pins down the quantity of sovereign debt B_2. The right-hand side contains the difference in the price of sovereign and private debt divided by the Frisch elasticity. If the prices are the same, we have $\tau_2 = \tau_1$, so the local goverment borrows just enough to perfectly smooth taxes, as this minimizes the tax distortion. Now consider an increase in ε^a, then $\tau_2 > \tau_1$ and B_2 increases. The increase in sovereign debt is inversely proportional to the Frisch elasticity because it determines the costs of deviating from perfect tax smoothing. For $\eta \to 0$, taxes become lump-sum, so tax distortions go to zero, and the response goes to infinity.[19] Conversely, as $\eta \to \infty$ the costs of not smoothing taxes are very high, so the response goes to zero. Essentially, η determines how (price-)substitutes the two types of debt are. In this model, we can think of η as a measure of the country's tax capacity. Then, the implication is that for countries with high tax capacity (low η), policies that only restrict one type of debt would be less effective. Therefore, a joint design of fiscal rules and macroprudential policies becomes more important.

3 General model

In this section, I generalize the previous results in a fully dynamic model with heterogeneous households, aggregate risk, and multiple state-contingent assets.

Environment and households. The time horizon is infinite $t = 1,2,...,\infty$. There is a stochastic aggregate state s_t, with history up to t denoted by $s^t = (s_1,...,s_t)$. I assume s_t is continuous and let $\pi(s_t|s^{t-1})$ be the transition density. The local economy is populated by a finite set of household types $\mathcal{I}$. Type $i \in \mathcal{I}$ has mass π^i, welfare weight ω^i, and utility $U^i(c^i,y^i,s)$, which is a function of its consumption, c^i, income, y^i, and the aggregate state s. I denote, for $x \in \{c,y\}$, $U^i_x = \frac{\partial U^i}{\partial x}$ and $U^i_{xx} = \frac{\partial U^i}{\partial^2 x}$, and assume U^i is twice continuously differentiable, and satisfies $U^i_c > 0 > U^i_{cc}$, $U^i_y < 0$ and $U^i_{yy} < 0$. Moreover, I assume that $\frac{\partial^2 U^i}{\partial y \partial s} > 0$ so that we can interpret higher s as increasing the households' productivity. The government and all the household types have a common discount factor $\beta \in (0,1)$.

Assets. There is a set $\mathcal{K}$ of assets available to the households.[20] Each asset $k \in \mathcal{K}$ has payoff $R_k(s^t,s_{t+1})$ and price $q_k(s^t)$, and $a^i_k(s^t)$ denotes type i's position on asset k. I as-

[19] In this case, and without default or debt constraints, the government would go infinitely long in one type of debt and short in the other. Therefore, distortionary taxation ensures that this kind of arbitrage is not optimal.

[20] I model the financial market in a similar fashion as Berger *et al.* (2023).

sume that, for all $k \in \mathcal{K}$ and s_{t-1}, the payoff $R_k(s^{t-1},.)$ and asset price $q_k(.)$ functions are continuous and almost everywhere differentiable.[21] The households may also hold domestic sovereign debt $b^i(s^t) \geq 0$ and they face the following portfolio constraints

$$\mathcal{H}^i\left(b^i(s^t), \{a^i_k(s^t)\}_{k\in\mathcal{K}}, s^t\right) \geq 0, \tag{29}$$

where $\mathcal{H}^i$ is a vector-valued function. The constraints do not include asset prices, so I abstract from pecuniary externalities. Define the household's financial wealth net of the payments from domestic sovereign debt as

$$I^i_{t+1}(s^{t+1}) \equiv \sum_{k\in\mathcal{K}} \widetilde{R}_k(s^t, s_{t+1}) a^i_k(s^t), \tag{30}$$

where $\widetilde{R}_k(s^t, s_{t+1}) = R_k(s^t, s_{t+1}) + q_k(s^{t+1})$. The budget constraint of household i is

$$c^i(s^t) + \sum_k q_k(s^t)a^i_k(s^t) + Q^B(s^t)b^i_{t+1}(s^t) = (1-\tau^i(s^t))y^i(s^t) + I^i_t(s^t) + b^i_t(s^{t-1}), \tag{31}$$

where $\tau^i(s^t)$ is the type and state-specific income tax and $Q^B(s^t)$ is the equilibrium price of domestic sovereign debt in state s^t. To economize notation, let $X^i_{t+1}(s^{t+1}) = (I^i_{t+1}(s^{t+1}), b^i_{t+1}(s^t))$.

Foreign Lenders. There is a continuum of foreign lenders. I assume they are deep-pocketed, and so they will price all assets. Thus, the households and the local government take the asset prices $\{q^k(s^t)\}$ as given. The price of sovereign bonds $Q^B(B_{t+1}(s^t), \{X^i_{t+1}(s^{t+1})\}, s^t)$ depends on the default probability, but when that is zero, the price is $Q^B = q$.[22]

Government. The government taxes the households' income and borrows with one-period uncontingent bonds to finance a stochastic stream of government expenditures $\{G(s^t)\}$. The budget constraint of the government following repayment is

$$G(s^t) + Q^B((B_{t+1}(s^t), \{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}, s_{t+1}}, s^t)B_{t+1}(s^t) = \sum_i \pi^i\tau^i(s^t)y^i(s^t) + B_t(s^{t-1}). \tag{32}$$

[21] This assumption will be used in Lemma 1. It implies that we are not allowing for discrete jumps in the payoff function. However, it can accommodate, for example, standard put and call options of securities where the price of the underlying is a continuous function of the aggregate state s_{t+1}.

[22] Note that $\frac{1}{q} - 1$ may not necessarily be the rate at which the households can borrow.

As before, we have an implementability condition for the income tax

$$1-\tau^i(s^t) = -\frac{U^i_y(s^t)}{U^i_c(s^t)}. \tag{33}$$

Following repayment, the planning problem of the government is

$$V^R_t\left(B_t, \{X^i_t\}_{i\in\mathcal{I}}, s^t\right) = \max \sum_{i\in\mathcal{I}} \omega^i \pi^i U^i(c^i(s^t), y^i(s^t), s_t) + \beta \mathbb{E}_{s_{t+1}|s^t} V_{t+1}\left(B_{t+1}(s^t), \{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right)$$

subject to the constraints (29)-(33), and where

$$V_{t+1}\left(B_{t+1}(s^t), \{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) = \\ \max\{V^R_{t+1}\left(B_{t+1}(s^t), \{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right), V^D_{t+1}\left(\{I^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right)\},$$

and V^D_{t+1} is the value function after default. We can allow for general default penalties and different probabilities of regaining access to international markets after default. Later, I will also allow for changes in the assets available to the households, which nests private defaults following sovereign default. The budget constraints of the government and HH after default are

$$G^D(s^t) = \sum_i \pi^i \tau^i(s^t) y^i(s^t), \tag{34}$$

$$c^i(s^t) + \sum_k q_k(s^t) a^i_k(s^t) = (1-\tau^i(s^t)) y^i(s^t) + I^i_t(s^t) \tag{35}$$

where $G^D(s^t) \geq G(s^t)$ for all s_t.[23] I introduce a default productivity costs by assuming that utility after default is $U^i(c^i, y^i, s; D) = U^i(c^i, y^i, h(s))$, with $h(s) \leq s$. After default, the government regains access to financial markets with exogenous probability ι. Therefore, the value of default solves:

$$V^D_t\left(\{I^i_t(s^t)\}_{i\in\mathcal{I}}, s^t\right) = \max \sum_{i\in\mathcal{I}} \omega^i \pi^i U^i(c^i(s^t), y^i(s^t), h(s_t)) + \beta \mathbb{E}_{s_{t+1}|s^t} V^C_{t+1}\left(\{I^i_t(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) \tag{36}$$

subject to (29), (30) and (33)-(35), and the continuation value solves

$$V^C_{t+1}\left(\{I^i_t(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) = \iota V^R_{t+1}\left(0, \{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) + (1-\iota) V^D_{t+1}\left(\{I^i_t(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right),$$

with $X^i_{t+1}(s^{t+1}) = \{I^i_{t+1}(s^{t+1}), 0\}$.

[23]This assumption guarantees that the government would never default when $B_t = 0$.

3.1 Central authority's problem and general solvency constraints

As in the two-period model, we proceed by directly assuming that the default costs are high enough that default is never optimal for the central authority. That is, the central authority's problem imposes the constraints

$$V_{t+1}^{R}\left(B_{t+1}(s^t), \{X_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) \geq V_{t+1}^{D}\left(\{I_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) \ \forall s_{t+1}. \tag{37}$$

Moreover, I also solve a decentralized version of the problem with generalized solvency constraints. However, a more careful definition is required as, with state-contingent assets, the states s_{t+1} where constraint (37) binds may depend on the households' asset positions in a non-monotone way. That is, a priori, the constraint could bind in some state s'_{t+1} but not in $s_{t+1} < s'_{t+1}$, and conversely.

Define, for all $\{X_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}}$ and s^{t+1}, the cutoff level of sovereign debt $\overline{B}$ that guarantees there is no default implicitly as

$$V_{t+1}^{R}\left(\overline{B}(\{X_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}), \{X_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) = V_{t+1}^{D}\left(\{I_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right). \tag{38}$$

Then, I define the maximum sustainable sovereign debt as

$$B^{max}(\{X_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}, s_{t+1}}) = \max_{s_{t+1}} \overline{B}(\{X_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}). \tag{39}$$

Throughout, I will assume that the utility functions $U^i(c, y, s)$ and payoff functions $\widetilde{R}_k(s^t, s_{t+1})$ are such that the function $\overline{B}$ has a unique maximum. Notice that the function B^{max} depends on the asset positions of all household types in all aggregate states s_{t+1}. Hence, we impose the following solvency constraint

$$B_{t+1}(s^t) \geq B^{max}(\{X_{t+1}^{i}(s^{t+1})\}_{i\in\mathcal{I}, s_{t+1}}). \tag{40}$$

Again, this constraint generalizes the *solvency constraints that are not too tight* of Alvarez and Jermann (2000) to a limited commitment setting where constraints are a function of the portfolios of all household types. In Alvarez and Jermann (2000), the agents trade one-period state-contingent assets (Arrow securities), but after defaulting, the agents are excluded from financial markets and can only consume their flow endowment. Hence, the value of defaulting on one particular asset (here sovereign debt) does not depend on the rest of the portfolio. Instead, here, the value of default depends on the rest of the economy's portfolio because households cannot default in international markets. Thus, the solvency constraints must be written as a function of the households' positions.

The function B^{max} is always well-defined and bounded above by zero because

$$V^R_{t+1}\left(0, \{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) > V^D_{t+1}\left(\{I^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right)$$

for all $\left(\{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right)$.[24] The following lemma shows that this function is also differentiable and satisfies an envelope formula, which will be used to derive the optimal wedges. The envelope formula shows that B^{max} only depends on the positions in the state where the local government is most likely to default.

Lemma 1. *The function $B^{\max}$ is bounded above by zero and, letting $x \in \{b, I\}$, it satisfies*

$$\frac{\partial B^{max}(\{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}, s_{t+1}})}{\partial x^i_{t+1}(s^{t+1})} = \begin{cases} \frac{\partial -\overline{B}(\{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1})}{\partial x^i_{t+1}(s^{t+1})} & \text{if } s_{t+1} = s^*_{t+1} \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

where $s^*_{t+1} = \underset{s_{t+1}}{\operatorname{argmax}} \overline{B}(\{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1})$.

3.2 Optimal wedges

The wedges are defined as in the two-period model. The wedge on sovereign debt is

$$\tau^B(s^t) = 1 - \frac{\beta}{q\lambda^G(s^t)} \mathbb{E}_{s_{t+1}|s^t} \frac{\partial V^R(s^{t+1})}{\partial B_{t+1}(s^t)}, \tag{42}$$

where $\lambda^G(s^t)$ is the multiplier of the government budget constraint (32). Similarly, I define the wedge on asset position $a^i_k(s^t)$ by

$$\tau^{k,i}(s^t) = 1 - \frac{1}{q_k(s^t)\lambda^{HH,i}(s^t)} \left(\langle \mu^{\mathcal{H},i}(s^t), \mathcal{H}^i_{a^i_k}(s^t) \rangle + \beta \sum_{s^{t+1}} \widetilde{R}_k(s^t, s_{t+1}) \frac{\partial V^R_{t+1}(s^{t+1})}{\partial I^i_{t+1}(s^{t+1})} \right), \tag{43}$$

where $\lambda^{HH,i}(s^t)$ is the multiplier on type i's budget constraint (31), $\mu^{\mathcal{H},i}(s^t)$ is a vector of multipliers on the portfolio constraints (29), and $\langle , \rangle$ is the dot product operator. As in the lemma, I define $s^*_{t+1} = \underset{s_{t+1}}{\operatorname{argmax}} \overline{B}(\{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1})$ and let $s^{t+1,*} = (s^t, s^*_{t+1})$. Notice that s^*_{t+1} depends on all the positions $\{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}, s_{t+1}}$ and so is endogenous to the optimal wedges. However, I leave the dependence implicit for ease of notation. The following proposition generalizes the wedge formula of Proposition 1.

[24] This inequality is guaranteed by the assumptions $G^D(s^t) \geq G(s^t)$ and $h(s) \leq s$.

Proposition 5. *The optimal wedge on asset position $a_k^i(s^t)$ is:*

$$\tau^{k,i}(s^t) = \frac{\widetilde{R}_k(s^t, s_{t+1^*})}{q_k(s^t)} q \frac{\partial - \overline{B}(\{X_{t+1}^i(s^{t+1,*})\}_{i\in\mathcal{I}}, s^{t+1,*})}{\partial I_{t+1}^i(s^{t+1,*})} \frac{\lambda^G(s^t)}{\lambda^{HH,i}(s^t)} \times \tau^B(s^t). \tag{44}$$

The first term contains the ratio of returns in state s_{t+1}^* between sovereign debt and asset k. Then, similar to the two-period model, the second term depends on how much a higher financial wealth of agent i in state $s^{t+1,*}$ allows for an increase in sovereign debt while keeping the default incentives fixed. The third term is the shadow cost of transferring resources from the budget constraint of household type i to the government in state s_t. Recall that in the two-period model with no income effects, this term was just $\frac{1}{1-\eta\frac{\tau_1}{1-\tau_1}}$ where $\eta\frac{\tau_1}{1-\tau_1}$ is the deadweight loss of taxation. Therefore, the rationale for macroprudential policy described in the two-period models extends to a more general setting.

As in the two-period model, I denote by $U_c^i(s^{t+1,*}, R)$ and $U_c^i(s^{t+1,*}, D)$ the marginal utility of consumption in state $s^{t+1,*}$ following a repayment and a default, respectively. Similarly, $\tau(s^{t+1,*}, R)$ is the tax rate in state $s^{t+1,*}$ if the government repays. Assuming a utility function with no income effects, we can derive a formula akin to that in Proposition 2.

Proposition 6. *Assume $U^i(c^i, y^i, s) = u^i(c^i - v^i(\frac{y^i}{s}))$ and $\eta^i = \frac{(v^i)'(y)}{(v^i)''(y)y}$, the optimal wedge on asset position $a_k^i(s^t)$ can be written as:*

$$\tau^{k,i}(s^t) = \frac{\widetilde{R}_k(s^t, s_{t+1}^*)}{q_k(s^t)} q \left(1 - \frac{U_c^i(s^{t+1,*}, D)}{U_c^i(s^{t+1,*}, R)}\right) \frac{1 - \eta^i s_{t+1}^* \frac{\tau^i(s^{t+1,*}, R)}{1-\tau^i(s^{t+1,*}, R)}}{1 - \eta^i s_t \frac{\tau^i(s^t)}{1-\tau^i(s^t)}} \times \tau^B(s^t). \tag{45}$$

The first term in the formula is the same. As in the two-period model, the second term depends on the ratio of marginal utilities in the default and repayment states because this captures how much the agent's financial wealth affects the incentives to default. The third term captures the deviation from full tax-smoothing between states s^t and $s^{t+1,*}$.

We can also derive simple relations between the wedges across different assets and household types. For any $k, k' \in \mathcal{K}$, we have

$$\tau^{k',i}(s^t) = \frac{\frac{\widetilde{R}_{k'}(s^t, s_{t+1}^*)}{q_{k'}(s^t)}}{\frac{\widetilde{R}_k(s^t, s_{t+1}^*)}{q_k(s^t)}} \tau^{k,i}(s^t).$$

Hence, for any household type, the relative distortions in its portfolio only depend on the payments of each asset in the state s_{t+1}^*. In particular, the household should hold more of (or borrow less in) assets that pay more in state s^*_{t+1}. Similarly, for any $i, i' \in \mathcal{I}$, we can derive

$$\tau^{k,i'}(s^t) = \frac{\frac{\partial - \overline{B}(X^{i'}_{t+1}(s^{t+1,*}), s^{t+1,*})}{\partial I^{i'}_{t+1}(s^{t+1,*})} \frac{1}{\lambda^{HH,i'}(s^t)}}{\frac{\partial - \overline{B}(X^{i}_{t+1}(s^{t+1,*}), s^{t+1,*})}{\partial I^{i}_{t+1}(s^{t+1,*})} \frac{1}{\lambda^{HH,i}(s^t)}} \tau^{k,i}(s^t)$$
$$= \left(\frac{U^{i'}_c(s^{t+1,*}, R) - U^{i'}_c(s^{t+1,*}, D)}{U^{i}_c(s^{t+1,*}, R) - U^{i}_c(s^{t+1,*}, D)} \right) \frac{U^i_c(s^t)}{U^{i'}_c(s^t)} \tau^{k,i}(s^t),$$

where the second equality assumes a utility function with no income effects (i.e., $U^i(c^i, y^i, s) = u^i(c^i - v^i(\frac{y^i}{s})))$. Intuitively, a larger difference in marginal utilities in default and repayment implies that the gains from default for this agent are larger, and so its wedge should also be larger. Notice that the optimal wedges do not directly depend on the mass π^i or the Pareto weight ω^i of each household type. Because these are constant, the effect on the incentive to default of higher π^i or ω^i is exactly offset by the welfare cost of increasing distortions at t.

Optimal wedges to domestic sovereign debt. We can also derive a similar formula for the domestic holdings of sovereign debt. I define the wedge on position $b^i_{t+1}(s^t)$ as

$$\tau^{b,i}(s^t) = 1 - \frac{\beta}{q\lambda^{HH,i}(s^t)} \mathbb{E}_{s_{t+1}|s^t} \frac{\partial V^R(s^{t+1})}{\partial b^i_{t+1}(s^t)}. \tag{46}$$

The following proposition characterizes the optimal wedges on domestic sovereign debt holdings.

Proposition 7. *The optimal wedge on position $b^i_{t+1}(s^t)$ satisfies*

$$\tau^{b,i}(s^t) = \frac{\partial - \overline{B}(\{X^i_{t+1}(s^{t+1})\})}{\partial b^i_{t+1}(s^t)} \frac{\lambda^G(s^t)}{\lambda^{HH,i}(s^t)} \tau^B(s^t)$$
$$= \frac{1 - \eta^i s_{t+1,*} \frac{\tau^i(s^{t+1,*}, R)}{1 - \tau^i(s^{t+1,*}, R)}}{1 - \eta^i s_t \frac{\tau^i(s^t)}{1 - \tau^i(s^t)}} \tau^B(s^t),$$

where the second inequality assumes no income effects.

The first line shows that the rationale for distorting $b^i_{t+1}(s^t)$ is the same as for other assets. However, the second line is now different from the formula in Proposition 2. Intuitively, higher $b^i_{t+1}(s^t)$ directly increases the value of repaying by $U^i_c(s^{t+1,*},R)$ but does not affect the value of defaulting. Hence, we have a 1 instead of $1-\frac{U^i_c(s^{t+1,*},D)}{U^i_c(s^{t+1,*},R)}$. Notice that because $b^i_{t+1}(s^t)\geq 0$, $\tau^{b,i}(s^t)>0$ implies that holding domestic sovereign debt is subsidized. An important caveat of this formula is that because I keep the spreads on sovereign debt at zero, there are no interactions between the bond prices and the private sector's losses. In particular, I am abstracting from the doom loop dynamics that played a crucial role in the European sovereign debt crises (Farhi and Tirole (2018)). The formula only accounts for the commitment effects of domestic sovereign debt.

Asset restrictions and household default. The model also allows us to study the implications of household default in international markets following a sovereign default. I assume now that only a subset $\widetilde{\mathcal{K}}\subset\mathcal{K}$ of assets are available after sovereign default. Let $\widetilde{\mathcal{K}}^D=\mathcal{K}\setminus\widetilde{\mathcal{K}}$, then for asset $k^D\in\widetilde{\mathcal{K}}^D$, we say that the household defaults on the asset in international markets if $\widetilde{R}_{k^D}(s^t,s_{t+1})a^i_{k^D}(s^t)<0$. It is easy to verify that Proposition 5 extends to this setting for all $k\in\mathcal{K}$. For $k^D\in\widetilde{\mathcal{K}}^D$, without income effects, we now have

$$\tau^{k^D,i}(s^t)=\frac{\widetilde{R}_{k^D}(s^t,s^*)}{q_{k^D}(s^t)}q\frac{1-\eta^i s_{t+1,*}\frac{\tau^i(s^{t+1,*},R)}{1-\tau^i(s^{t+1,*},R)}}{1-\eta^i s_t\frac{\tau^i(s^t)}{1-\tau^i(s^t)}}\times\tau^B(s^t).^{25} \tag{47}$$

The formula is now as in Proposition 7 for domestic sovereign debt but with the ratio of returns between asset k^D and sovereign debt. As for domestic sovereign debt, assets $k^D\in\widetilde{\mathcal{K}}^D$ affect the value of repayment but not the value of default. Notice that if we consider two assets k, k^D that have the same prices and payoffs but with $k\in\widetilde{\mathcal{K}}$ and $k^D\in\widetilde{\mathcal{K}}^D$, we have $\tau^{k^D,i}(s^t)>\tau^{k,i}(s^t)$ because $1-\frac{U^i_c(s^{t+1,*},D)}{U^i_c(s^{t+1,*},R)}<1$. Therefore, macroprudential policies are more stringent if the households also default in international markets after sovereign default. Moreover, some domestic households may have a long position on these defaultable assets. In this case, saving in these assets is subsidized for the same commitment motive as domestic sovereign debt.

In Appendix D, I study an extension where I allow for private default without sovereign default. This case may resemble the Icelandic financial crisis, where the bank's liabilities were so large relative to the size of the local economy that a bailout from the sovereign was unfeasible. I show that the same private-debt-dependent fiscal rule can implement

[25] Notice that with household default, there could potentially exist some $\{X^i_{t+1}(s^{t+1})\}$ where the function B^{max} is unbounded. That is, the government could want to default at any level of sovereign debt. Hence, here we need to assume that for all $\{b^i,\{a^i_k(s^t)\}_{k\in\mathcal{K}}\}$ that satisfy the portfolio constraints $\mathcal{H}^i\left(b^i,\{a^i_k(s^t)\}_{k\in\mathcal{K}},s^t\right)\geq 0$, the function B^{max} is bounded.

the optimal allocation with the appropriately defined limit B^{max}. Whether the constraints preventing private or sovereign default bind, the formula in Proposition 1 continues to hold. However, when the no private default constraint binds, the macroprudential wedge is larger than the wedge on sovereign debt.

4 Quantitative analysis

In this final section, I solve and calibrate a quantitative version of the model. The objectives are: (i) quantify the effect of private debt on default probabilities, (ii) compute the private-debt-dependent fiscal rules, and (iii) quantify the size of the optimal macroprudential wedges. The fiscal rules can be computed directly from the value functions of the local government's problem. In turn, this allows us to back out the magnitude of the macroprudential wedges. Therefore, this approach will allow us to quantify the optimal policies without solving the full Ramsey problem of the central authority.

It is well known that standard quantitative sovereign debt models with short-term debt cannot match the high debt levels observed in the data. Following Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012), I extend the model by introducing long-term debt. I assume that sovereign bonds pay a geometrically declining sequence of coupons. That is, absent default, a bond pays $\lambda \in (0,1)$ at $t+1$, $\lambda(1-\lambda)$ at $t+2$, $\lambda(1-\lambda)^2$ at $t+3$... I assume foreign lenders are risk-neutral, and the risk-free rate is $\frac{1}{q}-1$, so the price of this bond satisfies

$$Q_t^B = q\mathbb{E}[\delta_{t+1}(\lambda + (1-\lambda)Q_{t+1}^B)], \tag{48}$$

where $\delta_{t+1} = 1$ if the government repays and $\delta_{t+1} = 0$ otherwise. To solve the model with long-term debt, I follow Dvorkin *et al.* (2021) and introduce extreme value shocks to the value of borrowing (see Appendix B.2).

The household side is kept similar to Section 2. So there is a representative household with utility function $U(c,y,s) = u(c - v(y/s))$, and that can borrow internationally with one-period uncontingent bonds at the risk-free price q. The household and the local government have common discount factor $\beta \in (0,1)$. After default, the household faces a productivity loss such that its productivity in state s is $s^D = h(s)$, for some function $h(s) \leq s$. I introduce the following borrowing limit on private debt

$$a_{t+1} \geq \overline{a}, \tag{49}$$

where $\bar{a} < 0$. Then, the budget constraint of the government is

$$G + Q^B(B_{t+1}, a_{t+1}, s_t)B_{t+1} = \lambda B_t + (1-\lambda)Q^B(B_{t+1}, a_{t+1}, s_t)B_t + \tau_t y_t, \tag{50}$$

where I assume that the flow government expenditures G are constant. As usual, the government maximizes the expected discounted utility of the household subject to the sequence of budget constraints of the government and the household, the bond prices (48), the borrowing limit on private debt, and the implementability conditions for the income tax (see Appendix B.1 for the formal definition of the recursive problem). For simplicity, I assume that after default, the government is permanently excluded from borrowing (both domestically and internationally). However, the household may still borrow internationally at the risk-free rate.

Parameterization. The utility function is parameterized by $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and $v(x) = \chi \frac{x^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}$, where I set $\gamma = 2$ as standard. For the Frisch elasticity, I set $\eta = \frac{1}{2}$ following Chetty *et al.* (2011) for the main calibration. However, as discussed, this parameter should be interpreted more broadly as a measure of the government's tax capacity. For this reason, I also recalibrate the model with $\eta = 3/4$ to quantify the effect of a lower tax capacity (see Appendix B.3 for the calibration details). I calibrate the disutility parameter χ to normalize the average output to one. The international risk-free rate is set to $\frac{1}{q} - 1 = 0.017$ which equals the average US quarterly interest rate. The household's discount factor β is calibrated to match the average default probability. I parameterize the productivity cost of default by $h(s) = s - \max\{\zeta_0 s + \zeta_1 s^2, 0\}$ following Chatterjee and Eyigungor (2012), where the parameters ζ_0 and ζ_1 are calibrated to match the average debt to GDP and average spreads, respectively.[26] The debt decay rate λ is chosen to generate an average maturity of 4 years (Chatterjee and Eyigungor (2012)).

I assume (log-) productivity follows an AR(1) process $log(z_t) = \rho log(z_{t-1}) + \varepsilon_t^z$, with $\mathbb{E}(\varepsilon_t^z) = 0$ and $\mathbb{E}((\varepsilon_t^z)^2) = \sigma^2$. The persistence parameter is set to $\rho = 0.95$ following Neumeyer and Perri (2005), and the standard deviation σ is calibrated to target the average standard deviation of GDP. I fix the flow government expenditures to $G = 0.14$ in order to replicate the average government expenditures to GDP. Finally, the private debt limit is set to $\bar{a} = 0.7$, which is approximately the value of private debt to GDP of Spain in

[26] Following Hatchondo and Martinez (2009), I compute spreads as follows: let $r^*(B_{t+1}, a_{t+1}, s_t) = \frac{\lambda}{Q^B(B_{t+1}, a_{t+1}, s_t)} - \lambda$, then the annualized spread is $R_s = \left(\frac{1+r^*(B_{t+1}, a_{t+1}, s_t)}{1+r_{free}}\right)^4 - 1$ where $r_{free} = \frac{1}{q} - 1$ is the risk free rate.

the years prior to the sovereign debt crises (2007-2011), see Arce (2023).[27]

Table 1 summarizes all the external and internally calibrated parameters. Table 2 summarizes the targeted moments in the data and model. The model does a reasonable job of approximating the targeted features of the data.

Table 1: Calibration parameters

External parameters

		Source
Relative risk aversion	$\gamma = 2$	Standard
Frisch elasticity	$\eta = 1/2$	Standard
Persistence productivity	$\rho = 0.95$	Neumeyer and Perri (2005)
Risk-free debt price	$q = \frac{1}{1+0.017}$	Average US interest rate (quarterly)
Debt decay rate	$\lambda = 0.05$	Average maturity
Government expenditures	$G = 0.14$	Gov. expenditures to GDP
Private debt limit	$\bar{a} = 0.7$	Private Debt to GDP Spain (2007-2011) (Arce (2023))

Internal parameters

		Moment matched
Discount factor	$\beta = 0.983$	Default probability
Disutility parameter	$\chi = 0.836$	Normalization avg. income
Productivity cost default	$\zeta_0 = -0.193$	Debt to GDP
Productivity cost default	$\zeta_1 = 0.201$	Mean Spread (Hatchondo and Martinez (2009))
Std. dev. productivity	$\sigma = 0.01$	Std. dev GDP

Table 2: Model fit

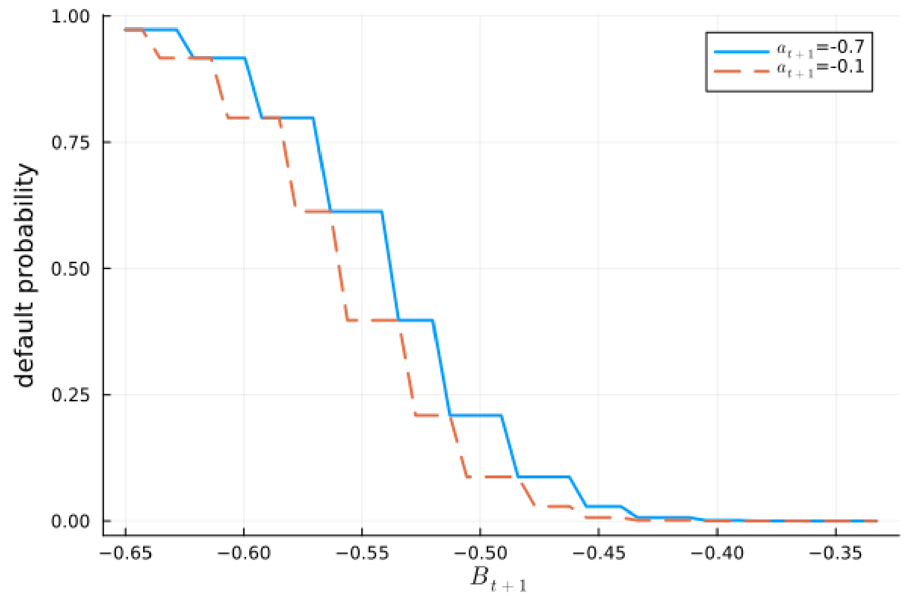
	Data	Model
Std. dev GDP	3.08	3.2
Default probability, %	4.5	3.7
Debt to GDP %	-32.5	-28.5
Mean Spread %	7.4	8.3
Avg. GDP (normalization)	1.0	1.0

Private debt and default probability. Figure 1 plots the default probability as a function of the quantity of sovereign debt for high ($a_t = -0.7$) and low ($a_t = -0.1$) levels of

[27]Notice that for both G and $\bar{a}$ we can assign the value directly because the average output is normalized to one.

private debt. When positive, the default probability is always higher when private debt is high, which verifies that the mechanism of the two-period model goes through in this quantitative model. The effect of private debt on the default probability is much larger in the calibration with low tax capacity (see Figure 5 in Appendix C).

Figure 1: Effects of private debt on default probability



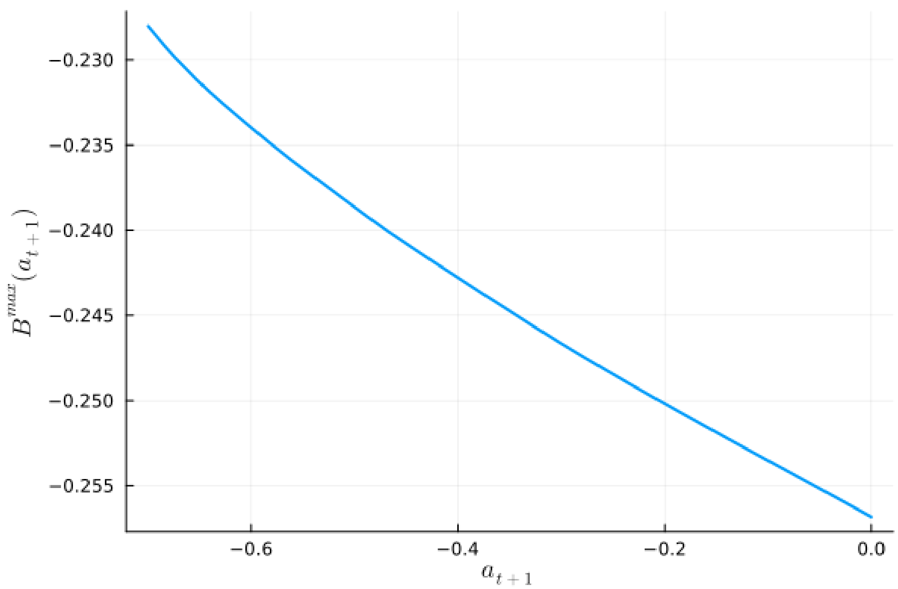
Private-debt-dependent fiscal rules. In Figure 2, I plot the private-debt-dependent fiscal rule for a fixed productivity level $s_t = 1$. That is, for every level of private debt in the x-axis, the y-axis contains the maximum level of sovereign debt that guarantees that the government would not default in any future state, which can be computed directly from the value function.[28] As expected, the borrowing limit is decreasing, so the government is allowed to borrow more when private debt is low. The slope will be steeper if private debt has a bigger effect on the default probabilities. So, decreasing private debt would allow for a larger increase in sovereign borrowing. Going from the (calibrated) borrowing limit on private debt ($a_{t+1} = -0.7$ or 70% of the average GDP) to zero private debt ($a_{t+1} = 0.0$) allows the government to borrow 11.2% more. Interestingly, it is approximately linear, so the following rule gives a good approximation: $B^{max} \approx -0.26 - 0.042a_{t+1}$. In Figure 4 in Appendix C, I plot several fiscal rules that allow for a positive default probability, as in Section 2.6. As expected, allowing for a positive default probability shifts the line up

[28]Notice that, numerically, the grid of productivity has a lower and upper bound. With the original AR(1) process with unbounded support, there may always be a low enough s such that the government may default for any $B_{t+1} < 0$.

so that the government can borrow more for every level of private debt. However, the slope of the fiscal rule appears to be approximately independent of the allowed default probability. The fiscal rule also takes a similar shape in the calibration with $\eta = 0.75$ (see Figure 6 in Appendix C).

The slope of the computed private-debt-dependent fiscal rule should be taken as a lower bound. With extra amplification channels of private debt, such as pecuniary externalities (Arce (2023)), we should see larger effects of private on the default incentives and a steeper slope. Similarly, the slope should also increase if the household can default in international markets after sovereign default.

Figure 2: Private-debt-dependent fiscal rule



Note: For every a_{t+1}, I find B^{max} by solving the root $V^{R,int}(B^{max}, a_{t+1}, \underline{s}) - V^{D,int}(a_{t+1}, \underline{s})$ where $V^{R,int}$ and $V^{D,int}$ are the (linearly) interpolated value functions and $\underline{s}$ is the lowest productivity level in the grid.

Size of the macroprudential wedges. Using Proposition 1, we can also infer the magnitude of the macroprudential wedges without solving for the allocation with no default constraints.[29] For average levels of output and productivity, the deadweight loss from the income tax is roughly $\eta \frac{\tau}{1-\tau} \approx 0.1$, and the derivative of the fiscal rule $\frac{\partial B^{max}}{\partial a} \approx 0.043$. This implies that the size of the macroprudential wedge relative to the wedge on sovereign

[29]Because the solvency constraints imply that the default probabilities are zero everywhere, the price of the long-term bond is always $Q = \frac{\lambda q}{1-q(1-\lambda)}$. So an increase in sovereign debt increases resources at t by $\frac{\lambda q}{1-q(1-\lambda)}$ and increases the repayment at $t+1$ by $\lambda + (1-\lambda)\frac{\lambda q}{1-q(1-\lambda)} = \frac{\lambda}{1-q(1-\lambda)}$. As a result, the same wedge formulas are also satisfied in this model with long-term debt.

debt is approximately $\frac{\tau^a}{\tau^B} \approx 4.7\%$. With a lower tax capacity ($\eta = 0.75$), the deadweight loss increases to 0.14, but the derivative of the fiscal rule decreases to 0.038, implying a similar ratio of wedges of approximately $\frac{\tau^a}{\tau^B} \approx 4.4\%$.[30] As for the fiscal rule, we should also interpret this value as a lower bound for the size of the optimal macroprudential wedges.

Macroprudential wedge in local government's problem ($\tau^{a,LG}$). As shown in Section 2.5, the local government wants to conduct macroprudential policies by itself when the default probability is positive. With long-duration sovereign debt, this macroprudential wedge is equal to

$$\tau_t^{a,LG} = \frac{1}{1-\eta\frac{\tau_t}{1-\tau_t}} q^{-1} \frac{\partial Q^B}{\partial a_{t+1}}(-(B_{t+1}-(1-\lambda)B_t)),$$

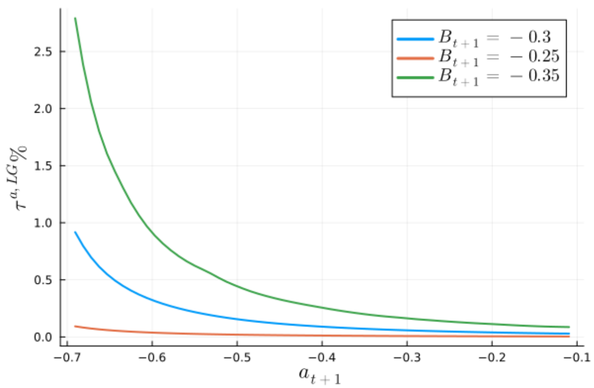
which now depends on the net debt issuance at t. I compute this wedge in Figure 3 (Panel 3a) along with the derivative $\frac{\partial Q^B}{\partial a_{t+1}}$ (Panel 3b) for multiple levels of sovereign and private debt starting from $B_t = -0.25$. The optimal wedge can grow substantially (up to 2.5%) when the sovereign debt issuance is high and a_{t+1} approaches the borrowing limit. In Figure 7 in Appendix C, I plot this wedge and the derivative $\frac{\partial Q^B}{\partial a}$ for the calibration with $\eta = 0.75$. The wedges are slightly higher in this case; in particular, they increase to 3% for high levels of sovereign and private debt. Although smaller, these wedges are of the same order of magnitude as the optimal macroprudential taxes computed by Arce (2023) for Spain between 2008 and 2015. Arce (2023) analyzes a model with pecuniary externalities in private borrowing but where the government can use lump-sum transfers. Hence, these results suggest that distortionary taxation is a potentially important amplification mechanism for the size of the optimal macroprudential interventions.[31]

[30] Note, however, that we should interpret this comparative static with caution as with $\eta = 0.75$, the model actually does a much better at matching the targeted levels of public debt and default probability, which could affect the estimated derivative of the fiscal rule.

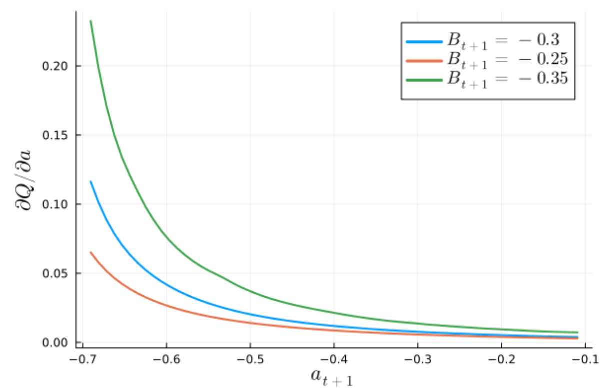
[31] In Figure 7 in Appendix C, I plot this wedge and the derivative $\frac{\partial Q^B}{\partial a}$ for the calibration with $\eta = 0.75$. The wedges are higher in this case, but the difference is very small.

Figure 3: Macroprudential wedges in local government's problem

(a) Macroprudential wedges (in %)



(b) Derivatives sovereign debt price with respect to private debt



Note: To compute the wedges I fix $B_t = -0.25$. For every B_{t+1}, I approximate the derivative $\frac{\partial Q^B}{\partial a_{t+1}}$ with a finite difference method.

5 Conclusion

The European Sovereign debt crises (2010-2012) showcased how excessive leverage of the private sector and financial instability can threaten sovereign debt sustainability and trigger a sovereign debt crisis. Notably, Ireland and Spain experienced large increases in sovereign debt spreads while having low levels of public debt in the preceding years and compiling with the limits imposed by the Stability and Growth Pact. Consequently, the shortcomings of Maastricht-type rules and the risks of insufficient macroprudential policy and supervision became clear.

Motivated by these events, this paper studies the optimal joint design of fiscal rules and macroprudential policies under sovereign default risk. These policies are designed by a central authority that internalizes externalities from sovereign default of a local government. I start by analyzing a stylized two-period model of a small open economy where both a local government and a representative household can borrow externally. The model yields two main insights. First, it provides a rationale for macroprudential policies based on the default externalities and distortionary taxation. Second, it shows that sovereign debt limits that are a function of the quantity of private debt (private-debt-dependent fiscal rules) can implement the optimal allocation.

These results are then generalized to a dynamic model with heterogeneous households, aggregate shocks, and multiple state-contingent securities. I show how to define the private-debt-dependent fiscal rules in this setting, which generalizes the solvency constraints that are not too tight of Alvarez and Jermann (2000).

Finally, I solve and calibrate a quantitative version of the model to compute the private-debt-dependent fiscal rules and the size of the macroprudential wedges. Using the results from the theory parts, I can compute the private-debt-dependent fiscal rules and infer the size of the macroprudential wedges without solving the Ramsey problem of the central authority. Future work could extend this approach of computing the fiscal rules and macroprudential to richer models. For example, it could study how household leverage interacts with the bank-sovereign doom loop and quantify how the optimal fiscal rules depend on both the households' and banks' balance sheets.

References

ABRAHÁM, A., CÁRCELES POVEDA, E., LIU, Y. and MARIMON, R. (2018). On the optimal design of a financial stability fund.

ACHARYA, V., DRECHSLER, I. and SCHNABL, P. (2014). A pyrrhic victory? bank bailouts and sovereign credit risk. *The Journal of Finance*, **69** (6), 2689–2739.

AGUIAR, M., AMADOR, M., FARHI, E. and GOPINATH, G. (2015). Coordination and crisis in monetary unions. *The Quarterly Journal of Economics*, **130** (4), 1727–1779.

ALVAREZ, F. and JERMANN, U. J. (2000). Efficiency, equilibrium, and asset pricing with risk of default. *Econometrica*, **68** (4), 775–797.

ARCE, F. (2023). Private overborrowing under sovereign risk.

ARELLANO, C., BAI, Y. and BOCOLA, L. (2017). *Sovereign default risk and firm heterogeneity*. Tech. rep., National Bureau of Economic Research.

AUCLERT, A. and ROGNLIE, M. (2014). Monetary union begets fiscal union. In *Working paper*.

BAILY, M. N. (1978). Some aspects of optimal unemployment insurance. *Journal of public Economics*, **10** (3), 379–402.

BERGER, D., BOCOLA, L. and DOVIS, A. (2023). Imperfect risk sharing and the business cycle. *The Quarterly Journal of Economics*, **138** (3), 1765–1815.

BERRIEL, R., GONZALEZ-AGUADO, E., KEHOE, P. J. and PASTORINO, E. (2023a). *Fiscal Federalism and Monetary Unions*. Tech. rep., National Bureau of Economic Research.

—, —, — and — (2023b). Is a fiscal union optimal for a monetary union? *Journal of Monetary Economics*.

BIANCHI, J. (2011). Overborrowing and systemic externalities in the business cycle. *American Economic Review*, **101** (7), 3400–3426.

BOCOLA, L. (2016). The pass-through of sovereign risk. *Journal of Political Economy*, **124** (4), 879–926.

BOVA, E., KINDA, T., MUTHOORA, P. and TOSCANI, F. (2015). Fiscal rules at a glance. *International Monetary Fund*.

BRONER, F., MARTIN, A. and VENTURA, J. (2021). On public spending and economic unions. *IMF Economic Review*, **69**, 122–154.

BRUNNERMEIER, M. K., GARICANO, L., LANE, P. R., PAGANO, M., REIS, R., SANTOS, T., THESMAR, D., VAN NIEUWERBURGH, S. and VAYANOS, D. (2016). The sovereign-bank diabolic loop and esbies. *American Economic Review*, **106** (5), 508–12.

— and REIS, R. (2019). *A crash course on the euro crisis*. Tech. rep., National Bureau of Economic Research.

CHARI, V. V. and KEHOE, P. J. (2007). On the need for fiscal constraints in a monetary union. *Journal of Monetary Economics*, **54** (8), 2399–2408.

CHATTERJEE, S. and EYIGUNGOR, B. (2012). Maturity, indebtedness, and default risk. *American Economic Review*, **102** (6), 2674–2699.

CHETTY, R. (2006). A general formula for the optimal level of social insurance. *Journal of Public Economics*, **90** (10-11), 1879–1901.

—, GUREN, A., MANOLI, D. and WEBER, A. (2011). Are micro and macro labor supply elasticities consistent? a review of evidence on the intensive and extensive margins. *American Economic Review*, **101** (3), 471–475.

CHODOROW-REICH, G., KARABARBOUNIS, L. and KEKRE, R. (2019). *The macroeconomics of the Greek depression*. Tech. rep., National Bureau of Economic Research.

DÁVILA, E. and KORINEK, A. (2018). Pecuniary externalities in economies with financial frictions. *The Review of Economic Studies*, **85** (1), 352–395.

DOVIS, A. and KIRPALANI, R. (2020). Fiscal rules, bailouts, and reputation in federal governments. *American Economic Review*, **110** (3), 860–88.

DVORKIN, M., SÁNCHEZ, J. M., SAPRIZA, H. and YURDAGUL, E. (2021). Sovereign debt restructurings. *American Economic Journal: Macroeconomics*, **13** (2), 26–77.

EIJFFINGER, S. C., KOBIELARZ, M. L. and URAS, B. R. (2018). Sovereign default, exit and contagion in a monetary union. *Journal of International Economics*, **113**, 1–19.

FARHI, E. and TIROLE, J. (2018). Deadly embrace: Sovereign and financial balance sheets doom loops. *The Review of Economic Studies*, **85** (3), 1781–1823.

— and WERNING, I. (2016). A theory of macroprudential policies in the presence of nominal rigidities. *Econometrica*, **84** (5), 1645–1704.

— and — (2017). Fiscal unions. *American Economic Review*, **107** (12), 3788–3834.

FERRARI, A., MARIMON, R. and SIMPSON-BELL, C. (2020). Fiscal and currency union with default and exit. *Available at SSRN 3673917*.

GOURINCHAS, P.-O., MARTIN, P. and MESSER, T. E. (2020). *The economics of sovereign debt, bailouts and the Eurozone crisis*. Tech. rep., National Bureau of Economic Research.

—, PHILIPPON, T. and VAYANOS, D. (2017). The analytics of the greek crisis. *NBER macroeconomics Annual*, **31** (1), 1–81.

HATCHONDO, J. C. and MARTINEZ, L. (2009). Long-duration bonds and sovereign defaults. *Journal of International Economics*, **79** (1), 117–125.

—, — and ROCH, F. (2022). Fiscal rules and the sovereign default premium. *American Economic Journal: Macroeconomics*, **14** (4), 244–73.

JESKE, K. (2006). Private international debt with risk of repudiation. *Journal of political Economy*, **114** (3), 576–593.

KAAS, L., MELLERT, J. and SCHOLL, A. (2020). Sovereign and private default risks over the business cycle. *Journal of International Economics*, **123**, 103293.

KIM, Y. J. and ZHANG, J. (2012). Decentralized borrowing and centralized default. *Journal of International Economics*, **88** (1), 121–133.

LANE, P. R. (2012). The european sovereign debt crisis. *Journal of economic perspectives*, **26** (3), 49–68.

LIU, Y., MARIMON, R., WICHT, A. *et al.* (2022). *Making sovereign debt safe with a financial stability fund*. Tech. rep.

LORENZONI, G. (2008). Inefficient credit booms. *The Review of Economic Studies*, **75** (3), 809–833.

MARTIN, P. and PHILIPPON, T. (2017). Inspecting the mechanism: Leverage and the great recession in the eurozone. *American Economic Review*, **107** (7), 1904–37.

MENDOZA, E. G. and YUE, V. Z. (2012). A general equilibrium model of sovereign default and business cycles. *The Quarterly Journal of Economics*, **127** (2), 889–946.

NEUMEYER, P. A. and PERRI, F. (2005). Business cycles in emerging economies: the role of interest rates. *Journal of monetary Economics*, **52** (2), 345–380.

SACHS, D., TSYVINSKI, A. and WERQUIN, N. (2020). Nonlinear tax incidence and optimal taxation in general equilibrium. *Econometrica*, **88** (2), 469–493.

SUBLET, G. (2022). Fiscal rules with discretion for an economic union.

TIROLE, J. (2015). Country solidarity in sovereign crises. *American Economic Review*, **105** (8), 2333–2363.

WRIGHT, M. L. (2006). Private capital flows, capital controls, and default risk. *Journal of International Economics*, **69** (1), 120–149.

A Proofs and Derivations

A.1 Two period model

Substituting the implementability condition (3), we have the following first-order conditions for problem local government's problem:

c_1:

$$u_1' = \lambda_1^{HH} \tag{51}$$

y_1:

$$u_1' v'(y_1) = \lambda_1^G \left(1 - v''(y_1)y_1 - v'(y_1)\right) + \lambda_1^{HH} \left(v''(y_1)y_1 + v'(y_1)\right) \tag{52}$$

Substituting the FOC for c_1, using $1 - \frac{1}{v'(y_1)} = -\frac{\tau_1}{1-\tau_1}$ and rearranging we have

$$\lambda_1^{HH} = \left(1 - \eta \frac{\tau_1}{1-\tau_1}\right) \lambda_1^G \tag{53}$$

Letting χ be the multiplier on the solvency constraint (19), the FOC for sovereign debt in the central authority's problem is

$$\lambda_1^G \left(1 - \frac{\chi}{q\lambda_1^G}\right) = q^{-1}\beta \frac{\partial V_2^R}{\partial B_2}, \tag{54}$$

and using the definition of the sovereign debt wedge we get $\tau^B = \frac{\chi}{q\lambda_1^G}$. Similarly, the FOC for private debt is

$$\lambda_1^{HH} \left(1 - \frac{\chi}{q\lambda_1^{HH}} \frac{\partial - B^{max}(a_2)}{\partial a_2}\right) = q^{-1}\beta \frac{\partial V_2^R}{\partial a_2} \tag{55}$$

and so the wedge for private debt is $\tau^a = \frac{\chi}{q\lambda_1^{HH}} \frac{\partial - B^{max}(a_2)}{\partial a_2}$.

Finally, the envelope conditions for the second period are:

$$\frac{\partial V_2^R}{\partial B_2} = \frac{u_2'(R)}{1 - \eta \frac{\tau_2}{1-\tau_2}} \tag{56}$$

$$\frac{\partial V_2^R}{\partial a_2} = u_2'(R) \tag{57}$$

$$\frac{\partial V_2^D}{\partial b_2} = u_2'(D) \tag{58}$$

Changes in default probabilities. First, notice that we can write the default probability as $\delta(B_2, a_2) = \mathbb{P}(\theta \leq \theta^*(B_2, a_2))$, where $\theta^*(B_2, a_2)$ satisfies

$$V_2^R(B_2, a_2) = V_2^D(a_2, \theta^*(B_2, a_2)). \tag{59}$$

Then note that, for $x \in \{B_2, a_2\}$, $\frac{d\delta(B_2,a_2)}{dx} = \frac{d\theta^*}{dx}\pi(\theta^*)$. Differentiating in the definition of θ^*, we have

$$\frac{d\theta^*}{dB_2} = \frac{\frac{\partial V_2^R}{\partial B_2}}{\frac{\partial V_2^D}{\partial \theta}}$$

$$\frac{d\theta^*}{da_2} = \frac{\frac{\partial V_2^R}{\partial a_2} - \frac{\partial V_2^D}{\partial a_2}}{\frac{\partial V_2^D}{\partial \theta}},$$

where $\frac{\partial V_2^D}{\partial \theta} = -1$. Using the envelope conditions (56)-(58), we obtain the expressions in the main text.

Proof of Proposition 1. Combining the wedges derived above to substitute for χ and using (11), we can derive the equation of the proposition. The inequality $\tau^B > \tau^a$ follows directly from the first wedge formula in Proposition 2 and $\frac{u_2'(D)}{u_2'(R)} < 1$. Moreover, using the definition of $B^{max}(a_2)$ and differentiating we have

$$\frac{\partial - B^{max}(a_2)}{\partial a_2} = \frac{\frac{\partial V^R}{\partial a_2} - \frac{\partial V^D}{\partial a_2}}{\frac{\partial V^R}{\partial B_2}} = \frac{u_2'(R) - u_2'(D)}{u_2'(R)} \left(1 - \eta \frac{\tau_2}{1 - \tau_2}\right). \tag{60}$$

Then, using $\frac{u_2'(D)}{u_2'(R)} < 1$ and that $\left(1 - \eta \frac{\tau_t}{1-\tau_t}\right) > 0$ for $t \in \{1,2\}$ at the increasing part of the Laffer curve (it cannot be optimal to be in the decreasing part of the Laffer curve in this model), it follows that $\tau^a > 0$ whenever $\tau^B > 0$.

Proof of Proposition 2. Substituting (60) into the wedge formula of Proposition 1 gives

$$\tau^a = \left(1 - \frac{u_2'(D)}{u_2'(R)}\right) \frac{1 - \eta \frac{\tau_2}{1-\tau_2}}{1 - \eta \frac{\tau_1}{1-\tau_1}} \tau^B. \tag{61}$$

From the wedge definitions, we have the tax smoothing formula: $\frac{1-\eta \frac{\tau_1}{1-\tau_1}}{1-\eta \frac{\tau_2}{1-\tau_2}} = \frac{(1-\tau^B)}{(1-\tau^a)}$. Substituting gives the first equation in the proposition.

To derive the second equation, we use the derivations above on the change in the default probabilities to show

$$\frac{\partial \delta(B_2, a_2)/\partial a_2}{\partial \delta(B_2, a_2)/\partial B_2}\Big|_{\delta(B_2,a_2)=0} = \frac{\frac{d\theta^*}{da_2}\pi(\theta^*)}{\frac{d\theta^*}{dB_s}\pi(\theta^*)}\Big|_{\theta^*=\underline{\theta}} = \frac{\frac{\frac{\partial V_2^R}{\partial a_2} - \frac{\partial V_2^D}{\partial a_2}}{\frac{\partial V_2^D}{\partial \theta}}}{\frac{\frac{\partial V_2^R}{\partial B_2}}{\frac{\partial V_2^D}{\partial \theta}}} = \frac{\frac{\partial V_2^R}{\partial a_2} - \frac{\partial V_2^D}{\partial a_2}}{\frac{\partial V_2^R}{\partial B_2}},$$

which is equal to $\frac{\partial - B^{max}(a_2)}{\partial a_2}$.

Proof of Proposition 3. I start by deriving the optimal wedges in the central authority's problem. Then, I show how we can obtain the same allocation (i.e., the same implicit wedges) with the private-debt-dependent fiscal rules and choosing the optimal θ^*. The FOCs of the central authority's problem are

$$\lambda^G(Q^B + \frac{\partial Q^B}{\partial B_2}B_2) + S\pi(\theta^*)\frac{\partial \theta^*}{\partial B_2} = \beta\frac{\partial V_2(B_2, a_2)}{\partial B_2},$$

and

$$\lambda^{HH}q + S\pi(\theta^*)\frac{\partial \theta^*}{\partial a_2} + \lambda^G\frac{\partial Q^B}{\partial a_2}B_2 = \beta\frac{\partial V_2(B_2, a_2)}{\partial a_2}.$$

Using $\frac{\partial \theta^*}{\partial B_2} = -\frac{u_2'(R)}{1-\eta\frac{\tau_2}{1-\tau_2}}$ and $\frac{\partial \theta^*}{\partial a_2} = u_2'(D) - u_2'(R)$, collecting terms and substituting the multipliers, we get that the wedges are

$$\tau^B = \frac{1}{(Q^B + \frac{\partial Q^B}{\partial B_2}B_2)}\frac{1-\eta\frac{\tau_1}{1-\tau_1}}{1-\eta\frac{\tau_2}{1-\tau_2}}\frac{u_2'(R)}{u_1'}\pi(\theta^*)S, \tag{62}$$

and

$$\tau^a = \frac{1}{qu_1'}(u_2'(R) - u_2'(D))\pi(\theta^*)S. \tag{63}$$

Combining the two equations, we can derive a similar relation between the two wedges that does not depend on S or $\pi(\theta^*)$:

$$\tau^a = \frac{Q^B + \frac{\partial Q^B}{\partial B_2}B_2}{q}\frac{1-\eta\frac{\tau_2}{1-\tau_2}}{1-\eta\frac{\tau_1}{1-\tau_1}}\left(1 - \frac{u_2'(D)}{u_2'(R)}\right)\tau^B, \tag{64}$$

which is the second equation in the proposition.

The FOC condition for $\tilde{\theta}$ gives

$$\chi = \frac{1}{\frac{\partial B^{max}}{\partial \theta^*}}\pi(\theta^*)(-S) = (-\frac{\partial V_2^R}{\partial B_2})\pi(\theta^*)(-S) = \frac{u_2'(R)}{1-\eta\frac{\tau_2}{1-\tau_2}}\pi(\theta^*)S$$

where the first equality follows from differentiating over θ^* in (26). From the first order conditions of B_2 and a_2 in the local governments problem with constraint (27) we obtain

$$\tau^B = \frac{1}{\frac{u_1'}{1-\eta\frac{\tau_1}{1-\tau_1}}(Q^B + \frac{\partial Q^B}{\partial B_2}B_2)}\chi,$$

and

$$\tau^a = \frac{1}{qu_1'}(-\frac{\partial B^{max}}{\partial a_2})\chi.$$

Substituting χ we recover the same wedges as above solving directly the central authority's problem (equations (62) and (63)), which shows that this fiscal rule implements the optimal allocation. Finally, combining the two to substitute for χ we also derive the first equation in the proposition.

Proof of Proposition 4. With ε^B and ε^a and without default, the FOC and envelope conditions give the following Euler equations

$$\frac{u_1'}{1-\eta\frac{\tau_1}{1-\tau_1}}q(1-\varepsilon^B) = \beta\frac{u_2'(R)}{1-\eta\frac{\tau_2}{1-\tau_2}} \tag{65}$$

$$u_1'q(1-\varepsilon^a) = \beta u_2'(R). \tag{66}$$

Combining the two

$$\frac{1-\eta\frac{\tau_1}{1-\tau_1}}{1-\eta\frac{\tau_2}{1-\tau_2}} = \frac{(1-\varepsilon^B)}{(1-\varepsilon^a)}. \tag{67}$$

Taking logs and rearranging, we can derive the expression in the main text.

A.2 General model

The lagrangian of the planning problem is, substituting the implementability conditions and letting $MRS^i(s^t) = -\frac{U^i_y(s^t)}{U^i_c(s^t)}$,

$$\begin{aligned}
\mathcal{L} = &\sum_{i\in\mathcal{I}} \omega^i \pi^i U^i(c^i(s^t), y^i(s^t), s_t) + \beta \mathbb{E}_{s_{t+1}|s^t} V^R_{t+1}\left(B_{t+1}(s^t), \{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}}, s^{t+1}\right) \\
&- \lambda^G(s^t)\left(G(s^t) + qB_{t+1}(s^t) - \sum_i \pi^i(1 - MRS^i(s^t))y^i(s^t) - B_t(s^{t-1})\right) \\
&- \sum_{i\in\mathcal{I}} \lambda^{HH,i}(s^t)\left(c^i(s^t) + \sum_k q_k(s^t)a^i_k(s^t) + qb^i_{t+1}(s^t) - MRS^i(s^t)y^i(s^t) - I^i_t(s^t) - b^i_t(s^{t-1})\right) \\
&+ \sum_{i\in\mathcal{I}} \int \mu^{I,i}(s^{t+1})\left(\sum_{k\in\mathcal{K}} \widetilde{R}_k(s^t, s_{t+1})a^i_k(s^t) - I^i_{t+1}(s^{t+1})\right) ds_{t+1} \\
&+ \sum_{i\in\mathcal{I}} \langle \mu^{\mathcal{H},i}(s^t), \mathcal{H}^i\left(b^i, \{a^i_k(s^t)\}_{k\in\mathcal{K}}, s^t\right)\rangle \\
&+ \chi(s^t)\left(B_{t+1}(s^t) - B^{max}(\{X^i_{t+1}(s^{t+1})\}_{i\in\mathcal{I}, s_{t+1}})\right)
\end{aligned}$$

The FOCs are

$c^i(s^t)$:

$$\omega^i \pi^i U^i_c(s^t) = \lambda^{HH,i}(s^t) + \left(\lambda^G(s^t)\pi^i - \lambda^{HH,i}(s^t)\right) MRS^i_c(s^t)y^i(s^t) \tag{68}$$

$y^i(s^t)$:

$$\omega^i \pi^i(-U^i_y(s^t)) = \lambda^G(s^t)\pi^i + \left(\lambda^{HH,i}(s^t) - \lambda^G(s^t)\pi^i\right)\left(MRS^i_y(s^t)y^i(s^t) + MRS^i(s^t)\right) \tag{69}$$

$B_{t+1}(s^t)$:

$$\lambda^G(s^t)\left(1 - \frac{\chi(s^t)}{q\lambda^G(s^t)}\right) = \beta q^{-1} \mathbb{E}_{s_{t+1}|s^t} \frac{\partial V^R(s^{t+1})}{\partial B_{t+1}(s^t)} \tag{70}$$

using the definition of the wedge on sovereign debt we get $\tau^B(s^t) = \frac{\chi(s^t)}{q\lambda^G(s^t)}$.

$a^i_k(s^t)$:

$$q_k(s^t)\lambda^{HH,i}(s^t) = \int \mu^{I,i}(s^{t+1})\widetilde{R}_k(s^t, s_{t+1})ds_{t+1} + \langle \mu^{\mathcal{H},i}(s^t), \mathcal{H}^i_{a^i_k}(s^t)\rangle \tag{71}$$

$I^i_{t+1}(s^{t+1})$:

$$\beta\pi(s_{t+1}|s^t)\frac{\partial V^R_{t+1}(s^{t+1})}{\partial I^i_{t+1}(s^{t+1})} - q\chi(s^t)\frac{\partial B^{max}(\{X^i_{t+1}(s^{t+1})\})}{\partial I^i_{t+1}(s^{t+1})} = \mu^{I,i}(s^{t+1}) \tag{72}$$

$b^i_{t+1}(s^t)$:

$$\beta \mathbb{E}_{s_{t+1}|s^t} \frac{\partial V^R_{t+1}(s^{t+1})}{\partial b^i_{t+1}(s^t)} = q\lambda^{HH,i}(s^t)\left(1 - \frac{\chi(s^t)}{q\lambda^{HH,i}(s^t)} \frac{\partial - B^{max}(\{X^i_{t+1}(s^{t+1})\})}{\partial b^i_{t+1}(s^t)}\right) \tag{73}$$

and so $\tau^{b,i}(s^t) = \frac{\chi(s^t)}{q\lambda^{HH,i}(s^t)} \frac{\partial - B^{max}(\{X^i_{t+1}(s^{t+1})\})}{\partial b^i_{t+1}(s^t)}$.

Proof of Lemma 1. As discussed in the main text, because $G^D(s^t) \geq G(s^t)$ and $h(s) \leq s$ and the local government's value is decreasing in the government expenditure and increasing in productivity, it cannot be optimal to default with $B_{t+1} = 0$, i.e.

$$V^R_{t+1}\left(0, \{X^i_{t+1}(s^{t+1})\}_{i \in \mathcal{I}}, s^{t+1}\right) > V^D_{t+1}\left(\{I^i_{t+1}(s^{t+1})\}_{i \in \mathcal{I}}, s^{t+1}\right).$$

Therefore, we must have $B^{max} \leq 0$.

To show the envelope formula, first notice that we can redefine the function $\overline{B}$ to depend on the positions in all states: $\overline{B}(\{X^i_{t+1}(s^{t+1})\}_{i \in \mathcal{I}, s_{t+1}}, s^{t+1})$. For exposition we set $b_{t+1}() = 0$ so we can write $\overline{B}$ only as a function of $I^i_{t+1}()$, but the same steps apply to $b_{t+1}()$. For an arbitrary $i' \in \mathcal{I}$, we consider variations $I^{i'}_{t+1} + \mu\gamma$ for $\mu > 0$ and some arbitrary function γ. Define the Gateaux derivative of B^{max} for dimension i' as

$$\widehat{B}^{max}_{i'}(\{I^i_{t+1}\}_i) = \lim_{\mu \to 0} \frac{B^{max}(\{I^i_{t+1}\}_{i \neq i'}, I^i_{t+1} + \mu\gamma) - B^{max}(\{I^i_{t+1}\}_{i \neq i'}, I^i_{t+1})}{\mu} \tag{74}$$

Notice also that for every, $\{I^i_{t+1}\}$, at the points where $\{I^i_{t+1}\}$ are differentiable, s^*_{t+1} solves the FOC

$$\left[\sum_i \frac{\partial \overline{B}}{\partial I^i_{t+1}(s_{t+1})} \frac{\partial I^i_{t+1}(s_{t+1})}{\partial s_{t+1}} + \frac{\partial \overline{B}}{\partial s_{t+1}}\right]\Big|_{s_{t+1} = s^*_{t+1}} = 0.$$

Hence, for any (differentiable) γ, the function $B^{max}(\{I^i_{t+1}\}_{i \neq i'}, I^i_{t+1} + \mu\gamma)$ parameterized by μ satisfies the – traditional– envelope theorem. Then, differentiating with respect to μ and evaluating at $\mu = 0$,

$$\widehat{B}^{max}_{i'}(\{I^i_{t+1}\}_i) = \int \frac{\partial \overline{B}(\{I^i_{t+1}\}_{i \in \mathcal{I}}, s^*_{t+1})}{\partial I^{i'}_{t+1}(s^t, \tilde{s}_{t+1})} \gamma(\tilde{s}_{t+1}) d\tilde{s}_{t+1} \tag{75}$$

where $s^*_{t+1} = \underset{s_{t+1}}{\text{argmax}} \overline{B}(\{X^i_{t+1}(s^{t+1})\}_{i \in \mathcal{I}}, s^{t+1})$. Then equation (75), implies, in particular, that

$$\frac{\partial B^{max}(\{I^i_{t+1}\}_i)}{\partial I^{i'}_{t+1}(s^{t+1})} = \frac{\partial \overline{B}(\{I^i_{t+1}\}_{i\in\mathcal{I}}, s^*_{t+1})}{\partial I^{i'}_{t+1}(s^t, s_{t+1})}.$$

We can show this following Sachs *et al.* (2020) by appropriately defining a sequence of smooth functions $\gamma_{s^*_{t+1},\epsilon}$ such that

$$\lim_{\epsilon\to 0}\gamma_{s_{t+1},\epsilon}(\tilde{s}_{t+1}) = \delta(\tilde{s}_{t+1} - s_{t+1}),$$

where δ is the Dirac delta function. Following Sachs *et al.* (2020), this can be obtained by defining an absolutely integrable and smooth function $\gamma_{s_{t+1}}(\tilde{s}_{t+1})$ with compact support and $\int \gamma_{s_{t+1}}(\tilde{s}_{t+1})d\tilde{s}_{t+1} = 1$ and letting $\gamma_{s_{t+1},\epsilon}(\tilde{s}_{t+1}) = \epsilon^{-1}\gamma_{s_{t+1}}(\frac{\tilde{s}_{t+1}}{\epsilon})$. Then, by the dominated convergence theorem

$$\begin{aligned}\lim_{\epsilon\to 0}\int \frac{\partial \overline{B}(\{I^i_{t+1}\}_{i\in\mathcal{I}}, s^*_{t+1})}{\partial I^{i'}_{t+1}(s^t, \tilde{s}_{t+1})}\gamma_{s_{t+1},\epsilon}(\tilde{s}_{t+1})d\tilde{s}_{t+1} &= \int \frac{\partial \overline{B}(\{I^i_{t+1}\}_{i\in\mathcal{I}}, s^*_{t+1})}{\partial I^{i'}_{t+1}(s^t, \tilde{s}_{t+1})}\lim_{\epsilon\to 0}\gamma_{s_{t+1},\epsilon}(\tilde{s}_{t+1})d\tilde{s}_{t+1} \\ &= \frac{\partial \overline{B}(\{I^i_{t+1}\}_{i\in\mathcal{I}}, s^*_{t+1})}{\partial I^{i'}_{t+1}(s^t, s_{t+1})}.\end{aligned}$$

Finally, if $s_{t+1} \neq s^*_{t+1}$, we have $\frac{\partial \overline{B}(\{I^i_{t+1}\}_{i\in\mathcal{I}}, s^*_{t+1})}{\partial I^{i'}_{t+1}(s^t, s_{t+1})} = 0$, and if $s_{t+1} = s^*_{t+1}$:

$$\frac{\partial \overline{B}(\{I^i_{t+1}\}_{i\in\mathcal{I}}, s^*_{t+1})}{\partial I^{i'}_{t+1}(s^t, s_{t+1})} = -\frac{\frac{\partial V^R_{t+1}(s^{t+1,*})}{\partial I^i_{t+1}(s^{t+1,*})} - \frac{\partial V^D_{t+1}(s^{t+1,*})}{\partial I^i_{t+1}(s^{t+1,*})}}{\frac{\partial V^R_{t+1}(s^{t+1,*})}{\partial B_{t+1}}},$$

so the derivative exists because $\frac{\partial V^R_{t+1}(s^{t+1,*})}{\partial B_{t+1}} \neq 0$.

Proof of Proposition 5. Combining the FOCs (71) and (72)

$$q_k(s^t)\lambda^{HH,i}(s^t)\left(1 - \int \frac{\widetilde{R}_k(s^t, s_{t+1})}{q_k(s^t)} q \frac{\partial - B^{max}(\{X^i_{t+1}(s^{t+1})\})}{\partial I^i_{t+1}(s^{t+1})}\frac{1}{\lambda^{HH,i}(s^t)}\chi(s^t)ds_{t+1}\right) \tag{76}$$

$$= \beta\mathbb{E}_{s_{t+1}|s^t}\widetilde{R}_k(s^t, s_{t+1})\frac{\partial V^R_{t+1}(s^{t+1})}{\partial I^i_{t+1}(s^{t+1})} + \langle \mu^{\mathcal{H},i}(s^t), \mathcal{H}^i_{a^i_k}(s^t)\rangle \tag{77}$$

and so $\tau^{k,i}(s^t) = \int \frac{\widetilde{R}_k(s^t,s_{t+1})}{q_k(s^t)} q \frac{\partial - B^{max}(\{X^i_{t+1}(s^{t+1})\})}{\partial I^i_{t+1}(s^{t+1})} \frac{1}{\lambda^{HH,i}(s^t)} \chi(s^t) ds_{t+1}$. By Lemma 1 we can write

$$\tau^{k,i}(s^t) = \frac{\widetilde{R}_k(s^t,s^*_{t+1})}{q_k(s^t)} q \frac{\partial - \overline{B}(\{X^i_{t+1}(s^{t+1,*})\})}{\partial I^i_{t+1}(s^{t+1.*})} \frac{1}{\lambda^{HH,i}(s^t)} \chi(s^t)$$

Using the FOC for $B_{t+1}(s^t)$ (equation (70)) and the definition $\tau^B(s^t)$ to substitute for $\chi(s^t)$ we derive the equation in the proposition.

Proof of Proposition 6. Differentiating in the definition of $\overline{B}$

$$\frac{\partial - \overline{B}(\{X^i_{t+1}(s^{t+1,*})\}_{i\in\mathcal{I}}, s^{t+1,*})}{\partial I^i_{t+1}(s^{t+1,*})} = \frac{\frac{\partial V^R_{t+1}(s^{t+1,*})}{\partial I^i_{t+1}(s^{t+1,*})} - \frac{\partial V^D_{t+1}(s^{t+1,*})}{\partial I^i_{t+1}(s^{t+1,*})}}{\frac{\partial V^R_{t+1}(s^{t+1,*})}{\partial B_{t+1}}} = \frac{\lambda^{HH,i}(s^{t+1,*},R) - \lambda^{HH,i}(s^{t+1,*},D)}{\lambda^G(s^{t+1,*})}. \tag{78}$$

Under the utility function $u^i(c^i - v^i(\frac{y^i}{s_t}))$, the FOC conditions (68) and (69) write

$$\omega^i \pi^i u^i_c(s^t) = \lambda^{HH,i}(s^t) \tag{79}$$

$$\omega^i \pi^i u^i_c(s^t) \frac{(v^i)'(\frac{y^i(s^t)}{s_t})}{s_t} = \lambda^G(s^t)\pi^i + \left(\lambda^{HH,i}(s^t) - \lambda^G(s^t)\pi^i\right)\left(\frac{(v^i)''(\frac{y^i(s^t)}{s_t})y^i(s^t)}{s_t^2} + \frac{(v^i)'(\frac{y^i(s^t)}{s_t})}{s_t}\right). \tag{80}$$

Following similar steps as in the two-period model, and using $1 - \frac{s_t}{(v^i)'(\frac{y^i(s^t)}{s_t})} = -\frac{\tau^i(s^t)}{1-\tau^i(s^t)}$ we get

$$\lambda^{HH,i}(s^t) = \lambda^G(s^t)\pi^i \left(1 - \eta^i s_t \frac{\tau^i(s^t)}{1-\tau^i(s^t)}\right). \tag{81}$$

Then combining (44), (78) and (81) we can derive

$$\tau^{k,i}(s^t) = \frac{\widetilde{R}_k(s^t,s^*)}{q_k(s^t)} q \left(1 - \frac{U^i_c(s^{t+1,*},D)}{U^i_c(s^{t+1,*},R)}\right) \frac{1 - \eta^i s_{t+1,*}\frac{\tau^i(s^{t+1,*},R)}{1-\tau^i(s^{t+1,*},R)}}{1 - \eta^i s_t \frac{\tau^i(s^t)}{1-\tau^i(s^t)}} \times \tau^B(s^t). \tag{82}$$

Proof of Proposition 7. By Lemma 1, we have $\frac{\partial B^{max}}{\partial b^i_{t+1}(s^t)} = \frac{\partial \overline{B}(s^{t+1,*})}{\partial b^i_{t+1}(s^t)}$. From the definition of $\overline{B}$,

$$\frac{\partial \overline{B}}{\partial b^i_{t+1}(s^t)} = \frac{\frac{\partial V^R_{t+1}(s^{t+1,*})}{\partial b^i_{t+1}(s^t)}}{\frac{\partial V^R_{t+1}(s^{t+1,*})}{\partial B_{t+1}(s^t)}} = \frac{\lambda^{HH,i}(s^{t+1,*},R)}{\lambda^{G,i}(s^{t+1,*},R)}. \tag{83}$$

Combine the wedges $\tau^B(s^t)$ and $\tau^{b,i}(s^t)$ in the FOC (73) to substitute for $\chi(s^t)$ gives

$$\tau^{b,i}(s^t) = \frac{\partial - \overline{B}}{\partial b^i_{t+1}(s^t)} \frac{\lambda^G(s^t)}{\lambda^{HH,i}(s^t)} \tau^B(s^t) = \frac{\lambda^{HH,i}(s^{t+1,*}, R)}{\lambda^{HH,i}(s^t)} \frac{\lambda^G(s^t)}{\lambda^G(s^{t+1,*}, R)} \tau^B(s^t). \tag{84}$$

And using (81) gives

$$\tau^{b,i}(s^t) = \frac{1 - \eta^i s_{t+1,*} \frac{\tau^i(s^{t+1,*},R)}{1-\tau^i(s^{t+1,*},R)}}{1 - \eta^i s_t \frac{\tau^i(s^t)}{1-\tau^i(s^t)}} \tau^B(s^t). \tag{85}$$

B Numerical implementation

B.1 Recursive government problem.

Using $1 - \tau = \frac{1}{s} v'(\frac{y}{s})$, the local government's value following repayment solves

$$V^R(B, a, s) = \max_{c,y,B',a'} u(c - v(\frac{y}{s})) + \beta \mathbb{E}_{s'|s}\left[V(B', a', s')\right]$$
$$s.t \;\; G + Q^B(B', a', s)B' = \lambda B + (1-\lambda)Q^B(B', a', s)B + (1 - \frac{1}{s}v'(\frac{y}{s}))y$$
$$c + qa' = a + \frac{1}{s}v'(\frac{y}{s})y$$
$$a' \geq \underline{a}$$

and bond prices given by equation (48),

where $V(B', a', s') = \max\{V^R(B', a', s'), V^D(a', s')\}$ and the value of default solves

$$V^D(a, s) = \max_{c,y,a'} u(c - v(\frac{y}{h(s)})) + \beta \mathbb{E}_{s'|s}\left[V^D(a', s')\right]$$
$$s.t \;\; G = (1 - \frac{1}{h(s)}v'(\frac{y}{h(s)}))y$$
$$c + qa' = a + \frac{1}{h(s)}v'(\frac{y}{h(s)})y$$
$$a' \geq \underline{a}.$$

B.2 Extreme value taste shocks.

Following Dvorkin *et al.* (2021), to solve the model with long-term debt, I introduce extreme value type I taste shocks to the value of borrowing. Let ϵ be the vector of shocks to the value of borrowing. Assume they are i.i.d with scale parameter ξ and location parameter equal to zero. We have one shock for every pair of public and private debt $\epsilon(B_i, a_j) = \epsilon_{i,j}$ in the grid. Let $S = \{B, a, s\}$ denote the aggregate state variable, and $W^R(S, B', a')$ the value of choosing B', a' in the repayment state if the aggregate state is S. Then the ex-ante value for the government (before the realization of the taste shock) is:

$$\begin{aligned} V^{E,R}(S) &= \mathbb{E} \max_{B',a'} \{W^R(S, B', a') + \epsilon(B', a')\} \\ &= \xi \log \sum_{B',a'} exp\left(W^R(S, B', a')\right)^{\frac{1}{\xi}} \\ &= V^R(S) + \xi log \sum_{B',a'} exp\left(\frac{W^R(S, B', a') - V^R(S)}{\xi}\right), \end{aligned} \tag{86}$$

where $V^R(S) = \max_{B',b'} W^R(S, B', a')$. Moreover, the probability of choosing B', a' is

$$\pi(B', a', S) = \frac{exp\left(W^R(S, B', a')\right)^{\frac{1}{\xi}}}{\sum_{\hat{B}',\hat{b}'} exp\left(W^R(S, \hat{B}', \hat{a}')\right)^{\frac{1}{\xi}}} = \frac{exp\left(\frac{W^R(S,B',a') - V^R(S)}{\xi}\right)}{\sum_{\hat{B}',\hat{b}'} exp\left(\frac{W^R(S,B',a') - V^R(S)}{\xi}\right)}, \tag{87}$$

and the bond price is given by

$$Q^B(B', a', x) = q\mathbb{E}_{x'|x}\left[(1 - \mathbb{P}(Default|B', a', s'))\left(\lambda + (1-\lambda)\sum_{B'',a''} \pi(B'', a'', S')Q^B(B'', a'', s')\right)\right]. \tag{88}$$

Note that, without taste shocks to the relative values of defaulting and repaying, we have $\mathbb{P}(Default|B', a', s')) \in \{0, 1\}$.

B.3 Calibration with high Frisch elasticity

Table 3 contains the internally calibrated parameters, and Table shows 4 the model fit with $\eta = 0.75$. As before, the model does a good job of matching the targeted moments.

Table 3: Internal parameters with $\eta = 0.75$

Parameters		
		Moment matched
Discount factor	$\beta = 0.9765$	Default probability
Disutility parameter	$\chi = 0.8398$	Normalization avg. income
Productivity cost default	$\zeta_0 = -0.1831$	Debt to GDP
Productivity cost default	$\zeta_1 = 0.1919$	Mean Spread (Hatchondo and Martinez (2009))
Std. dev. productivity	$\sigma = 0.0087$	Std. dev GDP

Table 4: Model fit with $\eta = 0.75$

	Data	Model
Std. dev GDP	3.08	3.2
Default probability, %	4.5	4.2
Debt to GDP %	−32.5	−30.4
Mean Spread %	7.4	8.5
Avg. GDP (normalization)	1.0	1.1

C Extra figures

Figure 4: Private-debt-dependent fiscal rules with positive default probability

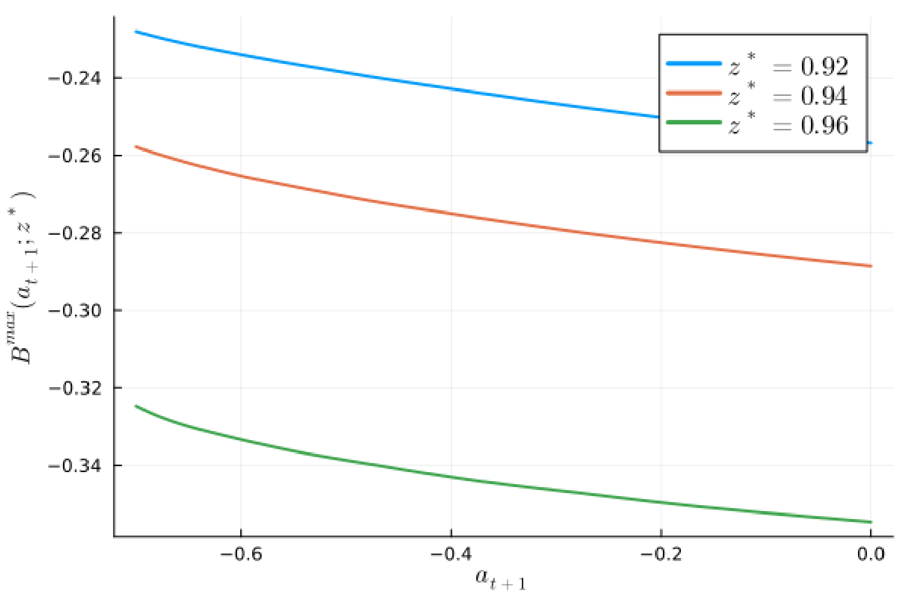


Figure 5: Effects of private debt on default probabilities with $\eta = 0.75$

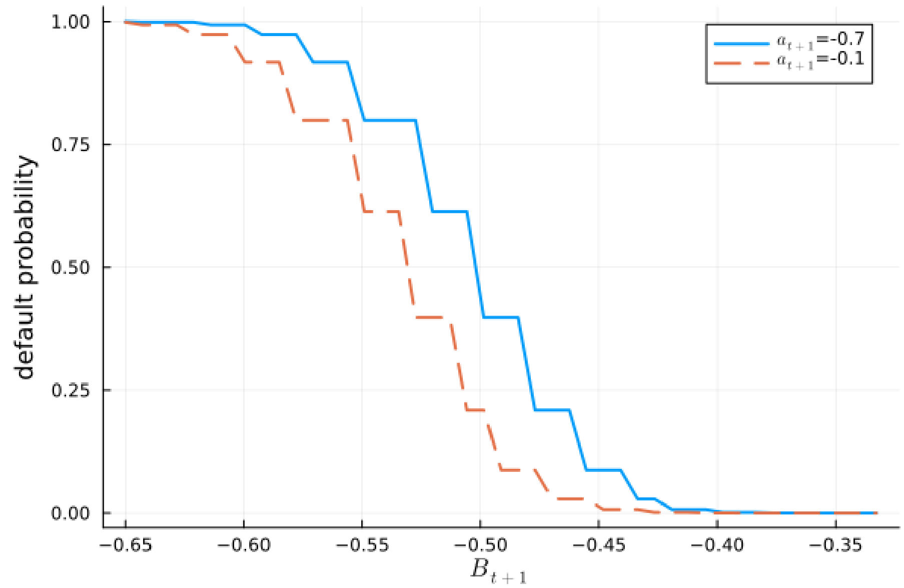
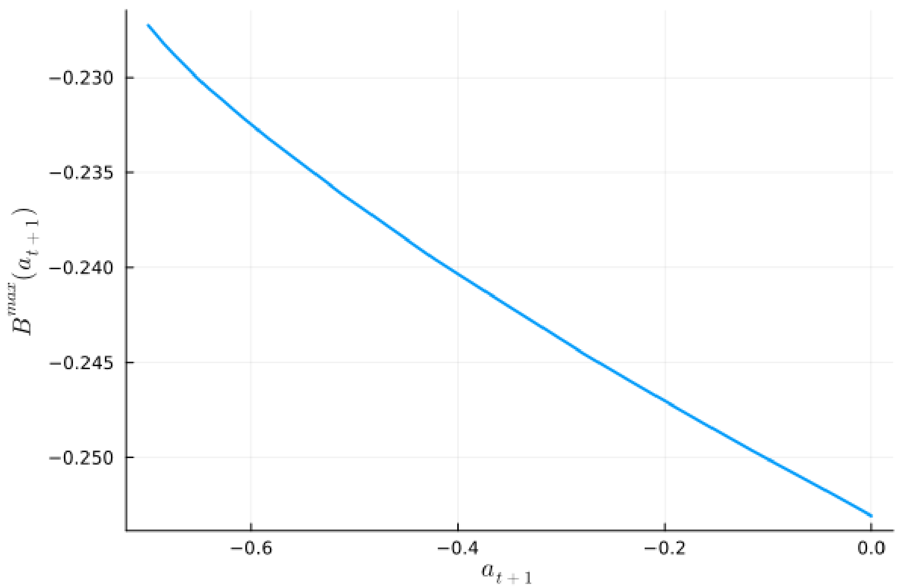


Figure 6: Private-debt-dependent fiscal rule with $\eta = 0.75$

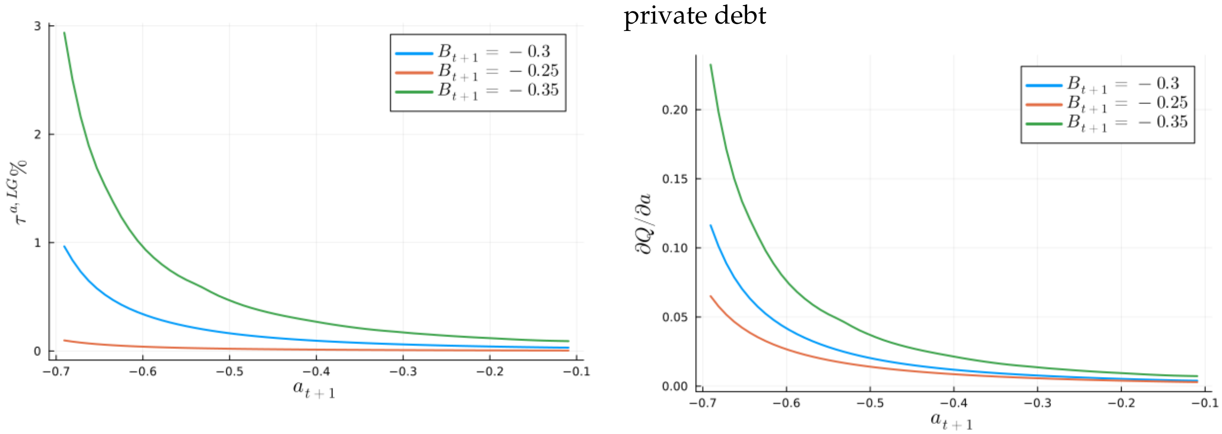


Note:

Figure 7: Macroprudential wedges in local government's problem with $\eta = 0.75$

(a) Macroprudential wedges (in %)

(b) Derivatives sovereign debt price with respect to private debt



D Extension: private default without sovereign default

In this section, I consider an extension of the two-period model where I allow for private default without sovereign default. That is, I allow the local government to default on sovereign debt and/or make the household default. To this end, I assume there are stochastic utility penalties θ_B and θ_a after a default on sovereign and private debt, respectively. Moreover, I assume that the random variables θ_B and θ_a are independent and both drawn from the same continuous density $\pi()$.

The value of defaulting on private debt is defined as

$$V_2^{D,a}(B_2,\theta_a) = \max_{c_2,y_2} u\left(c_2 - v\left(y_2\right)\right) - \theta_a$$

$$\text{s.t} \; -B_2 = \tau_2 y_2,$$

$$c_2 = (1-\tau_2)y_2$$

and with the usual implementability condition. Similarly, we now denote the value of default on sovereign debt as $V_2^{D,B}(a_2,\theta_B)$, which is defined as in the main text. For simplicity, we assume that private default also carries an externality S and assume it is high enough such that default is never optimal. Hence, the central authority imposes the constraints

$$V_2^R(B_2,a_2) \geq V_2^{D,B}(a_2,\theta_B) \;\forall\theta_B$$

$$V_2^R(B_2,a_2) \geq V_2^{D,a}(B_2,\theta_a) \;\forall\theta_a$$

which is equivalent to $V_2^R(B_2,a_2) \geq \max\{V_2^{D,B}(a_2,\underline{\theta_B}), V_2^{D,a}(B_2,\underline{\theta_a})\}$. Therefore, in regions of the state space, (B_2,a_2), where $V_2^{D,B}(a_2,\underline{\theta_B}) \geq V_2^{D,a}(B_2,\underline{\theta_a})$, the no sovereign default constraint may bind, and conversely in the rest of the state space. In any case, we can still use the fiscal rule $B \geq B^{max}(a_2)$, where now B^{max} is implicitly defined as

$$\begin{cases} V_2^R(B^{max}(a_2),a_2) = V_2^{D,B}(a_2,\underline{\theta_B}) \text{ if } V_2^{D,B}(a_2,\underline{\theta_B}) \geq V_2^{D,a}(B_2,\underline{\theta_a}) \\ V_2^R(B^{max}(a_2),a_2) = V_2^{D,a}(B^{max}(a_2),\underline{\theta_a}) \text{ otherwise.} \end{cases}$$

Notice that we could also use the following constraint that imposes a limit on private debt as a function of the quantity of sovereign debt $a_2 \geq a^{max}(B_2)$. If $V_2^{D,B}(a_2,\underline{\theta_B}) \neq V_2^{D,a}(B_2,\underline{\theta_a})$, the function B^{max} is differentiable and we have the same equation relating the wedges as in Proposition 1. Therefore, when $V_2^{D,B} \geq V_2^{D,a}$, the equation $\frac{\tau^a}{1-\tau^a} = \left(1 - \frac{u_2'(D)}{u_2'(R)}\right)\frac{\tau^B}{1-\tau^B}$ of Proposition 2 also holds.

Conversely, if $V_2^{D,B} < V_2^{D,a}$, we have

$$\frac{\partial B^{max}}{\partial a_2} = \frac{\frac{\partial V_2^R}{\partial a_2}}{\frac{\partial V_2^{D,a}}{\partial B_2} - \frac{\partial V_2^R}{\partial B_2}} = \frac{u_2'(R)}{u_2'(D,a) - u_2'(R)}(1 - \eta\frac{\tau_2}{1-\tau_2}),$$

where $u_2'(D,a)$ denotes the marginal utility at period two after private default. The second equality follows because the tax rate τ_2 is the same whether there is private default or not due to the no income effects assumption. Substituting into the formula of Proposition 1, using $\frac{1-\tau^a}{1-\tau^B} = \frac{1-\eta\frac{\tau_2}{1-\tau_2}}{1-\eta\frac{\tau_1}{1-\tau_1}}$ and collecting terms

$$\frac{\tau^a}{1-\tau^a} = (1 - \frac{u_2'(D,a)}{u_2'(R)})^{-1}\frac{\tau^B}{1-\tau^B}.$$

This formula is akin to that of Proposition 2 but with the term $(1 - \frac{u_2'(D,a)}{u_2'(R)})^{-1}$ instead of $\left(1 - \frac{u_2'(D)}{u_2'(R)}\right)$ because now we are in the converse scenario where sovereign debt is restricted to avoid too large restrictions on private debt. Finally, notice that because $\frac{u_2'(D,a)}{u_2'(R)} \in (0,1)$, the macroprudential wedge is larger than the wedge on sovereign debt $\tau^a > \tau^B$ whenever they are positive.