A Theory of Conglomerate Mergers

Online Appendix

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A Product Differentiation

Consider highly concentrated markets where $n_A = n_B = 2$. In each market, two products are horizontal differentiated à la Hotelling, and consumers' preferences over the two products are perfectly correlated. There are two firms, A_1 and B_1, located at one end while the other two, A_2 and B_2, are located at another end. Let α_j and β_j denote the margins for products A_j and B_j, respectively.

In market A, a consumer located at $x \in [0,1]$ derives a net utility of $w_A - \alpha_1 - tx$ from buying A_1 and $w_A - \alpha_2 - t(1-x)$ from A_2, where $t > 0$ indicates the degree of product differentiation. Similarly, in market B, the same consumer obtains a net utility of $w_B - \beta_1 - tx$ from B_1 and $w_B - \beta_2 - t(1-x)$ from B_2. We assume that $w_A, w_B > 3t$, which ensures full market coverage.

We assume further that consumption synergies are uniformly distributed over $[0,1]$, with $F(s) = s$ for the sake of tractability in the analysis. Since the two markets, A and B, are symmetric in this case, we focus on the symmetric equilibrium where $\alpha_1 = \beta_1 \equiv \rho_1$ and $\alpha_2 = \beta_2 \equiv \rho_2$.

Before a merger, the two markets are independent. Consumers located at $x < \hat{x} \equiv \frac{1}{2} - \frac{\rho_1 - \rho_2}{2t}$ will purchase the combination $\{A_1, B_1\}$, while others will buy $\{A_2, B_2\}$. Firms A_1 and B_1 earn a profit of $\rho_1 \hat{x}$, while A_2 and B_2 earn $\rho_2(1-\hat{x})$. Their best responses are given by $\rho_1 = 2t\hat{x}$ and $\rho_2 = 2t(1-\hat{x})$. Solving for the best responses leads to the Hotelling price margins $\rho_1^* = \rho_2^* = t$, and each firm earns a profit of $t/2$.

Consider a merger between A_1 and B_1. Suppose the conglomerate offers stand-alone products A_1 and B_1, as well as the bundle $A_1 - B_1$. Consumers prefer A_1 to A_2 if:

$$x \leq x_A \equiv \frac{1}{2} - \frac{\alpha_1 - \alpha_2}{2t}, \tag{1}$$

and prefer B_1 to B_2 if:

$$x \leq x_B \equiv \frac{1}{2} - \frac{\beta_1 - \beta_2}{2t}.$$

Without loss of generality, let's assume $x_A \leq x_B$. Consumers with $x < x_A$ buy the product portfolio $\{A_1, B_1\}$ if $s < \sigma_1 \equiv \mu - \alpha_1 - \beta_1$, whereas consumers with $x_A < x < x_B$ buy the portfolio $\{A_2, B_2\}$ if:

$$s < \sigma_{21}(x) \equiv \mu - \alpha_2 - \beta_1 + 2t\left(x - \frac{1}{2}\right), \tag{2}$$

and consumers with $x > x_B$ buy $\{A_2, B_1\}$ if:

$$s < \sigma_2(x) \equiv \mu - \alpha_2 - \beta_2 + 4t\left(x - \frac{1}{2}\right). \tag{3}$$

Conversely, consumers buy the bundle $A_1 - B_1$ if:

$$s > \sigma(x) \equiv \max\{\sigma_1, \sigma_{21}(x), \sigma_2(x)\},$$

where $\sigma(x)$ is continuous and satisfies $\sigma_1 = \sigma_{21}(x_A) \leq \sigma_{21}(x_B) = \sigma_2(x_B) \leq \sigma_2(1)$.

The analysis takes into account various configurations determined by the comparison between σ_1 and $\sigma_2(1)$ within the range of $s \in [0,1]$, as well as the boundaries of 0 and 1. Specifically, we focus on the scenario where $0 < \sigma_1 < \sigma_2(1) < 1$, with $0 < x_A < x_B < 1$, and this case is illustrated in Figure A.0.

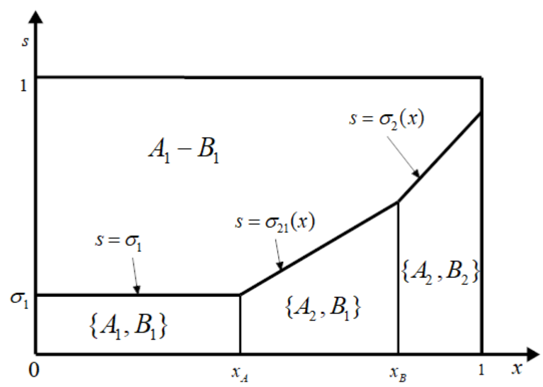

Figure A.0

We proceed to characterize the symmetric equilibria, where the conglomerate M sets symmetric stand-alone margins ($\alpha_1 = \beta_1 = \rho_1$), and the two stand-alone firms also charge symmetric prices ($\alpha_2 = \beta_2 = \rho_2$). Consequently, the thresholds x_A and x_B are also symmetric:

$$x_A = x_B = \hat{x} \equiv \frac{1}{2} - \frac{\rho_1 - \rho_2}{2t}. \tag{4}$$

Throughout the analysis, we make the assumption that w_A and w_B are sufficiently large to guarantee that the entire market is served in equilibrium. The following lemma demonstrates that, without loss of generality, we can further narrow our focus to the range where $\sigma_1 < 1$ and $\hat{x} < 1$:

Lemma 1 *Without loss of generality, we can concentrate on candidate equilibria where $\sigma_1 < 1$ and $\hat{x} < 1$.*

Proof. Consider a candidate equilibria in which $\sigma_1 \geq 1$, or $\mu > 1 + 2\rho_1$, implying that no consumer buys the bundle. We first note that the portfolio $\{A_1, B_1\}$ must have a positive market share. To see why, suppose the portfolio $\{A_2, B_2\}$ attracts all consumers, implying that M attracts no consumers and obtains zero profit. It must be the case that $\rho_2 \geq 0$; otherwise, the stand-alone firms would incur a loss, and they would benefit from raising their prices to at least cover costs. However, in that case, M could profitably deviate by charging ρ_1 slightly above ρ_2 (along with a prohibitively high μ, for instance). This deviation would attract (almost) half of the consumers and generate positive profits.

Thus, we must have $\hat{x} \equiv \frac{1}{2} - \frac{\rho_1 - \rho_2}{2t} > 0$ and $\rho_1 \geq 0$ (otherwise, M would incur losses and could profitably deviate by increasing ρ_1 to cost). Now, let's suppose that M reduces the margin on the bundle to $\mu' = 1 + 2\rho_1 - \varepsilon$, where ε is a positive but arbitrarily small value, so that $\sigma_1' = \mu' - 2\rho_1 = 1 - \varepsilon > 0$. This deviation does not affect the demand from consumers with $s < \sigma_1'$, but it induces those with $s > \sigma_1'$ and $x < \hat{x}$ to switch from $\{A_1, B_1\}$ to the bundle, generating an additional margin of $\mu' - 2\rho_1 = \sigma_1' > 0$. Moreover, it also induces consumers with $x > \hat{x}$ and $s > \sigma_2(x)$ to switch from $\{A_2, B_2\}$ to the bundle, generating additional profits (as $\mu' = 2\rho_1 + \sigma_1' > 2\rho_1 \geq 0$). Thus, the deviation is profitable, which contradicts the assumption.

Consider a candidate equilibrium in which $\hat{x} \geq 1$, implying that no consumer purchases from the stand-alone firms. Once again, we observe that the portfolio $\{A_1, B_1\}$ must have a positive market share. To see why, suppose that all consumers buy the bundle, resulting in M obtaining μ. Now, consider a deviation where M charges $\rho_1' = (\mu - \varepsilon)/2$ and $\mu' = \mu - \varepsilon + \sigma_1'$, where $\varepsilon > 0$ and $\sigma_1' \in (0, 1)$. In this deviation, all consumers with $s < \mu' - 2\rho_1' = \sigma_1'$ then purchase the portfolio $\{A_1, B_1\}$ at a total margin of $2\rho_1' = \mu - \varepsilon$ (as this gives them a higher net surplus compared to the candidate equilibrium). Additionally, all consumers with $s > \sigma_1'$ buy the bundle (as s then exceeds the premium $\mu' - 2\rho_1'$). By doing so, M obtains $\mu'(1 - \sigma_1') + 2\rho_1'\sigma_1' = \mu - \varepsilon + \sigma_1'(1 - \sigma_1')$, which exceeds the profit of the candidate equilibrium, μ, for any $\sigma_1' \in (0, 1)$ and $\varepsilon < \sigma_1'(1 - \sigma_1')$.

Hence, we must have $\sigma_1 = \mu - 2\rho_1 > 0$ (and, as mentioned earlier, $\sigma_1 < 1$). We now demonstrate that ρ_1 cannot be negative. To see this, consider a candidate equilibrium where $\rho_1 < 0$. If $\hat{x} > 1$, then a slight increase in ρ_1 would not affect the sales of the bundle but would reduce the loss incurred on the portfolio $\{A_1, B_1\}$, leading to a contradiction. If, instead, $\hat{x} = 1$, consider a small deviation to $\mu' = \mu + \varepsilon$ and $\rho_1' = \left(\mu' - \sigma_1'\right)/2$ (such that $\sigma_1' \equiv \mu' - 2\rho_1' = \sigma_1$). This deviation induces some consumers with $x \in (1-\varepsilon, 1]$ to switch to the rival portfolio $\{A_2, B_2\}$. Specifically, those with $s < \sigma_1$ switch away from the loss-making portfolio $\{A_1, B_1\}$, resulting in an increased profit for M by $(-2\rho_1) \times \sigma_1 \times \varepsilon$. On the other hand, those with $s \in (\sigma_1, \sigma_1 + 4t\varepsilon)$ stop buying the bundle, reducing M's profit by $\mu \times 4t\varepsilon^2$. Since the gain is linear and the loss is quadratic in ε, it follows that, for sufficiently small ε, the deviation has a positive net impact on M's profit.

Therefore, we must have $\rho_1 \geq 0$. However, in that case, any stand-alone firm can profitably attract some consumers (specifically, those close to $x = 1$ and $s = 0$) by charging slightly more than ρ_1, leading to a contradiction. ■

Lemma 1 ensures that M sells the bundle in equilibrium (i.e., $\sigma_1 < 1$). As a result, we can focus on two types of equilibria: "true" mixed bundling, where M sells the portfolio $\{A_1, B_1\}$ alongside the bundle (i.e., $\sigma_1 > 0$), and "de facto" pure bundling, where M charges unattractive stand-alone prices despite not committing to offering only the bundle (i.e., $\sigma_1 \leq 0$).

We proceed to characterize the candidate equilibria for each possible configuration before addressing their existence.

A.1 Candidate symmetric equilibria

We begin by considering the true mixed bundling configuration, where M sells the bundle as well as two stand-alone products: $\sigma_1 \in (0, 1)$. Let $\bar{x}$ denote the location threshold such that $\sigma_2(\bar{x}) = 1$:

$$\bar{x} \equiv \frac{1}{2} + \frac{2\rho_2 - \mu + 1}{4t}.$$

A.1.1 Configuration 1: $0 < \sigma_1 < \sigma_2(1) \leq 1$

This configuration arises when $0 < \hat{x} < 1 \leq \bar{x}$, as illustrated by Figure A.1 below. The conglomerate M's profit can be expressed as:

$$\Pi_M = \mu \times \{(1 - \sigma_1)\hat{x} + \int_{\hat{x}}^{1} [1 - \sigma_2(x)] dx\} + 2\rho_1 \times \sigma_1 \hat{x} = \mu[1 - \int_{\hat{x}}^{1} \sigma_2(x)\, dx] - \sigma_1^2 \hat{x}$$

where the second equality follows from $\mu - 2\rho_1 = \sigma_1$, $\hat{x} = \frac{1}{2} - \frac{\rho_1 - \rho_2}{2t}$, and $\sigma_2(x) = \sigma_1 + 4t(x - \hat{x})$.

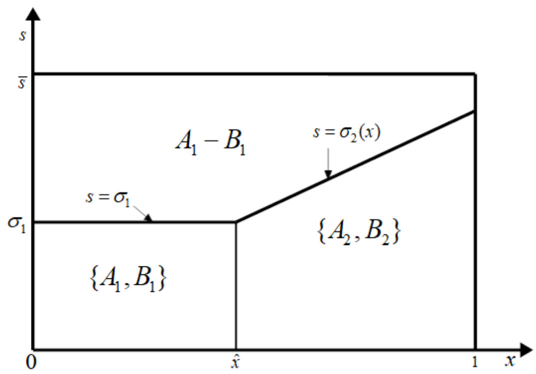

Figure A.1

Differentiating this profit function with respect to ρ_1 and using $\sigma_2(\hat{x}) = \sigma_1$, we obtain:

$$\frac{\partial \Pi_M}{\partial \rho_1} = -\mu \frac{\sigma_1}{2t} + 4\sigma_1 \hat{x} + \frac{\sigma_1^2}{2t} = \frac{\sigma_1}{2t}(\sigma_1 - \mu) + 4\sigma_1 \hat{x} = -\frac{\sigma_1}{t}\rho_1 + 4\sigma_1 \hat{x} = \frac{\sigma_1}{t}(4t\hat{x} - \rho_1).$$

Solving for the first-order conditions for ρ_1 gives:

$$\rho_1 = 4t\hat{x}.$$

Consider a slight decrease of A_2's margin α_2 from $\alpha_2 = \rho_2$. The profit of the stand-alone firm A_2 then becomes:

$$\Pi_A = \alpha_2 [\int_{x_A}^{x_B} \sigma_{21}(x)\, dx + \int_{x_B}^{1} \sigma_2(x)\, dx],$$

where x_A, $\sigma_{21}(x)$, and $\sigma_2(x)$ are given by (1), (2) and (3), respectively. Differentiating this profit function with respect to α_2 and evaluating the derivative at $\alpha_2 = \beta_2 = \rho_2$ leads to:

$$\begin{aligned} \left.\frac{\partial \Pi_A}{\partial \alpha_2}\right|_{\alpha_2=\beta_2=\rho_2} &= \int_{\hat{x}}^{1} [\sigma_1 + 4t(x - \hat{x})]\, dx - \rho_2 \frac{\sigma_1}{2t} - \rho_2 \int_{\hat{x}}^{1} dx \\ &= (\frac{\sigma_1}{2t} + 1 - \hat{x})[2t(1 - \hat{x}) - \rho_2]. \end{aligned}$$

Solving for the first-order conditions for $\alpha_2 = \rho_2$ gives:

$$\rho_2 = 2t(1 - \hat{x}).$$

Combining these two FOCs leads to the equilibrium margins:

$$\rho_1^* = \rho_1^m \equiv \frac{3}{2}t, \ \rho_2^* = \rho_2^m \equiv \frac{5}{4}t,$$

and the equilibrium threshold:

$$\hat{x}^* = \hat{x}^m \equiv \frac{3}{8}.$$

Moreover, differentiating Π_M with respect to μ yields:

$$\frac{\partial \Pi_M}{\partial \mu} = 1 - \int_{\hat{x}}^{1} \sigma_2(x)\, dx - \mu(1 - \hat{x}) - 2\sigma_1 \hat{x}.$$

Evaluating the derivative at $\hat{x}^m$ and ρ_1^m leads to:

$$\begin{aligned} \mu^* &= \mu^m \equiv \frac{1}{2} + \frac{107}{64}t, \\ \sigma_1^* &= \sigma_1^m \equiv \frac{1}{2} - \frac{85}{64}t. \end{aligned}$$

Then, the assumption $\sigma_1^m > 0$ holds if:

$$t < t_m \equiv \frac{32}{85}.$$

Conversely, whenever this condition holds, the other assumption also holds:

$$1-\sigma_2(1)=1-[\sigma_1^*+4t(1-\hat{x}^*)]=\frac{32-75t}{64}>\left.\frac{32-75t}{64}\right|_{t=t_m}=\frac{1}{17}>0.$$

Therefore, this candidate equilibrium arises whenever $t<t_m$. We demonstrate in Online Appendix B that this equilibrium indeed exists under this condition.

A.1.2 Configuration 2: $0<\sigma_1<1\leq\sigma_2(1)$

This case arises when $0<\hat{x}<\bar{x}\leq 1$, as illustrated by Figure A.2 below. M's profit is then equal to:

$$\Pi_M=\mu\times\{(1-\sigma_1)\hat{x}+\int_{\hat{x}}^{\bar{x}}[1-\sigma_2(x)]dx\}+2\rho_1\times\sigma_1\hat{x}=\mu[\bar{x}-\int_{\hat{x}}^{\bar{x}}\sigma_2(x)dx]-\sigma_1^2\hat{x}.$$

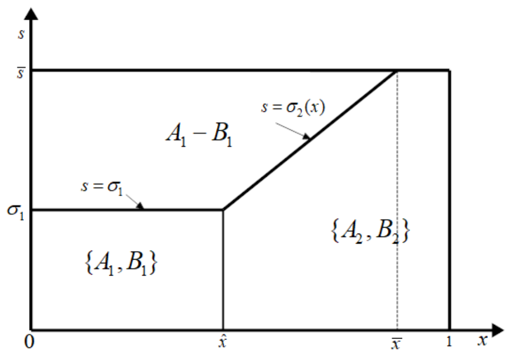

Figure A.2

We demonstrate that this configuration cannot arise in equilibrium. The proof consists of two steps: first, we solve for the first-order conditions and derive the boundaries of parameters for the best responses; second, we show that there is no interior solution for the best responses in this configuration.

We first derive the first-order conditions for ρ_1 and ρ_2 and identify the boundary of σ_1 under the candidate equilibrium. The derivative of Π_M with respect to ρ_1 remains the same as before:

$$\frac{\partial\Pi_M}{\partial\rho_1}=\frac{\sigma_1}{t}(4t\hat{x}-\rho_1).$$

Solving for the first-order conditions for ρ_1 then gives: $\rho_1=4t\hat{x}$.

Following a slight decrease in α_2 from $\alpha_2=\rho_2$, A_2's profit becomes:

$$\Pi_A=\alpha_2[\int_{x_A}^{x_B}\sigma_{21}(x)\,dx+\int_{x_B}^{\bar{x}}\sigma_2(x)\,dx+(1-\bar{x})],$$

where x_A, $\sigma_{21}(x)$, and $\sigma_2(x)$ are given by (1), (2), and (3), respectively, while:

$$\bar{x} \equiv \frac{1}{2} + \frac{\alpha_2 + \beta_2 - \mu + 1}{4t}.$$

Differentiating this profit with respect to α_2 and evaluating the derivative at $\alpha_2 = \beta_2 = \rho_2$ yields:

$$\begin{aligned}
\left.\frac{\partial \Pi_A}{\partial \alpha_2}\right|_{\alpha_2=\beta_2=\rho_2} &= \int_{\hat{x}}^{\bar{x}} [\sigma_1 + 4t(x-\hat{x})]\,dx + (1-\bar{x}) - \rho_2[\frac{\sigma_1}{2t} + \int_{\hat{x}}^{\bar{x}} dx] \\
&= [2t(\bar{x}-\hat{x}) - \rho_2]\,(\frac{\sigma_1}{2t} + \bar{x} - \hat{x}) + 1 - \bar{x} \\
&= \frac{1}{4t}\left\{\left[\frac{1-\sigma_1}{2} - \rho_2\right](1+\sigma_1) + 4t(1-\hat{x}) - (1-\sigma_1)\right\}.
\end{aligned}$$

Solving for the first-order conditions for $\alpha_2 = \rho_2$ and using $4t(\bar{x}-\hat{x}) = 1-\sigma_1$ then leads to:

$$\rho_2 = \frac{1-\sigma_1}{2} + \frac{4t(1-\hat{x}) - (1-\sigma_1)}{1+\sigma_1}.$$

Combine the two FOCs and using (4), we obtain the equilibrium margins:

$$\rho_1 = \frac{2t(5+\sigma_1) - (1-\sigma_1)^2}{5+3\sigma_1},\ \rho_2 = \frac{20t - 3(1-\sigma_1)^2}{2(5+3\sigma_1)},$$

and equilibrium thresholds:

$$\hat{x} = \frac{2t(5+\sigma_1) - (1-\sigma_1)^2}{4t(5+3\sigma_1)},\ \bar{x} = \frac{t(5+\sigma_1) + 2\left(1-\sigma_1^2\right)}{2t(5+3\sigma_1)}. \tag{5}$$

This candidate equilibrium arises under the condition $\bar{x} \le 1$, which amounts to:

$$t(5+\sigma_1) + 2\left(1-\sigma_1^2\right) \le 2t(5+3\sigma_1) \Longleftrightarrow (1+\sigma_1)(2-5t-2\sigma_1) \le 0.$$

The above condition is equivalent to:

$$\sigma_1 \ge \underline{\sigma}_1(t) \equiv 1 - \frac{5t}{2}.$$

Therefore, this candidate equilibrium can arise if $\sigma_1 \in (\max\{\underline{\sigma}_1(t), 0\}, 1)$. Note that $\underline{\sigma}_1(t) > 0$ if and only if $t < \frac{2}{5}$.

Meanwhile, $\sigma_1 > \underline{\sigma}_1$ also implies:

$$\hat{x} = \frac{1}{2} - \frac{(1-\sigma_1)^2 + 4t\sigma_1}{4t(5+3\sigma_1)} > \frac{1}{2} - \frac{\left(\frac{5t}{2}\right)^2 + 4t\left(1-\frac{5t}{2}\right)}{4t\left(5+3\left(1-\frac{5t}{2}\right)\right)} = \frac{3}{8}.$$

The inequality follows from the fact that the numerator, $(1-\sigma_1)^2 + 4t\sigma_1$, decreases in σ_1 for $\sigma_1 > \underline{\sigma}_1(t)$, while the denominator, $4t(5+3\sigma_1)$, increases in σ_1, then the second term decreases in σ_1.

We show that the first-order derivative for μ, as evaluated at the candidate equilibrium, is always negative. Differentiating M's profit with respect to μ, while keeping ρ_1 and ρ_2 fixed and using $\bar{x} = (1 + 2\rho_2 - \mu)/4t$, yields:

$$\begin{aligned}
\frac{\partial \Pi_M}{\partial \mu}(\mu, \rho_1; \rho_2) &= \bar{x} - \int_{\hat{x}}^{\bar{x}} \sigma_2(x)dx - 2\sigma_1\hat{x} - \mu \int_{\hat{x}}^{\bar{x}} dx \\
&= \hat{x} + \frac{1-\sigma_1}{4t} - \sigma_1 \frac{(1-\sigma_1)}{4t} - \frac{(1-\sigma_1)^2}{8t} - 2\sigma_1\hat{x} - (\sigma_1 + 8t\hat{x}) \frac{(1-\sigma_1)}{4t} \\
&= -\hat{x} - \frac{1-\sigma_1}{8t}(3\sigma_1 - 1),
\end{aligned}$$

where the second equality comes from $\bar{x} - \hat{x} = (1-\sigma_1)/4t$, $\mu = \sigma_1 + 2\rho_1$, and the first-order condition $\rho_1 = 4t\hat{x}$.

Moreover, using $\hat{x} = \hat{x}(\sigma_1)$ given by (5), we obtain:

$$\left.\frac{\partial \Pi_M}{\partial \mu}(\mu, \rho_1; \rho_2)\right|_{\hat{x}=\hat{x}(\sigma_1)} = \frac{7 - 20t - (21+4t)\sigma_1 + 5\sigma_1^2 + 9\sigma_1^3}{8t(5+3\sigma_1)} \equiv \frac{\phi(\sigma_1)}{8t(5+3\sigma_1)}.$$

The function $\phi(\cdot) \equiv 7 - 20t - (21+4t)\sigma_1 + 5\sigma_1^2 + 9\sigma_1^3$ is strictly convex in σ_1 for $\sigma_1 \geq 0$ (as $\phi''(\sigma) = 54\sigma + 10$ is positive) and satisfies $\phi(1) = -24t < 0$. Thus, it is negative in the entire range $\sigma_1 \in [\max\{\underline{\sigma}_1(t), 0\}, 1]$ if and only if it is negative at the lower bound of the range.

If $t \geq 2/5$, $\max\{\underline{\sigma}_1(t), 0\} = 0$. Then:

$$\phi(0) = 7 - 20t \leq -1 < 0,$$

where the first inequality stems from $t \geq 2/5$. If, instead, $t < 2/5$, $\max\{\underline{\sigma}_1(t), 0\} = \underline{\sigma}_1(t) > 0$. Then:

$$\phi(\underline{\sigma}_1) = -\frac{t}{8}(32 - 75t)(16 - 15t) < 0,$$

where the inequality stems from $t < 2/5$.

Hence, for any $\mu > 0$ and any (ρ_1, ρ_2) satisfying the associated first-order conditions, the first-order condition for μ is negative, implying that M can benefit from reducing μ.

We now proceed to the candidate equilibria with de facto pure bundling, where $\sigma_1 \leq 0$. We need to consider several configurations with different values of $\sigma_2(0)$ and $\sigma_2(1)$. However, we can straightforwardly rule out two irrelevant cases. First, $\sigma_2(0) < \sigma_2(1) < 0$, in which case the stand-alone firms do not sell their products, resulting in $\rho_2 = 0$. Then, $\sigma_2(1) = \mu - 2\rho_2 + 2t = \mu + 2t < 0$ implies $\mu < 0$, leading to a loss for the conglomerate from selling the bundle. Second, $0 < 1 < \sigma_2(0) < \sigma_2(1)$. In this case the conglomerate does not sell any products, and it must set

$\mu = 0$ (otherwise, the conglomerate could attempt to sell the bundle by continuously reducing μ). However, this would imply $\sigma_2(0) = 0 - 2\rho_2 - 2t < 0$, which contradicts the assumption.

We consider the following four configurations: configurations 3, 4, 5, and 6, respectively.

A.1.3 Configuration 3: $\sigma_2(0) \leq 0 < \sigma_2(1) \leq 1$

Let $\tilde{x} \equiv \frac{1}{2} - \frac{\mu - \alpha_2 - \beta_2}{4t}$ denote the threshold such that $\sigma_2(\tilde{x}) = 0$. This case arises when $0 < \tilde{x} < 1 < \bar{x}$. Consider a candidate equilibrium in which:

- firm M charges prohibitively high (infinite, for example) margins for stand-alone products, selling the bundle only to consumers with $s > \sigma_2(x)$,
- firm M charges a margin of the bundle such that $0 < \sigma_2(1) < 1$, and
- stand-alone firms charge a margin such that $\sigma_2(0) < 0$.

Since $\tilde{x} = \sigma_2^{-1}(0) > 0$, all consumers with $x \leq \tilde{x}$ will purchase the bundle. Using:

$$\sigma_2(x) = \mu - \alpha_2 - \beta_2 + 4t\left(x - \frac{1}{2}\right) = 4t(x - \tilde{x}),$$

the demand for the stand-alone firms can be expressed as:

$$D_{\{A_2B_2\}} = \int_{\tilde{x}}^{1} \sigma_2(x)\, dx = \int_{\tilde{x}}^{1} 4t(x - \tilde{x})\, dx = 2t(1 - \tilde{x})^2,$$

and the demand for the bundle is equal to $1 - D_{\{A_2B_2\}}$.

This configuration is demonstrated in Figure A.3 below.

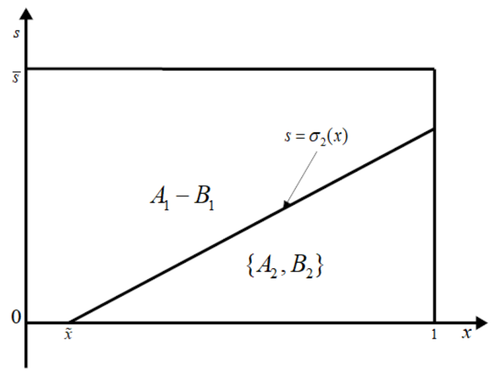

Figure A.3

Then, the profits of the conglomerate and stand-alone firms are given respectively by:

$$\Pi_M = \mu\left(1 - D_{\{A_2B_2\}}\right) = \mu\left(1 - 2t\left(1-\tilde{x}\right)^2\right), \tag{6}$$

and:

$$\begin{aligned}\Pi_A &= \alpha_2 D_{\{A_2B_2\}} = \alpha_2 2t\left(1-\tilde{x}\right)^2, \\ \Pi_B &= \beta_2 D_{\{A_2B_2\}} = \beta_2 2t\left(1-\tilde{x}\right)^2.\end{aligned}$$

Each profit function is strictly concave in the firm's own margin:

$$\begin{aligned}\frac{\partial^2 \Pi_M}{\partial \mu} &= -2\left(1-\tilde{x}\right) - \frac{\mu}{4t} < 0, \\ \frac{\partial^2 \Pi_A}{\partial \alpha_2^2} &= -2\left(1-\tilde{x}\right) - \frac{\alpha_2}{4t} < 0, \\ \frac{\partial^2 \Pi_B}{\partial \beta_2^2} &= -2\left(1-\tilde{x}\right) - \frac{\beta_2}{4t} < 0.\end{aligned}$$

Then, the best responses are characterized by the following first-order conditions, which are expressed as the functions of the threshold $1-\tilde{x}$:

$$\begin{aligned}\mu &= \frac{1}{1-\tilde{x}} - 2t\left(1-\tilde{x}\right), \\ \alpha_2 &= \beta_2 = 2t\left(1-\tilde{x}\right).\end{aligned}$$

We now solve for the equilibrium threshold $1-\tilde{x}$, which determines the equilibrium margins. Note that:

$$\begin{aligned}\sigma_2\left(\tilde{x}\right) &= \gamma - \alpha_2 - \beta_2 + 4t\left(\tilde{x} - \frac{1}{2}\right) \\ &= \frac{1}{1-\tilde{x}} - 6t\left(1-\tilde{x}\right) + 4t\left(\tilde{x} - \frac{1}{2}\right),\end{aligned}$$

then $\sigma_2\left(\tilde{x}\right) = 0$ amounts to:

$$10t\left(1-\tilde{x}\right)^2 - 2t\left(1-\tilde{x}\right) - 1 = 0.$$

The above is a quadratic function of $1-\tilde{x}$, and the solution is given by:

$$1-\tilde{x} = \frac{t + \sqrt{t^2+10t}}{10t}.$$

The equilibrium margins are (where the superscript p stands for pure bundling):

$$\begin{aligned}\mu^* &= \breve{\mu}^p \equiv \frac{4\sqrt{t^2+10t} - 6t}{5}, \\ \rho_2^* &= \breve{\rho}_2^p = \frac{\sqrt{t^2+10t} + t}{5},\end{aligned}$$

and the corresponding threshold is:

$$\tilde{x}^* = \tilde{x}^p = \frac{9t - \sqrt{t^2 + 10t}}{10t}.$$

It is straightforward to verify that $\tilde{x}^p > 0$ and $\sigma_2(1) < 1$ under condition $\frac{32}{85} < t < \frac{5}{12}$.

The conglomerate's equilibrium profit:

$$\Pi_M^* = \breve{\mu}^p \left(1 - 2t\left(1 - \tilde{x}^p\right)^2\right) = \left(\frac{4\sqrt{t^2 + 10t} - 6t}{5}\right)\left(\frac{201 - t - \sqrt{t^2 + 10t}}{25}\right),$$

and stand-alone firms' profits are given by:

$$\Pi_A^* = \Pi_B^* = \breve{\rho}_2^p \times 2t\left(1 - \tilde{x}^p\right)^2 = \frac{\left(\breve{\rho}_2^p\right)^3}{2t}.$$

This equilibrium arises when $0 < \sigma_2(1) \leq 1$, which requires $t \leq t_p \equiv \frac{5}{12}$. Furthermore, to prevent the conglomerate's deviation leading to mixed bundling with $0 < \sigma_1 \leq \sigma_2(1) \leq 1$, it is required that $t > t_m = \frac{32}{85}$ (see the proof for existence of the equilibrium). Therefore, this equilibrium exists if $t_m < t < t_p$. In the analysis of equilibrium existence, we demonstrate that there are no profitable deviations within this parameter range.

A.1.4 Configuration 4: $\sigma_2(0) \leq 0 < 1 \leq \sigma_2(1)$

This case arises when $0 < \tilde{x} < \bar{x} < 1$. In the candidate equilibrium:

- firm C charges prohibitively high (infinite, for example) stand-alone margins, selling the bundle only to consumers with $s > \sigma_2(x)$,
- firm C charges a high margin on the bundle such that $\sigma_2(1) > 1$, and
- the stand-alone firms charge a margin such that $\sigma_2(0) \leq 0$.

Recall that $\tilde{x}$ and $\bar{x}$ denote the location thresholds such that $\sigma_2(\tilde{x}) = 0$ and $\sigma_2(\bar{x}) = 1$. The demand for the conglomerate and the stand-alone firms can be expressed respectively by $\bar{x} - \tilde{D}_{\{A_2B_2\}}$ and $1 - \bar{x} + \tilde{D}_{\{A_2B_2\}}$, where:

$$\tilde{D}_{\{A_2B_2\}} \equiv \int_{\tilde{x}}^{\bar{x}} \sigma_2(x)\, dx = \frac{1}{8t}.$$

The demand in this configuration is illustrated in Figure A.4 below.

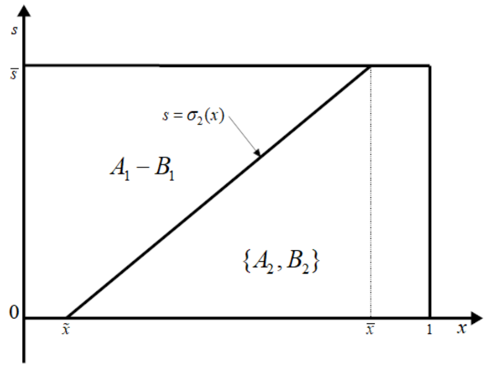

Figure A.4

The equilibrium margins are determined by the first-order conditions. Using $\sigma_2(\tilde{x}) = 0$ and $\sigma_2(\bar{x}) = 1$, we have:

$$\tilde{x} = \frac{1}{2} + \frac{\alpha_2 + \beta_2 - \mu}{4t}, \; \bar{x} = \frac{1}{2} + \frac{1 + \alpha_2 + \beta_2 - \mu}{4t}. \tag{7}$$

The conglomerate's profit can be expressed as:

$$\Pi_M = \mu\left(\bar{x} - \tilde{D}_{\{A_2 B_2\}}\right) = \mu\left(\frac{1}{2} + \frac{\alpha_2 + \beta_2 - \mu}{4t} + \frac{1}{8t}\right), \tag{8}$$

while the stand-alone firms' profits are given by:

$$\begin{aligned} \Pi_A &= \alpha_2\left(1 - \bar{x} + \tilde{D}_{\{A_2 B_2\}}\right) = \alpha_2\left(\frac{1}{2} + \frac{\mu - \alpha_2 - \beta_2}{4t} - \frac{1}{8t}\right), \\ \Pi_B &= \beta_2\left(1 - \bar{x} + \tilde{D}_{\{A_2 B_2\}}\right) = \beta_2\left(\frac{1}{2} + \frac{\mu - \alpha_2 - \beta_2}{4t} - \frac{1}{8t}\right). \end{aligned} \tag{9}$$

Each profit function is strictly concave in the firm's own margin. Thus, the best responses are characterized by the following first-order conditions:

$$\begin{aligned} \text{For } \mu &: \frac{1}{2} + \frac{\alpha_2 + \beta_2 - 2\gamma_1}{4t} + \frac{1}{8t} = 0, \\ \text{For } \alpha_2 &: \frac{1}{2} + \frac{\mu - 2\alpha_2 - \beta_2}{4t} - \frac{1}{8t} = 0, \\ \text{For } \beta_2 &: \frac{1}{2} + \frac{\mu - \alpha_2 - 2\beta_2}{4t} - \frac{1}{8t} = 0. \end{aligned}$$

Solving these equations leads to a unique candidate equilibrium:

$$\begin{aligned} \mu^* &= \tilde{\mu}^p \equiv \frac{5t}{2} + \frac{1}{8}, \\ \alpha_2^* &= \beta_2^* = \tilde{\rho}_2^p \equiv \frac{3t}{2} - \frac{1}{8}. \end{aligned}$$

The corresponding thresholds are then given by:

$$\bar{x}^* = \bar{x}^p = \frac{5}{8} + \frac{5}{32t}, \; \tilde{x}^* = \tilde{x}^{ps} = \frac{5}{8} - \frac{3}{32t}.$$

The equilibrium profits are:

$$\begin{aligned} \Pi_M^* &\equiv \frac{(\tilde{\mu}^p)^2}{4t} = t\left(\frac{5}{4} + \frac{1}{16t}\right)^2, \\ \Pi_A^* &= \Pi_B^* \equiv \frac{(\tilde{\rho}_2^p)^2}{4t} = t\left(\frac{3}{4} - \frac{1}{16t}\right)^2. \end{aligned} \tag{10}$$

It is easy to verify that $\bar{x} \leq 1$ if and only if $t \geq t_p = \frac{5}{12}$. Therefore, this equilibrium occurs when $t \geq t_p$. In the analysis of existence, we demonstrate that no profitable deviation exists from this candidate equilibrium within this parameter range.

A.1.5 Configuration 5: $0 < \sigma_2(0) < \sigma_2(1) \leq 1$

This case arises when $\tilde{x} < 0 < 1 \leq \bar{x}$. In this candidate equilibrium:

- firm C charges prohibitively high stand-alone margins such that it sells the bundle only to consumers with $s > \sigma_2(x)$,
- firm C charges a margin on the bundle such that $\sigma_2(1) \leq 1$, and
- the stand-alone firms charge a margin such that $\sigma_2(0) > 0$.

The demand for the conglomerate and the stand-alone firms are given by $1 - D_{\{A_2 B_2\}}$ and $D_{\{A_2 B_2\}}$, respectively, where:

$$D_{\{A_2 B_2\}} = \int_0^1 \sigma_2(x)\, dx = (\mu - \alpha_2 - \beta_2).$$

The profits of the conglomerate and the stand-alone firms can be expressed by:

$$\begin{aligned} \Pi_M &= \mu\left(1 - D_{\{A_2 B_2\}}\right) = \mu\left(1 - (\mu - \alpha_2 - \beta_2)\right), \\ \Pi_A &= \alpha_2 D_{\{A_2 B_2\}} = \alpha_2(\gamma - \alpha_2 - \beta_2), \\ \Pi_B &= \beta_2 D_{\{A_2 B_2\}} = \beta_2(\gamma - \alpha_2 - \beta_2). \end{aligned}$$

The demand under this configuration is demonstrated in Figure A.5.

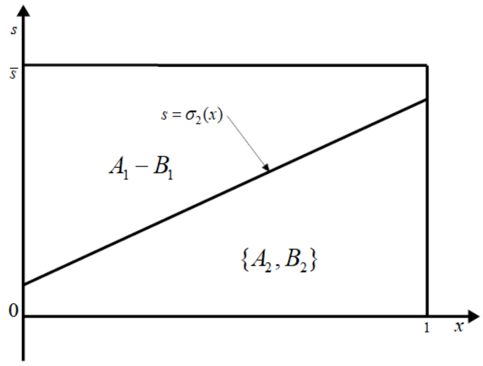

Figure A.5

Solving for the best responses leads to:

$$\mu^* = \frac{3}{4},\ \rho_2^* = \frac{1}{4}. \tag{11}$$

The equilibrium exists if $\sigma_2(0) = \gamma - \alpha_2 - \beta_2 - 2t > 0$ and $\sigma_2(1) < 1$, and these two conditions hold when $t < 1/8$. The equilibrium profits are: $\Pi_M^* = \frac{9}{16}$ and $\Pi_A^* = \Pi_B^* = \frac{1}{16}$.

Now we demonstrate that the aforementioned candidate equilibrium cannot be sustained. Let's consider a deviation in which the conglomerate matches the rivals' margins for the stand-alone products: $\rho_1 = \rho_2$. This allows the conglomerate to cater to its loyal consumers located at $x \leq \hat{x} = 1/2$, who will purchase either the bundle $A_1 - B_1$ or the portfolio $\{A_1, B_1\}$. However, such a deviation will lead some consumers to choose the stand-alone products $\{A_1, B_1\}$ instead of the bundle. The net gain from this deviation can be calculated as follows:

$$\Delta \equiv \mu \int_0^{\hat{x}} \sigma_2(x)\, dx - \sigma_1^2 \hat{x}.$$

Using the equilibrium margins $\mu = \frac{3}{4}$ and $\rho_2^* = \frac{1}{4}$, and noting that $\hat{x} = 1/2$ and $\sigma_1 = \frac{1}{4}$, we obtain:

$$\Delta = \frac{3}{4}\int_0^{1/2} \left(\frac{1}{4} + 4t\left(x - \frac{1}{2}\right)\right) dx - \frac{1}{32} = \frac{1}{16} - \frac{3t}{8} > 0,$$

where the inequality holds when $t < 1/8$. Hence, the candidate equilibrium cannot be sustained in this configuration under the conditions $0 < \sigma_2(0) < \sigma_2(1) < 1$.

A.1.6 Configuration 6: $0 < \sigma_2(0) < 1 < \sigma_2(1)$

This case arises when $\tilde{x} < 0 < \bar{x} < 1$. In this candidate equilibrium:

- firm C charges prohibitively high stand-alone margins such that it sells the bundle only to consumers with $s > \sigma_2(x)$,
- firm C charges a high margin on the bundle such that $\sigma_2(1) > 1$, and
- the stand-alone firms charge a margin such that $\sigma_2(0) > 0$.

Using $\sigma_2(\bar{x}) = 1$, we can calculate:

$$\bar{x} = \frac{1 + 2t + \alpha_2 + \beta_2 - \mu}{4t}.$$

The demand faced by the conglomerate and the stand-alone firms is given by $\bar{x} - \hat{D}_{\{A_2B_2\}}$ and $1 - \bar{x} + \hat{D}_{\{A_2B_2\}}$, respectively, where:

$$\hat{D}_{\{A_2B_2\}} = \int_0^{\bar{x}} \sigma_2(x)\, dx = \bar{x}\left(\frac{1 - 2t + \mu - 2\rho_2}{2}\right).$$

The conglomerate's profit can be expressed as:

$$\Pi_M = \mu\left(\bar{x} - \hat{D}_{\{A_2B_2\}}\right) = 2t\gamma_1\, \bar{x}^2,$$

and the stand-alone firm's profits can be written as:

$$\Pi_A = \alpha_2\left(1 - 2t\, \bar{x}^2\right),\ \Pi_B = \beta_2\left(1 - 2t\, \bar{x}^2\right).$$

The demand is demonstrated in Figure A.6.

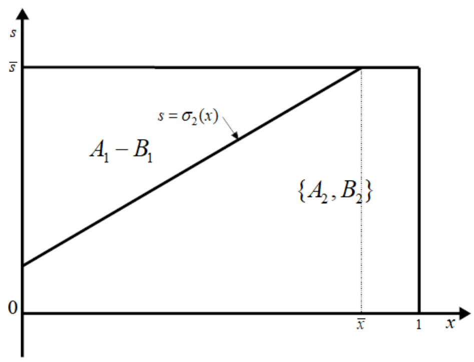

Figure A.6

We now solve for the candidate equilibrium. Differentiating Π_A with respect to α_2, we obtain:

$$\frac{\partial \Pi_A}{\partial \alpha_2} = 1 - 2t\, \bar{x}^2 - \rho_2 \bar{x}.$$

Taking a further derivative with respect to α_2, we have:

$$\frac{\partial^2 \Pi_A}{\partial \alpha_2^2} = -\ 2\bar{x} - \rho_2 \frac{1}{4t} < 0.$$

The profit function is concave and the best response ρ_2 is characterized by the first-order condition:

$$\rho_2 = \frac{1 - 2t\ \bar{x}^2}{\bar{x}}.$$

Now, differentiating Π_M with respect to μ, we have:

$$\frac{\partial \Pi_M}{\partial \mu} = 2t\ \left(\bar{x}^2 - \mu \bar{x} \frac{1}{2t}\right) = \bar{x}\ \ (2t\bar{x} - \mu)\,.$$

There is a unique interior solution for the first-order condition, given by $\mu = 2t\bar{x}$.

Now we demonstrate that this interior optimum cannot be supported in the current configuration. From

$$\bar{x} = \frac{1 + 2t + 2\rho_2 - \mu}{4t},$$

we can derive:

$$4t\bar{x} = 1 + 2t + 2\rho_2 - \mu = 1 + 2t + \frac{2 - 4t\ \bar{x}^2}{\bar{x}} - 2t\bar{x}.$$

By rearranging and using $\omega \equiv 1/t$, we have $10\ \bar{x}^2 - (\omega + 2)\,\bar{x} - 2\omega = 0$. Solving for $\bar{x}$ leads to:

$$\bar{x} = \frac{(\omega + 2) + \sqrt{(\omega + 2)^2 + 80\omega}}{20}.$$

On the one hand, for $\bar{x} < 1$ to hold, we require $\omega \leq \frac{8}{3}$. On the other hand, $\sigma_2(0) = 1 - 4t\bar{x} > 0$ requires $\bar{x} < \omega/4$, which implies $\omega > \frac{20}{3}$. These two conditions contradict each other. Therefore, there is no interior optimum in this configuration, and the candidate equilibrium cannot be supported for $0 < \sigma_2\,(0) < 1 < \sigma_2\,(1)$.

The equilibrium outcomes are summarized below:

Proposition 1 *Consider a conglomerate merger between two markets where products are horizontally differentiated à la Hotelling with perfect correlation across markets. Suppose the merged entity does not commit to pure bundling.*

- *When products are weak differentiated (i.e., $t < t_m = \frac{32}{85}$), there exists a unique symmetric equilibrium where the merged entity engages in mixed bundling. The equilibrium is characterized in Configuration 1.*

- *When products are mildly differentiated (i.e., $t_m < t < t_p = \frac{5}{12}$), a unique symmetric equilibrium exists where the merged firm engages in de facto pure bundling. The equilibrium is characterized in Configuration 3.*
- *When products are strongly differentiated (i.e., $t > t_p$), a unique symmetric equilibrium exists where the merged firm engages in de facto pure bundling. The equilibrium is characterized in Configuration 4.*

A.2 Existence of Equilibria

We have thoroughly characterized the candidate equilibria under product differentiation and identified three configurations, each having a unique candidate equilibrium. We will now examine the existence of each candidate equilibrium by verifying all potential deviations. As the second-order conditions hold for each candidate equilibrium, there are no profitable local deviations within their respective configurations. However, since a firm can unilaterally deviate to other configurations, we must consider all possible "global deviations". We will begin by analyzing unilateral deviations by the conglomerate firm and subsequently assess deviations by the stand-alone firms.

A.2.1 Deviations by the Conglomerate

We show that the conglomerate cannot benefit from any deviations, given the stand-alone firms' margins $\alpha_2^* = \beta_2^* = \rho_2^*$. Firm M has the flexibility to adjust three margins under deviation, namely α_1, β_1, and μ, allowing for numerous possibilities. However, the following Lemma enables us to focus on "symmetric" deviations and limit the number of relevant deviations:

Lemma 2 *Without loss of generality, we can focus on deviations that result in $x_A = x_B \in [0, 1]$ and $\sigma_1 \in [0, 1)$.*

Proof. Without loss of generality, we can focus on deviations that satisfy $\alpha_1 \geq \beta_1$, which implies $x_A \leq x_B$. Furthermore, we can narrow our focus to deviations that result in $x_A \geq 0$ and $x_B \leq 1$. If $x_A < 0$, firm M does not sell product A on a stand-alone basis. In this case, replacing α_1 with the "limit" margin $\alpha_1' = \rho_2^* + t$, leading to $x_A' = 0$, would not affect its profit. On the other hand, if $x_B > 1$, it means that marginal consumers are not interested in B_2. Therefore, firm M could increase its profit by slightly increasing μ and β_1, as it would not affect consumer demand (the choice between the relevant portfolios $\{A_1, B_1\}$ and $\{A_1, B_2\}$ only depends on α_1, and the choice between the bundle and these portfolios only depends on $\mu - \beta_1$). Hence, we can restrict our attention to deviations that lead to $0 \leq x_A \leq x_B \leq 1$.

Let's consider a deviation where $0 \leq x_A < x_B \leq 1$, implying $\alpha_1 > \beta_1$. In this case, the portfolio $\{A_1, B_2\}$ is strictly dominated by the portfolio $\{A_2, B_1\}$, which provides the same utility but at a lower total price (since $\alpha_2^* = \beta_2^*$). As a result, consumers will choose either the bundle (if their consumption synergies are significant) or one of the three stand-alone portfolios: $\{A_2, B_1\}$, $\{A_2, B_2\}$, or $\{A_1, B_1\}$. Specifically, consumers with $x < x_A$ will opt for $\{A_1, B_1\}$ if $s < \sigma_1$, while consumers with $x_A < x < x_B$ will choose $\{A_2, B_2\}$ if:

$$s < \sigma_{21}(x) = \mu - \rho_2^* - \beta_1,$$

and consumers with $x > x_B$ will buy $\{A_2, B_1\}$ if:

$$s < \sigma_2(x) = \mu - 2\rho_2^*.$$

Conversely, consumers choose the bundle if:

$$s > \sigma(x) \equiv \max\{\sigma_1, \sigma_{21}(x), \sigma_2(x)\}.$$

Note that, by construction, $\sigma(x)$ is continuous and increasing in x, and $\sigma(x) = \sigma_1$ for $x \leq x_A$, $\sigma(x) = \sigma_{21}(x)$ for $x_A \leq x \leq x_B$, and $\sigma(x) = \sigma_2(x_B)$ for $x \geq x_B$.

Let's now replace α_1 and β_1 with:

$$\alpha_1' = \beta_1' = \frac{\alpha_1 + \beta_1}{2},$$

and denote $\hat{x} = \frac{x_A + x_B}{2}$ as the resulting threshold for the choice between the stand-alone offers. In other words, among the stand-alone offers, consumers will prefer the conglomerate if $x < \hat{x}$ and will favor the stand-alone firms if $x > \hat{x}$. Since this change does not alter $\sigma_2(x)$, it does not affect the demand of consumers with consumption synergies exceeding $\sigma_2(x_B)$, as they choose between the bundle and $\{A_2, B_2\}$.

Furthermore, suppose there exists consumers with consumption synergies below σ_1 (when $\sigma_1 > 0$). Then:

- consumers with $x < x_A$ or $x > x_B$ are not affected: the first ones continue buying $\{A_1, B_1\}$, and the latter continue purchasing $\{A_2, B_2\}$;

- consumers with $x_A < x < \hat{x}$ switch from $\{A_2, B_1\}$ to the bundle (Area G_1 in Figure A.7), whereas consumers with $\hat{x} < x < x_B$ switch from $\{A_2, B_1\}$ to $\{A_2, B_2\}$ (Area L_1 in Figure A.7). As there are an equal number of consumers in both categories (i.e., $x_B - \hat{x} = \hat{x} - x_A$), the net impact on M's profit is:

$$(\mu - \beta_1) - \beta_1 > \mu - \alpha_1 - \beta_1 = \sigma_1 > 0,$$

  where the first inequality follows from $\alpha_1 > \beta_1$, and the second inequality arises due to the existence of consumers with $0 \leq s < \sigma_1$.

Finally, suppose there exists consumers with consumption synergies $s \in (\sigma_1, \sigma_2(x_B))$. Let:

$$x'_A(s) = \sigma_{21}^{-1}(s)$$

denote the relevant threshold for the choice between the bundle and $\{A_2, B_1\}$ when facing the margins α_1 and β_1. Furthermore, let:

$$\hat{x}'(s) = \frac{x'_A(s) + x_B}{2}$$

denote the similar threshold when facing the margins α'_1 and β'_1. Then:

- consumers with $x < x'_A(s)$ or $x > x_B$ are not affected: the former continue purchasing the bundle, and the latter continue buying $\{A_2, B_2\}$;

- consumers with $x'_A(s) < x < \hat{x}'(s)$ switch from $\{A_2, B_1\}$ to the bundle (with net gain $\mu - \beta_1$ from each consumer; see Area G_2 in Figure A.7), while these with $\hat{x}'(s) < x < x_B$ switch from $\{A_2, B_1\}$ to $\{A_2, B_2\}$ (with a net loss β_1 from each consumer; see Area L_2 in Figure A.7). As there are an equal number of consumers in both categories (i.e., $x_B - \hat{x}' = \hat{x}' - x'_A(s)$ and the Area G_2 and Area L_2 are equal), the net impact on M's profit is:

$$(\mu - \beta_1) - \beta_1 > 0,$$

  where the inequality follows from the existence of consumers with $0 \leq s < \sigma_2(x_B) = \mu - 2\beta_1$.

The gain and loss of demand resulting from this modification are illustrated in Figure A.7 (for the case with $\sigma_1 > 0$ and $\sigma_2(1) < 1$), where Area G_1 and Area G_2 stand for the gain of demand and Area L_1 and Area L_2 are the loss of demand from this modification. Since $\hat{x}$ is the midpoint between x_A and x_B, Area G_1 is equal to Area L_1, and Area G_2 is equal to Area L_2.

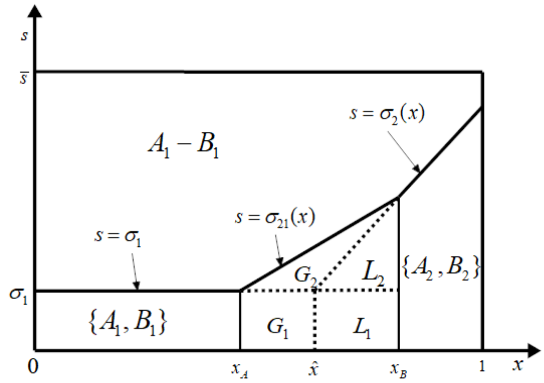

Figure A.7

Thus, we can focus on deviations involving symmetric stand-alone margins: $\alpha_1 = \beta_1$, which implies $x_A = x_B$ and $\sigma_2(x_A) = \sigma_2(x_B) = \sigma_1$. Furthermore, we can narrow our attention to deviations satisfying $\sigma_1 \geq 0$. To understand why, consider a deviation where $\alpha_1 = \beta_1$ and $\sigma_1 < 0$. In this case, consumers only choose between the bundle and the offers of the stand-alone firms. Let $\tilde{x} \equiv \sigma_2^{-1}(0)$ denote the lowest threshold above which some consumers may prefer the stand-alone firms. Raising α_1 and β_1 to $\alpha_1' = \beta_1' = \mu/2$ would then yield $\sigma_1' = 0$ and $x_A' = x_B' = \tilde{x}$, thereby leaving M's profit unchanged. Consumers with $x < \tilde{x}$ would still choose the bundle on which M charges the same margin μ, whereas consumers with $x > \tilde{x}$ would opt for the stand-alone firms if $s < \sigma_2(x)$ and revert to the bundle if $s > \sigma_2(x)$.

Finally, we show that we can further narrow our focus to deviations satisfying $\sigma_1 < 1$. To see this, consider a deviation with μ and $\alpha_1 = \beta_1$, resulting in $\sigma_1 \geq 1$. In this case, consumers do not purchase the bundle but instead choose either $\{A_1, B_1\}$ (if $x < \hat{x} \equiv x_A = x_B$) or $\{A_2, B_2\}$ (if $x > \hat{x}$). Now, if we reduce the margin for the bundle to $\mu = 1 + \alpha_1 + \beta_1 - \varepsilon$, where ε is a positive but arbitrarily small number, it does not affect the demand from consumers with $s < 1 - \varepsilon$, who continue to choose either $\{A_1, B_1\}$ or $\{A_2, B_2\}$. However, among the consumers with $s > 1 - \varepsilon$, this change encourages those with $x < \hat{x}$ to switch from $\{A_1, B_1\}$ to the bundle and also prompts some consumers with $x > \hat{x}$ (specifically, those for whom $s > \sigma_2(x)$) to switch from $\{A_2, B_2\}$ to the bundle. It's important to note that for this deviation to be profitable, the initial deviation must satisfy $\alpha_1 + \beta_1 > 0$. Then, the condition $\sigma_1 \geq 1$ implies $\mu \geq \alpha_1 + \beta_1 > 0$. Consequently, the alternative deviation resulting in $\sigma_1 < 1$ is strictly more profitable than any deviation with $\sigma_1 \geq 1$. ■

Now we examine the existence for all three types of equilibrium under weak, mild, and strong product differentiation, respectively.

Weak Product Differentiation ($t < t_m$) We first verify the existence of the equilibrium under weak differentiation (i.e., $t \leq t_m = \frac{32}{85}$). The candidate equilibrium with mixed bundling is characterized in Configuration 1. In this equilibrium, the conglomerate offers a price discount for the stand-alone products such that $\sigma_1 > 0$. The equilibrium arises under the conditions $0 < \sigma_1 < \sigma_2(1) \leq 1$ and $x_A = x_B = \hat{x} \in (0,1)$. Denoting by $\omega \equiv \frac{1}{t}$, this equilibrium occurs in the parameter region $\omega \geq \frac{85}{32}$.

Now, we examine the relevant deviations by the conglomerate given $\alpha_2^* = \beta_2^* = \rho_2^m = \frac{5t}{4}$. The relevant cut-off thresholds are now given by:

$$\begin{aligned} \hat{x} &\equiv \frac{1}{2} - \frac{\rho_1 - \rho_2^m}{2t} = \frac{5}{8} - \frac{\rho_1}{2t}, \\ \sigma_2(x) &= \mu - 2\rho_2^m + 4t\left(x - \frac{1}{2}\right) = \mu - \frac{9}{2}t + 4tx. \end{aligned}$$

The next Lemma 2 shows that we can further ignore deviations for which $\sigma_1 = 0$ (i.e., any deviations leading to pure bundling):

Lemma 3 *Without loss of generality, we can focus on deviations that lead to $x_A = x_B \in [0,1]$ and $\sigma_1 \in (0,1)$.*

Proof. From Lemma 2, we can restrict our attention to deviations of the form $\alpha_1 = \beta_1 = \rho_1$. Consider such a deviation and further suppose $\rho_1 = \mu/2$, which implies $\sigma_1 = \mu - 2\rho_1 = 0$. Two cases can arise based on whether $\sigma_2(1)$ is below or above 1. Let $\underline{\mu}$ be the threshold value of μ such that $\sigma_2(1) = \mu - \frac{1}{2}t = 1$.

Suppose $\sigma_2(1) \geq 1$, which occurs when:

$$\mu \geq \underline{\mu} = 1 + \frac{t}{2}.$$

Recall that $\tilde{x}$ and $\bar{x}$ denote the location thresholds such that $\sigma_2(\tilde{x}) = 0$ and $\sigma_2(\bar{x}) = 1$, respectively. The demands for the conglomerate and for stand-alone firms are equal to $\bar{x} - \tilde{D}_{\{A_2B_2\}}$ and $1 - \bar{x} + \tilde{D}_{\{A_2B_2\}}$, where:

$$\tilde{D}_{\{A_2B_2\}} = \int_{\tilde{x}}^{\bar{x}} \sigma_2(x)\, dx = \frac{1}{8t}.$$

Using $\sigma_2(\tilde{x}) = 0$ and $\sigma_2(\bar{x}) = 1$, we have:

$$\tilde{x} = \frac{1}{2} + \frac{2\hat{\rho}_2 - \mu}{4t} = \frac{9}{8} - \frac{\mu}{4t} \text{ and } \bar{x} = \frac{9}{8} + \frac{1-\mu}{4t}.$$

Firm M's profit is given by:

$$\Pi_M = \mu\left(\bar{x} - \tilde{D}_{\{A_2B_2\}}\right) = \mu\left(\frac{9}{8} - \frac{\mu}{4t} + \frac{1}{8t}\right),$$

which is strictly concave in in μ. Solving for the best response yields:

$$\mu^d = \frac{1}{4} + \frac{9}{4}t.$$

However, $t \leq \frac{32}{85}$ implies $\mu^d < \underline{\mu}$. Hence, the maximum profit is achieved at the boundary $\mu = \underline{\mu}$, which is equal to:

$$\Pi_M = \underline{\mu}\left(\frac{9}{8} - \frac{\underline{\mu}}{4t} + \frac{1}{8t}\right) = \left(1 + \frac{t}{2}\right)\left(1 - \frac{1}{8t}\right) = t\left(\omega + \frac{1}{2}\right)\left(1 - \frac{\omega}{8}\right).$$

Comparing with the equilibrium profit:

$$\Pi_M = \frac{1}{4} + \frac{107}{64}t - \frac{2375t^2}{4096} = t\left(\frac{\omega}{4} + \frac{107}{64} - \frac{2375}{4096\omega}\right),$$

we have:

$$\frac{(\Pi_M - \Pi_M)}{t} = \frac{N(\omega)}{4096\omega},$$

where $N(\omega) \equiv 512\omega^3 - 2816\omega^2 + 4800\omega - 2375$. Note that $N\left(\frac{85}{32}\right) > 0$ and $N'(\omega) > 0$ for $\omega \geq \frac{85}{32}$ imply $N(\omega) > 0$ for $\omega \geq \frac{85}{32}$. It follows that $\Pi_M > \Pi_M$, and such deviation is not profitable.

Consider now the case where $\sigma_2(1) < 1$, which occurs when:

$$\mu < \underline{\mu} = 1 + \frac{t}{2}.$$

Then, firm M's profit can be expressed as:

$$\Pi_M(\mu) = \mu\left[1 - 2t(1-\tilde{x})^2\right],$$

where $\tilde{x}$ is such that $\sigma_2(\tilde{x}) = \sigma_1 = 0$ and satisfies:

$$4t(1-\tilde{x}) = \sigma_2(1).$$

Differentiating Π_M with respect to μ yields:

$$\begin{aligned} \frac{\partial \Pi_M}{\partial \mu} &= 1 - 2t(1-\tilde{x})^2 - \mu(1-\tilde{x}) \\ &> 1 - \frac{1+\underline{\mu}}{8t} = \frac{15t-4}{16t} > 0, \end{aligned}$$

where the first inequality comes from $1-\tilde{x} = \sigma_2(1)/4t < 1/4t$ and $\mu < \underline{\mu}$, and the last inequality follows from $t > \frac{32}{85}$. Hence, any deviation involving $\sigma_1 = 0$ and $\mu < \underline{\mu}$ is dominated by the deviation with $\sigma_1 = 0$ and $\mu = \underline{\mu}$ (i.e.$\sigma_2(1) = 1$), and the latter is not profitable as we have already shown. ■

Relevant Deviations

It follows from Lemmas 2 and 3 that we can restrict our attention to deviations with symmetric stand-alone margins, of the form $\alpha_1 = \beta_1 = \rho_1$, and for which $0 < \sigma_1 = \mu - 2\rho_1 < 1$. Two cases can arise, depending on whether $\sigma_2(1)$ is above or below 1. For $0 < \sigma_1 < \sigma_2(1) \leq 1$, we know that the candidate equilibrium maximizes the conglomerate's profit in this configuration, and no such deviation can be profitable.

Hence, we only need to consider the deviation leading to $0 < \sigma_1 < 1 < \sigma_2(1)$. This case arises when:

$$\mu \geq \underline{\mu} = 1 + \frac{t}{2}.$$

Recall that:

$$\bar{x} = \sigma_2^{-1}(\overline{s}) = \frac{2\rho_2^m - \mu + \overline{s} + 2t}{4t} = \frac{9}{8} + \frac{\overline{s} - \mu}{4t},$$

which represents the largest location for which a consumer may be buying the bundle (any consumer with $x > \bar{x}$ will patronize the stand-alone firms, as $s \leq \overline{s} < \sigma_2(x)$). The demand for firm M's stand-alone offering is given by:

$$D_{\{A_1B_1\}}(\mu, \rho_1) = \sigma_1 \hat{x},$$

whereas the demand for the bundle is:

$$D_{A_1-B_1}(\mu, \rho_1) = (1 - \sigma_1)\hat{x} + \int_{\hat{x}}^{\bar{x}} (1 - \sigma_2(x))\, dx.$$

Therefore, firm M's profit can be expressed as:

$$\Pi_M(\mu, \rho_1) = \mu D_{A_1-B_1} + 2\rho_1 D_{\{A_1B_1\}} = \mu \left[\bar{x} - \int_{\hat{x}}^{\bar{x}} \sigma_2(x) dx\right] - \sigma_1^2 \hat{x}.$$

We first characterize the optimal deviation under the configuration $0 < \sigma_1 < 1 < \sigma_2(1)$. Differentiating with respect to ρ_1 (noting that $\bar{x}$ does not depend on ρ_1 and using $\sigma_2(\hat{x}) = \sigma_1$) yields:

$$\frac{\partial \Pi_M}{\partial \rho_1}(\mu, \rho_1) = \frac{\sigma_1}{t}(4t\hat{x} - \rho_1) = \frac{\sigma_1}{t}(2t + 2\rho_2^m - 3\rho_1).$$

It follows from $\sigma_1 > 0$ that $\Pi_M(\mu, \rho_1)$ is strictly quasi-concave in ρ_1 and is maximal for:

$$\rho_1 = \rho_1^d = \frac{3}{2}t.$$

The condition $\sigma_1 > 0$ then amounts to (denote by $\mathring{\mu}$ the threshold of μ such that $\sigma_1 = 0$):

$$\mu > \mathring{\mu} \equiv 2\rho_1^d = 3t.$$

Then:

$$\underline{\mu} - \mathring{\mu} = 1 - \frac{5}{2}t > 1 - \frac{5}{2}t_m = \frac{1}{17} > 0.$$

The above condition implies that, for any $\mu > \underline{\mu} > \mathring{\mu}$:

$$\sigma_1 > \underline{\tau}_1 \equiv \underline{\mu} - 2\rho_1^m = 1 - \frac{5t}{2} > \frac{1}{17}.$$

Thus, the optimal deviation must satisfy $\rho_1 = \rho_1^d$, implying:

$$\hat{x} = \frac{1}{2} - \frac{\rho_1^d - \hat{\rho}_2}{2t} = \frac{3}{8}.$$

Now, we show that the optimal deviation is not profitable. Using:

$$\begin{aligned}
\bar{x} &= \hat{x} + \frac{1-\sigma_1}{4t}, \\
\mu &= \sigma_1 + 2\rho_1^d = \sigma_1 + 8t\hat{x}, \\
\int_{\hat{x}}^{\bar{x}} \sigma_2(x)dx &= (\bar{x} - \hat{x})\frac{1+\sigma_1}{2} = \frac{(1-\sigma_1)(1+\sigma_1)}{8t},
\end{aligned}$$

we can further express M's profit as a function of σ_1:

$$\begin{aligned}
\Pi_M(\sigma_1) &= (\sigma_1 + 8t\hat{x})\left(\hat{x} + \frac{1-\sigma_1}{4t} - \frac{(1-\sigma_1)(1+\sigma_1)}{8t}\right) - \sigma_1^2\hat{x} \\
&= (\sigma_1 + 8t\hat{x})\left(\hat{x} + \frac{(1-\sigma_1)^2}{8t}\right) - \sigma_1^2\hat{x}.
\end{aligned}$$

The relevant range is $\sigma_1 \in (\underline{\tau}_1, 1)$, and the derivative is given by

$$\Pi_M'(\sigma_1) = \frac{1 - 8t\hat{x} - 4\sigma_1 + 3\sigma_1^2}{8t} = \frac{1 - 3t - 4\sigma_1 + 3\sigma_1^2}{8t}.$$

Then, $\Pi_M'(\sigma_1)$ is convex and satisfies:

$$\begin{aligned}
\Pi_M'(1) &= \frac{-3}{8} < 0, \\
\Pi_M'(\underline{\tau}_1) &= -\frac{(32 - 75t)}{32} < 0,
\end{aligned}$$

where the second inequality comes from $t \leq \frac{32}{85}$. It follows that $\Pi_M(\sigma_1) < 0$ in the relevant range $\sigma_1 \in (\underline{\tau}_1, 1)$. Thus, any deviation leading to $\sigma_1 > \underline{\tau}_1$ is dominated by a deviation leading to $\sigma_1 = \underline{\tau}_1$ (i.e., $\sigma_2(1) = 1$). However, we know that any deviation leading to $0 < \sigma_1 < \sigma_2(1) = 1$ is not profitable.

Mild Product Differentiation ($t_m < t < t_p$) The equilibrium with de facto pure bundling occurs under mild product differentiation occurs (i.e., $\frac{32}{85} \leq t \leq t_p = \frac{5}{12}$). The candidate equilibrium is characterized in Configuration 4 with $\sigma_1 \leq 0 < \sigma_2(1) \leq 1$. Using $\omega = \frac{1}{t}$, the relevant parameter range is:

$$\frac{12}{5} \leq \omega \leq \frac{85}{32}. \tag{12}$$

Firm M's equilibrium profit can be rewritten as:

$$\Pi_M = \breve{\mu}^p \left(1 - 2t\left(1 - \tilde{x}^p\right)^2\right) = t\left(\frac{4\sqrt{1+10\omega} - 6}{5}\right)\left(\frac{20\omega - 1 - \sqrt{1+10\omega}}{25\omega}\right). \tag{13}$$

Thanks to Lemma 1, without loss of generality, we can focus on deviations with $\alpha_1 = \beta_1 = \rho_1$, given that the margins of stand-alone firms are $\alpha_2^* = \beta_2^* = \breve{\rho}_2^p = t\frac{\sqrt{1+10\omega}+1}{5}$. Consumers' relevant options include the bundle and the two stand-alone portfolios offered by firm M and by the stand-alone firms. Specifically, consumers with $x < \hat{x}$ will purchase $\{A_1, B_1\}$ if $s < \sigma_1$, where:

$$\hat{x} = \frac{1}{2} - \frac{\rho_1 - \breve{\rho}_2}{2t} = \frac{\sqrt{1+10\omega}+6}{10} - \frac{\rho_1}{2t}.$$

Alternatively, these consumers will opt for the bundle if $s > \sigma_1$. Conversely, consumers with $x > \hat{x}$ will buy $\{A_2, B_2\}$ if $s < \sigma_2(x)$, where:

$$\sigma_2(x) = \mu - 2\breve{\rho}_2 + 4t\left(x - \frac{1}{2}\right) = \mu - \frac{2t\left(\sqrt{1+10\omega}+6\right)}{5} + 4tx, \tag{14}$$

These consumers will opt for the bundle if $s > \sigma_2(x)$.

The following Lemma demonstrates that we can further disregard deviations where $\sigma_1 = 0$ (i.e., any deviations leading to pure bundling):

Lemma 4 *Without loss of generality, we can focus on deviations leading to $x_A = x_B \in [0,1]$ and $\sigma_1 \in (0,1)$.*

Proof. From Lemma 2, we can limit our focus to deviations of the form $\alpha_1 = \beta_1 = \rho_1$. Consider a deviation with $\rho_1 = \mu/2$, such that $\sigma_1 = \mu - 2\rho_1 = 0$. Two scenarios can then arise, depending on whether $\sigma_2(1)$ falls below or above 1. Let's denote by $\underline{\mu}$ the threshold value of μ for which $\sigma_2(1) = \mu - 2\breve{\rho}_2 + 2t = 1$.

If $\sigma_2(1) \leq 1$, then the demands for the conglomerate and for the stand-alone firms are respectively equal to $1 - D_{\{A_2B_2\}}$ and $D_{\{A_2B_2\}}$, where $D_{\{A_2B_2\}} = 2t\left(1 - \tilde{x}\right)^2$, and $\tilde{x}$ is such that $\sigma_2(\tilde{x}) = 0$. Consequently, the profit of firm M, as given by (6) (with the margins of the stand-alone firms set to their equilibrium values: $\alpha_2 = \beta_2 = \breve{\rho}_2^p$), is strictly concave in μ and reaches its maximum for $\mu^* = \breve{\mu}$.

Let's consider the case where $\sigma_2(1) > 1$. Using (14), this condition arises when:

$$\mu > \underline{\mu} = 1 + 2\check{\rho}_2 - 2t = 1 + \frac{t\left(2\sqrt{1+10\omega} - 8\right)}{5}.$$

Firm M's profit can be expressed as:

$$\Pi_M(\mu) = \mu\left(\bar{x} - D_{\{A_2B_2\}}\right) = \mu\left(\frac{1}{2} + \frac{2\check{\rho}_2 - \mu}{4t} + \frac{1}{8t}\right),$$

where $\bar{x}$ is such that $\sigma_2(\bar{x}) = 1$.

Differentiating Π_M with respect to μ, we obtain:

$$\begin{aligned}
\Pi_M(\mu) &= \frac{1}{2} + \frac{2\check{\rho}_2 - 2\gamma_1}{4t} + \frac{1}{8t} \\
&< \frac{3}{2} - \frac{3\omega}{8} - \frac{\check{\rho}_2}{2t} \\
&= \frac{3}{2} - \frac{3\omega}{8} - \frac{\sqrt{1+10\omega}+1}{10} \leq 0,
\end{aligned}$$

where the first inequality comes from $\mu > \underline{\mu}$, and the last inequality follows from $\omega \geq \frac{12}{5}$. Hence, any deviation involving $\sigma_1 = 0$ and $\mu > \underline{\mu}$ is dominated by the deviation with $\sigma_1 = 0$ and $\mu = \underline{\mu}$ (i.e. $\sigma_2(1) = 1$). However, we have already shown that deviations leading to $\sigma_2(1) = 1$ is not profitable. ■

It follows from Lemmas 2 and 4 that we can restrict our attention to deviations with symmetric stand-alone margins $\alpha_1 = \beta_1 = \rho_1$ satisfying $0 < \sigma_1 = \mu - 2\rho_1 < 1$. We examine two relevant cases that depend on whether $\sigma_2(1)$ lies above or below 1.

(1): Deviations leading to $0 < \sigma_1 \leq \sigma_2(1) \leq 1$

Using (14), this case arises when $\mu \leq \underline{\mu}$. The demand for firm M's stand-alone products is given by:

$$D_{\{A_1B_1\}}(\mu, \rho_1) = \sigma_1\hat{x},$$

whereas the demand for the bundle is given by:

$$D_{A_1-B_1}(\mu, \rho_1) = (1-\sigma_1)\hat{x} + \int_{\hat{x}}^{1}[1-\sigma_2(x)]\,dx.$$

Then, firm M's profit can be expressed as:

$$\Pi_M(\mu, \rho_1) = \mu D_{A_1-B_1} + 2\rho_1 D_{\{A_1B_1\}} = \mu\left[1 - \int_{\hat{x}}^{1}\sigma_2(x)\,dx\right] - \sigma_1^2\hat{x}.$$

First, we solve for the optimal deviation. By differentiating Firm M's profit with respect to ρ_1 and using $\sigma_2(\hat{x}) = \sigma_1$, we get:

$$\frac{\partial\Pi_M}{\partial\rho_1}(\mu, \rho_1) = \frac{\sigma_1}{t}(4t\hat{x} - \rho_1) = \frac{\sigma_1}{t}(2t + 2\check{\rho}_2 - 3\rho_1).$$

Given that $\sigma_1 > 0$, it follows that $\Pi_M(\mu, \rho_1)$ is strictly quasi-concave in ρ_1 and reaches its maximum at:

$$\rho_1 = \rho_1^d = \frac{2t + 2\check{\rho}_2}{3} = \frac{2t\left(\sqrt{1+10\omega}+6\right)}{15}.$$

The condition $\sigma_1 > 0$ then implies:

$$\mu > \mathring{\mu} = 2\rho_1^d = \frac{4t\left(\sqrt{1+10\omega}+6\right)}{15},$$

where $\mathring{\mu}$ denotes the threshold of μ such that $\sigma_1 = 0$.

We will now demonstrate that the optimal deviation is not profitable. To do this, we need to compare $\mathring{\mu}$ with $\underline{\mu}$, considering two scenarios: either either $\mathring{\mu} > \underline{\mu}$ or $\mathring{\mu} \leq \underline{\mu}$.

Suppose $\mathring{\mu} \geq \underline{\mu}$. Then, $\mu \leq \underline{\mu} < \mathring{\mu}$ implies $\sigma_1 < 0$. Since $\Pi_M(\mu, \rho_1)$ is continuous and strictly quasi-concave in ρ_1, it follows that any deviation leading to $0 < \sigma_1 < 1 \leq \sigma_2(1)$ is dominated by the deviation that leads to $\sigma_1 = 0$ (maintaining μ and adjusting ρ_1 to $\mu/2$). However, according to Lemma 4, deviations where $\sigma_1 = 0$ are not profitable.

Suppose $\mathring{\mu} < \underline{\mu}$. For any $\mu \in (\mathring{\mu}, \underline{\mu}]$, $\sigma_1 > 0$ and the optimal deviation must satisfy $\rho_1 = \rho_1^d$, which implies:

$$\hat{x} = \frac{1}{2} - \frac{\rho_1^d - \check{\rho}_2^p}{2t} = \frac{1}{6} + \frac{\check{\rho}_2^p}{6t}.$$

Using $\sigma_2(x) = \mu - 2\check{\rho}_2 + 4t\left(x - \frac{1}{2}\right) = \sigma_1 + 4t(x - \hat{x})$, we can rewrite Π_M as a function of σ_1:

$$\Pi_M(\sigma_1) = (\sigma_1 + 8t\hat{x})\left[1 - \sigma_1(1-\hat{x}) - 2t(1-\hat{x})^2\right] - \sigma_1^2\hat{x}.$$

The relevant range is $\sigma_1 \in (0,1)$ and the derivative is given by:

$$\Pi_M'(\sigma_1) = 1 - 2\sigma_1 - 2t(1-\hat{x})^2 - 8t\hat{x}(1-\hat{x}),$$

which satisfies:

$$\Pi_M'(1) = -1 - 2t(1-\hat{x})^2 - 8t\hat{x}(1-\hat{x}) < 0,$$

and:

$$\begin{aligned}
\Pi_M'(0) &= 1 - 2t(1-\hat{x})(1+3\hat{x}) \\
&= \frac{160\omega - 383 - 8\sqrt{1+10\omega}}{150} \\
&\leq \frac{160 \times \frac{85}{32} - 383 - 8\sqrt{1 + 10 \times \frac{85}{32}}}{150} = 0,
\end{aligned}$$

where the inequality comes from $\omega \leq \frac{85}{32}$.

Given that the second-order derivative is negative, it follows that $\Pi_M(\sigma_1) < 0$ within the relevant range of $\sigma_1 \in (0,1)$. Therefore, any deviation leading to $\sigma_1 > 0$ is once again dominated by the deviation resulting in $\sigma_1 = 0$. However, according to Lemma 4, there are no profitable deviations when $\sigma_1 = 0$.

(2): Deviation leading to $0 < \sigma_1 < 1 < \sigma_2(1)$

This case arises when $\mu > \underline{\mu}$. Recall that:

$$\bar{x} = \sigma_2^{-1}(1) = \frac{2\check{\rho}_2 - \mu + 1 + 2t}{4t} = \frac{3}{5} + \frac{\sqrt{1+10\omega}}{10} + \frac{\omega}{4} - \frac{\mu}{4t},$$

which represents the maximum location x at which a consumer will choose to buy the bundle. The demand for M's stand-alone products is given by:

$$D_{\{A_1B_1\}}(\mu, \rho_1) = \sigma_1 \hat{x},$$

while the demand for the bundle is:

$$D_{A_1-B_1}(\mu, \rho_1) = (1-\sigma_1)\hat{x} + \int_{\hat{x}}^{\bar{x}} [1 - \sigma_2(x)]\, dx.$$

Therefore, firm M's profit can be expressed as:

$$\Pi_M(\mu, \rho_1) = \mu D_{A_1-B_1} + 2\rho_1 D_{\{A_1B_1\}} = \mu \left[\bar{x} - \int_{\hat{x}}^{\bar{x}} \sigma_2(x) dx\right] - \sigma_1^2 \hat{x}.$$

First, we characterize the optimal deviation. Differentiating Π_M with respect to ρ_1 (noting that $\bar{x}$ does not depend on ρ_1, and using $\sigma_2(\hat{x}) = \sigma_1$), we obtain:

$$\frac{\partial \Pi_M}{\partial \rho_1}(\mu, \rho_1) = -\mu \frac{\sigma_1}{2t} + \frac{\sigma_1^2}{2t} + 4\sigma_1 \hat{x} = \frac{\sigma_1}{t}(4t\hat{x} - \rho_1).$$

Given that $\sigma_1 > 0$, it follows that $\Pi_M(\mu, \rho_1)$ is strictly quasi-concave in ρ_1 and is maximized when:

$$\rho_1 = \rho_1^d = \frac{2t + 2\check{\rho}_2}{3}.$$

The condition $\sigma_1 > 0$ then corresponds to $\mu > \mathring{\mu} = 2\rho_1^d$. Similarly, we need to consider two cases, depending on whether $\mathring{\mu} > \underline{\mu}$ or $\mathring{\mu} \leq \underline{\mu}$ (recall that $\underline{\mu}$ is the threshold of μ such that $\sigma_2(1) = 1$).

Suppose $\mathring{\mu} > \underline{\mu}$. Given that $\Pi_M(\mu, \rho_1)$ is continuous and strictly quasi-concave in ρ_1, it follows that any deviation involving $\mu \in [\underline{\mu}, \mathring{\mu})$ (i.e., $\sigma_1 < 0$) is dominated by a deviation leading to $\sigma_1 = 0$. But deviations where $\sigma_1 = 0$ is not profitable from Lemma 4.

We now focus on the case where $\mu > \mathring{\mu}$ (i.e., $\sigma_1 > 0$). From the above analysis, for any such μ, the optimal deviation involves $\rho_1 = \rho_1^d$, which implies:

$$\hat{x} = \frac{1}{2} - \frac{\rho_1^d - \check{\rho}_2^p}{2t} = \frac{1}{6} + \frac{\check{\rho}_2}{6t}.$$

Using:

$$\begin{aligned}
\bar{x} &= \hat{x} + \frac{1-\sigma_1}{4t}, \\
\mu &= \sigma_1 + 2\rho_1^d = \sigma_1 + 8t\hat{x}, \\
\int_{\hat{x}}^{\bar{x}} \sigma_2(x)dx &= (\bar{x} - \hat{x}) \frac{1+\sigma_1}{2} = \frac{(1-\sigma_1)(1+\sigma_1)}{8t},
\end{aligned}$$

we can express M's profit as a function of σ_1:

$$\Pi_M(\sigma_1) = (\sigma_1 + 8t\hat{x}) \left(\hat{x} + \frac{1-\sigma_1}{4t} - \frac{(1-\sigma_1)(1+\sigma_1)}{8t} \right) - \sigma_1^2 \hat{x},$$

where the relevant range is $\sigma_1 \in (0,1)$.

Differentiating $\Pi_M(\sigma_1)$ with respect to σ_1, we have:

$$\Pi_M'(\sigma_1) = \frac{1 - 8t\hat{x} - 4\sigma_1 + 3\sigma_1^2}{8t},$$

which is convex and satisfies:

$$\Pi_M(1) = -\hat{x} < 0,$$

and:

$$\Pi_M(0) = \frac{1 - 2\rho_1^d}{8t} = \frac{1}{8} \left(\omega - \frac{4\sqrt{1+10\omega} + 24}{15} \right) < 0,$$

where the inequality comes from $\omega \leq \frac{85}{32}$.

It follows that $\Pi_M(\sigma_1) < 0$ in the relevant range $\sigma_1 \in (0,1)$. Hence, any deviation leading to $\sigma_1 > 0$ is once again dominated by a deviation that leads to $\sigma_1 = 0$, which is not profitable from Lemma 4**.**

Suppose $\hat{\mu} \leq \underline{\mu}$. Note that, given $\rho_1 = \rho_1^d$, $\Pi_M(\mu) = \Pi_M(\sigma_1) < 0$ with the relevant range $\sigma_1 \in (0,1)$, which corresponds to the relevant range of $\mu \in (\hat{\mu}, \bar{\mu})$ with $\bar{\mu} = 2\rho_1^d + 1$. Also note that $\omega \leq \frac{85}{32}$ implies:

$$\bar{\mu} - \underline{\mu} = \frac{4t + 4\check{\rho}_2^p}{3} - 2\check{\rho}_2^p + 2t = \frac{t}{15}\left(48 - 2\sqrt{1+10\omega}\right) > 0.$$

Therefor, for $\bar{\mu} \geq \mu > \underline{\mu} > \mathring{\mu}$, any deviation leading to $\mu > \underline{\mu}$ is dominated by a deviation leading to $\mu = \underline{\mu}$ (and $\sigma_2(1) = 1$). However, we have already demonstrated that such a deviation is not profitable.

Strong Product Differentiation ($t > t_p$) Another equilibrium with de facto pure bundling arises under strong product differentiation (i.e., $t \geq t_p = \frac{5}{12}$), and the candidate equilibrium is characterized in Configuration 5 with $\sigma_2(1) \geq 1$ and $\sigma_1 \leq 0$. We can rewrite the relevant parameter region as:

$$t \geq \frac{5}{12} \iff \omega = \frac{1}{t} \leq \frac{12}{5}. \tag{15}$$

Now, we demonstrate that the conglomerate cannot benefit from deviations, given that the stand-alone firms set their margins $\alpha_2^* = \beta_2^* = \tilde{\rho}_2^p = \frac{3t}{2} - \frac{1}{8}$. Thanks to Lemma 2, we can restrict our attention to relevant deviations with $x_A = x_B \in [0,1]$ and $\sigma_1 \in [0,1)$. Under such deviations, the thresholds are given by:

$$\hat{x} = \frac{1}{2} - \frac{\rho_1 - \tilde{\rho}_2^p}{2t} = \frac{5}{4} - \frac{\omega}{16} - \frac{\rho_1}{2t},$$

and:

$$\sigma_2(x) = \mu - 2\tilde{\rho}_2^p + 4t\left(x - \frac{1}{2}\right) = \mu + \frac{1}{4} - 5t + 4tx. \tag{16}$$

The following Lemma demonstrates that we can further disregard deviations for which $\sigma_1 = 0$:

Lemma 5 *Without loss of generality, we can focus on deviations leading to $x_A = x_B \in [0,1]$ and $\sigma_1 \in (0,1)$.*

Proof. From Lemma 2, we can restrict our attention to deviations with $\alpha_1 = \beta_1 = \rho_1$. Let's consider a deviation with $\rho_1 = \mu/2$, implying $\sigma_1 = \mu - 2\rho_1 = 0$. Two cases can then arise, depending on whether $\sigma_2(1)$ lies below or above 1. Let $\underline{\gamma}_1$ denote the threshold of μ such that $\sigma_2(1) = \mu + \frac{1}{4} - t = 1$.

Suppose $\sigma_2(1) \geq 1$. Then, the demand for the conglomerate and for the stand-alone firms is respectively equal to $\bar{x} - \tilde{D}_{\{A_2B_2\}}$ and $1 - \bar{x} + \tilde{D}_{\{A_2B_2\}}$, where $\tilde{D}_{\{A_2B_2\}} = \omega/8$. Consequently, the profit of the conglomerate is given by:

$$\Pi_M = \mu\left(\bar{x} - \tilde{D}_{\{A_2B_2\}}\right) = \mu\left(\frac{1}{2} - \frac{\mu - 2\tilde{\rho}_2}{4t} + \frac{\omega}{8}\right),$$

which is strictly concave in μ and is maximized at $\mu = \tilde{\mu}^p$.

Suppose $\sigma_2(1) < 1$. Using (16), this implies:

$$\mu < \underline{\mu} = t + \frac{3}{4}.$$

The merged firm M's profit can be expressed as:

$$\Pi_M(\mu) = \mu\left[1 - 2t(1 - \tilde{x})^2\right],$$

where $\tilde{x}$ is such that $\sigma_2(\tilde{x}) = 0$ and satisfies:

$$4t(1-\tilde{x}) = \sigma_2(1).$$

Differentiating $\Pi_M(\mu)$ with respect to μ, we obtain:

$$\Pi_M'(\mu) = 1 - 2t(1-\tilde{x})^2 - \mu(1-\tilde{x}) > \frac{12t-5}{16t} \geq 0,$$

where the first inequality comes from $1 - \tilde{x} = \sigma_2(1)/4t < 1/4t$ and $\mu < \underline{\mu}$, while the last inequality follows from (15). Hence, any deviation involving $\sigma_1 = 0$ and $\mu < \underline{\mu}$ is dominated by the deviation with $\sigma_1 = 0$ and $\mu = \underline{\mu}$ ($\sigma_2(1) = 1$). However, we have already shown that the deviation leading to $\sigma_2(1) = 1$ is not profitable. ■

As inferred from Lemmas 2 and 5, we can narrow our focus to deviations where the stand-alone margins are symmetric, i.e., $\alpha_1 = \beta_1 = \rho_1$, which satisfy $0 < \sigma_1 = \mu - 2\rho_1 < 1$. Under this condition, two cases can arise depending on whether $\sigma_2(1)$ is greater or lesser than 1.

(1): Deviation leading to $0 < \sigma_1 < 1 \leq \sigma_2(1)$

This case arises when $\mu \geq \underline{\mu}$. Note that:

$$\bar{x} = \sigma_2^{-1}(\bar{s}) = \frac{2\tilde{\rho}_2 - \mu + \bar{s} + 2t}{4t} = \frac{5}{4} + \frac{3\omega}{16} - \frac{\mu}{4t}.$$

Firm M's profit can be expressed as:

$$\Pi_M(\mu, \rho_1) = \mu \left[\bar{x} - \int_{\hat{x}}^{\bar{x}} \sigma_2(x) dx \right] - \sigma_1^2 \hat{x}.$$

First, we characterize the optimal deviation. Differentiating Π_M with respect to ρ_1, we have:

$$\frac{\partial \Pi_M}{\partial \rho_1}(\mu, \rho_1) = \frac{\sigma_1}{t}(4t\hat{x} - \rho_1) = \frac{\sigma_1}{t}\left(5t - \frac{1}{4} - 3\rho_1\right).$$

Given that $\sigma_1 > 0$, it follows that $\Pi_M(\mu, \rho_1)$ is strictly quasi-concave in ρ_1 and is maximized at:

$$\rho_1 = \rho_1^d = \frac{5t}{3} - \frac{1}{12}.$$

Then, the condition $\sigma_1 > 0$ implies:

$$\mu > \mathring{\mu} = 2\rho_1^d = \frac{10t}{3} - \frac{1}{6}.$$

Then:

$$\mathring{\mu} - \underline{\mu} = \frac{7t}{3} - \frac{11}{12} \geq \frac{t}{18} > 0,$$

where the first inequality comes from (15).

Given that $\Pi_M(\mu, \rho_1)$ is continuous and strictly quasi-concave in ρ_1, it follows that any deviation involving $\mu \in \left[\underline{\mu}, \mathring{\mu}\right]$ and $\sigma_1 \in (0,1)$ is dominated by a deviation leading to $\sigma_1 = 0$, which is not profitable from Lemma 5**.**

Then, we only need to examine the case where $\mu > \mathring{\mu}$ (i.e., $\sigma_1 > 0$). From the previous discussion, for any such μ, the optimal deviation must satisfy $\rho_1 = \rho_1^d$, which implies:

$$\hat{x} = \frac{1}{2} - \frac{\rho_1^d - \tilde{\rho}_2}{2t} = \frac{5}{12} - \frac{\omega}{48}.$$

Firm M's profit can then be rewritten as a function of σ_1:

$$\Pi_M(\sigma_1) = (\sigma_1 + 8t\hat{x})\left(\frac{8t\hat{x} + (1-\sigma_1)^2}{8t}\right) - \sigma_1^2\hat{x},$$

where the relevant range is $\sigma_1 \in (0,1)$. The derivative is given by:

$$\Pi_M'(\sigma_1) = \frac{1 - 8t\hat{x} - 4\sigma_1 + 3\sigma_1^2}{8t},$$

which is convex and satisfies (using (15)):

$$\begin{aligned} \Pi_M'(0) &= \frac{7\omega - 20}{48} \leq -\frac{1}{15} < 0, \\ \Pi_M'(1) &= -\hat{x} < 0. \end{aligned}$$

This implies that $\Pi_M(\sigma_1) < 0$ in the relevant range $\sigma_1 \in (0,1)$. Thus, any deviation leading to $\sigma_1 > 0$ is once again dominated by a deviation leading to $\sigma_1 = 0$, which is not profitable according to Lemma 5**.**

(2): Deviation leading to $0 < \sigma_1 \leq \sigma_2(1) < 1$

This case arises when $\mu < \underline{\mu}$. Firm M's profit can be expressed as:

$$\Pi_M(\mu, \rho_1) = \mu\left[1 - \int_{\hat{x}}^{1} \sigma_2(x)\,dx\right] - \sigma_1^2\hat{x}.$$

Differentiating Π_M with respect to ρ_1 and using $\sigma_2(\hat{x}) = \sigma_1$, we obtain:

$$\frac{\partial \Pi_M}{\partial \rho_1}(\mu, \rho_1) = -\mu\frac{\sigma_1}{2t} + \frac{\sigma_1^2}{2t} + 4\sigma_1\hat{x} = \frac{\sigma_1}{t}(4t\hat{x} - \rho_1).$$

For $\sigma_1 > 0$, the optimal deviation must lead to $\rho_1 = \rho_1^d = 4t\hat{x}$. However, $\mu < \underline{\mu} < \mathring{\mu} = 2\hat{\rho}_1$ implies $\sigma_1 < 0$. Since $\Pi_M(\mu, \rho_1)$ is continuous and strictly quasi-concave in ρ_1, it follows that any such deviation is dominated by a deviation leading to $\sigma_1 = 0$, which, as we've previously established, is not profitable.

A.2.2 Deviations by Stand-alone Firms

We demonstrate that the stand-alone firms cannot benefit from deviations from three types of equilibria.

Weak Product Differentiation ($t < t_m$) This equilibrium arises when $t < \frac{32}{85}$. The candidate equilibrium is characterized in Configuration 1, where equilibrium margins are given by $\mu^* = \mu^m = \frac{1}{2} + \frac{107}{64}t$, $\alpha_1^* = \beta_1^* \equiv \rho_1^m = \frac{3}{2}t$, and $\alpha_2^* = \beta_2^* \equiv \rho_2^m = \frac{5}{4}t$. Given the symmetric positions of the two firms, we need to consider two situations. First, one firm, say, stand-alone firm A_2 reduces α_2, resulting in $x_A < x_B = \hat{x}$. Second, stand-alone firm B_2 increases β_2, leading to $x_A = \hat{x} < x_B$.

Deviations by Stand-alone Firm A_2 Suppose firm A_2 sets $\alpha_2 < \alpha_2^* = \rho_2^m$. Consumers with $x_A < x \leq x_B = \hat{x}$ prefer the mix of $\{A_2, B_1\}$ than other options if:

$$s \leq \sigma_{21}(x) = \mu^* - \alpha_2 - \beta_1^* + 2t\left(x - \frac{1}{2}\right) = \frac{1}{2} - \frac{53}{64}t + 2tx - \alpha_2.$$

Moreover, consumers prefer $\{A_2, B_2\}$ than other options if:

$$s \leq \sigma_2(x) = \mu^* - \alpha_2 - \beta_2^* + 4t\left(x - \frac{1}{2}\right) = \frac{1}{2} - \frac{101}{64}t + 4tx - \alpha_2.$$

We will examine the following four relevant deviations respectively:

(1). Deviations leading to $\sigma_2(1) \leq 1$ and $0 \leq x_A \leq x_B = \hat{x}$

This case arises when:

$$\begin{aligned}\sigma_2(1) &= \frac{1}{2} + \frac{155}{64}t - \alpha_2 \leq 1, \\ 0 &\leq x_A = \frac{\alpha_2}{2t} - \frac{1}{4} \leq \frac{3}{8}.\end{aligned}$$

This requires $\alpha_2 \in [\underline{\alpha}_2, \alpha_2^*]$, where $\underline{\alpha}_2 = \min\{\frac{155}{64}t - \frac{1}{2}, \frac{t}{2}\}$. Firm A_2' profit function can be written as:

$$\begin{aligned}\Pi_A(\alpha_2) &= \alpha_2\left[\int_{x_A}^{\hat{x}} \sigma_{21}(x)dx + \int_{\hat{x}}^{1} \sigma_2(x)dx\right] \\ &= \frac{\alpha_2\left(64\alpha_2^2 - 2\alpha_2(32 + 75t) + 5t(32 + 15t)\right)}{256t}.\end{aligned}$$

Differentiating Π_A with respect to α_2, we obtain:

$$\Pi_A^{'}(\alpha_2) = \frac{(4\alpha_2 - 5t)(48\alpha_2 - 32 - 15t)}{2561t}.$$

Solving for the first-order condition leads to:

$$\alpha_2^1 = \frac{5t}{4} \text{ and } \alpha_2^2 = \frac{32+15t}{48},$$

where $\alpha_2^1 = \alpha_2^* < \alpha_2^2$.

Differentiating $\Pi_A^{'}(\alpha_2)$ further with respect to α_2 leads to:

$$\Pi_A^{''}(\alpha_2) = \frac{384\alpha_2 - 128 - 300t}{256t},$$

and

$$\Pi_A^{'''}(\alpha_2) = \frac{3}{2t}.$$

Note that $t < \frac{32}{85}$ implies:

$$\Pi_A^{''}(\alpha_2^1) = \frac{180t - 128}{256t} < 0.$$

It is straightforward to verify that $\Pi_A^{'}(\underline{\alpha}_2) > 0$. Hence, the profit function is strictly increasing in the relevant range $\alpha_2 \in \left[\min\{\frac{155}{64}t - \frac{1}{2}, \frac{t}{2}\}, \frac{5}{4}t\right]$ and is maximized at $\alpha_2^1 = \alpha_2^*$.

(2). Deviation leading to $\sigma_2(1) \leq 1$ and $x_A < 0$

This case occurs when:

$$\frac{155}{64}t - \frac{1}{2} \leq \alpha_2 < \frac{t}{2}.$$

Firm A_2' profit function can be expressed as:

$$\Pi_A(\alpha_2) = \alpha_2\left[\int_0^{\hat{x}} \sigma_{21}(x)dx + \int_{\hat{x}}^1 \sigma_2(x)dx\right] = \frac{\alpha_2(-16\alpha_2 + 8 + 9t)}{16}.$$

The first-order derivative of $\Pi_A(\alpha)$ is given by:

$$\Pi_A^{'}(\alpha_2) = \frac{-32\alpha_2 + 8 + 9t}{16}.$$

Then, the profit function is strictly concave:

$$\Pi_A^{''}(\alpha_2) = -2 < 0.$$

It is straightforward to verify that $\Pi_A^{'}(\alpha_2) > 0$ for $\alpha_2 < \frac{t}{2}$, given that $t < \frac{32}{85}$. Therefore, the profit function is strictly increasing in the relevant range and is maximized at $\alpha_2 = \frac{t}{2}$. The maximum profit from deviation is equal to:

$$\Pi_A^d = \frac{t(81+t)}{32} = \frac{t(8\omega+1)}{32\omega}.$$

Comparing it with the equilibrium profit:

$$\Pi_A^* = \frac{25t(32-5t)}{2048} = \frac{25t(32\omega-5)}{2048\omega},$$

it follows that:

$$\frac{\left(\Pi_A^* - \Pi_A^d\right)\omega}{t} = \frac{25(32\omega - 5)}{2048} - \frac{(8\omega + 1)}{32} = \frac{9(32\omega - 21)}{2048} > 0.$$

Hence, such deviation is not profitable.

(3). Deviation leading to $\sigma_2(1) > 1$ and $x_A \geq 0$

This case occurs under the following conditions:

$$\frac{t}{2} \leq \alpha_2 < \bar{\alpha}_2 = \frac{155}{64}t - \frac{1}{2}.$$

This requires:

$$\frac{32}{123} < t < \frac{32}{85},$$

or equivalently:

$$\frac{85}{32} < \omega < \frac{123}{32}.$$

In this case, firm A_2' profit function can be written as:

$$\begin{aligned}
\Pi_A(\alpha_2) &= \alpha_2 \left[\int_{x_A}^{\hat{x}} \frac{\sigma_{21}(x)}{1} dx + \int_{\hat{x}}^{\bar{x}} \frac{\sigma_2(x)}{1} dx + (1 - \bar{x}) \right] \\
&= \frac{\alpha_2 \left[1024\left(4\alpha_2^2 - 12\alpha_2 - 1\right) + 320t(2\alpha_2 + 95) - 14425t^2\right]}{32768t}.
\end{aligned}$$

The profit function is cubic with respect to α_2 and the derivative satisfies:

$$\Pi_A^{'''}(\alpha_2) = \frac{3}{4t}.$$

Taking the first-order derivative leads to:

$$\Pi_A'(\alpha_2) = \frac{1024\left(12\alpha_2^2 - 24\alpha_2 - 1\right) + 320t(4\alpha_2 + 95) - 14425t^2}{32768t}.$$

Solving for FOC yields two optimal values:

$$\begin{aligned}
\alpha_2^1 &= \frac{1921 - 10t - \sqrt{39936 - 95040t + 43375t^2}}{192}, \\
\alpha_2^2 &= \frac{1921 - 10t + \sqrt{39936 - 95040t + 43375t^2}}{192},
\end{aligned}$$

where $\alpha_2^1 < \alpha_2^2$.

We now show that both values are out of the range for α_2, and thus the optimal deviation must be at the boundary. To see this, note that:

$$\begin{aligned}
\frac{\alpha_2^1 - \bar{\alpha}}{t} &= \frac{3}{2}\omega - \frac{475}{192} - \frac{1}{192}\sqrt{39\,936\omega^2 - 95\,040\omega + 43375} \\
&= \frac{288\omega - 475 - \sqrt{39\,936\omega^2 - 95\,040\omega + 43375}}{192}.
\end{aligned}$$

The above expression is positive since:

$$\begin{aligned} M(\omega) &\equiv (288\omega-475)^2-\left(39\,936\omega^2-95\,040\omega+43375\right). \\ &= 43\,008\left(\omega-\frac{465}{224}\right)^2-\frac{21\,600}{7}. \\ &> 43\,008\left(\frac{85}{32}-\frac{465}{224}\right)^2-\frac{21\,600}{7}=11400, \end{aligned}$$

where we used $\omega>\frac{85}{32}$ to obtain the inequality.

Hence, the profit function is strictly increasing in the relevant range for $\alpha_2\leq\bar{\alpha}_2$ and is maximized at $\bar{\alpha}_2$. The maximum profit is equal to:

$$\begin{aligned} \Pi_A^d(\bar{\alpha}_2) &= \frac{(155t-32)\left(3072-4800t+5575t^2\right)}{1048576t} \\ &= \frac{t(155-32\omega)\left(3072\omega^2-4800\omega+5575\right)}{1048576\omega}. \end{aligned}$$

It is less than the equilibrium profit Π_A^*, since $\omega\geq\frac{85}{32}$ implies:

$$\frac{\Pi_A^*-\Pi_A^d(\bar{\alpha}_2)}{t}=\frac{3(32\omega-75)^2}{1048576\omega}(32\omega-55)>0.$$

(4). Deviation leading to $\sigma_2(1)>1$ and $x_A\leq 0$

This case can occur in the following range: $\alpha_2\in\left(0,\bar{\alpha}_2=\min\left\{\frac{155t}{64}-\frac{1}{2},\frac{t}{2}\right\}\right)$. It requires:

$$\frac{32}{155}<t<\frac{32}{85},$$

or equivalently:

$$\frac{85}{32}<\omega<\frac{155}{32}.$$

In this case, firm A_2' profit function can be written as:

$$\begin{aligned} \Pi_A(\alpha_2) &= \alpha_2\left[\int_0^{\hat{x}}\sigma_{21}(x)dx+\int_{\hat{x}}^{\bar{x}}\sigma_2(x)dx+(1-\bar{x})\right] \\ &= \frac{\alpha_2\left[64t(411-202\alpha_2)-1024(2\alpha_2+1)^2-5593t^2\right]}{32768t}. \end{aligned}$$

The profit function is cubic with respect to α_2 and the derivative satisfies:

$$\Pi_A^{'''}(\alpha)=-\frac{3}{4t}.$$

The first-order derivative of $\Pi_A(\alpha_2)$ is given by:

$$\Pi_A^{'}(\alpha_2)=\frac{64t(411-404\alpha_2)-1024(2\alpha_2+1)(6\alpha_2+1)-5593t^2}{32768t}.$$

Solving for the relevant optimum leads to

$$\begin{aligned}\alpha_2 &= \frac{\sqrt{1024+104768t+24025t^2}-64-202t}{192}\\ &= t\frac{\sqrt{1024\omega^2+104768\omega+24025}-64\omega-202}{192}.\end{aligned}$$

Comparing with the upper bound $\bar{\alpha}_2$, we have:

$$\begin{aligned}\frac{\alpha_2-\bar{\alpha}_2}{t} &= k_1(\omega), \text{ for } \omega\in(\frac{123}{32},\frac{155}{32})\\ \frac{\alpha_2-\bar{\alpha}_2}{t} &= k_2(\omega), \text{ for } \omega\in(\frac{85}{32},\frac{123}{32}),\end{aligned}$$

where:

$$\begin{aligned}k_1(\omega) &\equiv \frac{\alpha_2-\left(\frac{155t}{64}-\frac{1}{2}\right)}{t}=\frac{\sqrt{1024\omega^2+104768\omega+24025}-(667-32\omega)}{192},\\ k_2(\omega) &\equiv \frac{\alpha_2-\frac{t}{2}}{t}=\frac{\sqrt{1024\omega^2+104768\omega+24025}-(64\omega+298)}{192}.\end{aligned}$$

Note that $k_1(\omega)$ is positive in the relevant range:

$$\begin{aligned}k_1(\omega) &\equiv 1024\omega^2+104768\omega+24025-(667-32\omega)^2\\ &= 147\,456\omega-420\,864>147\,456\times\frac{123}{32}-420\,864>0.\end{aligned}$$

Moreover, $k_2(\omega)$ is also positive in the relevant range:

$$k_2(\omega)\equiv 1024\omega^2+104768\omega+24025-(64\omega+298)^2=66\,624\omega-64\,779-3072\omega^2>0.$$

This is true since $k_2'(\omega)=66\,624-6144\omega>0$ for $\omega\in(\frac{85}{32},\frac{123}{32})$, and $k_2\left(\frac{85}{32}\right)=90516>0$. It follows that the profit function is strictly increasing in the relevant range $\alpha_2\in[0,\bar{\alpha}_2)$ and is maximized at $\bar{\alpha}_2$. The maximum profit under deviation is:

$$\Pi_A^d(\bar{\alpha}_2)=\max\left\{\Pi_A^d(\frac{155t}{64}-\frac{1}{2}),\Pi_A^d(\frac{t}{2})\right\}.$$

We now show that $\Pi_A^d(\bar{\alpha}_2)$ is less than the equilibrium profit. Note that:

$$\frac{\Pi_A^*-\Pi_A^d(\bar{\alpha}_2)}{t}=\min\left\{\frac{\Pi_A^*-\Pi_A^d(\frac{155t}{64}-\frac{1}{2})}{t}=d_1(\omega),\frac{\Pi_A^*-\Pi_A^d(\frac{t}{2})}{s}=d_2(\omega)\right\},$$

where:

$$\begin{aligned}d_1(\omega) &\equiv \frac{(18195-12128\omega)}{4096\omega}+\frac{\omega}{2},\\ d_2(\omega) &\equiv \frac{3(3027+448\omega)}{65536\omega}+\frac{\omega}{64}.\end{aligned}$$

It follows that $d_1(\omega)$ is positive in the relevant range $\omega\in(\frac{123}{32},\frac{155}{32})$ and $d_2(\omega)$ is also positive in the relevant range $\omega\in(\frac{85}{32},\frac{123}{32})$.

Deviations by Stand-alone Firm B_2 Lastly, we confirm that the standalone firm B_2 cannot profit from any deviations. Consider the deviation where firm B_2 increases β_2 such that $x_A = \hat{x} \leq x_B$. It is never optimal to deviate such that $x_B > 1$, as firm B_2 would then face no demand. For simplicity, we focus on cases where $x_A = \hat{x} \leq x_B \leq 1$. Furthermore, increasing β_2 reduces $\sigma_2(x) = \mu - \alpha_2^* - \beta_2 + 4t(x - \frac{1}{2})$, eliminating the possibility of $\sigma_2(1) > 1$ since $\sigma_2(1) \leq 1$ in the equilibrium.

Hence, the relevant deviation arises under the following constraints:

$$\max\{\frac{155}{64}t - \frac{1}{2}, \frac{5t}{4}\} \leq \beta_2 \leq \frac{5t}{2}.$$

In this case, firm B_2's profit can be written as:

$$\Pi_B(\beta_2) = \beta_2 \left[\int_{x_B}^{1} \sigma_2(x)dx\right] = \frac{\beta_2(5t - 2\beta_2)(32 - 5t)}{256t}.$$

The profit function is quadratic and its derivative is given by:

$$\Pi_B'(\beta_2) = \frac{(5t - 4\beta_2)(32 - 5t)}{256t}.$$

Solving for the optimum leads to:

$$\beta_2 = \beta_2^* = \frac{5t}{4},$$

and the profit function is exactly equal to the equilibrium profit Π_B^*. Thus, there are no profitable deviations for firm B_2.

Mild Product Differentiation ($t_m < t < t_p$) In this candidate equilibrium, the conglomerate only sells the bundle and the margins of its stand-alone products must satisfy $\alpha_1 + \beta_1 > \mu^* = \check{\mu}^p$. Without loss of generality, we focus on the candidate equilibrium with $\alpha_1, \beta_1 = \infty$ (any other candidate equilibrium with lower stand-alone prices for the conglomerate would only generate fewer sales and profits for a deviating stand-alone firm) and on deviations by firm A_2 (given the symmetric positions of the two standalone firms, they face the same deviation profits). As firm M charges prohibitively high stand-alone prices, the demand for A_2 can only come from consumers buying $\{A_2, B_2\}$, which requires $s \leq \sigma_2(x)$, where:

$$\begin{aligned}\sigma_2(x) &= \check{\mu}^p - \alpha_2 - \check{\rho}_2^p + 4t\left(x - \frac{1}{2}\right) \\ &= \frac{3\sqrt{t^2 + 10t} - 17t}{5} + 4tx - \alpha_2.\end{aligned}$$

Recall that the equilibrium margins are given by:

$$\begin{aligned}\mu^* &= \check{\mu}^p \equiv \frac{4\sqrt{t^2+10t}-6t}{5} = \frac{t\left(4\sqrt{1+10\omega}-6\right)}{5}, \\ \alpha_2^* &= \beta_2^* = \check{\rho}_2^p = \frac{\sqrt{t^2+10t}+t}{5} = \frac{t\left(\sqrt{1+10\omega}+1\right)}{5}.\end{aligned}$$

and the equilibrium arises in the parameter range $\frac{12}{5} \le \omega \le \frac{85}{32}$.

As before, let $\tilde{x}$ and $\bar{x}$ denote the location thresholds such that $\sigma_2(\tilde{x}) = 0$ and $\sigma_2(\bar{x}) = 1$. By construction, $\bar{x} > \tilde{x}$ and, as long as $\tilde{x} \ge 0$ and $\bar{x} \ge 1$, firm A_2's profit is given by $\Pi_A = \alpha_2 2t(1-\tilde{x})^2$, which is strictly concave in α_2. It follows that there is no profitable deviation in this range. We now consider deviations leading to either $\tilde{x} < 0$ or $0 < \tilde{x} < \bar{x} < 1$.

(1). Deviation leading to $\tilde{x} < 0$

This case arises when $\sigma_2(0) > 0$. This condition holds when:

$$\alpha_2 < \frac{3\sqrt{t^2+10t}-17t}{5} = \frac{t}{5}\left(3\sqrt{1+10\omega}-17\right) < 0,$$

where the last equality comes from $\omega \le \frac{85}{32}$. Hence, this deviation cannot profitable.

(2). Deviation leading to $0 \le \tilde{x} < \bar{x} < 1$

This case arises when $\sigma_2(1) > 1$, which holds when:

$$\alpha_2 < \bar{\alpha}_2 = \frac{3\sqrt{t^2+10t}+3t}{5} - 1.$$

Using $\sigma_2(\tilde{x}) = 0$ and $\sigma_2(\bar{x}) = 1$, we have:

$$\tilde{x} = \frac{1}{2} + \frac{\alpha_2+\beta_2-\mu}{4t} \text{ and } \bar{x} = \frac{1}{2} + \frac{1+\alpha_2+\beta_2-\mu}{4t},$$

and:

$$\int_{\tilde{x}}^{\bar{x}} \sigma_2(x)\,dx = \frac{1}{8t}.$$

We can rewrite firm A_2's profit as:

$$\begin{aligned}\Pi_A(\alpha_2) &= \alpha_2\left(\int_{\tilde{x}}^{\bar{x}} \sigma_2(x)dx + 1 - \bar{x}\right) \\ &= \alpha_2\left(\frac{3\sqrt{t^2+10t}+3t}{20t} - \frac{1}{8t} - \frac{\alpha_2}{4t}\right).\end{aligned}$$

Differentiating Π_A with respect to α_2, we obtain:

$$\Pi_A'(\alpha_2) = \frac{3}{20} + \frac{3\sqrt{t^2+10t}}{20t} - \frac{1}{8t} - \frac{\alpha_2}{2t}.$$

Then, $\alpha_2 < \bar{\alpha}_2$ implies:

$$\begin{aligned}\Pi_A'(\alpha_2) &> \frac{3}{20} + \frac{3\sqrt{t^2+10t}}{20t} - \frac{1}{8t} - \frac{\bar{\alpha}_2}{2t} \\ &= \frac{31}{8t} - \frac{3\sqrt{t^2+10t}+3t}{20t} \\ &= \frac{3}{40}\left(5\omega - 2 - 2\sqrt{1+10\omega}\right) \geq 0,\end{aligned}$$

where the last inequality follows from $\omega \geq \frac{12}{5}$. It follows that $\Pi_A(\alpha_2)$ is increasing in the relevant range $\alpha_2 < \bar{\alpha}_2$. Thus, any deviation leading to $\bar{x} < 1$ is strictly dominated by the deviation leading to $\bar{x} = 1$, which is not profitable as we have already shown.

Strong Product Differentiation ($t > t_p$) This equilibrium with de factor pure bundling arises when $t > \frac{5}{12}$. The candidate equilibrium is characterized in Configuration 5 where $\sigma_2(1) \geq 1$. Without loss of generality, we focus on the candidate equilibrium with α_1, $\beta_1 = \infty$ and on deviations by firm A_2. As firm M charges prohibitively high stand-alone prices, the demand for A_2 can only come from consumers buying $\{A_2, B_2\}$, which requires $s \leq \sigma_2(x)$, where:

$$\sigma_2(x) = \mu^* - \alpha_2 - \beta_2^* + 4t\left(x - \frac{1}{2}\right) = \frac{1}{4} - t + 4tx - \alpha_2.$$

The equilibrium margins are given by:

$$\tilde{\mu}^p = \frac{5t}{2} + \frac{1}{8} \text{ and } \tilde{\rho}_2^p = \frac{3t}{2} - \frac{1}{8}.$$

Let $\tilde{x}$ and $\bar{x}$ denote as before the location thresholds such that $\sigma_2(\tilde{x}) = 0$ and $\sigma_2(\bar{x}) = 1$. Then $\sigma_2(1) \geq 1$ amounts to $\bar{x} \leq 1$. By construction, $\bar{x} > \tilde{x}$ and, as long as $\tilde{x} \geq 0$ and $\bar{x} \leq 1$, firm A_2's profit remains given by (9), and is strictly concave in α_2. It follows that there is no profitable deviation in this range. We now consider deviations leading to either $\tilde{x} < 0$ and $\bar{x} > 1$.

(1). Deviation leading to $\bar{x} > 1$

This case arises when $\sigma_2(1) < 1$, which amounts to:

$$\alpha_2 > \underline{\alpha}_2 \equiv 3t - \frac{3}{4}.$$

Using:

$$4t(1-\tilde{x}) = \sigma_2(1) = \frac{1}{4} + 3t - \alpha_2,$$

firm A_2's profit can be written as:

$$\Pi_A(\alpha_2) = \alpha_2 \int_{\tilde{x}}^{1} \sigma_2(x)dx = \alpha_2 2t(1-\tilde{x})^2 = \alpha_2 \frac{\sigma_2^2(1)}{8t}.$$

Differentiating Π_A with respect to α_2, we get:

$$\begin{aligned}\Pi_A'(\alpha_2) &= \frac{\sigma_2(1)}{8t}(\sigma_2(1) - 2\alpha_2) \\ &= \frac{\sigma_2(1)}{8t}\left(\frac{1}{4} + 3t - 3\alpha_2\right) \\ &\leq \frac{3\sigma_2(1)}{4t}\left(\frac{5}{12} - t\right) \leq 0,\end{aligned}$$

where the first inequality comes from $\alpha_2 \geq \underline{\alpha}_2$ and the second inequality follows from $t > \frac{5}{12}$. Hence, any deviation leading to $\bar{x} > 1$ (or, equivalently, to $\alpha_2 \geq \underline{\alpha}_2$) is dominated by the deviation that leads to $\alpha_2 = \underline{\alpha}_2$, which is not profitable as we have already shown.

(2). Deviation leading to $\tilde{x} < 0$

This case arises when $\sigma_2(0) > 0$, which amounts to $\alpha_2 < \bar{\alpha}_2 \equiv \frac{1}{4} - t$. Since $t > \frac{5}{12}$ implies $\bar{\alpha}_2 < 0$, this deviation cannot be profitable.

A.3 Welfare Comparison

We compare firms' profits and consumer surplus before and after the merger. We consider three types of equilibria respectively. Recall that before the merger, firms compete à la Hotelling in each market and set equilibrium margins equal to t. Each firm earns a profit equal to $\frac{t}{2}$. Total consumer surplus before the merger can be expressed as:

$$S_0 \equiv \int_0^{1/2} (w - 2tx - 2t)\,dx + \int_{1/2}^{1} (w - 2t(1-x) - 2t)\,dx = w - \frac{5}{2}t.$$

For simplifying the exposition, we will use $\omega = 1/t$.

A.3.1 Weak Product Differentiation

The conglomerate engages mixed bundling in the equilibrium when $t < \frac{32}{85}$ (equivalent to $\omega > \frac{85}{32}$). Using $\omega = 1/t$, we can rewrite firm M's equilibrium profit as:

$$\Pi_M(\omega) = t\left(\frac{\omega}{4} + \frac{107}{64} - \frac{2375}{4096\omega}\right),$$

It appears that $\Pi_M(\omega)$ increases with ω, and $\omega > \frac{85}{32}$ implies that the merger is profitable:

$$\Pi_M(\omega) > \Pi_M\left(\frac{85}{32}\right) = \frac{36}{17}t > t.$$

It is straightforward to check that the stand-alone firms are worse off after the merger:

$$\Pi_A^*(\omega) = \Pi_B^*(\omega) = \frac{25(32\omega - 5)t}{2048\omega} < \frac{25(32\omega)t}{2048\omega} = \frac{25t}{64} < \frac{t}{2}.$$

We now examine consumer surplus. Following the merger, the conglomerate charges μ^m for the bundle, while the margins for the stand-alone products are $\alpha_1^* = \beta_1^* = \rho_1^m$ and $\alpha_2^* = \beta_2^* = \rho_2^m$, respectively. The cut-off threshold is $\hat{x} = \frac{3}{8}$. Consumers with $x \leq \hat{x}$ and $s \leq \sigma_1$, who purchase $\{A_1, B_1\}$, obtain a surplus of $w - 2tx - 2\rho_1^m$. Those with $\hat{x} < x \leq 1$ and $s \leq \sigma_2(x)$ who buy $\{A_2, B_2\}$, receive a surplus of $w - 2t(1-x) - 2\rho_2^m$. The remaining consumers, who opt for the bundle, receive a surplus of $w + s - 2tx - \mu^m$.

Using $\sigma_1 = \mu^m - 2\rho_1^m$ and $\sigma_2(x) = \mu^m - 2\rho_2^m + 2t(2x-1) = \sigma_1 + 4t(x - \hat{x}^m)$, the total consumer surplus post-merger can be expressed as:

$$
\begin{aligned}
S_1 &= \int_0^{\hat{x}^m}\int_0^{\sigma_1}(w - 2tx - 2\rho_1^m)\,dsdx + \int_{\hat{x}^m}^1\int_0^{\sigma_2(x)}(w - 2t(1-x) - 2\rho_2^m)\,dsdx \\
&\quad + \int_0^{\hat{x}^m}\int_{\sigma_1}^1 (w + s - 2tx - \mu^m)\,dsdx + \int_{\hat{x}^m}^1\int_{\sigma_2(x)}^1 (w + s - 2tx - \mu^m)\,dsdx \\
&= \int_0^{\hat{x}^m}\int_0^{\sigma_1}(w - 2tx - 2\rho_1^m)\,dsdx + \int_{\hat{x}^m}^1\int_0^{\sigma_2(x)}(w - 2t(1-x) - 2\rho_2^m)\,dsdx \\
&\quad + \int_0^{\hat{x}^m}\int_{\sigma_1}^1 (w - 2tx - \mu^m)\,dsdx + \int_{\hat{x}^m}^1\int_{\sigma_2(x)}^1 (w - 2tx - \mu^m)\,dsdx + \int_0^{\hat{x}^m}\int_{\sigma_1}^1 sdsdx + \int_{\hat{x}^m}^1\int_{\sigma_2(x)}^1 sdsdx \\
&= w - \int_0^{\hat{x}^m}(2tx + \mu^m)\,dx - \int_{\hat{x}^m}^1 (2tx + \mu^m)\,dx + \frac{1}{2} + \int_{\hat{x}^m}^1 \frac{\sigma_2^2(x)}{2}dx + \frac{\sigma_1^2}{2}\hat{x}^m \\
&= w - t - \mu^m + \frac{1}{2} + \frac{\sigma_1^2}{2} + 2t\sigma_1(1 - \hat{x}^m)^2 + \frac{8t^2(1 - \hat{x}^m)^3}{3}.
\end{aligned}
$$

Substituting $\mu^m = \frac{1}{2} + \frac{107}{64}t$, $\rho_1^m = \frac{3}{2}t$, $\sigma_1 = \mu^m - 2\rho_1^m = \frac{1}{2} - \frac{85}{64}t$, and $\hat{x}^m = \frac{3}{8}$ into the above expression leads to:

$$
S_1 = w + \frac{t}{24\,576\omega}\left(3072\omega^2 - 72\,384\omega + 12\,175\right).
$$

Comparing with the total consumer surplus before the merger, we obtain:

$$
\Delta_1(\omega) \equiv \frac{S_1 - S_0}{t} = \frac{1}{24\,576}\left(3072\omega - 10\,944 + \frac{12\,175}{\omega}\right).
$$

Note that $\Delta_1\left(\frac{85}{32}\right) = \frac{239}{3264}$ when $\omega = \frac{85}{32}$. Furthermore, $\omega > \frac{85}{32}$ implies:

$$
\Delta_1'(\omega) = \frac{1}{24\,576}\left(3072 - \frac{12\,175}{\omega^2}\right) > \Delta_1'\left(\frac{85}{32}\right) = \frac{95}{1734} > 0.
$$

It follows that the merger increases total consumer surplus in this equilibrium.

A.3.2 Mild Product Differentiation

The conglomerate engages in de facto pure bundling in the equilibrium when $\frac{32}{85} < t < \frac{5}{12}$ (equivalently $\frac{12}{5} < \omega < \frac{85}{32}$). The equilibrium margins for the bundle and the stand-alone products are given by:

$$\breve{\mu}^p = \frac{\left(4\sqrt{10\omega+1}-6\right)t}{5},\ \breve{\rho}_2^p = \frac{\left(\sqrt{10\omega+1}+1\right)t}{5},$$

while the corresponding threshold is:

$$\tilde{x}^p = \frac{9-\sqrt{10\omega+1}}{10}.$$

In equilibrium, firm M's profit is given by:

$$\Pi_M(\omega) = \frac{t\left(4\sqrt{10\omega+1}-6\right)}{125} \times \frac{\left(20\omega-1-\sqrt{10\omega+1}\right)}{\omega}.$$

It is easy to check that both terms increase in ω. Then, $\omega > \frac{12}{5}$ implies:

$$\Pi_M(\omega) > \Pi_M\left(\frac{12}{5}\right) = \frac{49}{25}t > t.$$

Thus, the merger increases the conglomerate's total profit.

We now demonstrate that the stand-alone firms earn lower profits after the merger than before. The equilibrium profits for the stand-alone firms are given by:

$$\Pi_A^*(\omega) = \Pi_B^*(\omega) = \frac{\breve{\rho}_2^3}{2t} = \frac{t\left(\sqrt{10\omega+1}+1\right)^3}{250\omega} = \frac{t\Phi(\omega)}{250},$$

where:

$$\Phi(\omega) \equiv \frac{\left(\sqrt{10\omega+1}+1\right)^3}{\omega}.$$

Note that $\omega > \frac{12}{5}$ implies:

$$\Phi'(\omega) = \frac{\left(\sqrt{10\omega+1}+1\right)^2}{\omega^2}\left(\frac{5\omega-1-\sqrt{10\omega+1}}{\sqrt{10\omega+1}}\right) > 0.$$

It follows that Π_A^* increases with ω. Then, $\omega < \frac{85}{32}$ implies:

$$\Pi_A^*(\omega) < \Pi_A^*\left(\frac{85}{32}\right) = \frac{t}{250}\Phi\left(\frac{85}{32}\right) = \frac{25}{68}t < \frac{t}{2}.$$

Next, we assess the consumer surplus following the merger. In this scenario, the conglomerate offers the pure bundle only. Consumers with $x \geq \tilde{x}$ and $s \leq \sigma_2(x)$ purchase $\{A_2, B_2\}$, and they achieve a surplus of $w - 2t(1-x) - 2\breve{\rho}_2^p$. In contrast, others who purchase the bundle receive a surplus of $w + s - 2tx - \breve{\mu}^p$. The total consumer surplus can be calculated as follows, using the expression $\sigma_2(x) = \check{\mu}^p - 2\check{\rho}_2^p + 2t(2x-1) = 4t(x - \tilde{x}^p)$ for $x \geq \tilde{x}^p$:

$$\begin{aligned}
S_2 &= \int_0^{\tilde{x}^p}\int_0^1 (w + s - 2tx - \check{\mu}^p)\, dsdx + \int_{\tilde{x}^p}^1\int_{\sigma_2(x)}^1 (w + s - 2tx - \check{\mu}^p)\, dsdx \\
&\quad + \int_{\tilde{x}^p}^1\int_0^{\sigma_2(x)} (w - 2t(1-x) - 2\check{\rho}_2^p)\, dsdx \\
&= w - \int_0^{\tilde{x}^p} (2tx + \check{\mu}^p)\, dx - \int_{\tilde{x}^p}^1 (2tx + \check{\mu}^p)(1 - \sigma_2(x))\, dx - \int_{\tilde{x}^p}^1 (2t(1-x) + 2\check{\rho}_2^p)\, \sigma_2(x)\, dx \\
&\quad + \frac{1}{2} - \int_{\tilde{x}^p}^1 \frac{\sigma_2^2(x)}{2} dx \\
&= w - t - \check{\mu}^p + \frac{1}{2} + \frac{8t^2}{3}(1 - \tilde{x})^3.
\end{aligned}$$

Substituting the value of $\check{\mu}^p$ and $\check{\rho}_2^p$, we obtain:

$$S_2 = w - \frac{t\left(4\sqrt{10\omega+1} - 1\right)}{5} + \frac{1}{2} + \frac{8t^2}{3}\left(\frac{1+\sqrt{10\omega+1}}{10}\right)^3.$$

Comparing with the total surplus before the merger, we have:

$$\begin{aligned}
\Delta_2(\omega) &\equiv \frac{S_2 - S_0}{t} = \frac{27}{10} + \frac{1}{2}\omega - \frac{4\sqrt{10\omega+1}}{5} + \frac{8}{3\omega}\left(\frac{1+\sqrt{10\omega+1}}{10}\right)^3 \\
&= \frac{1}{750}\left(2085 + \frac{6\sqrt{10\omega+1}}{\omega} + \frac{2(10\omega+1)^{\frac{3}{2}}}{\omega} + 375\omega - 600\sqrt{10\omega+1} + \frac{8}{\omega}\right).
\end{aligned}$$

Note that:

$$\begin{aligned}
\Delta_2\left(\frac{12}{5}\right) &= \frac{6}{5} + \frac{7}{2} - \frac{24}{5} + \frac{10}{9}\left(\frac{6}{10}\right)^3 = \frac{7}{50}, \\
\Delta_2\left(\frac{85}{32}\right) &= \frac{85}{64} + \frac{7}{2} - 5 + \frac{8}{3 \times \frac{85}{32}}\left(\frac{1+\frac{21}{4}}{10}\right)^3 = \frac{239}{3264}.
\end{aligned}$$

Moreover, $\frac{12}{5} < \omega < \frac{85}{32}$ implies:

$$\Delta_2'(\omega) = \frac{1}{750}\left(-\frac{30\omega+6}{\omega^2\sqrt{10\omega+1}} - \frac{\sqrt{10\omega+1}\,(17\omega+2)}{\omega^2} + 375 - \frac{3000}{\sqrt{10\omega+1}} - \frac{8}{\omega^2}\right) < 0.$$

It follows that $\Delta_2(\omega) > 0$ for $\frac{12}{5} < \omega < \frac{85}{32}$. Thus, the merger increases total consumer surplus in this equilibrium.

A.3.3 Strong Product Differentiation

The conglomerate offers the pure bundle only when $t > 5/12$ (equivalently $\omega < \frac{12}{5}$). The equilibrium margins are given by:

$$\tilde{\mu}^p = t\left(\frac{5}{2} + \frac{\omega}{8}\right), \tilde{\rho}_2^p = t\left(\frac{3}{2} - \frac{\omega}{8}\right),$$

while the corresponding thresholds are:

$$\bar{x}^p = \frac{5}{8} + \frac{5\omega}{32}, \; \tilde{x}^{ps} = \frac{5}{8} - \frac{3\omega}{32}.$$

The conglomerate benefits from the merger since:

$$\Pi_M = t\left(\frac{5}{4} + \frac{\omega}{16}\right)^2 > t.$$

The stand-alone firms' profits after the merger become:

$$\Pi_A^* = \Pi_B^* = t\left(\frac{3}{4} - \frac{\omega}{16}\right)^2.$$

Comparing with the profit before merger, $\frac{t}{2}$, the merger increases their profits if:

$$\omega < 12 - 8\sqrt{2} \simeq 0.686\,29,$$

or equivalently:

$$t > t_r \equiv \frac{3 + 2\sqrt{2}}{4}.$$

Thus, the merger reduces the stand-alone firms' profits when $\frac{5}{12} < t < t_r$ but increases their profits when $t > t_r$.

We now compare the total consumer surplus. Consumers with $\tilde{x} \leq x$ and $s \leq \sigma_2(x)$ will choose to buy $\{A_2, B_2\}$, and they obtain a surplus of $w - 2t(1-x) - 2\tilde{\rho}_2^p$. On the other hand, those who purchase the bundle receive a surplus of $w + s - 2tx - \tilde{\mu}^p$. The total consumer surplus can be expressed as (using $\sigma_2(x) = \tilde{\mu}^p - 2\tilde{\rho}_2^p + 2t(2x-1) = 4t(x - \tilde{x}^{ps})$ for $x \geq \tilde{x}^{ps}$):

$$\begin{aligned}
S_3 &= \int_0^{\tilde{x}^{ps}} \int_0^1 (w + s - 2tx - \tilde{\mu}^p)\, ds dx + \int_{\tilde{x}^{ps}}^{\bar{x}^p} \int_{\sigma_2(x)}^1 (w + s - 2tx - \tilde{\mu}^p)\, ds dx \\
&\quad + \int_{\tilde{x}^{ps}}^{\bar{x}^p} \int_0^{\sigma_2(x)} (w - 2t(1-x) - 2\tilde{\rho}_2^p)\, ds dx + \int_{\bar{x}^p}^1 \int_0^1 (w - 2t(1-x) - 2\tilde{\rho}_2^p)\, ds dx \\
&= w - \int_0^{\tilde{x}^{ps}} (2tx + \tilde{\mu}^p)\, dx - \int_{\tilde{x}^{ps}}^{\bar{x}^p} (2tx + \tilde{\mu})(1 - \sigma_2(x))\, dx + \frac{1}{2}\bar{x}^p - \int_{\tilde{x}^{ps}}^{\bar{x}^p} \frac{\sigma_2^2(x)}{2} dx \\
&\quad - \int_{\tilde{x}^{ps}}^{\bar{x}^p} (2t(1-x) + 2\tilde{\rho}_2^p)\, \sigma_2(x)\, dx - \int_{\bar{x}^p}^1 (2t(1-x) + 2\tilde{\rho}_2^p)\, dx \\
&= w - t(\bar{x}^p)^2 - \tilde{\mu}^p \bar{x}^p + \frac{1}{2}\bar{x}^p + \frac{8t^2(\bar{x}^p - \tilde{x}^{ps})^3}{3} - 2\tilde{\rho}_2^p(1 - \bar{x}^p) - t(1 - \bar{x}^p)^2.
\end{aligned}$$

Substituting $\bar{x}^p = \frac{5}{8} + \frac{5\omega}{32}$, $\bar{x}^p - \tilde{x}^{ps} = \omega/4$, $\tilde{\mu}^p = t\left(\frac{5}{2} + \frac{\omega}{8}\right)$, and $\tilde{\rho}_2^p = t\left(\frac{3}{2} - \frac{\omega}{8}\right)$ into the above expression, we obtain:

$$S_3 = w + t\left(\frac{19}{1536}\omega^2 + \frac{21}{64}\omega - \frac{103}{32}\right).$$

Comparing with the total consumer surplus before the merger, we get:

$$\begin{aligned}\Delta_3(\omega) &\equiv \frac{S_3 - S_0}{t} = \frac{19}{1536}\omega^2 + \frac{21}{64}\omega - \frac{23}{32} \\ &= \frac{19}{1536}\left(\left(\omega + \frac{252}{19}\right)^2 - \frac{84\,480}{361}\right).\end{aligned}$$

Note that $\Delta_3(\omega)$ is a quadratic function. Furthermore, we have $\Delta_3\left(\frac{12}{5}\right) = \frac{7}{50}$, and $\Delta_3(0) = -\frac{23}{32}$. Then, $\Delta_3 > 0$ if:

$$\omega > \bar{\omega} \equiv \frac{16\sqrt{330} - 252}{19},$$

or equivalently:

$$t < t_s \equiv \frac{19}{16\sqrt{330} - 252} \simeq 0.49.$$

Thus, the merger increases consumer surplus when $\frac{5}{12} < t < t_s$. Conversely, the merger reduces consumer surplus when $t > t_s$.

B Merger Dynamics

In this section, we begin by proving Lemma 1 in the text.

Proof. Intuitively, in the absence of mergers, firms always have an incentive to initiate the first merger, as portfolio differentiation allows for profit generation. Conversely, if a conglomerate L already exists, attempting to form a second conglomerate with an identical portfolio would trigger unprofitable Bertrand-type competition between the two conglomerates. We now prove that there cannot be two medium-sized conglomerates in equilibrium.

Let's assume the existence of two conglomerates, M_1 and M_2, where M_1 is formed by the first merger between firms A_1 and B_1, and M_2 is formed by the second merger between firms B_2 and C_2. Firm M_1 offers the bundle $A_1 - B_1$ at a margin μ_M^1 while firm M_2 offers the bundle $B_2 - C_2$ at the margin μ_M^2.

When consumers combine the bundle $A_1 - B_1$ with a stand-alone product C_j, they obtain a net consumer value of $w + s - \mu_M^1 - \gamma_j$, where γ_j represents the margin for C_j. On the other hand, mixing the bundle $B_2 - C_2$ with a stand-alone product A_j yields a net surplus of $w + s - \mu_M^2 - \alpha_j$, where α_j represents the margin for A_j. Consumers prefer the first portfolio over the second portfolio if and only if $\mu_M^1 + \gamma_j \leq \mu_M^2 + \alpha_j$. This triggers Bertrand-type competition between these two portfolios, resulting in zero margins for all products.

Suppose $\mu_M^1 + \gamma_j > \mu_M^2 + \alpha_j \geq 0$. In this case, the merged firm M_2 can benefit from raising its margin μ_M^2. Similarly, suppose $\mu_M^2 + \alpha_j > \mu_M^1 + \gamma_j > 0$. In this scenario, firm M_1 can benefit from raising its margin μ_M^1 slightly. Finally, suppose $\mu_M^2 + \alpha_j = \mu_M^1 + \gamma_j > 0$. In this situation, either M_1 or M_2 can benefit from undercutting the rival. Therefore, $\mu_M^1 + \gamma_j > 0$ or $\mu_M^2 + \alpha_j > 0$ cannot be an equilibrium outcome. Since firms will not charge a negative margin, in equilibrium, we must have $\mu_M^1 = \mu_M^2 = \alpha_j = \gamma_j = 0$. Thus, if there is already a conglomerate M_1 formed, it is never profitable for the remaining stand-alone firms to form another conglomerate M_2. ■

Hence, without loss of generality, we can focus on market configurations that have at most one conglomerate of any size. This means we consider market configurations where there is one large conglomerate L offering all three products and one medium-sized conglomerate M offering two products. In this case, we can analyze the medium-sized conglomerate merger involving two firms A_j and B_j without any loss of generality.

In the subsequent analysis, we will examine the equilibria in three scenarios: highly concentrated markets with $n = 2$, mildly concentrated markets with $n = 3$, and dispersed markets with $n \geq 4$. For each scenario, we will consider different market configurations resulting from the mergers. These configurations include two conglomerates L and M, only one conglomerate L, and only one conglomerate M. We characterize the equilibrium under different bundling options and calculate the equilibrium profits of the merged firms. By comparing the profits obtained under different configurations, we can determine the subgame perfect Nash equilibrium for the dynamic merger game.

As a result of conglomerate mergers, there will be three possible product portfolios available to consumers:

- Portfolio $\mathcal{P}_L$ includes the bundle $\mathcal{B}_L$ offered by the conglomerate L. Consumers derive a gross value of $w + 2s$ from consuming this portfolio.
- Portfolio $\mathcal{P}_M$ consists of a bundle $\mathcal{B}_M$ (e.g., $A_j - B_j$) offered by the conglomerate M and a stand-alone product C_j. Consumers derive a gross value of $w + s$ from consuming this portfolio.
- Portfolio $\mathcal{P}_S$ comprises three stand-alone products, namely A_j, B_j, and C_j. Consumers derive a gross value of w from consuming this portfolio.

We assume that the aggregate social value generated by the three products w is sufficiently large such that $w > 3/4$. This assumption ensures that all consumers will purchase all three products in equilibrium. We will also introduce μ_L and μ_M as the margins for the bundles $\mathcal{B}_L$ and $\mathcal{B}_M$, respectively.

B.1 Highly concentrated markets

We first analyze the merger decisions in highly concentrated markets with $n_i = 2$ for each market $i = A, B, C$. We consider three market configurations in the candidate equilibria: both L and M are present, only L is formed, and only M is formed.

B.1.1 Configuration 1: two conglomerates

In the first market configuration, both conglomerates L and M have been formed by the end of Stage 2. Suppose conglomerate M is formed by two firms from market A and market B, respectively. After the two mergers, there is only one stand-alone firm remaining in market C, while there are no stand-alone firms in the other markets. We analyze the equilibrium prices and profits in Stage 3.

Mixed bundling by both L and M Suppose both merged firms, L and M, continue to supply stand-alone products in addition to their bundles. In this case, all three markets for stand-alone products remain competitive, and the stand-alone products are supplied at cost. Consumers have three product portfolios to choose from:

- opting for portfolio $\mathcal{P}_L$ provides a net utility of $w + 2s - \mu_L$;
- purchasing portfolio $\mathcal{P}_M$ yields a net value of $w + s - \mu_M$;
- selecting portfolio $\mathcal{P}_S$ results in a net surplus of w.

Then, consumers with $s < \mu_M$ opt for portfolio $\mathcal{P}_S$, these with $\mu_M < s < \mu_L - \mu_M$ purchase portfolio $\mathcal{P}_M$, and those with $s > \mu_L - \mu_M$ buy portfolio $\mathcal{P}_L$. In the candidate equilibrium where all three options attract consumers, the profits of the two merged firms are as follows:

$$\begin{aligned} \Pi_L &= \mu_L \left[1 - F\left(\mu_L - \mu_M\right)\right] = \mu_L \left(1 + \mu_M - \mu_L\right), \\ \Pi_M &= \mu_M \left[F\left(\mu_L - \mu_M\right) - F\left(\mu_M\right)\right] = \mu_M \left(\mu_L - 2\mu_M\right). \end{aligned}$$

By solving for the equilibrium margins, we find:

$$\mu_L^{mm} \equiv \frac{4}{7}, \ \mu_M^{mm} \equiv \frac{1}{7},$$

where the superscript mm stands for "mixed bundling by L and mixed bundling by M". The equilibrium profits of the merged firms are:

$$\Pi_L^{mm} \equiv \frac{16}{49},\ \Pi_M^{mm} \equiv \frac{2}{49}.$$

Pure bundling by both L and M Suppose conglomerate L commits to offering only the pure bundle $\mathcal{B}_L$, while conglomerate M commits to offering only the pure bundle $\mathcal{B}_M$. In this case, there are no stand-alone firms in markets A and B. However, in market C, there remains a single stand-alone firm that charges a positive margin γ. Consumers have two portfolios to choose from. Choosing portfolio $\mathcal{P}_L$ yields a net utility of $w + 2s - \mu_L$, while opting for portfolio $\mathcal{P}_M$ provides a net value of $w + s - \mu_M - \gamma$.

In the candidate equilibrium where both options attract consumers, those with $s > \mu_L - \mu_M - \gamma$ choose portfolio $\mathcal{P}_L$, while others opt for portfolio $\mathcal{P}_M$. The profits of the three relevant firms are given as follows:

$$\begin{aligned}
\Pi_L &= \mu_L \left[1 - F\left(\mu_L - \mu_M - \gamma\right)\right] = \mu_L \left(1 + \mu_M + \gamma - \mu_L\right), \\
\Pi_M &= \mu_M F\left(\mu_L - \mu_M - \gamma\right) = \mu_M \left(\mu_L - \mu_M - \gamma\right), \\
\Pi_S &= \gamma F\left(\mu_L - \mu_M - \gamma\right) = \gamma \left(\mu_L - \mu_M - \gamma\right).
\end{aligned}$$

The equilibrium margins are given by

$$\mu_L^{pp} \equiv \frac{3}{4},\ \gamma^{pp} = \mu_M^{pp} \equiv \frac{1}{4},$$

where the superscript pp stands for "pure bundling by L and pure bundling by M". The equilibrium profits are:

$$\Pi_L^{pp} \equiv \frac{9}{16},\ \Pi_S^{pp} = \Pi_M^{pp} \equiv \frac{1}{16}.$$

We examine whether all three options attract consumers in equilibrium. Consumers opting for portfolio $\mathcal{P}_M$ obtain a net surplus:

$$w - \mu_M^{pp} - \gamma^{pp} + s = w - \frac{1}{2} + s > w - \frac{1}{2}.$$

On the other hand, buying only the stand-alone firm's product C_j would give them:

$$\frac{w}{3} - \gamma^{pp} = \frac{w}{3} - \frac{1}{4}.$$

The assumption $w > \frac{3}{4}$ implies $w - \frac{1}{2} > \frac{w}{3} - \frac{1}{4} > 0$, which ensures the existence of the above interior equilibrium.

Pure bundling by L only Suppose conglomerate L commits to offering only $\mathcal{B}_L$, while conglomerate M continues to supply stand-alone products A_j and B_j alongside its bundle $\mathcal{B}_M$. Conglomerate M charges positive margins α and β for the stand-alone products A_j and B_j, respectively. Consumers have three options:

- buying portfolio $\mathcal{P}_L$ yields a net utility $w + 2s - \mu_L$;
- purchasing portfolio $\mathcal{P}_M$ provides a net value $w + s - \mu_M - \gamma$;
- opting for portfolio $\mathcal{P}_S$ results in a net surplus $w - \alpha - \beta - \gamma$.

In the candidate equilibrium in which all options attract consumers, consumers with $s < \mu_M - \alpha - \beta$ opt for portfolio $\mathcal{P}_S$, these with $s > \mu_L - \mu_M - \gamma$ buy portfolio $\mathcal{P}_L$, while others opt for portfolio $\mathcal{P}_M$. The profits of the relevant firms in the candidate equilibrium are as follows:

$$\begin{aligned}
\Pi_L &= \mu_L \left[1 - F\left(\mu_L - \mu_M - \gamma\right)\right] = \mu_L \left(1 - \mu_L + \mu_M + \gamma\right), \\
\Pi_M &= \mu_M \left[F\left(\mu_L - \mu_M - \gamma\right) - F\left(\mu_M - \alpha - \beta\right)\right] + (\alpha + \beta) F\left(\mu_M - \alpha - \beta\right) \\
&= \mu_M F\left(\mu_L - \mu_M - \gamma\right) - \left(\mu_M - \alpha - \beta\right) F\left(\mu_M - \alpha - \beta\right), \\
\Pi_S &= \gamma F\left(\mu_L - \mu_M - \gamma\right) = \gamma \left(\mu_L - \mu_M - \gamma\right).
\end{aligned}$$

In this case, M can increase its profits by squeezing the demand for its stand-alone products. By replacing $\alpha+\beta < \mu_M$ with $\alpha'+\beta' = \mu_M$, M's profit increases by $\left(\mu_M - \alpha - \beta\right) F\left(\mu_M - \alpha - \beta\right) > 0$. Thus, conditional on L engaging in pure bundling, M strictly prefers to do the same even if it did not commit to it. As a result, the equilibrium margins and profits are the same as if M had committed to pure bundling:

$$\mu_L^{pm} \equiv \mu_L^{pp} = \frac{3}{4}, \; \gamma^{pm} = \gamma^{pp} = \mu_M^{pm} \equiv \mu_M^{pp} = \frac{1}{4},$$

and

$$\begin{aligned}
\Pi_L^{pm} &\equiv \Pi_L^{pp} = \frac{9}{16}, \\
\Pi_S^{pm} &= \Pi_S^{pp} = \Pi_M^{pm} \equiv \Pi_M^{pp} = \frac{1}{16}.
\end{aligned}$$

Pure bundling by M only Suppose M commits to offering only the pure bundle $\mathcal{B}_M$, while L continues to offer stand-alone products along with the bundle $\mathcal{B}_L$. L becomes the sole supplier of stand-alone products in markets A and B, charging positive margins α and β respectively. However, it is not optimal for L to offer the stand-alone product in market C since there is already a stand-alone firm C_j in that market. Therefore, L commits not to offer the stand-alone product in market C, and the remaining stand-alone firm C_j charges a positive margin γ.

Consumers can choose from three options:

- buying portfolio $\mathcal{P}_L$ yields a net utility $w + 2s - \mu_L$;
- purchasing portfolio $\mathcal{P}_M$ provides a net value $w + s - \mu_M - \gamma$;
- opting for portfolio $\mathcal{P}_S$ results in a net surplus $w - \alpha - \beta - \gamma$.

The profits of the three firms in the candidate equilibrium where all three options attract consumers are as follows:

$$\begin{aligned}
\Pi_L &= \mu_L \left[1 - F\left(\mu_L - \mu_M - \gamma\right)\right] + (\alpha + \beta) F\left(\mu_M - \alpha - \beta\right) \\
&= \mu_L \left(1 + \mu_M + \gamma - \mu_L\right) + (\alpha + \beta)\left(\mu_M - \alpha - \beta\right), \\
\Pi_M &= \mu_M \left[F\left(\mu_L - \mu_M - \gamma\right) - F\left(\mu_M - \alpha - \beta\right)\right] \\
&= \mu_M \left(\mu_L + \alpha + \beta - 2\mu_M - \gamma\right), \\
\Pi_S &= \gamma \left(\mu_L - \mu_M - \gamma\right)
\end{aligned}$$

The equilibrium margins are:

$$\mu_L^{mp} \equiv \frac{12}{17},\ \mu_M^{mp} \equiv \frac{2}{17},\ \alpha^{mp} + \beta^{mp} \equiv \frac{1}{17},\ \gamma^{mp} \equiv \frac{5}{17},$$

where the superscript mp stands for "mixed bundling by L and pure bundling by M". In this equilibrium, the profits of the firms are:

$$\Pi_L^{mp} \equiv \frac{145}{289},\ \Pi_M^{mp} \equiv \frac{8}{289},\ \Pi_S^{mp} \equiv \frac{25}{289}.$$

Bundling decisions Based on the analysis, it is evident that pure bundling is a dominant strategy for the merged entity L. Regardless of M's bundling decision, L prefers pure bundling over mixed bundling or mixed bundling by M. The profit comparison supports this:

$$\begin{aligned}
\Pi_L^{pp} - \Pi_L^{mp} &= \frac{9}{16} - \frac{145}{289} = \frac{281}{4624} > 0, \\
\Pi_L^{pp} - \Pi_L^{mm} &= \frac{9}{16} - \frac{16}{49} = \frac{185}{784} > 0.
\end{aligned}$$

Thus, L always engages in pure bundling. Furthermore, L is better off by offering only one bundle $\mathcal{B}_L$. On the other hand, given that L engages in pure bundling, M is better off also opting for pure bundling.

In conclusion, when there are two merged firms, L and M, they both commit to pure bundling strategies: L provides only $\mathcal{B}_L$, while M offers only $\mathcal{B}_M$. The resulting equilibrium profits are:

$$\Pi_L^{pp} = \frac{9}{16},\ \Pi_S^{pp} = \Pi_M^{pp} = \frac{1}{16}.$$

B.1.2 Configuration 2: large conglomerate only

In the second configuration where only the large conglomerate L has been formed, there is no benefit for L to supply the stand-alone products after the merger. In this case, L has the option to offer either one bundle or two bundles. We will consider the two possible bundling options: L offering one bundle $\mathcal{B}_L$ only or L offering two bundles $\mathcal{B}_L$ and $\mathcal{B}_M$.

L offers one bundle $\mathcal{B}_L$ only Suppose L offers one bundle $\mathcal{B}_L$ only. In each market there remains one stand-alone product which charges a positive margin α, β, and γ respectively. There are two options available to consumers:

- buying portfolio $\mathcal{P}_L$ yields a net utility $w + 2s - \mu_L$;
- purchasing portfolio $\mathcal{P}_S$ offers yields a net value $w - \alpha - \beta - \gamma$.

Consumers with $s > (\mu_L - \alpha - \beta - \gamma)/2$ buy $\mathcal{P}_L$ while the others opt for $\mathcal{P}_S$. In the candidate equilibrium where both options attract consumers, the profits of the conglomerate L and the stand-alone firm in each market (A, B, C) are given respectively by:

$$\begin{aligned}
\Pi_L &= \mu_L \left[1 - F\left(\frac{\mu_L - \alpha - \beta - \gamma}{2}\right)\right] = \frac{1}{2}\mu_L\left(2 + \alpha + \beta + \gamma - \mu_L\right), \\
\Pi_A &= \alpha F\left(\frac{\mu_L - \alpha - \beta - \gamma}{2}\right) = \frac{1}{2}\alpha\left(\mu_L - \alpha - \beta - \gamma\right), \\
\Pi_B &= \beta F\left(\frac{\mu_L - \alpha - \beta - \gamma}{2}\right) = \frac{1}{2}\beta\left(\mu_L - \alpha - \beta - \gamma\right), \\
\Pi_C &= \gamma F\left(\frac{\mu_L - \alpha - \beta - \gamma}{2}\right) = \frac{1}{2}\gamma\left(\mu_L - \alpha - \beta - \gamma\right).
\end{aligned}$$

Solving for the equilibrium margins yields:

$$\mu_L^l \equiv \frac{8}{5},\ \alpha^l = \beta^l = \gamma^l \equiv \frac{6}{5},$$

where the superscript l stands for "large bundle only". The corresponding profits are:

$$\Pi_L^l \equiv \frac{32}{25},\ \Pi_A^l = \Pi_B^l = \Pi_C^l = \Pi_S^l \equiv \frac{2}{25}.$$

L **offers both bundles** $\mathcal{B}_L$ **and** $\mathcal{B}_M$ Suppose now L offers both bundles $\mathcal{B}_L$ and $\mathcal{B}_M$ (A_1-B_1). Consumers face three options:

- buying portfolio $\mathcal{P}_L$ yields a net utility $w+2s-\mu_L$;
- purchasing portfolio $\mathcal{P}_M$ provides a net value $w+s-\mu_M-\gamma$;
- opting for portfolio $\mathcal{P}_S$ leads to a net surplus $w-\alpha-\beta-\gamma$.

Consumers with $s>\mu_L-\mu_M-\gamma$ buy $\mathcal{P}_L$, these with $\mu_M-\alpha-\beta<s<\mu_L-\mu_M-\gamma$ buy $\mathcal{P}_M$, and the others purchase $\mathcal{P}_S$. In the candidate equilibrium where all three options attract consumers, the relevant firms' profits are given respectively by:

$$\begin{aligned}
\Pi_L &= \mu_L\left[1-F\left(\mu_L-\mu_M-\gamma\right)\right]+\mu_M\left[F\left(\mu_L-\mu_M-\gamma\right)-F\left(\mu_M-\alpha-\beta\right)\right] \\
&= \mu_L\left(1+\mu_M+\gamma-\mu_L\right)+\mu_M\left(\mu_L-2\mu_M-\gamma+\alpha+\beta\right), \\
\Pi_A &= \alpha F\left(\mu_M-\alpha-\beta\right)=\alpha\left(\mu_M-\alpha-\beta\right), \\
\Pi_B &= \beta F\left(\mu_M-\alpha-\beta\right)=\beta\left(\mu_M-\alpha-\beta\right), \\
\Pi_C &= \gamma F\left(\mu_L-\mu_M-\gamma\right)=\gamma\left(\mu_L-\mu_M-\gamma\right).
\end{aligned}$$

Solving for equilibrium margins leads to

$$\mu_L^{lm}=\frac{17}{12},\ \mu_M^{lm}=\frac{3}{4},\ \alpha^{lm}=\beta^{lm}=\frac{1}{4},\ \gamma^{lm}=\frac{1}{3},$$

where the superscript lm stands for "large and medium-sized bundles". The corresponding equilibrium profits are:

$$\Pi_L^{lm}=\frac{145}{144},\ \Pi_A^{lm}=\Pi_B^{LM}=\frac{1}{16},\ \Pi_C^{lm}=\frac{1}{9}.$$

Offering the medium-sized bundle $\mathcal{B}_M$ on top of the large bundle $\mathcal{B}_L$ can indeed lead to cannibalization and increased competition with the stand-alone firms. This can negatively impact L's profits. Comparing L's profits under the two cases, we find that L is worse off in offering both bundles after the merger: $\Pi_L^L=\frac{32}{25}>\Pi_L^{LM}=\frac{145}{144}$.

B.1.3 Configuration 3: medium-sized conglomerate only

Consider the third configuration in which a medium-sized conglomerate, M, has been formed after the merger between firms A_j and B_j by the end of Stage 2. In this configuration, only one stand-alone firm remains in markets A and B, while two stand-alone firms remain in market C.

As a result, the remaining stand-alone firm in market A (resp. B) charges a positive margin α (resp. β), while the margin for the stand-alone product in market C remains zero. The conglomerate M does not benefit from supplying the stand-alone products.

Suppose M commits itself to pure bundling. Consumers face two options:

- buying $\mathcal{P}_M$ yields a net utility $w + s - \mu_M$;
- purchasing $\mathcal{P}_S$ provides a net value $w - \alpha - \beta$.

Consumers with $s > \mu_M - \alpha - \beta$ buy $\mathcal{P}_M$ while the others opt for $\mathcal{P}_S$. In the candidate equilibrium where both option attract consumers, the profits for the conglomerate and stand-alone firms in markets A and B are given respectively by:

$$\begin{aligned}
\Pi_M &= \mu_M \left[1 - F\left(\mu_M - \alpha - \beta\right)\right] = \mu_M \left(1 + \alpha - \beta - \mu_M\right), \\
\Pi_A &= \alpha F\left(\mu_M - \alpha - \beta\right) = \alpha \left(\mu_M - \alpha - \beta\right), \\
\Pi_B &= \beta F\left(\mu_M - \alpha - \beta\right) = \beta \left(\mu_M - \alpha - \beta\right).
\end{aligned}$$

Solving for equilibrium margins leads to:

$$\mu_M^m = \frac{3}{4},\ \alpha^m = \beta^m = \frac{1}{4},$$

where the superscript m stands for "medium-sized bundle". The equilibrium profits are:

$$\Pi_M^m = \frac{9}{16},\ \Pi_A^m = \Pi_B^m = \frac{1}{16}.$$

B.1.4 Merger decisions

We analyze the merger decisions in the subgame perfect Nash equilibrium and consider the six possible candidate equilibria based on Lemma 1:

- (a). Conglomerate L formed in Stage 1 followed by M in Stage 2;
- (b). Conglomerate L formed in Stage 1 and no merger in Stage 2;
- (c). Conglomerate M formed in Stage 1 followed by L in Stage 2;
- (d). Conglomerate M formed in Stage 1 and no merger in Stage 2;
- (e). No merger in Stage 1 followed by L in Stage 2;
- (f). No merger in Stage 1 followed by M in Stage 2.

We proceed by eliminating dominated candidate equilibria to identify the subgame perfect Nash equilibrium.

Suppose a large conglomerate has already emerged in Stage 1, and consider a subsequent merger decision between two firms present in markets A and B in Stage 2. It is evident that the second merger is never profitable if the conglomerate L is committed to providing both bundles $\mathcal{B}_L$ and $\mathcal{B}_M$. Suppose L is committed to offering the large bundle $\mathcal{B}_L$ only. In this case, by remaining independent, each firm obtains profits $\Pi^l_A = \Pi^l_B = \frac{2}{25}$. On the other hand, if a merger occurs to form conglomerate M, where both merged firms engage in pure bundling, the resulting profit is $\Pi^{pp}_M = \frac{1}{16}$. However, considering the comparison

$$\Pi^{pp}_M - 2\Pi^L_S = \frac{1}{16} - 2 \times \frac{2}{25} = -\frac{39}{400} < 0,$$

we conclude that the two firms are better off remaining as stand-alone entities. Hence, we can rule out candidate equilibrium (a).

Suppose a medium conglomerate M between two firms in markets A and B has been formed in Stage 1, and consider a subsequent merger decision between three firms present in markets A, B, and C in Stage 2. Since mixed bundling is dominated, we only need to consider pure bundling by M. By remaining stand-alone, each firm in market A and B earns $\Pi^m_A = \Pi^m_B = \frac{1}{16}$, while firms in market C earns zero profit. On the other hand, a subsequent merger forming a larger conglomerate yields a total profit $\Pi^{pp}_L = \frac{9}{16}$. Comparing this with three times the profit of a stand-alone firm $3\Pi^M_S = \frac{3}{16}$, we have

$$\Pi^{pp}_L - 3\Pi^M_S = \frac{9}{16} - 3 \times \frac{1}{16} = \frac{3}{8} > 0.$$

This indicates that the merger is profitable. Therefore, we can exclude candidate equilibrium (d) as it is not viable.

Suppose that the three firms decide not to merge in Stage 1, anticipating a potential merger in Stage 2. It is evident that forming the large conglomerate L would be more profitable than forming the medium conglomerate M in Stage 2. Therefore, we can rule out candidate equilibrium (f) as it is not a viable option.

We now turn our attention back to Stage 1 and examine the merger decisions made by the three randomly selected firms in the remaining candidate equilibria (b), (c), and (e).

In candidate equilibrium (b), all three firms decide to form the conglomerate L without any subsequent mergers. In this configuration, the merged entity L earns a total profit of $\Pi^l_L = \frac{32}{25}$.

In candidate equilibrium (c), two firms choose to merge and form the conglomerate M, while one firm remains as a stand-alone entity. This is followed by a subsequent merger to form the conglomerate L. In this equilibrium, the merged entity M earns a profit of $\Pi_M^{pp} = \frac{1}{16}$, while the stand-alone firm earns a profit of $\Pi_S^{pp} = \frac{1}{16}$, resulting in an aggregate profit $\frac{1}{8}$.

In candidate equilibrium (e), all three firms opt to remain as stand-alone entities in Stage 1. Then, in Stage 2, a subsequent merger occurs, leading to the formation of the conglomerate L. In this case, each stand-alone firm earns a profit of $\Pi_S^l = \frac{2}{25}$, resulting in an aggregate profit of $\frac{6}{25}$.

Comparing the aggregate profits for the three selected firms and assuming efficient merger bargaining, we find that candidate equilibrium (b) yields the highest total profit among the three. Therefore, it dominates the other equilibria. Consequently, based on the analysis, we can conclude that there exists a dominant candidate equilibrium, namely (b), in which the conglomerate L is formed in Stage 1 without any subsequent mergers. The merged entity L will offer a pure bundle $\mathcal{B}_L$ only, as summarized below:

Proposition 2 *Consider a two-stage dynamic merger game involving three product markets, where each market is served by two identical stand-alone firms. There exists a unique SPNE in which only the large conglomerate will be formed in Stage 1, with no subsequent mergers. In this equilibrium, conglomerate L offers a pure bundle consisting of three products and charges a margin $\mu_L^l = \frac{8}{5}$, resulting in a profit $\Pi_L^l = \frac{32}{25}$, while each stand-alone firm charges a margin of $\frac{6}{5}$ and earns a profit $\Pi_S^l = \frac{2}{25}$.*

B.2 Mildly concentrated markets

We now analyze the merger dynamics in the context of mildly concentrated markets, where there are three firms operating in each market. We will consider the same three market configurations as discussed previously.

B.2.1 Configuration 1: two conglomerates

Consider the first market configuration, where both conglomerates L and M have been formed by the end of Stage 2. Suppose conglomerate M is formed by one from market A and the other from market B. As a result of the mergers, there is now only one stand-alone firm remaining in markets A and B, while in market C there are still two stand-alone firms, and the margin for the stand-alone product in market C remains zero.

We will now analyze the equilibrium prices and profits under different bundling decisions, following the same approach as before.

Mixed bundling When there are three firms in each market and at least one of the conglomerates does not commit to pure bundling, Bertrand-type competition leads to all stand-alone prices being driven down to cost. In this case, the equilibrium outcomes are exactly the same as under mixed bundling in highly concentrated markets. The profits of conglomerates L and M, denoted by superscript m for "mixed bundling by at least one conglomerate," are as follows:

$$\Pi_L^m \equiv \frac{16}{49},\ \Pi_M^m \equiv \frac{2}{49}.$$

Pure bundling by both L and M Suppose that both conglomerates commit themselves to pure bundling. In this case, the remaining stand-alone firm in market A (resp. B) will charge a positive margin α (resp. β). However, competition among the two stand-alone firms in market C drives their margins to zero. Consumers face three options:

- buying $\mathcal{P}_L$ yields a net utility equal to $w + 2s - \mu_L$;
- purchasing $\mathcal{P}_M$ yields a net value $w + s - \mu_M$;
- opting for $\mathcal{P}_S$ provides a net surplus $w - \alpha - \beta$.

Consumers with $s > \mu_L - \mu_M$ buy $\mathcal{P}_L$, these with $\mu_M - \alpha - \beta < s < \mu_L - \mu_M$ purchase $\mathcal{P}_M$, and those with $s < \mu_M - \alpha - \beta$ buy $\mathcal{P}_S$.

In the candidate equilibrium where all three options attract consumers, the profits of relevant firms are given respectively by:

$$\begin{aligned}
\Pi_L &= \mu_L \left[1 - F\left(\mu_L - \mu_M\right)\right] = \mu_L \left(1 + \mu_M - \mu_L\right), \\
\Pi_M &= \mu_M \left[F\left(\mu_L - \mu_M\right) - F\left(\mu_M - \alpha - \beta\right)\right] = \mu_M \left(\mu_L + \alpha + \beta - 2\mu_M\right), \\
\Pi_A &= \alpha F\left(\mu_M - \alpha - \beta\right) = \alpha \left(\mu_M - \alpha - \beta\right), \\
\Pi_B &= \beta F\left(\mu_M - \alpha - \beta\right) = \beta \left(\mu_M - \alpha - \beta\right).
\end{aligned}$$

Solving for equilibrium leads to:

$$\mu_L^{pp} \equiv \frac{10}{17},\ \mu_M^{pp} \equiv \frac{3}{17},\ \alpha^{pp} = \beta^{pp} \equiv \frac{1}{17}.$$

The corresponding equilibrium profits are:

$$\Pi_L^{pp} \equiv \frac{100}{289},\ \Pi_M^{pp} \equiv \frac{18}{289},\ \Pi_A^{pp} = \Pi_B^{pp} \equiv \frac{1}{289}.$$

Bundling decisions It can be concluded from the analysis that pure bundling is a weakly dominant strategy for both conglomerates.

- If conglomerate M chooses mixed bundling, conglomerate L is indifferent between mixed and pure bundling. In either case, the margins for stand-alone products are zero, and the conglomerates earn their profits solely from their bundles. However, if conglomerate M chooses pure bundling, conglomerate L prefers pure bundling as well, as shown below:

$$\Pi_L^{pp} - \Pi_L^m = \frac{100}{289} - \frac{16}{49} = \frac{276}{14161} > 0.$$

- Similarly, if conglomerate L chooses mixed bundling, conglomerate M is indifferent between mixed and pure bundling. However, if conglomerate L chooses pure bundling, conglomerate M prefers pure bundling as well, as shown by:

$$\Pi_M^{pp} - \Pi_M^m = \frac{18}{289} - \frac{2}{49} = \frac{304}{14161} > 0.$$

Suppose two conglomerates, L and M, have been formed. There are two Nash equilibria in pure strategies: one in which both L and M commit themselves to pure bundling, and one in which none of them does. However, eliminating weakly dominated strategies pins down a unique outcome, in which both conglomerates commit themselves to pure bundling. The related equilibrium profits are:

$$\Pi_L^{pp} = \frac{100}{289},\ \Pi_M^{pp} = \frac{18}{289},\ \Pi_A^{pp} = \Pi_B^{pp} = \frac{1}{289}.$$

B.2.2 Configuration 2: larger conglomerate only

Consider the second market configuration, where only the large conglomerate L has been formed by the end of Stage 2. In this scenario, there are at least two stand-alone firms remaining in each market. Regardless of the bundling decision made by L, Bertrand-type competition drives stand-alone prices down to the cost.

Suppose L offers only one bundle $\mathcal{B}_L$. Consumers face two options:

- buying $\mathcal{P}_L$ yields a net utility $w + 2s - \mu_L$;

- purchasing $\mathcal{P}_S$ provides a net value w.

Consumers with $s > \mu_L/2$ buy $\mathcal{P}_L$, whereas the others opt for $\mathcal{P}_S$. In the candidate equilibrium where both options attract consumers, the conglomerate's profit is given by

$$\Pi_L = \mu_L \left[1 - F\left(\frac{\mu_L}{2}\right)\right] = \frac{1}{2}\mu_L (2 - \mu_L).$$

Solving for the equilibrium margin and profit leads to

$$\mu_L^l = 1, \ \Pi_L^l = \frac{1}{2}.$$

Suppose now L offers both bundles $\mathcal{B}_L$ and $\mathcal{B}_M$ $(A_1 - B_1)$. There are three options available to consumers:

- buying $\mathcal{P}_L$ yields a net utility $w + 2s - \mu_L$;
- purchasing $\mathcal{P}_M$ yields a net value $w + s - \mu_M$;
- buying $\mathcal{P}_S$ provides a net value w.

Consumers with $s > \mu_L - \mu_M$ buy $\mathcal{P}_L$, these with $\mu_M \leq s \leq \mu_L - \mu_M$ purchase $\mathcal{P}_M$, and the others opt for $\mathcal{P}_S$.

Providing bundle $\mathcal{B}_M$ does not affect the margins for stand-alone products, which remain at zero. However, offering $\mathcal{B}_M$ has the effect of diverting some consumers from option 1 to option 2. Since option 2 generates less consumer value than option 1, the conglomerate L does not benefit from offering bundle $\mathcal{B}_M$. This can be shown by examining L's profit when both bundles are offered:

$$\begin{aligned}\Pi_L &= \mu_L [1 - F(\mu_L - \mu_M)] + \mu_M [F(\mu_L - \mu_M) - F(\mu_M)] \\ &= \mu_L (1 - \mu_L + \mu_M) + \mu_M (\mu_L - 2\mu_M).\end{aligned}$$

The first-order condition for μ_M leads to $\mu_L = 2\mu_M$ (the second-order derivative is negative), which implies that at the optimum, $F(\mu_L - \mu_M) = F(\mu_M)$. Thus, L will set a sufficiently high margin μ_M to ensure that no consumers choose to buy bundle $\mathcal{B}_M$. Consequently, the resulting equilibrium remains the same as before: $\mu_L^l = 1$ and $\Pi_L^l = \frac{1}{2}$.

B.2.3 Medium-sized conglomerate only

Suppose that only a medium-sized conglomerate, M, has been formed. In this case, regardless the bundling decision by M, the margin for the stand-alone products remains zero in all three markets. Consumers face two options:

- purchasing $\mathcal{P}_M$ yields a net value $w + s - \mu_M$;
- buying $\mathcal{P}_S$ provides a net value w.

Consumers with $s > \mu_M$ opt for the first option, whereas the others opt for the second option. The conglomerate M's profit is:

$$\Pi_M = \mu_M \left[1 - F\left(\mu_M\right)\right] = \mu_M \left(1 - \mu_M\right).$$

Solving for the equilibrium margin and profit leads to:

$$\mu_M^m = \frac{1}{2},\ \Pi_M^m = \frac{1}{4}.$$

B.2.4 Merger decisions

We now proceed to analyze the merger decisions. Similar to our previous analysis, we consider the following six candidate equilibria:

- (a). Conglomerate L formed in Stage 1 followed by M in Stage 2;
- (b). Conglomerate L formed in Stage 1 and no merger in Stage 2;
- (c). Conglomerate M formed in Stage 1 followed by L in Stage 2;
- (d). Conglomerate M formed in Stage 1 and no merger in Stage 2;
- (e). No merger in Stage 1 followed by L in Stage 2;
- (f). No merger in Stage 1 followed by M in Stage 2.

By eliminating dominated candidate equilibria, we can determine the subgame perfect Nash equilibrium.

Suppose conglomerate L has been formed in Stage 1. Consider a subsequent merger decision between two firms from markets A and B in Stage 2. If these firms choose to remain stand-alone, they will make zero profit due to intense Bertrand competition. However, if they decide to merge and form conglomerate M, it results in a positive profit. In this case, if both conglomerates commit to pure bundling, L will earn a profit of $\Pi_L^{pp} = \frac{100}{289}$. If at least one merged firm engages in mixed bundling, L's profit will be $\Pi_L^m = \frac{16}{49}$.

On the other hand, if L commits to providing both bundles $\mathcal{B}_L$ and $\mathcal{B}_M$, the second merger will not occur. In this scenario, L's profit will be $\Pi_L^l = \frac{1}{2}$. Comparing the profits, we have

$$\Pi_L^L = \frac{1}{2} > \Pi_L^{pp} = \frac{100}{289} > \Pi_L^m = \frac{16}{49}.$$

Thus, L will commit to offering two bundles to prevent the subsequent merger. Therefore, candidate equilibrium (a) cannot arise in the subgame perfect Nash equilibrium.

Suppose a medium-sized conglomerate M has already formed in Stage 1. In this case, a subsequent merger to form conglomerate L is always profitable, which rules out candidate equilibrium (d). Additionally, choosing not to merge in Stage 1 and anticipating a merger in Stage 2 is a dominated strategy. Any selected firm would strictly prefer to participate in the merger rather than remaining stand-alone because all stand-alone firms earn zero profits when there is only one conglomerate. Therefore, we can rule out candidate equilibria (e) and (f).

When comparing candidate equilibria (b) and (c), conglomerate L is strictly better off in case (b). As a result, in the scenario where there are three firms in each market, there exists a unique subgame perfect Nash equilibrium in which the large conglomerate L is formed in Stage 1 and commits itself to offering two bundles. No subsequent mergers occur in this equilibrium.

B.3 Dispersed markets

In dispersed markets where each market has at least four firms ($n_i \geq 4$ for $i = A, B, C$), regardless of the number of conglomerates formed, the competition among stand-alone firms drives all stand-alone prices down to cost. In this context, both pure bundling and mixed bundling strategies lead to the same equilibrium outcomes, resulting in zero profits for stand-alone firms. The conglomerates, on the other hand, earn positive profits solely from their bundles and not from the stand-alone products.

B.3.1 Configuration 1: two conglomerates

If both conglomerates have been formed in dispersed markets, the situation is indeed similar to the case of fully mixed bundling in more concentrated markets. Therefore, the conglomerates' profit are given by:

$$\Pi_L^m = \frac{16}{49} \text{ and } \Pi_M^m = \frac{2}{49}.$$

B.3.2 Configuration 2: larger conglomerate only

Suppose only the large conglomerate L has been formed. Then the equilibrium outcome is the same as with mildly concentrated markets, in which L earns a profit

$$\Pi_L^l = \frac{1}{2}.$$

B.3.3 Configuration 3: medium-sized conglomerate only

Suppose only a medium-sized conglomerate M has emerged. Once again, the equilibrium outcome is the same as with mildly concentrated markets, in which M earns a profit

$$\Pi_M^m = \frac{1}{4}.$$

B.3.4 Merger decisions

Since $\Pi_L^l > \Pi_L^m$, the large conglomerate L is better off when only one conglomerate has been formed. Consequently, L will strategically commit itself to offering two bundles after the merger, in order to preempt the subsequent merger. The equilibrium outcome is the same as in the case with mildly concentrated market.

Summarizing the above analysis leads to the following proposition:

Proposition 3 *Consider a two-stage dynamic merger game involving three product markets where each market is served by at least three identical stand-alone firms. There exists a unique SPNE in which only the large conglomerate is formed in Stage 1, with no subsequent mergers. In this equilibrium, the conglomerate L commits itself to offering two bundles $\mathcal{B}_L$ and $\mathcal{B}_M$ and charges the margins $\mu_L^l = 1$ and $\mu_M^l \geq 1/2$ for bundles $\mathcal{B}_L$ and $\mathcal{B}_M$ respectively, making a profit $\Pi_L^l = 1/2$.*

B.4 Welfare Analysis

In highly concentrated markets where each market has only two firms, after the first merger to form the large conglomerate L, there are no subsequent mergers. In this equilibrium, L commits itself to offering the pure large bundle $\mathcal{B}_L$ only. However, the merger leads to an increase in prices for the stand-alone products. The equilibrium prices are determined as follows: $\mu_L^l \equiv \frac{8}{5}$ and $\alpha^l = \beta^l = \gamma^l = \frac{2}{5}$, while the cut-off threshold is given by $\left(\mu_L^l - \alpha^l - \beta^l - \gamma l\right)/2 = \frac{1}{5}$. The welfare loss resulting from the price increases outweighs the welfare gain from consumption synergies. Consequently, the total consumer surplus reduces after the merger:

$$S_2^l = \int_0^{\frac{1}{5}} \left(w - \alpha^l - \beta^l - \gamma^l\right) ds + \int_{\frac{1}{5}}^1 \left(w + 2s - \mu_L^l\right) ds = w - \frac{14}{25} < w.$$

In less concentrated markets where each market has at least three firms ($n_i \geq 3$), after the first merger, conglomerate L will commit itself to offering two pure bundles, and there will be no subsequent mergers. Unlike in highly concentrated markets, the merger in this case does not result in higher prices for the stand-alone products. As a result, the merger actually increases the total consumer surplus.

However, the large conglomerate's preemptive behaviour harms consumers. It is straightforward to see that, in dispersed markets with $n_i \geq 4$, the subsequent merger to form a medium-sized conglomerate M is welfare enhancing. This merger does not result in higher prices for the stand-alone products and, in fact, provides consumers with more options. By committing to its preemptive strategy, conglomerate L reduces competition and limits consumer choice, ultimately harming consumer welfare.

In the mildly concentrated markets with $n_i = 3$, the formation of conglomerate M in the second merger raises the prices for the stand-alone products. However, the net welfare effect of this merger is not straightforward and requires further analysis. To assess the welfare impact, we compare the consumer surplus under two candidate equilibria.

In the equilibrium where only the large conglomerate L exists, the margins for the stand-alone products remain zero, while the margin for the bundle is $\mu_L^L = 1$. The relevant cut-off threshold is $\mu_L^L/2 = 1/2$. In this equilibrium, the total consumer surplus is given by:

$$S_3^l = \int_0^{\frac{1}{2}} w ds + \int_{\frac{1}{2}}^1 \left(w + 2s - \mu_L^l\right) ds = w + \frac{1}{4}.$$

By contrast, if L commits itself to the large bundle $\mathcal{B}_L$ only, there will be two conglomerates L and M in equilibrium. The corresponding margins are:

$$\mu_L^{pp} \equiv \frac{10}{17},\ \mu_M^{pp} \equiv \frac{3}{17},\ \alpha^{pp} = \beta^{pp} \equiv \frac{1}{17},$$

while the cut-off thresholds are:

$$\mu_L - \mu_M = \frac{7}{17},\ \mu_M - \alpha - \beta = \frac{1}{17}.$$

In this equilibrium, the total consumer surplus is given by:

$$\begin{aligned} S_3^{lm} &= \int_0^{\frac{1}{17}} \left(w - \frac{2}{17}\right) ds + \int_{\frac{1}{17}}^{\frac{7}{17}} \left(w + s - \frac{3}{17}\right) ds + \int_{\frac{7}{17}}^{1} \left(w + 2s - \frac{10}{17}\right) ds. \\ &= w + \frac{144}{289}. \end{aligned}$$

Since $\frac{144}{289} > \frac{1}{4}$, L's preemption reduces total consumer surplus.

Proposition 4 *Consider a two-stage dynamic merger game involving three product markets. When each market is served by two stand-alone firms, the conglomerate merger with pure bundling increase the prices for stand-alone products and reduces total consumer welfare. When each market is served by more than three stand-alone firms, the conglomerate merger increases total consumer surplus. However, the large conglomerate's preemptive strategy to prevent the second merger reduces consumer welfare.*

C Monopoly in Market A

We provide an illustrative example for the equilibrium outcome and welfare comparison. Assume $F(s) = s$ and $G(\omega) = \omega/\bar{u}$. Before the merger, then the monopoly margin for product A is $\alpha_m = \bar{u}/2$.

Equilibrium outcome

After the merger, the merged firm maximizes the following profit by choosing μ and α:

$$\Pi_M = \mu \left[1 - \frac{1}{\bar{u}} \int_0^{\alpha} (\mu - \omega)\, d\omega - \left(1 - \frac{\alpha}{\bar{u}}\right)(\mu - \alpha)\right] + \alpha \left(1 - \frac{\alpha}{\bar{u}}\right)(\mu - \alpha).$$

The equilibrium margin α^* is the solution to $\alpha = 2\lambda(\alpha) = 2(\bar{u} - \alpha)$, which gives:

$$\alpha^* = \frac{2}{3}\bar{u}.$$

On the other hand, the equilibrium margin for the bundle μ^* is determined by the following first-order condition:

$$2(\mu - \alpha)\left(1 - \frac{\alpha}{\bar{u}}\right) = 1 - \frac{1}{\bar{u}} \int_0^{\alpha} (\mu - \omega)\, d\omega - \mu\frac{\alpha}{\bar{u}}.$$

Solving for the equilibrium μ^* leads to:

$$\mu^* = \frac{1}{2} + \frac{1}{3}\bar{u}.$$

This equilibrium exists only if $\mu^* > \alpha^*$, which holds when $\bar{u} < \frac{3}{2}$.

Welfare Analysis

Before the merger, consumers obtain a net value of $\omega + w_B - \alpha_m$ if $\omega > \alpha_m$ and w_B if $\omega < \alpha_m$. After the merger:

- consumers with $\omega < \alpha^*$ and $s < \alpha^* - \omega + \mu^* - \alpha^*$ purchase the stand-alone product B_i only and receive w_B;
- consumers with $\omega < \alpha^*$ and $s > \alpha^* - \omega + \mu^* - \alpha^*$ purchase the bundle and receive $\omega + w_B + s - \alpha^* - \mu^* - \alpha^*$;
- consumers with $\omega \geq \alpha^*$ and $s < \mu^* - \alpha^*$ mix A and B_i and get $\omega + w_B - \alpha^*$;
- consumers with $\omega \geq \alpha^*$ and $s > \mu^* - \alpha^*$ opt for the bundle and receive $\omega + w_B + s - \alpha^* - \mu^* - \alpha^*$.

For comparison, consider three different parameter regions: $\omega \geq \alpha^*$, $\alpha_m \leq \omega < \alpha^*$, and $\omega < \alpha_m$. Then:

- consumers with $\omega \geq \alpha^*$ and $s < \mu^* - \alpha^*$ are worse off since $\omega + w_B - \alpha^* < \omega + w_B - \alpha_m$;
- consumers with $\omega \geq \alpha^*$ and $s > \mu^* - \alpha^*$ are better off if $s > \mu^* - \alpha_m > \mu^* - \alpha^*$, and these with $\mu^* - \alpha^* < s < \mu^* - \alpha_m$ are worse off.
- consumers with $\alpha_m \leq \omega < \alpha^*$ and $s < \mu^* - \omega$ now opt for the stand-alone product B_i only, and they are worse off since $w_B < \omega + w_B - \alpha_m$;
- consumers with $\alpha_m \leq \omega < \alpha^*$ and $s > \mu^* - \omega$ are better off if $s > \mu^* - \alpha_m > \mu^* - \alpha^*$, and these with $\mu^* - \alpha^* < s < \mu^* - \alpha_m$ are worse off.
- consumers with $\omega < \alpha_m$ and $s < \mu^* - \omega$ receive the same net utility as before, w_B;
- consumers with $\omega < \alpha_m$ and $s > \mu^* - \omega$ are better off since $\omega + w_B + s - \mu^* > w_B$.

The merger does not have a negative impact on consumers who do not purchase A before the merger, e.g., these with $\omega < \alpha_m$. However, it harms consumers who choose to combine A and B_j both before and after the merger. These consumers have $\omega \geq \alpha_m$ and low consumption synergies such that $s < \{\mu^* - \alpha^*, \mu^* - \omega\}$. Furthermore, the merger also harms consumers with $\omega \geq \alpha_m$ and moderate consumption synergies such that $\omega \geq \alpha_m$, who switch to purchasing the bundle after the merger. On the other hand, the merger benefits consumers with sufficiently high consumption synergies.

Total Consumer Surplus

We show now, under uniform distributions, the merger leads to an increase in total consumer surplus.

The total consumer surplus before the merger is given by:

$$S_b = w_B + \frac{1}{\bar{u}} \int_{\alpha_m}^{\bar{u}} (\omega - \alpha_m)\, d\omega = w_B + \frac{\bar{u}}{8}.$$

Denoting by $\tau(\omega) \equiv \mu - \omega$ the cut-off threshold, the total consumer surplus after the merger can be expressed as:

$$\begin{aligned} S_a &= w_B + \frac{1}{\bar{u}} \int_0^{\alpha} \int_{\tau(\omega)}^{1} (s - \tau(\omega))\, ds d\omega + \frac{1}{\bar{u}} \int_{\alpha}^{\bar{u}} \int_{\mu-\alpha}^{1} (s - \tau(\omega))\, ds d\omega + \frac{1}{\bar{u}} \int_0^{\mu-\alpha} \int_{\alpha}^{\bar{u}} (\omega - \alpha)\, ds d\omega \\ &= w_B + \frac{1}{2\bar{u}} \int_0^{\bar{u}} (1 - \tau(\omega))^2\, d\omega - \frac{1}{2\bar{u}} \int_{\alpha}^{\bar{u}} (\mu - \alpha - \tau(\omega))^2\, d\omega + \frac{\mu - \alpha}{\bar{u}} \int_{\alpha}^{\bar{u}} (\omega - \alpha)\, d\omega. \end{aligned}$$

Substituting $\alpha^* = \frac{2}{3}\bar{u}$ and $\mu^* - \alpha^* = \frac{1}{2} - \frac{1}{3}\bar{u}$, we obtain:

$$\begin{aligned} S_a &= w_B + \frac{1}{2\bar{u}} \int_0^{\bar{u}} \left(\frac{1}{2} - \frac{1}{3}\bar{u} + \omega\right)^2 d\omega - \frac{1}{2\bar{u}} \int_{\frac{2}{3}\bar{u}}^{\bar{u}} \left(\omega - \frac{2}{3}\bar{u}\right)^2 d\omega + \frac{1}{\bar{u}} \left(\frac{1}{2} - \frac{1}{3}\bar{u}\right) \int_{\frac{2}{3}\bar{u}}^{\bar{u}} \left(\omega - \frac{2}{3}\bar{u}\right) d\omega \\ &= w_B + \frac{5}{162}\bar{u}^2 + \frac{1}{9}\bar{u} + \frac{1}{8}. \end{aligned}$$

Comparing to S_b, when $\bar{u} < 3/2$, we have:

$$S_a - S_b = \frac{5}{162}\bar{u}^2 + \frac{1}{9}\bar{u} + \frac{1}{8} - \frac{\bar{u}}{8} = \frac{5}{162}\bar{u}^2 - \frac{1}{72}\bar{u} + \frac{1}{8} > 0.$$

Therefore, the merger results in an increase in total consumer surplus under mixed bundling.