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Matching Unskilled/Skilled Workers to Firms Facing Budget Constraints

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Abstract

We study a matching model in which firms face budget constraints. If the production function only depends on a firm's technology, a weak stable matching always exists; furthermore, when a strong stable matching does not exist, there is a nearby budget vector for firms such that a strong stable matching exists for the problem with perturbed budgets. If the production function is multiplicative, one can reach a strong stable matching by changing the budget of firms such that the total budget remains the same and each firm's budget change is bounded by the value of at most one worker for that firm.

Keywords: Matching Theory, Market Design, Labor Market

JEL classification: D47, C78, C71

1 Introduction

We study a many-to-one matching problem with salaries in which firms face budget constraints. Kelso and Crawford (1982) study the same matching with salaries in cases in which firms have access to a perfect credit market. They show that a stable matching always exists if workers are regarded as gross substitutes. But Mongell and Roth (1986) show that when firms face a budget constraint, workers are not gross substitutes from a firm's

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point of view; therefore, a stable matching may not exist. We study the same problem that Mongell and Roth (1986) researched under two special forms of firms' valuation of workers. We show that the existence of a stable matching is not guaranteed, even in these special cases; however, it is possible to achieve a stable matching by changing the budget constraints. In one case, the required budget change is minimal; i.e., we can bound the budget change to any small positive number. On the other hand, for the second case, the required upper bound for the budget change of a firm is equal to the value of a worker to that firm, while the sum of the budget changes for all firms is zero.

First, assume that workers are homogeneous from each firm's standpoint; i.e., each firm values all workers equally. There are many applications in which a firm cannot differentiate between candidates or in which the marginal gain of hiring different workers is the same; however, the marginal product of a worker is different at different firms. The marginal gain to a hospital of hiring a general practitioner is the same no matter which candidate the hospital hires, but based on the location of the hospital and the scarcity of general practitioners, the value of a general practitioner to different hospitals can be unequal. Schools that hire fresh-out-of-school teachers cannot differentiate among candidates; however, the value of a teacher is not the same for schools in different districts or cities. A district attorney's office that is required by law to hire an attorney for indigent criminal defendants does not differentiate among attorneys, as the law specifies only a minimum set of requirements for an assigned counsel, so the actual quality of the attorney does not affect the value of the attorney to the DA's office, but the number of attorneys and cases in a district can cause the value of an assigned counsel to vary between two districts of the same state.

More generally, a firm that hires unskilled workers does not differentiate among workers, as the marginal product of workers in a job slot is the same, even though different job slots have a different marginal product of labor for the same worker. We show that in this setting, a weakly stable matching always exists, where all workers get hired with equal salaries. We prove that a strong stable matching, even in this special case, need not exist.

However, we can find nearby budgets for firms such that a strong stable matching exists. More specifically, we show that when a matching problem with a specified budget vector does not have strong stable matching, then there is a sequence of budget vectors converging to the specified budget vector such that each element of the sequence has a strong stable matching.

Second, assume that all firms agree on their valuation of workers; however, the productivity of a worker at a firm depends on the firm technology as well as on worker productivity. We consider the multiplicative functional form for the valuation of a worker at a firm.[1] In this setting, there is a complementarity between worker's skills and the firm's technology. We call this case matching of skilled workers with firms, as opposed to the first case, which we call matching of unskilled workers with firms.

Schools may value a teacher by his/her score card report; hence, a school's valuation of every teacher's quality may be the same; however, the productivity of the teacher depends on the specific school as well as on the teacher's quality. Hospitals agree on the quality of a surgeon based on his/her quality report card.[2] However, the value of a surgeon depends on the specific hospital that he/she works in. We provide an example in this case such that the assumption of gross substitution fails and Kelso and Crawford (1982)'s algorithm for finding a stable matching does not work. We show that changing the budgets of firms such that the total budgets of all firms remain the same and each firm's budget change is bounded by, at most, one worker's value, allows us to find a stable matching. Suppose that one entity is allocating budgets to several branches, as in the cases of public schools in various locations in a city or a hospital with a variety of specialties; then, changing the budgets without affecting the sum of all the budgets seems reasonable.[3] For example,

[1]This production function is a common functional form for a super modular production function in matching literature and CEO compensation literature; see, e.g., Mailath et al. (2017) and Tervio (2008).

[2]For example, there is a quality "report card" for surgeons performing coronary artery bypass graft (CABG) surgery in Pennsylvania; see Kolstad (2013).

[3]Nguyen and Vohra (2022) state that "If a 'planner' knows the excess demand a priori, they can withhold that amount to 'add back in' to ensure that each agent's demand is satisfied which amounts to 'burning' some of the supply to ensure feasibility. "

suppose we have a proposal for budgets for all the public schools in a state. We provide a budget vector that is close to the proposed budget vector and has a stable matching. Note that the sum of all the budgets in the new budget vector is equal to the sum of the budgets in the proposed budgets.

2 Related literature

Gale and Shapley (1962) study the college admissions problem as a two-sided matching problem with no monetary transfer. They introduce the deferred-acceptance algorithm, which results in a stable matching. They assume that colleges' preferences are responsive. Kelso and Crawford (1982) model a labor market as a one-to-many matching problem with salaries. They drop the responsive preference assumption and replace it with a gross substitutes assumption. Note that this assumption rules out a budget constraint for firms. Moreover, Kelso and Crawford (1982) show that without this assumption, a stable matching may not exist. Mongell and Roth (1986) explore a matching model with budget constraints. They show that, if firms do not have access to a perfect credit market, a stable match may not exist.

The matching problem with constraints is important because the standard theories are not always applicable to practical problems. Supply constraints, such as capacity constraints for firms or schools, are the only constraints in the standard theory of matching. However, other constraints are observed in practical matching markets. Kojima (2017) explains that the medical market in Japan limits the number of medical doctors in some regions. With this restriction, a stable match may not exist. Another example of matching with constraints is the national resident matching program with couples, where a stable matching need not exist. Nguyen and Vohra (2018) show that perturbing the capacity of hospitals ensures that a stable matching exists. They prove that this perturbation is less than 2 for each hospital's capacity and less than 4 for aggregate capacities. Abizada (2016) studies matching of colleges and students in which colleges have fixed budgets and offer stipends to students. He assumes that stipends are offered in only three different amounts and that the colleges' preferences depend only on the students, not the stipends. He shows that stable matching may not exist in the presence of budget constraints. However, a pairwise stable allocation always exists. Hamada et al. (2017) investigate the same problem where each student has one type that is the same for all colleges. Moreover, colleges have budget constraints and colleges' preferences are responsive over students and independent of wages. Hamada et al. (2017) focus on discrete sets of wages instead of a continuum. They conclude that a pairwise stable matching always exists (the same result reached by Abizada (2016)). Kawase and Iwasaki (2017) perturbed budgets in order to find a stable matching. Kawase and Iwasaki (2017) provide an example in which, if firms' budgets are constrained at particular intervals, a stable matching does not exist. They introduce a compatibility as a new assumption, which is restrictive in our setting. Under this assumption, they conclude that a stable matching exists if the budget change is bounded by a certain amount.

Under some restrictions, stable matchings correspond to competitive equilibrium (CE). A rich literature has studied finding CE in a market with divisible or indivisible goods. Budish (2011) shows that if agents have equal budgets, the CE for indivisible goods may not exist. However, he proves that existence is not an issue if we can perturb budgets by a small amount. Another problem in this literature is the Fisher market, where each buyer has a linear-additive utility and a monetary budget. However, budgets have no intrinsic value; they are useful only for buying products. Babaioff et al. (2017) explore the Fisher market with indivisible goods and two buyers. They show that if you perturb both budgets, CE does exist. Segal-Halevi (2018) proves that even after changing budgets, CE with four buyers and four goods may not exist; there exists a positive measure of budget space such that CE does not exist in that space. In our model, each salary that a firm pays has a disutility. Hence, money has an intrinsic value. Moreover, we assume a continuum of salaries. Furthermore, we use stability as our equilibrium concept, not just pairwise stability.

In recent work, Jagadeesan and Teytelboym (2022) study the existence of stable matching in the presence of budget constraints, which is the closest paper to our setting with a different approach.[4] They define the net substitutability condition under which a stable outcome exists, showing that net substitutability is more general than the gross substitutability condition in Kelso and Crawford (1982) and can incorporate hard budget constraints. However, as we show in example 2, the net substitutability condition is not satisfied in our setting. Moreover, Jagadeesan and Teytelboym (2022) show the existence of a stable matching without perturbing the budget, which is a different approach than the one we use here.[5] Furthermore, our definition of a strong stable match is equivalent to the Jagadeesan and Teytelboym (2022) definition of competitive equilibrium, which may not exist in their setting even if the net substitutability condition is satisfied; their definition of stable matching, which is the solution concept that they work with, is weaker than our definition of weakly stable matching. Nguyen and Vohra (2022), Nguyen and Vohra (2018), and Azevedo et al. (2013) take the same approach as our paper, but in different contexts. Similar to our paper, they change the resource constraint to prove the existence of an equilibrium. More specifically, they use an approximate market clearing condition that becomes negligible as the market grows large.

3 Model

There is a finite set of workers W, and a finite set of firms F. Each firm $f \in F$ has a budget $b_f \in \mathbb{R}_+$; denote the budget of all firms as $B = (b_w)_{f \in F}$. Each firm can hire (match) zero, one, or more workers and has to pay a non-negative salary to the hired workers.[6] A worker cannot work in two different firms (cannot match with two different firms). [7] Formally, a

[4] The first draft of our paper was publicly available in 2020.

[5] Similarly, Baldwin et al. (2020) use Hicksian demands to define net substitutability and show the existence of equilibrium in the presence of income effect, absent hard budget constraints.

[6] We assume the outside option of firms and workers to be zero.

[7] Workers cannot work part-time in one firm and part-time in the other firm.

many-to-one matching is an assignment of workers to firms with specified salaries, denoted by $\mu : W \to (F \times S) \cup \{(\emptyset, 0)\}$, where each worker is either matched with one firm or is unmatched with a salary equal to zero. The set of all possible salaries is represented as $S = \mathbb{R}_+$. A firm can hire more than one worker; denote the set of workers assigned to firm f by $\mu^o(f) := \{(w,s)|(f,s) = \mu(w) \text{ for some } w \in W\}$. Denote the budget of all firms by $B \in \mathbb{R}_+^{|F|}$.

The set of workers' values is $V = \{v_{wf}\}$ where v_{wf} is the value of worker $w \in W$ for firm $f \in F$. Each firm has a utility function $u_f : \cup_{A \subset W}(A \times S^A) \to \mathbb{R}_+$, which under matching μ is equal to

$$u_f(\mu^o(f)) = \sum_{(w,s_w) \in \mu^o(f)} v_{wf} - s_w,$$

if $\sum_{(w,s_w)\in\mu^o(f)} s_w \leq b_f$. Otherwise, the utility is $-\infty$; firms face a hard budget constraint. Denote worker w's utility with $u_w : (F \cup \{\emptyset\}) \times S \to \mathbb{R}_+$, which is equal to the worker's salary if he/she is employed and zero if he/she is unemployed.

Firms have additive and quasi-linear utility functions. Money is not neutral; it plays an important role in the firms' utility functions. Each worker's importance to firms equals the net worker's value, which is the difference between the worker's value and his/her salary; the group of workers does not have a separate value for firms. Salary is the only important thing for workers. They do not care about the firm where they are employed or about the group of their coworkers.

A matching is *feasible* if $\forall f \in F : \sum_{(w,s_w)\in\mu^o(f)} s_w \leq b_f$. A matching μ is *individually rational*, if (1) it is feasible; (2) for each $f \in F$, $u_f(\mu^o(f)) \geq u_f(C)$ for all $C \subset \mu^o(f)$; and (3) $u_w(\mu(w)) \geq u_w(\emptyset, 0)$. A matching μ is *strongly stable* if it is individually rational and, for any firm f and $A \subset W$, if there is a feasible salary vector $(s_w)_{w\in A}$ with $u_f(\{(w,s_w)|w \in A\}) > u_f(\mu^o(f))$, then there exists a $w \in A$ such that $u_w(\mu(w)) > u_w(f, s_w)$. A matching μ is *weakly stable* if it is individually rational and, for any firm f and $A \subset W$, if there is a feasible salary vector $(s_w)_{w\in A}$ with $u_f(\{(w,s_w)|w \in A\}) > u_f(\mu^o(f))$, then there exists a

$w \in A$ such that $u_w(\mu(w)) \geq u_w(f, s_w)$.

Kelso and Crawford (1982) define gross substitutability as:[8]

Definition 1 *Workers are gross substitutes for each firm f, if for any two vectors of salaries, $\vec{s} = (s_w)_{w \in W}$ and $\vec{s'} = (s'_w)_{w \in W}$, where $\vec{s} \leq \vec{s'}$ and $s_w = s'_w$, then hiring worker w at salary s implies hiring that worker at salary s'.*

The algorithm provided by Kelso and Crawford (1982) assumes that workers are gross substitutes for each firm f. The following example shows that when firms face budget constraints, workers may not be gross substitutes for each firm.

Example 1 *There are two firms: $F = \{1, 2\}$ with budgets $b_1 = b_2 = 50$. There are three workers, $W = \{1, 2, 3\}$, with values $v_{11} = 100, v_{12} = 75, v_{21} = 100, v_{22} = 75, v_{31} = 20$, and $v_{32} = 15$.[9] Let $\vec{s} = (20, 20, 10)$ and $\vec{s'} = (25, 25, 10)$. Note that $\vec{s} \leq \vec{s'}$ and $s_3 = s'_3$. Firm 1 chooses worker 3 when the salary vector is $\vec{s}$ but does not choose worker 3 when the salary vector is $\vec{s'}$. Therefore, Kelso and Crawford (1982)'s gross substitutes condition is violated.*

Jagadeesan and Teytelboym (2022) define net substitutability similarly to the gross substitutability of Kelso and Crawford (1982) by changing the Marshallian demand for a fixed income to Hicksian demand for a fixed utility level. However, the following example shows that the net substitutability condition is not satisfied in our setting:

Example 2 *Consider a firm and three workers, which are the buyer and three trades in the spirit of Jagadeesan and Teytelboym (2022). The buyer values the first trade at $v_1 = 5$, the second trade at $v_2 = 1$, and the third trade at $v_3 = 4$. We add the value of each trade to find the value of a set of trades. The income of the buyer is 2. First, we find the Hicksian demand at the utility level of 5 utils when the prices are $(0.9, 0.4, 0.4)$. The buyer's Hicksian*

[8] Kelso and Crawford (1982) have a richer utility function, but firms in their model do not have a budget constraint. In their setting, utilities of the firms depend not only on individual workers' values and salaries but also on the group of workers that firms hire. Utilities of the workers depend on the salaries they receive and the firms where they work.

[9] These values are multiplicative, same as our skilled worker section: $v_{11} = 5 \times 20 = 100, v_{12} = 5 \times 15 = 75, v_{21} = 5 \times 20 = 100, v_{22} = 5 \times 15 = 75$, and $v_{31} = 1 \times 20 = 20, v_{32} = 1 \times 15 = 15$.

demand is $\{2,3\}$*. Next, we increase the price of the third trade to* 0.55*. At this new price vector* $(0.9, 0.4, 0.55)$*, and the utility level of* 5 *utils, the Hicksian demand of the buyer is* $\{1\}$*, which violates the net substitutability condition.*

4 Unskilled workers

First, we study the existence of weak and strong stable matchings in a setting in which workers are homogeneous from each firm's standpoint, i.e., each firm values all workers equally. However, different firms may have different valuations. We use the following definitions of strict demand and pseudo demand to find the number of workers matched with each firm in a strong stable matching and a weak stable matching respectively.

Definition 2 *Define the strict demand* $D_f(s)$ *for firm* $f \in F$ *as*

$$D_f(s) = \begin{cases} 0 & v_f < s \\ \{0, 1, ..., \lfloor \frac{b_f}{s} \rfloor\} & v_f = s \\ \lfloor \frac{b_f}{s} \rfloor & v_f > s. \end{cases}$$

Define the pseudo demand $D'_f(s)$ *for firm* $f \in F$ *as:*

$$D'_f(s) = \begin{cases} 0 & v_f < s \\ \{0, 1, ..., \lfloor \frac{b_f}{s} \rfloor\} & v_f = s \\ \{\frac{b_f}{s} - 1, \frac{b_f}{s}\} & v_f > s \text{ and } \frac{b_f}{s} \in \mathbb{N} \\ \lfloor \frac{b_f}{s} \rfloor & v_f > s \text{ and } \frac{b_f}{s} \notin \mathbb{N} \end{cases}$$

The following theorem shows that when matching unskilled workers to firms, the existence of weak stable matching with equal salaries is guaranteed. Subsequently, we show that the existence of a strong stable matching is an issue in this special case. Finally, we prove that there is always a nearby budget vector such that, with the new budget vector, a strong stable matching exists.

Theorem 1

Suppose each firm values all workers equally, i.e., ($v_{wf} = v_f, \forall w \in W, f \in F$):

i) There exists a weakly stable matching with equal salary $s_w = s$ for all $w \in W$. Moreover, the number of matches for firm $f \in F$ is an element of the pseudo demand $D'_f(s)$.

ii) A strongly stable matching may not exist.

iii) For any budget vector B, a sequence of budget vectors $\{B_n\}_{n\in\mathcal{N}}$ exists such that $\{B_n\}_{n\in\mathcal{N}}$ converges to B and every element of the sequence has a strongly stable matching. In any of these strongly stable matchings, salaries of workers are equal and the number of matches of firms are an element of their strict demands.

The intuition behind Theorem 1 i is very close to the intuition for competitive equilibrium Arrow-Debreu (1951). However, here workers are indivisible objects. There exists a salary s, such that the allocation based on the requested pseudo demand forms a weakly stable matching. Moreover, the market clears on the worker side; all workers find a job. Although, there may exist some firms with a low willingness to pay for the workers (low value), in which they are left with no workers.

The non existence of a strong stable matching in Theorem 1 ii is due to the degree of flexibility of the pseudo demand. In a weakly stable matching firm f is satisfied with $b_f/s - 1$ amount of workers when b_f/s is an integer number. In this situation if firm f wants to hire one more worker has to pay an additional salary s, and has to spend the entire budget. The last worker that firm f wants to hire is already matched to a firm with the same salary s, so firm f's offer is not very interesting to the this worker, and at the same time firm f cannot pay even one dolor more than s. In contrast to the strongly stable matching, firm f is happy with $b_f/s-1$ workers under weakly stable matching.

In fact, the issue raises for strongly stable matching when at least for two firms the ratio of budgets to the salary become integers. This can be solved by perturbing budgets. Using this idea Theorem 1 iii shows there exist a sequence of budgets that converges to the initial budgets such that for each element of this sequence the ratio of budgets to the salary of at most one firm is integer.

In order to prove this theorem, we first state two lemmas.[10] The first lemma shows that, when each firm values all workers equally, we can restrict our attention to matchings with equal salaries.

Lemma 1

Suppose each firm values all workers equally, i.e., ($v_{wf}=v_f, \forall w \in W, f \in F$). If μ is a weakly (strongly) stable matching, then a weakly (strongly) stable matching $\mu^{'}$ exists such that 1) the allocation of $\mu^{'}$ is the same as the allocation in μ, and 2) salaries of workers in $\mu^{'}$ are equal.

The intuition of Lemma 1 is as follows: Assume there exist a worker w_l outside of firm f with a lower salary compare to a worker w_h inside firm f. Firm f can fire the existing worker w_h (that receives a higher salary) and hire the outside worker w_l with a salary that is average of salaries of w_l and w_h.

Define the number of workers assigned to firm f, under matching μ and salary s by $N_f(\mu(s))$. Lemma 2, by using the strict demand and the pseudo demand, characterizes weak and strong stable matchings. Using Lemma 1 allows us focus on matching μ with salary s for all workers.

Lemma 2

[10] The Appendix provides proofs not given in the main text.

Suppose each firm values all workers equally, i.e., ($v_{wf} = v_f, \forall w \in W, f \in F$). Suppose all salaries in matching μ are equal to s.[11]

1) Matching μ is strongly stable if and only if:

i) The number of workers assigned to firm f, $N_f(\mu(s))$, is an element of the firm f strict demand. Formally, $N_f(\mu(s)) \in D_f(s)$.

ii) The market clears: $\sum_{f\in F} N_f(\mu(s)) = |W|$.

2) Matching μ is weakly stable if and only if:

i) The number of workers assigned to firm f, $N_f(\mu(s))$, is an element of the firm f's pseudo demand. Formally, $N_f(\mu(s)) \in D^{'}_f(s)$.

ii) The market clears: $\sum_{f\in F} N_f(\mu(s)) = |W|$.

Lemma 2 argues that if firms request a demand according to the pseudo (strict) demand then the matching is weakly (strongly) if and only if the market clears on the workers side. The intuition is that a firm can fire a worker with a positive income and hire an unemployed worker with a very small salary. The offer is interesting for the unemployed worker since the outside option of workers is zero. Therefore there should not exist unmatched workers on a stable matching.

Using these two lemmas, we can prove Theorem 1:

Proof:

Define $T(s)$ as the sum of all elements (Minkowski addition) of $D_f(s)$ for all $f \in F$. Formally,

$$T(s) = \{\sum_{f\in F} a_f | a_f \in D_f(s)\}.$$

Observe that $T(s)$ is a set. For example, suppose there are two firms, firm 1 and firm 2, and some workers. Let $s = 5$, $V_1 = 5$ (the value of the first firm), $b_1 = 10$, $V_2 = 10$, and

[11] With Lemma 1 we can focus on matchings with equal salaries.

$b_2 = 10$. Then $D_1(s) = \{0, 1, 2\}$, and $D_2(s) = \{1, 2\}$. Therefore,

$$T(s) = \{1, 2, 3, 4\}.$$

Note that $T(s)$ is a set of consecutive natural numbers. Similarly, define

$$T^{'}(s) = \{\sum_{f \in F} a^{'}_f | a^{'}_f \in D^{'}_f(s)\}.$$

i) Using Lemma 2, we know that if there exists a salary s and $N_f(\mu(s)) \in D^{'}_f(s)$ such that $\sum_{f \in F} N_f(\mu(s)) = |W|$, then a weakly stable matching exists. The idea is to find s, and $N_f(\mu(s)) \in D^{'}_f(s)$. If we find $T^{'}(s)$, such that $|W| \in T^{'}(s)$, then by definition of $T^{'}(s)$, we have $N_f(\mu(s) \in D^{'}_f(s)$, such that $\sum_{f \in F} N_f(\mu(s)) = |W|$. Now, we need to show that for any $|W| \in \mathbb{N}$, there exists s such that $|W| \in T^{'}(s)$.

By contradiction, assume that there exists $k \in \mathbb{N}$ exists such that there does not exist a salary s, such that $k \in T^{'}(s)$. By decreasing s, the maximum element of set $D^{'}_f(s)$ for all $f \in F$ is weakly increasing and not bounded from above. Hence, set $T^{'}(s)$ is not bounded from above. Therefore, for any $k \in \mathbb{N}$ we can find an s small enough that $\max\{T^{'}(s)\} > k$. Therefore, without loss of generality, we can assume that there exists a salary s such that $k + 1 \in T^{'}(s)$, and $k \notin T^{'}(s)$. This is true because $T^{'}(s)$ is not bounded from above. When $k + 1 \in T^{'}(s)$ and $k \notin T^{'}(s)$, we have that

$$k + 1 = \sum_{f \in F} \min\{D^{'}_f(s)\}.$$

Define

$$s_{sup} = \sup\{s | k + 1 \in T^{'}(s)\}.$$

Note that s_{sup} exists since $T^{'}(s) = 0$, for a large enough s. Consider two cases:

1) First assume that $k + 1 \in T^{'}(s_{sup})$, then $k + 1 = \sum_{f \in F} \min\{D^{'}_f(s_{sup})\}$. For $\epsilon_1 \geq 0$

by assumption, $k+1 \notin T^{'}(s_{sup}+\epsilon_1)$. Note that the maximum of sets $D^{'}_f(s)$ for all $f \in F$ are weakly decreasing in s, therefore

$$k+1 > \sum_{f \in F} \max\{D^{'}_f(s_{sup}+\epsilon_1)\}.$$

Hence, at least one firm, call it f^*, exists such that for an arbitrary small $\epsilon_1 \geq 0$

$$\min\{D^{'}_{f^*}(s_{sup})\} > \max\{D^{'}_{f^*}(s_{sup}+\epsilon_1)\}.$$

Now either $v_{f^*} = s_{sup}$ or $\frac{b_f}{s_{sup}} \in \mathbb{N}$. Otherwise $D^{'}_{f^*}(.)$ cannot change from s_{sup} to $s_{sup}+\epsilon_1$, for a small enough ϵ_1. Let $v_{f^*} = s_{sup}$, then

$$0 = \min\{D^{'}_{f^*}(s_{sup})\} > \max\{D^{'}_{f^*}(s_{sup}+\epsilon_1)\} = 0,$$

a contradiction. Let $\frac{b_f}{s_{sup}} \in \mathbb{N}$, then

$$\frac{b_f}{s_{sup}} - 1 = \min\{D^{'}_{f^*}(s_{sup})\} > \max\{D^{'}_{f^*}(s_{sup}+\epsilon_2)\} = \lfloor \frac{b_f}{s_{sup}+\epsilon_1} \rfloor = \frac{b_f}{s_{sup}} - 1,$$

a contradiction.

2) Second, assume $k+1 \notin T^{'}(s_{sup})$. Then there exists $m \geq 1$ such that $k-m = \sum_{f \in F} \max\{D^{'}_f(s_{sup})\}$. Observe that for any $\epsilon_2 \geq 0$ such that $k+1 \in T^{'}(s_{sup}-\epsilon_2)$, we have that $k-m \notin T^{'}(s_{sup}-\epsilon_2)$, because if $k+1, k-m \in T^{'}(s_{sup}-\epsilon_2)$; then we must have that $k \in T^{'}(s_{sup}-\epsilon_2)$ as well, which is a contradiction. Maximums of sets $D^{'}_f(s)$ for all $f \in F$ are weakly decreasing in s, therefore

$$k-m < \sum_{f \in F} \min\{D^{'}_f(s_{sup}-\epsilon_2)\}.$$

Hence, at least one firm, call it f^*, exists such that for an arbitrary small $\epsilon_2 \geq 0$,

$$\max\{D'_{f^*}(s_{sup})\} < \min\{D'_{f^*}(s_{sup} - \epsilon_2)\}.$$

Now either $v_{f^*} = s_{sup}$ or $\frac{b_f}{s_{sup}} \in \mathbb{N}$. Otherwise, $D'_{f^*}(.)$ cannot change from s_{sup} to $s_{sup} - \epsilon_2$, for a small enough ϵ_2. Let $v_{f^*} = s_{sup}$, then

$$\lfloor \frac{b_{f^*}}{s_{sup}} \rfloor = \max\{D'_{f^*}(s_{sup})\} < \min\{D'_{f^*}(s_{sup} - \epsilon_2)\} = \lfloor \frac{b_{f^*}}{s_{sup} - \epsilon_2} \rfloor = \lfloor \frac{b_{f^*}}{s_{sup}} \rfloor,$$

a contradiction. Let $\frac{b_f}{s_{sup}} \in \mathbb{N}$, then

$$\frac{b_{f^*}}{s_{sup}} = \max\{D'_{f^*}(s_{sup})\} < \min\{D'_{f^*}(s_{sup} - \epsilon_2)\} = \lfloor \frac{b_{f^*}}{s_{sup} - \epsilon_2} \rfloor = \frac{b_{f^*}}{s_{sup}},$$

a contradiction.

ii) Consider the following example: There are two firms: $F = \{1, 2\}$ with budgets $b_1 = b_2 = 6$. There are three workers $W = \{1, 2, 3\}$ with values $v_{11} = v_{12} = v_{21} = v_{22} = v_{31} = v_{32} = 5$. Note that in any strongly stable matching, there are no unemployed workers: If $s_w > 0$ for a hired worker, then hiring the unemployed worker instead of worker w with salary $\frac{s_w}{2}$ strictly increases the firm's utility and the unemployed worker's utility. If the salary for all hired workers is zero, then hiring the unemployed worker in addition to the hired worker with salary $\epsilon > 0$ strictly increases the firm's utility and the unemployed worker's utility. Suppose that there is a strongly stable matching in which one firm hires at least two workers. The salary of one of these workers must be less than or equal to 3. Without loss of generality, suppose that firm 1 hires at least workers 1 and 2, where $s_1 \leq 3$. Consider the following cases:

(1) Firm 1 hires all three workers. In this case, firm 2 hiring worker 1 with salary 4 strictly increases their utilities and is a blocking group.

(2) Firm 2 hires worker 3 with salary $s_3 > 3$. In this case, firm 2 hiring only worker 1 with salary s_1', where $s_1 < s_1' < s_3$, strictly increases their utilities and is a blocking group.

(3) Firm 2 hires worker 3 with salary $s_3 \leq 3$. in this case, firm 2 hiring workers 1 and 3 with salaries $s_1' = 3$ and s_3 strictly increases firm 2's utility and does not decrease the utility of workers 1 and 3 and is a blocking group.

Therefore, there is no strongly stable matching.

iii) First we show that for all budget vectors $B \in \mathbb{R}_+^{|F|}$, a sequence of vectors B_k exists such that: (1) $lim_{k\to\infty} B_k = B$; (2) for all $k \in \mathbb{N}$, the proportion of each two different arrays of B_k is not in $\mathbb{Q}$. Note that a sequence of vectors Q_k exist that converge to B and that for each k, every array of Q_k is rational. Using Q_k, we can construct B_k. Let b_{ki} and q_{ki} be array i in B_k and Q_k respectively. Define

$$b_{ki} = q_{ki}(1 + \frac{i\sqrt{2}}{k}).$$

The ratio of array i to j ($i \neq j$) of B_k is not rational. Suppose not; define

$$p = \frac{q_{ki}(1 + \frac{i\sqrt{2}}{k})}{q_{kj}(1 + \frac{j\sqrt{2}}{k})}$$

and $p' = p\frac{q_{kj}}{q_{ki}}$. Observe that

$$\frac{k - kp'}{p'j - i} = \sqrt{2},$$

which is a contradiction, because the left side is a rational number but the right side is an irrational number.[12]

[12]Observe that:

$$\frac{k - kp'}{p'j - i} = \sqrt{2}, \iff \frac{(j-i)\sqrt{2}}{(1 + \frac{j\sqrt{2}}{k})} = \sqrt{2}(p'j - i) \iff \frac{(j-i)\sqrt{2}}{(1 + \frac{j\sqrt{2}}{k})} = \sqrt{2}\frac{(j + \frac{ij\sqrt{2}}{k} - i - \frac{ij\sqrt{2}}{k})}{(1 + \frac{j\sqrt{2}}{k})}.$$

Using the above argument, we show that if for all $f, f' \in F : \frac{b_f}{b_{f'}} \notin \mathbb{Q}$, a strongly stable matching exists, which concludes the result. Using Lemma 2, we know that if there exists a salary s and $N_f(\mu(s)) \in D_f(s)$ such that $\sum_{f \in F} N_f(\mu(s)) = |W|$, then a strongly stable matching exists. The idea is to find s, and $N_f(\mu(s)) \in D_f(s)$. If we find $T(s)$, such that $|W| \in T(s)$, then by definition of $T(s)$, we have $N_f(\mu(s) \in D_f(s)$, such that $\sum_{f \in F} N_f(\mu(s)) = |W|$. We need to show that for any $|W| \in \mathbb{N}$, there exists s such that $|W| \in T(s)$. The rest of the argument is almost the same as part i and is provided in Appendix A.

■

5 Skilled workers

So far, we have considered the case in which $v_{wf} = v_f$. The second case we want to investigate is one in which each worker's value at a firm depends on both the worker's quality and the firm's technology. We consider the multiplicative functional form for the value of a worker at a firm. Assume that each worker $w \in W$ has an intrinsic value V_w (talent, productivity, knowledge, etc). Let the technology of firm $f \in F$ be V_f. We consider the case in which the value of the worker w for firm f is $v_{wf} = V_w.V_f$, i.e., there is complementarity between firms and workers. The efficient firm can combine its technology (V_f) and human resources (V_w) to produce a highly valued output ($V_f.V_w$), as there are complementaries between firm's technology and worker's quality. For simplicity, assume that $V_1 < V_2 < ... < V_{|F|}$. We say salaries are fair when the salaries that workers receive for each unit of productivity are the same for different workers. Formally

Definition 3 *Workers have fair salaries if for all $w, w' \in W$:*

$$\frac{S_w}{V_w} = \frac{S_{w'}}{V_{w'}}.$$

We introduce a *fair salary algorithm*, which results in a strongly stable matching by changing the budget of firms so that the total budget remains the same and each firm's budget change is bounded by the value of at most one worker to that firm. The *fair salary algorithm*, has two stages, in the first stage we find salaries, and the in the second stage we introduce the matching mechanism.

Stage one, finding salaries:

In stage one we find salaries. Define

$$\alpha(\overrightarrow{x}) = \frac{\sum_{f\in F} x_f b_f}{\sum_{w\in W} V_w},$$

where x_f is the f-th array of vector $\overrightarrow{x} \in [0,1]^{|F|}$. Let $y \in [0,1]$, and define $\overrightarrow{y_f} \in [0,1]^{|F|}$ for $f \in F$ a vector such that all the elements before f are zero, all the elements after f are one, and the fth element is equal to y, i.e.,

$$\overrightarrow{y_f}.e_i = \begin{cases} 1 & \text{if } i > f \\ 0 & \text{if } i < f \\ y & \text{if } i = f. \end{cases}$$

Observe that if $y = 1$ and $f = 1$, then $\overrightarrow{y_f} = \overrightarrow{1}$.

Lemma 3 *If* $\alpha(\overrightarrow{1}) > V_1$*:*

1. *Either there exists* $f^* \in F$ *and* $y \in [0,1]$ *such that:*

$$V_{f^*} = \alpha(\overrightarrow{y_{f^*}}),$$

2. *or* $f^* \in F$ *exists such that*

$$V_{f^*-1} < \alpha(\overrightarrow{1_{f^*}}) \leq V_{f^*}.$$

Using the Lemma 3, we determine the salary, productivity ratio. Define $\alpha^* = \alpha(\overrightarrow{1})$, if $\alpha(\overrightarrow{1}) \leq V_1$. If $\alpha(\overrightarrow{1}) > V_1$, then define $\alpha^* = \alpha(\overrightarrow{y_{f^*}})$, where $f^* \in F$, and $y \in [0,1]$ satisfies Lemma 3.[13] Choose $S_w = \alpha^* V_w$ for all $w \in W$.

When $\alpha(\overrightarrow{1}) \leq V_1$, then α^* (the ratio of the salary to the productivity) is in a way that all firms can hire workers. The reason is that value net salary for all firms are positive; i.e.

$$V_w V_f - \alpha^* V_w = (V_f - \alpha^*) V_w \geq 0,$$

for all $f \in F$, and $w \in W$. If $\alpha(\overrightarrow{1}) > V_1$, then based on Lemma 3 there exists f^* such that $V_{f^*-1} < \alpha^* \leq V_{f^*}$. This means that firms $f' \in F$ such that $V_{f'} < V_{f^*}$ do not hire any worker at this wage, because $V_f V_w - \alpha^* V_w = (V_f - \alpha^*) V_w < 0$. Firms $f'' \in F$ such that $V_{f''} \geq V_{f^*}$ have incentive to hire workers, because $V_f V_w - \alpha^* V_w = (V_f - \alpha^*) V_w \geq 0$.

Next, we introduce the fair salary algorithm which finds a strongly stable matching with fair salaries by changing the budgets of firms.

Stage two, the matching mechanism:

1. Remove firms that are not efficient; $V_f < \alpha^*$ (if these firms exist).

2. Change b_{f^*} to $b'_{f^*} = y_{f^*} b_{f^*}$.

3. Among firms that are efficient enough ($V_f \geq \alpha^*$), choose one firm, match workers as much as possible until the budget constraint does not allow the matching of even one more worker. Choose another firm and match some worker/workers (or zero workers) from the rest of the workers until the budget constraint does not allow the matching of more worker. Continue until there are no more firms in this group of firms ($V_f \geq \alpha^*$). Remove all the firms that have exhausted their budgets completely.

4. Choose one worker among the unmatched workers (if one exists), and match that worker with one of the remaining firms. Update the budget of the firm to the sum of

[13] If there are multiple $f^* \in F$, and $y \in [0,1]$, which satisfy Lemma 3, select the firm with highest value as f^*.

the salaries that it has to pay, and remove both the firm and the worker. Repeat the same process for another unmatched worker. Continue until no unmatched worker remains.

5. Add $b_{f^*} - b'_{f^*}$ to the budget of firm f^*.

The following examples show how this algorithm works. In Example 3 $\alpha(\overrightarrow{1})$ is lower than the minimum technology of the firms (V_1), and in Example 4 $\alpha(\overrightarrow{1})$ is higher than the minimum technology of the firms.

Example 3 *There are three firms: $F = \{f_1, f_2, f_3\}$ with values $V_{f_1} = 4$, $V_{f_2} = 6$, and $V_{f_3} = 7$, and budgets $b_{f_1} = 50$, $b_{f_2} = 75$, and $b_{f_3} = 25$. There are three types of workers with values $4, 5,$ and 6. Assume that there are five workers of each type: 15 workers in total.*

Now we run the ***fair salary algorithm****. First, we need to find α^*.*

$$\alpha(\overrightarrow{1}) = \frac{b_{f_1} + b_{f_2} + b_{f_3}}{5 \times 4 + 5 \times 5 + 5 \times 6} = \frac{150}{75} = 2 < V_{f_1}.$$

Therefore $\alpha^* = \alpha(\overrightarrow{1}) = 2$. Choose $S_w = 2V_w$ for all $w \in W$.

Skip this step since $V_1 > \alpha^*$. Now we run the matching mechanism

1. *Skip this step since $\alpha^* = \alpha(\overrightarrow{1})$.*

2. *Start from f_1, and match all the type one workers (w_1) to f_1. This costs $5 \times 8 = 40$, so f_1 still has 10 budgets that can hire one worker of the second type (w_2). The new worker costs 10, and f_1 spends the entire budget. Now move to f_2. For f_2, match all the remaining second type, w_2, which costs $4 \times 10 = 40$, and match two workers of the third type, w_3, which costs 2×12. Finally, firm f_2 spends 64, and has 11 left in its budget. We cannot match any more workers with the remaining budget. So, move to the third firm, f_3. Match two workers of the third type, w_3, to f_3, which costs 24 out*

of the budget. At this point, the remaining budget for f_3 does not have enough left to hire any more workers. Because f_1 has finished its budget, the algorithm removes it.

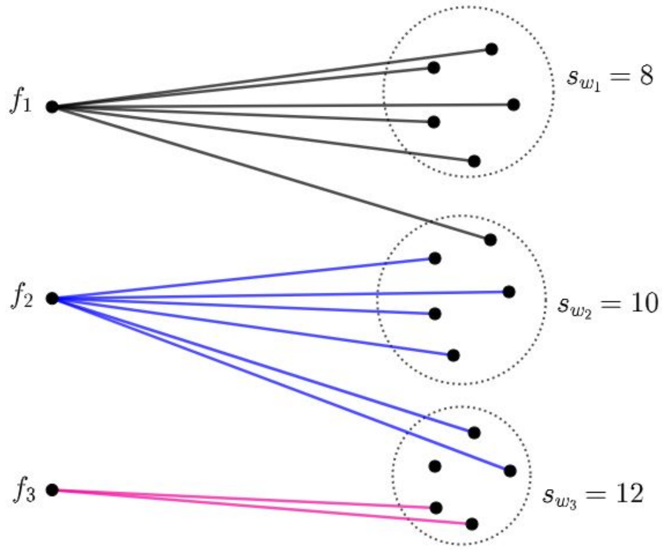

Figure 1: The outcome after step 4 in example 3.

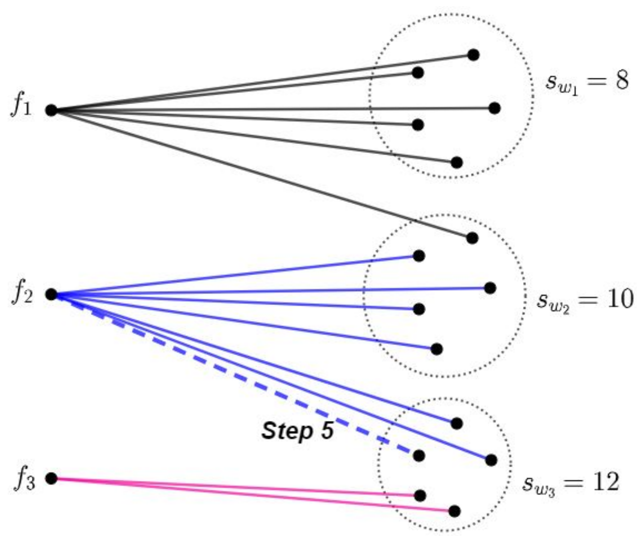

Figure 2: The outcome after step 5 in example 3.

3. *There is only one worker who has not been matched, a worker of the third type. Match this worker to f_2. Finally, update the f_2 budget from* 75 *to* 76*, and the f_3 budget from* 25 *to* 24*.*

4. *Skip this step since $\alpha^* = \alpha(\overrightarrow{1})$.*

Note that the outcome of the algorithm may not be unique. For instance, in step 5, the algorithm can match the last worker to f_3; there is no restriction for that. However, all the outcomes have the properties of theorem 2. As we can see in this example, workers receive a fair salary $s_w/v_w = 2$ for all $w \in W$. Moreover, the f_1 budget does not change, the f_2 budget increases by one, and the f_3 budget decreases by one.

EXAMPLE 4 *There are three firms: $F = \{f_1, f_2, f_3\}$ with values $V_{f_1} = 4$, $V_{f_2} = 6$, and $V_{f_3} = 7$, and budgets $b_{f_1} = 115$, $b_{f_2} = 150$, and $b_{f_3} = 50$. There are three types of workers with values $4, 5,$ and 6. Assume that there are five workers of each type: 15 workers in total.*

*Now we run the **fair salary algorithm**. First, we need to find α^*.*

$$\alpha(\overrightarrow{1}) = \frac{b_{f_1} + b_{f_2} + b_{f_3}}{5 \times 4 + 5 \times 5 + 5 \times 6} = \frac{315}{75} = 4.2 > V_{f_1}.$$

$$\alpha((\frac{100}{115}, 1, 1)) = \frac{\frac{100}{115} b_{f_1} + b_{f_2} + b_{f_3}}{5 \times 4 + 5 \times 5 + 5 \times 6} = \frac{315}{75} = 4 = V_{f_1}.$$

Therefore $\alpha^* = \alpha((\frac{100}{115}, 1, 1)) = 4$, and $f^* = f_1$. Choose $S_w = 4V_w$ for all $w \in W$. Now we run the matching mechanism.

1. *There is no inefficient firm since $V_1 = \alpha^*$.*

2. *Change b_{f_1} to $b^{'}_{f_1} = y_{f_1} b_{f_1} = 100$.*

3. *Start from f_1, and match all the type one workers (w_1) to f_1. This costs $5 \times 16 = 80$, so f_1 still has 20 left in its budget, which is enough to hire one worker of the second type (w_2). The new worker costs 20, and f_1 spends the entire budget. Now, move to f_2. For f_2, match all the remaining workers of the second type, w_2, which costs $4 \times 20 = 80$, and match two workers of the third type, w_3, which costs 2×24. Finally, firm f_2 spends 128, and has 22 left in its budget. Firm f_2 cannot match any more workers with the remaining budget. So, move to f_3. Match two of the third type (w_3)*

of workers to f_3, which costs 48 out of the budget of 50. With the remaining budget, f_3 cannot hire any more workers. Because f_1 has exhausted its budget completely, the algorithm removes f_1.

4. *There is only one worker who has not been matched, a worker of the third type. Match this worker to f_2. Finally update the budget of f_2 from 150 to 152, and the budget of f_3 from 50 to 48.*

5. *Add $b_{f_1} - b'_{f_1} = 115 - 100$ to the budget of firm f_1. Therefore, $b_{f_1} = 115$.*

Note that f_1 has not spent all of its budget, i.e., the budget is 115, but the expenditure is 100. This fact does not create any problem, because f_1 is indifferent between hiring and not hiring workers at these wages.

Theorem 2 *Using the fair salary algorithm results in a strongly stable matching with fair salaries in which*

1. *The budget of firm $f \in F$ changes at most $\max_{w\in W}\{\alpha^* V_w\}$,*

2. *The sum of the budgets of all the firms remains the same.*[14]

Given a proposed budget vector, the fair salary algorithm is a method for finding a nearby budget vector with a strongly stable matching in which the total of all the budgets is the same. For markets where a central authority assigns budgets, this mechanism is very helpful. For example, when a state or federal government assigns budgets to schools or other branches of the government, this algorithm, without changing the total budgets, results in a strongly stable matching.

Corollary 1 *The algorithm does not change the budget of firm f more than $\max_{w\in W}\{v_{wf}\}$.*

[14] Formally, by changing b_f to b'_f such that $|b_f - b'_f| \leq \max_{w\in W}\{\alpha^* V_w\}$, $b'_f \geq 0$, and $\sum_{f\in F} b_f = \sum_{f'\in F} b'_f$.

Proof: The algorithm does not change budgets of firm $f \in F$ if $V_f < V_{f^*}$. For firm $f \in F$, such that $V_f \geq V_{f^*}$, we know that $\alpha^* \leq V_{f^*} \leq V_f$; therefore, $\alpha^* V_w \leq \max_{w \in W}\{\alpha^* V_w\} \leq V_f V_w$. ■

Corollary 2 *If $\alpha^* < V_1$ (all firms are efficient enough):*

1. *The algorithm creates a worker-optimal strongly stable matching among all the strongly stable matchings that are budget neutral,*

2. *The algorithm generates highest salaries for all workers among strongly stable matchings with fair salaries that are budget neutral.*

Proof: When $\alpha^* < V_1$, it means that

$$\alpha^* = \frac{\sum_{f \in F} b_f}{\sum_{w \in W} V_w}.$$

1. We have to show there is no vector of budgets $B^{'}$ and a strongly stable matching with a vector of salaries $S^{'}$ such that 1) $\sum_{f \in F} b_f = \sum_{f \in F} b^{'}_f$, 2) $S^{'}_w \geq S_w$ for all $w \in W$, and 3) $S^{'}_w > S_W$ for some $w \in W$. By contradiction, assume that there is a strongly stable matching with the above properties. Then

$$\sum_{f \in F} b^{'}_f \geq \sum_{w \in W} S^{'}_w > \sum_{w \in W} S_w = \alpha^* \sum_{w \in W} V_w = \sum_{f \in F} b_f.$$

   Hence, $\sum_{f \in F} b^{'}_f > \sum_{f \in F} b_f$, a contradiction.

2. Suppose not: There exists a vector of budgets $B^{'}$, and a strongly stable matching with a vector of salaries $S^{'}$ such that 1) $\sum_{f \in F} b_f = \sum_{f \in F} b^{'}_f$, 2) $S^{'}_w = \alpha V_w$ for all $w \in W$ and some $\alpha \in \mathbb{R}_+$, and 3) $S^{'}_w > S_w$ for some $w \in W$. Therefore, it must be

the case that $\alpha > \alpha^*$. Observe that

$$\sum_{f\in F} b'_f \geq \sum_{w\in W} S'_w > \sum_{w\in W} S_w = \alpha^* \sum_{f\in F} b_f.$$

Hence, $\sum_{f\in F} b'_f > \sum_{f\in F} b_f$, a contradiction.

■

6 Conclusion

We study the problem of finding stable matching in cases in which firms hire workers with a specified salary while facing a budget constraint. We know that, in this setting, a stable matching need not exist; therefore, we study two special cases. Compared with the literature, in our model, money has an intrinsic value for firms, salaries can be any positive real number, and our equilibrium concept is stability.

We show that the problem of existence, when workers are homogeneous from the firms' point of view, can be resolved in two ways: (i) Use weak stable matching as the solution concept, which we prove always exists, or (ii) allow small perturbation of firms' budgets. We show that there is always a nearby budget vector with a strong stable matching.

In the second case, where we consider the multiplicative functional form for the valuation of a worker at a firm, a stable matching may not exist. We introduce an algorithm that results in a strong stable matching with fair salaries by changing the firms' budgets. The budget change for each firm is bounded by the value of at most one worker to that firm; moreover, the sum of all new budgets equals the sum of all previous budgets, i.e., there is no need for additional funds to find a strong stable matching.

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A Mathematical Appendix

Proof of Lemma 1:

We consider two cases. In the first, not all workers are hired by one firm in μ. In the second case, all workers are hired by one firm in μ.

1) Assume that not all workers are hired by one firm in μ. We show that salaries in μ should be equal. Suppose there exist two workers w_1, and w_2, with different salaries ($s_1 \neq s_2$); then we have two cases:

   i) Workers w_1 and w_2 are matched with different firms. Without loss of generality, let w_1 match with f_1, and w_2 match with f_2. Assume $s_1 > s_2$. Consider a salary s such that $s_1 > s > s_2$. Now, f_1 and w_2 with salary s form a blocking pair: a contradiction.

   ii) Workers w_1 and w_2 are matched with one firm; assume f_1. We know there exists a worker $w^{'}$ with salary $s^{'}$ who is matched with $f^{'} \neq f_1$. Note that either $s^{'} \neq s_1$ or $s^{'} \neq s_2$. Then, suppose $s^{'} \neq s_1$. If $s^{'} > s_1$, then firm $f^{'}$ and worker w_1 with salary s form a blocking pair, where $s^{'} > s > s_1$. On the other hand, if $s^{'} < s_1$, then firm f_1 and worker $w^{'}$ with salary s form a blocking pair, where $s^{'} < s < s_1$: a contradiction.

2) Suppose all workers are matched with one firm (f_1, for instance) in μ. Let s_w be the salary of worker $w \in W$ in μ. Matching μ is weakly (strongly) stable, so for every hired worker and every firm $f \in F$ (if it exists), except in the case of f_1, either $s_w \geq v_f$ or $s_w \geq b_f$ ($s_w > b_f$ in the case of a strongly stable matching). Define $s_m = min_{w \in W} s_w$. Define $\mu^{'}$ such that all hired workers are matched to firm f_1 with salary s_m. This matching is a weakly (strongly) stable matching, because for all hired workers w and all firms f other than f_1, either $s_m \geq v_f$ or $s_m \geq b_f$ ($s_w > b_f$ if the matching is strongly stable).

■

Proof of Lemma 2:

Let $s_w = s$ for all $w \in W$. Define problem $\mathbb{P}_f$ for firm $f \in F$:

$$\max_{A \subset W} \sum_{w \in A} (v_{fw} - s_w) = \max_{A \subset W} \sum_{w \in A} (v_f - s),$$

$s.t.$

$$\sum_{w \in A} s_w \leq b_f.$$

Before starting the first part of the lemma, we show that the matching with salary s for all workers is strongly stable if and only if every firm f is assigned to the solution of problem $\mathbb{P}_f$ and each worker is assigned to exactly one firm.

First, we show that if every firm f is assigned to the solution of problem $\mathbb{P}_f$ and each worker is assigned to exactly one firm, with salary s, then this matching, μ, is strongly stable. Matching μ is individually rational for workers because $s \geq 0$. It is individually rational for firms because it is the solution to the firm's optimization problem. A firm f and $A \subset W$ cannot make a blocking pair. By contradiction, assume that there is a feasible salary vector $(s'_w)_{w \in A}$ with $u_f(\{(w, s'_w) | w \in A\}) > u_f(\mu^o(f))$, and for all $w \in A$, $u_w(\mu(w)) \leq u_w(f, s'_w)$. This implies that $s'_w \geq s$ for all $w \in A$. Therefore,

$$u_f(\{(w, s) | w \in A\}) \geq u_f(\{(w, s'_w) | w \in A\}).$$

However, $u_f(\{(w, s'_w) | w \in A\}) > u_f(\mu^o(f))$, hence,

$$u_f(\{(w, s) | w \in A\}) > u_f(\mu^o(f)).$$

This contradicts the assumption that $\mu^o(f)$ is an optimal decision for firm f when salaries are s.

On the other hand, if the matching μ with salary s for all workers is strongly stable, we can conclude that every firm solves problem $\mathbb{P}_f$, and each worker is assigned to exactly one firm. The matching μ is strongly stable, therefore, for all $f \in F$, there is no $A \subset W$ with the same salary s for all workers such that $u_f(\{(w,s)|w \in A\}) > u_f(\mu^o(f))$. This implies that $\mu^o(f)$ is the solution of problem $\mathbb{P}_f$ for all $f \in F$. μ is a matching, so each worker matches with at most one firm. Moreover, there cannot be an unmatched worker because any firm $f \in F$ with $\mu^o(f)$ and the unmatched worker with salary zero forms a blocking pair.

Now we want to use the above result. Instead of strongly stable matching, we use a matching with a fixed salary s in which firm $f \in F$ is assigned to the solution of problem $\mathbb{P}_f$ for all $f \in F$, and each worker matches to exactly one firm.

1) **Strongly stable $\Rightarrow$ i), and ii):** Since the values and salaries of workers for firm f are the same, firm f is indifferent between workers. The number of workers who firm f can hire based on the solution of problem $\mathbb{P}_f$ is $D_f(s)$. [15] Each worker should be assigned to exactly one firm, thus the sum of demands should be equal to the number of workers.

Strongly stable $\Leftarrow$ i), and ii): A matching μ with salary s in which the number of workers who match to firm f is an element of $D_f(s)$ implies that firm f solves problem $\mathbb{P}_f$. This is because firms are indifferent among workers and the only relevant parameter is the number of workers they hire. The market clearing condition ensures that each worker matches to exactly one firm.

2) **Weakly stable $\Leftarrow$ i), and ii)**: Suppose that each firm is assigned to the number of workers equal to an element of the firm's pseudo demand and the market clears. Observe that $s \geq 0$ so the matching is individually rational for workers. Moreover, it is individually rational for firms: If $v_f < s$, firm f does not choose any worker; if $v_f = s$, the firm gets zero utility from workers; if $v_f > s$, each worker has a strict positive value and the firm prefers more workers. By contradiction, suppose that there is a firm f, a set $A \subset W$, and a feasible salary vector $(s'_w)_{w \in A}$ such that $u_f(\{(w, s'_w)|w \in A\}) > u_f(\mu^o(f))$ and for every

[15]Note that when $v_f = s$, firm f is indifferent to the hiring of workers, so it can hire any number of workers between 0 and $\lfloor \frac{b_f}{s} \rfloor$.

$w \in A$ salaries are higher; i.e., $s'_w > s$. Consider three cases:

1. $v_f \leq s$: If A is not empty, then $u_f(\{(w, s'_w)|w \in A\} < 0$. However, $u_f(\mu^o(f)) = 0$, which is a contradiction.[16]

2. $v_f > s$ and $\frac{b_f}{s} \in \mathbb{N}$: If salaries are higher than s, then firm f can hire at most $\frac{b_f}{s} - 1$ workers. Define $\tilde{s} = \min\limits_{w \in A} s'_w$. Note that we have $(\lfloor \frac{b_f}{\tilde{s}} \rfloor)(v_f - \tilde{s}) \geq u_f(\{(w, s'_w)|w \in A\})$. Putting these together, we have:

$$(\lfloor \frac{b_f}{\tilde{s}} \rfloor)(v_f - \tilde{s}) \geq u_f(\{(w, s'_w)|w \in A\}) > u_f(\mu^o(f)) \geq (\frac{b_f}{s} - 1)(v_f - s).$$

   Therefore, we have

$$(\lfloor \frac{b_f}{\tilde{s}} \rfloor)(v_f - \tilde{s}) > (\frac{b_f}{s} - 1)(v_f - s).$$

   This is a contradiction because $\lfloor \frac{b_f}{\tilde{s}} \rfloor \leq \frac{b_f}{s} - 1$ and $v_f - \tilde{s} \leq v_f - s$.

3. $v_f > s$ and $\frac{b_f}{s} \notin \mathbb{N}$: Note that $v_f(\mu^o(f)) \geq (\lfloor \frac{b_f}{s} \rfloor)(v_f - s)$. Define $\tilde{s} = \min\limits_{w \in A} s'_w$. Note that $(\lfloor \frac{b_f}{\tilde{s}} \rfloor)(v_f - \tilde{s}) \geq u_f(\{(w, s_w)|w \in A\})$. Putting these together, we have:

$$(\lfloor \frac{b_f}{\tilde{s}} \rfloor)(v_f - \tilde{s}) \geq u_f(\{(w, s_w)|w \in A\}) > u_f(\mu^o(f)) \geq (\lfloor \frac{b_f}{s} \rfloor)(v_f - s),$$

   which is a contradiction.

Weakly stable $\Rightarrow$ i), and ii): Suppose matching μ with salary s for each worker is weakly stable. We show that all firms are assigned to their pseudo demands. Consider four cases:

1. $v_f < s$: Firm f does not choose any worker due to individual rationality, which is equal to the pseudo demand.

2. $v_f = s$: Firm f cannot choose more than $\lfloor \frac{b_f}{s} \rfloor$. Firm f is indifferent between hiring and not hiring workers. Note that any number less than $\lfloor \frac{b_f}{s} \rfloor$ satisfies pseudo demand.

[16] This case includes both $v_f < s$, and $v_f = s$, where in both cases $u_f(\mu^o(f)) = 0$.

3. $v_f > s$ and $\frac{b_f}{s} \in \mathbb{N}$: Firm f should match to at least $\frac{b_f}{s} - 1$ workers. Otherwise, it matches to $\frac{b_f}{s} - 2$ or fewer workers. In that case, firm f with $\frac{b_f}{s} - 1$ workers and salary $s + \epsilon$ (ϵ small enough) make a blocking pair.

4. $v_f > s$ and $\frac{b_f}{s} \notin \mathbb{N}$: Firm f should match to at least $\lfloor \frac{b_f}{s} \rfloor$ workers. Otherwise, it matches to $\lfloor \frac{b_f}{s} \rfloor - 1$ or fewer workers. In that case, firm f with $\lfloor \frac{b_f}{s} \rfloor$ workers and salary $s + \epsilon$ (ϵ small enough) make a blocking pair.

When matching μ is weakly stable, then there is no unmatched worker, because a firm can replace one worker with the unmatched worker with a salary less than s. Therefore, the market clears. ■

Proof of Theorem 1:

Here is the rest of the proof of part iii of Theorem 1. By contradiction, assume $k \in \mathbb{N}$ exists such that there does not exist a salary s such that $k \in T(s)$. By decreasing s, the maximum element of set $D_f(s)$ for all $f \in F$ is weakly increasing and not bounded from above; hence, set $T(s)$ is not bounded from above. Therefore, for any $k \in \mathbb{N}$, we can find an s small enough that the $\max\{T(s)\} > k$. Therefore, without loss of generality, we can assume that a salary s exists such that $k + 1 \in T(s)$, and $k \notin T(s)$. This is because $T(s)$ is not bounded from above. $k + 1 \in T(s)$, and $k \notin T(s)$, so

$$k + 1 = \sum_{f \in F} \min\{D_f(s)\}.$$

Define

$$s_{sup} = \sup\{s | k + 1 \in T(s)\}.$$

Note that s_{sup} exists because $T(s) = 0$, for a large enough s. Consider two cases:

1) Assume $k + 1 \in T(s_{sup})$, then $k + 1 = \sum_{f \in F} \min\{D_f^{'}(s_{sup})\}$. For $\epsilon_1 \geq 0$ by assumption, $k \notin T(s_{sup} + \epsilon_1)$. The maximum of sets $D_f(s)$ for all $f \in F$ are weakly decreasing in s;

therefore,

$$\sum_{f\in F}\min\{D_f(s_{sup})\}-1=k>\sum_{f\in F}\max\{D_f(s_{sup}+\epsilon_1)\}.$$

There are two cases: First, there exist two firms, f^* and f^{**}, in which the strict demand decreases by at least one:

$$\min\{D_f(s_{sup})\}>\max\{D_f(s_{sup}+\epsilon_1)\},$$

for $f\in\{f^*,f^{**}\}$, and for $\epsilon_1>0$. Second, there exists one firm f^* in which the strict demand decreases by at least two:

$$\min\{D_{f^*}(s_{sup})\}-1>\max\{D_{f^*}(s_{sup}+\epsilon_1)\},$$

for $\epsilon_1>0$. For both cases, $v_{f^*}\neq s_{sup}$ (and $v_{f^{**}}\neq s_{sup}$ in the first case). Otherwise,

$$0=\min\{D_{f^*}(s_{sup})\}>\max\{D_{f^*}(s_{sup}+\epsilon_1)\}=0,$$

which is a contradiction.[17] Observe that for both cases $\frac{b_{f^*}}{s_{sup}}\in\mathbb{N}$, (and $\frac{b_{f^{**}}}{s_{sup}}\in\mathbb{N}$ for the first case). Otherwise, $D_{f^*}(.)$ cannot change from s_{sup} to $s_{sup}+\epsilon_1$. Consider the first case, $\frac{b_{f^*}}{s_{sup}}\in\mathbb{N}$ and $\frac{b_{f^{**}}}{s_{sup}}\in\mathbb{N}$, therefore, $\frac{b_{f^{**}}}{b_{f^*}}\in\mathbb{Q}$. This is a contradiction because for all $f,f'\in F:\frac{b_f}{b_{f'}}\notin\mathbb{Q}$. Consider the second case, $\frac{b_{f^*}}{s_{sup}}\in\mathbb{N}$ and

$$\min\{D_{f^*}(s_{sup})\}-1>\max\{D_{f^*}(s_{sup}+\epsilon_1)\}.$$

This is impossible because $D_{f^*}(s_{sup})$ cannot decrease more than one from s_{sup} to $s_{sup}+\epsilon_1$, for $\epsilon_1>0$ small enough.

2) Second, assume that $k+1\notin T(s_{sup})$, then there exists $m\geq 1$ such that $k-m=$

[17] Similarly, for the first case $0=\min\{D_{f^{**}}(s_{sup})\}>\max\{D_{f^{**}}(s_{sup}+\epsilon_1)\}=0$, which is a contradiction.

$\sum_{f\in F}\max\{D'_f(s_{sup})\}$. Note that for $\epsilon_2 \geq 0$ such that $k+1 \in T(s_{sup}-\epsilon_2)$ we have that $k-m \notin T(s_{sup}-\epsilon_2)$. Because if $k+1, k-m \in T(s_{sup}-\epsilon_2)$ then it must be true that $k \in T(s_{sup}-\epsilon_2)$, too, which is a contradiction.

The maximum of sets $D_f(s)$ for all $f \in F$ are weakly decreasing in s; therefore,

$$\sum_{f\in F}\max\{D_f(s_{sup})\}+m = k < \sum_{f\in F}\min\{D_f(s_{sup}-\epsilon_2)\}.$$

Since $m \geq 1$, there must be two cases: First, there exist two firms, f^* and f^{**}, in which the strict demand changes by at least one:

$$\max\{D_f(s_{sup})\} < \min\{D_f(s_{sup}-\epsilon_2)\},$$

for $f \in \{f^*, f^{**}\}$, and for $\epsilon_2 > 0$. Second, there exists one firm f^* in which the strict demand changes by at least two:

$$\max\{D_{f^*}(s_{sup})\}+1 < \min\{D_{f^*}(s_{sup}-\epsilon_2)\},$$

for $\epsilon_2 > 0$. For both cases, $v_{f^*} \neq s_{sup}$ (and $v_{f^{**}} \neq s_{sup}$ in the first case). Otherwise,

$$\lfloor\frac{b_{f^*}}{s_{sup}}\rfloor = \max\{D_{f^*}(s_{sup})\} < \min\{D_{f^*}(s_{sup}-\epsilon_2)\} = \lfloor\frac{b_{f^*}}{s_{sup}}\rfloor,$$

for ϵ_2 small enough, a contradiction.[18] Observe that for both cases, $\frac{b_{f^*}}{s_{sup}} \in \mathbb{N}$, (and $\frac{b_{f^{**}}}{s_{sup}} \in \mathbb{N}$ for the first case), otherwise, $D_{f^*}(.)$ (and for the first case $D_{f^{**}}(.)$) cannot change from s_{sup} to $s_{sup}-\epsilon_2$. Consider the first case: $\frac{b_{f^*}}{s_{sup}} \in \mathbb{N}$ and $\frac{b_{f^{**}}}{s_{sup}} \in \mathbb{N}$. Therefore, $\frac{b_{f^{**}}}{b_{f^*}} \in \mathbb{Q}$, which is a contradiction, because for all $f, f' \in F$: $\frac{b_f}{b_{f'}} \notin \mathbb{Q}$. Consider the

[18] Similarly, for the first case, $\lfloor\frac{b_{f^{**}}}{s_{sup}}\rfloor = \max\{D_{f^{**}}(s_{sup})\} < \min\{D_{f^{**}}(s_{sup}+\epsilon_1)\} = \lfloor\frac{b_{f^{**}}}{s_{sup}}\rfloor$, which is a contradiction.

second case: $\frac{b_{f^*}}{s_{sup}} \in \mathbb{N}$, and

$$\min\{D_{f^*}(s_{sup})\} - 1 > \max\{D_{f^*}(s_{sup} + \epsilon_1)\}.$$

This is impossible because $D_{f^*}(s_{sup})$ cannot change more than one from s_{sup} to $s_{sup} - \epsilon_2$, for $\epsilon_2 > 0$ small enough.

$\blacksquare$

Proof of Lemma 3:

By definition, $\alpha(\overrightarrow{1}) = \alpha(\overrightarrow{1_1}) > V_1$. If $\alpha(\overrightarrow{1_2}) \leq V_1$, then by continuity, $y \in [0, 1]$ exists such that

$$V_1 = \alpha(\overrightarrow{y_1}).$$

If $\alpha(\overrightarrow{1_2}) > V_1$, and $\alpha(\overrightarrow{1_2}) \leq V_2$, then part 2 of the Lemma 3 is satisfied. Otherwise, $\alpha(\overrightarrow{1_2}) > V_2$. If $\alpha(\overrightarrow{1_3}) \leq V_2$, then by continuity, $y \in [0, 1]$ exists such that

$$V_2 = \alpha(\overrightarrow{y_2}).$$

If $\alpha(\overrightarrow{1_3}) > V_2$, and $\alpha(\overrightarrow{1_3}) \leq V_3$, then part 2 of the Lemma 3 is satisfied. Otherwise, $\alpha(\overrightarrow{1_3}) > V_3$. By repeating this argument, we prove either Lemma 3 or $\alpha(\overrightarrow{1_{|F|}}) > V_{|F|}$. By definition, $\alpha(\overrightarrow{0_{|F|}}) = \alpha(\overrightarrow{0}) = 0$. Hence, by continuity, if $\alpha(\overrightarrow{1_{|F|}}) > V_{|F|}$, then $y \in [0, 1]$ exists such that

$$V_{|F|} = \alpha(\overrightarrow{y_{|F|}}).$$

This concludes the argument. $\blacksquare$

Proof of Theorem 2:

Observe that the workers receive fair salaries because:

$$\frac{S_w}{V_w} = \frac{S_{w'}}{V_{w'}} = \alpha^*,$$

for all $w,\, w^{'} \in W$.

The algorithm generates a strongly stable matching. First, we show that firms do not have any incentive to change their allocations at these wages. The argument is as follows. In step two, for not efficient firms, all $f \in F$ such that $V_f < V_{f^*}$, hiring a worker at these wages generates a negative profit, so the algorithm removes these firms from the market. In step three, if $b_f \neq b_{f'}$, then $y_{f^*} < 1$, which means $V_{f^*} = \alpha^*$. In this case, f^* makes a zero profit, so f^* is indifferent between hiring and not hiring workers. In step four, efficient firms ($V_f \geq \alpha^*$) that have exhausted their budgets completely hire zero or positive numbers of workers. These firms are indifferent between workers because the value of spending one dollar for hiring a worker is the same among different workers and is

$$\frac{V_f V_w - S_w}{S_w} = \frac{V_f V_w - \alpha^* V_w}{\alpha^* V_w} = \frac{V_f - \alpha^*}{\alpha^*}.$$

These firms do not have any incentive to change their allocations. After step four and before starting step five, there remain some unmatched workers and some firms, which are efficient, but still have not finished spending their budgets.[19] It is not possible to match all workers without spending all budgets; in other words, it is not possible to exhaust all budgets without matching all workers, because salaries are such that $\sum_{w \in W} S_w = \sum_{w \in W} \alpha^* V_w = b^{'}_{f^*} + \sum_{f > f^*} b_f$.[20] In step five, after matching workers and updating budgets, all firms $f \in F$ such that $V_f > \alpha^*$ have reached their budgets, and do not have any incentive to change their allocations, as they are indifferent between workers. In step six, changing the budget of firm f^* does not create an incentive for f^* to change the allocation. This is due to the fact that if $b_f \neq b_{f'}$, then f^* is indifferent between hiring workers or not hiring them. Note that if an efficient firm strictly prefers another vector of salaries to the current one, then

[19]Note that the algorithm may have finished at step four (matched all workers and finished spending all budgets of all firms $f \in F$ such that $V_f > \alpha^*$).

[20]Note that the number of unmatched workers is less than or equal to the number of firms that still have some amount left in their budgets (the remaining firms). Because the sum of the remaining budgets is equal to the sum of the salaries of the unmatched workers, if the number of unmatched workers is larger than the number of remaining firms, then at least one firm could afford the salary of one unmatched worker.

it has to pay a salary less than the current one to at least one of the workers, because all efficient firms have exhausted their budgets completely.

1. First, we analyze firms f such that $V_f > V_{f^*}$. In step five, when the algorithm matches w to f, firm f's remaining budget is less than S_w; therefore, it increases at most $S_w = \alpha^* V_w$, because the algorithm does not match two or more workers to a firm at step five. After matching all workers, there are some firms that do not receive any worker at step five. The algorithm decreases their budgets. Their budgets do not change by more than the salary of a worker, because in step four they could not hire any worker. The argument for firm f^* is as follows. In step two, the algorithm changes the budget from b_{f^*} to b'_{f^*}, but in step six, it adds the difference back into its budget. In step five, the same as it does at other efficient firms, the budget b'_{f^*} changes by, at most, $\max_{w \in W} \alpha^* V_w$. Hence, after step six, budget b_{f^*} does not change by more than $\max_{w \in W} \alpha^* V_w$.

2. The sum of budgets at the end of step five is equal to the sum of salaries. This is because the algorithm hires all workers at the specified salaries. The sum of all the salaries at the end of step five is equal to the sum of the budgets at step four; i.e., $\sum_{w \in W} S_w = \sum_{w \in W} \alpha^* V_w = b'_{f^*} + \sum_{f > f^*} b_f$. Finally, in step six, by adding $b_{f^*} - b'_{f^*}$ to the budget of firm f^*, the sum of all these firms' budgets will be $b_{f^*} + \sum_{f > f^*} b_f$.

■