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DEBT MANAGEMENT GAME AND DEBT CEILING

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Abstract. We introduce a non zero-sum game between a government and a legislative body to study the optimal level of debt. We succeed in characterising Nash equilibria in the class of Skorokhod-reflection policies which implies that the legislator imposes a debt ceiling. In addition, we highlight the importance of the time preferences in the magnitude of the optimal level of the statutory debt ceiling. In particular, we show that laissez-faire policy can be optimal for high values of the legislator's discount rate.

Keywords: non-zero-sum game, singular stochastic control, free-boundary problem, debt-to- GDP ratio.

MSC2010 subject classification:

1. Introduction

There is probably no more topical issue in macroeconomics than the determination of the optimal level of debt that favours both its sustainability and the long-term growth of an economy. Although paramount and extensively studied in the literature (see [1] and [11] for a general presentation), the question of the optimal debt level has not yet received clear theoretical foundations. This lack of a consensual theoretical framework has led to the implementation of exogenous mechanisms to monitor the level of debt. In USA, one of these mechanisms is the statutory debt ceiling which restricts the amount of debt a government can be permitted to issue[1]. These have proven to be more or less effective and resulted in a constant upward revision of the debt ceiling.

The traditional analysis of public debt has shown that high public debt has a negative effect on long-term economic growth giving an argument to debt ceiling advocates. Indeed, a high level of debt generates high risk premiums that reflect creditors' doubts about the government's ability to refinance itself. Being unable not only to repay its debts but also to pay for the excess of its expenditures over its revenues, the government must then immediately balance its budget by taking exceptional measures, like increasing taxes and/or cutting its investments, which can have a dramatic impact on growth, or alternatively negotiate the lift of the debt ceiling to avoid the huge costs of the aforementioned measures. The latter negotiation could be challenging, when the governing party is different from the majority party in the House of Representatives (and/or the Senate), as is often the case in the United States. On the other hand, the positive effect of a high level of public debt on growth should not be overlooked since public investments in social policies, education, healthcare, justice, research, and infrastructure help private initiatives to develop effectively. As Blanchard observed in his presidential address to the American Economic Association [2], as long as the interest rate is lower than the growth rate, a large deficit can be allowed without increasing the debt-to-GDP ratio. The subject of this paper is the study of the debt management game between two groups with

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[1]Within the European Community, a similar mechanism exists since the Maastricht treaty set 60% as the upper bound for the debt-to-GDP ratio for members of the European Union.

opposing interests; on one hand, a government inclined to increase the debt and, on the other hand, a legislature concerned with imposing a mechanism limiting the level of debt.

So far, the theoretical literature on public debt management has focused on the stochastic control problem faced by a single decision maker (say a government), whose objective is to balance the pros and cons of a high level of public debt. In [5] and [6], the debt-to-GDP ratio (also called "debt ratio") evolves as a controlled one-dimensional geometric Brownian motion and the government can only reduce its debt level through singular and bounded variation controls, respectively. The objective is to minimize the expected total costs resulting from the instantaneous cost of the debt level and the intervention costs. In [12], the optimal debt ratio reduction problem is considered as well, but this time the government takes into consideration the evolution of the inflation rate of the country as well. The latter evolves as an uncontrolled diffusion process which makes the problem a fully two-dimensional singular stochastic control problem. In [13], the authors consider a model where the drift of the debt ratio process is affected by an exogenous macroeconomic shock modelled by a continuous-time Markov chain and the government is allowed to both decrease and increase the debt ratio. The stochastic control problem associated to the minimization of debt management costs is then solved using probabilistic arguments. What all these papers have in common is that an optimal level of debt ceiling is defined endogenously. However, they do not consider the potential political game between a government and a legislative body (e.g. the Congress), whose political interests may be divergent.

Our paper focuses on this alternative point of view. Instead of considering the case of a single decision-maker, we consider the following two-player game. The first player is a government that has an interest in increasing the level of debt for political reasons, such as increasing its chances of re-election. The second player is a legislative body that may, for example, impose a debt limit as a hard constraint on governments, in view of restraining government spending to avoid potential debt crises. Unlike the aforementioned papers, the optimal debt level management policy by the government must now be viewed as a best reply to a mechanism imposed by the legislative body to monitor spending, while allowing further borrowing. In particular, the legislative body would like governments to ideally keep the country's debt ratio at low levels, in order to maintain a low probability of default and feasible borrowing from the markets. When the level of debt collides with the constraints of the mechanism, exceptional measures must be put in place to keep the debt ratio below its limit. For instance, a liberalisation policy could be applied to increase the level of GDP. This may include the privatisation of some governmental institutions and assets, the increase of labour market flexibility, the lowering of tax rates for businesses, the reduction of restrictions on both domestic and foreign capital, etc. In order to fine-tune such a mechanism, the legislative body must therefore take into account the government's debt issuance policy, and devise their mechanism as a best response. This framework then results in a non-zero-sum game between a government, whose mandate is to manage its public debt issuance policy, and a legislative body, that sets rules to contain debt issuance. In mathematical terms, both players exert a monotone control to set the path of the debt ratio. The first player (government) can only increase the level of debt ratio by exerting its control, while the second player (legislative body) can only decrease the level of debt ratio. Each player wants to minimize their own total cost functional and we allow the rate of increase/reduction of each player to be unbounded and have an instantaneous effect on the debt ratio. This therefore leads to the formulation of a stochastic non-zero-sum game of singular controls.

The main contribution of this paper is to show the existence of a unique Nash equilibrium in the class of Skorokhod-reflection type policies, which in economic terms means that the mechanism is a debt ceiling b. This is the level at which the legislative body will demand the decrease of the debt ratio and the adoption of liberalisation policies by the government. The idea is to study separately the two coupled constrained stochastic control problems faced by the two players, namely the government and the legislative body. The government's problem is as follows. Assuming that the legislative body imposes a debt ceiling b or equivalently a Skorokhod reflection policy for the debt ratio process at b, what should be the optimal policy for issuing new public debt? We establish a connection between this constrained stochastic control problem and a free-boundary problem, that we solve by a classical guess-and-verify approach. As a result, we show that the best debt issuance policy is to reflect the debt ratio process upwards at a level $a(b)$. The problem of the legislative body is then the following. Considering that the government is going to use a debt issuance policy of reflecting the debt ratio process at a level a, should it impose a debt ceiling $b(a)$? In particular, if there is already a statutory exogenous debt ceiling, is it optimal to raise it and by how much? We show that a debt ceiling should indeed be imposed for a specific range of the legislative body's time preference rates.

The remainder of our analysis then focuses on finding an equilibrium in the game. To that end, we define a Nash equilibrium in the class of Skorokhod-reflection policies as a pair (a^*, b^*), such that $a^* = a(b^*)$ and $b^* = b(a^*)$. We prove the existence and uniqueness of such a Nash equilibrium (a^*, b^*), hence the optimal policies prescribe that the debt ratio is kept inside the interval $[a^*, b^*]$ with the minimal cost, associated to Skorokhod reflection policies. It is worth noting that, for another range of the legislative body's time preference rates, the legislative body should optimally not intervene, and an associated Nash equilibrium without debt ceiling is proved to hold. Interestingly, we prove that the optimality of adopting such a strategy relies solely on the legislative body's time preferences compared to the parameter constellation in the model – it does *not depend* on the actions of the opposing player (government) – see specifically our results in Section 4.1. When the discount rate of future costs is high, the consideration of the risk of a debt crisis in the future is too low to make the implementation of a debt ceiling mechanism optimal. We thus split our search for Nash equilibria based on the magnitude of the legislative body's time preferences.

Our paper is closely related to [10] where the authors have proved the existence of a Nash equilibrium in the same class of Skorokhod-reflection policies by establishing a new connection with a non-zero-sum stopping game. In order to justify this connection between non-zero-sum games of monotone controls and those of stopping, it is necessary that both players have the same discount factor (or equivalently time preferences) and that the running cost is a differentiable function. Unlike [10], we assume here that the two players have different discount rates reflecting a different time preference and that the running cost is not a differentiable function, which requires a different methodology to prove the existence of a Nash equilibrium.

The paper is organized as follows. Section 2 describes the setting and the two problems faced by the two players. In Section 3, we solve the government constrained control problem by distinguishing two cases: the legislative body does not intervene or forces the government to keep its debt ratio below a debt ceiling $b > 0$. In both cases, we show that the government can devise an optimal debt issuance policy when the debt ratio is sufficiently low. In Section 4, we solve the constrained control problem of the legislative body and find its best response strategy to the above governmental policy. We show that the magnitude of the legislative body's time preference rate plays a crucial role, which is a key result of our analysis. In particular, if it is relatively small, a debt ceiling mechanism is optimal, while if it is relatively large, the legislative body should not set a debt ceiling at all. We prove the existence and uniqueness of a Nash Equilibrium in the class of Skorokhod-reflection policies in Section 5. Finally, Section 6 is devoted to a comparative statics analysis; we explore how the optimal debt issuance policy and debt ceiling mechanism are affected by changes in the model parameters. In particular, we are able to quantify the transition between a legislative body's optimal intervention and non-intervention regimes.

2. Setting and Problem Formulation

2.1. **Motivation for the model.** The nominal debt grows at rate r, i.e. it evolves according to

$$dD_t = rD_t dt, \qquad t \geq 0,$$

in the absence of any intervention, where we denote by $r \in \mathbb{R}$ the interest rate on government debt. When designing its economic policy, the government can choose to increase the current level of the debt by a new issuance. Denoting by ξ_t the cumulative percentage of debt increase up to time $t \geq 0$, the dynamics of the adjusted debt reads as

$$dD_t = rD_t dt + D_t d\xi_t, \qquad t \geq 0,$$

where the latter integral (and others of the same type in the following motivation) will be defined later in Section 2.2. The GDP follows the stochastic dynamics

$$dG_t = gG_t dt + \sigma G_t d\widehat{W}_t, \qquad t \geq 0,$$

in the absence of any intervention, where we denote by $g \in \mathbb{R}$ the growth rate of the GDP and by $\widehat{W}$ a standard one-dimensional Brownian motion. A legislative body can implement a liberalisation policy in order to boost the GDP.

Denoting by η_t the cumulative percentage of GDP increase up to time $t \geq 0$, the dynamics of the adjusted GDP read as

$$dG_t = gG_t dt + \sigma G_t d\widehat{W}_t + G_t d\eta_t, \qquad t \geq 0,$$

Hence, we may conclude that the dynamics of the debt-to-GDP ratio, obtained via the use of Itô's formula on $X := D/G$, evolves according to

$$dX_t = (r-g)X_t dt + \sigma X_t(\sigma dt - d\widehat{W}_t) + X_t d\xi_t - X_t d\eta_t, \qquad t \geq 0,$$

Without loss of generality, a change of measure to one under which $W_t := \sigma t - \widehat{W}_t$ is a Brownian motion, will allow for the following problem formulation.

2.2. **Problem formulation.** Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space accommodating a one-dimensional Brownian motion $W := (W_t)_{t\geq 0}$. We denote by $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ the filtration generated by W augmented by P-null sets. In absence of any interventions, the debt-to-GDP ratio (also called "debt ratio") evolves according to the stochastic differential equation (SDE)

$$dX_t^0 = (r-g)X_t^0 dt + \sigma X_t^0 dW_t, \qquad X_t^0 = x > 0. \tag{2.1}$$

The classical macroeconomic dynamics of the debt ratio, see e.g. [3], are simply the deterministic version of (2.1) with $\sigma = 0$. When increasing the current debt ratio level by $\varepsilon > 0$ percentage points, the debt ratio exhibits a jump

$$\Delta X_t = X_t - X_{t-} = \varepsilon X_{t-}.$$

Hence, for small $\varepsilon > 0$, we can associate a governmental intervention on the debt ratio with $X_t = (1+\varepsilon)X_{t-} \approx e^{\varepsilon} X_{t-}$. Furthermore, interpreting an intervention $\Delta\xi_t$ as a sequence of N individual interventions of size $\varepsilon = \Delta\xi_t/N$ we have $X_t = e^{N\varepsilon}X_{t-} = e^{\Delta\xi_t}X_{t-}$, for N large enough. We can thus model the controlled debt ratio dynamics (by arguing similarly for the interventions of the legislative body) via

$$dX_t^{\xi,\eta} = (r-g)X_t^{\xi,\eta} dt + \sigma X_t^{\xi,\eta} dW_t + X_t^{\xi,\eta} \circ_u d\xi_t - X_t^{\xi,\eta} \circ_d d\eta_t, \qquad t \geq 0, \tag{2.2}$$

where the operators $\circ_u$ and $\circ_d$ are defined as

$$\begin{aligned} X_t^{\xi,\eta} \circ_u d\xi_t &= X_t^{\xi,\eta} d\xi_t^c + X_{t-}^{\xi,\eta} \int_0^{\Delta\xi_t} e^u du = X_t^{\xi,\eta} d\xi_t^c + X_{t-}^{\xi,\eta}\big[e^{\Delta\xi_t} - 1\big], \\ X_t^{\xi,\eta} \circ_d d\eta_t &= X_t^{\xi,\eta} d\eta_t^c + X_{t-}^{\xi,\eta} \int_0^{\Delta\xi_t} e^{-u} du = X_t^{\xi,\eta} d\eta_t^c + X_{t-}^{\xi,\eta}\big[1 - e^{\Delta\xi_t}\big]. \end{aligned} \tag{2.3}$$

Here, ξ^c (resp., η^c) denotes the continuous part of the process ξ (resp., η).

Using Itô's formula we can verify that the solution to (2.2) starting at time zero from level $x > 0$ is given by

$$X_t^{\xi,\eta} = x \exp\Big(\Big(r - g - \frac{1}{2}\sigma^2\Big)t + \sigma W_t + \xi_t - \eta_t\Big) = X_t^0 \exp(\xi_t - \eta_t), \qquad t \geq 0, \tag{2.4}$$

where X_t^0 denotes the solution to (2.1). Notice that the impact of interventions by the government and legislative body are of multiplicative structure and additive to the logarithm of the debt ratio.

In accordance with our reasoning above, ξ_t denotes the cumulative percentage amount of debt increase by the government and η_t denotes the cumulative percentage amount of debt decrease by the legislative body, up to time $t \geq 0$. It is therefore natural to model them as nondecreasing stochastic processes, adapted with respect to the available flow of information $\mathbb{F}$. Hence we take ξ and η in the set

$$\mathcal{U} := \Big\{ \upsilon : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ : \quad (\upsilon_t)_{t\geq 0} \ \mathbb{F}\text{-adapted, nondecreasing, càdlàg, and } \upsilon_{0-} = 0 \Big\}.$$

The problem of the government. In this framework, the government is facing a potential debt ceiling (or debt limit) as a hard constraint imposed by a legislative body, when the country's debt ratio is too high. In other words, the government has an exogenous factor, namely a debt ratio ceiling b, to take into consideration when designing its economic policy. This is the level at which a legislative body will demand the decrease of the debt ratio and the adoption of liberalisation policies by the government. In the following, we assume that having a debt level $X_t^{\xi,\eta}$ at time $t \geq 0$, the government incurs an instantaneous cost $h(X_t^{\xi,\eta})$. This may be interpreted as an opportunity cost resulting from having less room for financing public investments. We make the following standing assumption.

Assumption 2.1. *The instantaneous (running) cost function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:*

(i) *$x \mapsto h(x)$ is strictly convex, continuously differentiable and increasing on $[0, \infty)$;*
(ii) *the derivative h' of h satisfies $\lim_{x\to 0} h'(x) = 0$ and $\lim_{x\to\infty} h'(x) = +\infty$;*
(iii) *there exists $p > 1$, $K_1 > 0$ such that*

$$h(x) \leq K_1(1 + |x|^p), \quad x \in \mathbb{R}.$$

Remark 2.2. *It is worth noticing that a cost function of the form $h(x) = \frac{1}{2}x^2$ for $x > 0$ satisfies Assumption 2.1. Notice that $h(0) = 0$ together with $h'(0) = 0$ imply that any infinitesimal amount of debt does not generate holding costs for the country; indeed, $h(\varepsilon) \approx h'(0)\varepsilon = 0$. If one wishes to obtain closed-form solutions, a specific function h must be chosen according to Assumption 2.1; our choice will be precisely the above one.*

Moreover, whenever a legislative body decides to impose a debt ceiling mechanism, the government incurs a proportional cost to the amount of debt reduction (see also [5], [12] and [13]). This might be seen as a measure of the social and financial consequences, or repercussions for the financial stability of households and individuals, deriving from the enforcement of debt-reduction policies. The associated constant marginal cost $c_1 > 0$ allows to express it in monetary terms. Finally, the government's main aim is to increase the current level of debt ratio through public investments, e.g. investments in infrastructure, healthcare, education and research, etc. We assume that this has a positive political, social and financial effect, thus overall reduces the total expected "costs" of the government. The marginal benefit of increasing the debt ratio is a strictly positive constant $c_2 > 0$. From the point of view of the government, assuming that it discounts at a rate $\rho > 0$, the total expected cost functional, net of investment benefits, is thus given by

$$\mathcal{J}_{x,\eta}(\xi) := \mathsf{E}_x\Big[\int_0^\infty e^{-\rho t} h(X_t^{\xi,\eta})dt + c_1 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_d d\eta_t - c_2 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_u d\xi_t\Big], \tag{2.5}$$

where, for any $x \in \mathbb{R}_+$, E_x denotes the expectation under the measure $\mathsf{P}_x(\cdot) := \mathsf{P}(\cdot \mid X_{0-}^{\xi,\eta} = x)$.

The problem of the legislative body. On one hand, the legislative body (e.g. Congress) would like governments to ideally keep their country's debt ratio at low levels to maintain a low probability of default and a feasible borrowing from the markets. Even though countries that can print their own currency cannot default on their debts, there are many countries that do not control their own monetary policy, e.g. EU members who rely on the European Central Bank (ECB), or countries that hold large amounts of foreign denominated debts, e.g. Argentina (who defaulted on US government bonds). Several levels $m > 0$ defining the "healthy" region $[0, m]$ of relatively "low" debt ratio have been used in the last decades, e.g. $m = 60\%$ is the Maastricht Treaty's reference value of 1992 for all EU countries, or $m = 77\%$ is the threshold found by researchers at the World Bank [8] for developed economies and $m = 64\%$ for emerging markets.[2]

When the debt ratio $X^{\xi,\eta}$ exceeds this pre-specified value $m > 0$, the legislative body would face social and political pressure, which may lead to the implementation of liberalisation policies in order to decrease the level of $X^{\xi,\eta}$ via a control strategy η. This could, for example, be done by setting a debt ceiling b. This debt ceiling b is expected to be bigger than m, since imposing structural adjustment programs on countries or restricting further borrowing by governments, is costly for the legislative body and the associated marginal cost is $\kappa > 0$. From the point of view of such a legislative body, assuming that it discounts at a rate $\lambda > 0$, we model the expected cost functional as[3]

$$\mathcal{I}_{x,\xi}(\eta) := \mathsf{E}_x\Big[\int_0^\infty e^{-\lambda t}\alpha(X_t^{\xi,\eta} - m)^+ dt + \kappa \int_0^\infty e^{-\lambda t} X_t^{\xi,\eta} \circ_d d\eta_t\Big]. \tag{2.6}$$

When a government wants to reduce its public deficit, it has, in simple terms, a choice between increasing tax revenues while keeping expenditures constant, or reducing public expenditures with stable tax revenues. The second choice is usually the more difficult to make: public spending is sometimes structural (for example, the payment of civil servants' salaries) and therefore incompressible in the short term. This is why, when seeking to reduce public deficits, one most frequently turns to taxation. Hence, $\alpha > 0$ can be interpreted as a country tax compliance factor, the smaller α is the bigger is the willingness to pay tax, if needed in the future. When this factor is low as it is in Denmark for instance, the legislative body has thus less social pressure to reduce the debt ratio.

The non-zero-sum game of singular controls. In our analysis, we restrict our attention to controls producing finite payoffs, which includes the realistic assumption that both players will not use an economic policy leading to infinite cost and/or benefit of interventions. Moreover, we note that the definition of the integrals with respect to the controls, as specified in (2.3), requires some attention since simultaneous jumps of ξ and η may be difficult to handle.

Given that the debt ratio is always a positive number, we therefore consider pairs $(\xi, \eta) \in \mathcal{U} \times \mathcal{U}$ such that

$$\begin{aligned} &\mathsf{E}_x\Big[\int_0^\infty e^{-\rho t} h(X_t^{\xi,\eta}) dt + c_1 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_d d\eta_t + c_2 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_u d\xi_t\Big] < +\infty, \\ &\mathsf{E}_x\Big[\int_0^\infty e^{-\lambda t}\alpha(X_t^{\xi,\eta} - m)^+ dt + \kappa \int_0^\infty e^{-\lambda t} X_t^{\xi,\eta} \circ_d d\eta_t\Big] < +\infty, \\ &\mathsf{P}_x(\Delta\xi_t \cdot \Delta\eta_t > 0) = 0 \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R}_+. \end{aligned} \tag{2.7}$$

To that end, we define the class of controls $(\xi, \eta) \in \mathcal{A} := \mathcal{A}_\eta \times \mathcal{A}_\xi$, where

$$\mathcal{A}_\eta := \{\xi \in \mathcal{U} : \ (\xi, \eta) \in \mathcal{U} \times \mathcal{U} \text{ satisfies (2.7)}\} \quad \text{and} \quad \mathcal{A}_\xi := \{\eta \in \mathcal{U} : \ (\xi, \eta) \in \mathcal{U} \times \mathcal{U} \text{ satisfies (2.7)}\}$$

The problem introduced partly in (2.5) and (2.6) is therefore formulated as a non-zero-sum game between two players: The government (player 1) which aims at solving

$$V_1(x; \eta) := \inf_{\xi \in \mathcal{A}_\eta} \mathcal{J}_{x,\eta}(\xi), \qquad x \in \mathbb{R}_+, \tag{2.8}$$

[2]This study goes a step further to quantify the economic cost per percentage point the debt ratio exceeds m (see also [20] for an empirical study on the effect of high debt towards private investments' crowding out and a low subsequent growth)

[3]We again highlight the fact that the legislative body discounts with a different discount rate than the government, which can be interpreted as different time preferences. Moreover, the running cost function inside the first integral is non-differentiable, which clearly destroys the link between non-zero-sum games of singular controls and optimal stopping observed in [10].

for any fixed control process $\eta \in \mathcal{U}$, and the legislative body (player 2) which aims at solving

$$V_2(x;\xi) := \inf_{\eta \in \mathcal{A}_\xi} \mathcal{I}_{x,\xi}(\eta), \qquad x \in \mathbb{R}_+, \tag{2.9}$$

for any fixed control process $\xi \in \mathcal{U}$.

Definition 2.3. *A couple $(\xi^*, \eta^*) \in \mathcal{A}$ forms a Nash equilibrium if and only if*

$$\begin{cases} \mathcal{J}_{x,\eta^*}(\xi^*) \geq \mathcal{J}_{x,\eta^*}(\xi) & \text{for any } \xi \in \mathcal{A}_{\eta^*}, \\ \mathcal{I}_{x,\xi^*}(\eta^*) \geq \mathcal{I}_{x,\xi^*}(\eta) & \text{for any } \eta \in \mathcal{A}_{\xi^*}. \end{cases}$$

Each player's value of the game is then given by $V_1(x;\eta^*) = \mathcal{J}_{x,\eta^*}(\xi^*)$ and $V_2(x;\xi^*) = \mathcal{I}_{x,\xi^*}(\eta^*)$.

The following assumptions on the model's parameters will hold true in the rest of this paper.

Assumption 2.4. *The model's parameters satisfy:*

(i) $c_1 > c_2$;
(ii) $\rho > (p(r-g) + \frac{\sigma^2}{2}p(p-1))^+$, *where p is defined in Assumption 2.1;*
(iii) $\lambda > r - g$;
(iv) $m > 0$.

The condition in Assumption 2.4.(i) is typically assumed in the literature on bounded-variation stochastic control problems in order to ensure well-posedness of the optimisation problem (see, e.g., [15], [9] and [13]) and to avoid arbitrage opportunities. In economic terms, a possible interpretation is that the Keynesian multiplier is not high enough to offset the costs of liberalisation policies.

Assumption 2.4.(ii) reflects the fact that governments are more concerned about the present than the future, since they are in power for only a limited amount of years; hence discounting future costs and benefits at a sufficiently large rate. Moreover, combining this with Assumption (2.1).(iii), the trivial policy "never intervene on the debt ratio" is admissible, since it yields a finite expected cost. The latter is guaranteed also for the problem of the legislative body due to Assumption 2.4.(iii).

In this paper, we will devote our attention to the existence of Nash equilibria of the game (2.8)–(2.9) in the class of strategies, where at least one of the players chooses a Skorokhod reflection type policy at a constant threshold. To this end, we first recall the following well known results on Skorokhod reflection.

Lemma 2.5. *Let $a, b \in \mathbb{R}_+$ with $a < b$. For any $x \in [a,b]$ there exists a unique couple $(\xi(a), \eta(b)) \in \mathcal{A}$ that solves the Skorokhod reflection problem*

$$\text{Find } (\xi,\eta) \in \mathcal{A} \text{ s.t. } \begin{cases} X_t^{\xi,\eta} \in [a,b], \ P\text{-a.s. for } t > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} > a\}} d\xi_t = 0, \ P\text{-a.s. for any } T > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} < b\}} d\eta_t = 0, \ P\text{-a.s. for any } T > 0, \end{cases} \tag{$\mathbf{SP}$(a,b;x)}$$

and it follows that $supp\{d\xi_t(a)\} \cap supp\{d\eta_t(b)\} = \emptyset$.

Lemma 2.6. *For any $\eta \in \mathcal{U}$, $a \in \mathbb{R}_+$ and $x \geq a$ there exists a unique $\xi(a) \in \mathcal{A}_\eta$ solving the Skorokhod reflection problem*

$$\text{Find } \xi \in \mathcal{A}_\eta \text{ s.t. } \begin{cases} X_t^{\xi,\eta} \geq a, \ P\text{-a.s. for } t > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} > a\}} d\xi_t = 0, \ P\text{-a.s. for any } T > 0. \end{cases} \tag{$\mathbf{SP}$(a;x)}$$

Analogously, for any $\xi \in \mathcal{U}, b \in \mathbb{R}_+$ and $x \leq b$ there exists a unique $\eta(b) \in \mathcal{A}_\xi$ solving

$$\text{Find } \eta \in \mathcal{A}_\xi \text{ s.t. } \begin{cases} X_t^{\xi,\eta} \leq b, \ P\text{-a.s. for } t > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} < b\}} d\eta_t = 0, \ P\text{-a.s. for any } T > 0. \end{cases} \tag{$\mathbf{SP}$(b;x)}$$

Moreover, we define

$$\mathcal{M} := \{(\xi,\eta) \in \mathcal{A} : \ \xi \text{ solves } \mathbf{SP}(a;x) \text{ or } \eta \text{ solves } \mathbf{SP}(b;x) \text{ for some } a,b \in \mathbb{R}_+ \text{ and } x \in \mathbb{R}_+\} \tag{2.10}$$

and aim to prove the existence and uniqueness of a Nash equilibrium $(\xi,\eta) \in \mathcal{M}$, in different parameter configurations of the game. Indeed, we will show that if at least one player acts according to a Skorokhod reflection type policy as specified above, the game (2.8)–(2.9) admits a unique Nash equilibrium. Clearly, when not restricting at least one of the players to a Skorokhod reflection type policy there could also exist Nash equilibria outside of the set $\mathcal{M}$. However, as pointed out in previous contributions such as [10], it is impossible to rank different Nash equilibria without an additional optimality criterion.

3. The Optimal Governmental Debt Management Rule

In this section, we study the problem of the government choosing their investment economic policy ξ, taking into account that the legislative body (e.g. Congress) may or may not choose to intervene on the debt ratio. In the following, we distinguish between two cases, depending on the chosen control policy of the legislative body:

$$\text{(I)} \quad \eta_t = \overline{\eta}_t := 0, \qquad\qquad \text{(II)} \quad \eta_t = \eta^b_t := \mathbb{1}_{\{t>0\}}[(x-b)^+ + \eta_t(b)]. \tag{3.1}$$

In particular, the legislative body does not intervene in Case (I), while in Case (II) it imposes a debt ceiling mechanism, which forces the government to keep its debt ratio below a fixed level $b \in \mathbb{R}_+$ (via e.g. the adoption of liberalisation policies). In the latter definition, $\eta(b)$ uniquely solves the Skorokhod reflection problem $\mathbf{SP}(b;(x \wedge b))$. In the following, we aim at determining a best response (i.e. an optimal control strategy $\xi \in \mathcal{A}_\eta$) in both cases. For simplicity of exposition, we assume the running cost function $h(x) = x^2/2$ in (2.5) (cf. Remark 2.2) in the rest of the paper, which further yields that

$$\text{Assumption 2.4.(ii) with } p = 2 \quad \Leftrightarrow \quad \rho > 2(r-g) + \sigma^2. \tag{3.2}$$

3.1. The government's optimal strategy under no legislative body intervention: Case (I).

Let us assume that the legislative body does not intervene on the government's debt. The value function (2.8) thus rewrites as

$$\overline{V}_1(x) := \inf_{\xi \in \mathcal{A}_{\overline{\eta}}} \mathsf{E}_x\Big[\int_0^\infty e^{-\rho t} h(X^{\xi,\overline{\eta}}_t)dt - c_2 \int_0^\infty e^{-\rho t} X^{\xi,\overline{\eta}}_t \circ_u d\xi_t\Big]. \tag{3.3}$$

where we let $\overline{V}_1(x) := V_1(x;\overline{\eta})$. It follows from standard theory that we can associate the value function $\overline{V}_1$ of (3.3) with a suitable solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$\min\big\{(\mathcal{L}-\rho)u(x) + \tfrac{1}{2}x^2,\ u'(x) - c_2\big\} = 0 \tag{3.4}$$

for all $x \in \mathbb{R}_+$, where the second order linear operator $\mathcal{L}$ defined by its action on functions $f \in C^2$ is defined by

$$\mathcal{L}f := \tfrac{1}{2}\sigma^2 x^2 f'' + (r-g)xf'.$$

We guess that the government chooses to increase its debt ratio only when the current level is sufficiently small. Hence, we expect that there exists a critical x-level $\overline{a}$ at which the government increases their debt ratio via a Skorokhod reflection type policy. For any $x \in \mathbb{R}_+$, we thus consider the control

$$\xi^{\overline{a}}_t := \mathbb{1}_{\{t>0\}}[(\overline{a}-x)^+ + \xi_t(\overline{a})] \tag{3.5}$$

where $\xi(\overline{a})$ is the unique solution to the Skorokhod reflection problem **SP** $(\overline{a};(x\wedge\overline{a}))$. As a consequence, we can associate the given problem (3.3) with the free-boundary problem

$$
\begin{cases}
(\mathcal{L}-\rho)u(x) \geq \frac{1}{2}x^2, & x\in\mathbb{R}_+,\\
(\mathcal{L}-\rho)u(x) = \frac{1}{2}x^2, & \bar{a}<x,\\
u'(x)\geq c_2, & x\in\mathbb{R}_+,\\
u'(x) = c_2, & 0<x\leq\bar{a},\\
u''(\bar{a})=0, & \\
\lim_{x\to+\infty}\Big(u(x)-\dfrac{x^2}{2(\rho-2(r-g)-\sigma^2)}\Big)=0, &
\end{cases}
\tag{3.6}
$$

where we impose an additional smoothness condition at the boundary $\bar{a}$ and the latter one in order to guarantee uniqueness of the solution to the free-boundary problem. We begin solving the free-boundary problem (3.6), by constructing a solution to the ordinary differential equation and imposing the boundary conditions, to obtain a *candidate* value function

$$
\overline{U}_1(x) := \begin{cases}
\overline{U}_1(\overline{a})-c_2(\overline{a}-x), & 0<x\leq\overline{a},\\
\overline{D}_1(\overline{a})x^{\delta_2}+\frac{1}{2(\rho-2(r-g)-\sigma^2)}x^2, & \overline{a}<x,
\end{cases}
\tag{3.7}
$$

where

$$
\overline{D}_1(a) := -\frac{1}{(\rho-2(r-g)-\sigma^2)\delta_2(\delta_2-1)a^{\delta_2-2}}, \quad \text{and} \quad \overline{a} := \frac{(1-\delta_2)c_2(\rho-2(r-g)-\sigma^2)}{(2-\delta_2)},
\tag{3.8}
$$

with δ_2 denoting the negative root to the equation $\frac{1}{2}\sigma^2\delta(\delta-1)+(r-g)\delta-\rho=0$.

Lemma 3.1. *The function $\overline{U}_1$ of (3.7) solves the free-boundary problem (3.6) and satisfies the HJB equation (3.4).*

Proof. In view of the construction of $\overline{U}_1$, it remains to check whether (i) $(\mathcal{L}-\rho)\overline{U}_1(x)\geq-\frac{1}{2}x^2$ for $x\in(0,\overline{a})$ as well as (ii) $\overline{U}_1'(x)\geq c_2$ for $x\geq\overline{a}$.

We firstly notice that by construction $(\mathcal{L}-\rho)\overline{U}_1(\overline{a})=-\frac{1}{2}\overline{a}^2$. We then observe that $x\mapsto(\mathcal{L}-\rho)\overline{U}_1(x)$ decreases with slope $-c_2(\rho-(r-g))$, while $x\mapsto-\frac{1}{2}x^2$ decreases with slope $-x$. To conclude (i), it is thus sufficient to prove that $c_2(\rho-(r-g))>x$ for all $x\in(0,\overline{a})$, or equivalently that $c_2(\rho-(r-g))>\overline{a}$. The latter follows straightforwardly from the definition (3.8) of $\overline{a}$ and $\delta_2<0$.

Then, we show that $x\mapsto\overline{U}_1'(x)$ is increasing for $x\geq\overline{a}$, by computing

$$
\overline{U}_1''(x)=\frac{1}{\rho-2(r-g)-\sigma^2}\Big(1-\Big(\frac{x}{\overline{a}}\Big)^{\delta_2-1}\Big)>0,
$$

where the latter inequality follows from (3.2). Hence, given that $\overline{U}_1'(\overline{a})=c_2$ by construction, we conclude that (ii) holds true, which completes the proof. □

We now prove that indeed $\overline{V}_1=\overline{U}_1$ and thus obtain an optimal debt management rule.

Theorem 3.2 (Verification Theorem: Case (I))**.** *Assume that the legislative body does not intervene on the debt ratio and thus acts according to the policy $\overline{\eta}\equiv 0$. Then, the function $\overline{U}_1$ of (3.7) coincides with the government's value function $\overline{V}_1$ in (3.3) and the admissible $\xi_t^{\overline{a}}$ of (3.5), with $\overline{a}$ given by (3.8), is optimal for problem (3.3).*

Proof. The proof is similar to the one of Theorem 3.6 below and thus omitted for brevity. □

3.2. **The government's optimal strategy under legislative body interventions: Case (II).** We begin by fixing a constant $b \in \mathbb{R}_+$ and assume that the legislative body acts according to the control policy η^b of (3.1), thus keeping the debt ratio below the debt ceiling b according to a Skorokhod reflection type policy. From the government's point of view, we thus study the problem $V_1(x;\eta^b)$ defined in (2.8) and given by[4]

$$(3.9)\quad V_1(x;b) := \inf_{\xi\in\mathcal{A}_{\eta^b}} \mathsf{E}_x\Big[\int_0^\infty e^{-\rho t} h(X_t^{\xi,\eta^b})dt + c_1\int_0^\infty e^{-\rho t} X_t^{\xi,\eta^b} \circ_d d\eta_t^b - c_2 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta^b} \circ_u d\xi_t\Big].$$

Again, we can associate the latter value function with the solution to the HJB equation

$$(3.10)\qquad \min\big\{(\mathcal{L}-\rho)u(x;b) + \tfrac{1}{2}x^2,\ u'(x;b) - c_2\big\} = 0$$

for all $x \in (0,b)$ with boundary condition $u(0;b) = 0$ and Neumann boundary condition $u'(b;b) = c_1$.

We guess that the government increases their debt ratio only when the current level is sufficiently small. Hence, we expect that for any given debt ceiling $b \in \mathbb{R}_+$, there exists a critical debt-issuance level $a(b)$ at which the government increases their debt ratio with *minimal effort*, via a Skorokhod reflection type policy, where we stress the (possible) dependency on the debt ceiling threshold $b \in \mathbb{R}_+$. For any $x \in \mathbb{R}_+$, we thus consider the control

$$(3.11)\qquad \xi_t^{a(b)} := \mathbb{1}_{\{t>0\}}[(a(b)-x)^+ + \xi_t(a(b))],$$

where $\xi(a(b))$ is the unique control such that the couple $(\xi(a(b)), \eta(b))$ solves the Skorokhod reflection problem $\mathbf{SP}(a(b), b; (x \vee a(b)) \wedge b)$. As a consequence, we can associate the given problem (3.9) with the free-boundary problem

$$(3.12)\qquad \begin{cases} (\mathcal{L}-\rho)u(x;b) \geq -\frac{1}{2}x^2, & 0<x<b,\\ (\mathcal{L}-\rho)u(x;b) = -\frac{1}{2}x^2, & a(b)<x<b,\\ u'(x;b) \geq c_2, & 0<x<b,\\ u'(x;b) = c_2, & 0<x\leq a(b),\\ u'(x;b) = c_1, & b \leq x,\\ u''(a(b);b) = 0, \end{cases}$$

where we impose an additional smoothness condition at the free boundary $a(b)$. The forthcoming analysis is dedicated to determining the optimal debt-issuance threshold $a(b)$ and proving the optimality of the control (3.11) for the original debt ratio management problem of the government (3.9), which corresponds to (2.8) with $\eta = \eta^b$ defined in (3.1).

We begin with solving the free-boundary problem (3.12) by constructing a solution to the ordinary differential equation and imposing the boundary conditions to obtain a *candidate* value function

$$(3.13)\qquad U_1(x;b) = \begin{cases} U_1(a(b);b) - c_2(a(b)-x), & 0<x\leq a(b),\\ D_1(a(b))x^{\delta_1} + D_2(a(b))x^{\delta_2} + \frac{1}{2(\rho-2(r-g)-\sigma^2)}x^2, & a(b)<x<b,\\ U_1(b;b) + c_1(x-b), & b\leq x, \end{cases}$$

with

$$D_i(a) = \frac{(\delta_{3-i}-2)a - c_2(\delta_{3-i}-1)(\rho-2(r-g)-\sigma^2)}{(-1)^{i+1}\delta_i(\delta_1-\delta_2)(\rho-2(r-g)-\sigma^2)a^{\delta_i-1}}, \qquad \text{for } i=1,2,$$

and the constants δ_1, δ_2 denoting the positive and negative roots to the equation $\frac{1}{2}\sigma^2\delta(\delta-1) + (r-g)\delta - \rho = 0$, respectively. The optimal boundary is given by the solution $a(b) \in (0,b)$ to the equation

$$(3.14)\qquad F(a(b), b) = 0,$$

[4]For ease of notation, we denote by $V_1(x;y)$ and $V_2(x;y)$, $x,y \in \mathbb{R}_+$, the control value functions $V_1(x;\eta^y)$ and $V_2(x;\xi^y)$, i.e. when their opponents choose the Skorokhod reflection type strategies η^y and ξ^y, respectively.

where we define

$$F(a,b) :=[(2-\delta_2)a - c_2(1-\delta_2)(\rho-2(r-g)-\sigma^2)]\Big(\frac{b}{a}\Big)^{\delta_1-1}$$
$$+[(\delta_1-2)a - c_2(\delta_1-1)(\rho-2(r-g)-\sigma^2)]\Big(\frac{b}{a}\Big)^{\delta_2-1} - (\delta_1-\delta_2)[b-c_1(\rho-2(r-g)-\sigma^2)]. \tag{3.15}$$

We prove the existence and uniqueness of the optimal boundary in the following lemma.

Lemma 3.3. *Let $b \in \mathbb{R}_+$ and $F(\cdot,b)$ defined by (3.15) on $(0,b)$. There exists a unique $a(b) \in (0,b)$ solving $F(a(b),b)=0$ in (3.14), that satisfies $\frac{\partial}{\partial a}F(a(b),b) > 0$ and $a(b) < \widetilde{a}$, where*

$$\widetilde{a} := \frac{c_2(\delta_1-1)(1-\delta_2)(\rho-2(r-g)-\sigma^2)}{(\delta_1-2)(2-\delta_2)} = c_2(\rho-(r-g)) > 0. \tag{3.16}$$

Proof. We straightforwardly calculate

$$\lim_{a\downarrow 0} F(a,b) = -\infty \quad \text{and} \quad F(b,b) = (\delta_1-\delta_2)(c_1-c_2)(\rho-2(r-g)-\sigma^2) > 0,$$

where the latter inequality follows from (3.2). Then, the first derivative of $F(a,b)$ with respect to a is given by

$$\frac{\partial}{\partial a}F(a,b) = a^{-1}\Big[\Big(\frac{b}{a}\Big)^{\delta_1-1} - \Big(\frac{b}{a}\Big)^{\delta_2-1}\Big]\big[c_2(\delta_1-1)(1-\delta_2)(\rho-2(r-g)-\sigma^2) - (\delta_1-2)(2-\delta_2)a\big],$$

which implies

$$\frac{\partial}{\partial a}F(a,b) = \begin{cases} >0, & 0<a<\widetilde{a}\wedge b, \\ <0, & \widetilde{a}\wedge b < a < b, \end{cases} \quad \text{and} \quad \frac{\partial}{\partial a}F(b,b) = 0, \tag{3.17}$$

for $\widetilde{a}$ defined in (3.16); note that, the positivity of $\widetilde{a}$ follows from (3.2). As a by-product from the above, F crosses zero only once on $(0,b)$ and we can further conclude that $0 < a(b) < \widetilde{a} \wedge b$ and $\frac{\partial}{\partial a}F(a(b),b) > 0$. □

In the following, we make use of the partial derivatives

$$b\frac{\partial}{\partial b}F(a,b) = (\delta_1-1)[(2-\delta_2)a - c_2(1-\delta_2)(\rho-2(r-g)-\sigma^2)]\Big(\frac{b}{a}\Big)^{\delta_1-1}$$
$$+ (\delta_2-1)[(\delta_1-2)a - c_2(\delta_1-1)(\rho-2(r-g)-\sigma^2)]\Big(\frac{b}{a}\Big)^{\delta_2-1} - (\delta_1-\delta_2)b \tag{3.18}$$

and

$$b^2\frac{\partial^2}{\partial b^2}F(a,b) = (\delta_1-1)(\delta_1-2)\big[(2-\delta_2)a - c_2(\rho-2(r-g)-\sigma^2)(1-\delta_2)\big]\Big(\frac{b}{a}\Big)^{\delta_1-1}$$
$$+ (1-\delta_2)(2-\delta_2)\big[(\delta_1-2)a - c_2(\rho-2(r-g)-\sigma^2)(\delta_1-1)\big]\Big(\frac{b}{a}\Big)^{\delta_2-1}. \tag{3.19}$$

Furthermore, given that $\frac{\partial}{\partial a}F(a(b),b) > 0$ at the value $a(b)$ which satisfies (3.14) due to Lemma 3.3, we can obtain the monotonicity of $a(b)$ on $(0,\infty)$ through

$$a'(b) = -\frac{\frac{\partial}{\partial b}F(a(b),b)}{\frac{\partial}{\partial a}F(a(b),b)} \geq 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial b}F(a(b),b) \leq 0, \tag{3.20}$$

and we state the following key result.

Proposition 3.4. *Let $a(b)$ be the unique solution to (3.14) as in Lemma 3.3. Then,*

$$a(\cdot) \text{ is} \begin{cases} \text{increasing on } (0,\widehat{b}), \\ \text{decreasing on } (\widehat{b},\infty), \end{cases} \qquad \text{where } \widehat{b} \text{ is the unique solution to } \frac{\partial}{\partial b}F(a(\widehat{b}),\widehat{b}) = 0. \tag{3.21}$$

Moreover, $b \mapsto a(b)$ is concave on the interval $(0, \widehat{b})$ and $\lim_{b \to 0} a(b) = 0$ as well as $\lim_{b \to \infty} a(b) = \bar{a}$, where $\bar{a}$ is the optimal debt-issuance threshold defined by (3.8) in Case (I) of non-intervention by a legislative body.

Proof. Fix $b \in \mathbb{R}_+$ and consider the uniquely defined $a(b) \in (0, \widetilde{a} \wedge b)$ given by the solution to (3.14) according to Lemma 3.3. Regarding the monotonicity of $a(\cdot)$ and its limiting behaviour, we distinguish two cases depending on the location of the fixed $a(b)$ relative to $\bar{a}$ defined in (3.8).

Case (i): $a(b) \leq \bar{a}$. Given that $\bar{a} < \widetilde{a}$ due to (3.2) and the definitions (3.8) of $\bar{a}$ and (3.16) of $\widetilde{a}$, we observe that

$$\frac{\partial^2}{\partial b^2} F(a(b), x) < 0, \quad \text{for all } x > a(b) \quad \Rightarrow \quad x \mapsto \frac{\partial}{\partial b} F(a(b), x) \text{ is strictly decreasing on } (a(b), \infty).$$

Since $\frac{\partial}{\partial b} F(a(b), a(b)) = 0$ due to (3.18), we have $\frac{\partial}{\partial b} F(a(b), x) < 0$ for all $x > a(b)$. Combining this with the fact that $b > a(b)$, we conclude from (3.20) that $a'(b) > 0$ for all $b \in \mathbb{R}_+$ s.t. $a(b) \leq \bar{a}$. This yields that

$$a(\cdot) \text{ is increasing on } (0, \bar{b}], \qquad \text{where } \bar{b} \text{ is such that } a(\bar{b}) = \bar{a}. \tag{3.22}$$

Also, we observe that $a'(\bar{b}) > 0$ and $a(b) > \bar{a}$ for all $b > \bar{b}$.

Case (ii): $a(b) > \bar{a}$. We firstly note from Case (i) that this is realised when $b > \bar{b}$. We observe that $\frac{\partial^2}{\partial b^2} F(a(b), x) \leq 0$ if and only if $x \leq \widetilde{x}$, where

$$\widetilde{x} := \Big(\frac{(2 - \delta_2)(1 - \delta_2)(c_2(\delta_1 - 1)(\rho - 2(r - g) - \sigma^2) - (\delta_1 - 2)a(b))a(b)^{\delta_1 - \delta_2}}{(\delta_1 - 2)(\delta_1 - 1)((2 - \delta_2)a(b) - c_2(1 - \delta_2)(\rho - 2(r - g) - \sigma^2)} \Big)^{\frac{1}{\delta_1 - \delta_2}} > a(b),$$

which is well-defined since $a(b) \in (\bar{a}, \widetilde{a})$. To show the inequality via contradiction, assume that $\widetilde{x} \leq a(b)$. Then, $\frac{\partial^2}{\partial b^2} F(a(b), x) \geq 0$ and hence $x \mapsto \frac{\partial}{\partial b} F(a(b), x)$ is increasing for all $x \geq a(b)$. But since $\frac{\partial}{\partial b} F(a(b), a(b)) = 0$, it would follow that $x \mapsto F(a(b), x)$ is increasing on $(a(b), \infty)$, which is a contradiction to $F(a(b), b) = 0$, given that $a(b) < b$. Therefore,

$$\frac{\partial^2}{\partial b^2} F(a(b), x) \begin{cases} \leq 0, & a(b) \leq x \leq \widetilde{x}, \\ \geq 0, & x \geq \widetilde{x} \end{cases} \quad \Rightarrow \quad x \mapsto \frac{\partial}{\partial b} F(a(b), x) \text{ is} \begin{cases} \text{decreasing on } (a(b), \widetilde{x}), \\ \text{increasing on } (\widetilde{x}, \infty). \end{cases}$$

Combining this with the fact that $\frac{\partial}{\partial b} F(a(b), a(b)) = 0$ and $\lim_{x \to \infty} \frac{\partial}{\partial b} F(a(b), x) = +\infty$, we conclude that $\frac{\partial}{\partial b} F(a(b), x) = 0$ admits a unique solution on $(a(b), \infty)$, denoted by $x_m(b) \in (\widetilde{x}, \infty)$. Hence,

$$x \mapsto \frac{\partial}{\partial b} F(a(b), x) \begin{cases} < 0, & x \in (a(b), x_m(b)), \\ > 0, & x \in (x_m(b), \infty) \end{cases} \quad \Rightarrow \quad x \mapsto F(a(b), x) \text{ is} \begin{cases} \text{decreasing on } (a(b), x_m(b)), \\ \text{increasing on } (x_m(b), \infty). \end{cases}$$

Given that $F(a(b), a(b)) > 0$ (see proof of Lemma 3.3) and $\lim_{x \to \infty} F(a(b), x) = +\infty$, we conclude that there exist at most two solutions to $F(a(b), x) = 0$ and due to (3.20) that $a'(b)$ changes sign once. This implies – in view of the conclusion $a'(\bar{b}) > 0$ in Case (i) – that $a(\cdot)$ is either increasing on the whole $(\bar{b}, \infty)$, or it is increasing only on $(\bar{b}, \widehat{b})$ and then decreasing on $(\widehat{b}, \infty)$, where $\widehat{b} \in (\bar{b}, \infty)$ would be satisfying

$$a'(\widehat{b}) = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial b} F(a(\widehat{b}), \widehat{b}) = 0 \quad \Leftrightarrow \quad \widehat{b} = x_m(\widehat{b}).$$

In order to show that such a $\widehat{b}$ always exists, we study the system of equations $F(\widehat{a}, \widehat{b}) = 0 = \frac{\partial}{\partial b} F(\widehat{a}, \widehat{b})$ (cf. equations above), which is equivalent to

$$\begin{cases} J_{1,2}(\widehat{a}) = J_{1,1}(\widehat{b}), \\ J_{2,2}(\widehat{a}) = J_{2,1}(\widehat{b}), \end{cases} \qquad \text{where } \; J_{i,j}(x) := \frac{(\delta_i - 2)x - c_j(\delta_i - 1)(\rho - 2(r - g) - \sigma^2)}{x^{\delta_{3-i} - 1}} \tag{3.23}$$

It can be shown (see, e.g. [13]) that the system (3.23) admits a unique solution $(\widehat{a}, \widehat{b}) \in \mathbb{R}_+^2$, where

$$\widehat{a} = a(\widehat{b}), \text{ such that } a'(\widehat{b}) = 0 \text{ and } a(\cdot) \text{ is } \begin{cases} \text{increasing on } (\bar{b}, \widehat{b}), \\ \text{decreasing on } (\widehat{b}, \infty). \end{cases} \tag{3.24}$$

The monotonicity then follows by combining (3.22) and (3.24). Furthermore, this monotonicity together with the fact that for every choice of $b \in \mathbb{R}_+$, there always exists a best response $a(b)$, due to Lemma 3.3, that satisfies (3.14)–(3.15), then yields $\lim_{b\to\infty} a(b) = \bar{a}$.

Regarding the concavity of $a(\cdot)$ on the interval $(0, \widehat{b})$ we investigate the term

$$a''(b) = \frac{2F_a(a(b), b)F_b(a(b), b)F_{ab}(a(b), b) - F_b^2(a(b), b)F_{aa}(a(b), b) - F_a^2(a(b), b)F_{bb}(a(b), b)}{F_a^3(a(b), b)},$$

where we set $F_a(a, b) := \partial_a F(a, b)$ (and the other terms analogously). We first notice that, due to Lemma 3.3, we have $F_a(a(b), b) > 0$. Direct computation yields

$$\begin{aligned}
&2F_a(a(b), b)F_b(a(b), b)F_{ab}(a(b), b) - F_b^2(a(b), b)F_{aa}(a(b), b) - F_a^2(a(b), b)F_{bb}(a(b), b) \\
&= \Big\{ \frac{1}{a(b)b}(2-\delta_2)\Big(\frac{b}{a(b)}\Big)^{\delta_1 - 1} + \frac{1}{a(b)b}(\delta_1 - 2)\Big(\frac{b}{a(b)}\Big)^{\delta_2 - 1} - (\delta_1 - \delta_2)\frac{1}{a(b)^2} \Big\} \\
&\quad \times \Big\{ (\delta_1 - \delta_2)(\rho - 2(r-g) - \sigma^2)\big[c_2(\delta_1 - 1)(1-\delta_2)(\rho - 2(r-g) - \sigma^2) - (\delta_1 - 2)(2-\delta_2)a(b)\big] \\
&\qquad \times \frac{1}{b}\Big[c_2(\delta_1 - \delta_2)\Big(\frac{b}{a(b)}\Big)^{\delta_1 + \delta_2 - 2} - c_1(\delta_1 - 1)\Big(\frac{b}{a(b)}\Big)^{\delta_1 - 1} - c_1(1-\delta_2)\Big(\frac{b}{a(b)}\Big)^{\delta_2 - 1}\Big] \\
&\qquad + c_2(1-\delta_2)(\delta_1 - 1)(\rho - 2(r-g) - \sigma^2)\Big[\Big(\frac{b}{a(b)}\Big)^{\delta_1 - 1} - \Big(\frac{b}{a(b)}\Big)^{\delta_2 - 1}\Big]F_b(a(b), b)\Big\}
\end{aligned}$$

Some straightforward calculations reveal that the first term on the above right-hand side (second line) is strictly positive, while the term in the third line is strictly positive due to $a(b) \le \widetilde{a}$. Moreover, the term in the fourth line is strictly negative, which easily follows upon using $a(b) \le b$. Finally, we notice that for $b \le \widehat{b}$, we have $F_b(a(b), b) < 0$ (see (3.24)). Combining these facts, we conclude that indeed $a''(b) < 0$ for $b \in (0, \widehat{b})$ and the claim follows. □

Before we present the optimality of the controls, we first notice that the control policy $\xi^{a(b)}$ as in (3.11), combined with the policy η^b of (3.1), is indeed admissible. Clearly, the couple solves $\mathbf{SP}(a(b), b; x)$ and as such belongs to $\mathcal{A}$. Indeed, by arguing as in Lemma 4.1 in [21] one can easily show (2.7) and moreover, $\mathsf{P}_x(\Delta\xi^{a(b)} \cdot \Delta\eta^b > 0) = 0$ for all $t \ge 0$, by construction. Then, we have the following result for the associated candidate value function.

Proposition 3.5. *The function $U_1(x; b)$ of (3.13) solves the free-boundary problem (3.12), with $a(b)$ solving (3.14) as in Lemma 3.3, and satisfies the HJB equation (3.10).*

Proof. By construction, we have $(\mathcal{L}-\rho)U_1(x; b) = -\frac{1}{2}x^2$ for $x \in (a(b), b)$, $U_1'(x; b) = c_2$ for $x \in (0, a(b))$ and $U_1'(x; b) = c_1 > c_2$ for $x \ge a(b)$. Therefore, it remains to show that: (i) $(\mathcal{L} - \rho)U_1(x; b) \ge -\frac{1}{2}x^2$, for $x \in (0, a(b))$ and (ii) $U_1'(x; b) \ge c_2$, for $x \in (a(b), b)$. In the following, we fix $b \in \mathbb{R}_+$, so that $a(b)$ is the (fixed) unique solution to (3.14) according to Lemma 3.3.

For $x \in (0, a(b))$, we get

$$(\mathcal{L} - \rho)U_1(x; b) = (r-g)xc_2 - \rho U_1(a(b); b) + \rho c_2(a(b) - x).$$

Clearly, $x \mapsto (\mathcal{L} - \rho)U_1(x; b)$ decreases with slope $-c_2(\rho - (r-g))$, while $x \mapsto -\frac{1}{2}x^2$ decreases with slope $-x$. Since $(\mathcal{L} - \rho)U_1(a(b); b) = -\frac{1}{2}a(b)^2$, to prove (i), it is sufficient to then show that $c_2(\rho - (r-g)) > x$, for all $x \in (0, a(b))$. The latter is true due to (3.16), thus (i) holds true.

For $x \in (a(b), b)$, we can calculate

$$U_1'(x;b) = \frac{(\delta_2-2)a(b) - c_2(\delta_2-1)(\rho-2(r-g)-\sigma^2)}{(\delta_1-\delta_2)(\rho-2(r-g)-\sigma^2)}\Big(\frac{x}{a(b)}\Big)^{\delta_1-1} - \frac{(\delta_1-2)a(b) - c_2(\delta_1-1)(\rho-2(r-g)-\sigma^2)}{(\delta_1-\delta_2)(\rho-2(r-g)-\sigma^2)}\Big(\frac{x}{a(b)}\Big)^{\delta_2-1} + \frac{x}{\rho-2(r-g)-\sigma^2}.$$

Combining this with the definition (3.15) of F, we notice that

$$U_1'(x;b) \geq c_2 \quad \Leftrightarrow \quad F(a(b),x) \leq (c_1-c_2)(\delta_1-\delta_2)(\rho-2(r-g)-\sigma^2).$$

To prove (ii), given that

$$F(a(b),a(b)) = (c_1-c_2)(\delta_1-\delta_2)(\rho-2(r-g)-\sigma^2) \qquad \text{and} \qquad F(a(b),b)=0,$$

it is sufficient to have that $x \mapsto F(a(b),x)$ first decreases and changes sign at most once in $(a(b),b)$. The latter was shown in the proof of Lemma 3.4, thus (ii) holds true, and this completes the proof. □

We can now prove that indeed $V_1 = U_1$ and provide an optimal debt management rule.

Theorem 3.6 (Verification Theorem: Case (II))**.** *Assume that the legislative body acts according to the control policy η^b of (3.1). Then, the function U_1 of (3.13) coincides with the government's value function V_1 in (3.9) and the admissible $\xi^{a(b)}$ of (3.11) is optimal for problem (3.9).*

Proof. **Step 1.** Let $x \in \mathbb{R}_+$ and $\xi \in \mathcal{A}_{\eta^b}$ such that $(\xi,\eta^b) \in \mathcal{A}$. For $n \geq 1$, we let $\tau_n := \inf\{t \geq 0 : X_t^{0,\eta^b} \geq n\}$. Since $U_1 \in C^2(0,b)$, we can apply Itô-Meyer formula to the process $e^{-\rho\tau_n}U_1(X_{\tau_n}^{\xi,\eta^b};b)$ on $[0,\tau_n]$ and obtain

$$\begin{aligned} e^{-\rho\tau_n}U_1(X_{\tau_n}^{\xi,\eta^b};b) - U_1(x;b) &= \int_0^{\tau_n} e^{-\rho s}(\mathcal{L}-\rho)U_1(X_s^{\xi,\eta^b};b)ds + \sigma\int_0^{\tau_n} e^{-\rho s}U_1'(X_s^{\xi,\eta^b};b)dW_s \\ &\quad - \int_0^{\tau_n} e^{-\rho s}X_s^{\xi,\eta^b}U_1'(X_s^{\xi,\eta^b};b)d\eta_s^{c,b} + \int_0^{\tau_n} e^{-\rho s}X_s^{\xi,\eta^b}U_1'(X_s^{\xi,\eta^b};b)d\xi_s^c \\ &\quad + \sum_{s\leq\tau_n} e^{-\rho s}\big(U_1(X_s^{\xi,\eta^b};b) - U_1(X_{s-}^{\xi,\eta^b};b)\big). \end{aligned} \tag{3.25}$$

Clearly, for any $s \in (0,\tau_n]$ we have $0 < X_s^{\xi,\eta^b} < n \wedge b$ and thus continuity of U_1' implies that the second term in (3.25) is a martingale. Furthermore, since $(\xi,\eta^b) \in \mathcal{A}$, the last term in (3.25) rewrites as

$$\sum_{s<\tau_n} e^{-\rho s}\big(U_1(X_s^{\xi,\eta^b};b) - U_1(X_{s-}^{\xi,\eta^b};b)\big) = \sum_{s<\tau_n} e^{-\rho s}\big(U_1(X_s^{\xi,\eta^b};b) - U_1(X_{s-}^{\xi,\eta^b};b)\big)\big[\mathbb{1}_{\{\Delta\xi_s>0\}} + \mathbb{1}_{\{\Delta\eta_s^b>0\}}\big]$$

and

$$\begin{aligned} \sum_{s<\tau_n}\big(U_1(X_s^{\xi,\eta^b};b) - U_1(X_{s-}^{\xi,\eta^b};b)\big)\mathbb{1}_{\{\Delta\xi_s>0\}} &= \sum_{s<\tau_n} e^{-\rho s}\int_0^{\Delta\xi_s} e^u X_{s-}^{\xi,\eta^b}U_1'(e^u X_{s-}^{\xi,\eta^b};b)du \\ \sum_{s<\tau_n}\big(U_1(X_s^{\xi,\eta^b};b) - U_1(X_{s-}^{\xi,\eta^b};b)\big)\mathbb{1}_{\{\Delta\eta_s^b>0\}} &= -\sum_{s<\tau_n} e^{-\rho s}\int_0^{\Delta\eta_s^b} e^{-u} X_{s-}^{\xi,\eta^b}U_1'(e^{-u} X_{s-}^{\xi,\eta^b};b)du \end{aligned} \tag{3.26}$$

Taking expectations, rearranging terms and using the notation introduced in (2.3) we can thus write (3.25) as

$$\begin{aligned} U_1(x;b) &= \mathsf{E}_x\Big[e^{-\rho\tau_n}U_1(X^{\xi,\eta^b}_{\tau_n};b) - \int_0^{\tau_n} e^{-\rho s}(\mathcal{L}-\rho)U_1(X^{\xi,\eta^b}_s;b)ds + \int_0^{\tau_n} e^{-\rho s}X^{\xi,\eta^b}_s U_1'(X^{\xi,\eta^b}_s;b)\circ_d d\eta^b_s \\ &\qquad - \int_0^{\tau_n} e^{-\rho s}X^{\xi,\eta^b}_s U_1'(X^{\xi,\eta^b}_s;b)\circ_u d\xi_s\Big] \\ &\le \mathsf{E}_x\Big[e^{-\rho\tau_n}U_1(X^{\xi,\eta^b}_{\tau_n};b) + \int_0^{\tau_n} e^{-\rho s}h(X^{\xi,\eta^b}_s)ds + c_1\int_0^{\tau_n} e^{-\rho s}X^{\xi,\eta^b}_s \circ_u d\eta^b_s \\ &\qquad - c_2\int_0^{\tau_n} e^{-\rho s}X^{\xi,\eta^b}_s \circ_u d\xi_s\Big], \end{aligned} \tag{3.27}$$

where, in the latter inequality, we exploit the fact that U_1 solves the free-boundary problem (3.12), as stated in Proposition 3.5. By admissibility of ξ we have that the right-hand side of (3.27) is finite P-a.s., and we notice that Assumption 2.4.(ii) guarantees

$$\lim_{n\to\infty}\mathsf{E}_x\big[e^{-\rho\tau_n}U_1(X^{\xi,\eta^b}_{\tau_n};b)\big] = 0.$$

Then, noticing that $\tau_n\uparrow\infty$, P-a.s., and taking limits in (3.27), we can use Assumptions 2.1.(iii) and 2.4.(ii), employ the dominated convergence theorem, and conclude that

$$U_1(x;b) \le \mathsf{E}_x\Big[\int_0^\infty e^{-\rho s}h(X^{\xi,\eta^b}_s)ds + c_1\int_0^\infty e^{-\rho s}X^{\xi,\eta^b}_s\circ_d d\eta^b_s - c_2\int_0^\infty e^{-\rho s}X^{\xi,\eta^b}_s\circ_u d\xi_s\Big].$$

Since $\xi\in\mathcal{A}_{\eta^b}$ was arbitrary, we have $U_1(x;b)\le V_1(x;b)$ on $\mathbb{R}_+$.

Step 2. We can repeat the above arguments from Step 1, but now fix the control strategy $\xi^{a(b)}$ of (3.11). Since $X^{\xi^{a(b)},\eta^b}_t\in[a(b),b]$, P-a.s., for all $t>0$, and $U_1(X^{\xi^{a(b)},\eta^b}_t;b)=c_2$ on $\mathrm{supp}\{d\xi^{a(b)}_t\}$, the inequality in (3.27) becomes an equality and hence, employing dominated convergence arguments as before, we observe

$$\begin{aligned} U_1(x;b) &= \mathsf{E}_x\Big[\int_0^\infty e^{-\rho s}h(X^{\xi^{a(b)},\eta^b}_s)ds + c_1\int_0^\infty e^{-\rho s}X^{\xi^{a(b)},\eta^b}_s\circ_d d\eta^b_s - c_2\int_0^\infty e^{-\rho s}X^{\xi^{a(b)},\eta^b}_s\circ_u d\xi^{a(b)}_s\Big] \\ &\ge V_1(x;b). \end{aligned}$$

Step 3. Combining the results from Steps 1 and 2 then concludes that $U_1(x;b)=V_1(x;b)$ and $\xi^{a(b)}$ of (3.11) is an optimal control strategy for problem (3.9). □

4. The optimal debt ceiling

In this section, we study the control problem of the legislative body. As seen in Section 3, the best response of the government to either a legislative body non-intervention policy, or a debt ceiling mechanism (threshold-type policy) is given by a debt-issuance threshold-type policy. We now reverse the roles and assume that the government chooses to increase its debt ratio at a certain level $a\in\mathbb{R}_+$, i.e. to the debt-issuance control policy

$$\xi^a_t := \mathbb{1}_{\{t>0\}}[(a-x)^+ + \xi_t(a)], \tag{4.1}$$

where $\xi(a)$ uniquely solves the Skorokhod reflection problem $\mathbf{SP}(a;x\vee a)$. In the following, for any such level a, we study the problem (2.9) of finding a best response (i.e. an optimal control strategy $\eta\in\mathcal{A}_{\xi^a}$). We thus consider the problem[4]

$$V_2(x;a) := \inf_{\eta\in\mathcal{A}_{\xi^a}}\mathsf{E}_x\Big[\int_0^\infty e^{-\lambda t}\alpha(X^{\xi^a,\eta}_t - m)^+dt + \kappa\int_0^\infty e^{-\lambda t}X^{\xi^a,\eta}_t\circ_d d\eta_t\Big]. \tag{4.2}$$

Via standard arguments, we can associate the value function V_2 of (4.2) with a suitable solution to the HJB equation

$$\min\left\{(\mathcal{L}-\lambda)u(x;a)+\alpha(x-m)^+, \kappa - u'(x;a)\right\}=0, \tag{4.3}$$

for all $x \in (a,\infty)$ with Neumann boundary condition $u'(a;a)=0$. We presume that the legislative body may only decrease the debt ratio when the current level is sufficiently large. Therefore, if the legislative body chooses to intervene, we expect that for any given debt-issuance threshold $a \in \mathbb{R}_+$, there exists a critical debt ceiling level $b(a)$ at which the legislative body forces a decrease in the debt ratio via a Skorokhod reflection type policy, where we stress the (possible) dependency on the debt-issuance threshold $a \in \mathbb{R}_+$. On the other hand, also a non-intervention policy is conceivable. As it turns out, it is crucial in our analysis to distinguish two different cases, depending on the legislative body's time preference rate λ:

$$\text{(I)}\;\; \lambda > r-g+\frac{\alpha}{\kappa}, \qquad\qquad \text{(II)}\;\; \lambda < r-g+\frac{\alpha}{\kappa}.$$

In the forthcoming Sections 4.1 and 4.2 we study these cases separately, providing an optimal control strategy by the legislative body for each one of them.

4.1. **The legislative body's optimal strategy under high time preference rate: Case (I).**

Notice that the legislative body discounts future events with a relatively large discount factor in this case, and it is therefore appropriate to assume that the legislative body disregards the risk of future government insolvency at a greater extent compared to Case (II). We verify this intuition by showing that indeed, the optimal control policy of the legislative body prescribes *not to intervene* on the debt ratio at all.

To this end, we prove that the value function V_2 of (4.2) coincides with a suitable solution to a fixed-boundary problem

$$\begin{cases}(\mathcal{L}-\lambda)u(x;a) = -\alpha(x-m)^+, & a<x,\\ u'(x;a)<\kappa, & a<x,\\ u'(x;a)=0, & 0<x\le a.\end{cases} \tag{4.4}$$

We can solve the latter problem by constructing a solution to the ordinary differential equation and imposing the stated boundary condition, which lead to the *candidate* value function

$$\overline{U}_2(x;a) = \begin{cases}\overline{U}_2(a;a), & 0<x\le a;\\ \overline{D}_2(a)x^{\theta_2}+H(x), & a<x,\end{cases} \tag{4.5}$$

where

$$\begin{aligned}\overline{D}_2(a) &:= -\frac{\alpha}{\theta_2}a^{1-\theta_2}\int_0^\infty e^{-(\lambda-(r-g))t}\Phi(d_1(a,t))dt,\\ H(x) &:= \alpha\int_0^\infty \Big(xe^{-(\lambda-(r-g))t}\Phi(d_1(x,t)) - me^{-\lambda t}\Phi(d_2(x,t))\Big)dt,\\ d_1(x,t) &:= \frac{\log\left(\frac{x}{m}\right)+(r-g+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \quad \text{and} \quad d_2(x,t):=d_1(x,t)-\sigma\sqrt{t},\end{aligned}$$

with θ_2 denoting the negative root to the equation $\frac{1}{2}\sigma^2\theta(\theta-1)+(r-g)\theta-\lambda=0$ and $\Phi(\cdot)$ denoting the cumulative distribution function of a standard normal random variable.

Lemma 4.1. *The function $\overline{U}_2$ of (4.5) solves the free-boundary problem (4.4) and satisfies the HJB equation (4.3).*

Proof. In view of the construction of $\overline{U}_2$, it remains to check that $\frac{\partial}{\partial x}\overline{U}_2 < \kappa$, for all $x > a$. Straightforward calculations lead to

$$\frac{\partial}{\partial x}\overline{U}_2(x;a) = \alpha \int_0^{\infty} e^{-(\lambda-(r-g))t}\Big(\Phi(d_1(x,t)) - \Big(\frac{x}{a}\Big)^{\theta_2-1}\Phi(d_1(a,t))\Big)dt < \kappa,$$

where the inequality follows from the facts that $x > a$ and $\lambda > r - g + \frac{\alpha}{\kappa} > r - g$ under Case (I). Furthermore, Assumption 2.4.(iii) guarantees that $\frac{\partial}{\partial x}\overline{U}_2(x;a) \geq 0$, for all $x > a$. □

We now prove that indeed $V_2 = \overline{U}_2$ and that not intervening is an optimal debt management strategy.

Theorem 4.2 (Verification Theorem: Case (I))**.** *Assume that the government acts according to the control policy ξ^a of (4.1). Then, the function $\overline{U}_2$ of (4.5) coincides with the value function V_2 of (4.2) and the optimal policy for the legislative body prescribes not to act on the debt ratio, i.e. $\overline{\eta} := 0$.*

Proof. The proof follows the lines of the proof of Theorem 4.8 and is thus skipped for brevity. □

Remark 4.3. *Notice that the strategy of the legislative body not to intervene on the debt ratio is not triggered by some specific $a \in \mathbb{R}_+$. This comes solely from the fact that $\lambda > r - g + \frac{\alpha}{\kappa}$, independently of the governmental choice of a debt-issuance level $a \in \mathbb{R}_+$. On the other hand, the best response of the government to a legislative body non-intervention policy $\bar{\eta}$ from (3.1), has been treated in Section 3.1. In particular, the optimal control for problem $V_1(x;\overline{\eta})$ of (2.8) (cf. $\overline{V}_1(x)$ in (3.3)) is given by the Skorokhod reflection type policy $\xi^{\overline{a}}$ as in (3.11), with $\overline{a}$ defined in (3.8). Indeed, we prove that this pair of strategies leads to an equilibrium in Section 5.*

4.2. **The legislative body's optimal strategy under low time preference rate: Case (II).** While it is optimal for the legislative body to never intervene in Case (I), we show that in this case the best response to a governmental policy (4.1) requires intervention. Indeed, for any given governmental debt-issuance threshold $a \in \mathbb{R}_+$, this will prescribe keeping the debt ratio below a certain debt ceiling $b(a)$ with *minimal effort*, via a Skorokhod reflection type policy, where we stress the (possible) dependency on the debt-issuance threshold $a \in \mathbb{R}_+$. For any $x \in \mathbb{R}_+$, we thus consider the control

$$\eta_t^{b(a)} := \mathbb{1}_{\{t>0\}}[(x - b(a))^+ + \eta_t(b(a))] \tag{4.6}$$

where $\eta(b(a))$ is the unique control such that the couple $(\xi(a), \eta(b(a)))$ solves the Skorokhod reflection problem $\mathbf{SP}(a, b(a); (x \vee a) \wedge b(a))$. As a consequence, we can associate the given problem (4.2) with the free-boundary problem

$$\begin{cases} (\mathcal{L}-\lambda)u(x;a) \geq -\alpha(x-m)^+, & a < x, \\ (\mathcal{L}-\lambda)u(x;a) = -\alpha(x-m)^+, & a < x < b(a), \\ u'(x;a) \leq \kappa, & a < x, \\ u'(x;a) = \kappa, & b(a) \leq x, \\ u'(x;a) = 0, & 0 < x \leq a, \\ u''(b(a);a) = 0, \end{cases} \tag{4.7}$$

where we imposed an additional smoothness condition at the free boundary $b(a)$. The forthcoming analysis is dedicated to determining the optimal threshold $b(a)$ and proving the optimality of the control (4.6) for the original debt ratio management problem of the government (4.2), which corresponds to (2.9) with $\xi = \xi^a$ defined in (4.1).

We begin with solving the free-boundary problem (4.7) by constructing a solution to the ordinary differential equation and imposing the boundary conditions to obtain a *candidate* value function

$$U_2(x;a) := \begin{cases} U_2(a;a), & 0 < x \le a, \\ D_3(b(a))x^{\theta_1} + D_4(b(a))x^{\theta_2} + H(x), & a < x < b(a), \\ U_2(b(a));a) + \kappa(x - b(a)), & b(a) \le x, \end{cases} \tag{4.8}$$

where

$$D_i(b) := \frac{b^{1-\theta_{i-2}}}{\theta_{i-2}(\theta_2-\theta_1)}\Big[(\theta_{5-i}-1)\Big(k - \alpha\int_0^\infty e^{-(\lambda-(r-g))t}\Phi(d_1(b,t))dt\Big) + \alpha\int_0^\infty e^{-(\lambda-(r-g))t}\frac{1}{\sqrt{2\pi t}\sigma}e^{-\frac{1}{2}d_1(b,t)^2}dt\Big], \qquad i = 3,4,$$

and the constants θ_1, θ_2 are given by the positive and negative roots to the equation $\frac{1}{2}\sigma^2\theta(\theta-1) + (r-g)\theta - \lambda = 0$, respectively. The optimal boundary is given by the solution $b(a) \in (a,\infty)$ to the equation

$$G(a, b(a)) = 0, \tag{4.9}$$

where $G(a,\cdot)$ is defined on (a,∞) by

$$\begin{aligned} G(a,b) := &\Big[\Big(\frac{b}{a}\Big)^{1-\theta_2} - \Big(\frac{b}{a}\Big)^{1-\theta_1}\Big]\Big[\frac{\alpha}{\theta_1-\theta_2}\int_0^\infty e^{-(\lambda-(r-g))t}\frac{1}{\sqrt{2\pi t}\sigma}e^{-\frac{d_1(b,t)^2}{2}}dt\Big] \\ &+ \Big[(\theta_1-1)\Big(\frac{b}{a}\Big)^{1-\theta_2} - (\theta_2-1)\Big(\frac{b}{a}\Big)^{1-\theta_1}\Big]\Big[\frac{k}{\theta_1-\theta_2} - \frac{\alpha}{\theta_1-\theta_2}\int_0^\infty e^{-(\lambda-(r-g))t}\Phi(d_1(b,t))dt\Big] \\ &+ \alpha\int_0^\infty e^{-(\lambda-(r-g))t}\Phi(d_1(a,t))dt. \end{aligned}$$

After lengthy but straightforward calculations we get

$$\begin{aligned} G(a,b) = &\Big[(\theta_1-1)\Big(\frac{b}{a}\Big)^{1-\theta_2} + (1-\theta_2)\Big(\frac{b}{a}\Big)^{1-\theta_1}\Big]\Big(\frac{\kappa}{\theta_1-\theta_2} - \frac{\alpha}{(\theta_1-\theta_2)(\lambda-(r-g))}\mathbb{1}_{\{b\ge m\}}\Big) \\ &+ \frac{\alpha}{(\theta_1-\theta_2)(\lambda-(r-g))}\Big[(1-\theta_2)\Big(\frac{m}{a}\Big)^{1-\theta_1}\mathbb{1}_{\{b>m>a\}} + (\theta_1-1)\Big(\frac{m}{a}\Big)^{1-\theta_2}\mathbb{1}_{\{b>m>a\}}\Big] \\ &+ \frac{\alpha}{(\lambda-(r-g)}\mathbb{1}_{\{a\ge m\}}. \end{aligned} \tag{4.10}$$

In addition, we can derive the following expression for

$$\frac{\partial}{\partial b}G(a,b) = \Big[\Big(\frac{b}{a}\Big)^{1-\theta_2} - \Big(\frac{b}{a}\Big)^{1-\theta_1}\Big]\frac{(\theta_1-1)(1-\theta_2)}{(\theta_1-\theta_2)(\lambda-(r-g))b}\big[\kappa(\lambda-(r-g)) - \alpha\mathbb{1}_{\{b>m\}}\big], \tag{4.11}$$

which will be crucial for obtaining the existence and uniqueness of the optimal boundary.

Lemma 4.4. *Let $a \in \mathbb{R}_+$ and $G(a,\cdot)$ defined by (4.10) on (a,∞). There exists a unique $b(a) \in (a,\infty)$ solving $G(a,b(a)) = 0$ in (4.9), that satisfies $\frac{\partial}{\partial b}G(a,b(a)) < 0$ and $b(a) > m$.*

Proof. We conclude from the representation of G in (4.10) that

$$G(a,a) = k > 0, \qquad \text{and} \qquad \lim_{b\to\infty} G(a,b) = -\infty,$$

where the latter follows precisely from the fact that $\lambda < r - g + \frac{\alpha}{\kappa}$ in Case (II). Hence, there exists a solution to the equation (4.9). Moreover, it follows from (4.11) that

$$\frac{\partial}{\partial b}G(a,b) = \begin{cases} > 0, & a < b < a \vee m, \\ < 0, & a \vee m < b. \end{cases}$$

Therefore, for any $a \in \mathbb{R}_+$, there exists a unique $b(a) \in (a \vee m, \infty)$ such that $G(a,b(a)) = 0$. Moreover, due to (4.11), we conclude that $\frac{\partial}{\partial b}G(a,b(a)) < 0$. □

In what follows, we aim at deriving the monotonicity properties of the boundary function $a \mapsto b(a)$. To that end, we first present a useful lower bound.

Lemma 4.5. *Let $a \in \mathbb{R}_+$. Then*

$$b(a) \geq b_0 := \Big(\frac{\alpha}{\alpha - \kappa(\lambda - (r-g))}\Big)^{\frac{1}{1-\theta_2}} m > m. \tag{4.12}$$

Proof. Let b_0 be defined as in (4.12). Fix $a \in \mathbb{R}_+$ and recall that there exists a unique (fixed) $b(a) > a \vee m$, for any $a \in \mathbb{R}_+$, due to Lemma 4.4. To show (4.12), we examine two cases of a-values.

Case (i): $a \geq b_0$. In this case, we immediately have $b(a) > b_0$.

Case (ii): $a < b_0$. Notice that simple comparison arguments yield that $b_0 > m$ and can be shown that $G(a, b_0) \geq 0$. We then assume (aiming for contradiction) that $b(a) \in (a \vee m, b_0)$. Since $b \mapsto G(a, b)$ is strictly decreasing for $b > a \vee m$ (see proof of Lemma 4.4), it follows that

$$G(a, b(a)) > G(a, b_0) \geq 0,$$

which is a contradiction due to Lemma 4.4, thus $b(a) \geq b_0$. □

We are now in position to obtain the monotonicity of $b(\cdot)$ on $(0,\infty)$. Given that $\frac{\partial}{\partial b}G(a, b(a)) < 0$ at the value $b(a)$ which satisfies (4.9), due to Lemma 4.4, we have that

$$b'(a) = -\frac{\frac{\partial}{\partial a}G(a, b(a))}{\frac{\partial}{\partial b}G(a, b(a))} > 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial a}G(a, b(a)) > 0, \tag{4.13}$$

and we state the following key result.

Proposition 4.6. *Let $b(a)$ be the unique solution to (4.9) as in Lemma 4.4. The function $a \mapsto b(a)$ is strictly increasing on $\mathbb{R}_+$. In particular, it takes a linear form $b(a) = (1/\widetilde{q})a$, for all $a > m$, where $\widetilde{q} \in (0, 1)$ is given by the solution to*

$$(1-\theta_2)(\kappa(\lambda-(r-g))-\alpha)\widetilde{q}^{\theta_1-1} + (\theta_1-1)(\kappa(\lambda-(r-g))-\alpha)\widetilde{q}^{\theta_2-1} + \alpha(\theta_1-\theta_2) = 0. \tag{4.14}$$

Moreover, $a \mapsto b(a)$ is convex on the interval $(0, m)$, $\lim_{a\to 0} b(a) = b_0$, where b_0 is given by (4.12), and $\lim_{a\to\infty} b(a) = \infty$.

Proof. In order to obtain the desired results, we distinguish two cases.

Case (i). If $a > m$, then we notice that

$$\frac{\partial}{\partial a}G(a, b(a)) = \frac{(\theta_1-1)(1-\theta_2)(k(\lambda-(r-g))-\alpha)}{(\theta_1-\theta_2)(\lambda-(r-g))a}\Big[\Big(\frac{b}{a}\Big)^{1-\theta_1} - \Big(\frac{b}{a}\Big)^{1-\theta_2}\Big] > 0. \tag{4.15}$$

This implies, thanks to (4.13), that

$$a \mapsto b(a) \quad \text{is increasing on} \quad (m, \infty). \tag{4.16}$$

Furthermore, combining the expressions of the partial derivatives in (4.11) and (4.15) with the expression of $b'(\cdot)$ in (4.13) yields $b'(a) = \frac{b(a)}{a}$, which further implies that $b(a) = (1/\widetilde{q})a$, for some $\widetilde{q} \in (0, 1)$. The latter can be specified as the unique equation to the solution $G(\widetilde{q}, 1) = 0$, which is equivalent to (4.14).

Case (ii). If $a \leq m$, then we notice that

$$\begin{aligned}
&\frac{\partial}{\partial a}G(a, b(a)) \\
&\quad = \frac{2(\alpha-\kappa(\lambda-(r-g))}{(\theta_1-\theta_2)a\sigma^2}\Big[\Big(\frac{b(a)}{a}\Big)^{1-\theta_2} - \Big(\frac{b(a)}{a}\Big)^{1-\theta_1}\Big] + \frac{2\alpha}{(\theta_1-\theta_2)\sigma^2 a}\Big[\Big(\frac{m}{a}\Big)^{1-\theta_1} - \Big(\frac{m}{a}\Big)^{1-\theta_2}\Big] \\
&\quad \geq \frac{2(\alpha-\kappa(\lambda-(r-g))}{(\theta_1-\theta_2)a\sigma^2}\Big[\Big(\frac{b_0}{a}\Big)^{1-\theta_2} - \Big(\frac{b_0}{a}\Big)^{1-\theta_1}\Big] + \frac{2\alpha}{(\theta_1-\theta_2)\sigma^2 a}\Big[\Big(\frac{m}{a}\Big)^{1-\theta_1} - \Big(\frac{m}{a}\Big)^{1-\theta_2}\Big] \\
&\quad = \Big[\frac{2\alpha}{(\theta_1-\theta_2)\sigma^2 a} - \frac{2(\alpha-\kappa(\lambda-(r-g))b_0^{1-\theta_1}}{(\theta_1-\theta_2)\sigma^2 a}\Big] \geq 0,
\end{aligned}$$

where the first inequality follows from Lemma 4.5 and the second one from the definition (4.12) of b_0. This implies, thanks to (4.13), that

$$a \mapsto b(a) \quad \text{is increasing on} \quad (0,m]. \tag{4.17}$$

Regarding the convexity of $b(\cdot)$ on this interval, we examine the term

$$b''(a) = \frac{2G_a(a,b(a))G_b(a,b(a))G_{ab}(a,b(a)) - G_b^2(a,b(a))G_{aa}(a,b(a)) - G_a^2(a,b(a))G_{bb}(a,b(a))}{G_b^3(a,b(a))},$$

where $G_a(a,b) := \partial_a G(a,b)$ (and the other terms analogously). We first notice that, due to Lemma 4.4, we have $G_b(a,b(a)) < 0$. Upon using (4.10) and some direct calculation, we find

$$\begin{aligned}
&2G_a(a,b(a))G_b(a,b(a))G_{ab}(a,b(a)) - G_b^2(a,b(a))G_{aa}(a,b(a)) - G_a^2(a,b(a))G_{bb}(a,b(a)) \\
&= \frac{(\theta_1-1)^3(1-\theta_2)^3(\alpha-\kappa(\lambda-(r-g))\alpha}{(\lambda-(r-g))^3(\theta_1-\theta_2)^3a^2b(a)^2} \\
&\quad \times \Big\{(\alpha-\kappa(\lambda-(r-g))\Big[\Big(\frac{b(a)}{a}\Big)^{1-\theta_1} - \Big(\frac{b(a)}{a}\Big)^{1-\theta_2}\Big]^2\Big[\theta_2\Big(\frac{m}{a}\Big)^{1-\theta_2} - \theta_1\Big(\frac{m}{a}\Big)^{1-\theta_1}\Big] \\
&\qquad + \alpha\Big[\Big(\frac{m}{a}\Big)^{1-\theta_2} - \Big(\frac{m}{a}\Big)^{1-\theta_1}\Big]^2\Big[\theta_1\Big(\frac{b(a)}{a}\Big)^{1-\theta_1} - \theta_2\Big(\frac{b(a)}{a}\Big)^{1-\theta_2}\Big]\Big\}
\end{aligned}$$

While the first term on the above right-hand side (second line) is clearly positive, one can employ the fact that $b(a) > b_0 > m$ for all $a > 0$, with b_0 as in (4.12), to show that the second term is strictly negative. Consequently, we obtain $b''(a) > 0$ and thus the strict convexity of $b(\cdot)$ on $(0,m)$.

Moreover, we straightforwardly calculate $\lim_{a\downarrow 0} G(a,b_0) = 0$, which due to Lemma 4.4 and the monotonicity of $b(a)$ implies $\lim_{a\downarrow 0} b(a) = b_0$. □

Before we present the optimality of the controls, we first notice that the control policy $\eta^{b(a)}$ as in (4.6), combined with the policy ξ^a of (4.1), is indeed admissible. Clearly, the couple solves $\mathbf{SP}(a,b(a);x)$ and as such belongs to $\mathcal{A}$. Indeed, by arguing as in Lemma 4.1 in [21] one can easily show (2.7) and moreover, $\mathsf{P}_x(\Delta\xi^a \cdot \Delta\eta^{b(a)} > 0) = 0$ for all $t \geq 0$, by construction. Then, we have the following result for the associated candidate value function.

Proposition 4.7. *The function $U_2(x;a)$ of (4.8) solves the free-boundary problem (4.7), with $b(a)$ solving (4.9) as in Lemma 4.4, and satisfies the HJB equation (4.3).*

Proof. By construction, we have $(\mathcal{L}-\lambda)U_2(x;a) + \alpha(x-m)^+ = 0$ for $x \in (a,b(a))$ as well as $\frac{\partial}{\partial x}U_2(x;a) = \kappa$ for $x \geq b(a)$. It thus remains to show that: (i) $(\mathcal{L}-\lambda)U_2(x;a) + \alpha(x-m)^+ \geq 0$ for $x \geq b(a)$ and (ii) $U_2'(x;a) \leq \kappa$ for $x \in (a,b(a))$. In the following, we fix $a \in \mathbb{R}_+$, so that $b(a)$ is the (fixed) unique solution to (4.9) according to Lemma 4.4.

For $x \geq b(a)$, we notice from the expression (4.8) of U_2 that

$$(\mathcal{L}-\lambda)U_2(x;a) = (\mathcal{L}-\lambda)\Big(U_2(b(a);a) + \kappa(x-b(a))\Big) = (r-g)x\kappa - \lambda U_2(b(a);a) - \lambda\kappa(x-b(a))$$

which is a decreasing, linear function in x, with slope $-\kappa(\lambda-(r-g))$. We then observe that $x \mapsto -\alpha(x-m)$ also decreases linearly for $x \geq b(a) > m$, with a slope $-\alpha$ that is considered to satisfy $-\alpha < -\kappa(\lambda-(r-g))$, according to the parameter regime under Case (II). The claim (i) thus follows by noticing that for $x = b(a)$, we have $(\mathcal{L}-\lambda)U_2(b(a);a) = -\alpha(b(a)-m)^+ = -\alpha(b(a)-m)$.

For $x \in (a,b(a))$, we notice that $U_2'(x;a) = G(x,b(a))$. Since $x \mapsto G(x,b(a))$ is increasing (recall (4.13) and Proposition 4.6), we obtain via the representation (4.10) of G that

$$U_2'(x;a) = G(x,b(a)) \leq G(b(a),b(a)) = \kappa,$$

which concludes our claim. Furthermore, we have $\frac{\partial}{\partial x}U_2(x;a) \geq G(a,b(a)) = 0$ for $x > a$. □

Theorem 4.8 (Verification Theorem: Case (II))**.** *Assume that the government acts according to the control policy ξ^a of (4.1). Then, the function U_2 of (4.8) coincides with the value function V_2 of (4.2). Furthermore, the policy $\eta^{b(a)}$ of (4.6) with the optimal threshold determined via (4.9) is optimal.*

Proof. **Step 1.** We can proceed similarly as in Theorem 4.2. Let $x \geq 0$ and $\eta \in \mathcal{A}_{\xi^a}$ such that $(\xi^a, \eta) \in \mathcal{A}$. Since $U_2 \in C^2(a, \infty)$, we can apply Itô-Meyer formula, up to a localising sequence of stopping times given by $\tau_n := \inf\{t \geq 0 : X_t^{\xi^a,0} \geq n\}$ P-a.s., to the process $U_2(X^{\xi^a,\eta}; a)$ and obtain

$$\begin{aligned} e^{-\lambda\tau_n} U_2(X_{\tau_n}^{\xi^a,\eta}; a) - U_2(x; a) = &\int_0^{\tau_n} (\mathcal{L} - \lambda) U_2(X_s^{\xi^a,\eta}; a) ds + \sigma \int_0^{\tau_n} U_2'(X_s^{\xi^a,\eta}; b) dW_s \\ &- \int_0^{\tau_n} e^{-\lambda s} X_s^{\xi^a,\eta} U_2'(X_s^{\xi^a,\eta}; a) d\eta_s^c + \int_0^{\tau_n} e^{-\lambda s} X_s^{\xi^a,\eta} U_2'(X_s^{\xi^a,\eta}; a) d\xi_s^{c,a} \\ &+ \sum_{s < \tau_n} e^{-\lambda s} \big(U_2(X_s^{\xi^a,\eta}; a) - U_2(X_{s-}^{\xi^a,\eta}; a) \big) \end{aligned} \tag{4.18}$$

The second term in (4.18) is a martingale due to the continuity of U_2' and the fact that $a \leq X_s^{\xi^a,\eta} < n$ for any $s \in (0, \tau_n]$. Furthermore, we can proceed similarly as in (3.26) in order to rewrite the last term in (4.18) and obtain, after taking expectations and rearranging terms, that

$$\begin{aligned} U_2(x; a) = &\,\mathsf{E}_x \Big[e^{-\lambda\tau_n} U_2(X_{\tau_n}^{\xi^a,\eta}; a) - \int_0^{\tau_n} e^{-\lambda s} (\mathcal{L} - \lambda) U_2(X_s^{\xi^a,\eta}; a) ds \\ &+ \int_0^{\tau_n} e^{-\lambda s} X_s^{\xi^a,\eta} U_2'(X_s^{\xi^a,\eta}; a) \circ_d d\eta_s - \int_0^{\tau_n} e^{-\lambda s} X_s^{\xi^a,\eta} U_2'(X_s^{\xi^a,\eta}; a) \circ_u d\xi_s^a \Big] \\ \leq &\,\mathsf{E}_x \Big[e^{-\lambda\tau_n} U_2(X_{\tau_n}^{\xi^a,\eta}; a) + \int_0^{\tau_n} e^{-\lambda s} \alpha (X_s^{\xi^a,\eta} - m)^+ ds + \kappa \int_0^{\tau_n} e^{-\lambda s} X_s^{\xi^a,\eta} \circ_d d\eta_s \Big], \end{aligned} \tag{4.19}$$

where the latter inequality follows from the fact that U_2 solves the free-boundary problem (4.7) and $U_2'(X_s^{\xi^a,\eta}; a) = 0$ for all s in the support of $d\xi^a$. By admissibility of η, the right-hand side of (4.19) is finite P-a.s., and Assumption 2.4.(iii)

$$\lim_{n\to\infty} \mathsf{E}_x \big[e^{-\lambda\tau_n} U_2(X_{\tau_n}^{\xi^a,\eta}; a) \big] = 0.$$

Then, taking limits in (4.19) upon using that $\tau_n \uparrow \infty$, we can employ dominated convergence due to Assumption 2.4.(iii) and obtain

$$U_2(x; a) \leq \mathsf{E}_x \Big[\int_0^\infty e^{-\lambda s} \alpha (X_s^{\xi^a,\eta} - m)^+ ds + \kappa \int_0^\infty e^{-\lambda s} X_s^{\xi^a,\eta} \circ_d d\eta_s \Big].$$

We conclude that $U_2(x; a) \leq V_2(x; a)$ on $\mathbb{R}_+$.

Step 2. We can now repeat the steps from above, upon fixing the control strategy $\eta^{b(a)}$ of (4.6). Since $X_t^{\xi^a,\eta^{b(a)}} \in [a, b(a)]$ a.s. for all $t > 0$ and $U_2(X_t^{\xi^a,\eta^{b(a)}}; a) = \kappa$ on $\text{supp}\{d\eta^{b(a)}\}$, the inequality in (4.19) becomes an equality. Arguing as before, we thus obtain

$$U_2(x; a) = \mathsf{E}_x \Big[\int_0^\infty e^{-\lambda s} \alpha (X_s^{\xi^a,\eta^{b(a)}} - m)^+ ds + \kappa \int_0^\infty e^{-\lambda s} X_s^{\xi^a,\eta^{b(a)}} \circ_d d\eta_s^{b(a)} \Big] \geq V_2(x; a).$$

Step 3. Combining the results from Steps 1 and 2 then concludes that $U_2(x; a) = V_2(x; a)$ on $\mathbb{R}_+$ and $\eta^{b(a)}$ is an optimal control strategy in problem (4.2). □

5. Nash Equilibria in the model

Our results from Sections 3 and 4 – each player's best response to a Skorokhod-reflection type strategy is a Skorokhod-reflection type strategy or a no-intervention policy – suggest that we should aim at determining a Nash equilibrium via its Definition 2.3 in the class $\mathcal{M}$ of (2.10), in which at least one player acts according to a Skorokhod-reflection type policy. The analysis in this section focuses on this direction.

Recall the peculiarity arising from our results in Section 4, where we show that the legislative body may choose *not to intervene* on the debt ratio at all. Interestingly, we prove that the optimality of adopting such a strategy relies solely on their (individual) time preferences compared to the parameter constellation in the model – it does *not depend* on the actions of the opposing player (government) – see specifically our results in Section 4.1. We thus split our search for Nash equilibria in the forthcoming analysis based on the magnitude of the legislative body time preferences.

5.1. **The case of $\lambda > r-g+\alpha/\kappa$.** Our results in Section 4.1, suggest that the legislative body should restrain themselves from reflecting the debt ratio at any threshold, when their time preference rate λ is relatively large. In light of this non-intervention policy, it is natural to then examine what should the governmental strategy be as a best response. The characterisation of such a strategy is in fact the main aim of Section 3.1, which studies the optimal control (debt issuance policy) of the government when they are the sole player (there is no opponent).

We present the resulting Nash equilibrium in the following theorem.

Theorem 5.1 (Existence and uniqueness of Nash Equilibrium)**.** *Suppose that the model's parameters satisfy Assumptions 2.4 as well as $\lambda > r-g+\alpha/\kappa$. A unique Nash equilibrium of the game (2.8)–(2.9) in the set $\mathcal{M}$ of (2.10) can be characterised by the couple of controls $(\xi^{\overline{a}}, \overline{\eta}) = (\xi^{\overline{a}}, 0)$, with the former component defined as in (3.5) and the threshold $\overline{a}$ defined in (3.8).*

Proof. This simply follows from the notion of Nash equilibrium as introduced in Definition 2.3. □

5.2. **The case of $\lambda \in (r-g, r-g+\alpha/\kappa)$.** Our results from Sections 3 and 4 on each player's best response suggest that a Nash equilibrium could be characterised by Skorokhod-reflection type policies at finite thresholds. More precisely, while the government increases the debt ratio at $a(b)$ (as a best response to a debt ceiling $b \in \mathbb{R}_+$), the legislative body forces a debt ratio reduction at a debt ceiling $b(a)$ (as a best response to a governmental debt-issuance threshold $a \in \mathbb{R}_+$).

The aim of the following theorem is to prove that there always exists a pair (a^*, b^*) forming a fixed point of these best-response-maps, such that $a^* = a(b^*)$ and $b^* = b(a^*)$.

Theorem 5.2 (Existence of Nash Equilibrium)**.** *Suppose that the model's parameters satisfy Assumptions 2.4 as well as $\lambda \in (r-g, r-g+\alpha/\kappa)$. A Nash equilibrium in the game (2.8)–(2.9) can be characterised by the couple of Skorokhod-reflection type policies (ξ^{a^*}, η^{b^*}), as defined in (3.11) and (4.6), respectively. The pair of thresholds $(a^*, b^*) \in \mathbb{R}_+^2$ satisfies $a^* < b^*$ and solves the coupled system of equations $F(a^*, b^*) = 0 = G(a^*, b^*)$.*

Proof. Recall the function $a(b)$ from Proposition 3.4 and define the function

$$a_1(x) := a(x), \quad \text{for all } x \in \mathbb{R}_+. \tag{5.1}$$

Then, we recall from Proposition 4.6 that the unique solution to $G(a, \cdot) = 0$ for any fixed $a \in \mathbb{R}_+$, is given in terms of a strictly increasing function $a \mapsto b(a)$. We can therefore invert this function and define

$$a_2(x) := b^{-1}(x), \quad \text{such that } x \mapsto a_2(x) \text{ is strictly increasing on } \mathbb{R}_+. \tag{5.2}$$

Thanks to Proposition 4.6, we also know that $b(a) = (1/\widetilde{q})a$, for all $a > m$, which yields that

$$a_2(x) = \widetilde{q}x, \quad \text{for } x > m/\widetilde{q}. \tag{5.3}$$

For illustration, Figure 1 sketches the maps for different parameter specifications.

We can thus conclude that

$$\exists \text{ a Nash equilibrium} \quad \Leftrightarrow \quad \exists \text{ an intersection point } b^*, \text{ such that } a^* := a_1(b^*) = a_2(b^*). \tag{5.4}$$

In view of the definitions in (5.1)–(5.2), we have that $a^* = a(b^*)$ and $b^* = b(a^*)$. This would finally imply that (a^*, b^*) solves the system of equations $F(a^*, b^*) = 0 = G(a^*, b^*)$ and complete the proof.

In the remainder of the proof, we examine the existence of b^* in (5.4). On one hand, it follows from (5.1) and (3.21) in Proposition 3.4 that $a_1(\cdot)$ is bounded from above by $\widehat{a} := a(\widehat{b})$. On the other hand, it follows from (5.2)–(5.3) that $a_2(\cdot)$ is strictly increasing with $\lim_{b\to\infty} a_2(b) = +\infty$. We further know from Propositions 3.4 and 4.6 that the functions $a_1(\cdot)$ and $a_2(\cdot)$ have supports on $(0,\infty)$ and (b_0,∞), respectively, with $b_0 > m > 0$. due to (4.12). Clearly, there exists at least one $b^* \in (b_0,\infty)$ such that $a_1(b^*) = a_2(b^*)$, therefore (5.4) implies that there exists a Nash equilibrium. □

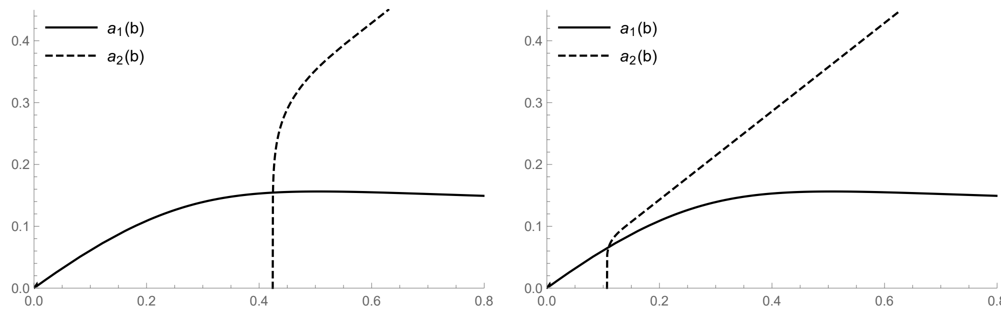

FIGURE 1. A Sketch of the maps $a_1(b)$ and $a_2(b)$ for different parameter specifications.

Theorem 5.3 (Uniqueness of Nash Equilibrium)**.** *Assume that $\lambda \in (r-g, r-g+\frac{\alpha}{\kappa})$. The Nash Equilibrium specified in Theorem 5.2 is unique in the class $\mathcal{M}$, in which at least one player plays a Skorokhod-reflection type policy.*

Proof. Recall from Theorem 5.2 that there exists an intersection point b^*, such that $a^* := a_1(b^*) = a_2(b^*)$, thus (a^*, b^*) is a Nash equilibrium (cf. (5.4)). In order to prove the uniqueness of this equilibrium (in the class of Skorokhod reflection-type strategies), we must prove the uniqueness of the intersection point b^*.

We begin by defining the function

$$a \mapsto b_0(a), \quad \text{for } a \in \mathbb{R}_+, \quad \text{such that } b_0(a) = (1/\widetilde{q})a \quad \text{with } \widetilde{q} \in (0,1) \text{ as in Proposition 4.6},$$

which can be inverted to define

$$a_0(b) := \widetilde{q}b, \quad \text{for } b \in \mathbb{R}_+.$$

Notice from Proposition 4.6 that $b(a) = b_0(a)$ for all $a > m$, hence in view of (5.2), we get $a_2(b) = a_0(b)$, for all $b > m/\widetilde{q}$. Moreover, Lemma 4.6 implies that $b \mapsto a_2(b)$ is strictly concave on $(b_0, \frac{m}{\widetilde{q}})$, where we notice that $m/\widetilde{q} > b_0$ due to the monotonicity of $b(a)$. Moreover, this implies $a_2(b) \le a_0(b)$ for all $b \ge b_0$.

As a first step, we prove that the curves $a_1(b)$ and $a_0(b)$ either admit no intersection or exactly one for $b > 0$. Since $a_0(b) = \widetilde{q}b$, any intersection clearly is of the form $(\widetilde{q}b, b)$. Plugging in points of this form into the function F of (3.15) we observe that

$$b \to F(\widetilde{q}b, b) = s(\widetilde{q})b + y(\widetilde{q}) \quad \text{is strictly increasing on } \mathbb{R}_+,$$

with

$$s(q) = (2-\delta_2)q^{2-\delta_1} + (\delta_1 - 2)q^{2-\delta_2} - (\delta_1 - \delta_2) > 0,$$
$$y(q) = (\rho - 2(r-g) - \sigma^2)\big[c_1(\delta_1 - \delta_2) - c_2(1-\delta_2)q^{1-\delta_1} - c_2(\delta_1 - 1)q^{1-\delta_2}\big].$$

Clearly, this implies (depending on the value $y(q)$) that either no intersection point exists or exactly one. Moreover, if no intersection of a_0 and a_1 exists, we conclude that $a_1'(0+) < a_0'(0+) = \widetilde{q}$.

Next, we come back to the uniqueness of an intersection of the maps $a_1(b)$ and $a_2(b)$. Recall that $b \mapsto a_1(b)$ is concave on $(0, \widehat{b})$, and $a \mapsto b(a)$ is convex, which implies that $b \mapsto a_2(b)$ is concave as well.

We now distinguish the following cases; an illustration for each one of them is given in Figure 2:

Case (i). No intersection exists of $a_1(b)$ and $a_0(b)$: In this case, we first notice that $a_2'(b) \geq a_0'(b) = \widetilde{q}$ for all b in the support of $a_2(b)$ due to the concavity of $a_2(b)$ and the fact that $a_2(b) \leq \widetilde{q}b$. Furthermore, we have $\widetilde{q} > a_1'(0+)$, and hence $\widetilde{q} > a_1'(b)$ for all $b > 0$. Combining these insights, it follows that $a_1(b) = a_2(b)$ for exactly one $b \in \mathbb{R}_+$.

For the following cases, we thus assume that there exists exactly one intersection of $a_1(b)$ and $a_0(b)$, which yields the point (a_0^*, b_0^*).

Case (ii). If $a_0^* \geq m$, the uniqueness of an intersection of $a_1(b)$ and $a_2(b)$ follows from the fact that $a_2(b) = a_0(b)$ for all $b \geq m/\widetilde{q}$. Hence, in this case, we obtain that the intersection of $a_1(b)$ and $a_2(b)$ is exactly given by $(a^*, b^*) = (a_0^*, b_0^*)$.

Case (iii). If $a_0^* < m$ and $b_0^* \geq \widehat{b}$, we again observe that $a_2(b) \leq \widetilde{q}b$, which implies that the intersection point is not realized on $b \leq b_0^*$. Therefore, since $a_1(b)$ is decreasing for $b \geq \widehat{b}$ and $a_2(b)$ is increasing, the intersection is unique and we denote it by (a^*, b^*).

Case (iv). If $a_0^* < m$ and $b_0^* < \widehat{b}$, the concavity of $a_1(b)$ implies $a_1'(b_0^*) < \widetilde{q}$ as well as $a_1'(b_0^*) > a_1'(b)$ for all $b \geq b_0^*$. Since $a_2'(b) > \widetilde{q}$ for all b, we must have $a_1(b) = a_2(b)$ for exactly one $b \in \mathbb{R}_+$ and we again denote the unique intersection by (a^*, b^*). □

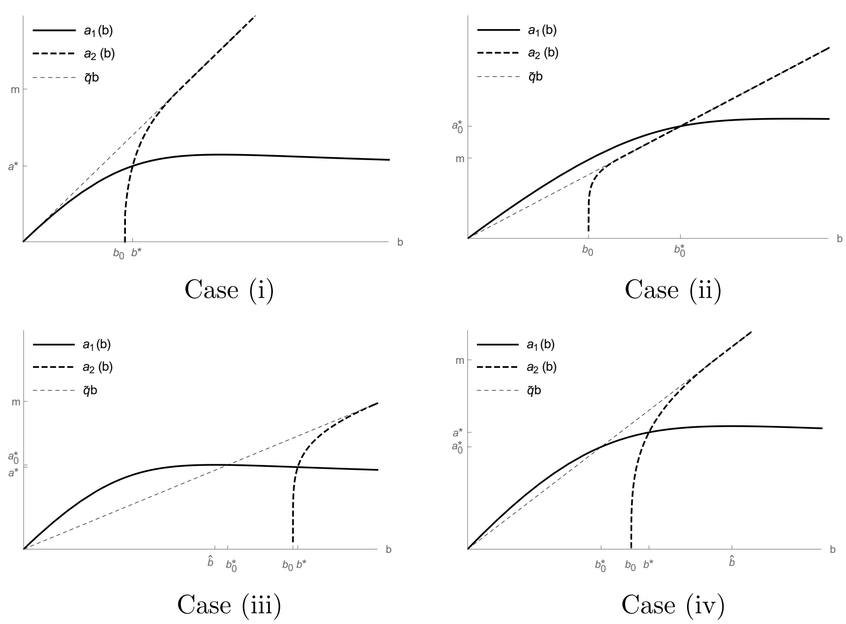

Figure 2. Case Study for the uniqueness of an intersection of $a_1(b)$ and $a_2(b)$

Remark 5.4. *It is interesting to notice that one can establish a connection between the equilibria determined in Theorems 5.1-5.3 by considering the limit as $\lambda \uparrow r-g+\alpha/\kappa$. Hence, in what follows, we stress the dependence of relevant quantities on the parameter λ. Recall that $a_2(b;\lambda) \leq a_0(b;\lambda) = \widetilde{q}(\lambda)b$ (see also Figure 2 for illustration), where $\widetilde{q} \equiv \widetilde{q}(\lambda) \in (0,1)$ is given by the solution to (4.14), which is equivalent to $G(\widetilde{q}(\lambda), 1) = 0$. By defining*

$$\widehat{G}(q) := (1-\theta_2)\kappa(\lambda-(r-g)) + (\theta_1-1)(\kappa(\lambda-(r-g))-\alpha)q^{\theta_2-1} + \alpha(\theta_1-1), \qquad q \in (0,1),$$

we observe that $G(q,1) \geq \widehat{G}(q)$ for all $q \in (0,1)$, since $\lambda < r-g+\frac{\alpha}{\kappa}$. Moreover, we notice that

$$\lim_{q \downarrow 0} G(q,1) = -\infty, \quad \lim_{q \downarrow 0} \widehat{G}(q) = -\infty, \quad G(1,1) = \widehat{G}(1) = (\theta_1-\theta_2)\kappa(\lambda-(r-g)) > 0$$

and that both $q \mapsto G(q,1)$ and $q \mapsto \widehat{G}(q)$ are monotonically increasing. Using these properties, we can denote by $\widehat{q} \equiv \widehat{q}(\lambda) \in (0,1)$ the unique solution to $\widehat{G}(q) = 0$ (which can be computed explicitly) and obtain

$$\widetilde{q}(\lambda) < \widehat{q}(\lambda) \quad \text{and} \quad \lim_{\lambda \uparrow r-g+\alpha/\kappa} \widehat{q}(\lambda) = 0 \quad \Rightarrow \quad \lim_{\lambda \uparrow r-g+\alpha/\kappa} \widetilde{q}(\lambda) = 0.$$

First of all, recall from Theorem 5.2 that for every $\lambda \in (r-g, r-g+\alpha/\kappa)$, there exists a unique equilibrium pair $(a^*(\lambda), b^*(\lambda))$, such that $a^*(\lambda) = a_1(b^*(\lambda)) = a_2(b^*(\lambda);\lambda)$. Since $a_2(b;\lambda) \leq \widetilde{q}(\lambda)b$, we notice that for any fixed $\widetilde{b} \in (0,\infty)$ we have $\lim_{\lambda \uparrow r-g+\alpha/\kappa} a_2(\widetilde{b};\lambda) = 0$. However, there is no such fixed $\widetilde{b} \in (0,\infty)$ that can give $a_1(\widetilde{b}) = \lim_{\lambda \uparrow r-g+\alpha/\kappa} a_2(\widetilde{b};\lambda) = 0$ to create an equilibrium pair $\lim_{\lambda \uparrow r-g+\alpha/\kappa}(a^*(\lambda), b^*(\lambda)) = (0,\widetilde{b})$. Hence, the only possibility for obtaining an equilibrium as $\lambda \uparrow r-g+\alpha/\kappa$, is for $b^*(\lambda) \to \infty$. Given that $b \mapsto a_1(b)$ is strictly decreasing on $(\widehat{b},\infty)$ and $\lim_{b\to\infty} a_1(b) = \bar{a}$ (independently of λ), we conclude that

$$\lim_{\lambda \uparrow r-g+\alpha/\kappa} a^*(\lambda) = \lim_{\lambda \uparrow r-g+\alpha/\kappa} a_2(b^*(\lambda);\lambda) = \lim_{\lambda \uparrow r-g+\alpha/\kappa} a_1(b^*(\lambda)) = \bar{a} \quad \text{and} \quad \lim_{\lambda \uparrow r-g+\alpha/\kappa} b^*(\lambda) = \infty.$$

6. Comparative Statics Analysis

In Section 5 we derived the existence and uniqueness of Nash equilibria under different parameter regimes in our model. It was revealed that the parameter λ, measuring the legislative body's time preferences, plays a crucial role on the characterisation of Nash equilibrium. While it is optimal not to intervene for large values of $\lambda > r-g+\alpha/\kappa$, the equilibrium strategies are characterised by a pair of thresholds (a^*, b^*) for intermediate values of $\lambda \in (r-g, r-g+\alpha/\kappa)$.

In this section, we study the sensitivity of these boundaries with respect to some of the model parameters. In order to highlight the transition from Nash equilibria that are characterised by thresholds (a^*, b^*) (cf. Theorems 5.2-5.3) to those that prescribe a non-intervention policy for the legislative body (cf. Theorem 5.1), as stated in Remark 5.4, we plot the equilibrium values of (a^*, b^*) as functions of λ in the following comparative statics.

Unless otherwise specified, we fix the following parameter set.

ρ	σ	r	g	α	m	c_1	c_2	κ
0.3	0.2	0.025	0.02	0.15	0.6	2	1.25	0.6

6.1. **Sensitivity with respect to λ.** To begin with, it is interesting to study the dependency of the equilibrium values a^* and b^* on the discount factor λ. The numerical sensitivity analysis exhibited in Figure 3 depicts the optimal intervention thresholds as functions of the legislative body's time preference rate, which here takes values on the interval $\lambda \in (r-g, r-g+\alpha/\kappa)$. The latter guarantees, as shown in the previous analysis, the optimality of a finite debt ceiling mechanism. Some remarks are worth mentioning. Clearly, the equilibrium debt ceiling b^* exhibits a monotonically increasing behaviour as a function of λ, with the peculiarity of an exploding behaviour $b^* \uparrow +\infty$ for $\lambda \uparrow r-g+\alpha/\kappa$. This illustrates our finding of the smooth transition from a debt ceiling mechanism to a non-intervention policy by the legislative body, as stated in Remark 5.4. It is interesting to notice that this monotonicity as well as limiting behaviour of b^* does not depend on the other parameters in the model, although some of them influence the interval bounds for the values of λ, for which the legislative body's optimal strategy is indeed a debt ceiling mechanism. More precisely, increasing (decreasing) the term $r-g$ shifts the interval to the right (left), while the fraction α/κ determines the length of the interval $(r-g, r-g+\alpha/\kappa)$. A short discussion on the implications of a shift in these parameters on the optimal strategy of the legislative body (depending on their time preference rate λ) is given in the subsequent sensitivity study. Last, we note that the intervention threshold a^*, characterising the optimal debt issuance policy by the government, again illustrates our finding of Remark 5.4, in the sense that $a^* \to \bar{a}$ for $\lambda \uparrow r-g+\alpha/\kappa$.

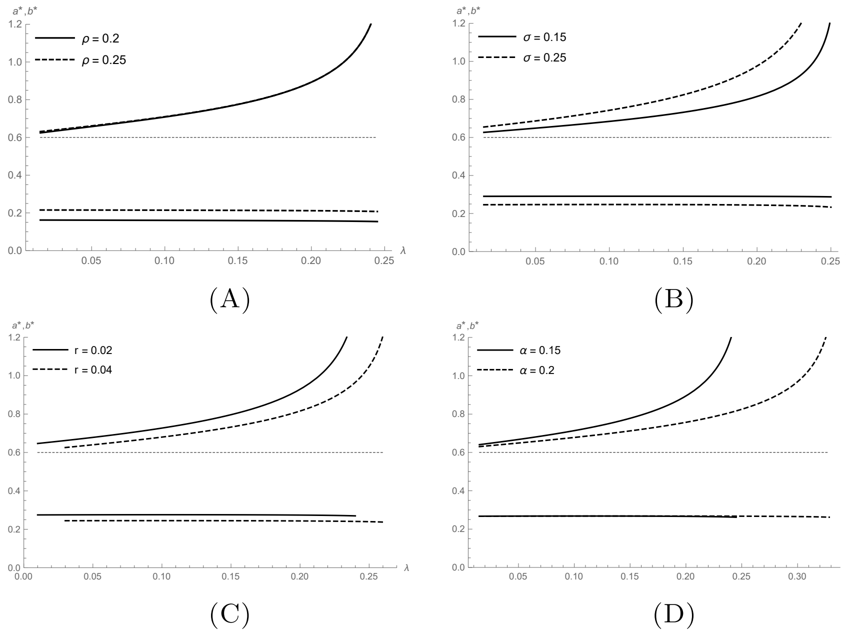

FIGURE 3. Sensitivity of the equilibrium values for a^* and b^* (as functions of λ) with respect to a change in some of the model parameters.

6.2. **Sensitivity with respect to governmental time preference rate ρ.** The discount factor ρ serves as a measure on how myopic a government is regarding its debt. Increasing ρ has the effect that the government discounts future costs and revenues more heavily, and thus cares less and less about the future compared to the present. We observe the sensitivity regarding a change in ρ in Figure 3A. Clearly, the government aims at increasing its debt ratio earlier, at a higher debt-issuance threshold a^*, whenever its subjective discount rate increases. The legislative body reacts to such an increase by increasing the debt ceiling as well, although we observe Figure 3A that the equilibrium value b^* is relatively robust with respect to a change in ρ.

6.3. **Sensitivity with respect to debt ratio volatility σ.** Increasing volatility increases the fluctuations of the debt ratio. The government and the legislative body adapt by acting on the debt ratio later, which is achieved by the government decreasing its optimal debt-issuance threshold and by the legislative body increasing the debt ceiling. We can observe this in Figure 3B.

6.4. **Sensitivity with respect to the interest rate r on government debt.** Increasing interest rates on public debt result in holding debt getting more costly for the government, which in turn increases the drift of the debt ratio. Clearly, it is optimal for the government to increase its debt at a later stage, which is achieved by decreasing its debt-issuance threshold, as observed in Figure 3C. In the equilibrium, the legislative body also decreases the debt ceiling, since countries with a higher cost of debt are more in danger of defaulting. Contrary, if interest rates decrease, the legislative body can be more flexible, since it is optimal to increase the debt ceiling. Intuitively, in such a case, the growth of GDP helps containing the debt ratio without interventions. Furthermore, we note that an increase in the interest rate r shifts the interval of λ-values, for which the optimal strategy of the legislative body prescribes to set a finite debt ceiling, to the right. It follows that a legislative body with a fixed time preference rate could change its optimal strategy from a non-intervention policy to a debt ceiling mechanism, if the government's interest rate on debt increases. Notice that an increase in the country's GDP growth rate has the contrary effect, thus implying that fast growing economies could allow a larger deficit without interventions from a legislative body.

6.5. **Sensitivity with respect to the tax compliance factor α.** The parameter α denotes the tax compliance factor, which measures the willingness to pay taxes within a country. This factor can be chosen freely, hence, the legislative body can account for the fact that some countries have a low probability of default, even though holding a lot of debt. In Figure 3D we observe the sensitivity of the equilibrium values a^* and b^* with respect to a change in α. Clearly, if the tax compliance factor increases (which implies decreasing willingness to pay tax), the legislative body faces stronger social and political pressure to act via the implementation of a debt ceiling mechanism. This has the consequence that a larger factor α (i) causes the legislative body to act earlier on the debt ratio (by decreasing the debt ceiling b^*) and (ii) enlarges the interval of time preference values $\lambda \in (r-g, r-g+\alpha/\kappa)$ for which the legislative body's optimal strategy is to impose a debt ceiling mechanism. The latter implies that, for a fixed time preference λ, a change in the factor α could incentivise the legislative body to switch from a laissez-faire policy to implementing a debt ceiling. On the other hand, if the assigned likelihood of the country's default decreases (in terms of a decrease in the parameter α), the legislative body is willing to postpone interventions by either implementing a larger debt ceiling b^* or even choosing a non-intervention policy.

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