Appendix to the paper "Extreme expectile estimation for short-tailed data"

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This appendix contains all necessary proofs and provides extra finite-sample results about our simulation study.

A Proofs of the main results

A.1 Auxiliary results

We first of all list a number of facts that will be used numerous times in our proofs: if condition $\mathcal{C}_2(\gamma, a, \rho, A)$ holds, then

- Condition $\mathcal{C}_2(\gamma, a, \rho, A)$ holds locally uniformly in z, see Remark B.3.8.1 on relationship (B.3.3) in de Haan and Ferreira (2006).
- The right endpoint of F is finite and will be denoted in the sequel by $x^\star$, see Theorem 1.2.1 on p.19 of de Haan and Ferreira (2006).
- One has $a(s)/(x^\star - U(s)) \to -\gamma$ as $s \to \infty$, see Lemma 1.2.9 on p.22 of de Haan and Ferreira (2006).
- The functions $x \mapsto \overline{F}(x^\star - 1/x)$ and $s \mapsto x^\star - U(s)$ are regularly varying with indices $1/\gamma$ and γ, respectively, see Theorem 1.2.1.2 on p.19 and Corollary 1.2.10.2 on p.23 of de Haan and Ferreira (2006).

Our first auxiliary result is a useful asymptotic inversion lemma that will be used several times.

Lemma A.1. *Suppose that condition $\mathcal{C}_2(\gamma, a, \rho, A)$ holds.*

(i) One has

$$\lim_{x \uparrow x^\star} \frac{U(1/\overline{F}(x)) - x}{a(1/\overline{F}(x))A(1/\overline{F}(x))} = 0.$$

In particular

$$\lim_{x \uparrow x^\star} \frac{1}{A(1/\overline{F}(x))} \left(\frac{x^\star - U(1/\overline{F}(x))}{x^\star - x} - 1 \right) = 0.$$

(ii) One has

$$\lim_{\tau \uparrow 1} \frac{1}{A((1-\tau)^{-1})} \left(\frac{\overline{F}(q_\tau)}{1-\tau} - 1 \right) = 0.$$

Proof of Lemma A.1. We only show (i); the proof of (ii) is similarly written by using an equivalent second-order condition on $\overline{F}$ (see de Haan and Ferreira, 2006, Theorem 2.3.8

p.48). Assume that A is positive; the proof for a negative A is similar. Condition $\mathcal{C}_2(\gamma, a, \rho, A)$ holds locally uniformly in z, so pick $\varepsilon \neq 0$ and apply this condition to obtain

$$\lim_{x\uparrow x^\star} \left| \frac{1}{A(1/\overline{F}(x))} \left(\frac{U([1+\varepsilon A(1/\overline{F}(x))]/\overline{F}(x)) - U(1/\overline{F}(x))}{a(1/\overline{F}(x))} - \frac{(1+\varepsilon A(1/\overline{F}(x)))^\gamma - 1}{\gamma} \right) \right. \\ \left. - \int_1^{1+\varepsilon A(1/\overline{F}(x))} v^{\gamma-1} \left(\int_1^v u^{\rho-1} \mathrm{d}u \right) \mathrm{d}v \right| = 0$$

and therefore

$$\lim_{x\uparrow x^\star} \frac{1}{a(1/\overline{F}(x))A(1/\overline{F}(x))} \left| U([1+\varepsilon A(1/\overline{F}(x))]/\overline{F}(x)) - U(1/\overline{F}(x)) \right. \\ \left. - a(1/\overline{F}(x)) \frac{(1+\varepsilon A(1/\overline{F}(x)))^\gamma - 1}{\gamma} \right| = 0.$$

Conclude that

$$\lim_{x\uparrow x^\star} \frac{U([1+\varepsilon A(1/\overline{F}(x))]/\overline{F}(x)) - U(1/\overline{F}(x))}{a(1/\overline{F}(x))A(1/\overline{F}(x))} = \varepsilon.$$

By definition of U as the left-continuous inverse of $1/\overline{F}$, one has $U([1+\varepsilon A(1/\overline{F}(x))]/\overline{F}(x)) \geq x$ when $\varepsilon > 0$ (resp. $\leq x$ when $\varepsilon < 0$), so

$$\limsup_{x\uparrow x^\star} \frac{x - U(1/\overline{F}(x))}{a(1/\overline{F}(x))A(1/\overline{F}(x))} \leq \lim_{x\uparrow x^\star} \frac{U([1+\varepsilon A(1/\overline{F}(x))]/\overline{F}(x)) - U(1/\overline{F}(x))}{a(1/\overline{F}(x))A(1/\overline{F}(x))} = \varepsilon$$

and a similar lower bound applies. Let $\varepsilon \to 0$ to complete the proof of the first convergence. The second convergence follows because $a(s)/(x^\star - U(s)) \to -\gamma$ as $s \to \infty$. $\square$

A fine understanding of the asymptotic behavior of extreme expectiles requires an asymptotic expansion of $\varphi^{(1)}(x) = \mathbb{E}((X-x)\mathbb{1}\{X > x\})$ for x close to the right endpoint $x^\star$. This is the focus of the below lemma, where we recall that more generally $\varphi^{(\kappa)}(x) = \mathbb{E}((X-x)^\kappa \mathbb{1}\{X > x\})$.

Lemma A.2. *Suppose that condition $\mathcal{C}_2(\gamma, a, \rho, A)$ holds.*

(i) Then, for any $\kappa \geq 1$, and as $x \uparrow x^\star$,

$$\frac{\varphi^{(\kappa)}(x)}{\overline{F}(x)[a(1/\overline{F}(x))]^\kappa} = \mathrm{O}(1).$$

(ii) As $x \uparrow x^\star$,

$$\frac{\varphi^{(1)}(x)}{\overline{F}(x)a(1/\overline{F}(x))} = \frac{1}{1-\gamma}\left(1 + \frac{1}{1-\gamma-\rho}A(1/\overline{F}(x)) + \mathrm{o}(|A(1/\overline{F}(x))|)\right).$$

(iii) As $x \uparrow x^\star$,

$$\frac{\varphi^{(2)}(x)}{\overline{F}(x)[a(1/\overline{F}(x))]^2} \\ = \frac{2}{(1-\gamma)(1-2\gamma)}\left(1 + \frac{3-4\gamma-2\rho}{(1-\gamma-\rho)(1-2\gamma-\rho)}A(1/\overline{F}(x)) + \mathrm{o}(|A(1/\overline{F}(x))|)\right).$$

Proof of Lemma A.2. Statement (i) is shown by writing

$$\frac{\varphi^{(\kappa)}(x)}{\overline{F}(x)[a(1/\overline{F}(x))]^{\kappa}} \leq \left(\frac{x^\star - x}{a(1/\overline{F}(x))}\right)^{\kappa} = \left(-\frac{1}{\gamma} \times \frac{x^\star - x}{x^\star - U(1/\overline{F}(x))}\right)^{\kappa} (1 + \mathrm{o}(1)) = \mathrm{O}(1)$$

by Lemma A.1(i) and because $a(s)/(x^\star - U(s)) \to -\gamma$ as $s \to \infty$.

To show (ii) and (iii), recall that $X \stackrel{d}{=} U(Y)$, where Y is a unit Pareto random variable. Write $\varphi^{(1)}(x) = \mathbb{E}((U(Y) - x)\mathbb{1}\{Y > 1/\overline{F}(x)\})$. By Lemma A.1(i),

$$\begin{aligned}
\varphi^{(1)}(x) &= \mathbb{E}((U(Y) - U(1/\overline{F}(x)))\mathbb{1}\{Y > 1/\overline{F}(x)\}) + \mathrm{o}(\overline{F}(x)a(1/\overline{F}(x))|A(1/\overline{F}(x))|) \\
&= \int_{1/\overline{F}(x)}^{\infty} (U(y) - U(1/\overline{F}(x)))\frac{\mathrm{d}y}{y^2} + \mathrm{o}(\overline{F}(x)a(1/\overline{F}(x))|A(1/\overline{F}(x))|) \\
&= \frac{1}{s}\int_{1}^{\infty} (U(sz) - U(s))\frac{\mathrm{d}z}{z^2} + \mathrm{o}(\overline{F}(x)a(1/\overline{F}(x))|A(1/\overline{F}(x))|) \qquad \text{(A.1)}
\end{aligned}$$

with $s = s(x) = 1/\overline{F}(x) \to \infty$ (as $x \uparrow x^\star$). Define now $a_\star$ and $A_\star$ as

$$a_\star(s) = \begin{cases} a(s)\left(1 - \dfrac{1}{\rho}A(s)\right), & \rho < 0, \\ a(s)\left(1 - \dfrac{1}{\gamma}A(s)\right), & \rho = 0, \end{cases} \quad \text{and } A_\star(s) = \begin{cases} \dfrac{1}{\rho}A(s), & \rho < 0, \\ A(s), & \rho = 0. \end{cases} \qquad \text{(A.2)}$$

By the set of uniform inequalities in Theorem 2.3.6 of de Haan and Ferreira (2006), there exist functions a_0 and A_0 such that A_0 is asymptotically equivalent to $A_\star$ and $a_0(s)/a_\star(s) = 1 + \mathrm{o}(|A_\star(s)|)$ as $s \to \infty$, and, for any $\varepsilon > 0$, the following inequality holds for s large enough:

$$\forall z \geq 1, \ \left|\frac{1}{A_0(s)}\left(\frac{U(sz) - U(s)}{a_0(s)} - \frac{z^\gamma - 1}{\gamma}\right) - \Psi_{\gamma,\rho}(z)\right| \leq \varepsilon z^{\gamma+\rho+\varepsilon},$$

$$\text{where } \Psi_{\gamma,\rho}(z) = \begin{cases} \dfrac{z^{\gamma+\rho} - 1}{\gamma + \rho}, & \rho < 0, \\ \dfrac{1}{\gamma}z^\gamma \log(z), & \rho = 0. \end{cases} \qquad \text{(A.3)}$$

Write then

$$\begin{aligned}
\frac{1}{s}\int_1^\infty (U(sz) - U(s))\frac{\mathrm{d}z}{z^2} = \frac{a_0(s)}{s}\int_1^\infty \frac{z^\gamma - 1}{\gamma}\frac{\mathrm{d}z}{z^2} \\
+ \frac{a_0(s)A_0(s)}{s}\int_1^\infty \frac{1}{A_0(s)}\left(\frac{U(sz) - U(s)}{a_0(s)} - \frac{z^\gamma - 1}{\gamma}\right)\frac{\mathrm{d}z}{z^2}
\end{aligned}$$

and use (A.3) in conjunction with the dominated convergence theorem together with straightforward calculations to get

$$\begin{aligned}
&\frac{1}{s}\int_1^\infty (U(sz) - U(s))\frac{\mathrm{d}z}{z^2} \\
&\quad = \frac{a_0(s)}{s}\left(\frac{1}{1-\gamma} + A_0(s)\left[\frac{1}{1-\gamma-\rho}\mathbb{1}\{\rho < 0\} + \frac{1}{\gamma(1-\gamma)^2}\mathbb{1}\{\rho = 0\} + \mathrm{o}(1)\right]\right).
\end{aligned}$$

Combine this last identity with (A.1), (A.2) and a straightforward calculation to conclude the proof of (ii). We turn to showing (iii). Start by writing

$$\begin{aligned}
(X - x)^2 &\stackrel{d}{=} (U(Y) - U(1/\overline{F}(x)))^2 \\
&\quad + (U(1/\overline{F}(x)) - x) \times (2(U(Y) - U(1/\overline{F}(x))) + (U(1/\overline{F}(x)) - x))
\end{aligned}$$

and use the results of the proof of (ii) along with Lemma A.1(i) to obtain

$$\begin{aligned}
\varphi^{(2)}(x) &= \mathbb{E}((U(Y)-U(1/\overline{F}(x)))^2\mathbb{1}\{Y>1/\overline{F}(x)\}) + \mathrm{o}(\overline{F}(x)[a(1/\overline{F}(x))]^2|A(1/\overline{F}(x))|) \\
&= \int_{1/\overline{F}(x)}^{\infty} (U(y)-U(1/\overline{F}(x)))^2 \frac{\mathrm{d}y}{y^2} + \mathrm{o}(\overline{F}(x)[a(1/\overline{F}(x))]^2|A(1/\overline{F}(x))|) \\
&= \frac{1}{s}\int_1^{\infty} (U(sz)-U(s))^2 \frac{\mathrm{d}z}{z^2} + \mathrm{o}(\overline{F}(x)[a(1/\overline{F}(x))]^2|A(1/\overline{F}(x))|) \qquad \text{(A.4)}
\end{aligned}$$

where again $s = s(x) = 1/\overline{F}(x) \to \infty$ as $x \uparrow x^{\star}$. Now

$$\begin{aligned}
\frac{1}{s}\int_1^{\infty} (U(sz)-U(s))^2 \frac{\mathrm{d}z}{z^2} &= \frac{[a_0(s)]^2}{s}\int_1^{\infty} \left(\frac{z^{\gamma}-1}{\gamma}\right)^2 \frac{\mathrm{d}z}{z^2} \\
&\quad + 2\frac{[a_0(s)]^2}{s}\int_1^{\infty} \left(\frac{U(sz)-U(s)}{a_0(s)} - \frac{z^{\gamma}-1}{\gamma}\right)\frac{z^{\gamma}-1}{\gamma}\frac{\mathrm{d}z}{z^2} \\
&\quad + \frac{[a_0(s)]^2}{s}\int_1^{\infty} \left(\frac{U(sz)-U(s)}{a_0(s)} - \frac{z^{\gamma}-1}{\gamma}\right)^2 \frac{\mathrm{d}z}{z^2}.
\end{aligned}$$

Combine (A.3) with the dominated convergence theorem and straightforward calculations to find

$$\begin{aligned}
&\frac{1}{s}\int_1^{\infty} (U(sz)-U(s))^2 \frac{\mathrm{d}z}{z^2} = \frac{[a_0(s)]^2}{s}\left(\frac{2}{(1-\gamma)(1-2\gamma)}\right. \\
&\left. + A_0(s)\left[\frac{2(2-2\gamma-\rho)}{(1-\gamma)(1-\gamma-\rho)(1-2\gamma-\rho)}\mathbb{1}\{\rho<0\} + \frac{2(2-3\gamma)}{\gamma(1-\gamma)^2(1-2\gamma)^2}\mathbb{1}\{\rho=0\} + \mathrm{o}(1)\right]\right).
\end{aligned}$$

Conclude the proof by combining (A.2) with (A.4) and further straightforward calculations. □

Inverting the limiting relationship (2.4), and providing an asymptotic expansion that strengthens (2.5), requires in particular an asymptotic expansion of $x^{\star} - \xi_{\tau}$. We do so in the following lemma.

Lemma A.3. *Suppose that condition $\mathcal{C}_2(\gamma, a, \rho, A)$ holds with $\rho < 0$.*

(i) The limit $C = \lim_{s\to\infty} s^{-\gamma}(x^{\star} - U(s))$ exists, is positive and finite, with

$$x^{\star} - U(s) = Cs^{\gamma}\left(1 + \frac{\gamma}{\rho(\gamma+\rho)}A(s) + \mathrm{o}(|A(s)|)\right)$$

$$\text{and} \quad a(s) = -\gamma C s^{\gamma}\left(1 + \frac{A(s)}{\rho} + \mathrm{o}(|A(s)|)\right) \quad \text{as } s \to \infty.$$

(ii) With the notation of (i), as $x \uparrow x^{\star}$,

$$\overline{F}(x) = C^{1/\gamma}(x^{\star}-x)^{-1/\gamma}\left(1 + \frac{1}{\rho(\gamma+\rho)}A(1/\overline{F}(x)) + \mathrm{o}(|A(1/\overline{F}(x))|)\right)$$

$$\text{and } \overline{F}(x)a(1/\overline{F}(x)) = -\gamma C^{1/\gamma}(x^{\star}-x)^{1-1/\gamma}\left(1 + \frac{\rho+1}{\rho(\gamma+\rho)}A(1/\overline{F}(x)) + \mathrm{o}(|A(1/\overline{F}(x))|)\right).$$

Proof of Lemma A.3. The main difficulty is to show (i). We start by the assertion on U. By Remark B.3.7 in de Haan and Ferreira (2006) with $(c_1, c_2) = (1, 0)$, the limit $c = \lim_{s\to\infty} s^{-\gamma}a(s) \in (0, \infty)$ exists and the function g defined by

$$g(s) = U(s) - c\frac{s^\gamma - 1}{\gamma}$$

satisfies

$$\lim_{s\to\infty} \frac{g(sz) - g(s)}{a(s)A(s)} = \frac{1}{\rho} \times \frac{z^{\gamma+\rho} - 1}{\gamma + \rho}, \text{ for all } z > 0.$$

By Theorem B.2.2 on p.373 of de Haan and Ferreira (2006) applied to $-\operatorname{sign}(A)g$, $c' = \lim_{s\to\infty} g(s)$ exists and

$$\lim_{s\to\infty} \frac{c' - g(s)}{a(s)A(s)} = -\frac{1}{\rho(\gamma + \rho)}.$$

The identity $c' = \lim_{s\to\infty} g(s)$ yields

$$c' = x^\star + \frac{c}{\gamma} \text{ and thus } c' - g(s) = x^\star - U(s) + c\frac{s^\gamma}{\gamma}.$$

The above convergence and the convergence $a(s)/(x^\star - U(s)) \to -\gamma$, as $s \to \infty$, then provide

$$\begin{aligned} x^\star - U(s) &= -c\frac{s^\gamma}{\gamma} - \frac{1}{\rho(\gamma+\rho)}a(s)A(s)(1 + \mathrm{o}(1)) \\ &= -c\frac{s^\gamma}{\gamma} + \frac{\gamma}{\rho(\gamma+\rho)}(x^\star - U(s))A(s)(1 + \mathrm{o}(1)). \end{aligned}$$

Set finally $C = -c/\gamma$ to find

$$x^\star - U(s) = Cs^\gamma \left(1 + \frac{\gamma}{\rho(\gamma+\rho)}A(s) + \mathrm{o}(|A(s)|)\right), \ s \to \infty,$$

as required. To show the assertion on a, set $h(s) = s^{-\gamma}a(s)$ and rewrite Equation (2.3.7) on p.44 of de Haan and Ferreira (2006) as

$$\lim_{s\to\infty} \frac{h(sz) - h(s)}{s^{-\gamma}a(s)A(s)} = \frac{z^\rho - 1}{\rho}, \text{ for all } z > 0.$$

By Theorem B.2.2 on p.373 of de Haan and Ferreira (2006) again,

$$\lim_{s\to\infty} \frac{c - h(s)}{s^{-\gamma}a(s)A(s)} = -\frac{1}{\rho}.$$

The conclusion in (i) is now immediate since $h(s) = s^{-\gamma}a(s)$ and $c/C = -\gamma$. The expansions in (ii) are obtained by taking $s = 1/\overline{F}(x)$ in the expansion of $x^\star - U(s)$ and then using Lemma A.1(i). □

The following lemma is the essential element in obtaining the joint asymptotic normality of the LAWS estimator and empirical quantile when the data generating process is $\alpha-$mixing. In the proof of this lemma and later on we shall use the following result: under condition $\mathcal{C}_2(\gamma, a, \rho, A)$, if $x_n, u_n \uparrow x^\star$ satisfy $(x^\star - x_n)/(x^\star - u_n) \to 1$, then

$$\frac{\overline{F}(x_n)}{\overline{F}(u_n)} \to 1 \text{ and } \frac{a(1/\overline{F}(x_n))}{a(1/\overline{F}(u_n))} \to 1. \tag{A.5}$$

The first convergence is found by using the regular variation property of $x \mapsto \overline{F}(x^\star - 1/x)$. The second one is then obtained by combining the convergence $a(s)/(x^\star - U(s)) \to -\gamma$, as $s \to \infty$, with the regular variation property of $s \mapsto x^\star - U(s)$.

Lemma A.4. *Assume that X satisfies condition $\mathcal{C}_2(\gamma, a, \rho, A)$. Suppose that F is continuous and that $(X_t)_{t\geq 1}$ is a strictly stationary sequence of copies of X satisfying conditions $\mathcal{M}$ and $\mathcal{D}$. Let finally $u_n \uparrow x^\star$ be such that $n\overline{F}(u_n) \to \infty$, $r_n\overline{F}(u_n) \to 0$, and $x_n, x_n' \uparrow x^\star$ be such that $(x^\star - x_n)/(x^\star - u_n) \to 1$ and $(x^\star - x_n')/(x^\star - u_n) \to 1$.*

(i) If $s_n \to \infty$ is such that $s_n = \mathrm{O}(r_n)$, one has

$$\frac{n}{s_n} \operatorname{Var}\left(\sum_{t=1}^{s_n} \sqrt{\frac{\overline{F}(u_n)}{n}} \left(\frac{\mathbb{1}\{X_t > x_n\}}{\mathbb{P}(X > x_n)} - 1 \right) \right) \to 1 + 2\sum_{t=1}^{\infty} R_t(1,1),$$

$$\frac{n}{s_n} \operatorname{Cov}\left(\sum_{t=1}^{s_n} \sqrt{\frac{\overline{F}(u_n)}{n}} \left(\frac{(X_t - x_n)\mathbb{1}\{X_t > x_n\}}{\mathbb{E}((X - x_n)\mathbb{1}\{X > x_n\})} - 1 \right), \sum_{t=1}^{s_n} \sqrt{\frac{\overline{F}(u_n)}{n}} \left(\frac{\mathbb{1}\{X_t > x_n'\}}{\mathbb{P}(X > x_n')} - 1 \right) \right)$$

$$\to 1 + (1 - \gamma^{-1}) \int_0^1 \sum_{t=1}^{\infty} [R_t(x^{-1/\gamma}, 1) + R_t(1, x^{-1/\gamma})] \, \mathrm{d}x, \ \textit{and}$$

$$\frac{n}{s_n} \operatorname{Var}\left(\sum_{t=1}^{s_n} \sqrt{\frac{\overline{F}(u_n)}{n}} \left(\frac{(X_t - x_n)\mathbb{1}\{X_t > x_n\}}{\mathbb{E}((X - x_n)\mathbb{1}\{X > x_n\})} - 1 \right) \right)$$

$$\to \frac{2(1-\gamma)}{1 - 2\gamma} + 2(1 - \gamma^{-1})^2 \iint_{(0,1]^2} \sum_{t=1}^{\infty} R_t(x^{-1/\gamma}, y^{-1/\gamma}) \, \mathrm{d}x \, \mathrm{d}y.$$

(ii) If the assumption $s_n \to \infty$ is dropped, then each of the three sequences in (i) stays bounded.

Proof of Lemma A.4. We prove both statements for the third sequence because the proofs for the first two sequences are simpler, and we start by preliminary calculations. By Lemma A.2(ii),

$$\frac{n}{s_n} \operatorname{Var}\left(\sum_{t=1}^{s_n} \sqrt{\frac{\overline{F}(u_n)}{n}} \left(\frac{(X_t - x_n)\mathbb{1}\{X_t > x_n\}}{\mathbb{E}((X - x_n)\mathbb{1}\{X > x_n\})} - 1 \right) \right)$$

$$= (1-\gamma)^2 \times \frac{1 + \mathrm{o}(1)}{\overline{F}(u_n)[a(1/\overline{F}(u_n))]^2} \times \frac{1}{s_n} \operatorname{Var}\left(\sum_{t=1}^{s_n} (X_t - x_n)\mathbb{1}\{X_t > x_n\} \right). \tag{A.6}$$

Then, combining Lemma A.2(ii) and (iii) with (A.5) and $s_n\overline{F}(u_n) = \mathrm{O}(r_n\overline{F}(u_n)) \to 0$,

$$\begin{aligned}
&\frac{1}{s_n} \operatorname{Var}\left(\sum_{t=1}^{s_n} (X_t - x_n)\mathbb{1}\{X_t > x_n\} \right) \\
&= \mathbb{E}((X - x_n)^2 \mathbb{1}\{X > x_n\}) - s_n \left[\mathbb{E}((X - x_n)\mathbb{1}\{X > x_n\})\right]^2 \\
&+ \frac{2}{s_n} \sum_{t=1}^{s_n - 1} (s_n - t)\mathbb{E}((X_1 - x_n)(X_{t+1} - x_n)\mathbb{1}\{X_1 > x_n, X_{t+1} > x_n\}) \\
&= \frac{2}{(1-\gamma)(1-2\gamma)} \overline{F}(u_n)[a(1/\overline{F}(u_n))]^2(1 + \mathrm{o}(1)) \\
&+ 2\sum_{t=1}^{\infty} \left(1 - \frac{t}{s_n}\right) \iint_{[x_n, x^\star)^2} \mathbb{P}(X_1 > v, X_{t+1} > v') \, \mathrm{d}v \, \mathrm{d}v' \mathbb{1}\{t < s_n\}.
\end{aligned} \tag{A.7}$$

It remains to control the integral in (A.7). Taking into account the continuity of F, the change of variables $(v, v') = (x_n - \gamma(x^\star - x_n)w, x_n - \gamma(x^\star - x_n)w') = (x^\star - (1 + \gamma w)(x^\star -$

$x_n), x^\star - (1+\gamma w')(x^\star - x_n))$, convergence $a(s)/(x^\star - U(s)) \to -\gamma$ as $s \to \infty$, Lemma A.1(i) and convergence (A.5) yield

$$\begin{aligned} &\frac{1}{\overline{F}(u_n)[a(1/\overline{F}(u_n))]^2} \iint_{[x_n,x^\star)^2} \mathbb{P}(X_1 > v, X_{t+1} > v')\,\mathrm{d}v\,\mathrm{d}v' \\ &= \iint_{[0,-1/\gamma)^2} \frac{1}{\overline{F}(u_n)} \mathbb{P}(\overline{F}(X_1) \le \overline{F}(x^\star - (1+\gamma w)(x^\star - x_n)), \\ &\qquad \overline{F}(X_{t+1}) \le \overline{F}(x^\star - (1+\gamma w')(x^\star - x_n)))\,\mathrm{d}w\,\mathrm{d}w'(1+\mathrm{o}(1)). \qquad (A.8) \end{aligned}$$

Since $x \mapsto \overline{F}(x^\star - 1/x)$ is regularly varying with index $1/\gamma$, one has

$$\forall w \in [0,-1/\gamma), \ \lim_{n\to\infty} \frac{\overline{F}(x^\star - (1+\gamma w)(x^\star - x_n))}{\overline{F}(x_n)} = (1+\gamma w)^{-1/\gamma}.$$

Then, we find, using Potter bounds (see Proposition B.1.9.5 on p.367 of de Haan and Ferreira, 2006) and the 1-homogeneity of the function R_t in condition $\mathcal{D}$ (as a direct consequence of its definition) along with (A.5) that, for any $w, w' \in [0,-1/\gamma)$ and $t \ge 1$,

$$\begin{aligned} &\frac{1}{\overline{F}(u_n)} \mathbb{P}(\overline{F}(X_1) \le \overline{F}(x^\star - (1+\gamma w)(x^\star - x_n)), \overline{F}(X_{t+1}) \le \overline{F}(x^\star - (1+\gamma w')(x^\star - x_n))) \\ &\to R_t((1+\gamma w)^{-1/\gamma}, (1+\gamma w')^{-1/\gamma}) \text{ as } n \to \infty. \qquad (A.9) \end{aligned}$$

We now assume that $s_n \to \infty$ and we prove (i). Fix $\varepsilon \in (0,-1/\gamma)$. From condition $\mathcal{D}$, Potter bounds and (A.5) again, we have, for n large enough,

$$\begin{aligned} &\frac{1}{\overline{F}(u_n)} \mathbb{P}(\overline{F}(X_1) \le \overline{F}(x^\star - (1+\gamma w)(x^\star - x_n)), \overline{F}(X_{t+1}) \le \overline{F}(x^\star - (1+\gamma w')(x^\star - x_n))) \\ &\le C\left(\rho(t)\sqrt{(1+\gamma w)^{-1/\gamma-\varepsilon}(1+\gamma w')^{-1/\gamma-\varepsilon}} + \overline{F}(u_n)(1+\gamma w)^{-1/\gamma-\varepsilon}(1+\gamma w')^{-1/\gamma-\varepsilon}\right) \\ &\qquad (A.10) \end{aligned}$$

for any $t \ge 1$ and any $w, w' \in [0,-1/\gamma)$, where C is a positive constant. Notice that for any $t < s_n$,

$$\begin{aligned} &\rho(t)\sqrt{(1+\gamma w)^{-1/\gamma-\varepsilon}(1+\gamma w')^{-1/\gamma-\varepsilon}} + \overline{F}(u_n)(1+\gamma w)^{-1/\gamma-\varepsilon}(1+\gamma w')^{-1/\gamma-\varepsilon} \\ &\to \rho(t)\sqrt{(1+\gamma w)^{-1/\gamma-\varepsilon}(1+\gamma w')^{-1/\gamma-\varepsilon}} \qquad (A.11) \end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned} &\sum_{t=1}^{\infty} \iint_{[0,-1/\gamma)^2} \Big(\rho(t)\sqrt{(1+\gamma w)^{-1/\gamma-\varepsilon}(1+\gamma w')^{-1/\gamma-\varepsilon}} \\ &\qquad + \overline{F}(u_n)(1+\gamma w)^{-1/\gamma-\varepsilon}(1+\gamma w')^{-1/\gamma-\varepsilon}\Big)\,\mathrm{d}w\,\mathrm{d}w'\,\mathbb{1}\{t < s_n\} \\ &\to \left(\int_{[0,-1/\gamma)^2} \sqrt{(1+\gamma w)^{-1/\gamma-\varepsilon}}\,\mathrm{d}w\right)^2 \sum_{t=1}^{\infty} \rho(t) < \infty \qquad (A.12) \end{aligned}$$

by splitting the sum and using the assumption that $s_n\overline{F}(u_n) = \mathrm{O}(r_n\overline{F}(u_n)) \to 0$. Combine Theorem 1 in Pratt (1960) with (A.8), (A.9), (A.10), (A.11) and (A.12) to get

$$\begin{aligned} &\frac{1}{\overline{F}(u_n)[a(1/\overline{F}(u_n))]^2} \sum_{t=1}^{\infty} \left(1 - \frac{t}{s_n}\right) \iint_{[x_n,x^\star)^2} \mathbb{P}(X_1 > v, X_{t+1} > v')\,\mathrm{d}v\,\mathrm{d}v'\mathbb{1}\{t < s_n\} \\ &\to \iint_{[0,-1/\gamma)^2} \sum_{t=1}^{\infty} R_t((1+\gamma w)^{-1/\gamma}, (1+\gamma w')^{-1/\gamma})\,\mathrm{d}w\,\mathrm{d}w'. \end{aligned}$$

Plug this into (A.7) and use a change of variables to complete the proof of (i). To show (ii), write $s_n \le C' r_n$ where C' is a positive constant, and note that, by (A.7),

$$\begin{aligned}
&\frac{1}{s_n}\operatorname{Var}\left(\sum_{t=1}^{s_n}(X_t - x_n)\mathbb{1}\{X_t > x_n\}\right) \\
&\le \frac{2}{(1-\gamma)(1-2\gamma)}\overline{F}(u_n)[a(1/\overline{F}(u_n))]^2(1+\mathrm{o}(1)) \\
&+ 2\sum_{t=1}^{\infty}\left(1-\frac{t}{C' r_n}\right)\iint_{[x_n,x^\star)^2}\mathbb{P}(X_1 > v, X_{t+1} > v')\,\mathrm{d}v\,\mathrm{d}v'\,\mathbb{1}\{t < C' r_n\}.
\end{aligned}$$

Follow then the proof of (i) and use (A.6) to obtain that the upper bound converges as $n \to \infty$. The desired conclusion is now immediate. □

Lemma A.5 below provides the asymptotic normality of the empirical survival function $\widehat{\overline{E}}_n$ at intermediate levels, when the data generating process is at least α−mixing. The asymptotic normality of the intermediate LAWS estimator will follow from that result.

Lemma A.5. *Assume that X satisfies condition $\mathcal{C}_2(\gamma, a, \rho, A)$. Suppose that F is continuous and that $(X_t)_{t\ge1}$ is a strictly stationary sequence of copies of X satisfying conditions $\mathcal{M}$ and $\mathcal{D}$. Let finally $u_n \uparrow x^\star$ be such that $n\overline{F}(u_n) \to \infty$, $r_n\overline{F}(u_n) \to 0$, and $x_n, x_n' \uparrow x^\star$ be such that $(x^\star - x_n)/(x^\star - u_n) \to 1$ and $(x^\star - x_n')/(x^\star - u_n) \to 1$.*

(i) If there is $\delta > 0$ such that $r_n(r_n/\sqrt{n\overline{F}(u_n)})^\delta \to 0$, then one has

$$\sqrt{n\overline{F}(u_n)}\left(\frac{\widehat{\varphi}_n^{(1)}(x_n)}{\varphi^{(1)}(x_n)} - 1, \frac{\widehat{\overline{F}}_n(x_n')}{\overline{F}(x_n')} - 1\right) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma}(\gamma) + 2\,\boldsymbol{D}(\gamma, R))$$

where the 2×2 symmetric matrices $\boldsymbol{\Sigma}(\gamma)$ and $\boldsymbol{D}(\gamma,R)$ are defined elementwise as $\Sigma_{11}(\gamma) = 2(1-\gamma)/(1-2\gamma)$, $\Sigma_{12}(\gamma) = \Sigma_{22}(\gamma) = 1$,

$$D_{11}(\gamma, R) = (1-\gamma^{-1})^2\iint_{(0,1]^2}\sum_{t=1}^{\infty}R_t(x^{-1/\gamma}, y^{-1/\gamma})\,\mathrm{d}x\,\mathrm{d}y$$

$$D_{12}(\gamma, R) = \frac{1}{2}(1-\gamma^{-1})\int_0^1\sum_{t=1}^{\infty}[R_t(x^{-1/\gamma}, 1) + R_t(1, x^{-1/\gamma})]\,\mathrm{d}x$$

and $D_{22}(\gamma, R) = \sum_{t=1}^{\infty}R_t(1,1)$.

(ii) If, choosing $\delta > 0$ as in (i), one has $\mathbb{E}(|\min(X,0)|^{2+\delta}) < \infty$ and $\sum_{l\ge1} l^{2/\delta}\alpha(l) < \infty$, then

$$\sqrt{n\overline{F}(u_n)}\left(\frac{\widehat{\overline{E}}_n(x_n)}{\overline{E}(x_n)} - 1, \frac{\widehat{\overline{F}}_n(x_n')}{\overline{F}(x_n')} - 1\right) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Sigma}(\gamma) + 2\,\boldsymbol{D}(\gamma, R)).$$

If X is bounded, then in (ii) assumption $\sum_{l\ge1} l^{2/\delta}\alpha(l) < \infty$ can be weakened to $\sum_{l\ge1}\alpha(l) < \infty$ and no integrability assumption on X is necessary.

If the X_i are in fact i.i.d. then both results hold with $\boldsymbol{D}(\gamma, R) = 0$ under the sole assumptions that X satisfies condition $\mathcal{C}_2(\gamma, a, \rho, A)$, $u_n \uparrow x^\star$ is such that $n\overline{F}(u_n) \to \infty$, and $x_n, x_n' \uparrow x^\star$ are such that $(x^\star - x_n)/(x^\star - u_n) \to 1$ and $(x^\star - x_n')/(x^\star - u_n) \to 1$, with the extra requirement that $\mathbb{E}(|\min(X,0)|^2) < \infty$ for (ii) only.

Proof of Lemma A.5. (i) Pick $(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}$. Clearly

$$\begin{aligned}
&\sqrt{n\overline{F}(u_n)}\left\{\lambda\left(\frac{\widehat{\varphi}_n^{(1)}(x_n)}{\varphi^{(1)}(x_n)}-1\right)+\mu\left(\frac{\widehat{\overline{F}}_n(x_n')}{\overline{F}(x_n')}-1\right)\right\}\\
&=\sum_{t=1}^n\lambda\times\sqrt{\frac{\overline{F}(u_n)}{n}}\left(\frac{(X_t-x_n)\mathbb{1}\{X_t>x_n\}}{\mathbb{E}((X-x_n)\mathbb{1}\{X>x_n\})}-1\right)+\mu\times\sqrt{\frac{\overline{F}(u_n)}{n}}\left(\frac{\mathbb{1}\{X_t>x_n'\}}{\mathbb{P}(X>x_n')}-1\right)\\
&=\sum_{t=1}^n\mathcal{X}_{n,t}(\lambda,\mu)
\end{aligned}$$

is a mean of identically distributed and centered random variables for every n. We start by the case when $(X_t)_{t\ge1}$ is an $\alpha-$mixing sequence. We aim to apply Lemma C.7(ii) in Davison et al. (2023), of which we check each condition. By Lemma A.4, and using condition $\mathcal{M}$,

$$\begin{aligned}
\frac{n}{r_n}\operatorname{Var}\left(\sum_{t=1}^{l_n}\mathcal{X}_{n,t}(\lambda,\mu)\right)&=\mathrm{O}(l_n/r_n)\to0,\\
\operatorname{Var}\left(\sum_{t=1}^{n-r_n\lfloor n/r_n\rfloor}\mathcal{X}_{n,t}(\lambda,\mu)\right)&=\mathrm{O}((n-r_n\lfloor n/r_n\rfloor)/n)=\mathrm{O}(r_n/n)\to0,\\
\lim_{n\to\infty}\frac{n}{r_n}\operatorname{Var}\left(\sum_{t=1}^{r_n}\mathcal{X}_{n,t}(\lambda,\mu)\right)&=\lambda^2(\Sigma_{11}(\gamma)+2D_{11}(\gamma,R))+2\lambda\mu(\Sigma_{12}(\gamma)+2D_{12}(\gamma,R))\\
&\quad+\mu^2(\Sigma_{22}(\gamma)+2D_{22}(\gamma,R)).
\end{aligned}$$

Besides, for any $\varepsilon>0$,

$$\begin{aligned}
\frac{n}{r_n}\mathbb{E}\left(\left|\sum_{t=1}^{r_n}\mathcal{X}_{n,t}(\lambda,\mu)\right|^2\mathbb{1}\left\{\left|\sum_{t=1}^{r_n}\mathcal{X}_{n,t}(\lambda,\mu)\right|>\varepsilon\right\}\right)&\le\varepsilon^{-\delta}\times\frac{n}{r_n}\mathbb{E}\left(\left|\sum_{t=1}^{r_n}\mathcal{X}_{n,t}(\lambda,\mu)\right|^{2+\delta}\right)\\
&=\mathrm{O}\left(nr_n^{1+\delta}\mathbb{E}(|\mathcal{X}_{n,1}(\lambda,\mu)|^{2+\delta})\right)\\
&=\mathrm{O}\left(r_n\left[\frac{r_n}{\sqrt{n\overline{F}(u_n)}}\right]^\delta\right)
\end{aligned}$$

by the Hölder inequality and Lemma A.2(i) and (ii). This converges to 0 by assumption, so Lemma C.7(ii) in Davison et al. (2023) applies and yields the desired conclusion in the $\alpha-$mixing framework thanks to the Cramér-Wold device. When the X_i are i.i.d., one may apply the standard Lyapunov central limit theorem (Billingsley, 1995, Theorem 27.3 p.362) instead: first of all

$$n\operatorname{Var}(\mathcal{X}_{n,1}(\lambda,\mu))\to\lambda^2\frac{2(1-\gamma)}{1-2\gamma}+2\lambda\mu+\mu^2$$

by Lemma A.2, because of (A.5). Then, by the Hölder inequality and Lemma A.2(i) and (ii),

$$n\,\mathbb{E}|\mathcal{X}_{n,1}(\lambda,\mu)|^4=\mathrm{O}\left(n\left(\frac{\overline{F}(u_n)}{n}\right)^2\left(\frac{\varphi^{(4)}(x_n)}{[\varphi^{(1)}(x_n)]^4}+\frac{1}{[\overline{F}(u_n)]^3}\right)\right)=\mathrm{O}\left(\frac{1}{n\overline{F}(u_n)}\right).$$

This converges to 0, so the Lyapunov central limit theorem applies and the proof of (i) is complete.

To show (ii), write

$$\log \frac{\widehat{\overline{E}}_n(x_n)}{\overline{E}(x_n)} = \log \frac{\widehat{\varphi}_n^{(1)}(x_n)}{\varphi^{(1)}(x_n)} - \log\left(\frac{2\widehat{\varphi}_n^{(1)}(x_n) + x_n - \overline{X}_n}{2\varphi^{(1)}(x_n) + x_n - \mathbb{E}(X)}\right).$$

Since

$$\begin{aligned}\frac{2\widehat{\varphi}_n^{(1)}(x_n) + x_n - \overline{X}_n}{2\varphi^{(1)}(x_n) + x_n - \mathbb{E}(X)} - 1 &= \frac{2(\widehat{\varphi}_n^{(1)}(x_n) - \varphi^{(1)}(x_n)) - (\overline{X}_n - \mathbb{E}(X))}{2\varphi^{(1)}(x_n) + x_n - \mathbb{E}(X)} \\ &= \mathrm{O}_{\mathbb{P}}\left(\frac{\varphi^{(1)}(x_n)}{\sqrt{n\overline{F}(u_n)}} \times \sqrt{n\overline{F}(u_n)}\left(\frac{\widehat{\varphi}_n^{(1)}(x_n)}{\varphi^{(1)}(x_n)} - 1\right)\right) + \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n}} \times \sqrt{n}(\overline{X}_n - \mathbb{E}(X))\right),\end{aligned}$$

it follows that

$$\frac{2\widehat{\varphi}_n^{(1)}(x_n) + x_n - \overline{X}_n}{2\varphi^{(1)}(x_n) + x_n - \mathbb{E}(X)} - 1 = \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{n\overline{F}(u_n)}}\right)$$

by Lemma A.2(i) and Corollary 1.2 on p.10 of Rio (2017) along with (1.25a) and (1.25b) on p.12 therein, when the X_i are $\alpha-$mixing and under the assumptions $\mathbb{E}(|\min(X,0)|^{2+\delta}) < \infty$ and $\sum_{l\geq 1} l^{2/\delta}\alpha(l) < \infty$. [When X is also bounded, condition $\sum_{l\geq 1}\alpha(l) < \infty$ is sufficient, see (1.24) on p.11 of Rio (2017).] In the case when the X_i are i.i.d., the usual central limit theorem can be applied instead in order to control $\overline{X}_n - \mathbb{E}(X)$, under the condition $\mathbb{E}(|\min(X,0)|^2) < \infty$. Hence, in both cases, the equality

$$\log \frac{\widehat{\overline{E}}_n(x_n)}{\overline{E}(x_n)} = \log \frac{\widehat{\varphi}_n^{(1)}(x_n)}{\varphi^{(1)}(x_n)} + \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{n\overline{F}(u_n)}}\right)$$

from which (ii) follows by applying (i). □

A.2 Proofs of the main results

Proof of Proposition 1. The starting point is to combine Equation (2.2) and Lemma A.2, in order to obtain

$$\begin{aligned}&x^\star - \mathbb{E}(X) - (x^\star - \xi_\tau) \\ &= \frac{2\tau - 1}{1-\tau} \times \frac{\overline{F}(\xi_\tau)a(1/\overline{F}(\xi_\tau))}{1-\gamma}\left(1 + \frac{1}{1-\gamma-\rho}A(1/\overline{F}(\xi_\tau)) + \mathrm{o}(|A(1/\overline{F}(\xi_\tau))|)\right)\end{aligned}$$

as $\tau \uparrow 1$. In other words,

$$\begin{aligned}\frac{\overline{F}(\xi_\tau)a(1/\overline{F}(\xi_\tau))}{1-\tau} &= (1-\gamma)[(x^\star - \mathbb{E}(X)) - (x^\star - \xi_\tau)] \times (1 - 2(1-\tau))^{-1} \\ &\times \left(1 - \frac{1}{1-\gamma-\rho}A(1/\overline{F}(\xi_\tau)) + \mathrm{o}(|A(1/\overline{F}(\xi_\tau))|)\right) \quad \text{as } \tau \uparrow 1. \qquad \text{(A.13)}\end{aligned}$$

Then, by Lemma A.3(i) with $s = 1/\overline{F}(q_\tau)$ and Lemma A.3(ii) with $s = 1/\overline{F}(\xi_\tau)$ combined with Lemma A.1, one has the alternative expansion

$$\begin{aligned}\frac{\overline{F}(\xi_\tau)a(1/\overline{F}(\xi_\tau))}{1-\tau} &= -\gamma\frac{(x^\star-\xi_\tau)^{1-1/\gamma}}{(x^\star-q_\tau)^{-1/\gamma}}\left(1+\frac{\rho+1}{\rho(\gamma+\rho)}A(1/\overline{F}(\xi_\tau))+\mathrm{o}(|A(1/\overline{F}(\xi_\tau))|)\right)\\ &\quad\times\left(1-\frac{1}{\rho(\gamma+\rho)}A((1-\tau)^{-1})+\mathrm{o}(|A((1-\tau)^{-1})|)\right)\quad\text{as } \tau\uparrow 1.\end{aligned}$$

A consequence of Equation (2.3) is that $1/\overline{F}(\xi_\tau) = \mathrm{o}((1-\tau)^{-1})$. Therefore, since $|A|$ is regularly varying with index $\rho < 0$, one has $A((1-\tau)^{-1}) = \mathrm{o}(|A(1/\overline{F}(\xi_\tau))|)$, from which it follows that

$$\frac{\overline{F}(\xi_\tau)a(1/\overline{F}(\xi_\tau))}{1-\tau} = -\gamma\frac{(x^\star-\xi_\tau)^{1-1/\gamma}}{(x^\star-q_\tau)^{-1/\gamma}}\left(1+\frac{\rho+1}{\rho(\gamma+\rho)}A(1/\overline{F}(\xi_\tau))+\mathrm{o}(|A(1/\overline{F}(\xi_\tau))|)\right) \tag{A.14}$$

as $\tau\uparrow 1$. Combine (A.13) and (A.14) to find

$$\begin{aligned}\frac{(x^\star-\xi_\tau)^{1-1/\gamma}}{(x^\star-q_\tau)^{-1/\gamma}} &= (x^\star-\mathbb{E}(X))(1-\gamma^{-1})\left[1-\frac{1}{x^\star-\mathbb{E}(X)}(x^\star-\xi_\tau)\right]\times(1-2(1-\tau))^{-1}\\ &\quad\times\left(1-\frac{1-\gamma}{\rho(\gamma+\rho)(1-\gamma-\rho)}A(1/\overline{F}(\xi_\tau))+\mathrm{o}(|A(1/\overline{F}(\xi_\tau))|)\right)\quad\text{as } \tau\uparrow 1.\end{aligned}$$

A consequence of Equation (2.4) is that $1-\tau = \mathrm{o}(x^\star-\xi_\tau)$. Hence

$$\begin{aligned}\frac{(x^\star-\xi_\tau)^{1-1/\gamma}}{(x^\star-q_\tau)^{-1/\gamma}} &= (x^\star-\mathbb{E}(X))(1-\gamma^{-1})\left[1-\frac{1}{x^\star-\mathbb{E}(X)}(x^\star-\xi_\tau)+\mathrm{o}(x^\star-\xi_\tau)\right]\\ &\quad\times\left(1-\frac{1-\gamma}{\rho(\gamma+\rho)(1-\gamma-\rho)}A(1/\overline{F}(\xi_\tau))+\mathrm{o}(|A(1/\overline{F}(\xi_\tau))|)\right)\quad\text{as } \tau\uparrow 1.\end{aligned} \tag{A.15}$$

Combine the above expansion with Equations (2.4) and (2.5) and the regular variation property of $|A|$ to get

$$\begin{aligned}&\frac{(x^\star-\xi_\tau)^{1-1/\gamma}}{(x^\star-q_\tau)^{-1/\gamma}}\\ &= (x^\star-\mathbb{E}(X))(1-\gamma^{-1})\\ &\times\left[1-(x^\star-\mathbb{E}(X))^{-1/(1-\gamma)}(1-\gamma^{-1})^{-\gamma/(1-\gamma)}(x^\star-q_\tau)^{1/(1-\gamma)}(1+\mathrm{o}(1))\right]\\ &\times\left(1-\frac{(1-\gamma)[(x^\star-\mathbb{E}(X))(1-\gamma^{-1})]^{-\rho/(1-\gamma)}}{\rho(\gamma+\rho)(1-\gamma-\rho)}A((1-\tau)^{-1}(x^\star-q_\tau)^{1/(1-\gamma)})(1+\mathrm{o}(1))\right)\end{aligned}$$

as $\tau\uparrow 1$. Then clearly

$$\begin{aligned}x^\star-\xi_\tau &= [(x^\star-\mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)}(x^\star-q_\tau)^{1/(1-\gamma)}\\ &\quad\times\left[1-[(x^\star-\mathbb{E}(X))(1-\gamma^{-1})]^{-1/(1-\gamma)}(x^\star-q_\tau)^{1/(1-\gamma)}(1+\mathrm{o}(1))\right]\\ &\quad\times\left(1+\frac{\gamma[(x^\star-\mathbb{E}(X))(1-\gamma^{-1})]^{-\rho/(1-\gamma)}}{\rho(\gamma+\rho)(1-\gamma-\rho)}A((1-\tau)^{-1}(x^\star-q_\tau)^{1/(1-\gamma)})(1+\mathrm{o}(1))\right)\end{aligned}$$

as $\tau\uparrow 1$, which is the first desired asymptotic expansion. The second statement is then a direct consequence of a combination of Lemma A.3(i), for $s = (1-\tau)^{-1}$, with this asymptotic expansion. □

Proof of Theorem 1. Fix $u, v \in \mathbb{R}$. To prove the desired joint convergence, it is sufficient to examine the convergence of the sequence

$$\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n} - \xi_{\tau_n}) \le u,\ \frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{q}_{\pi_n} - q_{\pi_n}) \le v\right).$$

First of all, since $a(s)/(x^\star - U(s)) \to -\gamma$ as $s \to \infty$, and $s \mapsto x^\star - U(s)$ is regularly varying, the assumption $\overline{F}(\xi_{\tau_n})/(1-\pi_n) \to 1$ yields

$$\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))} = \frac{\sqrt{n(1-\pi_n)}}{a((1-\pi_n)^{-1})}(1+\mathrm{o}(1)).$$

As a result, it is equivalent to analyze the asymptotic behavior of

$$\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n} - \xi_{\tau_n}) \le u,\ \frac{\sqrt{n(1-\pi_n)}}{a((1-\pi_n)^{-1})}(\widehat{q}_{\pi_n} - q_{\pi_n}) \le v\right).$$

The key observation in order to do so is that, for fixed $u, v \in \mathbb{R}$, if

$$x_n = x_n(u) = \xi_{\tau_n} + u\frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} \text{ and } x_n' = x_n'(v) = q_{\pi_n} + v\frac{a((1-\pi_n)^{-1})}{\sqrt{n(1-\pi_n)}}$$

then, following a simple calculation,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n} - \xi_{\tau_n}) \le u,\ \frac{\sqrt{n(1-\pi_n)}}{a((1-\pi_n)^{-1})}(\widehat{q}_{\pi_n} - q_{\pi_n}) \le v\right) \\
&= \mathbb{P}\left(\sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\widehat{\overline{E}}_n(x_n)}{\overline{E}(x_n)} - 1\right) \le \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\overline{E}(\xi_{\tau_n})}{\overline{E}(x_n)} - 1\right),\right. \\
&\qquad \left. \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\widehat{\overline{F}}_n(x_n')}{\overline{F}(x_n')} - 1\right) \le \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{1-\pi_n}{\overline{F}(x_n')} - 1\right)\right) \qquad \text{(A.16)}
\end{aligned}$$

because ξ_{τ_n} (resp. $\widehat{\xi}_{\tau_n}$) is the τ_nth quantile of the continuous distribution function E (resp. the distribution function $\widehat{E}_n$), and likewise q_{π_n} (resp. $\widehat{q}_{\pi_n}$) is the π_nth quantile of the distribution function F (resp. the distribution function $\widehat{F}_n$). We first handle the right-hand sides of both of the inequalities in (A.16). Note that

$$\frac{x^\star - x_n}{x^\star - \xi_{\tau_n}} - 1 = -u\frac{a(1/\overline{F}(\xi_{\tau_n}))}{x^\star - \xi_{\tau_n}} \times \frac{1}{\sqrt{n\overline{F}(\xi_{\tau_n})}} = \mathrm{O}\left(\frac{1}{\sqrt{n\overline{F}(\xi_{\tau_n})}}\right) \to 0 \qquad \text{(A.17)}$$

using the convergence $a(s)/(x^\star - U(s)) \to -\gamma$ as $s \to \infty$, and Lemma A.1(i). Note further that the function $\overline{E}$ is absolutely continuous on any compact interval, because

- The function $x \mapsto \varphi^{(1)}(x) = \int_x^\infty \overline{F}(y)\,dy$ is Lipschitz continuous,
- The denominator $x \mapsto \mathbb{E}(|X - x|) = 2\varphi^{(1)}(x) + x - \mathbb{E}(X)$ of $\overline{E}$ defines a Lipschitz continuous function that is bounded away from zero.

A straightforward calculation shows that $\overline{E}$ has Lebesgue derivative

$$\overline{E}'(x) = -\frac{\varphi^{(1)}(x) + \overline{F}(x)(x - \mathbb{E}(X))}{(2\varphi^{(1)}(x) + x - \mathbb{E}(X))^2}.$$

In particular, it comes as a consequence of Lemma A.2(ii) that $-\overline{E}'(x)/\overline{F}(x) \to 1/(x^\star - \mathbb{E}(X))$ as $x \uparrow x^\star$. Then

$$\begin{aligned}
\sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\overline{E}(\xi_{\tau_n})}{\overline{E}(x_n)} - 1\right) &= \sqrt{n\overline{F}(\xi_{\tau_n})}\int_{x_n}^{\xi_{\tau_n}} \frac{\overline{E}'(y)}{\overline{E}(x_n)}\mathrm{d}y \\
&= \sqrt{n\overline{F}(\xi_{\tau_n})}\int_{\xi_{\tau_n}}^{x_n} \frac{(1-\gamma)\overline{F}(y)}{\overline{F}(x_n)a(1/\overline{F}(x_n))}\mathrm{d}y(1+\mathrm{o}(1)) \\
&= (1-\gamma) \times \sqrt{n\overline{F}(\xi_{\tau_n})}\frac{x_n - \xi_{\tau_n}}{a(1/\overline{F}(\xi_{\tau_n}))}(1+\mathrm{o}(1)) \\
&\to (1-\gamma)u
\end{aligned} \tag{A.18}$$

as $n \to \infty$, by combining (A.5) with Lemma A.2(ii) and (A.17). Besides

$$\frac{x^\star - x_n'}{x^\star - \xi_{\tau_n}} - 1 = \left(\frac{x^\star - q_{\pi_n}}{x^\star - \xi_{\tau_n}} - 1\right) - v\frac{a((1-\pi_n)^{-1})}{x^\star - q_{\pi_n}} \times \frac{1+\mathrm{o}(1)}{\sqrt{n\overline{F}(\xi_{\tau_n})}} \to 0 \tag{A.19}$$

because of the convergence $(x^\star - q_{\pi_n})/(x^\star - \xi_{\tau_n}) \to 1$, granted by the assumption $\overline{F}(\xi_{\tau_n})/(1-\pi_n) \to 1$, the regular variation property of $s \mapsto x^\star - U(s)$ and convergence $a(s)/(x^\star - U(s)) \to -\gamma$ as $s \to \infty$, and Lemma A.1(i). Then

$$\begin{aligned}
\sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{1-\pi_n}{\overline{F}(x_n')} - 1\right) &= \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\overline{F}(q_{\pi_n})}{\overline{F}(x_n')} - 1\right) \\
&\quad + \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{1-\pi_n}{\overline{F}(q_{\pi_n})} - 1\right)\frac{\overline{F}(q_{\pi_n})}{\overline{F}(x_n')} \\
&= \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\overline{F}(q_{\pi_n})}{\overline{F}(x_n')} - 1\right) + \mathrm{o}(1)
\end{aligned}$$

because of Lemma A.1(ii), the asymptotic equivalence between $\overline{F}(q_{\pi_n})$, $\overline{F}(\xi_{\tau_n})$ and $\overline{F}(x_n')$ due (in part) to (A.19), assumption $\sqrt{n\overline{F}(\xi_{\tau_n})}A(1/\overline{F}(\xi_{\tau_n})) = \mathrm{O}(1)$ and the regular variation property of $|A|$. Note now that the convergence in Theorem 2.3.8 on p.48 of de Haan and Ferreira (2006), which is equivalent to the convergence granted by condition $\mathcal{C}_2(\gamma, a, \rho, A)$, is actually locally uniform by Theorem B.3.19 on p.401 of de Haan and Ferreira (2006), and therefore

$$\begin{aligned}
\sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\overline{F}(q_{\pi_n})}{\overline{F}(x_n')} - 1\right) &= \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\overline{F}(q_{\pi_n})}{\overline{F}(q_{\pi_n} + v\,a((1-\pi_n)^{-1})/\sqrt{n(1-\pi_n)})} - 1\right) \\
&\to v
\end{aligned}$$

using the asymptotic equivalence between $1-\pi_n$, $\overline{F}(q_{\pi_n})$ and $\overline{F}(\xi_{\tau_n})$. Conclude that

$$\sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{1-\pi_n}{\overline{F}(x_n')} - 1\right) \to v. \tag{A.20}$$

Combine (A.16), (A.18) and (A.20) to get

$$\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n}-\xi_{\tau_n})\le u,\ \frac{\sqrt{n(1-\pi_n)}}{a((1-\pi_n)^{-1})}(\widehat{q}_{\pi_n}-q_{\pi_n})\le v\right)$$
$$=\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{1-\gamma}\left(\frac{\widehat{\overline{E}}_n(x_n)}{\overline{E}(x_n)}-1\right)+\mathrm{o}(1)\le u,\ \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\widehat{\overline{F}}_n(x_n')}{\overline{F}(x_n')}-1\right)+\mathrm{o}(1)\le v\right).$$

Use Lemma A.5(ii) (this is allowed because of (A.17) and (A.19)) to conclude the proof. □

Proof of Corollary 2. Fix $v\in\mathbb{R}$. The proof relies on the identity

$$\left\{\sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\widehat{\overline{F}}_n(\widehat{\xi}_{\tau_n})}{\overline{F}(\xi_{\tau_n})}-1\right)\le v\right\}=\{\widehat{q}_{\pi_n}\le\widehat{\xi}_{\tau_n}\}$$

where

$$\pi_n=\pi_n(v)=1-\overline{F}(\xi_{\tau_n})\left(1+\frac{v}{\sqrt{n\overline{F}(\xi_{\tau_n})}}\right).$$

We therefore investigate the asymptotic behavior of $\mathbb{P}(\widehat{q}_{\pi_n}\le\widehat{\xi}_{\tau_n})$. To this end we write

$$\{\widehat{q}_{\pi_n}\le\widehat{\xi}_{\tau_n}\}$$
$$=\left\{\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{q}_{\pi_n}-q_{\pi_n})-\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n}-\xi_{\tau_n})\le\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\xi_{\tau_n}-q_{\pi_n})\right\}.$$

Now

$$\xi_{\tau_n}-q_{\pi_n}=-\left[U\left(\frac{1}{\overline{F}(\xi_{\tau_n})}\left(1+\frac{v}{\sqrt{n\overline{F}(\xi_{\tau_n})}}\right)^{-1}\right)-U\left(\frac{1}{\overline{F}(\xi_{\tau_n})}\right)\right]+\mathrm{o}\left(\frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}}\right)$$

by Lemma A.1(i) and condition $\sqrt{n\overline{F}(\xi_{\tau_n})}A(1/\overline{F}(\xi_{\tau_n}))=\mathrm{O}(1)$. Since condition $\mathcal{C}_2(\gamma,a,\rho,A)$ holds locally uniformly in z, a Taylor expansion and condition $\sqrt{n\overline{F}(\xi_{\tau_n})}A(1/\overline{F}(\xi_{\tau_n}))=\mathrm{O}(1)$ yield

$$\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\xi_{\tau_n}-q_{\pi_n})\to v.$$

As a consequence

$$\{\widehat{q}_{\pi_n}\le\widehat{\xi}_{\tau_n}\}=\left\{\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{q}_{\pi_n}-q_{\pi_n})-\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n}-\xi_{\tau_n})+\mathrm{o}(1)\le v\right\}.$$

The conclusion follows by applying Theorem 1. □

Proof of Theorem 2. Recall from Equation (2.4) that $1-\tau_n = \mathrm{o}(\overline{F}(\xi_{\tau_n}))$ and write

$$
\begin{aligned}
&\frac{\sqrt{n(1-\tau_n)}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}^{\star}_{1-p_n} - \xi_{1-p_n}) \\
&= \frac{\sqrt{n(1-\tau_n)}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n} - \xi_{\tau_n}) \\
&+ \sqrt{n(1-\tau_n)}\left(\frac{\widehat{\sigma}_n}{a(1/\overline{F}(\xi_{\tau_n}))} - 1\right)\frac{((1-\tau_n)/p_n)^{\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - 1}{\widehat{\gamma}_n} \\
&+ \sqrt{n(1-\tau_n)}\left(\frac{((1-\tau_n)/p_n)^{\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - 1}{\widehat{\gamma}_n} - \frac{((1-\tau_n)/p_n)^{\gamma/(1-\gamma)} - 1}{\gamma}\right) \\
&+ \sqrt{n(1-\tau_n)}\left(\frac{((1-\tau_n)/p_n)^{\gamma/(1-\gamma)} - (\overline{F}(\xi_{\tau_n})/\overline{F}(\xi_{1-p_n}))^{\gamma}}{\gamma}\right) \\
&- \sqrt{n(1-\tau_n)}\left(\frac{\xi_{1-p_n} - \xi_{\tau_n}}{a(1/\overline{F}(\xi_{\tau_n}))} - \frac{(\overline{F}(\xi_{\tau_n})/\overline{F}(\xi_{1-p_n}))^{\gamma} - 1}{\gamma}\right) \qquad \text{(A.21)} \\
&= \sqrt{n(1-\tau_n)}\left(\frac{((1-\tau_n)/p_n)^{\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - 1}{\widehat{\gamma}_n} - \frac{((1-\tau_n)/p_n)^{\gamma/(1-\gamma)} - 1}{\gamma}\right) \\
&+ \sqrt{n(1-\tau_n)}\left(\frac{((1-\tau_n)/p_n)^{\gamma/(1-\gamma)} - (\overline{F}(\xi_{\tau_n})/\overline{F}(\xi_{1-p_n}))^{\gamma}}{\gamma}\right) + \mathrm{o}_{\mathbb{P}}(1) \qquad \text{(A.22)}
\end{aligned}
$$

by Theorem 1 (for the control of the first term in (A.21)), the assumption on $\widehat{\sigma}_n$, the convergence of $\widehat{\gamma}_n$ to $\gamma < 0$, the assumption $(1-\tau_n)/p_n \to \infty$ (for the second term in (A.21)), Lemma A.1 and the arguments leading to the control of the nonrandom bias term IV in the proof of Theorem 4.3.1 on p.134 of de Haan and Ferreira (2006) (for the fifth term in (A.21)). Now

$$
\begin{aligned}
&\frac{((1-\tau_n)/p_n)^{\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - 1}{\widehat{\gamma}_n} - \frac{((1-\tau_n)/p_n)^{\gamma/(1-\gamma)} - 1}{\gamma} \\
&= \frac{((1-\tau_n)/p_n)^{\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - ((1-\tau_n)/p_n)^{\gamma/(1-\gamma)}}{\widehat{\gamma}_n} + [((1-\tau_n)/p_n)^{\gamma/(1-\gamma)} - 1](\widehat{\gamma}_n^{-1} - \gamma^{-1}) \\
&= \frac{((1-\tau_n)/p_n)^{\gamma/(1-\gamma)}}{\gamma}(((1-\tau_n)/p_n)^{\widehat{\gamma}_n/(1-\widehat{\gamma}_n) - \gamma/(1-\gamma)} - 1)(1+\mathrm{o}_{\mathbb{P}}(1)) + \frac{\widehat{\gamma}_n - \gamma}{\gamma^2}(1+\mathrm{o}_{\mathbb{P}}(1)) \\
&= \frac{\widehat{\gamma}_n - \gamma}{\gamma^2}(1+\mathrm{o}_{\mathbb{P}}(1)) + \mathrm{O}_{\mathbb{P}}\left(\frac{((1-\tau_n)/p_n)^{\gamma/(1-\gamma)}\log((1-\tau_n)/p_n)}{\sqrt{n(1-\tau_n)}}\right) \\
&= \frac{\widehat{\gamma}_n - \gamma}{\gamma^2} + \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}}\right) \qquad \text{(A.23)}
\end{aligned}
$$

because $\widehat{\gamma}_n$ is $\sqrt{n(1-\tau_n)}$−consistent and $x^{-c}\log x \to 0$ as $x \to \infty$ for any $c > 0$. Finally, combining Lemma A.3 with Corollary 1 results in

$$
\frac{\overline{F}(\xi_{\tau_n})}{\overline{F}(\xi_{1-p_n})} = \left(\frac{1-\tau_n}{p_n}\right)^{1/(1-\gamma)}\left(1 + \mathrm{O}((x^{\star} - q_{\tau_n})^{1/(1-\gamma)}) + \mathrm{O}(|A((1-\tau_n)^{-1})|)\right) \qquad \text{(A.24)}
$$

because $1/\overline{F}(\xi_\tau) = \mathrm{o}((1-\tau)^{-1})$ as $\tau \uparrow 1$, and therefore $A((1-\tau_n)^{-1}) = \mathrm{o}(|A(1/\overline{F}(\xi_{\tau_n}))|)$. Combine (A.22), (A.23) and (A.24) to complete the proof. □

Proof of Theorem 3. We decompose $\widetilde{\xi}^\star_{1-p_n} - \xi_{1-p_n}$ in the following way:

$$
\begin{aligned}
&\widetilde{\xi}^\star_{1-p_n} - \xi_{1-p_n} \\
&= \widehat{x}^\star - x^\star - (\widehat{x}^\star - \widetilde{\xi}^\star_{1-p_n} - (x^\star - \xi_{1-p_n})) \\
&= \widehat{x}^\star - x^\star \\
&- \{[(\widehat{x}^\star - \overline{X}_n)(1-\widehat{\gamma}_n^{-1})]^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)}]\}(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n)} \\
&- [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)}[(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n)} - (x^\star - q_{1-p_n})^{1/(1-\gamma)}] \\
&+ (x^\star - \xi_{1-p_n}) - [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)}(x^\star - q_{1-p_n})^{1/(1-\gamma)}.
\end{aligned}
$$

By Proposition 1 and Lemma A.3(i), it follows that

$$
\begin{aligned}
&\widetilde{\xi}^\star_{1-p_n} - \xi_{1-p_n} \\
&= \widehat{x}^\star - x^\star \\
&- \{[(\widehat{x}^\star - \overline{X}_n)(1-\widehat{\gamma}_n^{-1})]^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)}]\}(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n)} \\
&- [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)}[(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n)} - (x^\star - q_{1-p_n})^{1/(1-\gamma)}] \\
&+ \mathrm{O}(p_n^{-\gamma/(1-\gamma)}(p_n^{-\gamma/(1-\gamma)} + |A(p_n^{-1/(1-\gamma)})|)). \qquad \text{(A.25)}
\end{aligned}
$$

We control each of the three terms in (A.25). By the Skorokhod lemma, up to changing the probability space and with appropriate versions of the estimators involved, Theorem 4.5.1 on p.146 of de Haan and Ferreira (2006) provides

$$
\widehat{x}^\star - x^\star = \frac{a(n/k)}{\sqrt{k}} \times \frac{1}{\gamma^2}\left(\Gamma + \gamma^2 B - \gamma\Lambda - \lambda\frac{\gamma}{\gamma+\rho} + \mathrm{o}_{\mathbb{P}}(1)\right). \qquad \text{(A.26)}
$$

Writing

$$
\begin{aligned}
&[(\widehat{x}^\star - \overline{X}_n)(1-\widehat{\gamma}_n^{-1})]^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)} \\
&= \left[\left(\frac{\widehat{x}^\star - \overline{X}_n}{x^\star - \mathbb{E}(X)}\right)^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - 1\right][(x^\star - \mathbb{E}(X))(1-\widehat{\gamma}_n^{-1})]^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} \\
&+ [(x^\star - \mathbb{E}(X))(1-\widehat{\gamma}_n^{-1})]^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)},
\end{aligned}
$$

and combining (A.26) with the assumption that $\sqrt{k}(\overline{X}_n - \mathbb{E}(X)) \xrightarrow{d} 0$ and the delta-method, we find

$$
[(\widehat{x}^\star - \overline{X}_n)(1-\widehat{\gamma}_n^{-1})]^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)} = \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\right). \qquad \text{(A.27)}
$$

It follows from our assumptions that $\log(k/(np_n))/\sqrt{k} = \mathrm{O}(\log(n)/\sqrt{k}) \to 0$, and therefore

$$
\begin{aligned}
\widehat{x}^\star - \widehat{q}^\star_{1-p_n} &= -\widehat{a}(n/k)\frac{(k/(np_n))^{\widehat{\gamma}_n}}{\widehat{\gamma}_n} \\
&= -a(n/k)\frac{(k/(np_n))^{\gamma}}{\gamma}\left(1 + \frac{\log(k/(np_n))}{\sqrt{k}}\Gamma + \mathrm{o}_{\mathbb{P}}\left(\frac{\log n}{\sqrt{k}}\right)\right). \qquad \text{(A.28)}
\end{aligned}
$$

Recalling that, by Lemma A.3(i), $(k/n)^\gamma a(n/k) \to -\gamma C < \infty$ with $C = \lim_{s\to\infty} s^{-\gamma}(x^\star - U(s))$, it follows that

$$
\begin{aligned}
&(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n) - 1/(1-\gamma)} \\
&= \left(-a(n/k)\frac{(k/(np_n))^{\gamma}}{\gamma}\right)^{1/(1-\widehat{\gamma}_n) - 1/(1-\gamma)}\left(1 + \mathrm{o}_{\mathbb{P}}\left(\frac{\log n}{\sqrt{k}}\right)\right) \\
&= 1 - \frac{\log(p_n)}{\sqrt{k}} \times \frac{\gamma}{(1-\gamma)^2}(\Gamma + \mathrm{o}_{\mathbb{P}}(1)) + \mathrm{o}_{\mathbb{P}}\left(\frac{\log n}{\sqrt{k}}\right).
\end{aligned}
$$

Hence the asymptotic expansion

$$
\begin{aligned}
&(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n)} \\
&= \left(-a(n/k)\frac{(k/(np_n))^\gamma}{\gamma}\right)^{1/(1-\gamma)} \\
&\times \left(1 + \frac{\log(k/(np_n))}{\sqrt{k}} \times \frac{1}{1-\gamma}\Gamma - \frac{\log(p_n)}{\sqrt{k}} \times \frac{\gamma}{(1-\gamma)^2}\Gamma + \mathrm{o}_{\mathbb{P}}\left(\frac{\log n}{\sqrt{k}}\right)\right) \\
&= \left(-a(n/k)\frac{(k/(np_n))^\gamma}{\gamma}\right)^{1/(1-\gamma)} \left(1 + \frac{\log(k/(np_n^{1/(1-\gamma)}))}{\sqrt{k}} \times \frac{1}{1-\gamma}\Gamma + \mathrm{o}_{\mathbb{P}}\left(\frac{\log n}{\sqrt{k}}\right)\right) \qquad (A.29) \\
&= \mathrm{O}_{\mathbb{P}}\left(\left(-a(n/k)\frac{(k/(np_n))^\gamma}{\gamma}\right)^{1/(1-\gamma)}\right) \qquad (A.30)
\end{aligned}
$$

where the convergence $(k/n)^\gamma a(n/k) \to -\gamma C < \infty$ was used. Combining (A.27) and (A.30) results in particular in

$$
\begin{aligned}
&\{[(\widehat{x}^\star - \overline{X}_n)(1-\widehat{\gamma}_n^{-1})]^{-\widehat{\gamma}_n/(1-\widehat{\gamma}_n)} - [(x^\star - \mathbb{E}(X))(1-\gamma^{-1})]^{-\gamma/(1-\gamma)}]\}(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n)} \\
&= \mathrm{o}_{\mathbb{P}}\left(\left(-a(n/k)\frac{(k/(np_n))^\gamma}{\gamma}\right)^{1/(1-\gamma)} \frac{\log n}{\sqrt{k}}\right). \qquad (A.31)
\end{aligned}
$$

Using Lemma A.3(i) and the convergence $(k/n)^\gamma a(n/k) \to -\gamma C < \infty$ again in conjunction with the regular variation property of $|A|$,

$$
\begin{aligned}
x^\star - q_{1-p_n} &= \frac{x^\star - q_{1-p_n}}{a(1/p_n)}\frac{a(1/p_n)}{a(n/k)}a(n/k) \\
&= -a(n/k)\frac{(k/(np_n))^\gamma}{\gamma}\left(1 - \frac{1}{\sqrt{k}} \times \frac{\lambda}{\rho} + \mathrm{o}\left(\frac{1}{\sqrt{k}}\right)\right) \qquad (A.32)
\end{aligned}
$$

and so

$$
(x^\star - q_{1-p_n})^{1/(1-\gamma)} = \left(-a(n/k)\frac{(k/(np_n))^\gamma}{\gamma}\right)^{1/(1-\gamma)} \left(1 - \frac{1}{\sqrt{k}} \times \frac{\lambda}{\rho(1-\gamma)} + \mathrm{o}\left(\frac{1}{\sqrt{k}}\right)\right). \qquad (A.33)
$$

Combine (A.29) and (A.33) to obtain

$$
\begin{aligned}
&(\widehat{x}^\star - \widehat{q}^\star_{1-p_n})^{1/(1-\widehat{\gamma}_n)} - (x^\star - q_{1-p_n})^{1/(1-\gamma)} \\
&= \left(-a(n/k)\frac{(k/(np_n))^\gamma}{\gamma}\right)^{1/(1-\gamma)} \left(\frac{\log(k/(np_n^{1/(1-\gamma)}))}{\sqrt{k}}\frac{\Gamma}{1-\gamma} + \mathrm{o}_{\mathbb{P}}\left(\frac{\log n}{\sqrt{k}}\right)\right). \qquad (A.34)
\end{aligned}
$$

Finally, combine (A.26), (A.31) and (A.34) to complete the proof. □

Proof of Corollary 3. Under the assumptions of the result, and by Lemma A.3(i), $a(n/k)$ is asymptotically proportional to $n^{(1-\chi)\gamma}$, and $[a(n/k)(k/(np_n))^\gamma]^{1/(1-\gamma)}\log(np_n^{1/(1-\gamma)}/k)$ is asymptotically proportional to $n^{\omega\gamma/(1-\gamma)}\log n$. The assumption $\chi < 1 - \omega/(1-\gamma)$ therefore ensures

$$
\frac{a(n/k)}{\sqrt{k}} = \mathrm{o}\left(\frac{[a(n/k)(k/(np_n))^\gamma]^{1/(1-\gamma)}\log(np_n^{1/(1-\gamma)}/k)}{\sqrt{k}}\right).
$$

Moreover, it is a consequence of Proposition B.1.9.1 on p.366 of de Haan and Ferreira (2006) that $|A(s)| = \mathrm{o}(s^{\rho+\varepsilon})$ as $s \to \infty$ for any $\varepsilon > 0$, so $A(n^{\omega/(1-\gamma)}) = \mathrm{o}(n^{\omega\rho/(1-\gamma)+\delta})$ for any $\delta > 0$. Hence, using the assumption $\chi < 2\omega \min(-\gamma, -\rho)/(1-\gamma)$, the convergence

$$\frac{\sqrt{k}}{[a(n/k)(k/(np_n))^{\gamma}]^{1/(1-\gamma)} \log(np_n^{1/(1-\gamma)}/k)} \times n^{\omega\gamma/(1-\gamma)}(n^{\omega\gamma/(1-\gamma)} + |A(n^{\omega/(1-\gamma)})|) \to 0.$$

Apply Theorem 3 to complete the proof. □

Proof of Corollary 4. It was shown in the proof of Corollary 3 that $a(n/k)$ is asymptotically proportional to $n^{(1-\chi)\gamma}$, and $[a(n/k)(k/(np_n))^{\gamma}]^{1/(1-\gamma)} \log(np_n^{1/(1-\gamma)}/k)$ is asymptotically proportional to $n^{\omega\gamma/(1-\gamma)} \log n$. Under the assumptions of the result,

$$\frac{[a(n/k)(k/(np_n))^{\gamma}]^{1/(1-\gamma)} \log(np_n^{1/(1-\gamma)}/k)}{\sqrt{k}} = \mathrm{o}\left(\frac{a(n/k)}{\sqrt{k}}\right).$$

Apply Theorem 3 to complete the proof. □

Proof of Proposition 2. Write

$$\log \frac{1-\widehat{\tau}_n}{1-\tau_n} = \log\left(\frac{\widehat{x}^\star - \widehat{q}^\star_{1-p_n}}{x^\star - q_{1-p_n}}\right) - \log\left(\frac{\widehat{x}^\star - \overline{X}_n}{x^\star - \mathbb{E}(X)}\right) - \log\left(\frac{1-\widehat{\gamma}_n^{-1}}{1-\gamma^{-1}}\right) + \log\left(\frac{x^\star - q_{1-p_n}}{(x^\star - \mathbb{E}(X))(1-\gamma^{-1})} \frac{p_n}{1-\tau_n}\right). \tag{A.35}$$

Note that condition $\chi < \min(-2\omega\gamma, -2\rho/(1-2\rho))$ ensures in particular that $\sqrt{k}A(n/k) \to 0$. Combine (A.26), (A.28), (A.32), convergence $\sqrt{k}(\overline{X}_n - \mathbb{E}(X)) \xrightarrow{\mathbb{P}} 0$ and the delta-method to get

$$\begin{aligned} &\frac{\sqrt{k}}{\log(k/(np_n))}\left(\log\left(\frac{\widehat{x}^\star - \widehat{q}^\star_{1-p_n}}{x^\star - q_{1-p_n}}\right) - \log\left(\frac{\widehat{x}^\star - \overline{X}_n}{x^\star - \mathbb{E}(X)}\right) - \log\left(\frac{1-\widehat{\gamma}_n^{-1}}{1-\gamma^{-1}}\right)\right) \\ &= \frac{\sqrt{k}}{\log(k/(np_n))} \log\left(\frac{\widehat{x}^\star - \widehat{q}^\star_{1-p_n}}{x^\star - q_{1-p_n}}\right) + \mathrm{o}_{\mathbb{P}}(1) \xrightarrow{d} \Gamma. \end{aligned} \tag{A.36}$$

Combine now Proposition 1 and Lemma A.3 with the relationship $1-\tau = \mathrm{o}(x^\star - \xi_\tau)$ as $\tau \uparrow 1$ (coming as a consequence of Equation (2.4)) to get

$$\frac{(x^\star - \xi_\tau)\overline{F}(\xi_\tau)}{1-\tau} = (x^\star - \mathbb{E}(X))(1-\gamma^{-1})(1 + \mathrm{O}(x^\star - \xi_\tau) + \mathrm{O}(|A(1/\overline{F}(\xi_\tau))|)) \text{ as } \tau \uparrow 1.$$

With $\tau = \tau_n$ such that $\xi_\tau = \xi_{\tau_n} = q_{1-p_n}$ and using Lemma A.1(ii), we find

$$\frac{(x^\star - q_{1-p_n})}{(x^\star - \mathbb{E}(X))(1-\gamma^{-1})} \frac{p_n}{1-\tau_n} = 1 + \mathrm{O}(x^\star - q_{1-p_n}) + \mathrm{O}(|A(1/p_n)|) \text{ as } n \to \infty. \tag{A.37}$$

Assumptions $\chi + \omega - 1 > 0$ and $\chi < \min(-2\omega\gamma, -2\rho/(1-2\rho))$ ensure that $A(1/p_n) = \mathrm{o}(|A(n/k)|)$ and $\sqrt{k}(x^\star - q_{1-p_n}) \to 0$. Plug (A.36) and (A.37) into (A.35) to complete the proof. □

Proof of Theorem 4. We know that $\tau \mapsto \widehat{\xi}_\tau$ is the inverse of the distribution function $\widehat{E}_n = 1 - \widehat{\overline{E}}_n$ defined by

$$\widehat{\overline{E}}_n(x) = \frac{\widehat{\varphi}_n(x)}{2\widehat{\varphi}_n(x) + x - \frac{1}{n}\sum_{t=1}^n \widehat{\varepsilon}_t^{(n)}}, \text{ where } \widehat{\varphi}_n(x) = \widehat{\varphi}_n^{(1)}(x) = \frac{1}{n}\sum_{t=1}^n (\widehat{\varepsilon}_t^{(n)} - x)\mathbb{1}\{\widehat{\varepsilon}_t^{(n)} > x\}.$$

The conclusion of the proof of Theorem 1 contains the fact that for any fixed $u \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n} - \xi_{\tau_n}) \le u\right) \\
&= \mathbb{P}\left(\sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\widehat{\overline{E}}_n(x_n)}{\overline{E}(x_n)} - 1\right) \le \sqrt{n\overline{F}(\xi_{\tau_n})}\left(\frac{\overline{E}(\xi_{\tau_n})}{\overline{E}(x_n)} - 1\right)\right) \\
&= \mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{1-\gamma}\left(\frac{\widehat{\overline{E}}_n(x_n)}{\overline{E}(x_n)} - 1\right) + \mathrm{o}(1) \le u\right),
\end{aligned}$$

where

$$x_n = x_n(u) = \xi_{\tau_n} + u\frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}}.$$

Define also the unfeasible, innovation-based LAWS estimator of $\xi_{\tau_n}(\varepsilon)$ by

$$\widetilde{\xi}_{\tau_n} = \arg\min_{\theta\in\mathbb{R}} \sum_{t=1}^{n} \eta_{\tau_n}(\varepsilon_t - \theta).$$

Then $\tau \mapsto \widetilde{\xi}_\tau$ is the inverse of the distribution function $\widetilde{E}_n = 1 - \widetilde{\overline{E}}_n$ given by

$$\widetilde{\overline{E}}_n(x) = \frac{\widetilde{\varphi}_n(x)}{2\widetilde{\varphi}_n(x) + x - \overline{\varepsilon}_n}, \text{ where } \widetilde{\varphi}_n(x) = \widetilde{\varphi}_n^{(1)}(x) = \frac{1}{n}\sum_{t=1}^{n}(\varepsilon_t - x)\mathbb{1}\{\varepsilon_t > x\}$$

and $\overline{\varepsilon}_n$ is the sample mean of the ε_t, $1 \le t \le n$. We are going to prove that if $x_n = x_n(u)$ as above, then

$$\sqrt{n\overline{F}(\xi_{\tau_n})}\left|\frac{\widehat{\overline{E}}_n(x_n) - \widetilde{\overline{E}}_n(x_n)}{\overline{E}(x_n)}\right| \xrightarrow{\mathbb{P}} 0. \tag{A.38}$$

This will result in the fact that, for any fixed $u \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))}(\widehat{\xi}_{\tau_n} - \xi_{\tau_n}) \le u\right) = \mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{1-\gamma}\left(\frac{\widetilde{\overline{E}}_n(x_n)}{\overline{E}(x_n)} - 1\right) + \mathrm{o}_{\mathbb{P}}(1) \le u\right),$$

from which the conclusion will immediately follow by applying Lemma A.5(ii) to the i.i.d. sequence (ε_t).

Clearly

$$\begin{aligned}
\left|\widehat{\overline{E}}_n(x_n) - \widetilde{\overline{E}}_n(x_n)\right| &= \left|\frac{\widehat{\varphi}_n(x_n)}{2\widehat{\varphi}_n(x_n) + x_n - \frac{1}{n}\sum_{t=1}^{n}\widehat{\varepsilon}_t^{(n)}} - \frac{\widetilde{\varphi}_n(x_n)}{2\widetilde{\varphi}_n(x_n) + x_n - \overline{\varepsilon}_n}\right| \\
&\le \widehat{\varphi}_n(x_n)\frac{2|\widehat{\varphi}_n(x_n) - \widetilde{\varphi}_n(x_n)| + |\frac{1}{n}\sum_{t=1}^{n}\widehat{\varepsilon}_t^{(n)} - \overline{\varepsilon}_n|}{(2\widehat{\varphi}_n(x_n) + x_n - \frac{1}{n}\sum_{t=1}^{n}\widehat{\varepsilon}_t^{(n)})(2\widetilde{\varphi}_n(x_n) + x_n - \overline{\varepsilon}_n)} \\
&\quad + \frac{|\widehat{\varphi}_n(x_n) - \widetilde{\varphi}_n(x_n)|}{2\widetilde{\varphi}_n(x_n) + x_n - \overline{\varepsilon}_n}.
\end{aligned} \tag{A.39}$$

Our assumption on $|\widehat{\varepsilon}_t^{(n)} - \varepsilon_t|$ immediately entails

$$\left| \frac{1}{n} \sum_{t=1}^{n} \widehat{\varepsilon}_t^{(n)} - \overline{\varepsilon}_n \right| \leq \max_{1\leq t \leq n} |\widehat{\varepsilon}_t^{(n)} - \varepsilon_t| = \mathrm{o}_{\mathbb{P}}\left(\frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} \right). \tag{A.40}$$

In particular,

$$\frac{1}{n} \sum_{t=1}^{n} \widehat{\varepsilon}_t^{(n)} \xrightarrow{\mathbb{P}} 0 \tag{A.41}$$

from the law of large numbers and the fact that $a(x) \to 0$ as $x \uparrow e^{\star}$. Besides

$$\begin{aligned} |\widehat{\varphi}_n(x_n) - \widetilde{\varphi}_n(x_n)| &\leq \frac{1}{n} \sum_{t=1}^{n} |\widehat{\varepsilon}_t^{(n)} - \varepsilon_t| \mathbb{1}\{\varepsilon_t > x_n\} \\ &\quad + \frac{1}{n} \sum_{t=1}^{n} |\widehat{\varepsilon}_t^{(n)} - x_n| \left| \mathbb{1}\{\widehat{\varepsilon}_t^{(n)} > x_n\} - \mathbb{1}\{\varepsilon_t > x_n\} \right|. \end{aligned} \tag{A.42}$$

Denoting the empirical survival function of the ε_t by $\widetilde{\overline{F}}_n$, we obviously have

$$\frac{1}{n} \sum_{t=1}^{n} |\widehat{\varepsilon}_t^{(n)} - \varepsilon_t| \mathbb{1}\{\varepsilon_t > x_n\} \leq \widetilde{\overline{F}}_n(x_n) \max_{1\leq t \leq n} |\widehat{\varepsilon}_t^{(n)} - \varepsilon_t| = \mathrm{o}_{\mathbb{P}}\left(\frac{\overline{F}(\xi_{\tau_n}) a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} \right) \tag{A.43}$$

using our assumption on $|\widehat{\varepsilon}_t^{(n)} - \varepsilon_t|$, along with the Chebyshev inequality showing that $\widetilde{\overline{F}}_n(x_n)/\overline{F}(x_n) = 1 + \mathrm{o}_{\mathbb{P}}(1)$ and the asymptotic equivalence between $\overline{F}(x_n)$ and $\overline{F}(\xi_{\tau_n})$ due to a combination of (A.5) with (A.17) applied to the distribution of ε. Note then that, using again our assumption on $|\widehat{\varepsilon}_t^{(n)} - \varepsilon_t|$, one may define by induction a sequence of increasing integers N_k, for $k \geq 1$, such that for any $n > N_k$,

$$\mathbb{P}\left(\frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))} \max_{1\leq t \leq n} |\widehat{\varepsilon}_t^{(n)} - \varepsilon_t| > \frac{1}{k} \right) \leq \frac{1}{2^k}.$$

Setting $\delta_n = 1/k$ when $n \in \{N_k + 1, \ldots, N_{k+1}\}$ results in a nonrandom positive sequence (δ_n) converging to 0 and such that the event

$$A_n = \left\{ \frac{\sqrt{n\overline{F}(\xi_{\tau_n})}}{a(1/\overline{F}(\xi_{\tau_n}))} \max_{1\leq t \leq n} |\widehat{\varepsilon}_t^{(n)} - \varepsilon_t| \leq \delta_n \right\}$$

has probability arbitrarily close to 1 as $n \to \infty$. On A_n,

$$\begin{aligned} &\frac{1}{n} \sum_{t=1}^{n} |\widehat{\varepsilon}_t^{(n)} - x_n| \left| \mathbb{1}\{\widehat{\varepsilon}_t^{(n)} > x_n\} - \mathbb{1}\{\varepsilon_t > x_n\} \right| \\ &\leq \frac{1}{n} \sum_{t=1}^{n} |\widehat{\varepsilon}_t^{(n)} - x_n| \left(\mathbb{1}\{\varepsilon_t > x_{n,-}\} - \mathbb{1}\{\varepsilon_t > x_{n,+}\} \right) \\ &\leq \frac{1}{n} \sum_{t=1}^{n} (|\widehat{\varepsilon}_t^{(n)} - \varepsilon_t| + |\varepsilon_t - x_n|) \left(\mathbb{1}\{\varepsilon_t > x_{n,-}\} - \mathbb{1}\{\varepsilon_t > x_{n,+}\} \right) \\ &\leq 2\delta_n \frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} \times \frac{1}{n} \sum_{t=1}^{n} \left(\mathbb{1}\{\varepsilon_t > x_{n,-}\} - \mathbb{1}\{\varepsilon_t > x_{n,+}\} \right) \end{aligned} \tag{A.44}$$

where

$$x_{n,+} = x_n + \delta_n \frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} = \xi_{\tau_n} + (u+\delta_n)\frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}}$$

$$\text{and } x_{n,-} = x_n - \delta_n \frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} = \xi_{\tau_n} + (u-\delta_n)\frac{a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}}.$$

The upper bound in (A.44) is positive, so it is stochastically bounded from above by its expectation in view of the Markov inequality. This means that

$$\frac{1}{n}\sum_{t=1}^{n} |\widehat{\varepsilon}_t^{(n)} - x_n| \left| \mathbb{1}\{\widehat{\varepsilon}_t^{(n)} > x_n\} - \mathbb{1}\{\varepsilon_t > x_n\} \right| = o_{\mathbb{P}}\left(\frac{\overline{F}(\xi_{\tau_n}) a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} \right).$$

Combining this with (A.42) and (A.43) yields

$$|\widehat{\varphi}_n(x_n) - \widetilde{\varphi}_n(x_n)| = o_{\mathbb{P}}\left(\frac{\overline{F}(\xi_{\tau_n}) a(1/\overline{F}(\xi_{\tau_n}))}{\sqrt{n\overline{F}(\xi_{\tau_n})}} \right). \tag{A.45}$$

In particular, if $\varphi(x_n) = \mathbb{E}((\varepsilon - x_n)\mathbb{1}\{\varepsilon > x_n\})$,

$$\frac{\widetilde{\varphi}_n(x_n)}{\varphi(x_n)} \xrightarrow{\mathbb{P}} 1, \text{ so that } \frac{\widehat{\varphi}_n(x_n)}{\varphi(x_n)} \xrightarrow{\mathbb{P}} 1 \text{ and then } \widehat{\varphi}_n(x_n) \xrightarrow{\mathbb{P}} 0 \tag{A.46}$$

from Lemma A.5(i) in the i.i.d. setting. Finally, recalling that, from Lemma A.2(ii),

$$\begin{aligned}\overline{E}(x_n) = \frac{\varphi(x_n)}{2\varphi(x_n) + x_n - \mathbb{E}(\varepsilon)} &= \frac{\varphi(x_n)}{2\varphi(x_n) + x_n} \\ &\sim \frac{\varphi(x_n)}{e^\star} \sim \frac{\overline{F}(x_n)a(1/\overline{F}(x_n))}{(1-\gamma)e^\star} \sim \frac{\overline{F}(\xi_{\tau_n})a(1/\overline{F}(\xi_{\tau_n}))}{(1-\gamma)e^\star}\end{aligned}$$

as $n \to \infty$, (A.38) follows from combining (A.39), (A.40), (A.41), (A.45) and (A.46). □

B Further finite-sample results

We enclose here the full set of graphs we obtained in our numerical results for the six models we discuss in Section 3 and for the three sample sizes $n = 150, 300, 500$.

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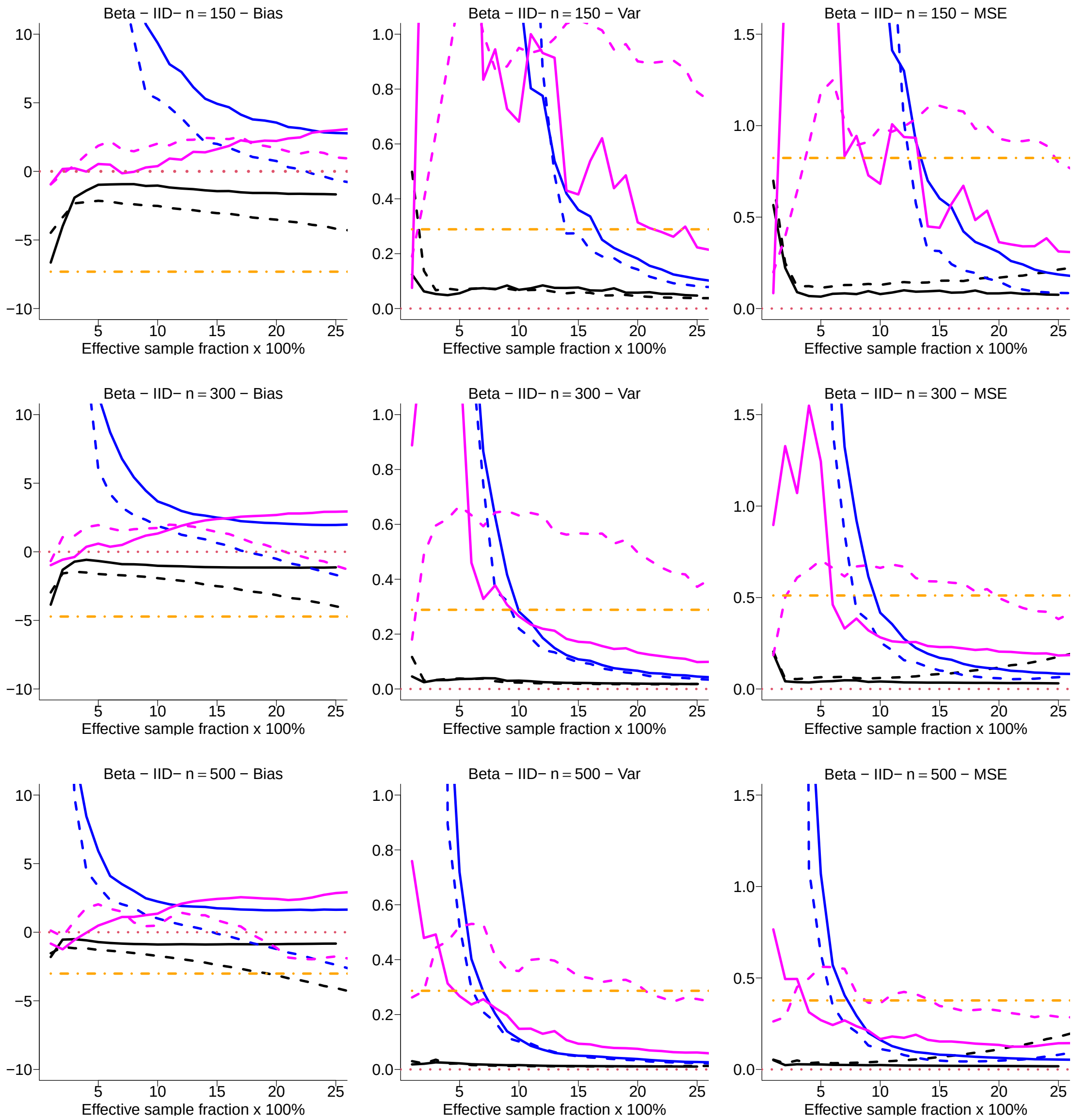

Figure B.1: Empirical relative bias, variance and MSE (left, middle and right), multiplied by 100, for the estimators of $\xi_{\tau'_n}$ obtained with i.i.d. observations from a Beta distribution (simulation setup (i)), $\tau'_n = 1 - 1/n$ and sample size $n = 150, 300, 500$ (top, middle, bottom). Purely empirical estimator $\widehat{\xi}_{\tau'_n}$ (orange line), extrapolating LAWS estimators $\widehat{\xi}^{\star}_{\tau'_n}$ (magenta lines) and $\overline{\xi}^{\star}_{\tau'_n}$ (blue lines), and extrapolating QB estimators $\widetilde{\xi}^{\star}_{\tau'_n}$ (black lines). The versions of the extrapolating estimators based on the GPML scale and shape parameter estimates are referred to using solid lines, and those based on the Moment estimators are referred to using dashed lines.

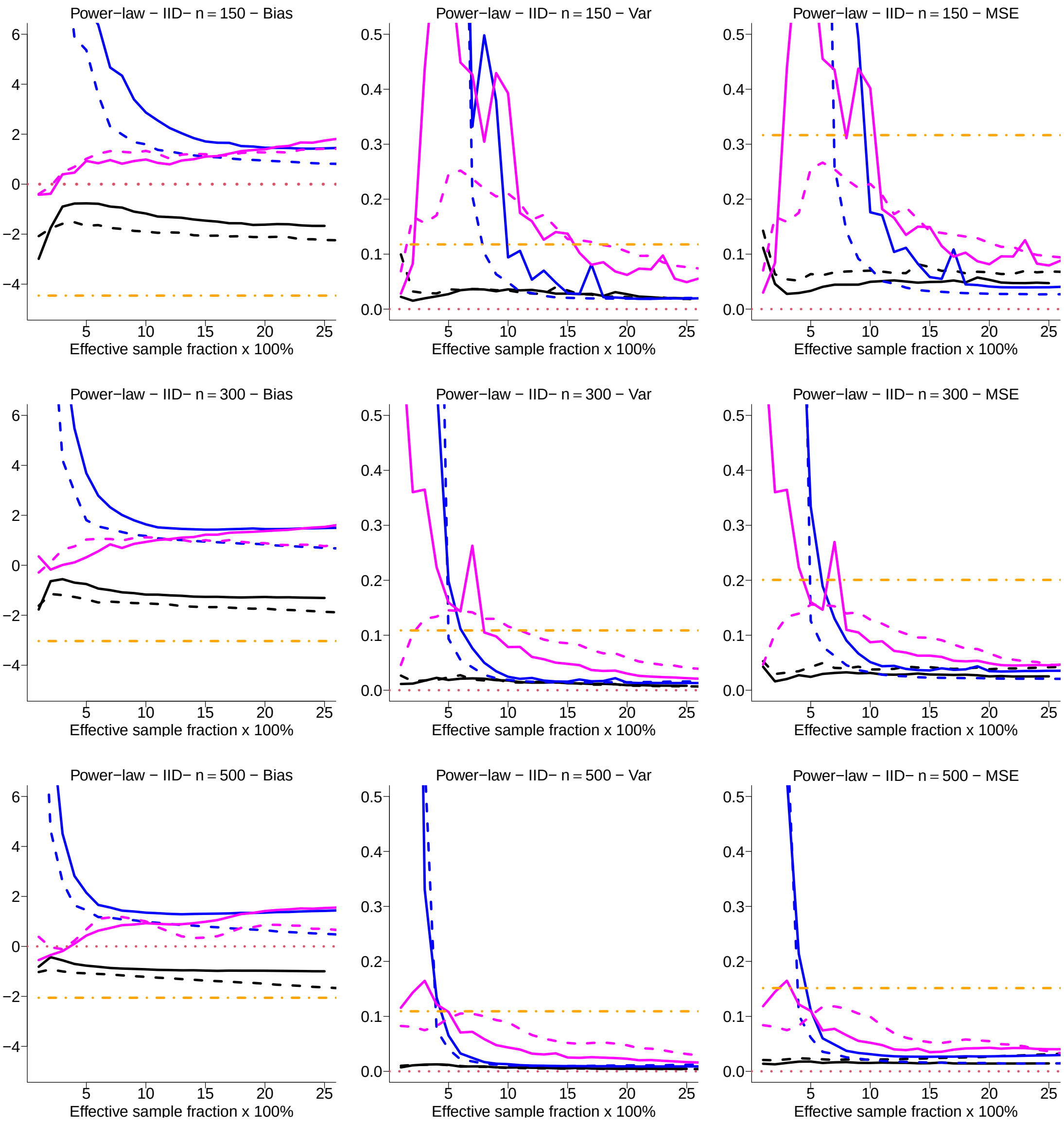

Figure B.2: Empirical relative bias, variance and MSE (left, middle and right), multiplied by 100, for the estimators of $\xi_{\tau_n'}$ obtained with i.i.d. observations from a power-law distribution (simulation setup (ii)), $\tau_n' = 1 - 1/n$ and sample size $n = 150, 300, 500$ (top, middle, bottom). Purely empirical estimator $\widehat{\xi}_{\tau_n'}$ (orange line), extrapolating LAWS estimators $\widehat{\xi}^{\star}_{\tau_n'}$ (magenta lines) and $\overline{\xi}^{\star}_{\tau_n'}$ (blue lines), and extrapolating QB estimators $\widetilde{\xi}^{\star}_{\tau_n'}$ (black lines). The versions of the extrapolating estimators based on the GPML scale and shape parameter estimates are referred to using solid lines, and those based on the Moment estimators are referred to using dashed lines.

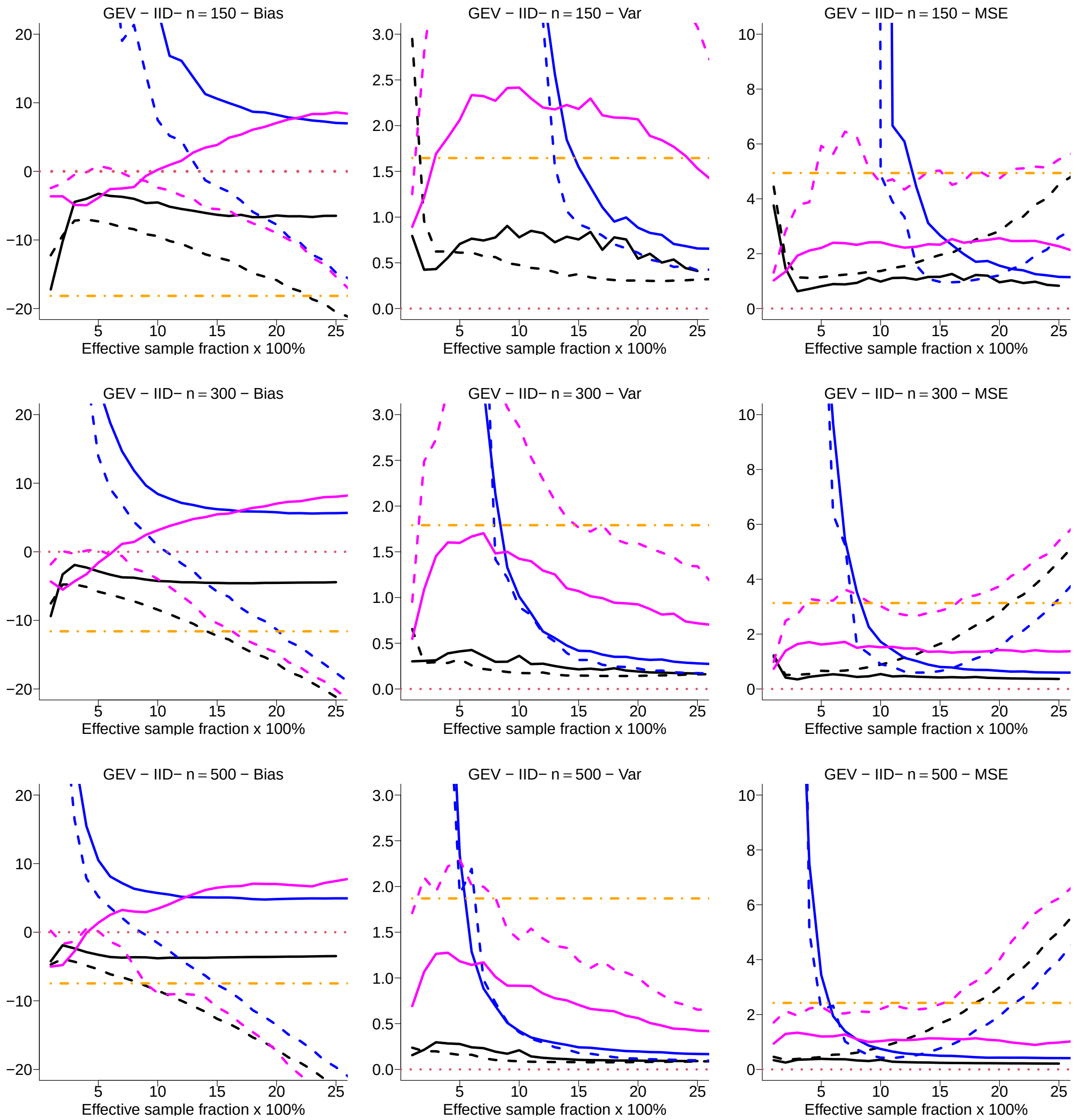

Figure B.3: Empirical relative bias, variance and MSE (left, middle and right), multiplied by 100, for the estimators of $\xi_{\tau_n'}$ obtained with i.i.d. observations from a GEV distribution (simulation setup (iii)), $\tau_n' = 1 - 1/n$ and sample size $n = 150, 300, 500$ (top, middle, bottom). Purely empirical estimator $\widehat{\xi}_{\tau_n'}$ (orange line), extrapolating LAWS estimators $\widehat{\xi}^{\star}_{\tau_n'}$ (magenta lines) and $\overline{\xi}^{\star}_{\tau_n'}$ (blue lines), and extrapolating QB estimators $\widetilde{\xi}^{\star}_{\tau_n'}$ (black lines). The versions of the extrapolating estimators based on the GPML scale and shape parameter estimates are referred to using solid lines, and those based on the Moment estimators are referred to using dashed lines.

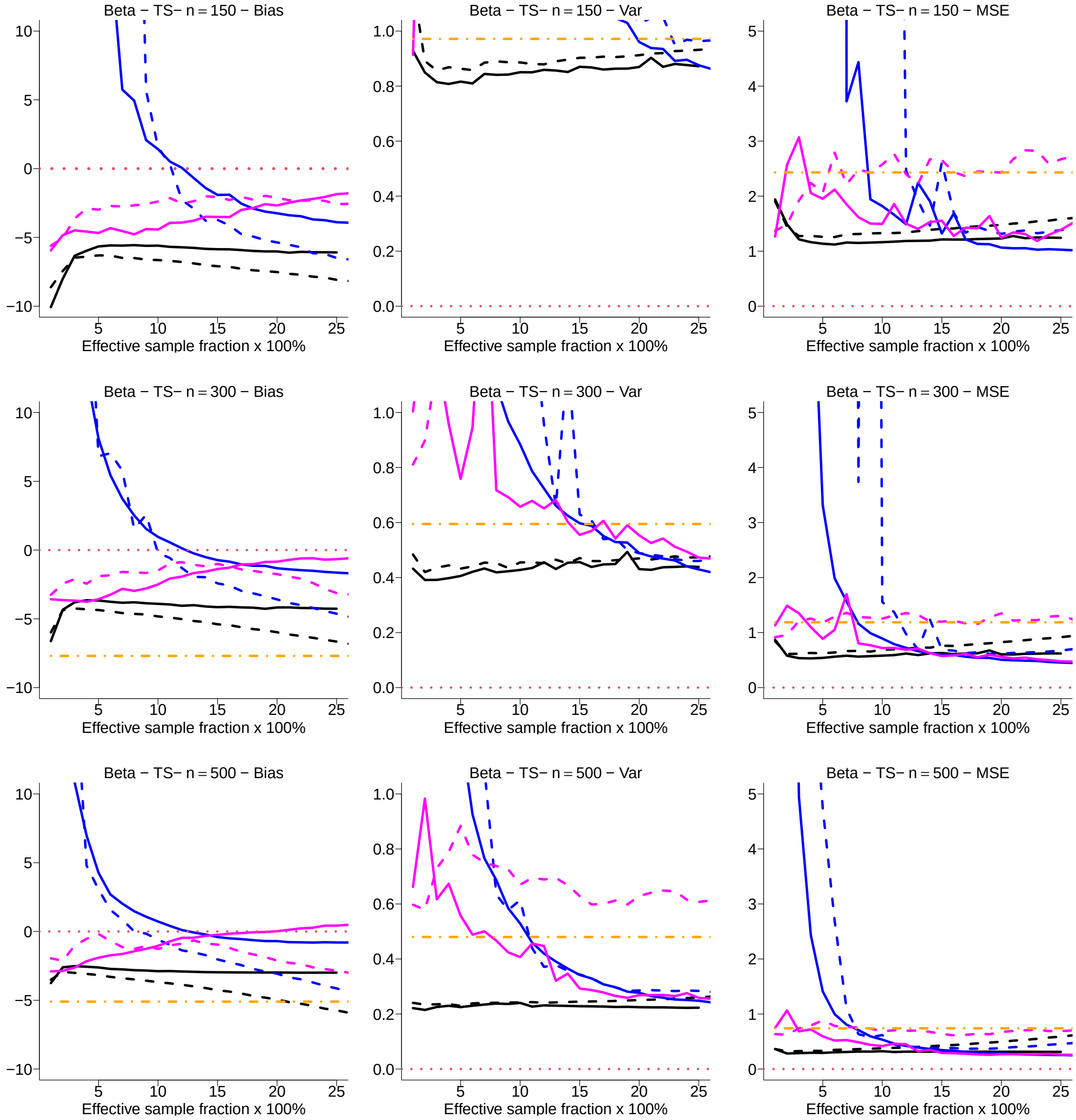

Figure B.4: Empirical relative bias, variance and MSE (left, middle and right), multiplied by 100, for the estimators of $\xi_{\tau_n'}$ obtained with nonlinear AR(1) observations from a Beta distribution (simulation setup (iv)), $\tau_n' = 1 - 1/n$ and sample size $n = 150, 300, 500$ (top, middle, bottom). Purely empirical estimator $\widehat{\xi}_{\tau_n'}$ (orange line), extrapolating LAWS estimators $\widehat{\xi}^{\star}_{\tau_n'}$ (magenta lines) and $\overline{\xi}^{\star}_{\tau_n'}$ (blue lines), and extrapolating QB estimators $\widetilde{\xi}^{\star}_{\tau_n'}$ (black lines). The versions of the extrapolating estimators based on the GPML scale and shape parameter estimates are referred to using solid lines, and those based on the Moment estimators are referred to using dashed lines.

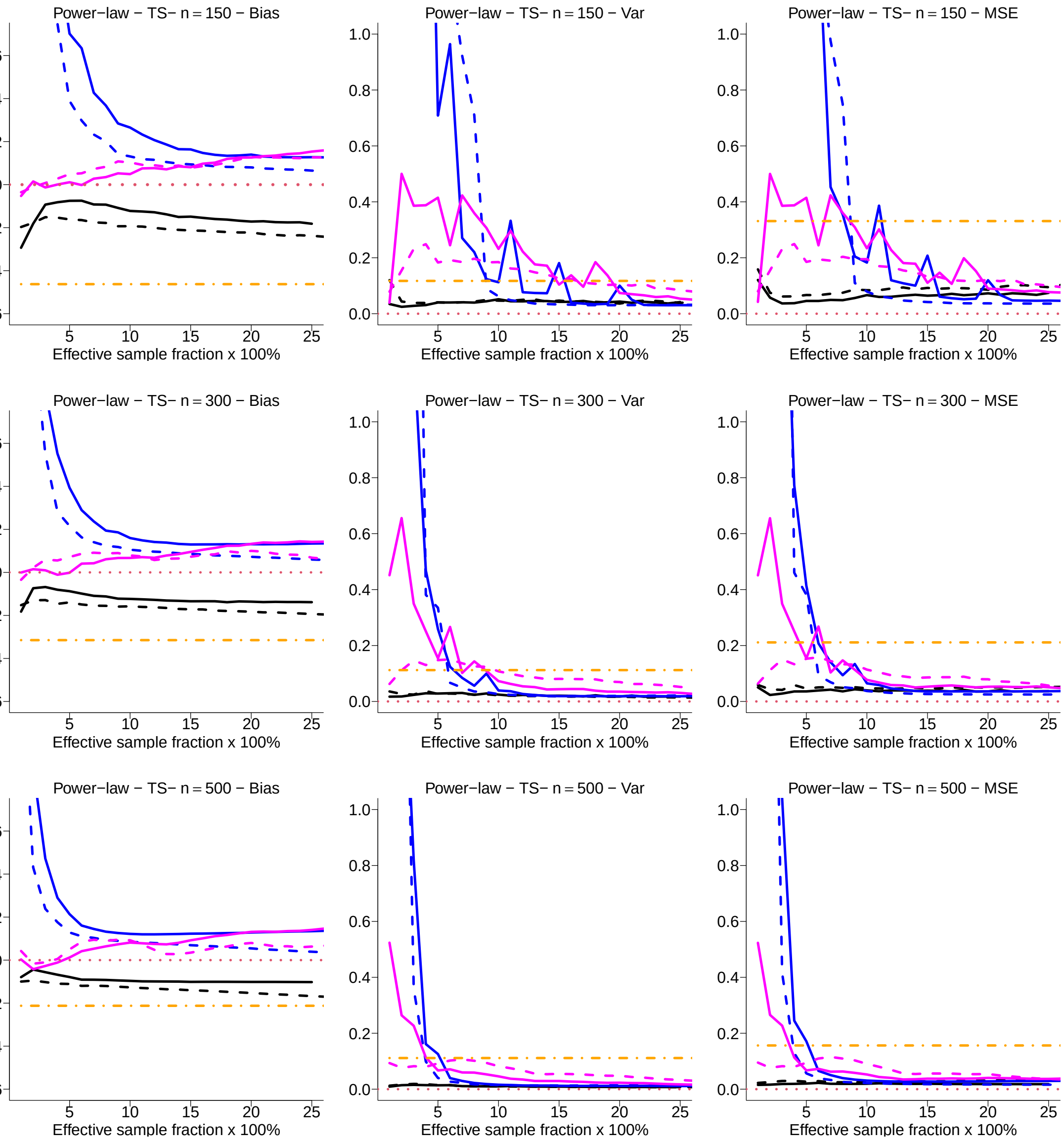

Figure B.5: Empirical relative bias, variance and MSE (left, middle and right), multiplied by 100, for the estimators of $\xi_{\tau_n'}$ obtained with nonlinear AR(1) observations from a power-law distribution (simulation setup (iv)), $\tau_n' = 1 - 1/n$ and sample size $n = 150, 300, 500$ (top, middle, bottom). Purely empirical estimator $\widehat{\xi}_{\tau_n'}$ (orange line), extrapolating LAWS estimators $\widehat{\xi}^{\star}_{\tau_n'}$ (magenta lines) and $\overline{\xi}^{\star}_{\tau_n'}$ (blue lines), and extrapolating QB estimators $\widetilde{\xi}^{\star}_{\tau_n'}$ (black lines). The versions of the extrapolating estimators based on the GPML scale and shape parameter estimates are referred to using solid lines, and those based on the Moment estimators are referred to using dashed lines.

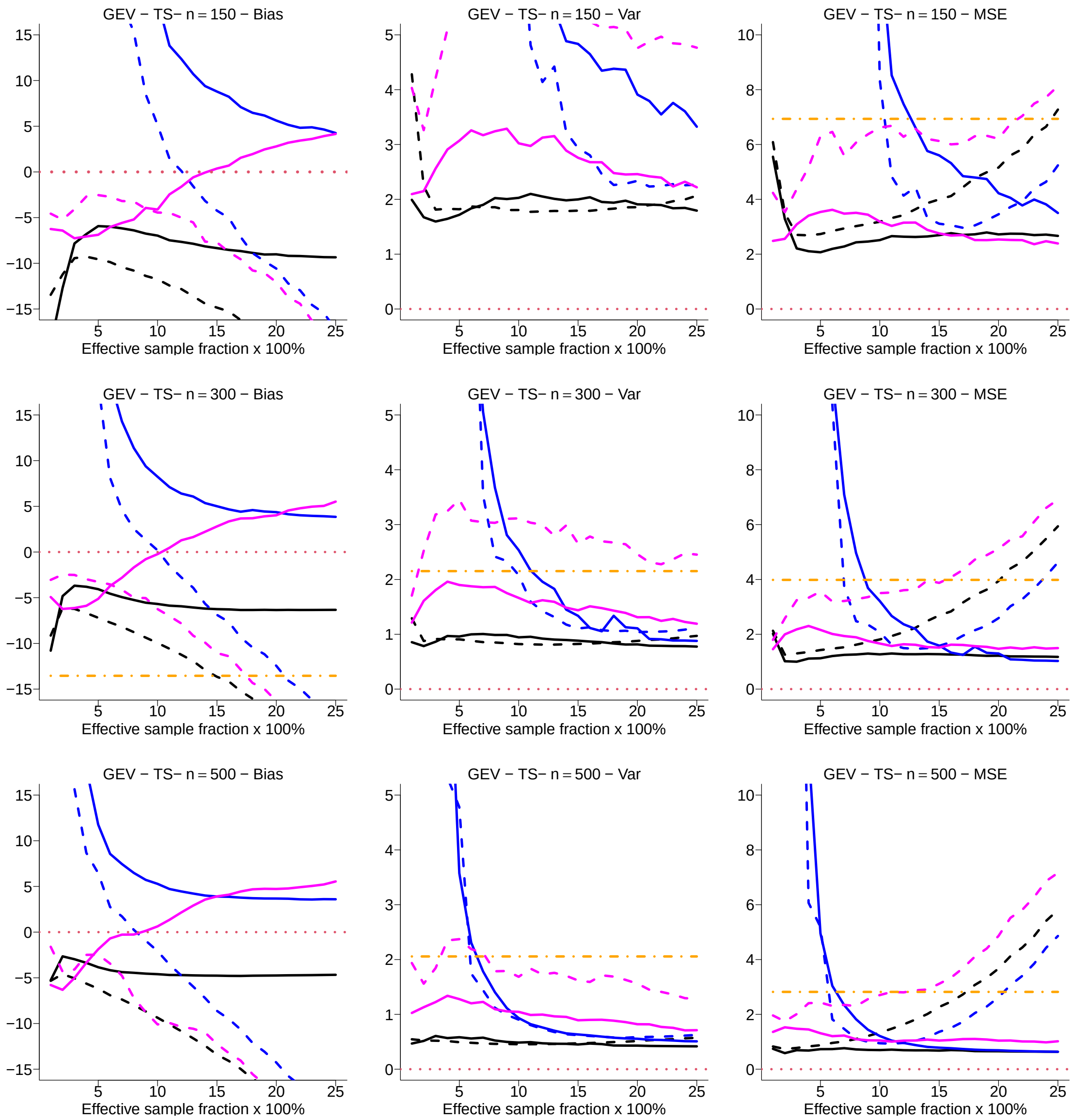

Figure B.6: Empirical relative bias, variance and MSE (left, middle and right), multiplied by 100, for the estimators of $\xi_{\tau_n'}$ obtained with nonlinear AR(1) observations from a GEV distribution (simulation setup (iv)), $\tau_n' = 1 - 1/n$ and sample size $n = 150, 300, 500$ (top, middle, bottom). Purely empirical estimator $\widehat{\xi}_{\tau_n'}$ (orange line), extrapolating LAWS estimators $\widehat{\xi}^{\star}_{\tau_n'}$ (magenta lines) and $\overline{\xi}^{\star}_{\tau_n'}$ (blue lines), and extrapolating QB estimators $\widetilde{\xi}^{\star}_{\tau_n'}$ (black lines). The versions of the extrapolating estimators based on the GPML scale and shape parameter estimates are referred to using solid lines, and those based on the Moment estimators are referred to using dashed lines.