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"Acting in the Darkness: Some Foundations for the *Precautionary Principle*"

Louise Guillouet and David Martimort

Toulouse School of Economics

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ACTING IN THE DARKNESS: SOME FOUNDATIONS FOR THE *PRECAUTIONARY PRINCIPLE*

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Louise Guillouet and David Martimort

Abstract. A decision-maker enjoys surplus from his current action but faces the possibility of an irreversible catastrophe, an event that follows a non-homogeneous Poisson process with a rate that depends on the stock of past actions. Passed a tipping point, the probability of a disaster increases. Only the distribution of possible values of the tipping point is known. For such a context that entails irreversibility, uncertainty and limited information, the *Precautionary Principle*, viewed as a constitutional commitment to an action plan, has repeatedly been invoked to guide decision-making. Although the optimal feedback rule should *a priori* determine actions in terms of both the stock of past actions and the current beliefs on whether the tipping point has been passed or not, an incomplete *Stock-Markov* feedback rule that only depends on stock suffices to implement the optimum. In such a *Stock-Markov Equilibrium*, the decision-maker conjectures that future selves stick to the same *Stock-Markov* feedback rule in the future, and observes deviations by previous selves if any. When deviations are non-observable and future selves have no evidence on how beliefs should change, equilibrium actions remain too low and beliefs are sticky. A commitment to ban actions below the equilibrium *Stock-Markov* feedback rule with observable deviations prevents such opportunistic deviations and restores the optimal trajectory.

keywords. *Precautionary Principle*, Regulation, Environmental Risk, Tipping Point, Uncertainty.

JEL codes. D83, Q55.

1. INTRODUCTION

On the Precautionary Principle. The major environmental and health issues that pertain to our modern *risk society* are most often due to our own production and consumption.[1] When dealing with such risks, decision-making is complicated by two features that make the standard tools of cost-benefit analysis of limited value. The first specificity is that consumption and production choices might entail a strong irreversibility component. The most salient example is given by global warming. Pollutants have been accumulating in the atmosphere from the beginning of the industrial era, leading to a

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[a]Columbia University, `louise.guillouet@columbia.edu`
[b]Toulouse School of Economics, `martimort.david@gmail.com`

[1]See Beck (1992).

steady increase in temperature. All current or planned efforts against global warming consist in controlling the growth rate of temperature, with little hope of reducing it. Another example is given by *GMO* crops whose production may profoundly modify the surrounding biotope without any possibility of engineering backwards because of irreversible mutations.[2] The second feature of those problems is that the costs and benefits of any decision have to be assessed under major uncertainty. Although the consequences of acting might be detrimental to the environment, the extent to which it is so and the probability of harmful events remain to a large extent unknown to decision-makers.

The policy guidelines that have been adopted to structure decision-making and regulation in those contexts greatly vary from one country to the other. To illustrate, while *GMOs* are authorized for human consumption in the U.S. without labelling, it is compulsory to label them in sixty four other countries throughout the world and they are actually forbidden in most of the E.U. To further guide decision-making, the so called *Precautionary Principle* has been repeatedly invoked. The original idea is due to the philosopher Hans Jonas' *Vorsorgeprinzip*, or *Principle of Foresight* - sometimes translated and referred to as the *Principle of Responsibility.* This *Principle* states that we should recognize the long-term irreversible consequences of present actions, and refrain from undertaking any such action if there is no proof that it would not negatively affect future generations' well-being.

The *Precautionary Principle* was acknowledged by the United Nations in 1992, during the *Earth Summit* held in Rio, and perhaps expressed less restrictively as: *"Where there are threats of serious and irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."* A similar principle was written in the French 2004 *Charter on Environment*,[3]that is now part of the French Constitution. Any risk, health or environmental regulation must thus comply with the legal framework that the *Principle* contributes to build.

At least since its inception, there has always been a lively debate, mainly led by philosophers and political scientists, on whether the *Precautionary Principle* provides a convenient guide for decision-making under uncertainty. Doubts exist on the fact that its adoption might actually do more harm, by hindering innovation and growth, than good, by protecting human health or the environment.[4] To illustrate the fuzziness of the concept, the legal scholar Cass Sunstein (2005) pointed out that the *Precautionary Principle* is sometimes understood as meaning not acting because more of the act is also associated to a greater harm while true precaution might instead require taking serious actions. Examples are investments in new technologies which have an uncertain, but promising future in terms of greenhouse gas emission such as hydrogen. Another front on which vagueness bites is probably the difficulty to agree on what is meant by *"full scientific certainty"* (or actually its absence) and how beliefs on the underlying risks are formed over time.

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[2]Other examples include hydraulic fracturing to exploit shale gas (which implies irreversible pollution of underground water reserves), authorizing the use of bisphenol A or glyphosate (which are both potential sources of cancers), relying excessively on antibiotic use and *de facto* creating antimicrobial resistance, relying exclusively on nuclear energy (with potential severe environmental destruction and health issues in case of an accident).

[3]Loi constitutionnelle n 2005-205 du 1 mars 2005 relative à la Charte de l'environnement

[4]See Gardiner (2006), Giddens (2011), O'Riordan (2013) for informed discussions and Immordino (2003) for a survey of the relevant economic literature.

In this paper, we take a broad perspective and shall view the *Precautionary Principle* as an upfront constitutional commitment to an action plan, in a dynamic world of irreversibility, uncertainty and limited information. Such commitment can only be justified on normative grounds if, in its absence, distortions away from the optimal trajectory of actions would arise. We will be agnostic on the direction of those distortions. It could be that, absent the constitutional constraint, actions are either too low or too high. In our context, we shall argue that distortions may find their source in the imperfect observability of decision-makers' actions over time, the scope for opportunistic behavior that it leaves, and the impact that it has on the evolution of beliefs.

MODEL. Consider the following set-up. A decision-maker (thereafter *DM*) chooses at any point in time an action that yields a flow surplus. The stock of past actions affects the arrival rate of an environmental disaster. Past actions have an irreversible impact. When the stock reaches a given tipping point, the rate jumps upwards.[5] A disaster is a major disruptive event. All opportunities for consumption/production disappear afterwards.

WHEN THE TIPPING POINT IS KNOWN. Suppose first that *DM* knows the tipping point. All actions taken earlier on contribute to approaching this tipping point; an *Irreversibility Effect.* In this scenario, the sole state variable is thus the stock of past actions. Because of discounting and because all past actions play the same role in approaching the tipping point, the optimal feedback rule, which is only function of the stock, requires lower actions as the stock increases during an early phase. Distortions below the myopic optimum are driven by the sole concern for irreversibility. Once the tipping point has been passed, actions no longer impact the arrival rate. *DM* maximizes current benefits by jumping to a higher myopic action. The benefit of low actions early on is to postpone the date at which the tipping point is reached. Yet postponing is also costly since actions can be shifted to the myopic optimum once the tipping point has been passed.

WHEN THE TIPPING POINT IS UNKNOWN. Suppose now that only the distribution of possible values for the tipping point is known.[6] *DM* now acts in the darkness, taking into account not only the irreversibility of his earlier actions but also his beliefs on whether the tipping point has been passed or not. When acting, *DM* only knows that there has been no disaster up to that date. Acting today changes how likely it is that the tipping point will be passed in the near future and thus affects *DM*'s posterior beliefs in case no disaster takes place. The state of the system is now best described by appending to the stock of past actions another state variable (that is called the *regime survival ratio* in what follows) that reflects beliefs on whether the tipping point has been passed or not. The optimal feedback rule now determines how the current action depends on both the existing stock of past actions and current beliefs. As *DM* becomes more pessimistic and believes that it is more likely that the tipping point has been passed, jumping towards the myopic optimum becomes more attractive.

[5]Tipping points models are frequently used in ecology and in climatology (Lenton et al., 2008). To illustrate, a recent report by the World Bank argues that *"As global warming approaches and exceeds 2-degrees Celsius, there is a risk of triggering nonlinear tipping elements. Examples include the disintegration of the West Antarctic ice sheet leading to more rapid sea-level rise.The melting of the Arctic permafrost ice also induces the release of carbon dioxide, methane and other greenhouse gases which would considerably accelerate global warming."* See http://whrc.org/project/arctic-permafrost.

[6]Kriegler et al. (2009) offers a view of what experts might think of those distributions of tipping points. Roe and Baker (2007) argues that whether past actions have already triggered a change of regimes might remain unknown for a while.

Stock-Markov Equilibria. Relying on such a complete feedback rule nevertheless raises some issues. First, conceiving an action plan which is contingent on all possible values of the stock and beliefs is a task that requires a huge degree of rationality and computational ability. On the optimal path, stock and beliefs indeed jointly evolve along a one-dimensional manifold. It makes it somewhat unattractive to compute an action plan for a two-dimensional space of state variables. Second, in the absence of hard scientific evidence on the state of the system, beliefs might be easily manipulated; an issue of prime importance in contexts where experts face difficulties in conveying evidence to policy-makers and interest groups may have a stake in blurring inference.[7]

In response to those caveats, we look at the properties of incomplete *Stock-Markov* feedback rules that only depend on stock. A so-called *Stock-Markov Equilibrium* (thereafter SME) is obtained when DM can only commit to an action over an infinitesimal period of time; anticipating that future selves abide to the same rule. The evolution of beliefs along the equilibrium path is in turn consistent with the feedback rule.

A priori, there is no reason to expect that such partial contingent plans would replicate the optimal trajectory, even if the objectives of DM's various selves that act at different points in time are aligned. The performances of such rules are actually strikingly different depending on whether deviations from the planned actions are observable or not.

Observable Deviations. Consider first a scenario where any current deviation is observable by future selves. In any such *SME*, actions remain below the myopic optimum because of the *Irreversibility Effect*. Yet, under uncertainty, the feedback rule must also account for the fact that increasing current action also modifies future beliefs. Because deviations are observable, DM's future selves will certainly believe that the tipping point is more likely to have been passed following a deviation that has increased the stock they inherited; a *Pessimism Stigma*. Thinking that the tipping point is more likely to have been passed, future selves have more incentives to increase actions.

Implementation of the Optimum. Importantly, an optimal trajectory can always be implemented as a *SME* when deviations are observable.[8] The intuition is simple. At the optimum, a complete feedback rule defines actions in terms of stock and beliefs. Since stock and beliefs evolve along a one-dimensional manifold along the optimal trajectory, the optimal feedback rule contingent on stock and beliefs naturally induces a *Stock-Markov* feedback rule *on path*. By construction, actions being the same with those two rules, beliefs evolve similarly. *Off path,* future selves can reconstruct the evolution of beliefs from the observed deviation and the conjecture that, beyond such deviation, all selves abide to the equilibrium feedback rule. This construction is key to align the choices of various selves acting at different points in time and follow the optimal trajectory. Local information that subsequent decision-makers may have on possible deviations keep in check the current decision-maker at any point in time.

Non-Observable Deviations. In contrast, consider the scenario where at each point in time, DM may adopt an opportunistic behavior and deviate from the equilibrium

[7] We shall leave aside the concerns about the reliability of information and how it can be manipulated or interpreted by groups of different backgrounds and experts. For some related discussion of those considerations, we refer to Hood, Rothstein and Baldwin (2003, Chapter 2).

[8] In passing, this argument also shows that such an equilibrium always exists since we demonstrate that the optimization problem has always a solution.

trajectory in a way that cannot be detected. That deviations are non-observable by future selves is meant to capture the fact that those selves have now limited information on the state of the system. Indeed, in our model, not only the stock but the whole history of past actions is necessary to construct correct beliefs. Limited observability of actions means thus limited information. In this context, decision-makers can only base their own decision on the current stock they inherit but not on the evolution of beliefs. Of course, they still conjecture that this evolution is consistent with the equilibrium feedback rule. At equilibrium, the *Stock-Markov* feedback rule must be optimal given those conjectures.

Because decision-makers cannot refrain from acting opportunistically if their actions are not observable, future selves can only form conjectures on how beliefs evolve. The equilibrium feedback rule now requires a prudent behavior. Actions are always too low in comparison with what the optimal trajectory would request. Along such a low-action trajectory, the tipping point is thought to be unlikely to have been passed yet; which in turn justifies adopting a prudent behavior in the first place. The *Precautionary Principle* then finds a rationale under those circumstances. Imposing an upfront commitment to the *Stock-Markov* feedback rule that would be optimal had deviations been observable, forces higher actions and still implements the optimum. In sharp contrast, there is no need to impose such rule when deviations are observable.

Organization. Section 2 reviews the literature. Section 3 presents the model. Section 4 analyzes the case where the tipping point is known. Section 5 deals with the scenario where only the distribution of the tipping point is known. Section 6 analyzes the properties of *SME* and shows that one such equilibrium implements the optimal trajectory when deviations are observable. Section 7 characterizes equilibrium feedback rules with non-observable deviations. Section 8 discusses the different perspectives that our model offers on the *Precautionary Principle*. Section 9 briefly recaps our results and discusses possible extensions. Proofs are relegated into Appendices.

2. LITERATURE REVIEW

Irreversibility, Uncertainty and Information. Arrow and Fisher (1974), Henry (1974) and Freixas and Laffont (1984) were the first to show how a decision-maker should take more preventive stances when the consequences of irreversible choices are uncertain. Epstein (1980) has discussed general conditions under which this *Irreversibility Effect* prevails. In those models, information is exogenous while in many contexts in environmental economics, actions also determine information structures. Hereafter, the probability of having passed the tipping point depends on the stock of past actions. Models with endogenous information structures are scarce. Freixas and Laffont (1984) have studied a scenario in which more flexible actions increase the quality of future information, thus confirming the existence of the *Irreversibility Effect*. Miller and Lad (1984) have challenged this view in a model of conservation in which irreversible actions might also be more informative. Salmi, Laiho and Murto (2019) study the trade-off faced by a decision-maker who must choose between acting now, which means taking a less informed decision but generating information that is useful in the sequel, and acting later, when being more informed. Greater actions accelerate the convergence of beliefs towards the true state.[9]

[9]Taking a broader perspective, it is fair to recognize that the general framework proposed by the irreversibility literature has been applied to the economics of climate change with mixed success. Some

The Economists' View of the Precautionary Principle. Gollier, Jullien and Treich (2000) have built on the insights of the irreversibility literature to give some economic content to the *Precautionary Principle*. These authors interpret the *Precautionary Principle* as the incentives of a decision-maker to reduce his action below the level that would otherwise be optimal without uncertainty, when this action is taken before any information is learned. Much in the spirit of Kolstad (1996), Gollier, Jullien and Treich (2000) build a two-period model of pollution accumulation with exogenous information and draw conclusions on specific forms of utility functions that induce more precaution. Asano (2010) has focused on the comparison of optimal environmental policies without and with ambiguity, showing that DM's lack of confidence forces him to hasten the adoption of a policy. In those models, DM's behavior is always optimal although constrained by informational requirements.[10] In Gollier, Jullien and Treich (2000), scientific information is modeled as an exogenous process and the authors are interested in comparing actions with and without such information. In contrast, we stress that beliefs on the state of the system are by and large endogenous and determined by the mere trajectory that is followed. In our context, the *Precautionary Principle* appears hereafter as a guide for actions in response to some opportunistic behavior by decision-makers that is only possible when information is limited. Interestingly, Gollier, Jullien and Treich (2000) have also stressed in their concluding remarks that more research is needed to understand the role of the *Principle* when opportunism is a concern. It is precisely this issue that we address hereafter.

On Tipping Points. A strand of the environmental economics literature has focused on analyzing tipping points. Sims and Finoff (2016) have analyzed how irreversibility in environmental damage and irreversibility in sunk cost investment interact. Tsur and Zemel (1995) have investigated a problem of optimal resource extraction when extraction affects the probability that the resource becomes obsolete passed a certain threshold. When this threshold is unknown, the initial state affects the optimal path and DM might end up exploiting resource less than under certainty. In a model of optimal control of atmospheric pollution, Tsur and Zemel (1996) have shown how uncertainty on a tipping point introduces a multiplicity of possible equilibrium values. Van der Ploeg (2014) has analyzed how uncertainty on tipping points may modify optimal carbon taxes. Liski and Salanié (2018) have also studied a model with unknown tipping points and uncertainty, but with different concerns like, for instance, the monotonicity of actions over time.

3. THE MODEL

Technology. DM runs a project which puts the environment at risk. Time is continuous. Let $r > 0$ be the discount rate. Let $\mathbf{x} = (x(\tau))_{\tau \geq 0}$ (resp. $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$) denote an action plan (resp. the continuation of such a plan from date t on).

The project may induce a catastrophe, an event that follows a Poisson process with a (non-homogeneous) rate $\theta(t)$. That rate depends on the stock $X(t) = \int_0^t x(\tau)d\tau$ of past

authors have argued that this literature suggests that current abatements of greenhouse gaz emissions should be greater when more information will be available in the future (Chichilnisky and Heal, 1993; Beltratti, Chichilnisky and Heal, 1995; Kolstad, 1996; Gollier, Jullien and Treich, 2000; among others). Others like Ulph and Ulph (2012) have pointed out that the sufficient conditions given by Epstein (1980) for the *Irreversibility Effect* to hold may fail even in simple models of global warming.

[10]This feature is shared by other models in the field like Immordino (2000) and Gonzales (2008).

actions that have already been taken before date t. More precisely, we postulate

$$\theta(t) = \theta_0 + \Delta \mathbb{1}_{\{X(t) > \overline{X}\}} \tag{3.1}$$

where $\overline{X}$ is a *tipping point*. Although it remains quite close to a homogeneous Poisson process, and indeed it is so before and after the tipping point, this specification features dependence on past actions. Indeed, when the stock of past actions $X(t)$ passes $\overline{X}$, the rate jumps from θ_0 to $\theta_1 > \theta_0$. Let $\Delta = \theta_1 - \theta_0 > 0$ measure this jump.

PREFERENCES. Action $x(t)$ yields a surplus (net of the action cost) at date t worth

$$u(x(t)) \equiv \zeta x(t) - \frac{x^2(t)}{2}.$$

where $\zeta > 0$. Feasible actions belong to an interval $\mathcal{X} = [0, 2\zeta]$ so that surplus remains non-negative under all circumstances below.

To capture its detrimental and irreversible impact, we assume that, if a catastrophe arises at date t, the flow surplus is no longer realized from that date on. A justification is that production may no longer be possible afterwards.[11]

A SPECIAL CASE AND SOME NOTATIONS. We start with the simplest scenario where DM has no control over the arrival rate of a disaster, i.e., the case of an homogeneous Poisson process and we assume that the tipping point was at $\overline{X} = 0$. Expected welfare can thus be written as:

$$\int_0^{+\infty} e^{-\lambda_1 t} u(x(t)) dt$$

where $\lambda_1 = r + \theta_1$ stands for the effective discount rate that applies once the possibility of a disaster is taken into account. Since he cannot influence the arrival rate of the disaster, DM maximizes current surplus at any point in time by choosing the *myopic action*

$$x^m(t) = \zeta \quad \forall t \geq 0.$$

For future reference, we denote the myopic payoff once the tipping point has been passed by

$$\mathcal{V}_\infty = \frac{u(\zeta)}{\lambda_1}.$$

4. WHEN THE TIPPING POINT IS KNOWN

Let $\overline{T}$ be the earliest date at which the tipping point is reached. With these notations at hands, we may rewrite DM's expected welfare as:

$$\int_0^{\overline{T}} e^{-\lambda_0 t} u(x(t)) dt + e^{-\lambda_0 \overline{T}} \int_{\overline{T}}^{+\infty} e^{-\lambda_1 (t - \overline{T})} u(x(t)) dt.$$

[11]This assumption is made for simplicity. A more general model would allow for an arbitrary number of disasters with possibly changes in the production/consumption structure following each of those events. This additional complexity would not add anything in terms of insights. Our model could also account for the possibility of incurring a damage flow D at the cost of some notational burden.

where $\lambda_0 = r + \theta_0$ stands for the effective discount rate before the tipping point. The first integral thus stems for welfare before the tipping point. The second integral stands for welfare after the tipping point, weighted by the probability of survival up to the date $\overline{T}$ at which the tipping point is reached, namely $e^{-\lambda_0 \overline{T}}$. The arrival rate of a catastrophe from that date on has now jumped up and payoffs beyond date $\overline{T}$ are more heavily discounted.

Dynamic Programming. Consider an action plan $\mathbf{x}_0 = \{x(\tau)\}_{\tau\geq 0}$ from date 0 onwards. If the stock at date 0 were X, the stock process $\hat{X}(\tau; X)$ would evolve as

$$\hat{X}(\tau; X) = X + \int_0^\tau x(s)ds. \tag{4.1}$$

After having passed the tipping point at date $\overline{T}$, DM always chooses the myopic optimal action ζ and gets, from that date on, a discounted continuation payoff worth $\mathcal{V}_\infty$. Let accordingly define the current value function $\mathcal{V}^k(X; \overline{X})$[12] as

$$\mathcal{V}^k(X; \overline{X}) \equiv \sup_{\mathcal{A}_0^k} \int_0^{\overline{T}} e^{-\lambda_0 \tau} u(x(\tau))d\tau + e^{-\lambda_0 \overline{T}} \mathcal{V}_\infty \tag{4.2}$$

where the set of feasible trajectories is

$$\mathcal{A}_0^k = \left\{ \mathbf{x}_0, \hat{X}(\cdot) \text{ s.t. (4.1) and } \hat{X}(\overline{T}; X) = \overline{X} \text{ for some } \overline{T} \geq 0 \right\}.$$ [13]

Equipped with these notations, we are now ready to further characterize the value function and the associated feedback rule.

Proposition 1 *The value function $\mathcal{V}^k(X; \overline{X})$ is continuously differentiable on $[0, \overline{X})$ and satisfies the following HBJ equation*

$$\dot{\mathcal{V}}^k(X; \overline{X}) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^k(X; \overline{X})}, \quad \forall X < \overline{X}. \tag{4.3}$$ [14]

$\mathcal{V}^k(X; \overline{X})$ is decreasing and strictly concave for $X \in [0, \overline{X})$ with the boundary condition

$$\mathcal{V}^k(X; \overline{X}) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}. \tag{4.4}$$

The optimal feedback rule is such that

$$\sigma^k(X; \overline{X}) = \begin{cases} \zeta + \underbrace{\dot{\mathcal{V}}^k(X; \overline{X})}_{\textit{Irreversibility Effect}} & \textit{for } X \in [0, \overline{X}), \\ \zeta & \textit{for } X \geq \overline{X}. \end{cases} \tag{4.5}$$

Moreover, $\sigma^k(X; \overline{X})$ is decreasing in X for $X \in [0, \overline{X})$.

[12] We prove in the Appendix that using this current value function suffices to characterize the optimum.

[13] It should be clear that the current value function $\mathcal{V}^k(X; \overline{X})$ and optimal decision-rule $\sigma^k(X; \overline{X})$ so obtained only depend on the distance $Y = \overline{X} - X$ to the tipping point. In other words, there exist two functions $\overline{\mathcal{V}}$ and $\overline{\sigma}$ such that $\mathcal{V}^k(X; \overline{X}) \equiv \overline{\mathcal{V}}(\overline{X} - X)$ and $\sigma^k(X; \overline{X}) \equiv \overline{\sigma}(\overline{X} - X)$. For the sake of comparing value functions and feedback rules under different scenarios, we nevertheless express the optimality conditions found below in terms of $\mathcal{V}^k(X; \overline{X})$ and $\sigma^k(X; \overline{X})$.

[14] At $X = \overline{X}$, this derivative is in fact a left-derivative but we use the same notation for simplicity.

Actions profile. The optimal action goes through two distinct phases. Before reaching the tipping point, DM chooses an action which remains below the myopic optimum. Actions that have been taken in the past have a long-lasting impact since they may contribute to passing the tipping point earlier on. Reducing such actions keeps the probability that a disaster arises earlier at a low level. More precisely, the quantity $-\dot{\mathcal{V}}^k(X;\overline{X})$ found on the r.-h. s. of (4.5) is in fact the Lagrange multiplier for the irreversibility constraint

$$\int_0^{\overline{T}} x(\tau)d\tau = \overline{X} - X. \tag{4.6}$$

As X increases without having yet reached $\overline{X}$, this irreversibility constraint becomes more demanding, and the value function is decreasing. Actions are reduced below the myopic optimum to account for this *Irreversibility Effect*.

The optimal action decreases over time before the tipping point. All actions taken during this first phase contribute the same to the overall stock. Because of discounting, DM prefers to choose higher actions earlier on and lower ones when approaching the tipping point. Expressed in terms of the value function, this monotonicity means that $\mathcal{V}^k(X;\overline{X})$ is strictly concave over this first phase. It is instead flat once the tipping point has been passed. By then, DM knows that his actions will no longer have any impact on the arrival rate of a disaster and thus chooses the myopic optimum.

Tipping Point. Because actions are now lower than the myopic optimum over the first phase, the tipping point is reached at a date[15]

$$\overline{T}^k = \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) \frac{1 - e^{-\lambda_0 \overline{T}^k}}{\lambda_0} > \overline{T}^m = \frac{\overline{X}}{\zeta} \tag{4.7}$$

where $\overline{T}^m$ is the time necessary to reach the tipping point when always adopting the myopic action. The intuition for this result is as follows. By pushing a bit further in the future the date $\overline{T}^k$ at which the tipping point is reached by a small amount $d\overline{T}$, DM incurs a welfare loss since, over the first phase, the action is below the myopic optimum. DM is therefore getting less than the optimal surplus over a longer period of time. Pushing a bit further in the future the date $\overline{T}^k$ also hardens the feasibility constraint. Finally, increasing $\overline{T}^k$ maintains the arrival rate of a disaster at its low level θ_0 over that extended period. By doing so, DM is less likely to losing the myopic surplus $u(\zeta)$ in case a disaster occurs.

Bounds. Next proposition provides bounds on payoffs and actions. As we will see below, those bounds will also prevail when the location of the tipping point remains uncertain.

Proposition 2 *$\mathcal{V}^k(X;\overline{X})$ and $\sigma^k(X;\overline{X})$ admit the following bounds*

$$\mathcal{V}_\infty \leq \mathcal{V}^k(X;\overline{X}) < \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty \quad \forall X, \tag{4.8}$$

$$\zeta\sqrt{\frac{\lambda_0}{\lambda_1}} \leq \sigma^k(X;\overline{X}) \leq \zeta \quad \forall X. \tag{4.9}$$

[15] See the Appendix for details.

Because the rate of arrival of a disaster remains low till the tipping point is passed, the value function remains above its long-term limit $\mathcal{V}_\infty$ reached beyond that point. The upper bound on the value function is the payoff corresponding to choosing always the myopic action but in the scenario where the tipping point would never be passed and the effective discount rate remains λ_0. The upper bound on actions is simply the myopic optimum $x^m = \zeta$. The lower bound $\zeta\sqrt{\frac{\lambda_0}{\lambda_1}}$ is the action that ends the phase where the *Irreversibility Effect* is at play.

5. UNCERTAINTY: VALUE FUNCTION AND FEEDBACK RULE

Suppose now that DM ignores where the tipping point lies. Switching to the myopic optimum once the tipping point has been passed is no longer possible since DM ignores whether this event occurred or not. Yet, DM must account for that possibility when choosing his action plan. Accordingly, let denote by F the distribution of possible values for the tipping point and by f its (positive) density function. This distribution has a finite support $\left[0, \overline{X}\right]$ (i.e., $\overline{X} < +\infty$) and, for simplicity, no mass point.[16]

5.1. *Preliminaries*

BELIEFS. Consider a history of past actions $\mathbf{x}^t$ with no disaster up to date t and a stock reached at that date given by $\hat{X}(t;0) = \int_0^t x(s)ds$. To evaluate DM's continuation payoff, we need to compute his posterior beliefs $f(\widetilde{X}|t,\mathbf{x}^t)d\widetilde{X}$ that the tipping point lies within the interval $\left[\widetilde{X}, \widetilde{X} + d\widetilde{X}\right]$ given that past history. This posterior density $f(\widetilde{X}|t,\mathbf{x}^t)$ should take into account that, if the tipping point lies at $\widetilde{X} \leq \hat{X}(t;0)$, the arrival rate has already jumped from θ_0 to θ_1 at an earlier date $T(\widetilde{X};0) \leq t$. If instead the tipping point is at $\widetilde{X} > \hat{X}(t;0)$, the arrival rate remains θ_0. A key variable to describe how the posterior density evolves is thus the probability of survival up to date t when the path of past actions is $\mathbf{x}^t$, namely

$$H(t,\mathbf{x}^t) = \int_0^{\hat{X}(t;0)} f(\widetilde{X})e^{-\theta_0 T(\widetilde{X};0)}e^{-\theta_1(t-T(\widetilde{X};0))}d\widetilde{X} + \int_{\hat{X}(t;0)}^{+\infty} f(\widetilde{X})e^{-\theta_0 t}d\widetilde{X}. \tag{5.1}$$

After manipulations, we obtain:

$$H(t,\mathbf{x}^t) = e^{-\theta_0 t}\left(1 - \Delta e^{-\Delta t}\int_0^t F(\hat{X}(\tau;0))e^{\Delta\tau}d\tau\right).\tag{5.2}$$[17]

When $\hat{X}(t;0)$ is close to 0, the likelihood of having passed the tipping point is also close to 0. The survival probability is then nearly that obtained when the arrival rate of a disaster is known to be θ_0 for sure. As $\hat{X}(t;0)$ increases towards $\overline{X}$, it becomes more likely that the tipping point has been passed and the survival probability accordingly decreases. Of course, the shape of the distribution function F matters to evaluate this

[16]In contrast, the RUNNING EXAMPLE below entails mass points but adapting the analysis is straightforward.

[17]See the Proof of Lemma B.1 in the Appendix.

probability. As F puts more mass around the origin, it is more likely that the tipping point has been passed early on and the survival probability diminishes.

For future reference, let define the *regime survival ratio* $\hat{Z}(t, \mathbf{x}^t)$ as

$$\hat{Z}(t, \mathbf{x}^t) = H(t, \mathbf{x}^t)e^{\theta_0 t} \quad \forall t \geq 0. \tag{5.3}$$

It is the ratio between the survival probability $H(t, \mathbf{x}^t)$ at date t following an history $\mathbf{x}^t$ and the survival probability $e^{-\theta_0 t}$ that would prevail had the tipping point never been passed.[18] This ratio actually reflects DM's beliefs on whether the tipping point has been passed or not. The faster the trajectory moves towards $\overline{X}$, the faster $\hat{Z}(t, \mathbf{x}^t) = 1 - \Delta e^{-\Delta t} \int_0^t F(\hat{X}(\tau; 0))e^{\Delta \tau} d\tau$ decreases. If the trajectory stays close to $X = 0$, $\hat{Z}(t, \mathbf{x}^t)$ decreases very slowly.

To further illustrate, consider values of the tipping point ahead of where DM currently stands, i.e., $\hat{X}(t; 0) \leq \widetilde{X}$. For such values, the posterior belief that the tipping point lies in the interval $\left[\widetilde{X}, \widetilde{X} + d\widetilde{X}\right]$ writes as

$$f(\widetilde{X}|t, \mathbf{x}^t)d\widetilde{X} = Proba\left(\left[\widetilde{X}, \widetilde{X} + d\widetilde{X}\right] |t, \mathbf{x}^t\right) = \frac{Proba\left(\left[\widetilde{X}, \widetilde{X} + d\widetilde{X}\right], t, \mathbf{x}^t\right)}{H\left(t, \mathbf{x}^t\right)}$$

$$= \frac{e^{-\theta_0 t}}{H(t, \mathbf{x}^t)} f(\widetilde{X})d\widetilde{X} = \frac{f(\widetilde{X})}{\hat{Z}(t, \mathbf{x}^t)} d\widetilde{X}.$$

A lower values of $\hat{Z}(t, \mathbf{x}^t)$ makes DM believe that, conditionally on having survived, it is very likely that the tipping point has not been passed yet.

Running Example. Suppose that F has Dirac masses q at 0 and $1 - q$ at $\overline{X}$. In other words, DM is uncertain whether the tipping point is passed right away or whether it will be later found at $\overline{X}$. For any $t > 0$ and history $\mathbf{x}^t$ that has not yet reached $\overline{X}$, the probability of survival is a convex combination of exponential discounting:

$$H(t, \mathbf{x}^t) = qe^{-\theta_1 t} + (1 - q)e^{-\theta_0 t}.$$

From this, it follows that the *regime survival ratio* before reaching $\overline{X}$ becomes

$$\hat{Z}(t, \mathbf{x}^t) = 1 - q + qe^{-\Delta t}.$$

■

Value Function. The value function $\hat{\mathcal{V}}(t, \mathbf{x}^t)$ is DM's continuation payoff starting from date t onwards given the past history $\mathbf{x}^t$, taking expectations over possible values of the tipping point according to the density function $f(\widetilde{X}|t, \mathbf{x}^t)$. Following such history, the stock has reached $X = \hat{X}(t; 0)$ at date t. For $\tau \geq t$, the stock (denoted with a slight abuse of notations by $\hat{X}(\tau; X, t)$) will evolve according to the stream of future actions $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$. Next Lemma provides a compact representation of the value function.

[18] Since the survival probability is bounded below by $e^{-\theta_1 t}$, the regime survival ratio itself lies within $(e^{-\Delta t}, 1]$.

Lemma 1 *The value function $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ satisfies*

$$\hat{Z}(t,\mathbf{x}^t)\hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t,\hat{X}(\tau;X,t)=X+\int_t^\tau x(s)ds} \int_0^{+\infty} e^{-\lambda_0\tau}\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})u(x(t+\tau))d\tau. \tag{5.4}$$

This definition implicitly captures the fact that the value function is computed at any point in time after having correctly updated beliefs on the distribution of possible values of the tipping point.

5.2. *Representation of the Value Function*

The representation (5.4) of the value function suggests that the state of the system is best described by adding to the stock X a second state variable, the *regime survival ratio* Z that reflects beliefs. Two trajectories that have reached the same stock X with the same beliefs Z at a given date should have the same continuation. Instead, two trajectories that have reached the same stock but with different beliefs might be pursued differently. If the regime switch is thought as having been likely, i.e., Z small, DM will certainly pursue with higher actions since he has less incentives to take a precautionary stance.

To the law of motion for the stock, namely

$$\dot{X}(\tau) = x(\tau), \tag{5.5}$$

we must now also add the law of motion for the regime survival ratio to complete the state of the system. From differentiating (5.3) and using (5.2), we get

$$\dot{Z}(\tau) = \Delta(1 - F(X(\tau)) - Z(\tau)). \tag{5.6}$$

Integrating (5.6) with the initial condition $Z(0) = Z$ yields

$$Z(\tau) = \underbrace{1 - \Delta e^{-\Delta\tau}\int_0^\tau F(X(s))e^{\Delta s}ds}_{\text{Memoryless Evolution}} - \underbrace{(1-Z)e^{-\Delta\tau}}_{\text{Pessimism Stigma}}. \tag{5.7}$$

This expression of $Z(\tau)$ highlights how the evolution of beliefs actually superposes two effects. Suppose that DM keeps no memory of what happened in the past. He is naively believing to start with $Z = 1$, only knowing about the current level of stock X and considering, from that point on, the ensuing trajectory $X(t)$ given by (5.5). The first term on the r.h.s. of (5.7) captures how such a naive DM would evaluate the consequences of pursuing this trajectory on future beliefs. Instead, whenever DM starts with some grain of pessimism inherited from past history, i.e., starting with a level of Z less than 1, this pessimism remains a stigma that is carried over in the future (although at a decreasing rate); an effect that is captured by the second term on the r.h.s. of (5.7).

Finally, (5.6) also implies that, once a trajectory $X(\tau)$ has reached the upper bound $\overline{X}$ at a date $\overline{T}$, the regime survival ratio evolves from then on as[19]

$$Z(\tau) = Z(\overline{T})e^{-\Delta(\tau-\overline{T})} \quad \forall\tau \geq \overline{T}. \tag{5.8}$$

[19] Once the stock level is beyond the support of F, the probability to be in the low-risk regime is 0.

Using (5.4) and (5.8), we can now get a representation of the value function in terms of the bi-dimensional state (X, Z). Let accordingly define the value function $\mathcal{V}^e(X, Z)$ for $X \geq 0$ and any $Z \in (0, 1]$ as

$$Z\mathcal{V}^e(X, Z) = \sup_{\mathcal{A}} \int_0^{\overline{T}} e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) d\tau + e^{-\lambda_0 \overline{T}} Z(\overline{T}) \mathcal{V}_\infty \tag{5.9}$$

where the set of admissible trajectories is

$$\mathcal{A} = \{\mathbf{x}, X(\cdot), Z(\cdot), \overline{T} \text{ s.t. } (5.5),\ (5.6),\ X(0) = X,\ X(\overline{T}) = \overline{X},\ Z(0) = Z\}.$$

Starting from any two-dimensional state (X, Z), DM looks for an optimal arc that reaches $\overline{X}$ at date $\overline{T}$. From that date on, DM knows for sure that the tipping point has been passed and chooses the myopic optimum. The tipping point might have been passed a long time ago but DM could not know it for sure.

The associated *feedback rule* $\sigma^e(X, Z)$ defines the trajectory both in terms of the overall stock $X^e(\tau; 0, 1)$ but also of the beliefs $Z^e(\tau; 0, 1)$ starting from the initial conditions $(X = 0, Z = 1)$. Provided that $X^e(\tau; 0, 1)$ is invertible, there is a one-to-one relationship between the current stock and beliefs. Even though the value function is computed for a broader set of values of those states variables, stock and beliefs evolve altogether along a one-dimensional manifold at the optimum. This remark plays a key role in what follows.

PROPOSITION 3 *The value function $\mathcal{V}^e(X, Z)$ satisfies:*

$$\frac{\partial \mathcal{V}^e}{\partial X}(X, Z) = -\zeta + \sqrt{2\lambda^e(X, Z)\mathcal{V}^e(X, Z) - 2\Delta(1 - F(X) - Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z)} \ \text{a.e.} \tag{5.10}$$

where

$$\lambda^e(X, Z) = \lambda_0 - \frac{\Delta(1 - F(X) - Z)}{Z} \tag{5.11}$$

together with the boundary conditions

$$\mathcal{V}^e(X, Z) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}, \forall Z \in (0, 1]. \tag{5.12}$$

The optimal complete feedback rule is

$$\sigma^e(X, Z) = \zeta + \frac{\partial \mathcal{V}^e}{\partial X}(X, Z). \tag{5.13}$$

The comparison of the HBJ equations with and without uncertainty is instructive. Although quite similar, Equations (4.3) and (5.10) bear two differences. The first one is related to how future payoffs are discounted. To see this, let us rewrite (5.9) as

$$\mathcal{V}^e(X, Z) = \sup_{\mathcal{A}} \int_0^{\overline{T}} e^{-\int_0^\tau \left(\lambda_0 - \frac{\dot{Z}(s)}{Z(s)}\right) ds} u(x(\tau)) d\tau + e^{-\int_0^{\overline{T}} \left(\lambda_0 - \frac{\dot{Z}(s)}{Z(s)}\right) ds} \mathcal{V}_\infty.$$

This expression showcases that, under uncertainty, the effective discount rate $\lambda^e(\tau) \equiv \lambda_0 - \frac{\dot{Z}(\tau)}{Z(\tau)}$ is time-dependent. Using the regime survival ratio as a state variable keeps track of this time-dependency. The choice of an action at date t has no direct impact on how this implicit discount rate evolves since the law of motion (5.6) for beliefs does not depend on current action. Yet, because stock and beliefs evolve over time, this implicit discount rate keeps on changing and DM must take this into account to assess how his future payoffs should be discounted. Specifically, DM is using $\lambda^e(\tau) \approx \lambda_0$ to discount future payoffs earlier on but, eventually, will switch to $\lambda^e(\tau) \approx \lambda_1$ later on. The hazard rate $-\dot{Z}(\tau)/Z(\tau)$ measures how information contained in the fact that no disaster has yet arisen is incorporated into this implicit discounting.

The second difference between Equations (4.3) and (5.10) comes from a novel term, not present under complete information, namely $-2\Delta(1 - F(X) - Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z)$ on the r.-h. s. of (5.10). Less optimistic stances, i.e., lower values of Z are associated with lower continuation values (i.e., $\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z) < 0$). Along the optimal trajectory, this new term is negative.[20] Being less optimistic and thinking that the tipping point has already been passed, DM certainly chooses to increase actions.

Finally, the comparison of the feedback rule (5.13) with its complete information counterpart (4.5) shows that the term $\frac{\partial \mathcal{V}^e}{\partial X}(X, Z)$ can again be interpreted as an opportunity cost of irreversibility. This cost now depends on beliefs. The consequences of such beliefs on actions can be further illustrated in the framework of our example.

Running Example (Continued). When $q = 0$, we have $F(X) = 0$ for all $X \in [0, \overline{X})$ and it is straightforward to check that the solution to (5.10) and (5.12) is $\mathcal{V}^e(X, Z) \equiv \mathcal{V}^k(X; \overline{X})$. When $q = 1$, we instead have $F(X) = 1$ for all $X \in (0, \overline{X}]$ and the solution to (5.10) and (5.12), for $Z = 1$ is then $\mathcal{V}^e(X, 1) \equiv \mathcal{V}_\infty$.

Although $\mathcal{V}^e(X, Z)$ cannot be expressed in closed form for $q > 0$, both the profile of optimal actions $\mathbf{x}^e$ along the trajectory starting from $X = 0$ and $Z = 1$, and the delay $\overline{T}^e$ till reaching the tipping point, can be solved explicitly.

Proposition 4 *Suppose that F has Dirac masses q at 0 and $1 - q$ at $\overline{X}$. The optimal trajectory starting from $X = 0$ and $Z = 1$ has the following features.*

- *The date $\overline{T}^e$ at which $\overline{X}$ is reached solves*

$$(5.14) \quad \overline{T}^e = \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right) e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e) \int_0^{\overline{T}^e} \frac{e^{\lambda_0 \tau}}{Z(\tau)} d\tau > \overline{T}^m$$

where the regime survival ratio is

$$(5.15) \quad Z(\tau) = 1 - q + q e^{-\Delta \tau} \quad \forall \tau \in [0, \overline{T}^e).$$

- *The optimal action is decreasing over $t \in [0, \overline{T}^e)$ and equal to the myopic optimum thereafter:*

$$(5.16) \quad x^e(\tau) = \begin{cases} \zeta \left(1 - e^{-\lambda_0(\overline{T}^e - \tau)} \frac{Z(\overline{T}^e)}{Z(\tau)} \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\right) < \zeta & \text{for } t \in [0, \overline{T}^e), \\ \zeta & \text{for } t \geq \overline{T}^e. \end{cases}$$

[20] Indeed, we have $-\dot{Z}(\tau)/Z(\tau) = -\frac{\Delta(1 - F(X(\tau)) - Z(\tau))}{Z(\tau)} > 0$.

With uncertainty, the *Irreversibility Effect* is still at play as long as the highest possible values of the tipping point has not been passed. Actions remain below the myopic optimum over that first phase.

Yet, actions are higher than when the tipping point is known to lie at $\overline{X}$ for sure. This result is illustrated by observing that the last actions before passing $\overline{X}$ has now been raised towards the myopic solution in comparison with the scenario where the tipping point is known to be at $\overline{X}$ for sure:

$$(5.17) \quad x^e(\overline{T}^{e-}) = \zeta\sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}} > \zeta\sqrt{\frac{\lambda_0}{\lambda_1}}.$$

Under uncertainty, the date at which the stock equals the maximum possible tipping point $\overline{X}$ is reached earlier on:

$$\overline{T}^e < \overline{T}^k.$$

Intuitively, there is now a chance that the tipping point has already been passed before, so that the best thing to do is to choose actions which are closer to the myopic optimum. It can also be readily checked that as q goes to 0 (resp. 1), $\overline{T}^e$ converges towards T^k (resp. $\overline{T}^m$). ■

6. *STOCK-MARKOV* EQUILIBRIA WITH OBSERVABLE DEVIATIONS

The value function $\mathcal{V}^e(X, Z)$ is a mere technical device to compute the optimal trajectory. So describing the state of the system allows to use dynamic programming techniques to compute a complete feedback rule $\sigma^e(X, Z)$ that guides behavior. There are two ways of thinking about this device. First, this rule may be viewed as a machine, to which DM commits upfront, that determines actions in response to the evolution of stock and beliefs along the trajectory. Second, and it is a consequence of the *Principle of Dynamic Programming*, such rule can alternatively be decentralized as a Perfect-Markov equilibrium with Markov-strategies based on (X, Z) even when DM cannot commit but perfectly anticipates that future selves will stick to that rule as well and are able to observe the whole past history of actions to reconstruct the state variable Z.

In this respect, we may wonder whether a more parsimonious decentralization of the optimal trajectory is achievable with incomplete *Stock-Markov* feedback rules that only depend on the stock X. Accordingly, we now thus consider *Stock-Markov* equilibria (thereafter SME) sustained with such *Stock-Markov* feedback rules. At any such equilibrium, DM sticks to the strategy $\sigma^o(X)$ today because he expects future selves to abide to that rule.[21] Our goal in this section is to investigate whether there exists an implementation of the optimal path of actions by means of such a SME. Of course, the answer depends in fine details on what information is available to future selves to adapt their behavior.

Along any *Stock-Markov* trajectory, the stock $X^o(\tau; X)$ thus evolves according to

$$(6.1) \quad \frac{\partial X^o}{\partial \tau}(\tau; X) = \sigma^o(X^o(\tau; X)) \text{ with } X^o(0; X) = X.$$

[21] Of course, this feedback rule should specify that $\sigma^o(X) = \zeta$ for $X \geq \overline{X}$ and this continuation will be kept implicit in what follows.

A priori, such incomplete feedback rule might not suffice to capture the whole state of the system. Yet, DM should choose an optimal action at any point in time keeping in mind how beliefs will evolve following such choice when such a feedback rule prevails in the future. In other words, DM should be able to reconstruct the regime survival ratio that applies, along the equilibrium path, for each possible level of the stock and, by that means, correctly infer how to discount future payoffs. Let denote by $Z^o(X)$ such function.

From (5.6), the regime survival ratio $Z(\tau; X)$, that starts from a value $Z^o(X)$ at date 0 and that is consistent with the *Stock-Markov* feedback rule $\sigma^o(X)$ from that date on, evolves as

$$\frac{\partial Z}{\partial \tau}(\tau; X) = \Delta(1 - F(X^o(\tau; X)) - Z(\tau; X)) \text{ with } Z(0; X) = Z^o(X). \tag{6.2}$$

Since conjectures on how the regime survival ratio evolves along the trajectory are correct, we must also have

$$Z(\tau; X) = Z^o(X^o(\tau; X)) \quad \forall \tau \geq 0, X \geq 0. \tag{6.3}$$

Taken together, those conditions dictate how the regime survival ratio evolves with the current stock along the trajectory. By differentiating (6.3) with respect to τ, we get

$$\sigma^o(X)\dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \quad \forall X \geq 0 \tag{6.4}$$

with the initial condition

$$Z^o(0) = 1. \tag{6.5}$$

We may now define a *Stock-Markov-value function* $\mathcal{V}^o(X)$ as DM's payoff function along such a *Stock-Markov* trajectory as

$$Z^o(X)\mathcal{V}^o(X) = \int_0^{+\infty} e^{-\lambda_0 \tau} Z^o(X^o(\tau; X))u(\sigma^o(X^o(\tau; X)))d\tau. \tag{6.6}$$

This definition showcases how future payoffs are discounted at a rate that depends on regime survival ratio along the *Stock-Markov* trajectory.

For further reference, let

$$\varphi^o(X) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^o(X^o(\tau; X)))d\tau \tag{6.7}$$

stems for DM's expected payoff once the tipping point has been passed for sure but, being ignorant of that event, DM still relies on the feedback rule $\sigma^o(X)$ to choose actions.

Impulse Deviations. To express the equilibrium requirement that sticking to the feedback rule $\sigma^o(X)$ is optimal at any point along the trajectory, we follow an approach which is similar in spirit although different in details to that developed in Karp and Lee (2003), Karp (2005, 2007), Ekeland, Karp and Sumaila (2015) and Ekeland and Lazrak (2006, 2008, 2010). These authors have analyzed various macroeconomic and growth models with time-inconsistency problems. Roughly speaking it consists in importing the notion of perfect-Markov equilibrium, familiar in discrete-time models, to a continuous time setting.The idea is to look at the benefits of deviating from the feedback rule for periods of commitment which are of arbitrarily small length; deriving from there conditions for the sub-optimality of such deviations.[22]

To this end, consider a possible deviation that would consist in committing to an action x for a period of length ε, reaching a stock level $X+x\varepsilon$, before jumping back to the above feedback rule σ^o. For such a deviation, actions evolve according to

$$y(x,\varepsilon,\tau;X)=\begin{cases} x & \text{if } \tau\in[0,\varepsilon],\\ \sigma^o(\hat{X}(x,\varepsilon,\tau;X)) & \text{if } \tau>\varepsilon \end{cases} \tag{6.8}$$

while the whole stock trajectory is modified as

$$\hat{X}(x,\varepsilon,\tau;X)=\begin{cases} X+x\tau & \text{if } \tau\in[0,\varepsilon],\\ X+x\varepsilon+\int_\varepsilon^\tau \sigma^o(\hat{X}(x,\varepsilon,s;X))ds & \text{if } \tau\geq\varepsilon. \end{cases} \tag{6.9}$$

By adopting the deviation (6.8)-(6.9), the regime survival ratio would also change as

$$\hat{Z}(x,\varepsilon,\tau;X)=1-\Delta e^{-\Delta\tau}\int_0^\tau F(\hat{X}(x,\varepsilon,s;X))e^{\Delta s}ds-(1-Z^o(X))e^{-\Delta\tau}. \tag{6.10}$$

From this, we may define DM's deviation payoff $\hat{\mathcal{V}}(x,\varepsilon;X)$ as

$$Z^o(X)\hat{\mathcal{V}}(x,\varepsilon;X)=\int_0^{+\infty} e^{-\lambda_0\tau}\hat{Z}(x,\varepsilon,\tau;X)u(y(x,\varepsilon,\tau;X))d\tau. \tag{6.11}$$

When ε is made arbitrarily small, we will refer to such deviations as *impulse deviations.*

That all future selves are able to observe any impulse deviation that the current decision-maker may entertain allows them to reconstruct the evolution of beliefs as expressed in (6.10). When considering a deviation, the current decision-maker should thus assess its consequences on his intertemporal payoff by applying the implicit discounting that follows from the evolution of beliefs so induced. This inference is clear in the expression of the continuation payoff on the right-hand side of (6.11).

In this context, a *Stock-Markov Equilibrium* $(\mathcal{V}^o(X),\sigma^o(X),Z^o(X))$ is a *Stock-Markov*-value function, a *Stock-Markov* feedback rule and a belief function that are immune to such impulse deviations.

Definition 1 $(\mathcal{V}^o(X),\sigma^o(X),Z^o(X))$ *is a SME with observable impulse deviations if the following conditions hold.*

1. *$\mathcal{V}^o(X)$ as defined by (6.6) cannot be improved upon by any impulse deviation of the form (6.8)-(6.9) for ε made arbitrarily small:*

$$\mathcal{V}^o(X)=\max_{x\in\mathcal{X}}\lim_{\varepsilon\to0^+}\hat{\mathcal{V}}(x,\varepsilon;X). \tag{6.12}$$

[22] To figure out how it could be done more formally, consider a discrete version of our model where DM would thus commit to an action over each period $[t,t+\varepsilon]$, $[t+\varepsilon,t+2\varepsilon]$, ...$[t+n\varepsilon,t+(n+1)\varepsilon]$ (with $n\in\mathbb{N}$). It is then natural to focus on stationary Markov-perfect subgame equilibria for such a discrete game. In such an equilibrium, DM follows a feedback rule $\sigma^o_\varepsilon(X)$ that defines his current action in terms of the existing stock only. Of course, the equilibrium requirement imposes that this feedback rule is a best-response given DM's anticipations of his own future actions, which should themselves follow the same feedback rule although, of course, the stock at future dates has evolved according to past actions.

2. $\sigma^o(X)$ is optimal for ε made arbitrarily small:

$$\sigma^o(X) \in \arg\max_{x\in\mathcal{X}} \lim_{\varepsilon\to 0^+} \hat{\mathcal{V}}(x,\varepsilon;X). \tag{6.13}$$

3. $Z^o(X)$ is consistent with the feedback rule $\sigma^o(X)$ and satisfies (6.4)-(6.5).

Item 1. requires to approximate the deviation payoff $\hat{\mathcal{V}}(x,\varepsilon;X)$ to the first order in ε and look for the optimal action that maximizes such approximation; an optimality condition that is expressed in Item 2.. Item 3. follows from the consistency condition (6.3) which states that the optimal evolution of beliefs is dictated by the *Stock-Markov* feedback rule.

Properties of $(\mathcal{V}^o(X),\sigma^o(X))$. Developing the equilibrium conditions suggested by Definition 2 gives us some important properties.

Proposition 5 *At any (continuously differentiable) SME, with observable impulse deviations, the Stock-Markov-value function $\mathcal{V}^o(X)$ satisfies the following functional equation*

$$\dot{\mathcal{V}}^o(X) = -\zeta - \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X) + \sqrt{2\lambda_0\mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2} \quad \forall X\in[0,\overline{X}) \tag{6.14}$$

together with the boundary condition

$$\mathcal{V}^o(X) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}. \tag{6.15}$$

The corresponding Stock-Markov feedback rule writes as

$$\sigma^o(X) = \zeta + \dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}(\mathcal{V}^o(X) - \varphi^o(X)). \tag{6.16}$$

The formula for the feedback rule in (6.16) bears some resemblance with its counterpart (4.5) that was found under complete information. To understand the changes, it is useful to rewrite the *Stock-Markov*-value function (6.6) so as to let appear an implicit discount rate as

$$\mathcal{V}^o(X) = \int_0^{+\infty} e^{-\int_0^\tau\left(\lambda_0 - \sigma^o(X^o(s;X))\frac{\dot{Z}^o(X^o(s;X))}{Z^o(X^o(s;X))}\right)ds} u(\sigma^o(X^o(\tau;X)))d\tau.$$[23]

Starting from a current stock X with current beliefs $Z^o(X)$ on the equilibrium path, consider thus an impulse deviation consisting in increasing by a marginal amount dx the current action $\sigma^o(X)$ over an interval of length ε, where ε is small enough. Since the current stock *de facto* increases by εdx, such impulse deviation reduces the *Stock-Markov*-value function by

$$-\dot{\mathcal{V}}^o(X)\varepsilon dx. \tag{6.17}$$

[23]Here we use the identity

$$\frac{Z^o(X^o(\tau;X))}{Z^o(X))} = e^{ln\left(\frac{Z^o(X^o(\tau;X))}{Z^o(X))}\right)} = e^{\int_0^\tau \frac{d}{ds}ln(Z^o(X^o(s;X)))ds} = e^{\int_0^\tau \sigma^o(X^o(s;X))\frac{\dot{Z}^o(X^o(s;X))}{Z^o(X^o(s;X))}ds}.$$

This impact can be further decomposed into three different components. First, increasing current action and moving towards the myopic optimum has a marginal benefit on payoff over the interval of time $[0, \varepsilon]$ which is approximatively worth

$$(\zeta - \sigma^o(X))\varepsilon dx. \tag{6.18}$$

Second, this increase in the current stock also decreases the implicit discount factor by an amount

$$\frac{\dot{Z}^o(X)}{Z^o(X)} e^{-\int_0^\tau \left(\lambda_0 - \sigma^o(X^o(s;X)) \frac{\dot{Z}^o(X^o(s;X))}{Z^o(X^o(s;X))}\right) ds} \varepsilon dx.$$

The corresponding loss on the continuation payoff is thus approximatively worth

$$-\frac{\dot{Z}^o(X)}{Z^o(X)} \mathcal{V}^o(X)\varepsilon dx. \tag{6.19}$$

Because it is observable by future selves, an impulse deviation today also has a long-lasting effect on future beliefs as highlighted by formula (6.10). A marginal change of the stock by εdx makes it more likely that the tipping point has been passed; which brings an extra grain of pessimism over the whole future trajectory. From (6.10), this impulse deviation increases the *Pessimistic Stigma* by a term which is worth

$$-\dot{Z}^o(X) e^{-\Delta\tau} \varepsilon dx.$$

The corresponding loss on the continuation payoff is thus

$$-\frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X)\varepsilon dx. \tag{6.20}$$

Gathering (6.17), (6.18), (6.19) and (6.20) above finally yields Condition (6.16).

Reciprocally, a triplet $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$ that satisfies (6.14), (6.15), (6.16) and the consistency requirements (6.4)-(6.5) forms a *SME*. This point is exploited in Proposition 6 below to show that an optimal arc can be implemented as a *SME*.

Implementing the Optimum. Our next result shows that there is no need to compute a complete value function and derive a complex two-dimensional feedback rule to describe an optimal trajectory. Provided that current deviations remain observable, playing a *Stock-Markov* equilibrium suffices.

Proposition 6 *Suppose that impulse deviations are observable, an optimal path can be implemented as a SME,*[24] $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$, *such that*

$$\mathcal{V}^o(X) = \mathcal{V}^e(X, Z^o(X)) \text{ and } \sigma^o(X) = \sigma^e(X, Z^o(X)) \quad \forall X \tag{6.21}$$

with $Z^o(X)$ *being consistent with the feedback rule* $\sigma^o(X)$ *and satisfying (6.4)-(6.5).*

[24] The difficulty in directly proving existence of a *SME* comes from the fact that the differential equation (6.14) for $\mathcal{V}^o(X)$ depends on *DM*'s payoff $\varphi^o(X)$ in case the tipping point has been passed which itself depends on the *Stock-Markov* feedback rule computed over the whole future trajectory. Local existence results are of little help given that non-local property. Proposition 6 overcomes this difficulty, in proving the existence of a *SME* indirectly from the existence of an optimal path.

To understand this proposition, we first observe that the evolution of beliefs along a *SME* is completely fixed by the feedback rule on path. If *DM* expects future selves to stick to a *Stock-Markov* rule that implements the optimal action profile, he also expects beliefs to be modified as expected at the optimum. Hence, when considering the possible benefits of an observable impulse deviation, there is nothing that distinguishes the current self when he is playing the SME defined in Proposition 6 from a planner who would be considering the impact of a marginal change of action at the same point in time on the future stream of payoffs. Because deviations are observable, future selves will modify beliefs as the current self or as a planner would also do and will accordingly choose the same action in the sequel.

Bounds. Proposition 7 below provides tight bounds on the *Stock-Markov*-value function and the feedback rule for any SME, and in particular the one, described in Section 6, that implements the optimal trajectory.

Proposition 7 *$\mathcal{V}^o(X)$, $\varphi^o(X)$ and $\sigma^o(X)$ admit the following bounds:*

$$\varphi^o(X) \le \mathcal{V}_\infty \le \mathcal{V}^o(X) \le \mathcal{V}_\infty \left(1 + \frac{\Delta}{\lambda_0}(1 - F(X))\right) \quad \forall X \in \left[0, \overline{X}\right], \tag{6.22}$$

$$\zeta\sqrt{\frac{\lambda_0}{\lambda_1}} \le \sigma^o(X) \le \zeta \quad \forall X \in \left[0, \overline{X}\right]. \tag{6.23}$$

These bounds are similar to those in the no-uncertainty scenario of Section 4. The dynamics with and without uncertainty are in fact similar. To illustrate, the lower bound on $\mathcal{V}^o(X)$ is readily obtained by following a non-equilibrium strategy consisting in adopting the myopic action under all circumstances. For X below but close enough to $\overline{X}$, the stock has already gone through most possible values of the tipping point. From (6.22), the *Stock-Markov*-value function converges towards $\mathcal{V}_\infty$ from above and is continuous at this point.[25] There, the current action has almost no longer any influence on the arrival rate of a disaster which is almost surely θ_1. On the other hand, the lower bound on possible actions is independent on the distribution F and again found for scenario where the tipping point is located at $\overline{X}$ for sure.

Running Example (Continued). Consider the trajectory starting from $X = 0$ and $Z = 1$. From the expression of the optimal action (5.16), the stock evolves as

$$X^e(\tau) = \begin{cases} \zeta\left(t - e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e)\left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\int_0^\tau \frac{e^{\lambda_0 s}}{Z(s)} ds\right) & \text{for } t \in [0, \overline{T}^e), \\ \overline{X} + \zeta(t - \overline{T}^e) & \text{for } t \ge \overline{T}^e. \end{cases} \tag{6.24}$$

Together with (5.15), this expression allows us to recover an almost closed form for $X^o(Z)$ (the inverse function of $Z^o(X)$) for $Z \in [1 - q + qe^{-\Delta\overline{T}^e}, 1]$ as

(6.25)

[25]The *Stock-Markov*-value function is not necessarily differentiable at $\overline{X}$ though it admits a right- and a left-derivative. This is so because the optimal action may have an upwards jump at that point; a case that arises when the distribution of tipping point has a mass point at $\overline{X}$ as in our Running Example. Continuity of the feedback rule at $\overline{X}$ holds when F has no mass point. See the Appendix for more details.

$$X^o(Z) = \zeta \left(-\frac{1}{\Delta} ln \left(1 + \frac{Z-1}{q}\right) - e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e) \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right) \int_0^{-\frac{1}{\Delta} ln\left(1+\frac{Z-1}{q}\right)} \frac{e^{\lambda_0 s}}{Z(s)} ds \right).$$

It can be readily verified that

$$\dot{X}^o(Z(\overline{T}^e)) = \frac{\zeta e^{\Delta \overline{T}^e}}{q\Delta} \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}.$$

We thus get $\lim_{q \to 1} \dot{X}^o(Z(\overline{T}^e)) = 0$ or, equivalently, $\lim_{q \to 1} \dot{Z}^o(\overline{X}^-) = -\infty$. Intuitively, when q is close to one, the function $Z^o(X)$ remains close to one for most values of X, only decreasing very quickly towards $e^{-\Delta \overline{T}^m}$ when X comes close to $\overline{X}$.

Finally, the lowest bound on actions that is found in (6.23) is easily confirmed. The optimal action at $\overline{X}^-$, namely $\sigma^o(\overline{X}^-) = x^e(\overline{T}^e)$ (which is expressed in (5.17)) indeed converges towards the lowest bound $\zeta\sqrt{\frac{\lambda_0}{\lambda_1}}$ as q goes to zero. ∎

7. *STOCK-MARKOV* EQUILIBRIA WITH NON-OBSERVABLE DEVIATIONS

Consider now a scenario where DM's impulse deviations are non-observable by future selves. Only the stock following such deviation remains so. In a *SME* with feedback rule $\sigma^{no}(X)$, the survival ratio is always thought to evolve according to

(7.1) $\quad \sigma^{no}(X)\dot{Z}^{no}(X) = \Delta(1 - F(X) - Z^{no}(X)) \quad \forall X \geq 0$[26]

with the initial condition

(7.2) $\quad Z^{no}(0) = 1.$

For any stock $X > \overline{X}$, the fact that $\sigma^{no}(X) = \zeta$ together with (7.1) immediately imply

(7.3) $\quad Z^{no}(X) = Z^{no}(\overline{X}) e^{-\frac{\Delta}{\zeta}(X - \overline{X})} \quad \forall X > \overline{X}.$

For any stock $X \leq \overline{X}$, we may now define DM's *Stock-Markov-value function with non-observable deviations* $\mathcal{V}^{no}(X)$ as a solution to the following optimization problem:

(7.4) $$Z^{no}(X)\mathcal{V}^{no}(X) = \sup_{\mathcal{A}} \int_0^{\overline{T}} e^{-\lambda_0 \tau} Z^{no}(X(\tau)) u(x(\tau)) d\tau + e^{-\lambda_0 \overline{T}} Z^{no}(\overline{X}) \mathcal{V}_\infty.$$

Impulse Deviations. An impulse deviation is again characterized by a modification of the action profile as specified in (6.8) and an evolution of the stock as in (6.9).

Two remarks are in order. First, it is key that each self only knows the current level of stock. Suppose instead, that a given self knows for how long the project has been run or at which point on line he is acting. Conjecturing that previous selves have abided to the *Stock-Markov* feedback rule and comparing with the current stock he is observing would allow him to infer that a deviation has taken place, without being able to tell at which point before. Assuming that only the current stock is observed avoids such inference and accordingly simplifies the analysis.

Second, because deviations are non-observable, future selves believe that the regime survival ratio still evolves as on path, i.e., as in (7.1). Instead, when deviating, DM knows that the correct evolution is given by (6.10). This difference in beliefs *a priori* implies that, beyond the commitment period whose length is infinitesimal, the discounted intertemporal streams of utilities evaluated with the current DM's beliefs and that of his future selfs have different values. To fix this issue, focus on the main consequences of non-observability in the simpler scenario and again simplify the analysis, we will assume that the current DM cares about the intertemporal payoff of his subsequent selves; *de facto* considering their beliefs in the computation of his future payoffs. From this, we may thus define DM's deviation payoff $\hat{\mathcal{V}}(x,\varepsilon;X)$ as

$$Z^{no}(X)\hat{\mathcal{V}}(x,\varepsilon;X) = \int_0^{+\infty} e^{-\lambda_0\tau} Z^{no}(\hat{X}(x,\varepsilon,\tau;X))u(y(x,\varepsilon,\tau;X))d\tau. \tag{7.5}$$

Definition 2 *$(\mathcal{V}^{no}(X), \sigma^{no}(X), Z^{no}(X))$ is a SME with non-observable deviations if the following conditions hold.*

1. *$\mathcal{V}^{no}(X)$ as defined by (6.6) cannot be improved upon by any impulse deviation of the form (6.8)-(6.9) for ε made arbitrarily small:*

$$\mathcal{V}^{no}(X) = \max_{x\in\mathcal{X}} \lim_{\varepsilon\to 0^+} \hat{\mathcal{V}}(x,\varepsilon;X). \tag{7.6}$$

2. *$\sigma^{no}(X)$ is optimal for ε made arbitrarily small:*

$$\sigma^{o}(X) \in \arg\max_{x\in\mathcal{X}} \lim_{\varepsilon\to 0^+} \hat{\mathcal{V}}(x,\varepsilon;X). \tag{7.7}$$

3. *$Z^{no}(X)$ is consistent with the feedback rule $\sigma^{no}(X)$ and satisfies(7.1)-(7.2).*

Items 1. and 2. imply that, when computing the value function (7.4), we are in fact looking for a feedback rule $\sigma^{no}(X)$ which is optimal given the correct expectations on $Z^{no}(X)$; an evolution of beliefs which is itself induced by such a rule according to (7.1) and (7.2).

Properties of $(\mathcal{V}^{no}(X), \sigma^{no}(X))$. Next proposition echoes our findings in Proposition 5 but now considering non-observable deviations.

Proposition 8 *At any (continuously differentiable) SME with non-observable impulse deviations, the Stock-Markov-value function $\mathcal{V}^{no}(X)$ satisfies the following HBJ differential equation*

$$\dot{\mathcal{V}}^{no}(X) = -\zeta - \frac{\dot{Z}^{no}(X)}{Z^{no}(X)}\mathcal{V}^{no}(X) + \sqrt{2\lambda_0\mathcal{V}^{no}(X)} \quad \forall X \in [0,\overline{X}) \tag{7.8}$$

together with the boundary condition

$$\mathcal{V}^{no}(X) = \mathcal{V}_\infty \quad \forall X \geq \overline{X}. \tag{7.9}$$

The corresponding Stock-Markov feedback rule writes as

$$\sigma^{no}(X) = \zeta + \dot{\mathcal{V}}^{no}(X) + \frac{\dot{Z}^{no}(X)}{Z^{no}(X)}\mathcal{V}^{no}(X) \quad \forall X \in [0,\overline{X}). \tag{7.10}$$

The formula (7.10) for the feedback rule with non-observable deviations is much like its counterpart (6.16) found when those deviations are observable. Yet, the term (6.20) is missing. To explain this omission, consider again increasing by a small amount dx the current action $\sigma^{no}(X)$ over an interval of length ε, where ε is also small enough, starting from a current stock X with current beliefs $Z^{no}(X)$. When this deviation is non-observable, future selves only evaluate its consequences through its impact on the stock they inherit, namely $X + \varepsilon dx$. The comparison with the scenario with observable deviations is thus straightforward.

First, this impulse deviation still impacts current surplus because the feedback rule $\sigma^{no}(X)$ requires a change in action at this new level of stock. This term is again given by (6.17). Second, this impulse deviation also increases the implicit discount rate; a term which is still captured by (6.18). Yet, with a non-observable deviation, the regime survival ratio $Z^{no}(X)$ is taken as given over the whole trajectory. Had such a deviation been observable, DM instead would have known that increasing current action also means that future beliefs will carry on some *Pessimistic Stigma* and this pessimism makes it more attractive for future selves to further increase actions later on. With non-observable deviations, this motive for raising actions disappears.

At equilibrium, the feedback rule now calls for excessively low actions in comparison with the optimal trajectory. Indeed, in any SME with observable deviations, we have

$$\sigma^o(X) > \zeta + \dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X).$$

With low actions early on, the conjectured evolution of beliefs remains quite optimistic. DM thinks that the tipping point remains unlikely to have been passed yet and, in response, adopts a prudent behavior. This prudent behavior is of course excessive in comparison with the optimal trajectory. Yet, it is self-fulfilling.

Running Example (Continued). The trajectory under a SME with non-observable impulse deviations can again be computed in (almost) closed form.

Proposition 9 *Suppose that F has Dirac masses q at 0 and $1-q$ at $\overline{X}$. The trajectory under a SME with non-observable impulse deviations starting from $X = 0$ and $Z = 1$ has the following features.*

- *The date $\overline{T}^{no} > \overline{T}^k$ at which $\overline{X}$ is reached solves*

(7.11)
$$\overline{T}^{no} = \sqrt{Z(T^{no})}e^{-\lambda_0\overline{T}^{no}}\left(\int_0^{\overline{T}^{no}}\frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}d\tau\right)\sqrt{\frac{\lambda_0}{\lambda_1}}+\lambda_0\int_0^{\overline{T}^{no}}\sqrt{Z(\tau)}e^{-\lambda_0\tau}\left(\int_0^{\tau}\frac{e^{\lambda_0 s}}{\sqrt{Z(s)}}ds\right)d\tau$$

where $Z(\tau)$ is still given by (5.15).

- *The action $x^{no}(\tau)$ satisfies*

(7.12)
$$x^{no}(\tau) = \begin{cases}\zeta\frac{e^{\lambda_0 t}}{\sqrt{Z(t)}}\left(\lambda_0\int_t^{\overline{T}^{no}}\sqrt{Z(\tau)}e^{-\lambda_0\tau}d\tau + \sqrt{Z(T^{no})}e^{-\lambda_0\overline{T}^{no}}\sqrt{\frac{\lambda_0}{\lambda_1}}\right) < \zeta & \text{for } t \in [0, \overline{T}^e),\\ \zeta & \text{for } t \geq \overline{T}^{no}.\end{cases}$$

To illustrate the tendency for choosing low actions when impulse deviations are non-observable, observe that the last action before jumping to the myopic optimum is always lower than in scenario with observable deviations (that, remember, implements the optimum):

$$x^{no}(\overline{T}^{no}) = \sqrt{\frac{\lambda_0}{\lambda_1}} < x^e(\overline{T}^e) = \sqrt{\frac{\lambda_0}{\lambda_1} + \frac{q\Delta e^{-\Delta \overline{T}^e}}{1 - q + qe^{-\Delta \overline{T}^e}}}.$$

■

8. FOUNDATIONS FOR A *PRECAUTIONARY PRINCIPLE*

The existing ambiguity on the meaning and content of the *Precautionary Principle* that we stressed in the Introduction may be due to the different perspectives that commentators, scholars and decision-makers have taken to justify or criticize its use.

The first perspective takes a positive stance and views the *Precautionary Principle* as providing a guide for actions in some informationally-constrained environments. This paper contributes on this front. Indeed, it should be stressed that, in our model, beliefs are entirely determined by the profile of past actions. That impulse deviations cannot be detected in the scenario of Section 7 thus means that future selves of the deviating decision-maker have no hard evidence on the evolution of such beliefs and can just form conjectures on such evolution based on the equilibrium feedback rule that is at play. Because it states that one should not act in the absence of evidence, the *Precautionary Principle* can then be interpreted as describing such equilibrium behavior. The *Principle* thus finds a rationale in informational contexts where decision-makers may act opportunistically at any point in time.

The second alternative approach is more normative and views the *Precautionary Principle* as a constitutional constraint on the set of feasible actions. This approach, which is the one that we favor, is much in spirit of the mechanism design literature on delegation (Melumad and Shibano, 1991; Alonso and Matousheck, 2008; Martimort and Semenov, 2008 among others). This literature has shown how such a ban can be used to align conflicting objectives in environments with agency problems. In our context, the conflict is between the scenarios with observable or with non-observable deviations that generate different incentives to act. To illustrate how a commitment to a rule might bite, consider a ban on actions below the feedback rule $\sigma^o(X)$. Presumably, enforcing this constraint requires to be able to detect actions that would fall below this floor but does not presume that upward deviations be observable. Because with non-observable deviations, decision-makers have less incentives to act than in the observable scenario, it should be clear that there then exists a *SME* with non-observable (upwards) deviations such that this floor is binding over the whole trajectory. The *Precautionary Principle* is thus a way of implementing the optimal trajectory in contexts where decision-makers are opportunistic and only limited information is available to keep them on check.

Some further remarks are in order. First, the constitutional constraint is *only* needed in the scenario with non-observable deviations. The equilibrium trajectory when deviations are observable was shown to be already enough to implement the optimum. The constitutional constraint would thus be useless had *DM*'s opportunistic behavior not been an issue but, at worst, it cannot harm. Second, the *Precautionary Principle* could be here interpreted by some readers as an *Action Principle* since equilibrium actions are too low in its absence. The word *Precaution* should then be viewed as expressing a broader concern of society for avoiding opportunistic behavior and adopting optimal strategies rather than suboptimal ones. It is not the risk of the underlying project *per se* that matters so much but the fact that the curse of unconstrained actions does not optimally manage that risk when decision-makers are opportunistic.

9. CONCLUDING REMARKS

We have considered a dynamic decision-making problem in a context that entails uncertainty and irreversibility. Current actions contribute to make it more likely to pass a tipping point and thus increase the likelihood of an environmental disaster but the location of such tipping point remains unknown through the process. We have shown that the optimal trajectory can be obtained with a feedback rule that should, *a priori*, depend on the stock of past actions as well as on the decision-maker's beliefs on whether the tipping point has been passed or not. Those beliefs are constructed from the whole history of past actions. From there, we have investigated the performances of incomplete *Stock-Markov* feedback rules, that depend only on stock. We showed that the optimal trajectory can still be implemented as a *Stock-Markov* equilibrium among selves of the decision-maker provided that impulse deviations are observable. Indeed, upon observing such deviations by the decision-maker at any point in time, future selves are able to reconstruct the evolution of beliefs and act accordingly to implement the optimal trajectory.

Instead, when decision-makers at each point in time may act opportunistically and such deviations are non-observable, the equilibrium feedback rule requires excessively prudent actions. Yet, this behavior is self-fulfilling. When actions have been kept low in the past, decision-makers remain optimistic on the fact that the tipping has not been passed yet and, in response, refrain from taking large actions. Beliefs evolve slowly because past actions also evolve slowly. In this scenario, decision-makers keep as given the evolution of beliefs that is consistent with the equilibrium actions and choose an action plan that respond to that evolution.

This framework has allowed us to discuss possible foundations and perspectives for the *Precautionary Principle* in a world of irreversibility, limited information and opportunistic behavior.

Taking stock of those findings, our framework could be modified along several interesting dimensions. First, signals on the location of the tipping point could be exogenously learned by the decision-maker as the trajectory evolves; maybe thanks to scientific progresses. As the trajectory comes closer to the tipping point, decision-makers may accumulate enough evidence on the decreasing distance to the regime switch. The history of past actions then would not suffice to determine beliefs. In this framework, a *Stock-Markov* equilibrium would certainly fail to replicate the optimal trajectory even when deviations are observable. Yet, we conjecture that, with observable deviations, the equilibrium feedback rule in a SME might still require higher actions as hard signals on the fact that the tipping point has been passed get known. With non-observable deviations, the evolution of beliefs being still taken as given, we conjecture that the equilibrium feedback rule is likely to make actions even more prudent than in the scenario investigated above but the extent by which it is so remains to unveil.

Second, society could be made of overlapping generations of agents with decision-makers at different points in time thus having different objectives. Again, this assumption certainly implies that the equilibrium trajectory might not be Pareto-optimal. Earlier selves might be consuming too much and the *Precautionary Principle* could be used to improve welfare of later generations. Relatedly, future selves may not be fully rational and put an excessive weight on the most recent information they might learn. In this case, any impulse deviation that increases current action will drive future beliefs towards over-pessimism and a fast move towards the myopic optimum. Under those conditions, the scenario with observable deviations might lead to excessive actions; a distortion that could be somewhat controlled by the *Precautionary Principle*.

Other political considerations could be at play. To illustrate, consider the possibility that rotating decision-makers with different preferences are democratically elected for periods of finite length. If a first decision-maker knows he is about to step down from power and be replaced with another decision-maker who favors higher actions, he might as well enact laws that stipulate limits on future actions. Now the *Precautionary Principle* is akin to a political constraint on future decision-makers. Such political considerations would also suggest that a decision-maker who instead does not care much about the catastrophe should force more prudent followers to adopt a minimal level of actions.

These extensions await for future research.

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APPENDIX A: KNOWN TIPPING POINT

PROOFS OF PROPOSITION 1 AND PROPOSITION 2: Consider an action plan $\mathbf{x}_t = \{x(\tau)\}_{\tau\geq t}$ from date t onwards. If the stock at date t is X, the stock process $\hat{X}(\tau; X, t)$ from that date on evolves as:

$$\hat{X}(\tau; X, t) = X + \int_t^\tau x(s)ds. \tag{A.1}$$

In the text, we slightly abuse notations and, for simplicity, write $\hat{X}(\tau; X) \equiv \hat{X}(\tau; X, 0)$, in which case the stock trajectory evolves as (4.1). Let define the value function $\widetilde{\mathcal{V}}^k(X, t; \overline{X})$, conditionally on having not yet faced a disaster, with a survival probability being $e^{-\theta_0 t}$ in this scenario where the value of the tipping point is known being at $\overline{X}$, as

$$\widetilde{\mathcal{V}}^k(X,t;\overline{X}) \equiv \sup_{\overline{T},\mathbf{x}_t,X(\cdot)\text{ s.t. (A.1) and }\hat{X}(\overline{T};X,t)=\overline{X}} \int_t^{\overline{T}} e^{-\lambda_0(\tau-t)}u(x(\tau))d\tau + e^{-\lambda_0(\overline{T}-t)}\mathcal{V}_\infty.$$[27]

First, observe that we can write $\widetilde{\mathcal{V}}^k(X,t;\overline{X}) = \mathcal{V}^k(X;\overline{X})$ for all $t \geq 0$, where the current value function $\mathcal{V}^k(X;\overline{X})$ is defined in (4.2).

Take now $X < \overline{X}$ and fix ε small enough so that $X + x\varepsilon < \overline{X}$. Denote $\mathcal{D}(\varepsilon) = \{x \text{ s.t. } X + x\varepsilon < \overline{X}\}$. By the *Principle of Dynamic Programming* when applied to (4.2), we must have

$$\mathcal{V}^k(X;\overline{X}) \equiv \sup_{x \in \mathcal{D}(\varepsilon)} \int_0^\varepsilon e^{-\lambda_0 \tau} u(x) d\tau + e^{-\lambda_0 \varepsilon} \mathcal{V}^k(X + x\varepsilon;\overline{X}).$$

Taking first-order Taylor approximations when $\mathcal{V}^k(X;\overline{X})$ is continuously differentiable in X, we may rewrite this problem as

$$\mathcal{V}^k(X;\overline{X}) = \sup_{x \in \mathcal{D}(\varepsilon)} \varepsilon u(x) + (1 - \lambda_0 \varepsilon)(\mathcal{V}^k(X;\overline{X}) + x\varepsilon \dot{\mathcal{V}}^o(X;\overline{X})).$$

The corresponding HBJ equation writes as

$$\lambda_0 \mathcal{V}^k(X;\overline{X}) = \max_x x\dot{\mathcal{V}}^k(X;\overline{X}) - \frac{1}{2}(x - \zeta)^2 + \lambda_1 \mathcal{V}_\infty \tag{A.2}$$

together with the boundary condition (4.4).

The maximand of the r.-h. s. of (A.2) is obtained for the optimal feedback rule (4.5). Inserting this feedback rule into (A.2) yields

$$\lambda_0 \mathcal{V}^k(X;\overline{X}) = \zeta \dot{\mathcal{V}}^k(X;\overline{X}) + \frac{(\dot{\mathcal{V}}^k(X;\overline{X}))^2}{2} + \lambda_1 \mathcal{V}_\infty. \tag{A.3}$$

Solving this second-degree polynomia for $\dot{\mathcal{V}}^k(X;\overline{X})$ and taking the root ensuring that $\sigma^k(X;\overline{X})$ as given by (4.5) remains positive yields (4.3).

Comparative Statics. Define

$$\widehat{\mathcal{V}}(X) = \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty. \tag{A.4}$$

From (4.3), we have $\dot{\mathcal{V}}^k(X;\overline{X}) \leq 0$ if and only if $\mathcal{V}^k(X) \leq \widehat{\mathcal{V}}(X)$. Observe that $\mathcal{V}^k(\overline{X};\overline{X}) < \widehat{\mathcal{V}}(\overline{X})$ because of (4.4). Moreover, $\mathcal{V}^k(X;\overline{X})$ were to cross $\widehat{\mathcal{V}}(X)$ at $X_1 < \overline{X}$, we would have $\dot{\mathcal{V}}^k(X_1;\overline{X}) = 0$. Observe that $\widehat{\mathcal{V}}(X)$ is a constant solution to (4.3). Suppose that $\mathcal{V}^k(X;\overline{X})$ were to cross $\widehat{\mathcal{V}}(X)$ at $X_1 < \overline{X}$. By Cauchy-Lipschitz Theorem, the only solution to (4.3) which is such $\mathcal{V}^k(X_1;\overline{X}) = \widehat{\mathcal{V}}(X_1)$ is such that $\mathcal{V}^k(X,\overline{X}) = \widehat{\mathcal{V}}(X)$ for all $X \in [0,\overline{X}]$. This would contradict the boundary condition (4.4). Hence, necessarily, $\mathcal{V}^k(X;\overline{X})$ remains always below $\widehat{\mathcal{V}}(X)$ and the r.-h. s. inequality of (4.8) holds. From (4.3), it then follows that $\dot{\mathcal{V}}^k(X;\overline{X}) < 0$ for $X < \overline{X}$. From (4.4), we thus have necessarily $\mathcal{V}^k(X;\overline{X}) > \mathcal{V}_\infty)$ for $X < \overline{X}$ and the l.-h. s. inequality of (4.8) also holds.

Turning now to the optimal action. The r.-h. s. inequality of (4.9) follows from (4.5) and $\dot{\mathcal{V}}^k(X;\overline{X}) < 0$ for $X < \overline{X}$. The l.-h. s. inequality follows from the l.-h. s. inequality in (4.8), together with (4.3) and (4.5).

Differentiating (A.3) with respect to X yields

$$(\dot{\mathcal{V}}^k(X;\overline{X}) + \zeta)\ddot{\mathcal{V}}^k(X;\overline{X}) = \lambda_0 \dot{\mathcal{V}}^k(X;\overline{X}) \tag{A.5}$$

[27]This expression of $\widetilde{\mathcal{V}}^k(X,t;\overline{X})$ is valid both for $X < \overline{X}$, and for $X \geq \overline{X}$ provided that we use the convention $\overline{T} = t$ in that latter case.

or

$$\text{(A.6)} \qquad \left(1 + \frac{\zeta}{\dot{\mathcal{V}}^k(X;\overline{X})}\right)\ddot{\mathcal{V}}^k(X;\overline{X}) = \lambda_0.$$

Because $\dot{\mathcal{V}}^k(X;\overline{X}) < 0$ for $X \in [0, \overline{X})$ and $\sigma^k(X;\overline{X}) = \dot{\mathcal{V}}^k(X;\overline{X}) + \zeta > 0$, we deduce that $\ddot{\mathcal{V}}^k(X;\overline{X}) < 0$ for $X \in [0, \overline{X})$ and thus $\sigma^k(X;\overline{X})$ is decreasing.

VERIFICATION THEOREM. It is routine and thus omitted. *Q.E.D.*

APPENDIX B: UNCERTAINTY.

Preliminaries

We start by presenting the evolution of the posterior density function $f(\widetilde{X}|t, \mathbf{x}^t)$. For future reference, notice that, as times passes, a stock process $\hat{X}(t;0)$ of the form (4.1) goes through various possible values $\widetilde{X}$ of the tipping point. We may thus also describe process by the time $T(\widetilde{X};0)$ at which this stock reaches a level $\widetilde{X}$.[28]

LEMMA B.1 *The posterior density function $f(\widetilde{X}|t, \mathbf{x}^t)$ conditional on not having a disaster up to date t following history $\mathbf{x}^t$ satisfies:*

$$\text{(B.1)} \qquad f(\widetilde{X}|t, \mathbf{x}^t) = \begin{cases} \frac{e^{-\theta_0 t}}{H(t,\mathbf{x}^t)} f(\widetilde{X}) & \textit{if } \hat{X}(t;0) \le \widetilde{X} \\ \frac{e^{-\theta_0 t} e^{-\Delta(t - T(\widetilde{X};0))}}{H(t,\mathbf{x}^t)} f(\widetilde{X}) & \textit{otherwise.} \end{cases}$$

PROOF OF LEMMA B.1: We first compute the probability of survival $H(t, \mathbf{x}^t)$, i.e., the probability that there has been no disaster till date t following history $\mathbf{x}^t$, as (5.1). The first term on the r.-h. s. of (5.1) stems for the probability that the tipping point is below $\hat{X}(t;0)$, and the rate of survival then jumps up to θ_1 at a date $T(\widetilde{X};0)$ before date t. The second term is the probability that the tipping point is above $\hat{X}(t;0)$ and the rate of arrival of a disaster is still θ_0. Denote these terms respectively by P_{1t} and P_{2t}. We immediately compute

$$\text{(B.2)} \qquad P_{2t} = (1 - F(\hat{X}(t;0)))e^{-\theta_0 t}.$$

Changing variables and letting $\hat{X}(\tau;0) = \widetilde{X}$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;0)d\tau = d\widetilde{X}$, we rewrite

$$P_{1t} = \int_0^{\hat{X}(t;0)} f(\widetilde{X}) e^{-\theta_0 T(\widetilde{X};0)} e^{-\theta_1(t - T(\widetilde{X};0))} d\widetilde{X} = \int_0^t f(\hat{X}(\tau;0)) \frac{\partial \hat{X}}{\partial \tau}(\tau;0) e^{-\theta_0 \tau} e^{-\theta_1(t-\tau)} d\tau.$$

Integrating by parts yields

$$\text{(B.3)} \qquad P_{1t} = e^{-\theta_0 t}\left(\left[F(\hat{X}(\tau;0))e^{\Delta(\tau - t)}\right]_0^t - \Delta \int_0^t F(\hat{X}(\tau;0))e^{\Delta(\tau-t)} d\tau\right).$$

Inserting (B.2) and (B.3) into (5.1) finally yields the expression of the probability of survival up to date t in (5.2). From this expression, we compute the conditional density

$$f(\widetilde{X}|t, \mathbf{x}^t) = \begin{cases} \frac{e^{-\theta_0 t}}{H(t,\mathbf{x}^t)} f(\widetilde{X}) & \text{if } \hat{X}(t;0) \le \widetilde{X} \\ \frac{e^{-\theta_0 T(\widetilde{X};0)} e^{-\theta_1(t - T(\widetilde{X};0))}}{H(t,\mathbf{x}^t)} f(\widetilde{X}) & \text{otherwise.} \end{cases}$$

Simplifying yields (B.1). *Q.E.D.*

[28] If $\hat{X}(t;0)$ is smooth, increasing and differentiable in t with no flat part, $T(\widetilde{X};0)$ is itself increasing and smooth and differentiable with a finite derivative.

Proofs of Lemma 1: Following an history of past actions $\mathbf{x}^t$, the stock $\hat{X}(\tau; X, t)$ will evolve as requested by (A.1) with a stream of future actions $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$. Let $T(\widetilde{X}; X, t)$ accordingly denote the inverse function defined for $\widetilde{X} \geq X$. The value function $\hat{\mathcal{V}}(t, \mathbf{x}^t)$ can be written as

$$\text{(B.4)} \qquad \hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t, X(\cdot) \text{ s.t. (A.1)}} \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-t)} u(x(\tau)) d\tau \right) f(\widetilde{X}|t,\mathbf{x}^t) d\widetilde{X}$$
$$+ \int_X^{+\infty} \left(\int_t^{T(\widetilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right.$$
$$\left. + e^{-\theta_0(T(\widetilde{X};X,t)-t)} \int_{T(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau - T(\widetilde{X};X,t))} u(x(\tau)) d\tau \right) f(\widetilde{X}|t,\mathbf{x}^t) d\widetilde{X}.$$

Taking into account the expression of the conditional density given in (B.1), we rewrite the expression of $\hat{\mathcal{V}}(t, \mathbf{x}^t)$ in (B.4) as

(B.5)

$$e^{\theta_0 t} H(t,\mathbf{x}^t)\hat{\mathcal{V}}(t,\mathbf{x}^t) \equiv \sup_{\mathbf{x}_t, X(\cdot) \text{ s.t. (A.1)}} \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-t)} u(x(\tau)) d\tau \right) e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X}$$
$$+ \int_X^{+\infty} \left(\int_t^{T(\widetilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right.$$
$$\left. + e^{-\theta_0(T(\widetilde{X};X,t)-t)} \int_{T(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau - T(\widetilde{X};X,t))} u(x(\tau)) d\tau \right) f(\widetilde{X}) d\widetilde{X}.$$

Let

$$\mathcal{I}_1 = \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-t)} u(x(\tau)) d\tau \right) e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X}$$

which rewrites as

$$\text{(B.6)} \qquad \mathcal{I}_1 = \left(\int_t^{+\infty} e^{-\lambda_1(\tau-t)} u(x(\tau)) d\tau \right) \left(\int_0^X e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X} \right).$$

Changing variables and letting $\hat{X}(\tau;0) = \widetilde{X}$ for $\tau \leq t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;0)d\tau = d\widetilde{X}$, we also rewrite

$$\int_0^X e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X} = \int_0^t e^{-\Delta(t-\tau)} f(\hat{X}(\tau;0)) \frac{\partial \hat{X}}{\partial \tau}(\tau;0) d\tau.$$

Integrating by parts, yields

$$\int_0^X e^{-\Delta(t-T(\widetilde{X};0))} f(\widetilde{X}) d\widetilde{X} = e^{-\Delta t} \left(\left[F(\hat{X}(\tau;0)) e^{\Delta \tau} \right]_0^t - \Delta \int_0^t F(\hat{X}(\tau;0)) e^{\Delta \tau} d\tau \right)$$
$$= F(X) - \Delta e^{-\Delta t} \int_0^t F(\hat{X}(\tau;0)) e^{\Delta \tau} d\tau$$

where the last equality follows from $\hat{X}(t;0) = X$. Inserting into (B.6) yields

$$\text{(B.7)} \qquad \mathcal{I}_1 = \left(\int_t^{+\infty} e^{-\lambda_1(\tau-t)} u(x(\tau)) d\tau \right) \left(F(X) - \Delta e^{-\Delta t} \int_0^t F(X(s;0)) e^{\Delta s} ds \right).$$

We now compute

$$\mathcal{I}_2 = \int_X^{+\infty} \left(\int_t^{T(\widetilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right.$$

$$\left. + e^{-\theta_0(T(\widetilde{X};X,t)-t)} \int_{T(\widetilde{X};X,t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-T(\widetilde{X};X,t))} u(x(\tau)) d\tau \right) f(\widetilde{X}) d\widetilde{X}.$$

Changing variables and letting $\hat{X}(\tau;X,t) = \widetilde{X}$ for $\tau \geq t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;X,t)d\tau = d\widetilde{X}$ and $\hat{X}(t;X,t) = X$, we also rewrite

$$\mathcal{I}_2 = \int_t^{+\infty} \left(\int_t^{\tau} e^{-\lambda_0(s-t)} u(x(s)) ds + e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) f(\hat{X}(\tau;X,t)) \frac{\partial \hat{X}}{\partial \tau}(\tau;X,t) d\tau.$$

Integrating by parts yields

$$\text{(B.8)} \quad \mathcal{I}_2 = \left[F(\hat{X}(\tau;X,t)) \left(\int_t^{\tau} e^{-\lambda_0(s-t)} u(x(s)) ds + e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) \right]_t^{+\infty}$$

$$-\Delta \int_t^{+\infty} F(\hat{X}(\tau;X,t)) e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \Bigg) d\tau.$$

Using that $\lim_{\tau \to +\infty} F(\hat{X}(\tau;X,t)) = 1$ if $\lim_{\tau \to +\infty} \hat{X}(\tau;X,t) = +\infty$ (which holds when the minimal action is positive at any point of time as we will see below), we get

$$\text{(B.9)} \quad \mathcal{I}_2 = \int_t^{+\infty} e^{-\lambda_0(s-t)} u(x(s)) ds - F(X) \int_t^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds$$

$$-\Delta \int_t^{+\infty} F(\hat{X}(\tau;X,t)) e^{\Delta(\tau-t)} \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) d\tau.$$

Integrating by parts, we obtain

$$\int_t^{+\infty} F(\hat{X}(\tau;X,t)) e^{\Delta\tau} \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) d\tau$$

$$= \left[\left(\int_t^{\tau} F(\hat{X}(s;X,t)) e^{\Delta s} ds \right) \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) \right]_t^{+\infty}$$

$$+ \int_\tau^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_t^{\tau} F(\hat{X}(s;X,t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau$$

$$= \int_\tau^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_t^{\tau} F(\hat{X}(s;X,t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Inserting into (B.9), we thus obtain

$$\text{(B.10)} \quad \mathcal{I}_2 = \int_t^{+\infty} e^{-\lambda_0(s-t)} u(x(s)) ds - F(X) \int_t^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds$$

$$-\Delta e^{-\Delta t} \int_t^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_t^{\tau} F(\hat{X}(s;X,t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Summing up (B.7) and (B.10) and taking into account that $\hat{X}(s;X,t)$ for $s \geq t$ is the continuation of the trajectory $\hat{X}(s;0)$, i.e., $\hat{X}(s;X,t) \equiv \hat{X}(s;0,0) = \hat{X}(s;0)$ (where the last equality slightly abuses notation) for $s \geq t$, yields

$$\mathcal{I} = \int_t^{+\infty} e^{-\lambda_0(\tau-t)} u(x(\tau))d\tau - \Delta e^{-\Delta t} \int_t^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_0^\tau F(\hat{X}(s;0))e^{\Delta s} ds \right) u(x(\tau))d\tau$$

and thus

$$\mathcal{I} = \int_t^{+\infty} e^{-\lambda_0(\tau-t)} \left(1 - \Delta e^{-\Delta \tau} \int_0^\tau F(\hat{X}(s;0))e^{\Delta s} ds \right) u(x(\tau))d\tau.$$

Changing variables and setting $\tau' = \tau - t$ yields

$$\text{(B.11)} \quad \mathcal{I} = \int_0^{+\infty} e^{-\lambda_0 \tau'} \left(1 - \Delta e^{-\Delta(\tau'+t)} \int_0^{\tau'+t} F(\hat{X}(s;0))e^{\Delta s} ds \right) u(x(\tau'+t))d\tau'.$$

Generalizing (5.2) to paths that go till date $t+\tau$, we observe that the probability of survival up to date $t+\tau$ can be expressed in terms of the action plan $\mathbf{x}^{t+\tau}$ followed up to that date (that plan includes all past actions taken up to date t, namely $\mathbf{x}^t$, and the actions planned from date t on $\mathbf{x}_t^{t+\tau}$) as

$$\text{(B.12)} \quad H(t+\tau, \mathbf{x}^{t+\tau}) = e^{-\theta_0(t+\tau)} \left(1 - \Delta e^{-\Delta(t+\tau)} \int_0^{t+\tau} F(\hat{X}(s;0))e^{\Delta s} ds \right).$$

Inserting into (B.11) and changing the name of dummy variables yields

$$\text{(B.13)} \quad \mathcal{I} = e^{\theta_0 t} \int_0^{+\infty} e^{-r\tau} H(t+\tau, \mathbf{x}^{t+\tau}) u(x(\tau+t))d\tau.$$

Inserting into (B.5) yields

$$e^{\theta_0 t} H(t, \mathbf{x}^t) \hat{\mathcal{V}}(t, \mathbf{x}^t) \equiv \sup_{\mathbf{x}_t, \hat{X}(\cdot)} \int_0^{+\infty} e^{-\lambda_0 \tau} e^{\theta_0(t+\tau)} H(t+\tau, \mathbf{x}^{t+\tau}) u(x(t+\tau))d\tau$$

$$\text{s.t. } \hat{X}(t+\tau;0) = X + \int_0^\tau x(t+s)ds \text{ and } X = \int_0^\tau \overline{x}(s)ds.$$

which can be written as (5.4) with the definition of $\hat{Z}(t+\tau, \mathbf{x}^{t+\tau})$ in (5.3). *Q.E.D.*

Value Function

Next proposition provides some properties of the value function $\mathcal{V}^e(X,Z)$.

PROPOSITION B.1 *There exists a solution to the optimization problem (5.9). $Z\mathcal{V}^e(X,Z)$ is non-increasing in X, convex in Z, Lipschitz-continuous in both arguments and thus a.e. differentiable.*

At a higher stock, $\mathcal{V}^e(X,Z)$ is necessarily lower since the irreversibility constraints become more stringent as X comes closer to $\overline{X}$. Convexity of $Z\mathcal{V}^e(X,Z)$ in Z somehow means that information is valuable for DM. From a technical viewpoint, this property implies that a standard result like Benveniste and Scheinkman (1979) that ensures (under some conditions) that the value function is differentiable when it is concave is not available here. Fortunately, Lispschitz-continuity ensures that such differentiability holds almost everywhere.

PROOF OF PROPOSITION B.1: We first define $\mathcal{W}^e(X,Z)$ as

$$\mathcal{W}^e(X,Z) = Z\mathcal{V}^e(X,Z).$$

EXISTENCE. Existence of a solution to the optimization problem (B.14) follows from applying Filipov-Cesari Theorem with free final time (see Seierstad and Sydsaeter, 1987, Theorem 12, p. 145). To check that all conditions for this theorem are satisfied, first observe that $\mathcal{X}$ is closed and bounded, while X is bounded above by $\overline{X}$ on the relevant interval and Z is also bounded since $Z \in [0,1]$. Denote

$$N(X,Z,\mathcal{X},\tau) = \{e^{-\lambda_0\tau}Zu(x)+\gamma \leq 0, x, \Delta(1-F(X)-Z); \gamma \leq 0, x \in \mathcal{X}\}.$$

Let us check that $N(X,Z,\mathcal{X},\tau)$ is convex for each (X,Z,τ). Take a pair $(x_1,x_2) \in N(X,Z,\mathcal{X},\tau)\times N(X,Z,\mathcal{X},\tau)$, i.e., there exist $\gamma_i \leq 0$ such that $e^{-\lambda_0\tau}Zu(x_i)+\gamma_i \leq 0$. Consider now $\lambda x_1 + (1-\lambda)x_2$ for $\lambda \in [0,1]$ and observe that

$$e^{-\lambda_0\tau}Zu(\lambda x_1+(1-\lambda)x_2) \leq e^{-\lambda_0\tau}Z(u(\lambda x_1+(1-\lambda)x_2)-\lambda u(x_1)-(1-\lambda)u(x_2))-\lambda\gamma_1-(1-\lambda)\gamma_2.$$

Define $\gamma = \lambda\gamma_1 + (1-\lambda)\gamma_2 + e^{-\lambda_0\tau}Z(\lambda u(x_1)+(1-\lambda)u(x_2)-u(\lambda x_1+(1-\lambda)x_2))$ and observe that $\gamma \leq 0$ since u is concave and $\gamma_i \leq 0$. Moreover, we have

$$e^{-\lambda_0\tau}Zu(\lambda x_1+(1-\lambda)x_2)+\gamma \leq 0.$$

Hence, $N(X,Z,\mathcal{X},\tau)$ is convex as requested. From Filipov-Cesari Theorem, an optimal arc thus exists. Let denote by $(X^e(\tau;X,Z), Z^e(\tau;X,Z), x^e(\tau;X,Z), \overline{T}^e(\tau;X,Z))$ such an arc.

PROPERTIES. Inserting (5.7) into the r.-h. s. of (B.14), we thus rewrite

$$\text{(B.14)}\quad \mathcal{W}^e(X,Z) = \max_{\mathbf{x},X(\cdot),T \text{ s.t. } (5.5), X(0)=X,\ X(T)=X} (Z-1)\left(\int_0^T e^{-\lambda_0\tau}e^{-\Delta\tau}u(x(\tau))d\tau\right.$$

$$\left.+\lambda_1\mathcal{V}_\infty\int_T^\infty e^{-\lambda_0\tau}e^{-\Delta\tau}d\tau\right) + \int_0^T e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F(X(s))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

$$+\int_T^{+\infty} e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F(X(s))e^{\Delta s}ds\right)\lambda_1\mathcal{V}_\infty d\tau.$$

Fixing an action path $\mathbf{x}$ and taking $X' \geq X$, the corresponding stocks satisfy $X(s;X) \leq X(s;X')$. The r.-h. s. of (B.14) is thus lower at X' for any action path. Taking the max-operator proves that $\mathcal{W}^e(X,Z)$ is non-increasing in X.

From (B.14), it also follows that $\mathcal{W}^e(X,Z)$ is convex as a maximum of linear functions of Z.

Consider an alternative pair (X',Z'). Because an arc which is optimal for (X',Z'), say $(X^e(\tau;X',Z'), Z^e(\tau;X',Z'), x^e(\tau;X',Z'), \overline{T}^e(X',Z'))$, is weakly suboptimal for (X,Z), the following inequality holds:

$$\mathcal{W}^e(X,Z) \geq (Z-1)\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_0\tau}e^{-\Delta\tau}u(x^e(\tau;X',Z')d\tau+\lambda_1\mathcal{V}_\infty\int_{\overline{T}^e(X',Z')}^\infty e^{-\lambda_0\tau}e^{-\Delta\tau}d\tau\right)$$

$$+\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F\left(X+\int_0^s x^e(s';X',Z')ds'\right)e^{\Delta s}ds\right)u(x^e(\tau;X',Z'))d\tau$$

$$+\int_{\overline{T}^e(X',Z')}^{+\infty} e^{-\lambda_0\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F\left(X+\int_0^s x^e(s';X',Z')ds'\right)e^{\Delta s}ds\right)\lambda_1\mathcal{V}_\infty d\tau.$$

We express the r.-h. s. in terms of $\mathcal{W}^e(X',Z')$ to get:

(B.15)

$$\mathcal{W}^e(X,Z)-\mathcal{W}^e(X',Z')\geq(Z-Z')\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_1\tau}u(x^e(\tau;X',Z'))d\tau+\lambda_1\mathcal{V}_\infty\int_{\overline{T}^e(X',Z')}^{\infty} e^{-\lambda_1\tau}d\tau\right)+$$

$$\Delta\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda_0\tau}\left(\int_0^\tau\left(F\left(X'+\int_0^s x^e(s';X',Z')ds'\right)\right.\right.\right.$$

$$\left.\left.\left.-F\left(X+\int_0^s x^e(s';X',Z')ds'\right)\right)e^{\Delta s}ds\right)u(x^e(\tau;X',Z'))d\tau\right)$$

$$+\Delta\left(\int_{\overline{T}^e(X',Z')}^{\infty} e^{-\lambda_0\tau}\left(\int_0^\tau\left(F\left(X'+\int_0^s x^e(s';X',Z')ds'\right)\right.\right.\right.$$

$$\left.\left.\left.-F\left(X+\int_0^s x^e(s';X',Z')ds'\right)\right)e^{\Delta s}ds\right)\lambda_1\mathcal{V}_\infty d\tau\right).$$

Permuting the roles of (X,Z) and (X',Z'), we deduce a similar inequality. Putting together those conditions implies

$$|\mathcal{W}^e(X,Z)-\mathcal{W}^e(X,Z)|\leq\mathcal{V}_\infty(\|f\|_\infty|X'-X|+|Z'-Z|).$$

From which, we deduce that there exists $k=2\mathcal{V}_\infty\max\{\|f\|_\infty,1\}$ such that

$$|\mathcal{W}^e(X,Z)-\mathcal{W}^e(X,Z)|\leq k||(X',Z')-(X,Z)||)$$

where $||\cdot||$ denotes the Euclidian norm. Thus, $\mathcal{W}^e(X,Z)$ is Lipschitz continuous and thus a.e. differentiable.

Q.E.D.

For future reference, we now define DM's payoff along an optimal arc $(X^e(\tau;X,Z),Z^e(\tau;X,Z))$ for the stock and the regime survival ratio starting from arbitrary initial conditions (X,Z) in case the regime switch has already occurred as

(B.16) $$\varphi^e(X,Z)=\int_0^{\overline{T}^e(X,Z)} e^{-\lambda_1\tau}u(\sigma^e(X^e(\tau;X,Z),Z^e(\tau;X,Z)))d\tau+e^{-\lambda_1\overline{T}^e(X,Z)}\mathcal{V}_\infty$$

where $\overline{T}^e(X,Z)$ is the date at which the highest possible value of the tipping point is reached, namely $X^e(\overline{T}^e(X,Z);X,Z)=\overline{X}$. Payoffs are discounted at a rate λ_1 once the tipping point has been passed. When $X\geq\overline{X}$, DM knows for sure that it has been the case. He adopts the myopic action with payoff $\mathcal{V}_\infty$ and beliefs evolve according to (5.8). Because $\varphi^e(X,Z)$ is computed when discounting payoffs at rate λ_1, while $\mathcal{V}^e(X)$ is computed by discounting at a lower rate λ_0 over a first phase, we necessarily have $\mathcal{V}^e(X,Z)\geq\varphi^e(X,Z)$. Although DM ignores having passed the tipping point, he knows that, if that happened, continuation payoffs are necessarily lower.

Proof of Proposition 3: Characterization.

PROPOSITION B.2 *At any point of differentiability, $\mathcal{W}^e(X,Z)$ that solves (B.14) satisfies the following HBJ partial differential equation:*

(B.17)

$$\lambda_0\mathcal{W}^e(X,Z)=\lambda_1\mathcal{V}_\infty Z+\zeta\frac{\partial\mathcal{W}^e}{\partial X}(X,Z)+\frac{1}{2Z}\left(\frac{\partial\mathcal{W}^e}{\partial X}(X,Z)\right)^2+\Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z).$$

The feedback rule is given by

(B.18) $$\sigma^e(X,Z)=\zeta+\frac{1}{Z}\frac{\partial\mathcal{W}^e}{\partial X}(X,Z).$$

Moreover, we have

(B.19) $$\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z)=\varphi^e(X,Z).$$

PROOF OF PROPOSITION B.2: For the sake of completeness and for future references, we remind below the well-known derivation of the *HBJ* equation satisfied by $\mathcal{W}^e(X,Z)$. Consider $Z\in[0,1]$. Using the *Dynamic Programming Principle*, $\mathcal{W}^e(X,Z)$ satisfies

(B.20) $$\mathcal{W}^e(X,Z)=\sup_{\mathcal{A}}\int_0^\varepsilon e^{-\lambda_0 t}Z(t)u(x(t))dt+e^{-\lambda_0\varepsilon}\mathcal{W}^e(X(\varepsilon;X,Z),Z(\varepsilon;X,Z)).$$

Consider now ε small enough and denote by x a fixed action over the interval $[0,\varepsilon]$. From (5.6) and (5.5), we get

$$X(\varepsilon;X,Z)=X+\varepsilon x+o(\varepsilon),Z(\varepsilon;X,Z)=Z+\varepsilon\Delta(1-F(X)-Z)+o(\varepsilon)$$

where $\lim_{\varepsilon\to 0}o(\varepsilon)/\varepsilon=0$.

When $\mathcal{W}^e(X,Z)$ is continuously differentiable, we can take a first-order Taylor expansion in ε of the maximand in (B.20) to write it as

$$\mathcal{W}^e(X,Z)+\varepsilon\left(Zu(x)+x\frac{\partial\mathcal{W}^e}{\partial X}(X,Z)+\Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z)-\lambda_0\mathcal{W}^e(X,Z)\right)+o(\varepsilon).$$

Inserting into (B.20) yields the following *HBJ* equation:

(B.21) $$\lambda_0\mathcal{W}^e(X,Z)=\sup_{x\in\mathcal{X}}\left\{Zu(x)+x\frac{\partial\mathcal{W}^e}{\partial X}(X,Z)+\Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z)\right\}.$$

FEEDBACK RULE. The maximand on the r.-h. s. of (B.21) is strictly concave. It immediately follows that the feedback rule $\sigma^e(X,Z)$ is given by (B.18) when interior. Simplifying (B.21) by using the feedback rule (B.18) finally yields (B.17).

PARTIAL DIFFERENTIAL EQUATION. Rewriting the optimality conditions in terms of $\mathcal{V}^e(X,Z)$, (B.17) becomes

$$\lambda_0\mathcal{V}^e(X,Z)=\lambda_1\mathcal{V}_\infty+\zeta\frac{\partial\mathcal{V}^e}{\partial X}(X,Z)+\frac{1}{2}\left(\frac{\partial\mathcal{V}^e}{\partial X}(X,Z)\right)^2+\frac{\Delta(1-F(X)-Z)}{Z}\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z).$$

Solving this second-degree equation and keeping the solution that gives a positive feedback rule yields

(B.22) $$\frac{\partial\mathcal{V}^{no}}{\partial X}(X,Z)=-\zeta+\sqrt{2\lambda_0\mathcal{V}^e(X,Z)-2\frac{\Delta(1-F(X)-Z)}{Z}\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z)}.$$

Denote the optimal solution to (B.14) by $(x^e(\tau;X,Z), X^e(\tau;X,Z), Z^e(\tau;X,Z), \overline{T}^e(X,Z))$. From (B.14), we can write

(B.23)

$$\mathcal{W}^e(X,Z) = \int_0^{\overline{T}^e(X,Z)} e^{-\lambda_0\tau} Z^e(\tau;X,Z) u(x^e(\tau;X,Z)) d\tau + Z^e(\overline{T}^e(X,Z);X,Z) e^{-\lambda_0 \overline{T}^e(X,Z)} \mathcal{V}_\infty.$$

Integrating (5.6), we obtain

$$\text{(B.24)}\quad \widetilde{Z}^e(\tau;X,Z) = (Z-1)e^{-\Delta\tau} + 1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X^e(s;X,Z)) e^{\Delta s} ds \quad \forall \tau \geq 0, X, Z \geq 0$$

Applying the Envelope Theorem to (B.14) thus yields

$$\text{(B.25)}\quad \frac{\partial \mathcal{W}^e}{\partial Z}(X,Z) = \varphi^e(X,Z)$$

or

$$Z\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z) + \mathcal{V}^e(X,Z) = \varphi^e(X,Z)$$

where $\varphi^e(X,Z)$ is defined as in (B.16). Inserting into (B.22) and simplifying yields

$$\frac{\partial \mathcal{V}^e}{\partial X}(X,Z) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^e(X,Z) - 2\frac{\Delta(1-F(X)-Z)}{Z}\varphi^e(X,Z)}$$

which can be written as (5.10).

Q.E.D.

Q.E.D.

Bounds. For future references, it is useful to provide simple bounds on $\mathcal{V}^e(X,Z)$.

Corollary B.1

$$\text{(B.26)}\quad Z\mathcal{V}_\infty \leq Z\mathcal{V}^e(X,Z) \leq \left(F(X) + (1-F(X))\frac{\lambda_1}{\lambda_0}\right)\mathcal{V}_\infty \quad \forall X \geq 0, \forall Z \in (0,1].$$

Proof of Corollary B.1: Observe that (5.6) and $F(X) \leq F(X^e(\tau;X,Z))) \leq 1$ imply

$$0 \leq \frac{d}{d\tau}\left(Z^e(\tau;X,Z)e^{\Delta\tau}\right) \leq \Delta(1-F(X))e^{\Delta\tau}.$$

Integrating between 0 and τ yields

$$0 \leq Ze^{-\Delta\tau} \leq Z^e(\tau;X,Z) \leq Ze^{-\Delta\tau} + (1-F(X))\left(1-e^{-\Delta\tau}\right).$$

From this and the fact that $0 \leq Z \leq 1$, it follows that

$$\text{(B.27)}\quad 0 \leq Ze^{-\Delta\tau} \leq Z^e(\tau;X,Z) \leq F(X)e^{-\Delta\tau} + 1 - F(X) \leq 1.$$

Henceforth, the whole trajectory $Z^e(\tau;X,Z)$ always remains in the stable domain $[0,1]$.

From the third inequality in (B.27), taking maximum on the r.-h. s. of (B.14), the r.-h. s. inequality of (B.26) follows. From the first inequality in (B.27), we immediately get the l.-h. s. inequality of (B.26). *Q.E.D.*

A VERIFICATION THEOREM. Proposition B.3 below shows that the conditions given Proposition 3 to characterize the extended value function by means of an HBJ equation together with boundary conditions are in fact sufficient. We follow Ekeland and Turnbull (1983, Theorem 1, p. 6) to derive a *Verification Theorem.*

PROPOSITION B.3 *Assume first that there exists a continuously differentiable function* $\mathcal{W}_0(X,Z)$ *which satisfies:*

(B.28)

$$\lambda_0\mathcal{W}_0(X,Z) \geq Z(t;X,Z)u(x)+x\frac{\partial \mathcal{W}_0}{\partial X}(X,Z)+\Delta(1-F(X)-Z(t;X,Z))\frac{\partial \mathcal{W}_0}{\partial Z}(X,Z) \quad \forall (x,X,Z);$$

and, second, that there exists an action profile X *and a path* $\overline{X}(t) = \int_0^t \overline{X}(\tau)d\tau$, $Z_0(t) = 1-\Delta e^{-\Delta t}\int_0^t F(\overline{X}(\tau))e^{\Delta\tau}d\tau$ *such that*

$$\text{(B.29)} \quad \lambda_0\mathcal{W}_0(\overline{X}(t),Z_0(t)) = Z_0(t)u(\overline{X}(t))$$
$$+\overline{X}(t)\frac{\partial \mathcal{W}_0}{\partial X}(\overline{X}(t),Z_0(t))+\Delta(1-F(\overline{X}(t))-Z_0(t))\frac{\partial \mathcal{W}_0}{\partial Z}(\overline{X}(t),Z_0(t)) \quad \forall t\geq 0.$$

Then X *is an optimal action profile with its associated path* $(\overline{X}(t),Z_0(t))$.

PROOF OF PROPOSITION B.3: Suppose that a function $\mathcal{W}^e(X,Z)$ that satisfies conditions in Proposition B.2 is continuously differentiable. It is our candidate for the function $\mathcal{W}_0(X,Z)$ in the statement of Proposition B.3. By definition (B.21), we have

$$\lambda_0\mathcal{W}^e(X,Z) = Zu(\sigma^e(X,Z))+\sigma^e(X,Z)\frac{\partial \mathcal{W}^e}{\partial X}(X,Z)+\Delta(1-F(X)-Z)\frac{\partial \mathcal{W}^e}{\partial Z}(X,Z), \quad \forall (X,Z)$$

and thus

$$\text{(B.30)} \quad \lambda_0\mathcal{W}^e(X,Z)\geq Zu(x)+x\frac{\partial \mathcal{W}^e}{\partial X}(X,Z)+\Delta(1-F(X)-Z)\frac{\partial \mathcal{W}^e}{\partial Z}(X,Z), \quad \forall (x,X,Z)$$

where the inequality comes from the fact that $\sigma^e(X,Z)$ maximizes the r.-h. s..

To get (B.29), we use again (B.21) but now applied to the path $(x^e(t),X^e(t),Z^e(t))$ where $X^e(t)$ is such that $\dot{X}^e(t) = x^e(t) = \sigma^e(X^e(t),Z^e(t))$ with $X^e(0)=0$ and $Z^e(t) = 1-\Delta e^{-\Delta t}\int_0^t F(x^e(\tau))e^{\Delta\tau}d\tau$.

Define now a value function $\widetilde{\mathcal{W}^e}(X,Z,t) = e^{-\lambda_0 t}\mathcal{W}^e(X,Z)$. By (B.30), we get

(B.31)

$$0\geq \frac{\partial\widetilde{\mathcal{W}^e}}{\partial t}(X,Z,t)+x\frac{\partial\widetilde{\mathcal{W}^e}}{\partial X}(X,Z,t)+\Delta(1-F(X)-Z)\frac{\partial\widetilde{\mathcal{W}^e}}{\partial Z}(X,Z,t)+e^{-\lambda_0 t}Zu(x) \quad \forall (x,X,Z).$$

Using $X^e(t)=\sigma^e(X^e(t),Z^e(t))$, $Z^e(t) = 1-\Delta e^{-\Delta t}\int_0^t F(x^e(\tau))e^{\Delta\tau}d\tau$ and (B.29), we get

$$\text{(B.32)} \quad 0 = \frac{\partial\widetilde{\mathcal{W}^e}}{\partial t}(X^e(t),Z^e(t),t)+x^e(t)\frac{\partial\widetilde{\mathcal{W}^e}}{\partial X}(X^e(t),Z^e(t),t)$$
$$+\Delta(1-F(X^e(t))-Z^e(t))\frac{\partial\widetilde{\mathcal{W}^e}}{\partial Z}(X^e(t),Z^e(t),t)+e^{-\lambda_0 t}Z^e(t)u(X^e(t)) \quad \forall t\geq 0.$$

Take now an arbitrary action plan $\mathbf{x}$ with the associated path $X(t) = \int_0^t x(\tau)d\tau$ and $Z(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X(\tau))e^{\Delta\tau}d\tau$. Eventually, this path crosses the upper bound $\overline{X}$ at some $\overline{T}^e$. Let us fix an arbitrary $t > 0$. Integrating (B.31) along the path $(x(\tau), X(\tau), Z(\tau))$, we compute

$$0 \geq \int_0^t \left(\frac{\partial \widetilde{\mathcal{W}^e}}{\partial \tau}(X(\tau), Z(\tau), \tau) + x(t)\frac{\partial \widetilde{\mathcal{W}^e}}{\partial X}(X(\tau), Z(\tau), \tau) \right.$$

$$\left. +\Delta(1 - F(X(\tau)) - Z(\tau))\frac{\partial \widetilde{\mathcal{W}^e}}{\partial Z}(X(\tau), Z(\tau), \tau) + e^{-\lambda_0\tau}Z(\tau)u(x(\tau))\right)d\tau$$

or

$$0 \geq \int_0^t \left(\frac{d\widetilde{\mathcal{W}^e}}{d\tau}(X(\tau), Z(\tau), \tau) + e^{-\lambda_0\tau}Z(\tau)u(x(\tau))\right)d\tau \quad \forall t \geq 0.$$

Integrating the first term on the r.-h. s., we thus get

$$\widetilde{\mathcal{W}^e}(0,0,0) \geq \widetilde{\mathcal{W}^e}(X(t), Z(t), t) + \int_0^t e^{-\lambda_0\tau}Z(\tau)u(x(\tau))d\tau \quad \forall \tau \geq 0.$$

Because $\widetilde{\mathcal{W}^e}(X, Z, t) = e^{-\lambda_0 t}\mathcal{W}^e(X, Z) \geq 0$ for all (X, Z, t), we obtain:

$$\mathcal{W}^e(0,0) \geq e^{-\lambda_0 t}\mathcal{W}^e(X(t), Z(t)) + \int_0^t e^{-\lambda_0\tau}Z(\tau)u(x(\tau))d\tau \quad \forall \tau \geq 0.$$

Because of the boundary conditions (B.26), $e^{-\lambda_0 t}\mathcal{W}^e(X(t), Z(t))$ converges towards zero as $t \to +\infty$ for any feasible path. Moreover, for any such feasible path $\int_0^{+\infty} e^{-\lambda_0\tau}Z(\tau)u(x(\tau))d\tau$ exists. Henceforth, we get:

$$\mathcal{W}^e(0,0) \geq \sup_{\mathbf{x}} \int_0^{+\infty} e^{-\lambda_0\tau}Z(\tau)u(x(\tau))dt$$

which shows that $(x^e(\tau), X^e(\tau), Z^e(\tau))$ is indeed an optimal path. *Q.E.D.*

Optimal Path

The intertemporal date 0-payoff $\mathcal{V}^e(0,1)$ is achieved by adopting the action profile $\sigma^e(X^e(\tau; 0, 1)$ for all $\tau \geq 0$ starting from the initial conditions $X = 0$ and $Z = 1$. Next Proposition provides necessary conditions for an optimal arc.

PROPOSITION B.4 *An optimal action path $x^e(t)$ satisfies the following necessary condition:*[29]

$$\text{(B.33)} \quad x^e(\tau) = \zeta - \frac{\Delta e^{\lambda_0\tau}}{Z^e(\tau)}\int_\tau^{\overline{T}^e} f(X^e(s))e^{\Delta s}\left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'}u(x^e(s'))ds'\right)ds$$

where, along the optimal trajectory, the probability of no-regime switch writes as

$$Z^e(t) = 1 - \Delta e^{-\Delta t}\int_0^t F(X^e(\tau))e^{\Delta\tau}d\tau.$$

The upper bound on possible values of the tipping point $\overline{X}$ is reached at a date $\overline{T}^e < \overline{T}^m$ such that

$$\text{(B.34)} \quad \overline{X} = \zeta\overline{T}^e - \int_0^{\overline{T}^e} \frac{\Delta e^{\lambda_0\tau}}{Z^e(\tau)}\left(\int_\tau^{\overline{T}^e} f(X^e(s))e^{\Delta s}\left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'}u(x^e(s'))ds'\right)ds\right)d\tau.$$

[29] We slightly abuse notations and omit the dependence on the initial conditions $(0, 1)$.

PROOF OF PROPOSITION B.4 : From (5.4), DM's intertemporal payoff writes as

$$\mathcal{V}^e(0,1) \equiv \sup_{\mathcal{A}} \int_0^T e^{-\lambda_0\tau} Z(\tau)u(x(\tau))d\tau + \int_T^{+\infty} e^{-\lambda_0\tau} Z(\tau)\lambda_1\mathcal{V}_\infty d\tau. \tag{B.35}$$

EXISTENCE. It immediately follows that there exists a solution to problem (B.35) from the argument for existence in the Proof of Proposition 3.

MAXIMUM PRINCIPLE. Observe that, for $\tau \geq T$, (5.6) implies

$$Z(\tau) = Z(T)e^{-\Delta(\tau-T)} \tag{B.36}$$

and thus the scrap value on the r.-h. s. of the maximand in (B.35) writes as

$$\int_T^{+\infty} e^{-\lambda_0\tau} Z(\tau)\lambda_1\mathcal{V}_\infty d\tau = Z(T)e^{-\lambda_0 T}\mathcal{V}_\infty. \tag{B.37}$$

We now define the Hamiltonian for this optimization problem as

$$\mathcal{H}^e(X, Z, x, \tau, \mu, \nu) = e^{-\lambda_0\tau} Zu(x) + \mu x + \nu\Delta(1 - F(X) - Z) \tag{B.38}$$

where μ and ν are respectively the costate variables for (4.1) and (5.6). The *Maximum Principle* with free final time and scrap value now gives us the following necessary conditions for optimality of an arc $(X^e(\tau), Z^e(\tau), x^e(\tau), \overline{T}^e)$. (See Seierstad and Sydsaeter, 1987, Theorem 11, p. 143).)

Costate variables. $\mu(\tau)$ and $\nu(\tau)$ are both continuously differentiable on $\mathbb{R}_+$ with

$$-\dot{\mu}(\tau) = \frac{\partial \mathcal{H}^e}{\partial X}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau))$$

or

$$\dot{\mu}(\tau) = \Delta f(X^e(\tau))\nu(\tau) \quad \forall \tau \in \left[0, \overline{T}^e\right]; \tag{B.39}$$

and

$$-\dot{\nu}(\tau) = \frac{\partial \mathcal{H}^e}{\partial Z}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau))$$

or

$$\dot{\nu}(\tau) = -e^{-\lambda_0\tau}u(x^e(\tau)) + \Delta\nu(\tau) \quad \forall \tau \in \left[0, \overline{T}^e\right]. \tag{B.40}$$

Transversality conditions. The boundary conditions $X^e(0) = 0$, $X^e(\overline{T}^e) = \overline{X}$ and $Z^e(0) = 1$ imply that there are no transversality conditions on $\mu(\tau)$ at both $\tau = 0$ and $\tau = \overline{T}^e$ and on $\nu(\tau)$ at $\tau = 0$ only while

$$\nu(\overline{T}^e) = 0. \tag{B.41}$$

Free-end point conditions. The optimality condition with respect to $\overline{T}$ writes as

$$\mathcal{H}^e(X^e(\overline{T}^e), Z^e(\overline{T}^e), x^e(\overline{T}^e), \overline{T}^e, \mu(\overline{T}^e), \nu(\overline{T}^e)) + \frac{d}{dT}\left(Z(T)e^{-\lambda_0 T}\right)_{T=\overline{T}^e} \mathcal{V}_\infty = 0. \tag{B.42}$$

Using (B.38), (B.41), (5.6) taken for $\overline{T}^e$ (with the fact that F has no mass point at $\overline{X}$), namely

$$\dot{Z}(\overline{T}^e) = -\Delta Z(\overline{T}^e), \tag{B.43}$$

Condition (B.42) rewrites as

$$e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e)\left(u(x^e(\overline{T}^{e-})) - \lambda_1 \mathcal{V}_\infty\right) + \mu(\overline{T}^e) x^e(\overline{T}^{e-}) = 0 \tag{B.44}$$

or

$$-\frac{1}{2} e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e)(x^e(\overline{T}^{e-}) - \zeta)^2 + \mu(\overline{T}^e) x^e(\overline{T}^{e-}) = 0 \tag{B.45}$$

where $x^e(\overline{T}^{e-})$ denotes the l.-h. side limit of $x^e(\tau)$ as $\tau \to \overline{T}^{e-}$.

Control variable $x^e(\tau)$.

$$x^e(\tau) \in \arg\max_{x \geq 0} \mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \mu(\tau), \nu(\tau)).$$

Because $\mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \tau, \mu(\tau), \nu(\tau))$ is strictly concave in x, an interior solution satisfies

$$\frac{\partial \mathcal{H}^e}{\partial x}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau)) = 0$$

or

$$x^e(\tau) = \zeta + e^{\lambda_0 \tau} \frac{\mu(\tau)}{Z^e(\tau)}. \tag{B.46}$$

Characterization. Inserting (B.46) taken for $\overline{T}^e$ into (B.45) yields

$$\frac{e^{\lambda_0 \overline{T}^e} \mu^2(\overline{T}^e)}{2Z^e(\overline{T}^e)} + \mu(\overline{T}^e)\zeta = 0.$$

The only solution consistent with a non-negative action at date $\overline{T}^e$ is thus

$$\mu(\overline{T}^e) = 0. \tag{B.47}$$

From there, it follows that the optimal action is continuous at $\overline{T}^e$, namely

$$x^e(\overline{T}^{e-}) = x^e(\overline{T}^{e+}) = \zeta. \tag{B.48}$$

The solution for (B.40) that satisfies the transversality condition (B.41) is

$$\nu(\tau) = e^{\Delta \tau} \int_\tau^{\overline{T}^e} e^{-\lambda_1 s} u(x^e(s)) ds. \tag{B.49}$$

Inserting into (B.39) and integrating yields

$$\mu(\tau) = \mu(\overline{T}^e) - \int_\tau^{\overline{T}^e} \Delta f(X^e(s)) e^{\Delta s} \left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'} u(x^e(s')) ds' \right) ds$$

or, using (B.47),

$$\mu(\tau) = -\int_\tau^{\overline{T}^e} \Delta f(X^e(s)) e^{\Delta s} \left(\int_s^{\overline{T}^e} e^{-\lambda_1 s'} u(x^e(s')) ds' \right) ds. \tag{B.50}$$

Inserting into (B.46), we obtain (B.33). Finally, the value of $\overline{T}^e$ is obtained when $\int_0^{\overline{T}^e} x^e(\tau) d\tau = \overline{X}$ or (B.34). That $\overline{T}^e < \overline{T}^m$ is immediate.

Q.E.D.

APPENDIX C: *SME* WITH OBSERVABLE IMPULSE DEVIATIONS

For further reference, we now state the following Lemmatas.

LEMMA C.1

$$\frac{\partial X^o}{\partial X}(\tau;X) = \frac{\sigma^o(X^o(\tau;X))}{\sigma^o(X)} = \frac{\frac{\partial X^o}{\partial \tau}(\tau;X)}{\sigma^o(X)}. \tag{C.1}$$

PROOF OF LEMMA C.1: Starting with the definition of $X^o(\tau;X)$ we get:

$$\frac{\partial X^o}{\partial \tau}(\tau;X) = \sigma^o(X^o(\tau;X)).$$

Differentiating with respect to X and using Schwartz' Lemma (for $X^o(\tau;X)$ twice continuously differentiable) yields

$$\frac{\partial}{\partial \tau}\log\left(\frac{\partial X^o}{\partial X}(\tau;X)\right) = \dot{\sigma}^o(X^o(\tau;X)).$$

Integrating and taking into account that $X^o(0;X) = X$ yields

$$\frac{\partial X^o}{\partial X}(\tau;X) = exp\left(\int_0^\tau \dot{\sigma}^o(X^o(s;X))ds\right). \tag{C.2}$$

Using the stationarity of the feedback rule and differentiating with respect to t yields

$$\dot{\sigma}^o(X^o(\tau;X)) = \frac{\frac{\partial^2 X^o}{\partial \tau^2}(\tau;X)}{\frac{\partial X^o}{\partial \tau}(\tau;X)}. \tag{C.3}$$

Inserting into (C.2) and integrating yields

$$\frac{\partial X^o}{\partial X}(\tau;X) = exp\left(ln\left(\frac{\frac{\partial X^o}{\partial \tau}(\tau;X)}{\frac{\partial X^o}{\partial \tau}(0;X)}\right)\right)$$

and thus

$$\frac{\partial X^o}{\partial X}(\tau;X) = \frac{\sigma^o(X^o(\tau;X))}{\sigma^o(X^o(0;X))}.$$

Noticing that $X^o(0;X) = X$ yields (C.1). *Q.E.D.*

LEMMA C.2

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = \sigma^o(X^o(\tau;X))\left(\frac{x}{\sigma^o(X)} - 1\right). \tag{C.4}$$

Proof of Lemma C.2: Take $\tau > \varepsilon$, we have

$$\hat{X}(x,\varepsilon,\tau;X) = X + x\varepsilon + \int_{\varepsilon}^{\tau} \sigma^o(\hat{X}(x,\varepsilon,s;X))ds$$

Now observe that, for $s \geq \varepsilon$, we have

$$\hat{X}(x,\varepsilon,s;X) = X^o(s-\varepsilon, X + x\varepsilon).$$

Hence, we rewrite

$$\hat{X}(x,\varepsilon,\tau;X) = X + x\varepsilon + \int_{\varepsilon}^{\tau} \sigma^o(X^o(s-\varepsilon, X + x\varepsilon))ds. \tag{C.5}$$

Differentiating with respect to ε yields

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = x - \sigma^o(X) + \int_0^{\tau} \dot{\sigma}^o(X^o(s;X))\left(-\frac{\partial X^o}{\partial s}(s;X) + x\frac{\partial X^o}{\partial X}(s;X)\right)ds. \tag{C.6}$$

Inserting (C.1) into (C.6) yields

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = x - \sigma^o(X) + \left(\frac{x}{\sigma^o(X)} - 1\right)\int_0^{\tau} \dot{\sigma}^o(X^o(s;X))\frac{\partial X^o}{\partial s}(s;X)ds.$$

Integrating the last term yields

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x,\varepsilon,\tau;X)|_{\varepsilon=0} = x - \sigma^o(X) + \left(\frac{x}{\sigma^o(X)} - 1\right)\left(\sigma^o(X^o(\tau,X)) - \sigma^o(X)\right). \tag{C.7}$$

Simplifying further yields (C.4). *Q.E.D.*

We first prove the following Lemma. on the properties of $Z(\tau;X)$ and $Z^o(X)$.

Lemma C.3 *$Z(\tau;X)$ and $Z^o(X)$ satisfy the following conditions*

$$\sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X) = \frac{\partial Z}{\partial \tau}(\tau;X) \quad \forall \tau \geq 0, X \geq 0, \tag{C.8}$$

$$\sigma^o(X)\dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \quad \forall X \geq 0 \text{ with } Z^o(0) = 1. \tag{C.9}$$

$Z^o(X) \geq 1 - F(X)$ for all X with equality at $X = 0$ only, and thus $\dot{Z}^o(X) \leq 0$ when $\sigma^o(X) > 0$.

Proof of Lemma C.3: Differentiating (6.3) with respect to τ yields

$$\frac{\partial Z}{\partial \tau}(\tau;X) = \dot{Z}^o(X^o(\tau;X))\sigma^o(X^o(\tau;X)). \tag{C.10}$$

Differentiating (6.3) with respect to X and using (C.1) now yields

$$\frac{\partial Z}{\partial X}(\tau;X) = \dot{Z}^o(X^o(\tau;X))\frac{\sigma^o(X^o(\tau;X))}{\sigma^o(X)}. \tag{C.11}$$

Gathering (C.10) and (C.11) yields (C.8). Using (C.8) and (6.3) and

$$Z(\tau;X) = (Z^o(X) - 1)e^{-\Delta\tau} + 1 - \Delta e^{-\Delta\tau}\int_0^{\tau} F(X^o(s;X))e^{\Delta s}ds \quad \forall \tau \geq 0, X \geq 0, \tag{C.12}$$

finally yields (C.9).

Consider $Z_0(X) = 1 - F(X)$. Observe that $\dot{Z}_0(X) < 0$ when $f(X) > 0$. Observe also that $\dot{Z}^o(0) = 0 > \dot{Z}_0(0)$ when $\sigma^o(0) > 0$. Hence, $Z^o(X) > Z_0(X)$ in a starred-right neighborhood of 0. Suppose that $Z^o(X)$ crosses again $Z_0(X)$ for the first time at some $X_1 > 0$, the same reasoning as above shows that $\dot{Z}^o(X_1) = 0 > \dot{Z}_0(X_1)$ when $\sigma^o(X) > 0$ and thus $Z^o(X) < Z_0(X)$ in a starred-left neighborhood of X_1; a contradiction. Hence, $Z^o(X) \geq Z_0(X)$ for all X with equality at $X = 0$ only. From (C.8), $\dot{Z}^o(X) \leq 0$. *Q.E.D.*

Next Lemma provides a characterization of any continuously differentiable *SME* with *Stock-Markov*-value function and feedback rule $(\mathcal{V}^o(X), \sigma^o(X)))$.

LEMMA C.4 *If $\mathcal{V}^o(X)$ is continuously differentiable, the following necessary conditions hold:*

$$\text{(C.13)} \quad 0 = \max_{x \in \mathcal{X}} \frac{\partial \hat{\mathcal{V}}}{\partial \varepsilon}(x, 0, X),$$

$$\text{(C.14)} \quad \sigma^o(X) \in \arg\max_{x \in \mathcal{X}} \frac{\partial \hat{\mathcal{V}}}{\partial \varepsilon}(x, 0, X).$$

PROOF OF LEMMA C.4: If $\mathcal{V}^o(X)$ is continuously differentiable, $\hat{\mathcal{V}}(x, \varepsilon; X)$ is itself continuously differentiable in ε, and a first-order Taylor expansion in ε yields

$$\text{(C.15)} \quad \hat{\mathcal{V}}(x, \varepsilon; X) = \mathcal{V}^o(X) + \varepsilon \frac{\partial \hat{\mathcal{V}}}{\partial \varepsilon}(x, 0, X) + o(\varepsilon).$$

Hence, (6.12) amounts to (C.13). Conjectures being correct at equilibrium, (C.14) also holds. *Q.E.D.*

PROOF OF PROPOSITION 5: We define

$$\text{(C.16)} \quad \mathcal{W}^o(X) = Z^o(X)\mathcal{V}^o(X)$$

where

$$\text{(C.17)} \quad \mathcal{W}^o(X) = \int_0^{+\infty} e^{-\lambda_0 \tau} Z(\tau; X) u(\sigma^o(X^o(\tau; X))) d\tau.$$

Next lemma turns to the properties of $\mathcal{V}^o(X)$ and $\varphi^o(X)$.

LEMMA C.5 *$\mathcal{V}^o(X)$ and $\varphi^o(X)$ satisfy the following system of first-order differential equations:*

$$\text{(C.18)} \quad \sigma^o(X)\left(\dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X)\right) = \lambda_0 \mathcal{V}^o(X) - u(\sigma^o(X)),$$

$$\text{(C.19)} \quad \sigma^o(X)\dot{\varphi}^o(X) = \lambda_1 \varphi^o(X) - u(\sigma^o(X)).$$

Proof of Lemma C.5: Differentiating (C.17) with respect to X yields

$$\dot{\mathcal{W}}^o(X) = \int_0^{+\infty} e^{-\lambda_0\tau} Z(\tau;X)u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial X}(\tau;X)d\tau$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\frac{\partial Z}{\partial X}(\tau;X)u(\sigma^o(X^o(\tau;X)))d\tau.$$

Using (C.1), we rewrite this condition as

$$\sigma^o(X)\dot{\mathcal{W}}^o(X) = \int_0^{+\infty} e^{-\lambda_0\tau} Z(\tau;X)u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial \tau}(\tau;X)d\tau \tag{C.20}$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X)u(\sigma^o(X^o(\tau;X)))d\tau.$$

Integrating by parts the first integral above, we find

(C.21)

$$\sigma^o(X)\dot{\mathcal{W}}^o(X) = \left[e^{-\lambda_0\tau}Z(\tau;X)u(\sigma^o(X^o(\tau;X)))\right]_0^{+\infty} + \lambda_0\int_0^{+\infty} e^{-\lambda_0\tau}Z(\tau;X)u(\sigma^o(X^o(\tau;X)))d\tau$$

$$+\int_0^{+\infty} e^{-\lambda_0\tau}\left(\sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X) - \frac{\partial Z}{\partial \tau}(\tau;X)\right)u(\sigma^o(X^o(\tau;X)))d\tau.$$

Using (C.8) and simplifying yields

$$\sigma^o(X)\dot{\mathcal{W}}^o(X) = \lambda_0\mathcal{W}^o(X) - Z^o(X)u(\sigma^o(X)) \quad \forall X. \tag{C.22}$$

Using the definition of $\mathcal{W}^o(X)$ in (C.16) and simplifying yields (C.18).

Using (6.7) and differentiating with respect to X yields

$$\dot{\varphi}^o(X) = \int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\frac{\partial X^o}{\partial X}(\tau;X)d\tau.$$

Using (C.1), we rewrite this condition as

$$\sigma^o(X)\dot{\varphi}^o(X) = \int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\frac{\partial X^o}{\partial \tau}(\tau;X)d\tau. \tag{C.23}$$

Integrating by parts we obtain

$$\int_0^{+\infty} e^{-\lambda_1\tau}u'(\sigma^o(X^o(\tau;X)))\frac{\partial X^o}{\partial \tau}(\tau;X)d\tau = \left[e^{-\lambda_1\tau}u(\sigma^o(X^o(\tau;X)))\right]_0^{+\infty}$$

$$+\lambda_1\int_0^{+\infty} e^{-\lambda_1\tau}u(\sigma^o(X^o(\tau;X)))d\tau = -u(\sigma^o(X)) + \lambda_1\varphi^o(X).$$

Inserting into (C.23) ends the proof. *Q.E.D.*

By adopting the deviation (6.8)-(6.9), the probability of no-regime switch would also change as (6.10). We can thus write the benefit of a deviation as

$$\mathcal{W}(\varepsilon,x;X) = \mathcal{W}_1(\varepsilon,x;X) + \mathcal{W}_2(\varepsilon,x;X) \tag{C.24}$$

where

$$\mathcal{W}_1(\varepsilon, x; X) = (Z^o(X) - 1)\left(\int_0^{\varepsilon} e^{-\lambda_1 \tau} u(x) d\tau + \int_{\varepsilon}^{+\infty} e^{-\lambda_1 \tau} u(\sigma^o(\hat{X}(x, \varepsilon, \tau; X))) d\tau\right) \tag{C.25}$$

and

$$\mathcal{W}_2(\varepsilon, x; X) = \int_0^{\varepsilon} e^{-\lambda_0 \tau}\left(1 - \Delta e^{-\Delta \tau} \int_0^{\tau} F(X + xs) e^{\Delta s} ds\right) u(x) d\tau \tag{C.26}$$

$$+ \int_{\varepsilon}^{+\infty} e^{-\lambda_0 \tau}\left(1 - \Delta e^{-\Delta \tau} \int_0^{\tau} F(\hat{X}(x, \varepsilon, \tau; X)) e^{\Delta s} ds\right) u(\sigma^o(\hat{X}(x, \varepsilon, \tau; X))) d\tau.$$

From (C.25), we deduce

$$\frac{\partial \mathcal{W}_1}{\partial \varepsilon}(0, x, X) = (Z^o(X) - 1)\Bigg(u(x) - u(\sigma^o(X)) \tag{C.27}$$

$$+ \int_0^{+\infty} e^{-\lambda_1 \tau} u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, s; X)|_{\varepsilon=0} d\tau \Bigg).$$

Using (C.4), this expression can be simplified as

$$\frac{\partial \mathcal{W}_1}{\partial \varepsilon}(0, x, X) = (Z^o(X) - 1)\Bigg(u(x) - u(\sigma^o(X)) \tag{C.28}$$

$$+ \left(\frac{x}{\sigma^o(X)} - 1\right) \int_0^{+\infty} e^{-\lambda_1 \tau} u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau \Bigg).$$

Integrating by parts, we also have

$$\int_0^{+\infty} e^{-\lambda_1 \tau} u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau \tag{C.29}$$

$$= \left[e^{-\lambda_1 \tau} u(\sigma^o(X^o(\tau; X))) \right]_0^{+\infty} + \lambda_1 \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^o(X^o(\tau; X))) d\tau.$$

$$= -u(\sigma^o(X)) + \lambda_1 \varphi^o(X) = \sigma^o(X) \dot{\varphi}^o(X)$$

where the last equality follows from (C.19). Inserting into (C.28) yields

$$\frac{\partial \mathcal{W}_1}{\partial \varepsilon}(0, x, X) = (Z^o(X) - 1)\left(u(x) - u(\sigma^o(X)) + (x - \sigma^o(X)) \dot{\varphi}^o(X) \right). \tag{C.30}$$

From (C.26) and (6.10), we deduce

$$\frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) \tag{C.31}$$

$$+ \int_0^{+\infty} e^{-\lambda_0 \tau} \left(Z(\tau; X) - (Z^o(X) - 1) e^{-\Delta \tau} \right) u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} d\tau$$

$$+ \int_0^{+\infty} e^{-\lambda_0 \tau} \left(- \Delta e^{-\Delta \tau} \int_0^{\tau} f(X^o(s; X)) \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, s; X)|_{\varepsilon=0} e^{\Delta s} ds \right) u(\sigma^o(X^o(\tau; X))) d\tau.$$

Using (C.4), this expression can be simplified as

$$\text{(C.32)}\quad \frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0,x,X) = u(x) - u(\sigma^o(X))$$

$$+\left(\frac{x}{\sigma^o(X)}-1\right)\left(\int_0^{+\infty} e^{-\lambda_0\tau}\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial \tau}(\tau;X)d\tau\right.$$

$$\left.+\int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^o(s;X))\frac{\partial X^o}{\partial \tau}(s;X)e^{\Delta s}ds\right)u(\sigma^o(X^o(\tau;X)))d\tau\right).$$

Differentiating (C.12) with respect to X and using (C.1) yields

$$\text{(C.33)}\quad \sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X) = \sigma^o(X)\dot{Z}^o(X)e^{-\Delta\tau} - \Delta e^{-\Delta\tau}\int_0^{\tau} f(X^o(s;X))\frac{\partial X^o}{\partial s}(s;X)e^{\Delta s}ds.$$

Using (C.33), we now rewrite

$$\text{(C.34)}\quad \int_0^{+\infty} e^{-\lambda_0\tau}\left(-\Delta e^{-\Delta\tau}\int_0^{\tau} f(X^o(s;X))\frac{\partial X^o}{\partial \tau}(s;X)e^{\Delta s}ds\right)u(\sigma^o(X^o(\tau;X)))d\tau$$

$$=\int_0^{+\infty} e^{-\lambda_0\tau}\left(\sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X)-\sigma^o(X)\dot{Z}^o(X)e^{-\Delta\tau}\right)u(\sigma^o(X^o(\tau;X)))d\tau.$$

Integrating by parts, we also have

$$\text{(C.35)}\quad \int_0^{+\infty} e^{-\lambda_0\tau}\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)u'(\sigma^o(X^o(\tau;X)))\dot{\sigma}^o(X^o(\tau;X))\frac{\partial X^o}{\partial \tau}(\tau;X)d\tau$$

$$=\left[e^{-\lambda_0\tau}\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)u(\sigma^o(X^o(\tau;X)))\right]_0^{+\infty}+$$

$$\int_0^{+\infty}\left(\lambda_0\left(Z(\tau;X)-(Z^o(X)-1)e^{-\Delta\tau}\right)-\frac{\partial Z}{\partial \tau}(\tau;X)-\Delta(Z^o(X)-1)e^{-\Delta\tau}\right)e^{-\lambda_0\tau}u(\sigma^o(X^o(\tau;X)))d\tau.$$

$$=\lambda_0\mathcal{W}^o(X)-u(\sigma^o(X))-\lambda_1(Z^o(X)-1)\varphi^o(X)-\int_0^{+\infty}e^{-\lambda_0\tau}\frac{\partial Z}{\partial \tau}(\tau;X)u(\sigma^o(X^o(\tau;X)))d\tau.$$

Using (C.34) and (C.35) and inserting into (C.32) yields

$$\frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0,x,X) = u(x) - u(\sigma^o(X))$$

$$+\left(\frac{x}{\sigma^o(X)}-1\right)\left(\lambda_0\mathcal{W}^o(X)-u(\sigma^o(X))-\lambda_1(Z^o(X)-1)\varphi^o(X)\right.$$

$$\left.+\int_0^{+\infty}e^{-\lambda_0\tau}\left(\sigma^o(X)\frac{\partial Z}{\partial X}(\tau;X)-\frac{\partial Z}{\partial \tau}(\tau;X)-\sigma^o(X)\dot{Z}^o(X)e^{-\Delta\tau}\right)u(\sigma^o(X^o(\tau;X)))d\tau\right).$$

Using (C.8) and simplifying yields

$$\text{(C.36)}\quad \frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0,x,X) = u(x) - u(\sigma^o(X))$$

$$+\left(\frac{x}{\sigma^o(X)}-1\right)\left(\lambda_0\mathcal{W}^o(X)-Z^o(X)u(\sigma^o(X))+(Z^o(X)-1)u(\sigma^o(X))-\sigma^o(X)\dot{Z}^o(X)\varphi^o(X)\right.$$

$$-\lambda_1(Z^o(X)-1)\varphi^o(X)\Big).$$

Using (C.22) and (C.19) and simplifying yields

(C.37)
$$\frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0,x,X)=u(x)-u(\sigma^o(X))+(x-\sigma^o(X))\left(\dot{\mathcal{W}}^o(X)-(Z^o(X)-1)\dot{\varphi}^o(X)-\dot{Z}^o(X)\varphi^o(X)\right).$$

Gathering (C.37) and (C.30) finally yields

$$\frac{\partial \mathcal{W}}{\partial \varepsilon}(0,x,X)=Z^o(X)\Big(u(x)-u(\sigma^o(X))\Big)+(x-\sigma^o(X))\Big(\dot{\mathcal{W}}^o(X)-\dot{Z}^o(X)\varphi^o(X)\Big).$$

Because $\frac{\partial \mathcal{W}}{\partial \varepsilon}(0,x,X)$ so obtained is strictly concave in x, the following first-order condition is necessary and sufficient for an interior optimum obtained from (C.13) and (C.14):

$$0=\frac{\partial^2 \mathcal{W}}{\partial \varepsilon \partial x}(0,\sigma^o(X),X)$$

Developing, we find

$$\text{(C.38)}\quad \sigma^o(X)=\zeta+\frac{\dot{\mathcal{W}}^o(X)}{Z^o(X)}-\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X).$$

which writes as (6.16).

Inserting (6.16) into (C.18), we now obtain

$$\sigma^o(X)\left(\sigma^o(X)-\zeta+\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)=\lambda_0\mathcal{V}^o(X)-\lambda_1\mathcal{V}_\infty+\frac{1}{2}(\sigma^o(X)-\zeta)^2.$$

Simplifying, we obtain

$$\text{(C.39)}\quad \left(\sigma^o(X)+\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2=2\lambda_0\mathcal{V}^o(X)+\left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2.$$

Taking then the highest root to (C.39), we obtain

$$\text{(C.40)}\quad \sigma^o(X)+\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)=\sqrt{2\lambda_0\mathcal{V}^o(X)+\left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2}.$$

Inserting (6.16) into (C.40) and simplifying finally yields (6.14).

Limiting Behavior. From (C.12) and the fact that $X^o(\tau;X)\geq \overline{X}$ for all $\tau\geq 0$ and $X\geq\overline{X}$, it follows that

$$\text{(C.41)}\quad Z(\tau;X)=Z^o(\overline{X})e^{-\Delta\tau}\quad \forall \tau\geq 0, X\geq \overline{X}.$$

Inserting into (6.6) immediately yields (6.15). From there, it immediately follows that

$$\text{(C.42)}\quad \sigma^o(X)=\zeta\quad \forall X\geq\overline{X}.$$

Q.E.D.

Proof of Proposition 6: Clearly (6.21) holds for $X \geq \overline{X}$. We turn to the more difficult case, $X \in [0, \overline{X})$. Consider the pair $(\mathcal{V}^e(X, Z^o(X)), \sigma^e(X, Z^o(X)))$ together with a belief index $Z^o(X)$ now defined as

$$\sigma^e(X, Z^o(X))\dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \tag{C.43}$$

with the boundary condition

$$Z^o(0) = 1. \tag{C.44}$$

Observe that, provided that $\sigma^e(X, Z)$ remains positive, such a $Z^o(X)$ is uniquely defined and satisfies the same properties as in Lemma C.3. In particular, $Z^o(X)$ is positive for all $X \in [0, \overline{X})$.

We shall prove that $\mathcal{V}^e(X, Z^o(X) \equiv \mathcal{V}^o(X)$, $\sigma^e(X, Z^o(X)) \equiv \sigma^o(X)$ and $Z^o(X)$ as defined above altogether form a *SME*. To ease notations, define accordingly $\mathcal{W}^o(X)$ as in (C.16).

First, notice that, from (B.21), it immediately follows that, for $X \in [0, \overline{X})$,

(C.45)
$$\lambda_0 \mathcal{W}^e(X, Z^o(X)) = \sup_{x \in \mathcal{X}} \left\{ Z^o(X)u(x) + x\frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \Delta(1 - F(X) - Z^o(X))\frac{\partial \mathcal{W}^e}{\partial Z}(X, Z^o(X)) \right\}$$

where we remind that $\mathcal{W}^e(X, Z^o(X)) = Z^o(X)\mathcal{V}^e(X, Z^o(X))$.

Using (B.25) and (C.43), we rewrite (C.45) as

(C.46)
$$\lambda_0 \mathcal{W}^e(X, Z^o(X)) = \sup_{x \in \mathcal{X}} \left\{ Z^o(X)u(x) + x\frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \sigma^e(X, Z^o(X))\dot{Z}^o(X)\varphi^e(X, Z^o(X)) \right\}$$

where the maximand above is achieved for

$$\sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)}\frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) \quad \forall X \in [0, \overline{X}). \tag{C.47}$$

Still using (B.25), we obtain the following expression of the total derivative of $\mathcal{W}^e(X, Z^o(X))$

$$\frac{d\mathcal{W}^e}{dX}(X, Z^o(X)) = \frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \dot{Z}^o(X)\varphi^e(X, Z^o(X)) \quad \forall X \in [0, \overline{X}). \tag{C.48}$$

Inserting (C.48) into (C.47) yields

$$\sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)}\left(\frac{d}{dX}\mathcal{W}^e(X, Z^o(X)) - \dot{Z}^o(X)\varphi^e(X, Z^o(X))\right) \quad \forall X \in [0, \overline{X}). \tag{C.49}$$

Also, (B.16) allows us to rewrite

$$\varphi^e(X, Z^o(X)) = \int_0^{+\infty} e^{-\lambda_1 \tau}u(\sigma^e(\widetilde{X}^e(\tau; X, Z^o(X)), \widetilde{Z}^e(\tau; X, Z^o(X))))d\tau. \tag{C.50}$$

At equilibrium, DM expects that the feedback rule $\sigma^o(X') = \sigma^e(X', Z^o(X'))$ prevails for all $X' > X$ and in particular for $X' = X^o(\tau; X)$ for $\tau > 0$, Observe that the future trajectory of stock and beliefs is thus such that $\widetilde{X}^e(\tau; X, Z^o(X)) = X^o(\tau; X)$ and $\widetilde{Z}^e(\tau; X, Z^o(X))) = Z^o(X^o(\tau; X))$ for all $\tau > 0$. Hence, we rewrite (C.50) as

$$\varphi^e(X, Z^o(X)) = \int_0^{+\infty} e^{-\lambda_1 \tau}u(\sigma^e(X^o(\tau; X), Z^o(X^o(\tau; X)))d\tau$$

or

(C.51) $\varphi^o(X) = \varphi^e(X, Z^o(X))$.

Inserting (C.51) into (C.49) yields

(C.52)

$$\sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)}\left(Z^o(X)\frac{d}{dX}\mathcal{V}^e(X, Z^o(X)) + \dot{Z}^o(X)(\mathcal{V}^e(X, Z^o(X)) - \varphi^o(X))\right) \quad \forall X \in [0, \overline{X}).$$

Rewriting (C.46), we obtain that $\mathcal{V}^e(X, Z^o(X))$ solves

(C.53) $$\lambda_0 Z^o(X)\mathcal{V}^e(X, Z^o(X)) = \sup_{x\in\mathcal{X}} Z^o(X)u(x)$$

$$+x\left(Z^o(X)\frac{d}{dX}\mathcal{V}^e(X, Z^o(X)) + \dot{Z}^o(X)(\mathcal{V}^e(X, Z^o(X)) - \varphi^o(X))\right) + \sigma^e(X, Z^o(X))\dot{Z}^o(X)\varphi^o(X)$$

where the maximum is achieved with $\sigma^e(X, Z^o(X))$ that satisfies (C.52).

From this, we now observe that $\mathcal{V}^o(X) \equiv \mathcal{V}^e(X, Z^o(X)$ and $\sigma^o(X) = \sigma^e(X, Z^o(X))$ altogether solve

(C.54) $$\lambda_0 Z^o(X)\mathcal{V}^o(X) = \sup_{x\in\mathcal{X}} Z^o(X)u(x) + x\Big(Z^o(X)\dot{\mathcal{V}}^o(X) + \dot{Z}^o(X)(\mathcal{V}^o(X) - \varphi^o(X))\Big)$$

$$+\sigma^o(X)\dot{Z}^o(X)\varphi^o(X)$$

where $\sigma^o(X)$, which achieves the maximum on the r.-h. s. above, satisfies

(C.55) $$\sigma^o(X) = \zeta + \frac{1}{Z^o(X)}\Big(Z^o(X)\dot{\mathcal{V}}^o(X) + \dot{Z}^o(X)(\mathcal{V}^o(X) - \varphi^o(X))\Big) \quad \forall X \in [0, \overline{X}).$$

Inserting (C.55) into (C.54), rearranging and simplifying yields that $\mathcal{V}^o(X) = \mathcal{V}^e(X, Z^o(X))$ indeed satisfies (6.14) as requested with any (continuously differentiable) *SME*. Moreover, and from (5.12), the boundary condition (6.15) holds. Hence, $(\mathcal{V}^e(X, Z^o(X)), \sigma^e(X, Z^o(X)))$ together with the associated index $Z^o(X)$ that satisfies (C.43)-(C.44) form a *SME*. *Q.E.D.*

Proof of Proposition 7: First, using (C.12) and noticing that $F(X) \le F(X^o(\tau; X)) \le 1$ for $\tau \ge 0$, we obtain the bounds

(C.56) $$Z^o(X)e^{\Delta\tau} \le Z(\tau; X) = Z^o(X^o(\tau; X)) \le 1 - F(X) + F(X)e^{-\Delta\tau} \quad \forall \tau \ge 0, X \ge 0.$$

Inserting into the definition of $\mathcal{V}^o(X)$ given in (6.6) and integrating, we obtain

(C.57) $$Z^o(X)\varphi^o(X) \le Z^o(X)\mathcal{V}^o(X) \le (1 - F(X))\frac{\lambda_1}{\lambda_0} + F(X)\varphi^o(X) \quad \forall X \ge 0.$$

Of course, we have

(C.58) $$\varphi^o(X) \le \mathcal{V}_\infty \quad \forall X \ge 0$$

which is the l.-h. s. inequality in (6.22). Inserting into (C.57) yields the r.-h. s. inequality in (6.22). The second inequality immediately follows from (6.21) and (B.26) taken for $Z = Z^o(X)$.

To obtain the r.-h. s. inequality in (6.23), first observe that (5.10), (5.13) and (6.21) imply

$$\sigma^o(X) \le \sqrt{2\lambda_1\mathcal{V}_\infty} = \zeta$$

as requested. To obtain the l.-h. s. inequality in (6.23), observe that $\dot{Z}^o(X) \le 0$ (from Lemma C.3) and $\varphi^o(X) \ge 0$ altogether imply

$$\sigma^o(X) \ge \sqrt{2\lambda_0\mathcal{V}^o(X)}.$$

Using the second left inequality in (6.22) yields the result. *Q.E.D.*

APPENDIX D: SME WITH NON-OBSERVABLE IMPULSE DEVIATIONS

PROOF OF PROPOSITION 8: Let first define

(D.1) $$\mathcal{W}^{no}(X) = Z^{no}(X)\mathcal{V}^{no}(X).$$

It is routine to show that, at any point of differentiability, $\mathcal{W}^{no}(X)$ satisfies the following HBJ equation for problem (7.4):

(D.2) $$\lambda_0 \mathcal{W}^{no}(X) = \max_{x\mathcal{X}} Z^{no}(X)u(x) + x\dot{\mathcal{W}}^{no}(X).$$

The maximand is obtained for an interior solution

(D.3) $$\sigma^{no}(X) = \zeta + \frac{\dot{\mathcal{W}}^{no}(X)}{Z^{no}(X)}.$$

Simplifying yields the SME feedback rule when impulse deviations are non-observable as in (7.10). Inserting (D.3) into (D.2) yields

$$\lambda_0 \mathcal{W}^{no}(X) = Z^{no}(X)\lambda_1 \mathcal{V}_\infty + \frac{(\dot{\mathcal{W}}^{no}(X))^2}{2Z^{no}(X)} + \zeta\dot{\mathcal{W}}^{no}(X).$$

Solving this second-degree equation in $\dot{\mathcal{W}}^{no}(X)$ yields

(D.4) $$\dot{\mathcal{W}}^{no}(X) = Z^{no}(X)\left(-\zeta + \sqrt{2\lambda_0 \frac{\mathcal{W}^{no}(X)}{Z^{no}(X)}}\right).$$

Rewriting this condition in terms of $\mathcal{V}^{no}(X)$ yields (7.8).

The boundary condition (7.9) is immediate. For future reference, observe that it also writes in terms of $\mathcal{W}^{no}(X)$ as

(D.5) $$\mathcal{W}^{no}(X) = \mathcal{Z}^{no}(X)\mathcal{V}_\infty \quad \forall X \geq \overline{X}.$$

Q.E.D.

EXISTENCE. Finally, our last result proves existence of a SME with non-observable impulse deviations. Its proof consists in studying the properties of the system of first-order differential equations satisfied by $(\mathcal{V}^{no}(X), Z^{no}(X))$ and showing that the boundary conditions at $X = 0$ and $X = \overline{X}$ for that system are satisfied.

PROPOSITION D.1 *A Stock-Markov-value function with non-observable deviations $\mathcal{V}^{no}(X)$ and an associated feedback rule $\sigma^{no}(X)$ always exist.*

PROOF OF PROPOSITION D.1: We consider the flow of the differential system made of (7.1) and (D.4) with the initial condition for $\mathcal{Z}^{no}(X)$ given by (7.2) together with an arbitrary initial condition for $\mathcal{W}^{no}(X)$ given by

(D.6) $$\mathcal{W}^{no}(0) \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right].$$

We look for such an initial value $\mathcal{W}^{no}(0)$ so that the terminal condition (D.5) is satisfied.

Observe that the system (7.1)-(D.4) is Lipschitz-continuous on the open domain

$$\text{(D.7)} \qquad \mathcal{W}^{no}(X) > 0$$

We now define $\widetilde{\mathcal{W}}^{no}(Y) = \mathcal{W}^{no}(X)$, $Z^{no}(Y) = Z^{no}(X)$, $\widetilde{\sigma}^{no}(Y) = \sigma^{no}(X)$ where $Y = 1-F(X) \in [0,1]$. Let also denote $R(Y) = f(F^{-1}(1-Y))$ for all $Y \in [0,1]$. First, notice that we also have $\dot{Z}^{no}(Y) = -\frac{\dot{Z}^{no}(X)}{R(Y)}$ and $\dot{\widetilde{\mathcal{W}}}^{no}(Y) = -\frac{\dot{\mathcal{W}}^{no}(X)}{R(Y)}$. Second, using (7.10) and (D.1), we rewrite

$$\text{(D.8)} \qquad \widetilde{\sigma}^{no}(Y) = \sqrt{2\lambda_0 \frac{\widetilde{\mathcal{W}}^{no}(Y)}{\mathcal{Z}^{no}(Y)}}.$$

We now transform the system of first-order differential equations (7.1)-(D.4) as

$$\text{(D.9)} \qquad \dot{\widetilde{\mathcal{W}}}^{no}(Y) = \frac{Z^{no}(Y)}{R(Y)}(\zeta - \widetilde{\sigma}^{no}(Y)),$$

$$\text{(D.10)} \qquad \dot{Z}^{no}(Y) = \frac{\Delta(Z^{no}(Y) - Y)}{R(Y)\widetilde{\sigma}^{no}(Y)}.$$

together with the following boundary conditions

$$\text{(D.11)} \qquad \widetilde{\mathcal{W}}^{no}(1) \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right], \quad Z^{no}(1) = 1$$

and

$$\text{(D.12)} \qquad \widetilde{\mathcal{W}}^{no}(0) = Z^{no}(0)\mathcal{V}_\infty.$$

Satisfying boundary conditions at the two end-points $Y = 0$ and $Y = 1$ requires a global analysis of the system. The first step consists in observing that the new system (D.9) can be transformed into an homogenous system expressed in terms of a variable $\tau \in \mathbb{R}_+$ such that (slightly abusing notations by not changing the names of variables although they now depend on τ)

$$\text{(D.13)} \qquad \dot{\widetilde{\mathcal{W}}}^{no}(\tau) = Z^{no}(\tau)(-\zeta + \widetilde{\sigma}^{no}(\tau)),$$

$$\text{(D.14)} \qquad \dot{Z}^{no}(\tau) = \frac{\Delta(Y - Z^{no}(\tau))}{\widetilde{\sigma}^{no}(Y)},$$

$$\text{(D.15)} \qquad \dot{Y}(\tau) = -R(Y(\tau))$$

together with the following boundary conditions

$$\text{(D.16)} \qquad \widetilde{\mathcal{W}}^{no}(0) \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right], \quad Z^{no}(0) = 1, \quad Y(0) = 1$$

and

$$\text{(D.17)} \qquad \lim_{\tau \to +\infty} \widetilde{\mathcal{W}}^{no}(\tau) - Z^{no}(\tau)\mathcal{V}_\infty = 0, \qquad \lim_{\tau \to +\infty} Y(\tau) = 0.$$

Observe that $Y(\tau)$ is decreasing. Moreover, direct integration of (D.15) together with the third condition in (D.16) yields

$$\text{(D.18)} \qquad \tau = \int_{Y(\tau)}^{1} \frac{dY}{R(Y)}.$$

Consider now the hyperplans

$$\mathcal{D}_0 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } \widetilde{\mathcal{W}} = \frac{\lambda_1 \mathcal{V}_\infty}{\lambda_0} Z\right\} \text{ and } \mathcal{D}_1 = \left\{(0, Z, Y) \in {\mathbb{R}_+}^3\right\}.$$

Observe that the segment for initial conditions

$$\mathcal{D}_3 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } \widetilde{\mathcal{W}} \in \left[0, \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty\right], \quad Z = 1, \quad Y = 1\right\}$$

lies in the cone of the positive orthant whose faces are the hyperplans $\mathcal{D}_0$ and $\mathcal{D}_1$. Observe that the hyperplan

$$\mathcal{D}_4 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } \widetilde{\mathcal{W}} = Z\mathcal{V}_\infty\right\}$$

belongs to that cone since $0 < \mathcal{V}_\infty < \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ and intersects $\mathcal{D}_0$ and $\mathcal{D}_1$ at the origin only.

Condition (D.18) shows that any trajectory is such that $Y(\tau)$ is decreasing and remains in the bandwith

$$\mathcal{D}_2 = \left\{(\widetilde{\mathcal{W}}, Z, Y) \in {\mathbb{R}_+}^3 \text{ s.t. } Y \in [0, 1]\right\}.$$

Moreover, Condition (D.18) also implies that a trajectory reaches $Y = 0$ in finite time if and only if $\int_0^1 \frac{dY}{R(Y)} < +\infty$. If instead $\int_0^1 \frac{dY}{R(Y)} = +\infty$, $Y = 0$ is only reached asymptotically.

Note that any solution to the system (D.13)-(D.14)-(D.15) with initial conditions (D.16) that would cross the hyperplan $\mathcal{D}_0$ at a time $\overline{T}$ crosses it from below (from the fact that $\dot{\widetilde{\mathcal{W}}}^{no}(\overline{T}) \leq 0$ and that direction is not in the hyperplan $\mathcal{D}_0$). Similarly, any solution to the system (D.13)-(D.14)-(D.15) with initial conditions (D.16) that would cross the hyperplan $\mathcal{D}_1$ at a time τ_1 reaches it from above (from the fact that $\dot{Z}^{no}(\tau_1) = +\infty$ and that direction is not in the hyperplan $\mathcal{D}_1$). Moreover, such trajectory stops there.

Because the system is continuous on the open positive cone defined by the faces $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$, any trajectory starting from the segment $\mathcal{D}_3$ can be extended till it reaches the boundaries of this domain in finite time (Nemytskii and Stepanov, 1989, p. 307). Because the flow of the system is continuous, the image of $\mathcal{D}_3$ which is connected and compact consists of a continuous line $\mathcal{L}$ that might lie on $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_2$. Observe that, for the initial condition $\widetilde{\mathcal{W}}(0) = \frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$, the trajectory immediately crosses $\mathcal{D}_0$ and goes out of the cone. Similarly, for the initial condition $\widetilde{\mathcal{W}}(0) = \frac{D}{\lambda_0}$, the trajectory immediately reaches $\mathcal{D}_1$ and stays there. By continuity of the flow of the differential system, trajectories with an initial condition $\widetilde{\mathcal{W}}(0)$ in a neighborhood of $\frac{\lambda_1}{\lambda_0}\mathcal{V}_\infty$ goes through $\mathcal{D}_0$ while trajectories with an initial condition $\widetilde{\mathcal{W}}(0)$ in a neighborhood of $\frac{D}{\lambda_0}$ reaches $\mathcal{D}_1$. Two cases may *a priori* arise. First, $\mathcal{L}$ may not go though the origin $(0, 0, 0)$. In this case, and by continuity, the part of $\mathcal{L}$ that lies on $\mathcal{D}_2$ necessarily crosses $\mathcal{D}_4$ somewhere and the boundary problem has a solution such that $\lim_{\tau\to+\infty} \widetilde{\mathcal{W}}(\tau) = \lim_{\tau\to+\infty} Z^{no}(\tau)\mathcal{V}_\infty > 0$ or, expressed in terms of original variables $\mathcal{W}^{no}(\overline{X}) = Z^{no}(\overline{X})\mathcal{V}_\infty > 0$. Second, $\mathcal{L}$ may go though the origin $(0, 0, 0)$. In this case, there is a trajectory that satisfies the boundary condition with $\lim_{\tau\to+\infty} \widetilde{\mathcal{W}}(\tau) = \lim_{\tau\to+\infty} Z^{no}(\tau)\mathcal{V}_\infty = 0$ or expressed in terms of original variables $\mathcal{W}^{no}(\overline{X}) = Z^{no}(\overline{X})\mathcal{V}_\infty = 0$.

Q.E.D.

APPENDIX E: RUNNING EXAMPLE

PROOF OF PROPOSITION 4: Observe that (5.7) rewrites now as

$$Z(\tau) = -(1-Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau}. \tag{E.1}$$

It is straightforward to check that $Z(\tau) \geq 1-q$ for all $\tau > 0$ when $Z \geq 1-q$. Since the optimal trajectory starts from $Z = 1$, this condition always holds.

This expression of $Z(\tau)$ allows us to rewrite the definition (5.9) for $\mathcal{V}^e(X,Z)$ in a quasi-explicit form as

$$Z\mathcal{V}^e(X,Z) = \max_{\mathbf{x},\overline{T}} \int_0^{\overline{T}} e^{-\lambda_0\tau}\left(-(1-Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau}\right) u(x(\tau))d\tau \tag{E.2}$$
$$+e^{-\lambda_0\overline{T}}\left(-(1-Z)e^{-\Delta\overline{T}} + 1 - q + qe^{-\Delta\overline{T}}\right)\mathcal{V}_\infty$$

subject to

$$\int_0^{\overline{T}} x(\tau)d\tau = \overline{X} - X. \tag{E.3}$$

Solving this problem is straightforward. Let denote by μ the multiplier for (E.3). We form the Lagrangean

$$\mathcal{L}(\mathbf{x},\overline{T}) = \int_0^{\overline{T}} e^{-\lambda_0\tau}\left(-(1-Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau}\right) u(x(\tau))d\tau$$
$$+e^{-\lambda_0\overline{T}}\left(-(1-Z)e^{-\Delta\overline{T}} + 1 - q + qe^{-\Delta\overline{T}}\right)\mathcal{V}_\infty + \mu\left(\overline{X} - X - \int_0^{\overline{T}} x(\tau)d\tau\right).$$

Pointwise optimization for this strictly concave objective yields the following expression of the optimal action at any point in time

$$\zeta - x^e(\tau) = \frac{\mu e^{\lambda_0\tau}}{Z(\tau)} \tag{E.4}$$

where, for simplicity, we omit the dependence on the state variables (X,Z).

Integrating over $\left[0,\overline{T}^e\right]$ yields

$$\zeta\overline{T}^e - (\overline{X} - X) = \mu\int_0^{\overline{T}^e} \frac{e^{\lambda_0\tau}}{Z(\tau)}d\tau. \tag{E.5}$$

Optimizing now with respect to $\overline{T}$ and assuming the quasi-concavity of the objective in $\overline{T}$ yields the following necessary first-order condition

$$e^{-\lambda_0\overline{T}^e}Z(\overline{T}^e)u(x^e(\overline{T}^{e-})) + \mathcal{V}_\infty e^{-\lambda_0\overline{T}^e}\left(-\lambda_0 Z(\overline{T}^e) + \dot{Z}(\overline{T}^e)\right) = \mu x^e(\overline{T}^{e-})$$

where $x^e(\overline{T}^{e-})$ denotes the l.h.-s limit of $x^e(\tau)$ at $\overline{T}^e$. Simplifying, we get

$$\zeta x^e(\overline{T}^{e-}) - \frac{(x^e(\overline{T}^{e-}))^2}{2} + \mathcal{V}_\infty\left(-\lambda_0 + \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}\right) = \mu\frac{e^{-\lambda_0\overline{T}^e}}{Z(\overline{T}^e)}x^e(\overline{T}^{e-})$$

Using (E.4) taken at $\tau = \overline{T}^e$, we rewrite the r.-h.s. and get

$$\zeta x^e(\overline{T}^{e-}) - \frac{(x^e(\overline{T}^{e-}))^2}{2} + \mathcal{V}_\infty \left(-\lambda_0 + \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}\right) = x^e(\overline{T}^{e-})(\zeta - x^e(\overline{T}^{e-}))$$

Simplifying further yields

$$x^e(\overline{T}^{e-}) = \zeta \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}.$$

From (E.4) taken at $\tau = \overline{T}^e$, we then get

$$\text{(E.6)} \qquad \mu \frac{e^{\lambda_0 \overline{T}^e}}{Z(\overline{T}^e)} = \zeta \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right).$$

Inserting (E.6) into (E.5) and (E.4) finally yields (E.7) and (E.8) respectively:

$$\text{(E.7)} \qquad \overline{T}^e = \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right) e^{-\lambda_0 \overline{T}^e} Z(\overline{T}^e) \int_0^{\overline{T}^e} \frac{e^{\lambda_0 \tau}}{Z(\tau)} d\tau,$$

$$\text{(E.8)} \qquad x^e(\tau) = \zeta \left(1 - e^{-\lambda_0(\overline{T}^e - \tau)} \frac{Z(\overline{T}^e)}{Z(\tau)} \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda_1}}\right)\right) \quad \forall \tau \in [0, \overline{T}^e).$$

Specializing this solution to the case $X = 0$ and $Z = 1$ yields the optimal trajectory described in (5.16) and (5.14) with $Z(\tau)$ being given by (5.15). Because $\frac{e^{\lambda_0 \tau}}{Z(\tau)}$ is increasing, $x^e(\tau)$ is itself decreasing over $[0, \overline{T}^e)$.

Specializing further to the case $q = 0$ yields the optimal trajectory when the tipping point is known being at $\overline{X}$ for sure. In this case, $\overline{T}^k$ is given by (4.7) while the optimal action is now

$$\text{(E.9)} \qquad x^k(\tau) = \begin{cases} \zeta \left(1 - e^{-\lambda_0(\overline{T}^k - \tau)} \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right)\right) < \zeta & \text{for } t \in [0, \overline{T}^k), \\ \zeta & \text{for } t \geq \overline{T}^k. \end{cases}$$

Because $Z(\tau)$ is decreasing, one has

$$\overline{T}^k < \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) e^{-\lambda_0 \overline{T}^k} \int_0^{\overline{T}^k} e^{\lambda_0 \tau} d\tau = \overline{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) \frac{1 - e^{-\lambda_0 \overline{T}^k}}{\lambda_0}.$$

Consider now the function $\delta(t) \equiv t - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) \frac{1 - e^{-\lambda_0 t}}{\lambda_0}$. We have $\delta(T^k) = T^m$, $\delta(0) = 0$ and $\delta'(t) = 1 - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) e^{-\lambda_0 t} > 0$. Hence, there is a unique positive root $0 < \overline{T}^k < T^m$ for (5.14). *Q.E.D.*

PROOF OF PROPOSITION 9: To find the optimal trajectory under commitment starting from $X = 0$, we want to maximize

$$\max_{\mathbf{x}, X(\cdot), \overline{T}} \int_0^{\overline{T}} e^{-\lambda_0 \tau} Z^{no}(X(\tau)) u(x(\tau)) d\tau + e^{-\lambda_0 \overline{T}} Z^{no}(\overline{X}) \mathcal{V}_\infty$$

subject to (5.5), $X(0) = X$, and $X(T) = \overline{X}$,

where $Z^{no}(X)$ is given by (7.1) and (7.2).

Let denote by μ the costate variable for (5.5). The Hamiltonian for this control problem is

$$\mathcal{H}^{no}(X, x, \tau, \lambda) = e^{-\lambda_0 \tau} Z^{no}(X) u(x) + \mu x. \tag{E.10}$$

The *Maximum Principle* with free final time and scrap value gives us the following necessary conditions for an optimal arc $(X^{no}(\tau), x^{no}(\tau), \overline{T}^{no})$. (See Seierstad and Sydsaeter, 1987, Theorem 11, p. 143).)

Costate variable. $\mu(\tau)$ is continuously differentiable on $\mathbb{R}_+$ with

$$-\dot{\mu}(\tau) = \frac{\partial \mathcal{H}^{no}}{\partial X}(X^{no}(\tau), x^{no}(\tau), \tau, \mu(\tau))$$

or

$$-\dot{\mu}(\tau) = e^{-\lambda_0 \tau} \dot{Z}^{no}(X^{no}(\tau)) u(x^{no}(\tau)) \quad \forall \tau \in \left[0, \overline{T}^{no}\right]. \tag{E.11}$$

Transversality conditions. The boundary conditions $X^{no}(0) = 0$ and $X^{no}(\overline{T}^{no}) = \overline{X}$ imply that there are no transversality conditions on $\mu(\tau)$ at both $\tau = 0$ and $\tau = \overline{T}^{no}$.

Control variable $x^{no}(\tau)$.

$$x^{no}(\tau) \in \arg\max_{x \geq 0} \mathcal{H}^{no}(X^{no}(\tau), x, \tau, \mu(\tau)).$$

Because $\mathcal{H}^{no}(X^{no}(\tau), x, \tau, \mu(\tau))$ is strictly concave in x, an interior solution satisfies

$$\frac{\partial \mathcal{H}^{no}}{\partial x}(X^{no}(\tau), x^{no}(\tau), \tau, \mu(\tau)) = 0$$

or

$$x^{no}(\tau) = \zeta + e^{\lambda_0 \tau} \frac{\mu(\tau)}{Z^{no}(X^{no}(\tau))}. \tag{E.12}$$

Free-end point conditions. The optimality condition with respect to $\overline{T}$ writes as

$$\mathcal{H}^{no}(X^{no}(\overline{T}^{no}), x^{no}(\overline{T}^{no}), \overline{T}^{no}, \mu(\overline{T}^{no})) - \lambda_0 Z^{no}(\overline{X}) e^{-\lambda_0 \overline{T}^{no}} \mathcal{V}_\infty = 0. \tag{E.13}$$

From (E.12), we get

$$x^{no}(\overline{T}^{no}) = \zeta + e^{\lambda_0 \overline{T}^{no}} \frac{\mu(\overline{T}^{no})}{Z^{no}(\overline{X})}. \tag{E.14}$$

Using (E.10), (E.14), inserting into (E.13) and simplifying yields

$$\zeta x^{no}(\overline{T}^{c-}) - \frac{1}{2}\left(x^{no}(\overline{T}^{c-})\right)^2 - \lambda_0 \mathcal{V}_\infty = x^{no}(\overline{T}^{c-})(\zeta - x^{no}(\overline{T}^{c-}))$$

or

(E.15) $x^{no}(\overline{T}^{c-}) = \zeta\sqrt{\frac{\lambda_0}{\lambda_1}}.$

where, to account for the discontinuity in action at $\overline{T}^{no}$, we denote by $x^{no}(\overline{T}^{c-})$ the l.-h. side limit of $x^{no}(\tau)$ as $\tau \to \overline{T}^{c-}$.

Characterization. Using (E.1) for the optimal arc starting from $Z = 1$, we get

(E.16) $Z(\tau) = 1 - q + qe^{-\Delta\tau}.$

Along the trajectory, we must have

(E.17) $Z^{no}(X^{no}(\tau)) = Z(\tau) \quad \forall\tau \leq \overline{T}^{no}.$

Differentiating, we get

(E.18) $\dot{Z}^{no}(X^{no}(\tau)) = \frac{\dot{Z}(\tau)}{x^{no}(\tau)} = -\frac{q\Delta e^{-\Delta\tau}}{x^{no}(\tau)}$

Now, we rewrite (E.12) as

$$\mu(\tau) = Z^{no}(X^{no}(\tau))(x^{no}(\tau) - \zeta)e^{-\lambda_0\tau}.$$

Differentiating w.r.t. τ and using (E.18) yields the following ordinary differential equation for $x^{no}(\tau)$:

$$\dot{x}^{no}(\tau) - \left(\lambda_0 - \frac{\dot{Z}(\tau)}{2Z(\tau)}\right)x^{no}(\tau) = -\lambda_0\zeta.$$

It is routine to check that the solution of this ordinary differential equation is of the form

(E.19) $x^{no}(\tau) = \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}\left(C_0 - \lambda_0\zeta\int_0^\tau e^{-\lambda_0 s}\sqrt{Z(s)}ds\right)$

for some constant C_0. The corresponding stock evolves according to

(E.20) $X^{no}(\tau) = C_0\int_0^\tau \frac{e^{\lambda_0 s}}{\sqrt{Z(s)}}ds - \lambda_0\zeta\int_0^\tau \frac{e^{\lambda_0 s}}{\sqrt{Z(s)}}\left(\int_0^s e^{-\lambda_0 s'}\sqrt{Z(s')}ds'\right)ds.$

Finally, the value of C_0 is obtained from the terminal condition $X^{no}(\overline{T}^{no}) = \overline{X} = \zeta\overline{T}^m$. We get:

(E.21) $\zeta\overline{T}^m = C_0\int_0^{\overline{T}^{no}} \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}d\tau - \lambda_0\zeta\int_0^{\overline{T}^{no}} \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}}\left(\int_0^\tau e^{-\lambda_0 s}\sqrt{Z(sds}\right)d\tau.$

Simplifying yields (7.11).

Inserting into (E.19), we obtain the expression of $x^{no}(\tau)$ for $\tau \leq \overline{T}^{no}$ given in (7.12). The expression $\tau \geq \overline{T}^{no}$ is straightforward.

Now, observing that $Z(\tau) \geq Z(\overline{T}^{no})$ for all $\tau \leq \overline{T}^{no}$, we obtain the following majoration of the r.-h. side of (7.11) as

$$\overline{T}^m < e^{-\lambda_0\overline{T}^{no}}\left(\int_0^{\overline{T}^{no}} e^{\lambda_0\tau}d\tau\right)\sqrt{\frac{\lambda_0}{\lambda_1}} + \lambda_0\int_0^{\overline{T}^{no}} e^{-\lambda_0\tau}\left(\int_0^\tau e^{-\lambda_0 s}ds\right)d\tau$$

or, after simplifying,

$$\overline{T}^m < \overline{T}^{no} - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right)\frac{1 - e^{-\lambda_0\overline{T}^{no}}}{\lambda_0}.$$

From there and (4.7), it follows that $\overline{T}^{no} > \overline{T}^k$. *Q.E.D.*