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Mixed-Strategy Equilibria in the War of Attrition under Uncertainty*

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Abstract

We study a generic family of two-player continuous-time nonzero-sum stopping games modeling a war of attrition with symmetric information and stochastic payoffs that depend on an homogeneous linear diffusion. We first show that any Markovian mixed strategy for player i can be represented by a pair (μ^i, S^i), where μ^i is a measure over the state space representing player i's stopping intensity, and S^i is a subset of the state space over which player i stops with probability 1. We then prove that, if players are asymmetric, then, in all mixed-strategy Markov-perfect equilibria, the measures μ^i have to be essentially discrete, and we characterize any such equilibrium through a variational system satisfied by the players' equilibrium value functions. This result contrasts with the literature, which focuses on pure-strategy equilibria, or, in the case of symmetric players, on mixed-strategy equilibria with absolutely continuous stopping intensities. We illustrate this result by revisiting the model of exit in a duopoly under uncertainty, and exhibit a mixed-strategy equilibrium in which attrition takes place on the equilibrium path though firms have different liquidation values.

Keywords: War of Attrition, Mixed-Strategy Equilibrium, Uncertainty.
JEL Classification: C61, D25, D83.

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1 Introduction

The war of attrition is a workhorse to model situations in which, at each point of time, each player has to decide whether to hold fast or to concede and forfeit a prize to its opponent. Examples of such situations include animal conflict (Maynard Smith (1974)), public good provision (Bliss and Nalebuff (1984)), exit from a declining industry (Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986)), labor strikes (Kennan and Wilson (1989)), delays in agreement to stabilization policies (Alesina and Drazen (1991)), competition between technological standards (Bulow and Klemperer (1999)), bargaining (Abreu and Gul (2000)), and investment in the presence of informational externalities (Décamps and Mariotti (2004)). Meanwhile, a growing literature attempts to test the theoretical predictions of these models and to estimate the welfare cost of delayed exit decisions (Takahashi (2015)).

Theoretical and empirical applications of war-of-attrition models face several challenges, however. The first is the multiplicity of equilibria, both in pure and mixed strategies, that characterize these models (Riley (1980), Hendricks, Weiss, and Wilson (1988)). It is therefore important to identify theoretical predictions of these models that are robust, in the sense that they hold in a large class of equilibria. The second is to account for observable asymmetries in players' characteristics, which many applied models disregard for simplicity. The third is to allow for stochastic payoffs, so as to capture uncertainty about the future evolution of, say, market conditions. The present paper is an attempt at resolving these issues in a unified framework. In so doing, it identifies a new class of equilibria in mixed strategies that have robust and novel empirical implications.

To this end, we study a generic model of the war of attrition with symmetric information, stochastic payoffs and potentially asymmetric players, which embeds the earlier models of Lambrecht (2001), Murto (2004), Steg (2015), and Georgiadis, Kim, and Kwon (2022). Two players initially present on a market face uncertainty about future market conditions—for instance, the future price of a relevant commodity, or the future state of market demand. Market conditions evolve according to an homogenous linear diffusion. Each player has the option to exit the market, which he may exert at any point in time. Specifically, both players continuously observe the evolution of market conditions; based on this information, each player then decides whether to remain in the market or to irreversibly exit, which terminates the game. In a Markovian way, the players' continuation payoffs when a player decides to exit the market only depend on current market conditions. Besides, there is a second-mover advantage in the sense that, if and when a player exits first, his continuation payoff is lower than the continuation payoff he would have obtained if the other player had exited first given the same market conditions. All payoff-relevant variables—the law of evolution of market conditions and the players' payoff functions—are assumed to be common knowledge. An example of this game is a war of attrition between two firms that may exit a market by liquidating their assets—say, because market demand deteriorates too much—but would meanwhile individually fare better as a monopolist than as a duopolist.

Given the payoff structure we postulate, it is natural to focus on Markov-perfect equilibria in which players' exit decisions at any point in time only depend on current market conditions (Maskin and Tirole (2001)). Our first contribution is to provide a precise definition of Markovian mixed strategies that allows for rich possibilities of randomization for the players. Specifically, our first main result, Theorem 1, shows that a randomized stopping time for any player i, as defined by Touzi and Vieille (2002) by introducing an auxiliary randomizing device à la Aumann (1964), is Markovian if and only if it can be represented by a pair (μ^i, S^i), where μ^i is a measure over the state space of the diffusion representing player i's stopping intensity, and S^i is a subset of the state space over which player i stops with probability 1; the interpretation is that player i exits the market with positive but finite intensity over the support of μ^i, and with infinite intensity over S^i. Well-known examples of this representation include pure strategies—that is, stopping times—as in Lambrecht (2001) and Murto (2004), in which the intensity measure μ^i is degenerate, and mixed strategies in which μ^i is absolutely continuous with respect to Lebesgue measure, as in Steg (2015) and Georgiadis, Kim, and Kwon (2022). These authors characterize pure-strategy Markov-perfect equilibria, and, in the case of symmetric players, mixed-strategy Markov-perfect equilibria in which players exit the market with the same absolutely continuous intensity measure. In the latter case, attrition is maximal in the sense that each player obtains the payoff he would obtain when facing a stubborn opponent threatening him never to exit the market.

These examples, however, do not exhaust the range of possibilities made available by our formalization of Markovian mixed strategies. In particular, it is possible to conceive of such strategies in which the measure μ^i is singular with respect to Lebesgue measure. Such strategies need not be artificial nor exotic. For instance, μ^i may be a Dirac measure at a given point x^i of the state space, weighted by some positive coefficient a^i. The interpretation is that, each time market conditions reach x^i, player i exits the market with finite intensity a^i, a strategy that can easily be obtained as the limit of Markovian mixed strategies defined on discretized state spaces with increasingly fine mesh, or, alternatively, as the limit of Markovian strategies with absolutely continuous intensity measures with supports that degenerate to x^i.

Our second main result, Theorem 2, precisely shows that, if players are asymmetric—for instance, if firms in a duopoly have different liquidation values—then any mixed-strategy Markov-perfect equilibrium involves strategies with discrete intensity measures. At each point in the support of these measures, the corresponding player is indifferent between exiting and remaining in the market. This implies that the state space is partitioned into intervals in which players alternate between being in a dominated position (with a continuation payoff close to the value he could secure if facing a stubborn opponent) or in a dominant position (with a continuation payoff significantly above that value). Our third main result, Theorem 3, finally characterizes these mixed-strategy Markov-perfect equilibria through a variational system satisfied by the two players' continuation value functions. Solving for these equilibria then becomes a rather simple numerical task.

As an illustration, and to show that these necessary and sufficient conditions can be satisfied—in the sense that our variational system has a solution—we provide conditions under which the war of attrition between two duopolists with different liquidation values has a mixed-strategy Markov-perfect equilibrium. In this equilibrium, the firm with the lowest liquidation value randomizes between remaining in the market and exiting at the exit threshold for market conditions that would be optimal if its opponent were stubborn. By contrast, the firm with the highest liquidation value exits with probability 1 if market conditions fall below an even lower threshold, the value of which is determined precisely so as to meet its opponent's indifference condition.

It may rightly be objected that our construction does not solve the multiplicity problem that plagues standard models of the war of attrition: if anything, we exhibit additional equilibria that have been disregarded in the literature (Georgiadis, Kim, and Kwon (2022)). A first answer to this objection is that, in the case of asymmetric players, any mixed-strategy Markov-perfect equilibrium must feature the discrete intensity measures we highlight in Theorem 2. Yet there may be many equilibria of this form. It is thus important to identify robust implications of such equilibria that differentiate them from those the literature has focused upon. In that respect, a robust property of the novel equilibria we identify in this paper is that, at any point at which a player randomizes between exiting or remaining in the market, the equilibrium value function of its opponent exhibits a kink. In our duopoly example, the market value of its opponent reaches a peak at this kink, while the randomizing firm's market value goes down to its liquidation value. Moreover, a robust testable implication of such equilibria is that, along any path of the diffusion process modeling the evolution of market conditions, the two firms' market values fluctuate stochastically over the attrition region, moving in opposite directions as long as none of them exits the market. This negative comovement of firms' market values stands in contrast with the symmetric mixed-strategy Markov-perfect equilibrium with an absolutely continuous intensity measure that arises when firms have identical liquidation values, in which firms' market values are constant and equal to their common liquidation value over the attrition region.

Related Literature

This paper belongs to the large literature on the war of attrition, starting with the seminal contribution of Maynard Smith (1974) on animal conflict. Ghemawat and Nalebuff (1985) study a war of attrition between duopolists who must decide when to exit from a declining industry. Hendricks, Weiss, and Wilson (1988) offer an exhaustive characterization of pure- and mixed-strategy equilibria in the war of attrition with symmetric information when players have potentially asymmetric payoffs that are deterministic functions of time. Riley (1980), Bliss and Nalebuff (1984), and Fudenberg and Tirole (1986) extend the analysis to asymmetric-information setups where, for instance, a firm is uncertain about its opponent's costs. In the same vein, Décamps and Mariotti (2004) study an investment game that has the structure of a war of attrition because a firm's investment generates additional information for its opponent about the return of a common-value project.

With the exception of the last paper—which, however, considers a very special Poisson information structure for signals—these papers confine their analysis to situations in which players' payoffs are deterministic. By contrast, a small literature, starting with Lambrecht (2001) and Murto (2004), examines the case where players in a war of attrition have symmetric information, but are uncertain about their future payoffs, which are driven by a diffusion process. Lambrecht (2001) analyzes the order in which firms go bankrupt within a given industry, and how this order is influenced by aggregate economic factors and firm-specific factors such as their financial structure. Murto (2004) studies a stochastic version of Ghemawat and Nalebuff (1985), and shows that a firm with a lower liquidation value may actually end up exiting the market first in equilibrium, despite being a priori more enduring than its opponent. Closest to the present paper in this literature is Georgiadis, Kim, and Kwon (2022). In a setting that extends Murto (2004), they show that, as soon as firms have different liquidation values, there exists no mixed-strategy Market-perfect equilibrium in which firms exit the market according to absolutely continuous intensity measures. They conclude that only pure-strategy Markov-perfect equilibria exist, and therefore that no attrition can actually take place in equilibrium. Our analysis shows that this conclusion is unwarranted once firms can exit the market according to Markovian randomized stopping times with singular intensity measures.

We have borrowed from Touzi and Vieille (2002) our concept of a randomized stopping time, which they introduced to show that continuous-time zero-sum Dynkin games admit a value. A technical contribution of the present paper is to provide a characterization of Markovian randomized stopping times in terms of an intensity measure and a stopping region. This characterization may prove useful for the study of general stochastic timing games.

The paper is organized as follows. Section 2 describes the model. Section 3 provides rigorous definitions of our strategy and equilibrium concepts, as well as preliminary properties of Markov-perfect equilibria. Section 4 heuristically shows how to construct a mixed-strategy Markov-perfect equilibrium involving a singular intensity measure for one of the players. Section 5 states our main characterization results. Proofs not given in the main text are collected in Appendices A–C.

2 The Model

2.1 A General Model of War of Attrition under Uncertainty

We study a war of attrition with symmetric information between two players, 1 and 2, facing uncertainty about future market conditions. In what follows, i (he) refers to an arbitrary player and j (she) to his opponent. Time is continuous and indexed by $t \geq 0$. Both players observe the evolution of market conditions; based on this information, each player decides whether to *hold fast*, that is, to remain in the market, or to *concede*, that is, to exit the market, an irreversible decision that effectively terminates the game.

The evolution of market conditions is modeled as a one-dimensional time-homogeneous diffusion process $X \equiv (X_t)_{t \geq 0}$ defined over the canonical space $(\Omega, \mathcal{F}, \mathbf{P}_x)$ of continuous trajectories with $X_0 = x$ under $\mathbf{P}_x$, that is solution in law to the stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad t \geq 0, \tag{1}$$

driven by some Brownian motion $W \equiv (W_t)_{t \geq 0}$. The state space for X is an interval $\mathcal{I} \equiv (\alpha, \beta)$, with $-\infty \leq \alpha < \beta \leq \infty$, and b and σ are continuous functions, with $\sigma > 0$ over $\mathcal{I}$. We assume that α and β are inaccessible (natural) endpoints for the diffusion. Therefore, X is regular over $\mathcal{I}$ and the SDE (1) admits a weak solution that is unique in law.

Player 1 chooses a (random) time τ^1 and player 2 chooses a (random) time τ^2. Both players discount future payoffs at a constant rate $r > 0$. For each $i = 1, 2$, the expected payoff of player i is[1]

$$J^i(x, \tau^1, \tau^2) = \mathbf{E}_x \Big[1_{\{\tau^i \leq \tau^j\}} \, \mathrm{e}^{-r\tau^i} R^i(X_{\tau^i}) + 1_{\{\tau^i > \tau^j\}} \, \mathrm{e}^{-r\tau^j} G^i(X_{\tau^j}) \Big]. \tag{2}$$

The payoff functions R^i and G^i in (2) are continuous over their domain $\mathcal{I}$ and satisfy $G^i \geq R^i$, with $G^i(x) > R^i(x)$ for x above some threshold $\alpha^i < \beta$.[2] Therefore, if player i concedes at time $\tau^i \leq \tau^j$, he obtains a payoff $R^i(X_{\tau^i})$, whereas, if player j concedes at time $\tau^j < \tau^i$ and $X_{\tau^j} > \alpha^i$, then player i obtains a strictly higher payoff $G^i(\tau^j)$ than the payoff $R^i(\tau^j)$ he would have obtained by conceding at time τ^j. The payoff functions R^i and G^i, $i = 1, 2$, are assumed to be common knowledge among the players; hence the only primitive source of uncertainty in the model is the diffusion process (1), whose realizations are observed by both players. We study the resulting war of attrition with symmetric information and uncertain payoffs under technical assumptions that we now present.

2.2 Technical Assumptions

We first recall useful properties of the solution X to the SDE (1). We next detail the assumptions on the payoff functions R^i and G^i and emphasize useful properties of the optimal stopping problem

$$V_{R^i}(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x [\mathrm{e}^{-r\tau} R^i(X_\tau)] \tag{3}$$

faced by player i when player j is *stubborn*, that is, plays $\tau^j = \infty$; here $\mathcal{T}$ denotes the set of all stopping times of the usual augmentation $(\mathcal{F}_t)_{t \geq 0}$ of the natural filtration generated by X over the canonical space.[3] We refer to (3) as player i's *stand-alone exit problem*, in which he cannot benefit from player j conceding.

Properties of the Diffusion X The infinitesimal generator of the diffusion X is defined for functions $u \in \mathcal{C}^2(\mathcal{I})$ by

$$\mathcal{L}u(x) \equiv b(x)u'(x) + \frac{1}{2}\,\sigma^2(x)u''(x), \quad x \in \mathcal{I}. \tag{4}$$

That $\sigma > 0$ over $\mathcal{I}$ ensures that the ordinary differential equation (ODE) $\mathcal{L}u - ru = 0$ admits a two-dimensional space of solutions in $\mathcal{C}^2(\mathcal{I})$, which is spanned by two positive fundamental

[1]By convention, we let $f(X_\tau) \equiv 0$ over $\{\tau = \infty\}$ for any Borel function f and any random time τ.

[2]Notice that one may have $\alpha^i \leq \alpha$. If $\alpha^i > \alpha$, then $G^i = R^i$ over $(\alpha, \alpha^i]$; this reflects that, for low values of x, it may be optimal for player i to exit the market even as a monopolist.

[3]The definition of $(\mathcal{F}_t)_{t \geq 0}$ is recalled in Appendix A.

solutions ψ and ϕ, respectively strictly increasing and strictly decreasing, that are uniquely defined up to a linear transformation. By Abel's theorem, the ratio

$$\gamma \equiv \frac{\psi'(x)\phi(x) - \psi(x)\phi'(x)}{S'(x)} > 0 \tag{5}$$

of the Wronskian of ψ and ϕ and of the derivative of the scale function of the diffusion X, which is uniquely defined up to an affine transformation by

$$S(x) \equiv \int_c^x \exp\left(-\int_c^y \frac{2b(z)}{\sigma^2(z)}\,\mathrm{d}z\right)\mathrm{d}y, \quad x \in \mathcal{I} \tag{6}$$

for some fixed $c \in \mathcal{I}$, is a constant independent of x. Because the boundaries α and β of $\mathcal{I}$ are natural, we know in particular that

$$\lim_{x\to\alpha^+} \psi(x) = 0, \quad \lim_{x\to\beta^-} \psi(x) = \infty, \quad \lim_{x\to\alpha^+} \phi(x) = \infty, \quad \lim_{x\to\beta^-} \phi(x) = 0. \tag{7}$$

Furthermore, letting $\tau_y \equiv \inf\{t \geq 0 : X_t = y\}$ be the hitting time of $y \in \mathcal{I}$ from $X_0 = x$, we have that

$$\mathbf{E}_x\left[\mathrm{e}^{-r\tau_y}\right] = \begin{cases} \frac{\psi(x)}{\psi(y)} & \text{if} \quad x \leq y, \\ \frac{\phi(x)}{\phi(y)} & \text{if} \quad x > y. \end{cases} \tag{8}$$

Assumptions on the Payoff Functions R^i and G^i For each $i = 1, 2$, we assume that $R^i \in \mathcal{C}^2(\mathcal{I})$, and that it satisfies

A1 For each $x \in \mathcal{I}$, $\mathbf{E}_x\left[\sup_{t\geq 0} \mathrm{e}^{-rt}|R^i(X_t)|\right] < \infty$.

A2 For each $x \in \mathcal{I}$, $\lim_{t\to\infty} \mathrm{e}^{-rt} R^i(X_t) = 0$, $\mathbf{P}_x$-almost surely.

A3 There exists $x_0^i \in \mathcal{I}$ such that $\mathcal{L}R^i - rR^i < 0$ over (α, x_0^i) and $\mathcal{L}R^i - rR^i > 0$ over (x_0^i, β).

A1 guarantees that the family $(\mathrm{e}^{-r\tau} R^i(X_\tau))_{\tau\in\mathcal{T}}$ is uniformly integrable. A1–A2 imply the useful growth property

$$\lim_{x\to\alpha^+} \frac{R^i(x)}{\phi(x)} = \lim_{x\to\beta^-} \frac{R^i(x)}{\psi(x)} = 0. \tag{9}$$

A3 intuitively captures the idea that, as long as the market conditions remain in the portion (α, x_0^i) of the state space, the gains from staying in the market decline if no player has conceded yet. This guarantees that the optimal stopping region $\{x \in \mathcal{I} : V_{R^i}(x) = R^i(x)\}$ for problem (3) is of the form $(\alpha, x_{R^i}]$ for some threshold $x_{R^i} < x_0^i$, so that

$$V_{R^i}(x) = \begin{cases} R^i(x) & \text{if} \quad x \leq x_{R^i}, \\ \frac{\phi(x)}{\phi(x_{R^i})} R^i(x_{R^i}) & \text{if} \quad x > x_{R^i}. \end{cases} \tag{10}$$

The smooth-fit property applies at x_{R^i}, that is, $R^{i\prime}(x_{R^i}) = \frac{\phi'(x_{R^i})}{\phi(x_{R^i})} R^i(x_{R^i})$ (Peskir and Shiryaev (2006), Dayanik and Karatzas (2003, Corollary 7.1)). It follows from standard optimal stopping theory that $(\mathrm{e}^{-rt}V_{R^i}(X_t))_{t\geq 0}$ is a supermartingale and that $\mathcal{L}V_{R^i} - rV_{R^i} \leq 0$ over $\mathcal{I} \setminus \{x_{R^i}\}$. The following lemma holds.

Lemma 1 *$V_{R^i} > 0$ over $\mathcal{I}$ and $R^i > 0$ over $(\alpha, x_{R^i}]$.*

We assume that $G^i \in \mathcal{C}^1(\mathcal{I})$, that G^i is piecewise $\mathcal{C}^2$ over $\mathcal{I}$, and that it satisfies

A4 For each $x \in \mathcal{I}$, $\mathbf{E}_x[\sup_{t\geq 0} \mathrm{e}^{-rt}G^i(X_t)] < \infty$.

A5 For each $x \in \mathcal{I}$, $\lim_{t\to\infty} \mathrm{e}^{-rt}G^i(X_t) = 0$, $\mathbf{P}_x$-almost surely.

A6 $G^i \geq V_{R^i}$ over $\mathcal{I}$ and $G^i(x) > V_{R^i}(x)$ if and only if $x > \alpha^i$ for some $\alpha^i < x_{R^i}$.

A7 $\mathcal{L}G^i - rG^i \leq 0$ everywhere $G^{i\prime\prime}$ is defined.

The interpretation of A7 is that player i would rather obtain the payoff $G^i(X_t)$ sooner than later. This is the case, for instance, when G^i is the value function of an ulterior optimal stopping problem faced by the winner of the war of attrition. From (3) and A6–A7, we have $G^i > R^i \vee 0$ over $\mathcal{I}$, so that, by Lemma 1, $G^i > 0$ over $\mathcal{I}$; hence A4 guarantees that the family $(\mathrm{e}^{-r\tau}G^i(X_\tau))_{\tau\in\mathcal{T}}$ is uniformly integrable. A4–A5 imply the useful growth property

$$\lim_{x\to\alpha^+} \frac{G^i(x)}{\phi(x)} = \lim_{x\to\beta^-} \frac{G^i(x)}{\psi(x)} = 0. \tag{11}$$

2.3 A Running Example: Exit in Duopoly

Consider the following model of exit in duopoly, in the spirit of Murto (2004) or Giorgiadis, Kim, and Kwon (2022). Two firms are initially present on the market. As long as both firms remain in the market, each earns a flow duopoly profit X_t, where X follows a geometric Brownian motion with drift $b < r$ and volatility σ,

$$\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \quad t \geq 0,$$

over the state space $\mathcal{I} \equiv (0, \infty)$. If firm i concedes at time τ^i, then its assets are liquidated for a value $l^i > 0$, while firm j enjoys from time τ^i on a flow monopoly profit mX_t for some $m > 1$, until it in turn decides to exit the market and receive its liquidation value $l^j > 0$. Thus the expected discounted profit of every firm i for given exit times τ^i and τ^j is

$$F^i(x, \tau^1, \tau^2) \equiv \mathbf{E}_x\left[\int_0^{\tau_1\wedge\tau_2} \mathrm{e}^{-rt}X_t\,\mathrm{d}t + 1_{\{\tau^i\leq\tau^j\}}\,\mathrm{e}^{-r\tau^i}l^i + 1_{\{\tau^i>\tau^j\}}\,\mathrm{e}^{-r\tau_j}V^i_m(X_{\tau^j})\right],$$

where V_m^i is firm i's value function as a monopolist,

$$V_m^i(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x \left[\int_0^\tau \mathrm{e}^{-rt} m X_t \, \mathrm{d}t + \mathrm{e}^{-r\tau} l^i \right].$$

Letting $E(x) \equiv \mathbf{E}_x\big[\int_0^\infty \mathrm{e}^{-rt} X_t \, \mathrm{d}t\big] = \frac{x}{r-b}$, $R^i \equiv l^i - E$, and $G^i \equiv V_m^i - E$, we obtain the expression (2) for $J^i(\cdot, \tau^1, \tau^2) \equiv F^i(\cdot, \tau^1, \tau^2) - E$. Standard computations (see, for instance, Dixit and Pindyck (1994)) yield

$$x_{R^i} = \frac{\rho^-}{\rho^- - 1} (r-b) l^i \quad \text{and} \quad \alpha^i = \frac{x_{R^i}}{m},$$

where

$$\rho^- \equiv \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Notice that $G^i(x) = R^i(x) = l^i - E(x)$ for all $x \in (\alpha, \alpha^i]$. It is easy to check that this specification satisfies A1–A7. We will use it in Section 4 to illustrate our results.

3 Mixed Strategies and Equilibrium Concept

Our key methodological contribution is to allow players to play randomized stopping times. We first recall the definition and basic properties of randomized stopping times. Imposing a Markov restriction leads to our first main result, which is a representation theorem for Markov randomized stopping times. We then define the concept of Markov-perfect equilibrium and give some important properties of best replies.

3.1 Randomized Stopping Times

One classical definition of a randomized stopping time consists, following Aumann (1964), in enlarging the probability space; this compensates for the absence of a natural measurable structure over the space of stopping times. For every player $i = 1, 2$, the corresponding enlarged probability space is $(\Omega^i, \mathcal{F}^i) \equiv (\Omega \times [0,1], \mathcal{F} \otimes \mathcal{B}([0,1]))$, endowed with the product probability $\mathbf{P}_x^i \equiv \mathbf{P}_x \otimes \mathit{Leb}$, where Leb denotes Lebesgue measure. We borrow the following definition from Touzi and Vieille (2002).

Definition 1 *A* randomized stopping time *for player $i = 1, 2$ is a $\mathcal{F} \otimes \mathcal{B}([0,1])$-measurable function $\gamma^i : \Omega^i \to \mathbb{R}_+$ such that, for Leb-almost every $u^i \in [0,1]$, $\gamma^i(\cdot, u^i) \in \mathcal{T}$. The process $\Gamma^i \equiv (\Gamma_t^i)_{t \geq 0}$ defined by*

$$\Gamma_t^i(\omega) \equiv \int_{[0,1]} 1_{\{\gamma^i(\omega, u^i) \leq t\}} \, \mathrm{d}u^i, \quad (\omega, t) \in \Omega \times \mathbb{R}_+, \tag{12}$$

is the conditional cumulative distribution function *(ccdf) of the randomized stopping time* γ^i*. The process* $\Lambda^i \equiv (\Lambda^i_t)_{t\geq 0}$ *defined by*

$$\Lambda^i_t(\omega) \equiv 1 - \Gamma^i_t(\omega), \quad (\omega, t) \in \Omega \times \mathbb{R}_+, \tag{13}$$

is the conditional survival function *(csf) of the randomized stopping time* γ^i*.*

It is immediate that the ccdf process Γ^i defined by (12) takes values in $[0, 1]$ and has nondecreasing and right-continuous trajectories. The following lemma shows that the process Γ^i is adapted and provides a useful representation.

Lemma 2 *The ccdf process* Γ^i *is* $(\mathcal{F}_t)_{t\geq 0}$*-adapted and, for* $\mathbf{P}_x$*-almost every* $\omega \in \Omega$*,*

$$\Gamma^i_t(\omega) = \mathbf{P}^i_x[\gamma^i \leq t \,|\, \mathcal{F}_t](\omega) \tag{14}$$

for all $x \in \mathcal{I}$ *and* $t \geq 0$*.*

We adopt the convention $\Gamma^i_{0-} \equiv 0$. This allows us in what follows to interpret integrals of the form $\int_{[0,\tau)} \cdot \, \mathrm{d}\Gamma^i_t$ in the Stieltjes sense for any ccdf Γ^i.

If the players use randomized stopping times γ^1 and γ^2, then their expected payoffs are defined over the product probability space $\Omega \times [0, 1] \times [0, 1]$ with canonical element (ω, u^1, u^2) endowed with the product probability $\overline{\mathbf{P}}_x \equiv \mathbf{P}_x \otimes Leb \otimes Leb$ by

$$J^i(x, \gamma^1, \gamma^2) \equiv \overline{\mathbf{E}}_x \Big[1_{\{\gamma^i \leq \gamma^j\}} \, \mathrm{e}^{-r\gamma^i} R^i(X_{\gamma^i}) + 1_{\{\gamma^i > \gamma^j\}} \, \mathrm{e}^{-r\gamma^j} G^i(X_{\gamma^j})\Big], \tag{15}$$

where $\gamma^1 \equiv \gamma^1(\omega, u^1)$ and $\gamma^2 \equiv \gamma^2(\omega, u^2)$, reflecting that player 1 and player 2 use the independent randomization devices u^1 and u^2, respectively.

Our next result shows that we may equivalently work with the family of ccdf processes Γ^i. As similar results appear elsewhere in the literature (Touzi and Vieille (2002), Riedel and Steg (2017)), its proof is relegated to Appendix A.

Lemma 3 *If the players use randomized stopping times with ccdf* Γ^1 *and* Γ^2*, then their expected payoffs write as*

$$J^i(x, \Gamma^1, \Gamma^2) = \mathbf{E}_x \left[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t) \Lambda^j_{t-} \, \mathrm{d}\Gamma^i_t + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t) \Lambda^i_t \, \mathrm{d}\Gamma^j_t\right]. \tag{16}$$

Moreover, any nondecreasing, right-continuous, $\mathcal{F}_t$*-adapted,* $[0, 1]$*-valued process* Γ^i *is the ccdf of the randomized stopping time* $\hat{\gamma}^i$ *defined by*

$$\hat{\gamma}^i(u^i) \equiv \inf\{t \geq 0 : \Gamma^i_t > u^i\}. \tag{17}$$

3.2 Markovian Randomized Stopping Times

Our goal in this paper is to characterize equilibria in which players concede according to Markov randomized strategies that only depend on current market conditions. Notice that such strategies have to be defined for any initial market conditions $x \in \mathcal{I}$. As a preliminary, we need the following standard definition (Revuz and Yor (1999, Chapter I, §3)).

Definition 2 *Let $Y \equiv (Y_t)_{t \geq 0}$ be the* coordinate process *over the canonical space Ω, defined by $Y_t(\omega) \equiv \omega_t$ for all $\omega \in \Omega$ and $t \geq 0$. Then, for each $t \geq 0$, the* shift operator *$\theta_t : \Omega \to \Omega$ is defined by $Y_s \circ \theta_t \equiv Y_{s+t}$ for all $s \geq 0$.*

In words, the effect of θ_t on a trajectory ω is to forget the part of the trajectory prior to time t and to shift back the remaining part by t units of time. We are now ready to define our notion of a Markovian randomized stopping time.

Definition 3 *A randomized stopping time for player $i = 1, 2$ with csf $\Lambda^i : \Omega \times \mathbb{R}_+ \to [0, 1]$ is* Markovian *if, for $\mathbf{P}_x$-almost every $\omega \in \Omega$ and for all $x \in \mathcal{I}$, $\tau \in \mathcal{T}$, and $s \geq 0$,*

$$\tau(\omega) < \infty \quad \textit{implies} \quad \Lambda^i_{\tau(\omega)+s}(\omega) = \Lambda^i_{\tau(\omega)}(\omega)\Lambda^i_s(\theta_{\tau(\omega)}(\omega)) \tag{18}$$

or, more compactly, $\Lambda^i_{\tau+s} = \Lambda^i_\tau \cdot \Lambda^i_s \circ \theta_\tau$ over the event $\{\tau < \infty\}$.

Definition 3 can be intuitively understood as follows. According to Definition 1 and Lemma 2, $\Lambda^i_{\tau+s}$ is the probability that player i concedes after time $\tau + s$ conditionally on $\mathcal{F}_{\tau+s}$. The Markov restriction then states that, at time τ, and conditionally on the fact that player i did not concede by then, the probability that he holds fast for at least s additional units of time should not depend on the trajectory prior to time τ. This probability is thus given by $\Lambda^i_s \circ \theta_\tau$, that is, the probability induced by the randomized strategy applied to the shifted trajectory. Formula (18) then follows from the standard formula for conditional probabilities.

Processes satisfying (18) are known as *strongly Markovian multiplicative functionals* of the Markov process X and are studied in the literature on general Markov processes (Blumenthal and Getoor (1968)). Combining a result by Sharpe (1971) with the classical representation result of additive functionals of regular diffusions (Borodin and Salminen (2002, Chapter II, Section 4, §23)), we can deduce the following representation result for Markovian randomized stopping times.

Theorem 1 *For any Markovian randomized stopping time for player $i = 1, 2$ with csf $\Lambda^i : \Omega \times \mathbb{R}_+ \to [0, 1]$, there exists a closed set $S^i \subset \mathcal{I}$ and a Radon measure*[4] *μ^i over $\mathcal{I} \setminus S^i$ such that, for each $x \in \mathcal{I}$, and for $\mathbf{P}_x$-almost every $\omega \in \Omega$ and each $t \geq 0$,*

$$\Lambda_t^i(\omega) = \mathrm{e}^{-\int_{\mathcal{I}} L_t^y(\omega)\, \mu^i(\mathrm{d}y)}\, 1_{\{\tau_{S^i}(\omega) > t\}}, \tag{19}$$

where

$$L_t^y \equiv \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(y-\varepsilon, y+\varepsilon)}(X_s) \sigma^2(X_s)\, \mathrm{d}s, \tag{20}$$

is the local time of X at (y, t), and

$$\tau_{S^i} \equiv \inf \{t \geq 0 : X_t \in S\}$$

is the hitting time of S^i by X. In particular, the mapping $t \mapsto \Lambda_t^i(\omega)$ is continuous over $[0, \tau_{S^i}(\omega))$ for $\mathbf{P}_x$-almost every $\omega \in \Omega$.

The interpretation of (19) is that player i concedes with probability 1 over S^i, and with positive but finite intensity over $\operatorname{supp} \mu^i$. The relation (19) allows us in the following to indifferently refer to a Markov strategy for player i as a ccdf Γ^i, a csf Λ^i, or a pair (μ^i, S^i); we shall use these notations interchangeably in the definition of players' payoffs. Three cases of the representation (19) are worth mentioning.

The Pure Stopping Case If $\mu^i \equiv 0$, then the Markov strategy $(0, S^i)$ is just the pure stopping time τ_{S^i}. This is the class of Markov strategies considered by Murto (2004).

The Absolutely Continuous Case If $\mu^i \equiv g^i \cdot Leb$ is absolutely continuous with density g^i with respect to Lebesgue measure, then, using the occupation time formula (Revuz and Yor (1999, Chapter VI, §1, Corollary 1.6)), the corresponding csf writes as

$$\Lambda_t^i = \mathrm{e}^{-\int_{\mathcal{I}} L_t^y g^i(y)\, \mathrm{d}y}\, 1_{\{\tau_{S^i} > t\}} = \mathrm{e}^{-\int_0^t g^i(X_s)\sigma^2(X_s)\, \mathrm{d}s}\, 1_{\{\tau_{S^i} > t\}}. \tag{21}$$

Outside S^i, this strategy consists for player i in conceding according to a Poisson process with stochastic intensity $\lambda^i(X_t) \equiv g^i(X_t)\sigma^2(X_t)$; that is, during a short time interval $[t, t + \mathrm{d}t)$, he concedes with probability 1 if $X_t \in S^i$ and with probability $\lambda^i(X_t)\, \mathrm{d}t$ otherwise. This is the class of Markov strategies considered by Giorgiadis, Kim, and Kwon (2022).

[4] Recall that a Radon measure over an open set $U \subset \mathbb{R}$ is a nonnegative Borel measure that is locally finite in the sense that every point of U has a neighborhood having finite μ-measure.

The Singular Case If $\mu^i \equiv a^i \delta_{x^i}$, where $a^i > 0$ and δ_{x^i} is the Dirac mass at $x^i \in \mathcal{I} \setminus S$, then the corresponding csf writes as

$$\Lambda_t^i = \mathrm{e}^{-a^i L_t^{x^i}} 1_{\{\tau_{S^i} > t\}}. \tag{22}$$

In particular, the mapping $t \mapsto \Lambda_t^i(\omega)$ is singular over $[0, \tau_{S^i}(\omega))$ for $\mathbf{P}_x$-almost every $\omega \in \Omega$ such that the trajectory of X crosses x^i; that is, its derivative is zero for *Leb*-almost every $t \in [0, \tau_{S^i}(\omega))$, though $\Lambda_t^i(\omega)$ is not constant as it decreases each time X crosses x^i. To the best of our knowledge, Markov strategies with singular csf have not been considered in the literature. Yet there is no reason to discard such strategies, as they naturally emerge as limits of more familiar ones. Here are two illustrations:

(i) First, discretize the state space (and possibly the time space) and consider Markov strategies for player i prescribing him to concede with positive intensity when the current state is x^i. Then, with appropriate normalizations, the natural limit of such strategies when the mesh of the discretization goes to 0 corresponds to a distribution with hazard rate proportional to the local time of the diffusion at x^i.

(ii) Second, consider the Markov strategy that, outside S^i, consists for player i in conceding according to a Poisson process with stochastic intensity $\lambda_\varepsilon^i(X_t) \equiv \frac{a^i}{2\varepsilon}\, \sigma^2(X_t)\, 1_{(x^i-\varepsilon, x^i+\varepsilon)}$ for $a^i > 0$ and some small $\varepsilon > 0$. By (21), the corresponding csf writes as

$$\Lambda_{\varepsilon,t}^i = \mathrm{e}^{-\int_0^t \lambda_\varepsilon^i(X_s)\,\mathrm{d}s} 1_{\{\tau_{S^i} > t\}}.$$

From the definition (20) of the local time $L_t^{x^i}$ of X at (x^i, t), we deduce that, for each $t \geq 0$, $\Lambda_{\varepsilon,t}^i$ converges $\mathbf{P}_x$-almost surely to Λ_t^i in (22) as ε goes to 0.

Let us finally mention an important property of a Markov strategy, such as (22), associated to a singular measure with an atom at x^i. Using the properties of the local time, one can check[5] that the total probability of conceding before time t starting from x^i is of order $\sqrt{t}$, whereas the same quantity is of order t for a Markov strategy, such as (21), associated to an absolutely continuous measure. As we will see in Sections 4–5, this particular singular behavior will create points of nondifferentiability in the players' equilibrium value functions.

3.3 Markov-Perfect Equilibrium and Properties of Best Replies

We are now ready to define our equilibrium concept and to provide some basic properties of best replies. Our first result, which we will repeatedly use in what follows, illustrates the

[5]This be obtained for example by adapting the method used in Lemma 15 in Peskir (2019).

standard fact that a player, given the behavior of his opponent, cannot improve his payoff merely by randomizing over pure strategies.

Lemma 4 *For each $x \in \mathcal{I}$ and for any pair of randomized stopping times with ccdf* (Γ^1, Γ^2),

$$J^1(x, \Gamma^1, \Gamma^2) \leq \sup_{\tau^1 \in \mathcal{T}} J^1(x, \tau^1, \Gamma^2),$$
$$J^2(x, \Gamma^1, \Gamma^2) \leq \sup_{\tau^2 \in \mathcal{T}} J^2(x, \Gamma^1, \tau^2).$$

This motivates the following definition.

Definition 4 *A* Markov-Perfect Equilibrium *(MPE) is a profile of Markov strategies* $((\mu^1, S^1), (\mu^2, S^2))$ *such that, for each $x \in \mathcal{I}$,*

$$J^1(x, (\mu^1, S^1), (\mu^2, S^2)) = \bar{J}^1(x, (\mu^2, S^2)) \equiv \sup_{\tau^1 \in \mathcal{T}} J^1(x, \tau^1, (\mu^2, S^2)),$$
$$J^2(x, (\mu^1, S^1), (\mu^2, S^2)) = \bar{J}^2(x, (\mu^1, S^1)) \equiv \sup_{\tau^2 \in \mathcal{T}} J^2(x, (\mu^1, S^1), \tau^2).$$

That is, for each $i = 1, 2$, (μ^i, S^i) is a perfect best reply *(pbr) for player i to (μ^j, S^j), and $\bar{J}^i(\cdot, (\mu^j, S^j))$ is player i's* best-reply value function *(brvf) to (μ^j, S^j).*

When no confusion can arise as to the strategy of player j, we write $\bar{J}^i$ instead of $\bar{J}^i(\cdot, (\mu^j, S^j))$. The next proposition provides useful general properties of pbr and brvf, and is key to establish our main results.

Proposition 1 *If (μ^i, S^i) is a pbr to (μ^j, S^j) with associated brvf $\bar{J}^i$, then $V_{R^i} \leq \bar{J}^i \leq G^i$. Furthermore,*

(i) $S^1 \cap S^2 \cap (\alpha^i, \beta) = \emptyset$;

(ii) $S^i \subset C^i \equiv \{x \in \mathcal{I} : \bar{J}^i(x) = R^i(x)\}$;

(iii) $\operatorname{supp} \mu^i \subset C^i \cup S^j$;

(iv) $\operatorname{supp} \mu^i \cup S^i \subset (\alpha, x_{R^i}]$;

(v) $(0, S^i)$ *is also a pbr to (μ^j, S^j) and more generally $(\tilde{\mu}^i, S^i)$ is a pbr to (μ^j, S^j) for any $\tilde{\mu}^i$ such that* $\operatorname{supp} \tilde{\mu}^i \subset C^i \cup S^j$;

Property (i) intuitively states that player i should never concede when market conditions x are such that player j concedes with probability 1 and player i's payoff from conceding is strictly less than the payoff from letting player j concede, that is, $x \in S^j$ and $G^i(x) > V_{R^i}(x)$. Property (ii) simply expresses the fact that player i's brvf coincides with R^i over the portion S^i of the state space over which he concedes with probability 1. Property (iii) states that player i's payoff is R^i when he concedes with positive intensity outside of player j's stopping region S^j. Property (iv) reflects that player i should never concede when market conditions are above the optimal threshold x_{R^i} for his stand-alone exit problem; intuitively, this is because waiting for X to drop down to x_{R^i} before conceding is player i's optimal strategy even in the worst-case scenario in which player j is stubborn, that is, $(\mu^j, S^j) = (0, \emptyset)$. Finally, property (v) states that, when conceding with positive intensity outside of S^i, player i should be indifferent between holding fast and conceding.

Remark Some authors (see, for instance, Murto (2004)) include, as a refinement in the definition of an MPE, the requirement that $(\alpha, \alpha^i] \subset S^i$ for all i. The rationale for this assumption is that, because $G^i = V_{R^i} = R^i$ over $(\alpha, \alpha^i]$, holding fast further below α^i would be weakly dominated for player i by conceding with probability 1 over this interval. For instance, being stubborn is a best reply for player i over (α, α^i) only if player j concedes with probability 1 over this interval, except perhaps over a set of Lebesgue measure 0. This behavior is not per se inconsistent with an MPE, but it is not consistent with trembling-hand perfection in the spirit of Selten (1975), see Ghemawat and Nalebuff (1985) for a discussion of a similar point in a deterministic model. Hereafter, we do not systematically impose this refinement, especially in Section 5 where this allows to simplify notation; however, we will indicate which MPEs can be modified so as to satisfy it.

We close this section with an important global regularity result.

Proposition 2 *If $((\mu^1, S^1), (\mu^2, S^2))$ is an MPE, then, for each $i = 1, 2$, player i's brvf $\bar{J}^i$ is continuous over $\mathcal{I}$.*

4 MPEs with Singular Strategies: Heuristics

We first recall within our general framework two standard MPEs, respectively in pure and mixed strategies, that have been emphasized in the literature. Based on these examples and on our representation theorem for Markovian randomized stopping times, we next describe a novel type of MPE involving a singular strategy for one of the two players. Our heuristic presentation leads to a variational system that turns out to fully characterize the candidate equilibrium. We use the running example of Section 2.3 to illustrate our findings.

4.1 A Pure-Strategy MPE

We say that player 1 is *as least as enduring* as player 2 if $\alpha^1 \leq \alpha^2$ and $x_{R^1} \leq x_{R^2}$; intuitively, player 1 is at least as willing to hold fast as player 2. Suppose then that player 1 threatens to hold fast maximally and concede only at $\tau^1 = \inf\{t \geq 0 : X_t \leq \alpha^1\}$. Then, because $\alpha^1 \leq \alpha^2$, we have $G^2(X_{\tau^1}) = R^2(X_{\tau^1})$ by definition of α^2. In light of (2)–(3), this implies that, for all $x \in \mathcal{I}$ and $\tau^2 \in \mathcal{T}$,

$$J^2(x, \tau^1, \tau^2) = \mathbf{E}_x\left[\mathrm{e}^{-r\tau^1\wedge\tau^2} R^2(X_{\tau^1\wedge\tau^2})\right] \leq V_{R^2}(x).$$

Thus a pbr for player 2 to τ^1 is to concede at $\tau^2 = \inf\{t \geq 0 : X_t \leq x_{R^2}\}$. As for player 1, if player 2 concedes at τ^2, then, for each $x \in \mathcal{I}$,

$$\mathbf{E}_x\left[\mathrm{e}^{-r\tau^2} G^1(X_{\tau^2})\right] \geq R^1(x).$$

For $x \leq x_{R^2}$, this follows from the fact that $G^1(x) \geq R^1(x)$ by A6, with a strict inequality if $x > \alpha^1$. For $x > x_{R^2}$, this follows from A6 again along with the fact that the process $(\mathrm{e}^{-rt} V_{R^1}(X_t))_{t\geq 0}$ is a martingale up to $\tau_{x_{R^1}}$, the hitting time of x_{R^1}, which is no less than τ^2 because $x_{R^1} \leq x_{R^2}$ by assumption. Thus a pbr for player 1 to τ^2 is to concede at τ^1. This implies the following result, which has many counterparts in the literature (see, for instance, Ghemawat and Nalebuff (1985), Décamps and Mariotti (2004), Murto (2004), Giorgiadis, Kim, and Kwon (2022)).

Proposition 3 *If player 1 is at least as enduring as player 2, then* $((0, (\alpha, \alpha^1]), (0, (\alpha, x_{R^2}]))$ *is a pure-strategy MPE.*

In the case where the asymmetry between the players is small, $((0, (\alpha, x_{R^1}]), (0, \emptyset))$ is also an MPE in which the more enduring player 1 follows his stand-alone optimal strategy because the less enduring player 2 is stubborn (Giorgiadis, Kim, and Kwon (2022)). However, this MPE does not satisfy Murto's (2004) trembling-hand-perfection refinement, because, for $x \in (\alpha^1, \alpha^2)$, player 2's strategy is no longer a best response when player 1 does not concede with probability 1 in any small enough neighborhood of x. Nevertheless, Murto (2004) shows that, when we allow player 1's stopping set S^1 to exhibit a gap, there may exist an MPE satisfying this refinement in which, when $x > x_{R^1}$, player 1 exits first when X reaches x_{R^1}.

4.2 A Mixed-Strategy MPE in the Symmetric Case

Suppose now that players are symmetric, in the weak sense that they are as enduring as each other, $\alpha^1 = \alpha^2 \equiv \alpha^*$ and $x_{R^1} = x_{R^2} \equiv x^*$. This is of course the case when the players have identical payoff functions, $R^1 = R^2$ and $G^1 = G^2$. The following result, which restates in our framework earlier results in the literature (Steg (2015), Georgiadis, Kim, and Kwon (2022))[6], characterizes a mixed-strategy MPE in which the players concede with absolutely continuous intensities over the interval $(\alpha^*, x^*]$.

Proposition 4 *If the players are as enduring as each other, then the strategy profile*

$$((\lambda^1(x)\sigma^{-2}(x)\,\mathrm{d}x, (\alpha, \alpha^*]), (\lambda^2(x)\sigma^{-2}(x)\,\mathrm{d}x, (\alpha, \alpha^*]))$$

defined, for each $i = 1, 2$, by

$$\lambda^i(x) \equiv \frac{rR^j(x) - \mathcal{L}R^j(x)}{G^j(x) - R^j(x)}\, 1_{\{\alpha^* < x \leq x^*\}}, \tag{23}$$

is a mixed-strategy MPE.

We know from Proposition 1(iv) that $\operatorname{supp} \mu^i \subset (\alpha^*, x^*]$. Following Theorem 5.1 in Steg (2015), the MPE constructed in Proposition 4 is such that each player stops with an intensity function λ^i with support $(\alpha^*, x^*]$. This intensity is constructed so that, at each point of this interval, each player is indifferent between holding fast and conceding. In equilibrium, the value function of each player coincides with the value function of his stand-alone exit problem (3). Thus, in expectation, the war of attrition yields no benefit to either player.

4.3 A Singular Mixed-Strategy MPE

When players are asymmetric but there is no uncertainty about their future payoffs, the war of attrition may admit mixed-strategy equilibria in which players strategies are described, over some interval of exit times, by absolutely continuous distributions (see, for instance, Hendricks, Weiss, and Wilson (1988)); this is certainly the case in the limiting case of our model where $\mu = \sigma = 0$, so that market conditions are constant. This result has no counterpart under Brownian uncertainty. Indeed, Georgiadis, Kim, and Kwon (2022) have shown that the construction of Proposition 4 does not extend to the case of asymmetric players: specifically, when players are not as enduring as each other, there exists no mixed-

[6] A related construction also appears in Kwon and Palczewski (2022). There, a symmetric Bayesian equilibrium is constructed in a model with asymmetric information and a continuum of types. The pure strategies, seen as randomized strategies assimilating the types as randomization devices, use absolutely continuous intensities depending on X and on an auxiliary belief process.

strategy MPE in which the players concede with absolutely continuous intensities. For all that, it would be incorrect to conclude that only pure-strategy MPEs exist, and thus that attrition cannot take place in an MPE of our model. This section argues for this claim by describing an MPE involving a singular strategy for one of the two players. For the sake of simplicity, the analysis below remains at a heuristical level. A full justification of our arguments is provided in Section 5.

From now on, assume as in Section 4.1 that player 1 is at least as enduring as player 2, that is, $\alpha^1 \leq \alpha^2$ and $x_{R^1} \leq x_{R^2}$. Consider then the following equation in x:

$$R^1(x_{R^1}) = \frac{\phi(x_{R^1})}{\phi(x)}\, G^1(x). \tag{24}$$

We show in Appendix B that (24) admits a unique solution $\underline{x}^2 \in (\alpha^1, x_{R^1})$. In words, the threshold $\underline{x}^2$ is such that, if player 2 threatens to concede only at $\tau^2 = \inf\{t \geq 0 : X_t \leq \underline{x}^2\}$, then, at x_{R^1}, player 1 is indifferent between conceding and obtaining $R^1(x_{R^1})$ immediately and waiting for player 2 to concede at τ^2 and obtaining $G^1(\underline{x}^2)$ only then. We claim that, if x_{R^2} is close enough to x_{R^1}, then there exists an MPE in which player 1 randomizes between holding fast and conceding at x_{R^1} and player 2 concedes only at τ^2. Using the representation for randomized stopping times provided in Theorem 1, this amounts to the existence of a constant $a^1 > 0$ such that the profile of strategy $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1)), (0, (\alpha, \underline{x}^2))$ is an MPE in which player 1 concedes, with positive but finite intensity, only at x_{R^1}.

4.3.1 Necessary Conditions

To establish this claim, we first assume that such an MPE exists, and we derive necessary conditions for the brvf $\bar{J}^1$ and $\bar{J}^2$. An obvious preliminary observation is that $\bar{J}^1 \geq R^1$ and $\bar{J}^2 \geq R^2$, because, when current market conditions are x, every player i can always guarantee himself the payoff $R^i(x)$ by exiting the market immediately.

Player 1 Player 1, whose strategy involves randomization at x_{R^1}, should be indifferent at x_{R_1} between conceding and holding fast until τ^2. This implies that his brfv $\bar{J}^1$ must be $\mathcal{C}^2$ over $(\underline{x}^2, \beta)$, with $\bar{J}^1(x_{R^1}) = R^1(x_{R^1})$ (value-matching). Because $\bar{J}^1 \geq R^1$, it follows in turn that $\bar{J}^{1\prime}(x_{R^1}) = R^{1\prime}(x_{R^1})$ as well. Moreover, by standard dynamic-programming arguments, $\bar{J}^1$ must satisfy the ODE $\mathcal{L}\bar{J}_1 - r\bar{J}_1 = 0$ over $(\underline{x}^2, \beta)$ (see, for instance, Dixit and Pindyck (1994)). This leads to

$$\bar{J}^1(x) = \frac{\phi(x)}{\phi(x_{R^1})}\, R^1(x_{R^1}), \quad x \in (\underline{x}^2, \beta). \tag{25}$$

In particular, $\bar{J}^1 = V_{R^1}$ over $[x_{R^1}, \beta)$: player 1 does not benefit from the war of attrition over $[x_{R^1}, \beta)$. By contrast, $\bar{J}^1 > V_{R^1}$ over $[\underline{x}^2, x_{R^1})$, reflecting that player 1 can hope that player 2 may concede at $\underline{x}^2$ before he himself concedes at x_{R^1}.

Player 2 Player 2 plays a pure strategy and hopes to benefit from player 1 conceding at x_{R^1}. We guess that $\bar{J}^2$ is $\mathcal{C}^2$ over $(\underline{x}^2, x_{R^1}) \cup (x_{R^1}, \beta)$, with $\bar{J}^2(\underline{x}^2) = R^2(\underline{x}^2)$ (value-matching) and $\bar{J}^{2\prime}(\underline{x}^2) = R^{2\prime}(\underline{x}^2)$ (smooth pasting), and that it satisfies the ODE $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over that region. There remains to characterize the behavior of $\bar{J}^2$ at x_{R^1}. Because player 1 randomizes at x_{R^1} between holding fast and conceding, we expect that $G^2(x_{R^1}) > \bar{J}^2(x_{R^1}) > R^2(x_{R^1})$. This, along with the properties of the local time highlighted in Section 3.2, implies that $\bar{J}^2$ is not differentiable at x_{R^1}. Indeed, starting from x_{R^1}, player 1 concedes in a small time interval of length $\mathrm{d}t$ with probability $\mathbf{E}_{x_{R^1}}[\Gamma_{\mathrm{d}t}] = a^1 c\sqrt{\mathrm{d}t} + o(\sqrt{\mathrm{d}t})$, where $\Gamma_{\mathrm{d}t} = 1 - \mathrm{e}^{-a^1 L^{x_{R^1}}_{\mathrm{d}t}}$ and c is a positive constant. If player 1 concedes, then player 2 benefits from the follower payoff $G^2(x_{R^1})$, while if player 1 holds fast, player 2 achieves the value $\bar{J}^2(X_{\mathrm{d}t})$. Thus

$$\bar{J}^2(x_{R^1}) = a^1 c\sqrt{\mathrm{d}t}\, G^2(x_{R^1}) + (1 - a^1 c\sqrt{\mathrm{d}t})\, \mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{d}t}\bar{J}^2(X_{dt})] + o(\sqrt{\mathrm{d}t}). \tag{26}$$

Now, suppose, by way of contradiction, that J^2 is $\mathcal{C}^2$ in a neighborhood of x_{R^1}. Then, from Itô's formula,

$$\mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{d}t}\bar{J}^2(X_{\mathrm{d}t})] = \bar{J}^2(x_{R^1}) + (\mathcal{L}\bar{J}^2 - r\bar{J}^2)(x_{R^1})\,\mathrm{d}t + o(\mathrm{d}t). \tag{27}$$

Plugging (27) into (26) yields $a^1 c[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})]\sqrt{dt} + o(\sqrt{\mathrm{d}t}) = 0$, a contradiction as $G^2(x_{R^1}) > \bar{J}^2(x_{R^1})$ and a^1 and c are positive constants. This is an indication that $\bar{J}^2$ is not differentiable at x_{R^1}; let us denote by $\Delta\bar{J}^{2\prime}(x_{R^1}) \equiv \bar{J}^{2\prime+}(x_{R^1}) - \bar{J}^{2\prime-}(x_{R^1})$ the corresponding derivative jump. From the Itô–Tanaka–Meyer formula, which generalizes Itô's formula to functions, such as $\bar{J}^2$, that can be written as the difference of two convex functions (Karatzas and Shreve (1991, Theorem 3.7.1 and Problem 3.6.24)), we have

$$\begin{aligned}\mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{d}t}\bar{J}^2(X_{\mathrm{d}t})] &= \bar{J}^2(x_{R^1}) + \mathbf{E}_{x_{R^1}}\bigg[\int_0^t \mathrm{e}^{-rs}(\mathcal{L}\bar{J}^2 - r\bar{J}^2)(X_s)\,\mathrm{d}s \\ &\qquad + \int_0^t \mathrm{e}^{-rs}\bar{J}^{2\prime-}(X_s)\sigma(X_s)\,\mathrm{d}W_s + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1})L^{x_{R^1}}_{\mathrm{d}t}\bigg] \\ &= \bar{J}^2(x_{R^1}) + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1})c\sqrt{dt} + o(\sqrt{\mathrm{d}t}),\end{aligned} \tag{28}$$

where the second equality follows from the fact that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over $(\underline{x}^2, x_{R^1}) \cup (x_{R^1}, \beta)$ and from the properties of local time. Plugging (28) into (26) yields

$$a^1[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})] + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1}) = 0. \tag{29}$$

Notice from $G^2(x_{R^1}) > J^2(x_{R^1})$ and (29) that $\Delta \bar{J}^{2\prime}(x_{R^1}) < 0$. Intuitively, player 2 gets more and more optimistic as X approaches x_{R^1}, but is disappointed if X crosses x_{R^1} but player 1 does not concede at x_{R^1}.

The Variational System Our discussion so far leads to the following variational system: find a constant $a > 0$, and two functions $w^1 \in \mathcal{C}^0((\alpha,\beta)) \cap \mathcal{C}^2((\alpha,\beta) \setminus \{\underline{x}^1\})$ and $w^2 \in \mathcal{C}^0((\alpha,\beta)) \cap \mathcal{C}^2((\alpha,\beta) \setminus \{\underline{x}^1, x_{R^1}\})$ such that

$$w^1 \geq R^1 \text{ over } (\alpha,\beta), \tag{30}$$
$$\mathcal{L}w^1 - rw^1 = 0 \text{ over } (\underline{x}^2, \beta), \tag{31}$$
$$w^1 = G^1 \text{ over } (\alpha, \underline{x}^2], \tag{32}$$
$$w^1(x_{R^1}) = R^1(x_{R^1}), \tag{33}$$
$$w^1(\beta^-) = 0, \tag{34}$$

$$w^2 \geq R^2 \text{ over } (\alpha,\beta), \tag{35}$$
$$\mathcal{L}w^2 - rw^2 = 0 \text{ over } (\underline{x}^2, x_{R^1}) \cup (x_{R^1}, \beta), \tag{36}$$
$$w^2(\underline{x}^2) = R^2(\underline{x}^2), \tag{37}$$
$$w^{2\prime}(\underline{x}^2) = R^{2\prime}(\underline{x}^2), \tag{38}$$
$$a[G^2(x_{R^1}) - w^2(x_{R^1})] + \frac{1}{2}\,\Delta w^{2\prime}(x_{R^1}) = 0, \tag{39}$$
$$w^2(\beta^-) = 0. \tag{40}$$

4.3.2 Sufficient Conditions

It is an implication of our main characterization result, Theorem 4, that, if $(a^1, \bar{J}^1, \bar{J}^2)$ is a solution to the variational system (30)–(40), then $\bar{J}^1$ is the brfv to $(0, (\alpha, \underline{x}^2))$ and w^2 is the brfv to $(a^1\delta_{x_{R^1}}, (\alpha, \alpha^1))$, so that, according to the construction in Section 4.3.1, $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1)), (0, (\alpha, \underline{x}^2))$ is an MPE. As for $\bar{J}^1$, we have already seen that (30)–(31) and (33) pin down a unique solution, given by (25), which satisfies $\bar{J}^1(\underline{x}^2) = G^1(\underline{x}^2)$ by definition of $\underline{x}^2$. As for $\bar{J}^2$, the analysis is a bit more delicate due to the presence of the derivative jump $\Delta\bar{J}^{2\prime}(x_{R^1})$ at x_{R^1}, which, by (39), is pinned down by the intensity a^1 with which player 1 exits at x_{R^1}. In our running example, it can be shown that, as long as the asymmetry between the players is small, assuming that $b > 0$ and m is sufficiently large, one can indeed find a positive value for a^1 such that (35)–(40) holds. The following result then holds[7].

[7]A numerical study that we performed suggests that, for firms' liquidation values $l^1 \leq l^2$ close to each other, the variational systems (30)-(34) and (35)-(40) admit a solution whatever the parameter values of the

Proposition 5 *In the running example, if the firms' liquidation values* $l^1 \leq l^2$ *are close to each other,* $b > 0$ *and* m *is sufficiently large, then there exists a mixed-strategy MPE* $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1)), (0, (\alpha, \underline{x}^2))$ *in which the more enduring firm 1 randomizes between holding fast and conceding at* x_{R^1} *while the less enduring firm 2 exits with probability 1 as soon as market conditions fall below* $\underline{x}^2 < x_{R^1}$*.*

The MPE constructed in Proposition 5 differs from the pure-strategy MPE of Proposition 3 in that, for $x \geq x_{R^1}$, it is the more enduring enduring player 1 who does not benefit from the war of attrition, in the sense that $\bar{J}^1 = V_{R^1}$ over $[x_{R^1}, \beta)$; by contrast, $\bar{J}^2 > V_{R^2}$ over this portion of the state space. The reason is that player 2 adopts a tougher stance by threatening to exit the market only at $\underline{x}^2 < x_{R^1} < x_{R^2}$, which makes player 1 indifferent between holding fast and conceding at x_{R^1}. By construction, this MPE satisfies the requirement that $(\alpha, \alpha^i] \in S^i$ for every player i, as in Murto (2004).

It should also be noted that, in this singular mixed-strategy MPE, we have $\max S^1 \vee \max S^2 = \underline{x}^2 < x_{R^1} < x_{R^2}$. This contrasts with pure-strategy MPEs, in which one always have $\max S^1 \vee \max S^2 \in \{x_{R^1}, x_{R^2}\}$. Thus mixing by player 1 delays the time at which a firm must necessarily exit the market. In particular, the difference with the pure-strategy MPE characterized by Murto (2004), in which the stopping set S^1 of player 1 exhibits a gap below $\max S^1 = x_{R^1}$ and player 1 is the first to exit the market at x_{R^1} when $x > x_{R^1}$, is that player 1 does not exit with probability 1 at x_{R^1}. This leads to a richer dynamics, whereby, on the equilibrium path, every player i can alternate between being in a dominated position (with a value close to V_{R^i}) or in a dominant position (with a value significantly above V_{R^i}); specifically, player 1 is in a dominant position when X is close to $\underline{x}^2$, while player 2 is in a dominant position when X is close to x_{R^1}. As we show in Theorem 3, this alternation phenomenon is a robust feature of any mixed-strategy MPE.

Figure 1 illustrates this point in the running example by plotting the firms' market-value functions $F^i \equiv \bar{J}^i + E$ in the singular mixed-strategy MPE constructed in Proposition 5. Notice that firm 1's value function coincides with its monopolist value function V_m^1 over $(\alpha, \underline{x}_2]$, because player 2 exits the market with probability 1 at any point of this interval. It can also be checked that $F^{2\prime-}(x_{R^1}) > 0 > F^{2\prime+}(x_{R^1})$, reflecting that player 2's market value reaches a local maximum when $X_t = x_{R^1}$. Finally, a testable implication of this MPE, which is also apparent from Figure 1, is that, along any path of X, the firms' market values move in opposite directions as long as none of them exits the market and market conditions do

model if $b > 0$, and for m in some compact interval $[1, C]$ if $b < 0$ for some constant C which increases with σ.

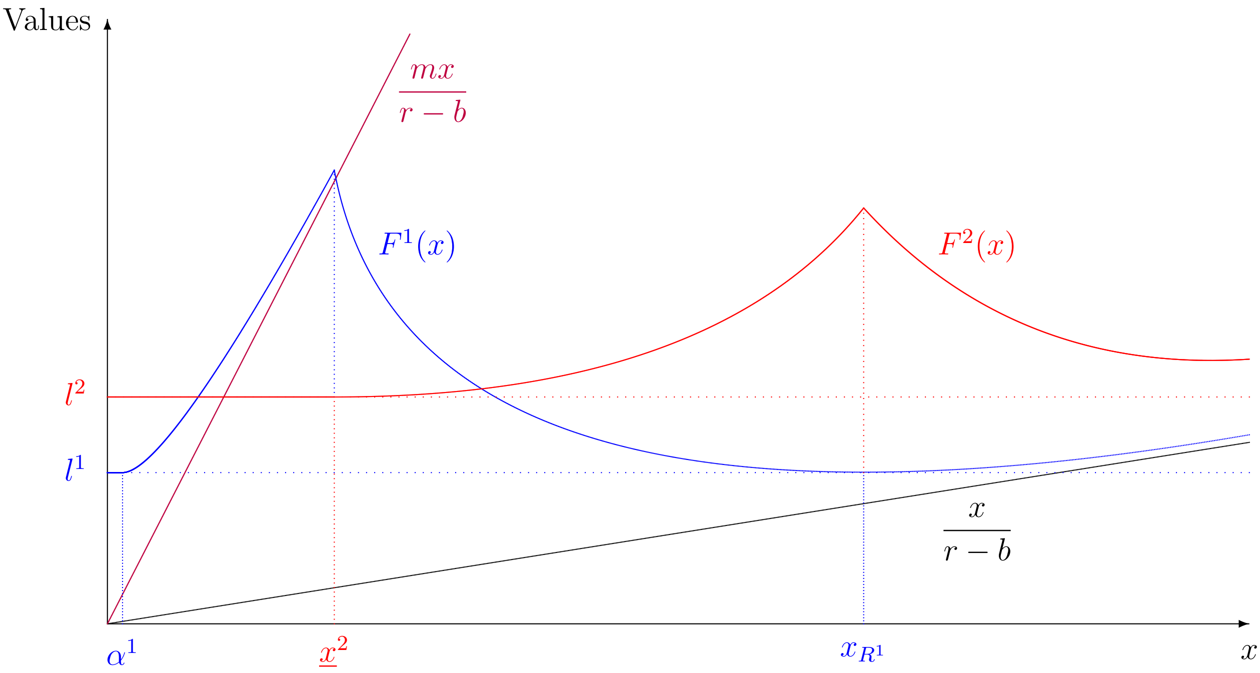

Figure 1: The value of never exiting the market in a duopoly (in black), the value of never exiting the market in a monopoly (in purple), and the more enduring firm 1's value (in blue) and the less ensuring firm 2's value (in red) in the singular mixed-strategy MPE.

not wander too much above x_{R^1}. Again, this negative comovement of firms' market values is a robust feature of any mixed-strategy MPE.

Our analysis also has novel welfare implications. As already noted, because firm 1 does not exit with probability 1 at x_{R^1}, firm 2's market value at x_{R^1}, $F^2(x_{R^1})$, must be less than its value as a monopolist, $V^2_m(x_{R^1})$. Because firm 1's market value satisfies $F^1 = V_{R^1} + E$ over $[x_{R^1}, \infty)$ and thus coincides with his stand-alone market value, this implies that, for $x > x_{R^1}$, the mixed-strategy MPE characterized in Proposition 5 is ex-ante Pareto dominated by the pure-strategy MPE $((0, (\alpha, x_{R^1}]), (0, \emptyset))$. Thus, even when firms have asymmetric liquidation values, wasteful attrition takes place with positive probability on the equilibrium path, in contrast with the conclusion of Georgiadis, Kim, and Kwon (2020).

5 Main Results

This section presents our main results, which extend Proposition 5 to our general model. We first provide a necessary condition for mixed-strategy MPEs in the case of asymmetric players. We then characterize such equilibria by a variational system satisfied by the two players' value functions.

5.1 The Alternating Structure of Mixed-Strategy MPEs

The proofs of our main results require an additional regularity assumption, which we will throughout maintain in what follows.

A8 The functions b, σ, and $R^{i\prime\prime}$ are locally Lipschitz.

By convention, let us set $\max \emptyset \equiv \alpha$ and, for any MPE $((\mu^1, S^1), (\mu^2, S^2))$, let $s^i \equiv \max S^i$. The following theorem then holds.

Theorem 2 *Suppose that $x_{R^1} \neq x_{R^2}$. Then, for any mixed-strategy MPE $((\mu^1, S^1), (\mu^2, S^2))$, the restrictions of the intensity measures μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$ are purely atomic.*

Theorem 2 confirms the basic insight of Georgiadis, Kim, and Kwon (2022), according to which there exists no mixed-strategy MPE with absolutely continuous intensity measures when $x_{R^1} \neq x_{R^2}$. Thus, if a mixed-strategy MPE exists at all, it must feature intensity measures that are singular with respect to Lebesgue measure. The additional information provided by Theorem 2 is that these measures must be discrete, which, for instance, rules out intensity measures with Cantor-set supports. The proof can be sketched as follows. Let us consider a mixed-strategy MPE $((\mu^1, S^1), (\mu^2, S^2))$, supposing one exists. Property (v) of Proposition 1 and dynamic-programming arguments imply that the brvf $\bar{J}^i$ of every player i satisfies the ODE $\mathcal{L}u - ru = 0$ over any interval where player j does not concede. It also follows from Proposition 1 that $\bar{J}^i \geq V_{R^i}$ and that $\bar{J}^i(q^i) = R^i(q^i)$ for all $q^i \in \operatorname{supp} \mu^i$. These observations imply that any point $q^i > s^1 \vee s^2$ in the support of μ^i is isolated in this set. Thus the only potential accumulation point of the support of μ^i is $s^1 \vee s^2$; it turns out that in this case $s^1 \vee s^2 = \alpha$, so that $S^1 = S^2 = \emptyset$. (This is where A8 is needed.) In turn, over an open interval around any point $q^i > s^1 \vee s^2$ in the support of μ^i, $\bar{J}^i$ coincides with the solution $T^i_{q^i}$ to the ODE $\mathcal{L}u - ru = 0$ that is tangent to R^i at q^i. At any such point q^i, player i is indifferent between holding fast and conceding.

Because the brvf of both players must satisfy these properties, a lot of structure is thereby induced on the supports of μ^1 and μ^2. First, it is easy to see that, because $\bar{J}^i \geq V_{R^i}$, we must have $\max \operatorname{supp} \mu^i \leq x_{R^i}$ for every player i. To fix ideas, let us suppose that $q^1_1 \equiv \max \operatorname{supp} \mu^1 \geq \max \operatorname{supp} \mu^2 \equiv q^2_1$. We can then show that in fact $x_{R^1} = q^1_1 > q^2_1$. It also follows that, for each $i = 1, 2$, and for any two consecutive points $q^i_n > q^i_{n+1} > s^1 \vee s^2$ in the support of μ^i, there must exists a single point $q^j \in (q^i_{n+1}, q^i_n)$ in the support of μ^j at which player j is indifferent between conceding or holding fast; as discussed above, this implies that $\bar{J}^j = T^j_{q^j}$ over (q^i_{n+1}, q^i_n). We thus obtain two decreasing sequences of randomization thresholds $(q_n^1)_{n=1}^{N^1}$ and $(q_n^2)_{n=1}^{N^2}$, with either $N^1 = N^2 = \infty$ or $0 \leq N^1 - N^2 \leq 1$, which are intertwined in the sense that $q_1^1 > q_1^2 > q_2^1 > q_2^2 > \ldots$ as long as these thresholds are defined, and which satisfy $\lim_{n\to\infty} q_n^1 = \lim_{n\to\infty} q_n^2 = \alpha$ if $N^1 = N^2 = \infty$. These two sequences characterize the restrictions of μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$. Thus any mixed-strategy MPE must fall into one of three categories.

Corollary 1 *Suppose that $x_{R^1} \neq x_{R^2}$, and let $((\mu^1, S^1), (\mu^2, S^2))$ be a mixed-strategy MPE such that, for every player i, $\operatorname{supp}\mu^i \cap (s^1 \vee s^2, \beta) = \{q_n^i : n = 1, \ldots, N^i\}$ for intertwined decreasing sequences of randomization thresholds $(q_n^1)_{n=1}^{N^1}$ and $(q_n^2)_{n=1}^{N^2}$ satisfying, with no loss of generality, $q_1^1 > q_1^2$. Then $q_1^1 = x_{R^1}$ and*

1. *if $N^1 = N^2 \equiv N \in \mathbb{N} \setminus \{0\}$, then $q_N^1 > q_N^2 > s^1 > s^2$;*
2. *if $N^1 = N^2 + 1 \equiv N \in \mathbb{N} \setminus \{0\}$, then $q_{N-1}^2 > q_N^1 > s^2 > s^1$, with $q_0^2 \equiv \beta$ by convention;*
3. *if $N^1 = N^2 = \infty$, then $\lim_{n\to\infty} q_n^1 = \lim_{n\to\infty} q_n^2 = s^1 = s^2 = \alpha$, so that $S^1 = S^2 = \emptyset$.*

In an MPE of type 1, player 1 exits the market with probability 1 at s^1, and player 2 has the lowest randomization threshold. In an MPE of type 2, player 1 has the lowest randomization threshold, and player 2 exits the market with probability 1 at s^2—the example of Section 4.3 is a case in point, with $N^1 = 1$ and $N^2 = 0$. In an MPE of type 3, neither player exits the market with probability 1 at any point of the state space, and players keep randomizing all the way down to α. It should be noted that an equilibrium of type 3 can exist only if $\alpha^1 = \alpha^2 = \alpha$; indeed, every player i such that $\alpha^i > \alpha$ would not be willing to delay exiting the market over (α, α^i) if his opponent were to do the same.

The upshot from Theorem 2 and Corollary 1 is that, when players have different stand-alone optimal exit thresholds, alternation is a robust feature of any mixed-strategy MPE, which generalizes the insights from Section 4.3. In the attrition region, players randomize between conceding and holding fast at isolated thresholds. In an equilibrium of type 1 and type 2, this process may persist until one player eventually reaches his stopping region and exits the market with probability 1. By contrast, in an equilibrium of type 3, exit must take place at some randomization threshold.

Corollary 1 fully characterizes equilibrium outcomes for an MPE of type 3, because any market conditions in $\mathcal{I}$ can be reached with positive probability from any initial market conditions $x \in \mathcal{I}$. The same holds true for MPEs of types 1 and 2, provided $x > x_{R^1}$ (with the convention that $q_1^1 > q_1^2$). Indeed, for any such MPE $((\mu^1, S^1), (\mu^2, S^2))$ and for each $x > x_{R^1}$, there exists an outcome-equivalent MPE $((\tilde{\mu}^1, \tilde{S}^1), (\tilde{\mu}^2, \tilde{S}^2))$ such that

$\text{supp}\,\tilde{\mu}^i = \text{supp}\,\mu^i \cap (s^1 \vee s^2, \beta)$ for every player i and $\tilde{S}^1 = (\alpha, s^1)$ and $\tilde{S}^2 = \emptyset$ (for an equilibrium of type 1), or $\tilde{S}^1 = \emptyset$ and $\tilde{S}^2 = (\alpha, s^2)$ (for an equilibrium of type 2).

By contrast, Corollary 1 does not determine equilibrium outcomes of MPEs of types 1 and 2 for lower initial market conditions. First, as in Murto (2004), it may be possible to construct MPEs in which the stopping regions S^1 and S^2 exhibit gaps. Second, these gaps may themselves include randomization thresholds at which players exit the market with positive but finite intensity.

5.2 The Characterization Result

The next Theorem gives a necessary and sufficient condition for the existence of an equilibrium of type 2. Similar results can be written for equilibria of types 1 and 3.

Theorem 3 *Let us consider two sequences of positive real numbers* $(q_i^1)_{1\leq i\leq n}$ *and* $(q_i^2)_{1\leq i\leq n-1}$ *and* $s^2 \in \mathcal{I}$ *with* $q_1^1 = x_{R^1} > q_1^2 > q_2^1 > \ldots > q_{n-1}^2 > q_n^1 > s^2$ *and two sequences of positive real numbers* $(a_i)_{1\leq i\leq n}$ *and* $(b_i)_{1\leq i\leq n-1}$. *Then, the strategy profile* $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{i=1}^n a_i\delta_{q_i^1}, \emptyset),\ (\sum_{i=1}^{n-1} b_i\delta_{q_i^2}, (\alpha, s^2]))$ *is a mixed-strategy MPE if and only if there exists a pair* (w^1, w^2) *solution to the variational systems* $\mathcal{VS}^1$ *and* $\mathcal{VS}^2$ *below.*

Precisely:

(i) $w^1 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{(q_i^2)_{1\leq i\leq n-1}, s^2\})$ *is solution to* $\mathcal{VS}^1$ *if:*

$$\begin{aligned}
\mathcal{L}w^1 - rw^1 &= 0 \quad \text{on } (s^2, \beta) \setminus \{(q_i^2)_{1\leq i\leq n-1}\} && (41)\\
w^1(x) &= G^1(x) \quad \text{on } x \leq s^2, && (42)\\
w^1(q_i^1) &= R^1(q_i^1),\ 1 \leq i \leq n && (43)\\
b_i[G^1(q_i^2) - w^1(q_i^2)] + \frac{1}{2}\Delta(w^1)'(q_i^2) &= 0,\ 1 \leq i \leq n-1 && (44)\\
\lim_{x\to\beta} w^1(x) &= 0, && (45)\\
w^1(x) &\geq R^1(x) \quad \text{on } \mathcal{I}. && (46)
\end{aligned}$$

(ii) $w^2 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{(q_i^1)_{1\le i\le n}\} \cup \{s^2\}))$ *is solution to* $\mathcal{VS}^2$ *if:*

$$\begin{aligned}
\mathcal{L}w^2 - rw^2 &= 0 \quad on\ (s^2, \beta) \setminus \{(q_i^1)_{1\le i\le n}\} && (47)\\
w^2(x) &= R^2(x) \quad on\ x \le s^2, && (48)\\
w^2(q_i^2) &= R^2(q_i^2),\ 1 \le i \le n-1 && (49)\\
(w^2)'(s^2) &= (R^2)'(s^2), && (50)\\
a_i[G^2(q_i^1) - w^2(q_i^1)] + \frac{1}{2}\Delta(w^2)'(q_i^1) &= 0,\ 1 \le i \le n && (51)\\
\lim_{x\to\beta} w^2(x) &= 0, && (52)\\
w^2 &\ge R^2 \quad on\ \mathcal{I}. && (53)
\end{aligned}$$

The proof is relegated to Appendix C and is based on the properties obtained from the proof of Theorem 2 together with classical methods employed in optimal stopping and stopping games. In particular, the conditions (44) and (51) are obtained by applying the Itô-Tanaka-Meyer formula.

Appendix A: Proofs for Section 3

We start with some definitions (Revuz and Yor (1999, Chapter I, §4)). The process X is defined over the canonical space $(\Omega, \mathcal{F})$ of continuous trajectories, and $\mathbf{P}_\mu$ denotes the law of the process X given an initial distribution $\mu \in \Delta(\mathcal{I})$ over $\mathcal{B}(\mathcal{I})$. We denote by $(\mathcal{F}^0_t)_{t\geq 0}$ the natural filtration $(\sigma(X_s; s \leq t))_{t\geq 0}$ generated by X, and we let $\mathcal{F}^0_\infty \equiv \sigma(\bigcup_{t\geq 0} \mathcal{F}^0_t)$. For each $\mu \in \Delta(\mathcal{I})$, we denote by $\mathcal{F}^\mu_\infty$ the completion of $\mathcal{F}^0_\infty$ with respect to $\mathbf{P}_\mu$, and, for each $t \geq 0$, we let $\mathcal{F}^\mu_t$ be the augmentation of $\mathcal{F}^0_t$ by the $\mathbf{P}_\mu$-null, $\mathcal{F}^\mu_\infty$-measurable sets. The usual augmented filtration $(\mathcal{F}_t)_{t\geq 0}$ is then defined by $\mathcal{F}_t \equiv \bigcap_{\mu\in\Delta(\mathcal{I})} \mathcal{F}^\mu_t$ for all $t \geq 0$. Because the process X is a Feller process in the sense of Revuz and Yor (1999, Chapter III, §2, Definition 2.5) and a standard process in the sense of Blumenthal and Getoor (1968, Chapter I, Definition 9.2), the filtration $(\mathcal{F}_t)_{t\geq 0}$ is actually right-continuous. As usual in this literature, we say that a property of the trajectories $\omega \in \Omega$ is satisfied *almost surely* if it is satisfied $\mathbf{P}_\mu$-almost surely for all $\mu \in \mathcal{I}$ or, equivalently, $\mathbf{P}_x$-almost surely for all $x \in \mathcal{I}$.

Proof of Lemma 1. The proof proceeds along the same lines as that of Lemma 1 in Décamps, Gensbittel, and Mariotti (2021). The result follows. ■

Proof of Lemma 2. For each $\mu \in \Delta(\mathcal{I})$, ω and u^i are independent under $\mathbf{P}^i_\mu \equiv \mathbf{P}_\mu \otimes Leb$, and hence

$$\Gamma^i_t(\omega) = \mathbf{P}^i_\mu[\gamma^i \leq t \,|\, \mathcal{F}](\omega)$$

almost surely for all $t \geq 0$. We may assume that $\gamma(\cdot, u^i) \in \mathcal{T}$ for all u^i, as we can replace γ^i by the constant stopping time 0 for all u^i in a Borel set of zero Lebesgue measure without modifying the process Γ^i. Therefore, for all $u^i \in [0, 1]$ and $t \geq 0$, we have $\{\omega \in \Omega : \gamma^i(\omega, u^i) \leq t\} \in \mathcal{F}_t$ as $\gamma(\cdot, u^i) \in \mathcal{T}$. Using Corollary 2 in Solan, Tsirelson, and Vieille (2012), this implies that Γ^i_t is measurable with respect to the augmentation of $\mathcal{F}_t$ by the $\mathbf{P}_\mu$-null, $\mathcal{F}^\mu_\infty$-measurable sets, which coincides with $\mathcal{F}^\mu_t$. As this is true for all $\mu \in \Delta(\mathcal{I})$, we deduce that Γ^i is adapted with respect to $\mathcal{F}_t$. In particular, letting $\mu \equiv \delta_x$ yields

$$\Gamma^i_t(\omega) = \mathbf{P}^i_x[\gamma^i \leq t \,|\, \mathcal{F}_t](\omega)$$

almost surely for all $t \geq 0$ by the law of iterated expectations. The result follows. ■

Proof of Lemma 3. Assume that, for each $i = 1, 2$, γ^i is a randomized stopping time with ccdf Γ^i. We have

$$\begin{aligned}
\overline{\mathbf{E}}_x\Big[1_{\{\gamma^i\le\gamma^j\}}\,\mathrm{e}^{-r\gamma^i}R^i(X_{\gamma^i})\Big] &= \int_0^1\int_0^1 \mathbf{E}_x\Big[1_{\{\gamma^i(u^i)\le\gamma^j(u^j)\}}\,\mathrm{e}^{-r\gamma^i(u^i)}R^i(X_{\gamma^i(u^i)})\Big]\,\mathrm{d}u^j\,\mathrm{d}u^i\\
&= \int_0^1 \mathbf{E}_x\left[\mathrm{e}^{-r\gamma^i(u^i)}R^i(X_{\gamma^i(u^i)})\int_0^1 1_{\{\gamma^i(u^i)\le\gamma^j(u^j)\}}\,\mathrm{d}u^j\right]\mathrm{d}u^i\\
&= \int_0^1 \mathbf{E}_x\Big[\mathrm{e}^{-r\gamma^i(u^i)}R^i(X_{\gamma^i(u^i)})\Lambda^j_{\gamma^i(u^i)-}\Big]\,\mathrm{d}u^i\\
&= \mathbf{E}_x\left[\int_0^1 \mathrm{e}^{-r\gamma^i(u^i)}R^i(X_{\gamma^i(u^i)})\Lambda^j_{\gamma^i(u^i)-}\,\mathrm{d}u^i\right]\\
&= \mathbf{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-rt}R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t\right],
\end{aligned}$$

where the second and fourth equalities follow from Fubini's theorem, and the third equality follows from the definition of Λ^j. The last equality follows from observing that, for $\mathbf{P}_x$-almost every $\omega\in\Omega$, $t\mapsto\Gamma^i_t(\omega)$ is the cdf of the random variable $\gamma^i(\omega,\cdot)$ defined on the probability space $([0,1],\mathcal{B}([0,1]),\mathit{Leb})$ and taking values in $[0,\infty]$, where $\Gamma^i_\infty(\omega)\equiv 1$ by convention; Fubini's theorem then implies that the random variable $u^i\mapsto \mathrm{e}^{-r\gamma^i(\omega,u^i)}R^i(X_{\gamma^i(\omega,u^i)})\Lambda^j_{\gamma^i(\omega,u^i)-}$ is Lebesgue integrable over $[0,1]$ for $\mathbf{P}_x$-almost every $\omega\in\Omega$,[8] and we can thus apply the usual formula for the expectation. The proof for the second term appearing in (15) and (16) is similar and thus omitted.

Let us then verify that (17) defines a randomized stopping time in the sense of Definition 1. That $\hat{\gamma}^i(u^i)\in\mathcal{T}$ for *Leb*-almost every $u^i\in[0,1]$ is standard (Jacod and Shiryaev (2003, Proposition I.1.28)). The random variable $(\omega,u^i)\mapsto\hat{\gamma}^i(u^i)(\omega)$ is $\mathcal{F}_\infty\otimes\mathcal{B}([0,1])$-measurable as it is nondecreasing and right-continuous with respect to u^i. That the ccdf associated with $\hat{\gamma}^i$ is Γ^i is proven in De Angelis, Ferrari, and Moriarty (2018, Lemma 4.1), who use this representation as the definition of a randomized stopping time. The result follows. ■

Proof of Theorem 1 If Γ is a Markovian strategy, then $\Lambda=1-\Gamma$ satisfies almost surely for all $\tau\in\mathcal{T}$, for all $s\ge 0$

$$\Lambda_{\tau+s}=\Lambda_\tau(\Lambda_s\circ\theta_\tau), \tag{A.1}$$

on the event $\{\tau<\infty\}$, where $\theta_.$ denotes the shift operator on Ω. In particular, this property applied at $\tau=s=0$ implies that $\Lambda_0=(\Lambda_0)^2$ and thus $\Lambda_0\in\{0,1\}$ a.s. In the terminology of Blumenthal and Getoor (1968), Λ is a strong right-continuous multiplicative functional of X adapted to $(\mathcal{F}_t)$. The set $E=\{x|\mathbf{P}_x(\Lambda_0=1)=1\}$ is called the set of permanent points of Λ. Using Blumenthal's $0-1$ law and the fact that $\Lambda_0\in\{0,1\}$, we have $\mathcal{I}\setminus E=\{x|\mathbf{P}_x(\Lambda_0=0)=1\}$. The stopping time $\tau=\inf\{t>0|\Lambda_t=0\}\in\mathcal{T}$ is called the lifetime of Λ. In order

[8] Recall that, by convention, this random variable is equal to 0 if $\gamma^i(\omega,u^i)=\infty$.

to apply the main result of Sharpe (1971), we need to prove that Λ is an exact multiplicative functional, in the sense of Definition III.4.13 in Blumenthal and Getoor (1968). It is sufficient to prove that (see Proposition III.5.9 in Blumenthal and Getoor (1968))

$$\forall x \notin E, \forall t > 0, \lim_{u \downarrow 0} \mathbf{E}_x[\Lambda_{t-u} \circ \theta_u] = 0.$$

Note that

$$1_{\{t-u \geq \tau_x \circ \theta_u\}} \Lambda_{t-u} \circ \theta_u = 0 \ \mathbf{P}_x - a.s.$$

so that for u sufficiently small:

$$\mathbf{E}_x[\Lambda_{t-u} \circ \theta_u] \leq \mathbf{P}_x(t-u < \tau_x \circ \theta_u) = \mathbf{E}_x[\mathbf{P}_{X_u}(t-u < \tau_x)] \leq \mathbf{E}_x[\mathbf{P}_{X_u}(t/2 < \tau_x)].$$

The map $y \to \mathbf{P}_y(\tau_x > t/2)$ is (universally) measurable and bounded, and has limit 0 when $y \to x$ since X is a regular diffusion. We conclude therefore by bounded convergence. This implies that E is open and thus that $\mathcal{I} \setminus E$ is closed (see Blumenthal and Getoor (1968) p.126 last paragraph, using that the fine topology coincides with the usual topology in our case).

We can therefore apply Theorem 7.1 in Sharpe (1971). The stopping time appearing in the general expression (7.1) in Sharpe (1971) is called the exact regularization of τ and is equal in our particular case to the hitting time of a Borel set B since the lifetime of X is $+\infty$ and X is continuous. Moreover, using that X is a regular diffusion, this stopping time is a.s. equal to the hitting time of the closure of B, which we denote by S. The product term in expression (7.1) of Sharpe (1971) is identically equal to 1 and thus disappears (see Theorems 5.1 and 5.2 in Sharpe (1971), having in mind that all semipolar sets for X are empty since X is a regular diffusion, and that X has continuous trajectories). Thanks to these remarks, there exists a continuous additive functional A with bounded 1-potential (which means that $x \to \mathbf{E}_x[\int_0^\infty e^{-t} dA_t]$ is bounded) and a positive Borel map f such that

$$\Lambda_t^i = 1_{t < \tau_S} e^{-\int_0^t f(X_s) dA_s} \ a.s.$$

Moreover, as stated in Theorem 7.1 in Sharpe (1971), the integral $\int_0^t f(X_s) dA_s$ is $\mathbf{P}_x$ a.s. finite for all $t < \tau_S$ except maybe for x in a τ_S-polar set. From the definition of a τ_S polar set (see the definition and Lemma 2.1 page 29 in Sharpe (1971)) we see that a τ_S-polar set in our case must be a subset of S.

Using the classical representation result (see Borodin and Salminen (2002) chapter I.2 paragraph 23), there exists a positive Radon measure μ (locally finite) on $\mathcal{I}$ such that $A_t = \int_{\mathcal{I}} L_t^x d\mu(x)$ a.s.. Therefore, we have for $t < \tau$

$$\tilde{A}_t = \int_0^t f(X_s) dA_s = \int_0^t \int_{\mathcal{I}} f(X_s) dL_s^x d\mu(x) = \int_{\mathcal{I}} L_t^x f(x) d\mu(x).$$

It follows that the measure $f(x)d\mu(x)$ on $\mathcal{I}\setminus S$ is a Radon measure, i.e. a non-negative Borel measure which is locally finite. Indeed, assume by contradiction that it is not locally finite, then there exists $x\in\mathcal{I}\setminus S$ such that for every $\varepsilon>0$ such that $(x-\varepsilon,x+\varepsilon)\subset\mathcal{I}\setminus S$, we have $\int_{(x-\varepsilon,x+\varepsilon)}f(x)d\mu(x)=\infty$. For all $t>0$, we have for all ω in a set of $\mathbf{P}_x$-probability 1 that $L_t^x(\omega)>0$, and therefore since the local time of X is a.s. jointly continuous, that $L_t^y(\omega)>0$ for $y\in[x-\varepsilon(\omega),x+\varepsilon(\omega)]$ for some $\varepsilon(\omega)>0$. This implies that if $0<t<\tau_S(\omega)$

$$\tilde{A}_t(\omega)=\int_{\mathcal{I}}L_t^x(\omega)f(x)d\mu(x)\geq\min_{y\in[x-\varepsilon(\omega),x+\varepsilon(\omega)]}L_t^y(\omega)\int_{(x-\varepsilon(\omega),x+\varepsilon(\omega))}f(x)d\mu(x)=+\infty.$$

Since $\mathbf{P}_x(\tau_S>0)=1$, this contradicts the statement of Theorem 7.1 in Sharpe (1971).

Reciprocally, if S is a closed subset of $\mathcal{I}$ and μ is a Radon measure on $\mathcal{I}\setminus S$, then the process

$$\Lambda_t=1_{t<\tau_S}e^{-\int_{\mathcal{I}}L_t^x d\mu(x)},$$

is well-defined and is a strong right-continuous multiplicative functional, and in particular satisfies (A.1). It can be checked directly that $\Gamma=1-\Lambda$ satisfies the assumptions of Lemma 3 and thus is the c.c.d.f. a randomized stopping time. ■

Proof of Lemma 4. We focus on player 1, the proof for player 2 being symmetrical. Observe from (16) that, for each $\tau^1\in\mathcal{T}$, player 1's payoff from playing τ^1 against Γ^2 is

$$J^1(x,\tau^1,\Gamma^2)=\mathbf{E}_x\left[\mathrm{e}^{-r\tau^1}R^1(X_{\tau^1})\Lambda^2_{\tau^1-}+\int_{[0,\tau^1)}\mathrm{e}^{-rt}G^1(X_t)\,\mathrm{d}\Gamma_t^2\right].\tag{A.2}$$

Letting $\hat{\gamma}^1$ be the randomized stopping time associated to the ccdf Γ^1 by (17), we have

$$\begin{aligned}J^1(x,\Gamma^1,\Gamma^2)&=\int_0^1\mathbf{E}_x\left[\mathrm{e}^{-r\hat{\gamma}^1(u^1)}R^1(X_{\gamma^1(u^1)})\Lambda^2_{\gamma^1(u^1)-}+\int_{[0,\hat{\gamma}^1(u^1))}\mathrm{e}^{-rt}G^1(X_t)\,\mathrm{d}\Gamma_t^2\right]\mathrm{d}u^1\\&=\int_0^1 J^1(x,\hat{\gamma}^1(u^1),\Gamma^2)\,\mathrm{d}u^1\\&\leq\sup_{u^1\in[0,1]}J^1(x,\hat{\gamma}^1(u^1),\Gamma^2)\\&\leq\sup_{\tau^1\in\mathcal{T}}J^1(x,\tau^1,\Gamma^2).\end{aligned}$$

where the first equality follows along the same steps as in the proof of Lemma 3, and the second equality follows from (A.2). The result follows. ■

Proof of Proposition 1 Before proving Proposition 1, we need a few technical results related to continuous additive functional of diffusions. More precisely, we consider processes A defined by $A_t=\int_{\mathcal{I}}L_t^x d\mu(x)$ where μ is a Radon measure on $\mathcal{I}\setminus S$ for some closed set S.

If $(a, b) \subset \mathcal{I} \setminus S$, the restriction of μ to (a, b) is a Radon measure, and A defines a continuous additive functional of the diffusion X killed at a and b, with state space (a, b).

If τ is the first exit time of (a, b) with $[a, b] \subset \mathcal{I}$, then

$$\mathbf{E}_x[A_\tau] = \int_{\mathcal{I}} \mathbf{E}_x[L_\tau^y] d\mu(y) = \int_{(a,b)} \mathbf{E}_x[L_\tau^y] d\mu(y) = \int_{(a,b)} 2(S'(y))^{-1} \Phi_{a,b}(x, y) d\mu(y), \qquad (A.3)$$

where

$$\Phi_{a,b}(x, y) = \frac{(S(x \wedge y) - S(a))(S(b) - S(x \vee y))}{S(b) - S(a)}$$

denotes the Green function (potential density) of the diffusion X killed at the boundaries a and b. (see Borodin and Salminen (2002) page 20-21). It is easy to check that $\mathbf{E}_x[A_\tau]$ is finite if and only if for some $x \in (a, b)$:

$$\int_a^x (S(y) - S(a)) d\mu(y) < \infty \text{ and } \int_x^b (S(b) - S(y)) d\mu(y) < \infty.$$

A more precise result can be shown (see Theorem 2.1 in Cetin (2018)). Precisely:

$$\begin{cases} A_{\tau_a} 1_{\tau_a < \tau_b} = +\infty \, a.s. & \text{if} \quad \int_a^x (S(y) - S(a)) d\mu(y) = +\infty \text{ for some } x \in (a, b) \\ A_{\tau_a} 1_{\tau_a < \tau_b} < +\infty \, a.s. & \text{otherwise} \end{cases} \qquad (A.4)$$

A symmetric result holds for b.

We will use the following lemma in the proof of Proposition 2.

Lemma 5 *Let $A_t = \int_{(a,b)} L_t^y d\mu(y)$ for some Radon measure μ on (a, b). Define the function h by*

$$\forall x \in (a, b), \; h(x) = \mathbf{E}_x[C_a e^{-A_{\tau_a}} 1_{\{\tau_a < \tau_b\}} + C_b e^{-A_{\tau_b}} 1_{\{\tau_b < \tau_a\}}],$$

with $C_a, C_b \geq 0$ and $\tau_y = \inf\{t \geq 0, X_t = y\}$.

Then, h is non-negative, S-convex and continuous on (a, b), and the limits $h(a+)$ and $h(b-)$ exists and are given by

$$h(a+) = \begin{cases} C_a & \textit{if} \quad \int_a^x (S(y) - S(a)) d\mu(y) < \infty \\ 0 & \textit{otherwise} \end{cases} \qquad (A.5)$$

$$h(b-) = \begin{cases} C_b & \textit{if} \quad \int_x^b (S(b) - S(y)) d\mu(y) < \infty \\ 0 & \textit{otherwise} \end{cases}. \qquad (A.6)$$

Proof of Lemma 5. h is clearly non-negative. S-convexity follows directly from the Markov property applied for $h(\lambda x_1 + (1 - \lambda) x_2)$ up to the stopping time $\tau_{x_1} \wedge \tau_{x_2}$. Precisely, we have

$$h(\lambda x_1 + (1 - \lambda) x_2) = \mathbf{E}_{\lambda x_1 + (1-\lambda) x_2}[h(x_1) e^{-A_{\tau_{x_1}}} 1_{\{\tau_{x_1} < \tau_{x_2}\}} + h(x_2) e^{-A_{\tau_{x_2}}} 1_{\{\tau_{x_2} < \tau_{x_1}\}}].$$

Using that $e^{-A_t} \leq 1$, we obtain that h is S-convex, i.e.

$$h(\lambda x_1 + (1-\lambda)x_2) \leq h(x_1)\frac{S(x_2) - S(\lambda x_1 + (1-\lambda)x_2)}{S(x_2) - S(x_1)} + h(x_2)\frac{S(\lambda x_1 + (1-\lambda)x_2) - S(x_1)}{S(x_2) - S(x_1)}].$$

S-convexity implies the continuity of h on (a,b).

If $\int_a^{x_0}(S(y) - S(a))d\mu(y) < \infty$ for some $x_0 \in (a,b)$, then by property (A.4) $h(x) = \mathbf{E}_x[C_b e^{-A_{\tau_b}} 1_{\{\tau_b < \tau_a\}}]$ and thus

$$0 \leq h(x) \leq C_b \mathbf{P}_x(\tau_b < \tau_a) \underset{x \to a}{\longrightarrow} 0.$$

It follows that $h(a+) = 0$.

If $\int_a^{x_0}(S(y) - S(a))d\mu(y) = \infty$ for some $x_0 \in (a,b)$, then by property (A.4) $e^{-A_{\tau_a}} > 0$ $\mathbf{P}_x$-a.s. If a_n is a decreasing sequence with limit a and $a_n < x$, applying the Markov property at τ_{a_n}, we have

$$h(x) = \mathbf{E}_x[h(a_n)e^{-A_{\tau_{a_n}}} 1_{\{\tau_{a_n} < \tau_b\}} + C_b e^{-A_{\tau_b}} 1_{\{\tau_b < \tau_{a_n}\}}].$$

Taking the limit along any subsequence such that $h(a_{n_k})$ converges to some z, we obtain

$$h(x) = \mathbf{E}_x[z e^{-A_{\tau_a}} 1_{\{\tau_a < \tau_b\}} + C_b e^{-A_{\tau_b}} 1_{\{\tau_b < \tau_a\}}],$$

and thus $z = C_a$ since $\mathbf{E}_x[e^{-A_{\tau_a}} 1_{\{\tau_a < \tau_b\}}] > 0$. Therefore $h(a+)$ exists and equals C_a. (A.6) follows by the same method. ■

Strong Markov property: We formalize in the next lemma a consequence of the strong Markov property that we will use several times throughout this Appendix.

Lemma 6 *If the players use Markovian randomized stopping times with ccdf Γ^1 and Γ^2, then for every stopping time T and every $x \in \mathcal{I}$, their expected payoffs write as*

$$J^i(x,\Gamma^1,\Gamma^2) = \mathbf{E}_x[\int_{[0,T)} e^{-rs}R^i(X_s)\Lambda^j_{s-}d\Gamma^i_s + \int_{[0,T)} e^{-rs}G^i(X_s)\Lambda^i_s d\Gamma^j_s + e^{-rT}\Lambda^j_{T-}\Lambda^i_{T-}J^i(X_T,\Gamma^1,\Gamma^2)] \tag{A.7}$$

Proof of Lemma 6. It follows from Lemma 3 that

$$\begin{aligned} J^i(x,\Gamma^1,\Gamma^2) = \mathbf{E}_x\Big[& \int_{[0,T)} e^{-rs}R^i(X_s)\Lambda^j_{s-}d\Gamma^i_s + \int_{[0,T)} e^{-rs}G^i(X_s)\Lambda^i_s d\Gamma^j_s \\ & + e^{-rT}R^i(X_T)\Lambda^j_{T-}(\Gamma^i_T - \Gamma^i_{T-}) + e^{-rT}G^i(X_T)\Lambda^i_T(\Gamma^j_T - \Gamma^j_{T-}) \\ & + \int_{(T,\infty)} e^{-rs}R^i(X_s)\Lambda^j_{s-}d\Gamma^i_s + \int_{(T,\infty)} e^{-rs}G^i(X_s)\Lambda^i_s d\Gamma^j_s\Big] \end{aligned}$$

Recall that the only jump of Λ^i occurs at τ_{S^i} at which time the process jumps to zero and remains equal to zero. On the one hand, we have

$$\begin{aligned}
&e^{-rT}R^i(X_T)\Lambda^j_{T-}(\Gamma^i_T-\Gamma^i_{T-})+e^{-rT}G^i(X_T)\Lambda^i_T(\Gamma^j_T-\Gamma^j_{T-})\\
&\quad=e^{-rT}R^i(X_T)\Lambda^j_{T-}\Lambda^i_{T-}1_{\tau_{S^j}\geq T=\tau_{S^i}}+e^{-rT}G^i(X_T)\Lambda^i_{T-}\Lambda^j_{T-}1_{\tau_{S^i}>T=\tau_{S^j}}\\
&\quad=e^{-rT}J^i(X_T)\Lambda^j_{T-}\Lambda^i_{T-}1_{\tau_{S^j}\geq T=\tau_{S^i}}+e^{-rT}J^i(X_T,\Gamma^1,\Gamma^2)\Lambda^i_{T-}\Lambda^j_{T-}1_{\tau_{S^i}>T=\tau_{S^j}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{(T,\infty)}e^{-rs}R^i(X_s)\Lambda^j_{s-}d\Gamma^i_s+\int_{(T,\infty)}e^{-rs}G^i(X_s)\Lambda^i_s d\Gamma^j_s\\
&\quad=e^{-rT}1_{T<\tau_{S^i}\wedge\tau_{S^j}}\left(\int_{(0,\infty)}e^{-rs}R^i(X_{T+s})\Lambda^j_{(T+s)-}d\Gamma^i_{T+s}+\int_{(0,\infty)}e^{-rs}G^i(X_{T+s})\Lambda^i_{T+s}d\Gamma^j_{T+s}\right)\\
&\quad=e^{-rT}1_{T<\tau_{S^i}\wedge\tau_{S^j}}\Lambda^j_T\Lambda^i_T\Big(\int_{(0,\infty)}e^{-rs}R^i(X_s\circ\theta_T)(\Lambda^j_{s-}\circ\theta_T)d(\Gamma^i_s\circ\theta_T)\\
&\qquad\qquad+\int_{(0,\infty)}e^{-rs}G^i(X_s\circ\theta_T)(\Lambda^i_s\circ\theta_T)d(\Gamma^2_s\circ\theta_T)\Big)\\
&\quad=e^{-rT}1_{T<\tau_{S^i}\wedge\tau_{S^j}}\Lambda^j_{T-}\Lambda^i_{T-}\Big(\int_{[0,\infty)}e^{-rs}R^i(X_s\circ\theta_T)(\Lambda^j_{s-}\circ\theta_T)d(\Gamma^i_s\circ\theta_T)\\
&\qquad\qquad+\int_{[0,\infty)}e^{-rs}G^i(X_s\circ\theta_T)(\Lambda^i_s\circ\theta_T)d(\Gamma^j_s\circ\theta_T)\Big),
\end{aligned}$$

where we used (18) in the second equality. Taking expectation and applying the Markov property at T, it leads to

$$\mathbf{E}_x\Big[\int_{(T,\infty)}e^{-rs}R^i(X_s)\Lambda^j_{s-}d\Gamma^i_s+\int_{(T,\infty)}e^{-rs}G^i(X_s)\Lambda^i_s d\Gamma^j_s\Big]=\mathbf{E}_x\Big[e^{-rT}1_{T<\tau_{S^i}\wedge\tau_{S^j}}\Lambda^j_{T-}\Lambda^i_{T-}J^i(X_T,\Gamma^1,\Gamma^2)\Big]$$

The conclusion follows by putting the pieces together and noticing that $e^{-rT}\Lambda^j_{T-}\Lambda^i_{T-}J^i(X_T,\Gamma^1,\Gamma^2)$ vanishes on the event $\{T>\tau_{S^i}\wedge\tau_{S^j}\}$. ■

Proof of Proposition 1. Let us fix Markov strategies (μ^i,S^i) for $i=1,2$. Let $A^i_t=\int_{\mathcal{I}}L^a_t d\mu^i(a)$ denote the continuous additive functional associated to μ^i, $\Lambda^i_t=e^{-A^i_t}1_{t<\tau_{S^i}}$ and $\Gamma^i_t=1-\Lambda^i_t$.

The payoff of player 1 for a pure stopping time τ^1 is given by

$$\begin{aligned}
&J^1(x,\tau^1,\Gamma^2)=\\
&\quad\mathbf{E}_x[e^{-r\tau^1}\Lambda^2_{\tau^1-}R^1(X_{\tau^1})+\int_{[0,\tau^1\wedge\tau_{S^2})}e^{-rs}G^1(X_s)d\Gamma^2_s+e^{-r\tau_{S^2}}\Lambda^2_{\tau_{S^2}-}G^1(X_{\tau_{S^2}})1_{\tau_{S^2}<\tau^1}].
\end{aligned}$$

Note that $\bar{J}^1$ is equal to R^1 on S^1 and to G^1 on $S^2 \setminus S^1$.

Using assumption A6, we first prove that $V_{R^1}(x) \leq \bar{J}^1(x) \leq G^1(x)$. These inequalities imply $\lim_{x\to\alpha+} \frac{\bar{J}^1(x)}{\phi(x)} = 0$ and $\lim_{x\to\beta-} \frac{\bar{J}^1(x)}{\psi(x)} = 0$.

For the first inequality, let $\tau_*^1 = \tau_{(\alpha, x_{R^1}]}$, and $\hat{\gamma}^2(u)$ be defined by (17). Using Lemma 3 and that $G^1 \geq V_{R^1}$ we obtain

$$\bar{J}^1(x) \geq J^1(x, \tau_*^1, \Gamma^2) = \int_0^1 J^1(x, \tau_*^1, \hat{\gamma}^2(u))du \geq$$
$$\int_0^1 \mathbf{E}_x[e^{-r\tau_*^1} R^1(X_{\tau_*^1}) 1_{\tau_*^1 \leq \hat{\gamma}^2(u)} + e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) 1_{\tau_*^1 > \hat{\gamma}^2(u)}]du.$$

For all u, we have on $\{\tau_*^1 > \hat{\gamma}^2(u)\}$,

$$e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) = \mathbf{E}_x[e^{-r\tau_*^1} R^1(X_{\tau_*^1}) | \mathcal{F}_{\hat{\gamma}^2(u)}] \ \mathbf{P}_x - a.s. \tag{A.8}$$

Therefore, using the tower property of conditional expectation

$$\mathbf{E}_x[e^{-r\tau_*^1} R^1(X_{\tau_*^1}) 1_{\tau_*^1 \leq \hat{\gamma}^2(u)} + e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) 1_{\tau_*^1 > \hat{\gamma}^2(u)}] = \mathbf{E}_x[e^{-r\tau_*^1} R^1(X_{\tau_*^1})] = V_{R^1}(x),$$

and we conclude that

$$\bar{J}^1(x) \geq \int_0^1 V_{R^1}(x)du = V_{R^1}(x).$$

For the second inequality, we have by assumption A6 $R^1 \leq V_{R^1} \leq G^1$. Therefore for every $\tau^1 \in \mathcal{T}$

$$\begin{aligned}
J^1(x, \tau^1, \Gamma^2) &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u))du \\
&\leq \int_0^1 \mathbf{E}_x[e^{-r\tau^1} G^1(X_{\tau^1}) 1_{\tau^1 \leq \hat{\gamma}^2(u)} + e^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) 1_{\tau^1 > \hat{\gamma}^2(u)}]du \\
&= \int_0^1 \mathbf{E}_x[e^{-r(\tau^1 \wedge \hat{\gamma}^2(u))} G^1(X_{\tau^1 \wedge \hat{\gamma}^2(u)})]du \\
&\leq \int_0^1 G^1(x)du = G^1(x),
\end{aligned}$$

where we used that $e^{-rt}G^1(X_t)$ is a supermartingale by assumption A7.

Let us prove (i): It is not optimal for player 1 to stop immediately in S^2 if $R^1 < G^1$. Therefore, for any PBR, $S^1 \cap S^2 \cap (\alpha^1, \beta) = \emptyset$ which proves (i).

Let us prove (ii): The inclusion $S^1 \subset C^1 = \{x|\bar{J}^1(x) = R^1(x)\}$ is obvious from the definition of a PBR.

We now prove (iii): $supp(\mu^1) \subset C^1 \cup S^2$.

Assume that $x \in supp(\mu^1) \setminus S^2$. Recall that by Lemma 3

$$\bar{J}^1(x) = \int_0^1 J^1(x, \hat{\gamma}^1(u), \Gamma^2)du,$$

where $\hat{\gamma}^1(u) = \inf\{t \geq 0 \,|\, \Gamma^1_t > u\}$. Moreover, since for all $u \in [0,1]$, $J^1(x, \hat{\gamma}^1(u), \Gamma^2) \leq \bar{J}^1(x)$, the latter inequality is an equality for all $u \in M$ where M has Lebesgue measure 1.

From the definition of Γ^1, we have $\hat{\gamma}^1(u) = \inf\{t \,|\, 1 - e^{-A^1_t} > u\} \wedge \tau_{S^1}$ for $u \in [0,1)$. We have $\hat{\gamma}^1(u) > 0$ $\mathbf{P}_x$-a.s. for all $u > 0$ as A^1 is continuous. Using that $x \in supp(\mu^1)$, we also have $\lim_{u \to 0} \hat{\gamma}^1(u) = 0$ $\mathbf{P}_x$-a.s.. Indeed, $\hat{\gamma}^1(u)$ is non-decreasing with respect to u for all ω and converges to $\hat{\gamma}^1(0) = \inf\{t \geq 0 | A^1_t > 0\} \wedge \tau_{S^1}$. Then, arguing as in Lemma 2.16 p. 416 in Revuz and Yor (1999), we obtain that $\hat{\gamma}^1(0) = \tau_{supp(\mu^1)} \wedge \tau_{S^1}$ a.s., and therefore that $\hat{\gamma}^1(0) = 0$ $\mathbf{P}_x$-a.s..

Precisely, the local times of X have a continuous version $(t, y) \to L^y_t$, and we know that for all $t > 0$, $L^x_t > 0$ $\mathbf{P}_x$-a.s.. We deduce that for every ω in a set of $\mathbf{P}_x$-probability one, $L^y_t > 0$ for y in a neighborhood of x, which implies

$$A^1_t = \int_{\mathcal{I}} L^y_t d\mu^1(y) > 0 \, \mathbf{P}_x - a.s.,$$

and thus $\hat{\gamma}^1(0) = 0$ $\mathbf{P}_x$-a.s..

For all $u \in M$:

$$\bar{J}^1(x) = J^1(x, \hat{\gamma}^1(u), \Gamma^2) = \mathbf{E}_x[\int_{[0,\hat{\gamma}^1(u))} e^{-rs} G^1(X_s) d\Gamma^2_s + e^{-r\hat{\gamma}^1(u)} \Lambda^2_{\hat{\gamma}^1(u)-} R^1(X_{\hat{\gamma}^1(u)})]$$

Taking the limit as $u \in M$ goes to zero, we deduce that $\bar{J}^1(x) = R^1(x)$ by bounded convergence.

Proof of (iv). We have

$$\bar{J}^1(x) \geq J^1(x, (0, (\alpha, x_{R^1}]), (\mu^2, S^2)) \geq J^1(x, (0, (\alpha, x_{R^1}]), (0, \emptyset)). \tag{A.9}$$

The first inequality in (A.9) follows from the fact that (μ^1, S^1) is a PBR to (μ^2, S^2). For the second one, recall that by assumption A6

$$G^1(x) \geq V_{R^1}(x) = \sup_\tau \mathbf{E}_x[e^{-r\tau} R^1(X_\tau)] = \mathbf{E}_x[e^{-r\tau^1_*} R^1(X_{\tau^1_*})],$$

where $\tau^1_* = \tau_{(\alpha, x_{R^1}]}$. We have

$$J^1(x, \tau^1_*, \Gamma^2) = \int_0^1 J^1(x, \tau^1_*, \hat{\gamma}^2(u))du$$
$$= \int_0^1 \mathbf{E}_x[e^{-r\tau^1_*} R^1(X_{\tau^1_*}) 1_{\tau^1_* \leq \hat{\gamma}^2(u)} + e^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) 1_{\tau^1_* > \hat{\gamma}^2(u)}]du.$$

On $\{\tau^1_* > \hat{\gamma}^2(u)\}$, using assumption A6, we have $\mathbf{P}_x$-a.s.

$$e^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) \geq e^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) = \mathbf{E}_x[e^{-r\tau^1_*} R^1(X_{\tau^1_*}) | \mathcal{F}_{\hat{\gamma}^2(u)}]. \tag{A.10}$$

Therefore, using the tower property of conditional expectation

$$J^1(x, \tau^1_*, \hat{\gamma}^2(u)) \geq \mathbf{E}_x[e^{-r\tau^1_*} R^1(X_{\tau^1_*})],$$

and the second inequality of (A.9) follows by integrating with respect to u.

The conclusion follows by noticing that $J^1(x, (0, (\alpha, x_{R^1}]), (0, \emptyset)) > R^1(x)$ for $x > x_{R^1}$ and applying (ii) and (iii).

Proof of (v). We have from Lemma 3

$$\bar{J}^1(x) = \int_0^1 J^1(x, \hat{\gamma}^1(u), \Gamma^2)du,$$

and for all $u \in [0, 1]$, $J^1(x, \hat{\gamma}^1(u), \Gamma^2) \leq \bar{J}^1(x)$, so that the latter inequality is an equality for all $u \in M$, where M has Lebesgue measure 1.

We will use that $\hat{\gamma}^1(u) \to \tau_{S^1}$ as $u \to 1$. This property follows from the definition of Γ^1 since we have $\tau_{S^1} = \inf\{t \geq 0 \,|\, \Gamma^1 = 1\}$, and $\Gamma^1_t < 1$ on $t < \tau_{S^1}$. Moreover $u \to \hat{\gamma}^1(u)$ is non-decreasing. We have for $u \in M$

$$\bar{J}^1(x) = J^1(x, \hat{\gamma}^1(u), \Gamma^2) = \mathbf{E}_x[\int_{[0, \hat{\gamma}^1(u))} e^{-rs} G^1(X_s) d\Gamma^2_s + e^{-r\hat{\gamma}^1(u)} \Lambda^2_{\hat{\gamma}^1(u)-} R^1(X_{\hat{\gamma}^1(u)})].$$

Taking the limit as $u \to 1$ with $u \in M$,we obtain by bounded convergence

$$\bar{J}^1(x) = \mathbf{E}_x[\int_{[0, \tau_{S^1})} e^{-rs} G^1(X_s) d\Gamma^2_s + e^{-r\tau_{S^1}} \Lambda^2_{\tau_{S^1}-} R^1(X_{\tau_{S^1}})] = J^1(x, \tau_{S^1}, \Gamma^2),$$

which concludes the proof of the first assertion.

For the second assertion, let $\tilde{\Gamma}^1$ be associated with $(\tilde{\mu}^1, S^1)$ and $\tilde{\gamma}^1(u) = \inf\{t \geq 0 \,|\, \tilde{\Gamma}^1_t > u\}$. By assumption, we have that for all u

$$\bar{J}^1(X_{\tilde{\gamma}^1(u)}) = R^1(X_{\tilde{\gamma}^1(u)}). \tag{A.11}$$

Therefore

$$J^1(x,\tilde{\Gamma}^1,\Gamma^2) = \int_0^1 \mathbf{E}_x[\int_{[0,\tilde{\gamma}^1(u))} e^{-rs}G^1(X_s)d\Gamma^2_s + e^{-r\tilde{\gamma}^1(u)}\Lambda^2_{\tilde{\gamma}^1(u)-}R^1(X_{\tilde{\gamma}^1(u)})]du.$$

Applying the Markov property at $\tilde{\gamma}^1(u)$ and using that $\bar{J}^1(.) = J^1(.,(0,S^1),\Gamma^2)$, we have

$$\begin{aligned}\bar{J}^1(x) &= \mathbf{E}_x[\int_{[0,\tilde{\gamma}^1(u))} e^{-rs}G^1(X_s)d\Gamma^2_s + e^{-r\tilde{\gamma}^1(u)}\Lambda^2_{\tilde{\gamma}^1(u)-}\bar{J}^1(X_{\tilde{\gamma}^1(u)})] \\ &= \mathbf{E}_x[\int_{[0,\tilde{\gamma}^1(u))} e^{-rs}G^1(X_s)d\Gamma^2_s + e^{-r\tilde{\gamma}^1(u)}\Lambda^2_{\tilde{\gamma}^1(u)-}R^1(X_{\tilde{\gamma}^1(u)})],\end{aligned}$$

where the second equality follows from (A.11). We conclude that

$$J^1(x,\tilde{\Gamma}^1,\Gamma^2) = \int_0^1 \bar{J}^1(x)du = \bar{J}^1(x),$$

which ends the proof. ■

Proof of Proposition 2

We first prove the following Lemma on the continuity of brvf.

Lemma 7 *If (μ^1, S^1) is a pbr to (μ^2, S^2) with associated brvf $\bar{J}^2$, then $\bar{J}^1$ is continuous on any interval $[a,b]$ such that $(a,b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$.*

Proof of Lemma 7 Given $x \notin S^1 \cup S^2$, define $T_\varepsilon = \tau_{x+\varepsilon} \wedge \tau_{x-\eta}$ such that $\varepsilon, \eta > 0$ and $[x-\eta, x+\varepsilon] \subset (S^1)^c \cap (S^2)^c$. Lemma 6 applied at T_ε implies:

$$\begin{aligned}\bar{J}^1(x) &= \mathbf{E}_x[\int_{[0,T_\varepsilon)} e^{-rs}R^1(X_s)\Lambda^2_{s-}d\Gamma^1_s + \int_{[0,T_\varepsilon)} e^{-rs}G^1(X_s)\Lambda^1_s d\Gamma^2_s + e^{-rT_\varepsilon}\Lambda^2_{T_\varepsilon-}\Lambda^1_{T_\varepsilon-}\bar{J}^1(X_{T_\varepsilon})] \\ &= \mathbf{E}_x[\int_{[0,T_\varepsilon)} e^{-rs}R^1(X_s)\Lambda^2_s d\Gamma^1_s + \int_{[0,T_\varepsilon)} e^{-rs}G^1(X_s)\Lambda^1_s d\Gamma^2_s + e^{-rT_\varepsilon}\Lambda^2_{T_\varepsilon}\Lambda^1_{T_\varepsilon}\bar{J}^1(X_{T_\varepsilon})].\end{aligned}$$

The last equality follows from the fact that on $\{t \le T_\varepsilon\}$, we have $\Lambda^i_{t-} = \Lambda^i_t$.

T_ε goes to zero in probability as ε goes to zero (η being fixed) (even a.s. since it is decreasing with ε) and $\mathbf{P}_x(T_\varepsilon = \tau_{x+\varepsilon})$ goes to 1. Therefore bounded convergence implies that $\bar{J}^1(x) = \bar{J}^1(x+)$. Similarly, we can prove $\bar{J}^1(x) = \bar{J}^1(x-)$.

Now, let us consider an interval $(a,b) \subset (S^1)^c \cap (S^2)^c$. $\bar{J}^1$ is continuous on (a,b) thanks to the preceding argument. Le us assume that $a > \alpha$ and belongs to S^1 so that $\bar{J}^1(a) = R^1(a)$, we will prove that $\bar{J}^1$ is right-continuous at a.

Using Proposition 1-(v), $\bar{J}^1(x) = J^1((0, S^1), (\mu^2, S^2))$. Applying Lemma 6 with $\tau = \tau_a \wedge \tau_b$, we have for $x \in (a, b)$:

$$\bar{J}^1(x) = \mathbf{E}_x[\int_{[0,\tau)} e^{-rs} G^1(X_s) d\Gamma_s^2 + e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau)]. \tag{A.12}$$

Since $a \notin S^2$, the measure μ^2 is locally finite at a. Therefore,

$$0 \leq \mathbf{E}_x[\int_{[0,\tau)} e^{-rs} G^1(X_s) d\Gamma_s^2] \leq C \mathbf{E}_x[1 - e^{-A_\tau^2}], \tag{A.13}$$

where C is an upper bound on G^1 on $[a, b]$. Applying Lemma 5 (with $C_a = C_b = 1$ and $\mu = \mu^2$), $\mathbf{E}_x[1 - e^{-A_\tau^2}]$ goes to 0 as x converges to a.

Lemma 5 also implies that $\mathbf{E}_x[e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau)]$ goes to $\bar{J}^1(a) = R^1(a)$ as x converges to a (by taking $\mu = \mu^2 + rLeb$), and thus $\bar{J}^1$ is right-continuous at a.

Let us now consider the case $a \in S^2$ so that $\bar{J}^1(a) = G^1(a)$, we will prove that $\bar{J}^1$ is right-continuous at a. Let $0 < \varepsilon < b - a$. As before with $\tau = \tau_a \wedge \tau_{a+\varepsilon}$, we have for $x \in (a, a + \varepsilon)$:

$$\bar{J}^1(x) = \mathbf{E}_x[\int_{[0,\tau)} e^{-rs} G^1(X_s) d\Gamma_s^2 + e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau)]. \tag{A.14}$$

If $\int_a^x (S(y) - S(a)) d\mu^2(y) < \infty$, the proof is completely similar as for the case $a \in S^1$. Let us assume that $\int_a^x (S(y) - S(a)) d\mu^2(y) = \infty$. Lemma 5 implies that $\mathbf{E}_x[e^{-r\tau} e^{-A_\tau^2} \bar{J}^1(X_\tau)]$ goes to zero as x converges to a (by taking $\mu = \mu^2 + rLeb$). On the other hand

$$\mathbf{E}_x[\int_{[0,\tau)} e^{-rs} G^1(X_s) d\Gamma_s^2] \geq (\min_{[a,a+\varepsilon]} G^1) \mathbf{E}_x[e^{-r\tau}(1 - e^{-A_\tau^2})] = (\min_{[a,a+\varepsilon]} G^1) \mathbf{E}_x[e^{-r\tau} - e^{-r\tau} e^{-A_\tau^2})].$$

By Lemma 5, the last expectation goes to 1 as x converges to a. We deduce that $\liminf_{x \to a+} \bar{J}^1(x) \geq (\min_{[a,a+\varepsilon]} G^1)$ and the conclusion follows by sending ε to zero. The cases $b \in S^1, b \in S^2$ can be proven in a similar way. Lemma 7 is proven ■.

We now prove a second lemma related to the pbr on the intervals $(\alpha, \alpha^i]$.

Lemma 8 *The following properties hold:*

- *If* (μ^i, S^i) *is a pbr to* (μ^j, S^j), *then* $(\alpha, \alpha^i] \subset S^1 \cup S^2$.
- *If* $((\mu^1, S^1), (\mu^2, S^2))$ *is a MPE, then either* $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^1$ *or* $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^2$.

Proof of Lemma 8 We prove the first point with $i = 1$. Note that $\bar{J}^1 = R^1 = V_{R^1} = G^1$ on $(\alpha, \alpha^1]$. Assume by contradiction that $x \in (\alpha, \alpha^1)$ and $x \notin S^1 \cup S^2$. Let $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$ with $b < \alpha^1$ which contains x. Since $(0, S^1)$ is a PBR to (μ^2, S^2), and using Lemma 6 at τ the exit time of (a, b) we have

$$
\begin{aligned}
\bar{J}^1(x) &= \mathbf{E}_x[\int_{[0,\tau)} e^{-rs} G^1(X_s) d\Gamma_s^2 + e^{-r\tau} \Lambda_{\tau-}^2 \bar{J}^1(X_\tau)] \\
&= \mathbf{E}_x[\int_{[0,\tau)} e^{-rs} R^1(X_s) d\Gamma_s^2 + e^{-r\tau} \Lambda_{\tau-}^2 R^1(X_\tau)] \\
&= \int_0^1 \mathbf{E}_x[e^{-r\hat{\gamma}^2(u,.)} R^1(X_{\hat{\gamma}^2(u,.)}) 1_{\hat{\gamma}^2(u,.)<\tau} + e^{-r\tau} 1_{\tau \leq \hat{\gamma}^2(u,.)} R^1(X_\tau)] du \\
&= \int_0^1 \mathbf{E}_x[e^{-r(\hat{\gamma}^2(u,.) \wedge \tau)} R^1(X_{\hat{\gamma}^2(u,.) \wedge \tau})] du \\
&< R^1(x),
\end{aligned}
$$

where the third equality is obtained as in Lemma 3, and the last inequality follows from assumption A3 together with the fact that for for all $u > 0$, $\tau \wedge \hat{\gamma}^2(u, .) > 0$ $\mathbf{P}_x$-almost surely since Γ^2 is continuous on $[0, \tau_{S^2})$ and $\tau_{S^2} > 0$ $\mathbf{P}_x$-almost surely. The above inequality contradicts $\bar{J}^1 \geq R^1$ and this concludes the proof of the first point.

Let us prove the second point. Assume without loss of generality that $\alpha^2 < \alpha^1$. By Proposition 1, we have $S^1 \cap S^2 \cap (\alpha^2, \alpha^1] = \emptyset$. However, the preceding point implies that $(\alpha^2, \alpha^1] \subset S^1 \cup S^2$. Using a connectedness argument, it must be that either $S^1 \cap (\alpha^2, \alpha^1]$ or $S^2 \cap (\alpha^2, \alpha^1]$ is empty. This concludes the proof of the second point. ■.

Let us now prove the Proposition. Assume without loss of generality that $\alpha^2 \leq \alpha^1$.

For $i = 1, 2$, the payoff at equilibrium $\bar{J}^i$ is continuous on $(\alpha, \alpha^1]$ by the preceding points since it is equal to R^i or G^i on this interval. Note that $\bar{J}^1$ is right-continuous at α^1 since $R^1 \leq \bar{J}^1 \leq G^1$, and that the same is true for $\bar{J}^2$ if $\alpha^2 = \alpha^1$.

By Lemma 7, $\bar{J}^i$ is continuous on any interval $[a, b]$ such that $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$. Moreover, $\bar{J}^i$ is also continuous on S^1 and on S^2. Therefore, if $\bar{J}^1$ is not right-continuous at x, it must be that $x \geq \alpha^1$, that $x \in S^1 \cup S^2$ and that for every $\varepsilon > 0$, $[x, x + \varepsilon)$ intersects both $S^1 \cup S^2$ and $\mathcal{I} \setminus (S^1 \cup S^2)$. Let us consider the case that $x \in S^1$ and $x > \alpha^1$ and assume by contradiction that $\bar{J}^i$ is not right-continuous at x. Since $(\alpha^1, \beta) \cap S^1 \cap S^2 = \emptyset$, we may choose ε sufficiently small so that $[x, x + \varepsilon) \cap S^2 = \emptyset$. If (a, b) is a connected component of the open set $[x, x+\varepsilon) \setminus S^1$, so that $a, b \in S^1$, then it must be that $\mu^2((a,b)) > 0$. By contradiction, if $\mu^2((a,b)) = 0$, then for $y \in (a,b)$:

$$\bar{J}^1(y) = J^1(y, (0, S^1), (\mu^2, S^2)) = \mathbf{E}_y[e^{-r\tau_{S^1}} R^1(X_{\tau_{S^1}})] < R^1(y),$$

where we used that $b \le x_{R^1}$ and assumption A3. This contradicts the fact that $\bar{J}^1 \ge R^1$. Therefore, there exists some $y \in (a,b)$ such that $\bar{J}^2(y) = R^2(y)$. As this is true for every connected component, it must be that there exists a decreasing sequence y_n with limit x such that $\bar{J}^2(y_n) = R^2(y_n)$ and a sequence of connected components (a_n, b_n) such that $y_n \in (a_n, b_n)$ whose length goes to zero. By Proposition 1, we have $\bar{J}^2(a_n) = G^2(a_n)$ and $\bar{J}^2(b_n) = G^2(b_n)$. Recall also that $G^2(x) - R^2(x) > 0$. On the interval (a_n, b_n), since $(0, S^2)$ is a best reply to (μ^1, S^1), we have by Lemma 6

$$\begin{aligned}
\bar{J}^2(y_n) &= \mathbf{E}_{y_n}[\int_{[0,\tau_n)} e^{-rs} G^2(X_s) d\Gamma^1_s + e^{-r\tau_n} \Lambda^1_{\tau_n -} \bar{J}^2(X_{\tau_n})] \\
&= \mathbf{E}_{y_n}[\int_{[0,\tau_n)} e^{-rs} G^2(X_s) d\Gamma^1_s + e^{-r\tau_n} \Lambda^1_{\tau_n -} G^2(X_{\tau_n})]
\end{aligned}$$

where $\tau_n := \tau_{a_n} \wedge \tau_{b_n}$. G^2 and R^2 being locally Lipschitz, there exists $\varepsilon > 0$ such that for all n sufficiently large

$$\forall y \in (a_n, b_n),\ G^2(y) > R^2(y_n) + \varepsilon.$$

We deduce that

$$\begin{aligned}
\bar{J}^2(y_n) &\ge (R^2(y_n) + \varepsilon) \mathbf{E}_{y_n}[\int_{[0,\tau_n)} e^{-rs} d\Gamma^1_s + e^{-r\tau_n} \Lambda^1_{\tau_n -}] \\
&\ge (R^2(y_n) + \varepsilon) \mathbf{E}_{y_n}[e^{-r\tau_n}].
\end{aligned}$$

We have $\mathbf{E}_{y_n}[e^{-r\tau_n}] = A_n \psi(y_n) + B_n \phi(y_n)$, where the coefficients A_n, B_n are such that

$$A_n \psi(a_n) + B_n \phi(a_n) = A_n \psi(b_n) + B_n \phi(b_n) = 1.$$

If follows easily that these coefficients are bounded, and therefore that $\mathbf{E}_{y_n}[e^{-r\tau_n}] \to 1$ by using that ψ, ϕ are locally Lipschitz. For n sufficiently large, it contradicts the fact that $\bar{J}^2(y_n) = R^2(y_n)$. We conclude that the functions $\bar{J}^i$ for $i = 1, 2$ are right-continuous at x. The cases ($x > \alpha^1$ and $x \in S^2$) and ($x = \alpha^1$ and $\alpha^1 > \alpha^2$ and $x \in S^1$) are similar.

To conclude that the payoff functions are right-continuous on $\mathcal{I}$, it remains only to prove that $\bar{J}^2$ is right-continuous at $x = \alpha^1$ if $x \in S^2$ and $\alpha^2 < \alpha^1$. Assume by contradiction that $\bar{J}^2$ is not right-continuous at x. Since $(\alpha^2, \beta) \cap S^1 \cap S^2 = \emptyset$, we may choose ε sufficiently small so that $[x, x+\varepsilon) \cap S^1 = \emptyset$ as well as

$$G^2 - R^2 > \varepsilon \text{ and } \mathcal{L}R^2 - rR^2 < -\varepsilon \text{ on } [x, x+\varepsilon).$$

Note that $\mu^1([x,x+\varepsilon)) < \infty$ since μ^1 is locally finite on $\mathcal{I}\setminus S^1$. If (a,b) is a connected component of the open set $(x,x+\varepsilon)\setminus S^2$, so that $a,b\in S^2$, then for $y\in(a,b)$ and $\tau=\tau_a\wedge\tau_b$:

$$\bar{J}^2(y) = J^2(y,(\mu^1,S^1),(0,S^2)) = \mathbf{E}_y[\int_{[0,\tau)} e^{-rs}G^2(X_s)d\Gamma^1_s + e^{-r\tau}\Lambda^1_{\tau-}R^2(X_\tau)] \geq R^2(y).$$

We deduce that

$$\begin{aligned}
\varepsilon\mathbf{E}_y[\Gamma^1_{\tau-}] &\geq \mathbf{E}_y[\int_{[0,\tau)} e^{-rs}(G^2(X_s)-R^2(X_s))d\Gamma^1_s]\\
&\geq \mathbf{E}_y[R^2(y) - e^{-r\tau}\Lambda^1_{\tau-}R^2(X_\tau) - \int_{[0,\tau)} e^{-rs}R^2(X_s)d\Gamma^1_s]\\
&= \int_0^1 \mathbf{E}_y[R^2(y) - e^{-r(\tau\wedge\hat{\gamma}^1(u,.))}R^2(X_{\tau\wedge\hat{\gamma}^1(u,.)})]du\\
&= -\int_0^1 \mathbf{E}_y[\int_0^{\tau\wedge\hat{\gamma}^1(u,.)} (\mathcal{L}R^2 - rR^2)(X_s)ds]du\\
&\geq \varepsilon\mathbf{E}_y[\int_0^\tau \Lambda^1_s ds]\\
&\geq \varepsilon\mathbf{E}_y[\Lambda^1_\tau].
\end{aligned}$$

Using that $\Gamma^1_{\tau-}=1-\Lambda^1_\tau$ and dividing by ε, we obtain

$$1 \geq 2\mathbf{E}_y[\Lambda^1_\tau] = 2\mathbf{E}_y[e^{-A^1_\tau}] \geq 2(1-\mathbf{E}_y[A^1_\tau]).$$

Using the formula (A.3), we have

$$\mathbf{E}_y[A^2_\tau] = \int_a^b 2(S'(z))^{-1}\Phi_{a,b}(y,z)d\mu^1(z) \leq C\mu^1((a,b)),$$

for some positive constant C using that $2(S'(x))^{-1}\Phi_{a,b}(y,x)$ is uniformly bounded on $[x,x+\varepsilon)$. We obtain finally

$$1 \geq 2(1-C\mu^1((a,b))).$$

There exists a sequence of connected components (a_n,b_n) of $(x,x+\varepsilon)\setminus S^2$ whose length goes to zero. Since μ^1 is locally bounded at x, it must be that $\mu^1((a_n,b_n))$ goes to zero, and the previous inequality implies $1\geq 2$, a contradiction. We conclude that $\bar{J}^2$ is right-continuous at x.

That the payoff functions are left-continuous follows from similar arguments.

We conclude that at equilibrium, the payoff functions of both players are continuous. ■.

Appendix B: Proofs for Section 4

The next Lemma solves equation (24)

Lemma 9 *The equation*

$$R^i(x_{R^i}) = \frac{\phi(x_{R^i})}{\phi(x)} G^i(x) \tag{B.1}$$

has a unique solution $\underline{x}^j \in (\alpha^i, x_{R^i})$ *and,* $R_i(x_{R^i}) < \frac{\phi(x_{R^i})}{\phi(x)} G^i(x)$ *on* $(\underline{x}^j, \beta)$.

The proof of Lemma 9 is a direct consequence of a useful and elegant change of variable introduced by Dayanik and Karatzas (2003) which we will use in several proofs. For each $x \in \mathcal{I}$, let us define $\zeta(x) := \frac{\phi(x)}{\psi(x)}$ which is strictly decreasing in x and maps $\mathcal{I}$ onto $(0, \infty)$. For any function $g : \mathcal{I} \to \mathbb{R}$, define the function $\hat{g}$ by

$$\hat{g}(y) := \frac{g}{\psi} \circ \zeta^{-1}(y), \quad y \in (0, \infty). \tag{B.2}$$

Observe that $\hat{\phi}(y) = y$ and $\hat{\psi}(y) = 1$ for every $y \in (0, \infty)$. A direct computation shows that, if $g \in \mathcal{C}^2(\mathcal{I})$, then for any $x \in \mathcal{I}$

$$\hat{g}''(\zeta(x)) = \frac{2\phi(x)^3}{(\gamma S'(x))^2 \sigma^2(x)} (\mathcal{L}g - rg)(x). \tag{B.3}$$

Thus, we deduce from assumption A7 that $(\hat{G}^i)'' \leq 0$ everywhere $(\hat{G}^i)''$ is defined. From assumption A3, $(\hat{R}^i)''(\zeta(x)) < 0$ for every $x \in (\alpha, x_0^i)$ or, equivalently, $(\hat{R}^i)''(y) < 0$ for every $y \in (\zeta(x_0^i), \infty)$. This implies that $(\hat{R}^i)''(y) < 0$ for every $y \in (\zeta(x_{R^i}), \infty)$ since $x_{R^i} < x_0^i$. We will use also the following remarks. From Lemma 1 and assumption A6, we have $G^i > 0$ over $\mathcal{I}$. Thus, $\hat{G}^i > 0$ over $(0, \infty)$, and (11) implies

$$\lim_{y \to 0} \hat{G}^i(y) = 0, \tag{B.4}$$

$$\lim_{y \to \infty} \frac{\hat{G}^i(y)}{y} = 0. \tag{B.5}$$

Proof of Lemma 9 Let define $f(x) := \frac{R^i(x_{R^i})}{\phi(x_{R^i})} \phi(x)$. Notice that $f = V_{R^i} \geq R^i$ on $[x_{R^i}, \beta)$ and that thanks to the smooth-fit property, f is tangent to R^i at x_{R^i}. Applying to f the change of variable formula (B.2), a direct computation shows that, $\underline{x}^i$ is a solution to (B.1) iff $\zeta(\underline{x}^i)$ is a solution to

$$\hat{f}(y) = \hat{G}^i(y), \tag{B.6}$$

which is equivalent to

$$\frac{R^i(x_{R^i})}{\phi(x_{R^i})} y = \hat{G}^i(y).$$

Using that $f = V_{R^i}$ on $[x_{R^i}, \beta)$, we deduce from assumption A6 that $\hat{f}(y) < \hat{G}^i(y)$ on $(0, \zeta(x_{R^i})]$. It follows that Equation (B.1) admits a unique solution $\underline{x}^j < x_{R^i}$ because $\hat{G}^i$ is positive concave and satisfies (B.5). Moreover, we have $\frac{\phi(x)}{\phi(x_{R^i})} R^i(x_{R^i}) < G^i(x)$ on $(\underline{x}^j, \beta)$ and $\frac{\phi(x)}{\phi(x_{R^i})} R^i(x_{R^i}) > G^i(x)$ on $(\alpha, \underline{x}^j)$. Finally, note that $\alpha^i \leq x_{R^i} \leq x_0^i$ and that $\hat{R}^i$ is strictly concave on $(\zeta(x_0^i), +\infty)$. Therefore, $\hat{f} > \hat{R}^i$ on $(\zeta(x_{R^i}), +\infty)$ since $\hat{f}$ is linear and tangent to $\hat{R}^i$ at $\zeta(x_{R^i})$. If $\alpha^i > \alpha$, because $G^i = R^i$ over $(\alpha, \alpha^i]$, it must be that $\alpha^i < \underline{x}^j$. ■

Proof of Proposition 5. We show in the next lemma that the variational systems (30)-(34) and (35)-(40) admit a solution. Proposition 5 is then a direct consequence of Theorem 3 as we explained in the main text.

Lemma 10 *In the running example, if the firms' liquidation values $l^1 \leq l^2$ are close to each other, m is sufficiently large and $b > 0$ then, there exists a constant $a > 0$, and two functions $w^1 \in \mathcal{C}^0((\alpha, \beta)) \cap \mathcal{C}^2((\alpha, \beta) \setminus \{\underline{x}^1\})$ and $w^2 \in \mathcal{C}^0((\alpha, \beta)) \cap \mathcal{C}^2((\alpha, \beta) \setminus \{\underline{x}^1, x_{R^1}\})$ solution to the variational systems (30)-(34) and (35)-(40).*

Proof of Proposition 5. Using notations of section 2.3, we have $\phi(x) = x^{\rho^-}$, $\psi(x) = x^{\rho^+}$ and

$$V_{R^i}(x) \equiv \sup_{\tau \in \mathcal{T}_X} \mathbf{E}_x[e^{-r\tau} R^i(X_\tau)] = \begin{cases} \frac{\phi(x)}{\phi(x_{R^i})} \left(-\frac{1}{r-\mu} x_{R^i} + l^i\right) & \text{if } \ x > x_{R^i}, \\ -\frac{1}{r-\mu} x + l^i & \text{if } \ x \leq x_{R^i}. \end{cases} \tag{B.7}$$

Also,

$$\begin{aligned} V_m^i(x) &= \sup_{\tau} \mathbf{E}_x\Big[\int_0^\tau e^{-rs} m X_s \, ds + l^i\Big] && \text{(B.8)} \\ &= \begin{cases} \frac{m}{r-\mu} x + \frac{\phi(x)}{\phi(\alpha^i)}\left(-\frac{m}{r-\mu}\alpha^i + l^i\right) & \text{if } \ x \geq \alpha^i, \\ l^i & \text{if } \ x \leq \alpha^i, \end{cases} && \text{(B.9)} \end{aligned}$$

with $\alpha^i = \frac{1}{m} x_{R^i} < x_{R^i}$ and $x_{R^i} = \frac{\rho^-(r-\mu)}{\rho^- - 1} l_i$. Thus,

$$\begin{aligned} G^i(x) &= (V_m^i - E)(x) && \text{(B.10)} \\ &= \begin{cases} \frac{m-1}{r-\mu} x + \frac{\phi(x)}{\phi(\alpha^i)}\left(-\frac{m}{r-\mu}\alpha^i + l^i\right) & \text{if } \ x \geq \alpha^i, \\ -\frac{1}{r-\mu} x + l^i & \text{if } \ x \leq \alpha^i. \end{cases} && \text{(B.11)} \\ &= \begin{cases} \frac{m-1}{r-\mu} x + l^i \frac{\phi(x)}{\phi(\alpha^i)(1-\rho^-)} & \text{if } \ x \geq \alpha^i, \\ -\frac{1}{r-\mu} x + l^i & \text{if } \ x \leq \alpha^i. \end{cases} && \text{(B.12)} \end{aligned}$$

Equation (24) that we write under the form

$$\frac{\phi(x)}{\phi(x_{R^1})} R^1(x_{R^1}) = G^1(x), \tag{B.13}$$

has a unique solution $\underline{x}^2 \in (\alpha^1, x_{R^1})$ which is explicit. Precisely, (B.13) is equivalent to

$$\frac{x^{\rho^-}}{\left(l^1(r-\mu)\frac{\rho^-}{\rho^--1}\right)^{\rho^-}}\left[l^1 - \frac{1}{(r-\mu)}l^1(r-\mu)\frac{\rho^-}{\rho^--1}\right] = \frac{m-1}{r-\mu}x + l^1\frac{x^{\rho^-}m^{\rho^-}}{\left(l^1(r-\mu)\frac{\rho^-}{\rho^--1}\right)^{\rho^-}(1-\rho^-)}.$$

Multiplying by $\left(l^1(r-\mu)\frac{\rho^-}{\rho^--1}\right)^{\rho^-}$ and dividing by xl^1, we obtain

$$x^{\rho^--1}\frac{1}{1-\rho^-} = \frac{m-1}{r-\mu}(l^1)^{\rho^--1}\left((r-\mu)\frac{\rho^-}{\rho^--1}\right)^{\rho^-} + \frac{x^{\rho^--1}m^{\rho^-}}{(1-\rho^-)}.$$

$$x^{\rho^--1}\frac{1-m^{\rho^-}}{1-\rho^-} = (m-1)(r-\mu)^{\rho^--1}(l^1)^{\rho^--1}\left(\frac{\rho^-}{\rho^--1}\right)^{\rho^-}.$$

$$x = l^1(r-\mu)\frac{\rho^-}{\rho^--1}\left[\frac{(1-m^{\rho^-})}{(-\rho^-)(m-1)}\right]^{\frac{1}{1-\rho^-}}.$$

We conclude that

$$\underline{x}^2 = l^1(r-\mu)\frac{\rho^-}{\rho^--1}\theta = \theta x_{R^1}.$$

with $\theta := \theta(m,\rho^-) = \left[\frac{(1-m^{\rho^-})}{(-\rho^-)(m-1)}\right]^{\frac{1}{1-\rho^-}} \in (1/m, 1)$.
It follows that the function w^1 defined by

$$w^1(x) = \begin{cases} \frac{\phi(x)}{\phi(x_{R^1})}\left(-\frac{1}{r-\mu}x_{R^1} + l^i\right) & \text{if } x > \underline{x}^2, \\ G^1(x) & \text{if } x \le \underline{x}^2. \end{cases} \tag{B.14}$$

is, by construction, solution to the variariational system (30)-(34).

If a solution w^2 to (35)-(40) exists, then letting T_x^2 denote the unique solution to $\mathcal{L}u - ru = 0$ which is tangent to R^2 at x, it must be that $w^2 = T^2_{\underline{x}^2}$ on $(\underline{x}^2, x_{R^1})$ and that $w^2 = A\phi$ on (x_{R^1}, ∞) for some constant A. Precisely, we have $T^2_{\underline{x}^2} = B\psi + C\phi$ with $B, C > 0$ given by

$$B = \frac{\phi'(\underline{x}^2)(\frac{1}{r-\mu}\underline{x}^2 - l^2) - \frac{1}{r-\mu}\phi(\underline{x}^2)}{\psi'(\underline{x}^2)\phi(\underline{x}^2) - \psi(\underline{x}^2)\phi'(\underline{x}^2)} > 0$$
$$C = \frac{\psi(\underline{x}^2)\frac{1}{r-\mu} + \psi'(\underline{x}^2)(-\frac{1}{r-\mu}\underline{x}^2 + l^2)}{\psi'(\underline{x}^2)\phi(\underline{x}^2) - \psi(\underline{x}^2)\phi'(\underline{x}^2)} > 0$$

Continuity at x_{R^1} yields

$$A = B\frac{\psi(x_{R^1})}{\phi(x_{R^1})} + C,$$

which leads to

$$\Delta(w^2)'(x_{R^1}) = B\left(\frac{\psi(x_{R^1})}{\phi(x_{R^1})}\phi'(x_{R^1}) - \psi'(x_{R^1})\right) < 0.$$

We deduce that, if

$$G^2(x_{R^1}) > T^2_{\underline{x}^2}(x_{R^1}) > T^2_{x_{R^2}}(x_{R^1}) \tag{B.15}$$

then, $w^2 = R^2 1_{(0,\underline{x}^2]} + T^2_{\underline{x}^2} 1_{[\underline{x}^2, x_{R^1}]} + A\phi 1_{(x_{R^1},\infty)}$ and, $a = -\frac{1}{2}\,\Delta w^{2\prime}(x_{R^1})/(G^2(x_{R^1}) - w^2(x_{R^1}))$ is a solution to the variational system (35)-(40). The first inequality in (B.15) ensures that $a > 0$, the second inequality ensures that $\underline{x}^2 < x_{R^2}$, and it also implies that $w^2 \geq R^2$ on $[x_{R^1}, \infty)$. The inequality $w^2 \geq R^2$ on $(\underline{x}^2, x_{R^1})$ follows from the convexity of $T^2_{\underline{x}^2}$. We show below that, if l^1, l^2 are close to each other, m is sufficiently large, and $b > 0$ then, (B.15) holds true.

Letting $\gamma = \rho^+ - \rho^-$, direct computations lead to

$$\begin{aligned}
B &= \frac{1}{\gamma}\left[-\frac{(1-\rho^-)}{r-\mu}(\underline{x}^2)^{1-\rho^+} - l^2\rho^-(\underline{x}^2)^{-\rho^+})\right] \\
&= \frac{\rho^-(\underline{x}^2)^{-\rho^+}}{\gamma}\left[\theta l^1 - l^2\right]
\end{aligned}$$

$$\begin{aligned}
C &= \frac{1}{\gamma}\left[\frac{1-\rho^+}{r-\mu}(\underline{x}^2)^{1-\rho^-} + l^2\rho^+(\underline{x}^2)^{-\rho^-}\right] \\
&= \frac{\rho^+(\underline{x}^2)^{-\rho^-}}{\gamma}\left[l^2 - \theta\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1}l^1\right]
\end{aligned}$$

Using that $\underline{x}^2 = \theta x_{R^1}$, we deduce that

$$\begin{aligned}
T^2_{\underline{x}^2}(x_{R^1}) &= B(x_{R^1})^{\rho^+} + C(x_{R^1})^{\rho^-} \\
&= \frac{\rho^-\theta^{-\rho^+}}{\gamma}\left[\theta l^1 - l^2\right] + \frac{\rho^+\theta^{-\rho^-}}{\gamma}\left[l^2 - \theta\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1}l^1\right]
\end{aligned}$$

We have $T^2_{x_{R^2}} = \frac{l^2}{(1-\rho^-)\phi(x_{R^2})}\phi$ so that

$$T^2_{x_{R^2}}(x_{R^1}) = \frac{l^2}{(1-\rho^-)\phi(x_{R^2})}\phi(x_{R^1}) = \frac{l^2}{(1-\rho^-)}\left(\frac{l^1}{l^2}\right)^{\rho^-}.$$

We also have (with equality if $x_{R^1} \geq \alpha^2$)

$$\begin{aligned}
G^2(x_{R^1}) &\leq \frac{m-1}{r-\mu}x_{R^1} + l^2\frac{\phi(x_{R^1})}{\phi(\alpha^2)(1-\rho^-)} \\
&= (m-1)\frac{\rho^-}{\rho^--1}l^1 + l^2\left(\frac{l^1}{l^2}\right)^{\rho^-}\frac{m^{\rho^-}}{1-\rho^-}.
\end{aligned}$$

Therefore, a sufficient condition for (B.15) is

$$
\begin{aligned}
(m-1)\frac{\rho^-}{\rho^--1}l^1 + l^2\left(\frac{l^1}{l^2}\right)^{\rho^-}\frac{m^{\rho^-}}{1-\rho^-}
&> \frac{\rho^-\theta^{-\rho^+}}{\gamma}\left[\theta l^1 - l^2\right] + \frac{\rho^+\theta^{-\rho^-}}{\gamma}\left[l^2 - \theta\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1}l^1\right] \\
&> \frac{l^2}{(1-\rho^-)}\left(\frac{l^1}{l^2}\right)^{\rho^-}.
\end{aligned}
$$

Let us rewrite these inequalities with $l^1 = l^2$:

$$
\begin{aligned}
(m-1)\frac{\rho^-}{\rho^--1} + \frac{m^{\rho^-}}{1-\rho^-}
&> \frac{\rho^-\theta^{-\rho^+}}{\gamma}\left[\theta - 1\right] + \frac{\rho^+\theta^{-\rho^-}}{\gamma}\left[1 - \theta\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1}\right] \\
&> \frac{1}{(1-\rho^-)}.
\end{aligned}
$$

We observe that, for m large enough

$$
(m-1)\frac{\rho^-}{\rho^--1} + \frac{m^{\rho^-}}{1-\rho^-} > \frac{1}{(1-\rho^-)}.
$$

Noticing that $1-\theta\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1} > 1 - \frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1} > 0$, we also have that

$$
\frac{\rho^-\theta^{-\rho^+}}{\gamma}\left[\theta - 1\right] + \frac{\rho^+\theta^{-\rho^-}}{\gamma}\left[1 - \theta\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1}\right] > \frac{1}{(1-\rho^-)}.
$$

We finally show that, for m large enough and $b > 0$,

$$
(m-1)\frac{\rho^-}{\rho^--1} + \frac{m^{\rho^-}}{1-\rho^-} > \frac{\rho^-\theta^{-\rho^+}}{\gamma}\left[\theta - 1\right] + \frac{\rho^+\theta^{-\rho^-}}{\gamma}\left[1 - \theta\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1}\right]. \tag{B.16}
$$

Recall that $\theta = \left[\frac{(1-m^{\rho^-})}{(-\rho^-)(m-1)}\right]^{\frac{1}{1-\rho^-}}$. We deduce that a sufficient condition for (B.16) is that, for m large enough

$$
m > \frac{\rho^- m^{\frac{\rho^+}{1-\rho^-}}}{\gamma}\left(m^{\frac{1}{\rho^--1}} - 1\right) + \frac{\rho^+ m^{\frac{\rho^-}{1-\rho^-}}}{\gamma}\left(1 - m^{\frac{1}{\rho^--1}}\frac{\rho^+-1}{\rho^+}\frac{\rho^-}{\rho^--1}\right).
$$

The latter inequality holds true for m large when $\rho^+ + \rho^- < 1$ which is equivalent to $b > 0$. A simple continuity argument ends the proof of the Lemma. ■

Appendix C: Proofs for Section 5

The proof of Theorem 2 is mainly based on Proposition 1 and on assertions (i), (ii) and (iii) of the following simple lemma. We will use later assertion (iv) in the proof of Theorem 3.

Lemma 11 *Let u be a C^2 function defined on an open interval $(a,b) \subset \mathcal{I}$ which satisfies*

$$\forall x \in (a,b),\ \mathcal{L}u(x) - ru(x) = 0.$$

Then, we have

(i) If $b = \beta$, $u(\beta-) = 0$, $u(a+) = R^i(a)$ and $u(x) \geq V_{R^i}(x)$ for all $x \in (a,\beta)$, then $a = x_{R^i}$.

(ii) If $u(x) \geq V_{R^i}(x)$ for all $x \in (a,b)$ then $\{x \in (a,b) \,|\, u(x) = R^i(x)\}$ contains at most one point.

(iii) If there exist two points $a', b' \in (a,b)$ with $b' \leq x_{R^i}$ and $u(a') = R^i(a')$, and $u(b') = R^i(b')$, then $u(x) < R^i(x)$ for all $x \in (a',b')$.

(iv) Assume that $\alpha < a \leq x_{R^i}$, $u \geq R^i$ on (a,b), $u(a) = R^i(a)$ and $u'(a+) > (R^i)'(a)$. Then, for every $\varepsilon > 0$ sufficiently small, the function f solution of $\mathcal{L}f - rf = 0$ on $(a-\varepsilon, a+\varepsilon)$ with $f(a-\varepsilon) = R^i(a)$ and $f(a+\varepsilon) = u(a+\varepsilon)$ satisfies $f(a) > u(a)$.

Proof of Lemma 11. Recall that a solution $u \in \mathcal{C}^2((a,b))$ of the equation $\mathcal{L}u(x) - ru(x) = 0$ writes under the form $u(x) = A\psi(x) + B\phi(x)$ where A and B are real numbers. Using the change of variables (B.2), the inequation $u(x) \geq V_{R^i}(x)$ on (a,b) is equivalent to $\hat{u}(z) = Az + B \geq \hat{V}_{R^i}(z)$ for all $z \in (\zeta(b), \zeta(a))$.

Recall that $\hat{V}_{R^i}$ is $\mathcal{C}^1$, that $\hat{V}_{R^i}(z) = C^i z > \hat{R}^i(z)$ for all $z \in (0, \zeta(x_{R^i}))$ for some $C^i > 0$, and that $\hat{V}_{R^i} = \hat{R}^i$ and is $\mathcal{C}^2$ and strictly concave on $[\zeta(x_{R^i}), \infty)$.

Proof of (i). The assumption $u(\beta-) = 0$ implies $B = 0$, and thus $\hat{u}(0+) = 0$. The assumption $\hat{u} \geq \hat{V}_{R^i}$ on $(0, \zeta(a))$, implies that $A \geq C^i$. If this inequality is strict, then $Az > C^i z \geq \hat{R}^i(z)$ for all $z > 0$, since $\hat{V}_{R^i}$ is concave, which would contradict the assumption $\hat{u}(\zeta(a)-) = A\zeta(a) = \hat{R}^i(\zeta(a))$. Therefore $A = C^i$ and from the properties of $\hat{V}_{R^i}$, the unique solution of $Az = \hat{R}^i(z)$ is $\zeta(x_{R^i})$.

Proof of (ii). Note that $V_{R^i} > R^i$ on (x_{R^i}, β), so that $\hat{V}_{R^i} > \hat{R}^i$ on $(0, \zeta(x_{R^i}))$. Therefore, if there exists $\bar{z} \in (\zeta(b), \zeta(a))$ such that $\hat{u}(\bar{z}) = \hat{R}^i(\bar{z})$, it must be that $\bar{z} \geq \zeta(x_{R^i})$. In such a case, $\hat{u}$ is a tangent to the concave $\mathcal{C}^1$ map $\hat{V}_{R^i}$ at $\bar{z}$. On the interval $[\zeta(x_{R^i}), +\infty)$, $\hat{V}_{R^i} = \hat{R}^i$ is strictly concave and thus $\hat{u}(z) > \hat{R}^i(z)$ for all $z \neq \bar{z}$ in $[\zeta(x_{R^i}), +\infty) \cap (\zeta(b), \zeta(a))$, which concludes the proof.

Proof of (iii). $\hat{u}$ is an affine map on the interval $[\zeta(b'), \zeta(a')]$ which is equal to $\hat{R}^i$ at both boundaries. The inequality follows directly from the fact that $\hat{R}^i$ is strictly concave on $[\zeta(x_{R^i}), \infty)$ given that $b' \leq x_{R^i}$.

Proof of (iv). We have $\hat{u}(z) = Az + B$ for some constants A, B on $[\zeta(b), \zeta(a)]$. Since $a \leq x_{R^i}$, the map $\hat{R}^i$ is concave on $[\zeta(a), +\infty)$. A direct computation shows that

$$A = \hat{u}'(\zeta(a)-) = \frac{\psi(a)u'(a+) - \psi'(a)u(a)}{\psi(a)^2\zeta'(a)}, \; (\hat{R}^i)'(\zeta(a)) = \frac{\psi(a)(R^i)'(a) - \psi'(a)R^i(a)}{\psi(a)^2\zeta'(a)},$$

so that

$$(\hat{R}^i)'(\zeta(a)) - \hat{u}'(\zeta(a)-) = \frac{(R^i)'(a) - u'(a+)}{\psi(a)\zeta'(a)} > 0,$$

using that $u(a) = R^i(a)$, $\zeta'(a) < 0$ and $u'(a+) > (R^i)'(a)$. We deduce that for ε small enough, $\hat{R}^i(\zeta(a-\varepsilon)) > A\zeta(a-\varepsilon) + B$. The function f satisfying $\mathcal{L}f - rf = 0$ on $(a-\varepsilon, a+\varepsilon)$, $f(a-\varepsilon) = R^i(a-\varepsilon)$ and $f(a+\varepsilon) = u(a+\varepsilon)$ is such that $\hat{f}(z) = A'z + B'$ for some constants A', B' with $\hat{f}(\zeta(a-\varepsilon)) = \hat{R}^i(\zeta(a-\varepsilon))$ and $\hat{f}(\zeta(a+\varepsilon)) = \hat{u}(\zeta(a+\varepsilon))$. We deduce that

$$A\zeta(a+\varepsilon) + B = A'\zeta(a+\varepsilon) + B \text{ and } A'\zeta(a-\varepsilon) + B' > A\zeta(a-\varepsilon) + B,$$

which imply $A'\zeta(a) + B' > A\zeta(a) + B$ or equivalently $f(a) > u(a)$. ∎

Proof of Theorem 2. For $i \in \{1, 2\}$, let us define $q_1^i := \max(\operatorname{supp}(\mu^i))$ with the convention $\max \emptyset = \alpha$. We assume without loss of generality that $q_1^1 \geq q_1^2$. From assertion (iv) of Proposition 1, we have that $x_{R^i} \geq q_1^i \vee s^i$. The proof consists of four steps and uses repeatedly Lemma 11. In the following, T_q^i denotes the curve solution to $\mathcal{L}u - ru = 0$ and tangent to R^i at q where $q \leq x_{R^i}$.

Step 1. We show that

(i) If $q_1^1 > s^1 \vee s^2$ then $q_1^1 = x_{R^1} > q_1^2$, and x_{R^1} is an isolated point of the support of μ^1.

(ii) If $q_1^2 > s^1 \vee s^2$ then, q_1^2 is an isolated point of the support of μ^2.

Note that the assumption $q_1^1 > s^1 \vee s^2$ implies $q_1^1 > \alpha$. From Proposition 1-(v), the strategy $(0, S^1)$ is a PBR to strategy (μ^2, S^2). Therefore, τ_{S^1} is an optimal solution to $\bar{J}^1(x, (\mu^2, S^2)) = \sup_{\tau^1 \in \mathcal{T}} J^1(x, \tau^1, (\mu^2, S^2))$. Letting $y = q_1^2 \vee s^1 \vee s^2$, we have $\tau_{S^1} \geq \tau_y$ $\mathbf{P}_x$-almost surely for all $x \in (y, \beta)$. We deduce that the best reply value function $\bar{J}^1$ satisfies, for any $x \in (y, \beta)$,

$$\begin{aligned}\bar{J}^1(x) := \bar{J}^1(x, (\mu^2, S^2)) &= J^1(x, (\mu^1, S^1), (\mu^2, S^2)) = J^1(x, (0, S^1), (\mu^2, S^2)) \\ &= \mathbf{E}_x\left[e^{-r\tau_y}\bar{J}^1(X_{\tau_y})\right] = \mathbf{E}_x\left[e^{-r\tau_y}\bar{J}^1(y)\right],\end{aligned}$$

where the strong Markov Property (A.7) yields the penultimate equality. It follows from (8) that

$$\lim_{x\to\beta} \bar{J}^1(x) = 0 \text{ and } \mathcal{L}\bar{J}^1(x) - r\bar{J}^1(x) = 0 \text{ on } (y,\beta). \tag{C.1}$$

Furthermore, we have that

$$\bar{J}^1(x) \geq V_{R^1}(x) \text{ on } (y,\beta), \text{ and, } \bar{J}^1(q_1^1) = R^1(q_1^1), \tag{C.2}$$

where the last equality follows from assertions (ii) and (iii) of Proposition 1. Finally, Lemma 11-(i) together with (C.1) and (C.2) imply that $q_1^1 = x_{R^1}$.

By way of contradiction, assume that $q_1^1 = q_1^2$. Applying the same arguments than above and exchanging the roles of the two players, we obtain $q_1^2 = x_{R^2} \neq x_{R^1}$, a contradiction. Therefore, $q_1^1 > q_1^2$ and thus $q_1^1 > y$. Lemma 11-(ii) together with (C.1) and (C.2) imply

$$q_1^1 = x_{R^1}, \text{ and, } (y,\beta) \cap \{x \in (\alpha,\beta) \,|\, \bar{J}^1(x) = R^1(x)\} = \{x_{R^1}\}. \tag{C.3}$$

Thus, $q_1^1 = x_{R^1}$ is an isolated point of the support of μ^1. Observe that $\bar{J}^1 = T^1_{q_1^1}$ on (y,β).

To prove (ii), we first define for $i = 1,2$

$$q_2^1 = \max(\mathrm{supp}(\mu^1) \setminus \{q_1^1\}) < q_1^1,$$

where the strict inequality follows from (i). Note that the assumption $q_1^2 > s^1 \vee s^2$ implies $y = q_1^2 > s^1 \vee s^2$. It follows that $q_1^2 > q_2^1$. Indeed, if $q_1^1 > q_2^1 \geq q_1^2$, then the strong Markov property implies that, for any $x \in (q_2^1, q_1^1)$, $\bar{J}^1(x) = \mathbf{E}_x[e^{-r\tau_{q_1^1}\wedge\tau_{q_2^1}} \bar{J}^1(X_{\tau_{q_1^1}\wedge\tau_{q_2^1}})]$. We deduce from a standard computation that $\bar{J}^1$ satisfies $\mathcal{L}\bar{J}^1 - r\bar{J}^1 = 0$ on (q_2^1, q_1^1). We also have $\bar{J}^1(q_1^1) = R^1(q_1^1)$, and $\bar{J}^1(q_2^1) = R^1(q_2^1)$. It follows from Lemma 11 - (iii) that $\bar{J}^1(x) < R^1(x)$ on (q_2^1, q_2^1), thus a contradiction.

Thus, we have that $q_1^1 > q_1^2 > q_2^1$. Then, given $x \in (q_2^1, q_1^1)$, the strong Markov property leads to $\bar{J}^2(x) = \mathbf{E}_x[e^{-r\tau_{q_1^1}\wedge\tau_{q_2^1}} \bar{J}^2(X_{\tau_{q_1^1}\wedge\tau_{q_2^1}})]$ with

$$\bar{J}^2(x) \geq V_{R^2}(x) \text{ on } (q_2^1, q_1^1), \text{ and } \bar{J}^2(q_1^2) = R^2(q_1^2). \tag{C.4}$$

We apply Lemma 11-(ii) to $\bar{J}^2$ and we get that

$$\bar{J}^2(x) > R^2(x) \text{ on } (q_2^1, q_1^1) \setminus \{q_1^2\}. \tag{C.5}$$

Thus, q_1^2 is an isolated point of the support of μ^2. Observe that $\bar{J}^2 = T^2_{q_1^2}$ on (q_2^1, q_1^1).

Define then

$$q_2^2 = \max(\mathrm{supp}(\mu^2) \setminus \{q_1^2\}) < q_1^2.$$

Step 2. We show that

(i) If $q_2^1 > s^1 \vee s^2$ then, q_2^1 is an isolated point of the support of μ^1.

(ii) If $q_2^2 > s^1 \vee s^2$ then, q_2^2 is an isolated point of the support of μ^2.

Let us consider that $q_2^1 > s^1 \vee s^2$ and let us show that q_2^1 is an isolated point of the support of μ^1. The same reasoning than in step 1-(ii) applies, and the inequality $q_2^1 > q_2^2$ is proven by contradiction using Lemma 11-(iii). Then, we have

$$\bar{J}^1(x) = \mathbf{E}_x[e^{-r\tau_{q_1^2}\wedge\tau_{q_2^2}}\bar{J}^1(X_{\tau_{q_1^2}\wedge\tau_{q_2^2}})] \geq V_{R^1}(x) \text{ on } (q_2^2, q_1^2) \text{ and } \bar{J}^1(q_2^1) = R^1(q_2^1), \tag{C.6}$$

and Lemma 11-(ii) implies

$$\bar{J}^1(x) > R^1(x) \text{ on } (q_2^2, q_1^2) \setminus \{q_2^1\}. \tag{C.7}$$

Thus, q_2^1 is an isolated point of the support of μ^1.

Observe that $\bar{J}^1 = T^1_{q_2^1}$ on (q_2^2, q_1^2). Because $\bar{J}^1$ is continuous from Proposition 2, $T^1_{q_1^1}$ and $T^1_{q_2^1}$ intersects at q_1^2 and we have that $\bar{J}^1(q_1^2) = T^1_{q_1^1}(q_1^2) = T^1_{q_2^1}(q_1^2)$.

Let us assume that $q_2^2 > s^1 \vee s^2$. An analogous reasoning yields that q_2^2 is an isolated point of the support of μ^2. The curve $T^2_{q_1^2}$ and $T^2_{q_2^2}$ intersects at q_2^1 and we have $\bar{J}^2(q_2^1) = T^2_{q_1^2}(q_2^1) = T^2_{q_2^2}(q_2^1)$.

Step 3. Proceeding by induction, let us define for all $k \geq 1$ and $i = 1, 2$

$$q_{k+1}^i = \max(\text{supp}(\mu_i) \setminus \{q_1^i, q_2^i, ..., q_k^i\}).$$

Assume that $(q_l^1)_{1\leq l\leq k}$ and $(q_l^2)_{1\leq l\leq k}$ are such that $q_1^1 > q_1^2 > q_2^1 > q_2^2 > ... > q_{k-1}^1 > q_{k-1}^2 > q_k^1 > q_k^2$ and that:

- for any $2 \leq l \leq k$, $\bar{J}^1 = T^1_{q_l^1}$ on (q_l^2, q_{l-1}^2) and, $T^1_{q_{l-1}^1}$ and $T^1_{q_l^1}$ intersects at q_{l-1}^2.
- for any $2 \leq l \leq k$, $\bar{J}^2 = T^2_{q_l^2}$ on (q_{l+1}^1, q_l^1) and, $T^2_{q_{l-1}^2}$ and $T^2_{q_l^2}$ intersects at q_l^1.

The same arguments as in step 2 show that

(i) If $q_{k+1}^1 > s^1 \vee s^2$ then, q_{k+1}^1 is an isolated point of the support of μ^1 and, $T^1_{q_k^1}$ and $T^1_{q_{k+1}^1}$ intersects at q_k^2.

(ii) If $q_{k+1}^2 > s^1 \vee s^2$ then, q_{k+1}^2 is an isolated point of the support of μ^2 and, $T^2_{q_k^2}$ and $T^2_{q_{k+1}^2}$ intersects at q_{k+1}^1.

Step 4. If for some $k \geq 1$ and $i = 1, 2$, we have $q_k^i \leq s^1 \vee s^2$, then applying the induction step a finite number of times, we conclude that the restrictions of μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$ have finite support, which concludes the proof.

Otherwise, we may apply the induction step infinitely many times, and we obtain two decreasing sequences $(q_n^1)_{n\geq 1}$, $(q_n^2)_{n\geq 1}$ with $q_1^1 > q_1^2 > q_2^1 > q_2^2 > ... > q_n^1 > q_n^2 > ...$ where $q_1^1 = x_{R^1}$, $q_1^2 < x_{R^1}$ is given and where for any $n \geq 2$,

- $T^1_{q_n^1}$ and $T^1_{q_{n-1}^1}$ intersects at q_{n-1}^2.
- $T^2_{q_{n-1}^2}$ and $T^2_{q_n^2}$ intersects at q_n^1.

Lemma 12 below shows that any decreasing sequences $(q_n^1)_{n\geq 1}$, $(q_n^2)_{n\geq 1}$ satisfying these properties converge to α. Thus, if for any $n \geq 1$, $q_n^i > s^1 \vee s^2$ then, $S^1 = S^2 = \emptyset$ and, $\text{supp}\mu^i = \{q_k^i,\ k \geq 1\}$, for $i = 1, 2$. This concludes the proof of Theorem 2. ■

Lemma 12 *The two decreasing sequences $(q_n^1)_{n\in\mathbb{N}}$ and $(q_n^2)_{n\in\mathbb{N}}$ converge to α.*

Proof of Lemma 12.

The proof makes extensive use of change of variable (B.2). As a preliminary, observe that, using the change of variable (B.2), assumption **A8** implies that $(\hat{R}^i)''$ is locally Lipschitz.

Let us assume by contradiction that $\lim q_i^n = \bar{q} > \alpha$.

Let $T^i(q)$ denote the curve solution to $\mathcal{L}u - ru = 0$ and tangent to R^i at q for $q \geq x_{R^i}$. This curve is above R^i for $x > x_{R^i}$ and is equal to $A(q)\psi + B(q)\phi$ for some positive coefficients $A(q), B(q)$.

Define $z^i = \zeta(x_{R^i})$ and note that for any $z \geq z^i$, then $\hat{T}_z^i = \widehat{T^i_{\zeta^{-1}(z)}}$ is the affine map tangent line to the function $\hat{R}^i$ at z given by:

$$\hat{T}_z^i(y) = A(\zeta^{-1}(z))y + B(\zeta^{-1}(z)) = \hat{R}^i(z) + \hat{R}'^i(z)(y - z).$$

We define $y_{2n} = \zeta(q_n^2)$ and $y_{2n-1} = \zeta(q_n^1)$ for $n \geq 1$.
As a consequence of the proof of Theorem 2, the following recursive relations are satisfied:

$$\hat{R}^1(y_{2n-1}) + (\hat{R}^1)'(y_{2n-1})(y_{2n} - y_{2n-1}) = \hat{R}^1(y_{2n+1}) + (\hat{R}^1)'(y_{2n+1})(y_{2n} - y_{2n+1}).$$

$$\hat{R}^2(y_{2n}) + (\hat{R}^2)'(y_{2n})(y_{2n+1} - y_{2n}) = \hat{R}^2(y_{2n+2}) + (\hat{R}^2)'(y_{2n+2})(y_{2n+1} - y_{2n+2}).$$

In words,

- The tangent line $\hat{T}^1_{y_{2n-1}}$ to the function $\hat{R}^1$ at y_{2n-1} and the tangent line $\hat{T}^1_{y_{2n+1}}$ to the function $\hat{R}^1$ at y_{2n+1} intersect at y_{2n}.
- The tangent line $\hat{T}^2_{y_{2n}}$ to the function $\hat{R}^2$ at y_{2n} and the tangent line $\hat{T}^2_{y_{2n+2}}$ to the function $\hat{R}^2$ at y_{2n+2} intersect at y_{2n+1}.

The above equations for $i = 1, 2$ can be written as (with $x < y < z$ three consecutive terms of the sequence (y_n))

$$\hat{R}^i(x) + (\hat{R}^i)'(x)(y - x) - \hat{R}^i(y) = \hat{R}^i(z) + (\hat{R}^i)'(z)(y - z) - \hat{R}^i(y) \tag{C.8}$$

Using Taylor's theorem with integral remainder, (C.8) is equivalent to

$$-\int_x^y (y - s)(\hat{R}^i)''(s)ds = -\int_y^z (s - y)(\hat{R}^i)''(s)ds. \tag{C.9}$$

Note that $(\hat{R}^i)'' < 0$ on $[y_1, \infty)$, so that the right-hand side of (C.9) is increasing in z. Therefore, given $y_1 \le x < y$, if a solution $z > y$ exists, it is unique.

By assumption, we have $y_n \to \bar{y} = \zeta(\bar{q}) < \infty$.

Since $(\hat{R}^i)''$ is locally Lipschitz, there exists K such that for all n and $i = 1, 2$,

$$\forall s, y \in [y_1, \bar{y}],\ |(\hat{R}^i)''(s) - (\hat{R}^i)''(y)| \le K|s - y|.$$

We deduce that

$$-\int_x^y (y - s)(\hat{R}^i)''(s)ds \ge -(\hat{R}^i)''(y)\frac{(y - x)^2}{2} - K\frac{(y - x)^3}{3}$$

$$-\int_y^z (s - y)(\hat{R}^i)''(s)ds \le -(\hat{R}^i)''(y)\frac{(z - y)^2}{2} + K\frac{(z - y)^3}{3}$$

From equation (C.9), we obtain

$$(z - y)^2 + 2K\frac{(z - y)^3}{3|(\hat{R}^i)''(y)|} \ge (y - x)^2 - 2K\frac{(y - x)^3}{3|(\hat{R}^i)''(y)|}$$

Let C such that for all $y \in [y_1, \bar{y}]$ and $i = 1, 2$

$$\frac{2K}{3|(\hat{R}^i)''(y)|} \le C.$$

Then, we have

$$(z - y)^2 + C(z - y)^3 \ge (y - x)^2 - C(y - x)^3.$$

Define $u_n = y_{n+1} - y_n$ and

$$g(u) = u^2 - Cu^3,\ h(u) = u^2 + Cu^3$$

so that the previous analysis leads to $h(u_{n+1}) \geq g(u_n)$.

By assumption $y_1 + \sum_{n\geq 1} u_n = \bar{y} < \infty$, which implies that $u_n \to 0$. Therefore, for n sufficiently large $g(u_n) > 0$ and we have $u_{n+1} \geq h^{-1}(\phi(u_n))$, where h^{-1} denotes the inverse of h restricted to $[0, \infty)$.

We have $h^{-1}(z) = \sqrt{z} - \frac{C}{2}z + o(z)$, so that $h^{-1}(g(u)) = u - Cu^2 + o(u^2)$. We deduce that

$$u_{n+1} \geq u_n - Cu_n^2 + o(u_n^2).$$

It follows that

$$\frac{1}{u_{n+1}} - \frac{1}{u_n} \leq \frac{1}{u_n}\left[\frac{1}{1 - Cu_n + o(u_n)} - 1\right] = C + o(1).$$

We obtain

$$\frac{1}{u_n} = \frac{1}{u_1} + \sum_{k=1}^{n-1}\left(\frac{1}{u_{k+1}} - \frac{1}{u_k}\right) \leq nC + o(n),$$

and finally

$$u_n \geq \frac{1}{nC} + o(\frac{1}{n}),$$

which is a contradiction as $y_1 + \sum_{n\geq 1} u_n = \bar{y} < \infty$. ■

Proof of Theorem 3. Let us assume that

$$((\mu^1, S^1), (\mu^2, S^2)) = \left(\left(\sum_{i=1}^{n} a_i \delta_{q_i^1}, \emptyset\right), \left(\sum_{i=1}^{n-1} b_i \delta_{q_i^2}, (\alpha, s^2]\right)\right)$$

is a mixed strategy MPE. Let Λ^i for $i = 1, 2$ denote the *ccdf* associated to the strategy (μ^i, S^i), so that

$$\Lambda_t^1 = e^{-\sum_{i=1}^{n} a_i L_t^{q_i^1}},\ \Lambda_t^2 = e^{-\sum_{i=1}^{n-1} b_i L_t^{q_i^2}} 1_{t<\tau_*^2},$$

where for the sake of lighter notation $\tau_*^2 = \tau_{(\alpha, s^2]}$.
Let us consider $\bar{J}^2$, the best reply value function to strategy (μ^1, S^1). We will show that $\bar{J}^2$ is a solution to the system $\mathcal{VS}^2$. From Proposition 1 we have that, $V_{R^2} \leq \bar{J}^2 \leq G^2$, thus $\bar{J}^2(x) \geq R^2(x)$ on $\mathcal{I}$. From Proposition 2, $\bar{J}^2$ is continuous on $\mathcal{I}$. From the proof of Proposition 2, we have that $\lim_{x\to\beta} \bar{J}^2(x) = 0$, that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ on (q_1^1, β) and on (s^2, q_n^1), and that $\bar{J}^2 = T_{q_l}^2$ on (q_{l+1}^1, q_l^1) for $1 \leq l \leq n-1$ where T_q^2 denotes the curve solution to

$\mathcal{L}u - ru = 0$ and tangent to R^2 at q. Therefore, if $((\mu^1, S^1), (\mu^2, S^2))$ is a MPE then, $\bar{J}^2$ satifies (47), (49), (52),(53). Also, $\bar{J}^2(x) = R^2(x)$ for $x \leq s^2$ because the payoff of player 2 is R^2 on S^2. Therefore, (48) is satisfied. Therefore, $\bar{J}^2$ is in $\mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{(q_i^1)_{1 \leq i \leq n}\} \cup \{s^2\}))$, and also $|(\bar{J}^2)'(x+)| < \infty$ and $|(\bar{J}^2)'(x-)| < \infty$ for $x \in \{(q_i^1)_{1 \leq i \leq n}\} \cup \{s^2\}$. It remains to prove the conditions (50) and (51) to show that $\bar{J}^2$ is a solution of the system $\mathcal{VS}^2$.

Let us prove that $\bar{J}^2$ satisfies (50). Since $\bar{J}^2 \geq R^2$ with equality at s^2, it must be that $(\bar{J}^2)'(s^2) \geq (R^2)'(s^2)$. Assume by contradiction that this inequality is strict. Let us consider the stopping time $T_\varepsilon := \inf\{t \geq 0 \,|\, X_t \notin (s^2 - \varepsilon, s^2 + \varepsilon)\}$ where $\varepsilon > 0$ is such that $\alpha < s^2 - \varepsilon < s^2 + \varepsilon < q_n^1$. Define $f(x) = \mathbf{E}_x[e^{-rT_\varepsilon} \bar{J}^2(X_{T_\varepsilon})]$ for $x \in [s^2 - \varepsilon, s^2 + \varepsilon]$. Noting that τ_*^2 is a best reply to (μ^1, S^1) and applying the strong Markov property, we obtain that f is the payoff of player 2 against (μ^1, S^2) when using the (non Markovian) stopping time $T_\varepsilon + \tau_{(\alpha, s^2]} \circ \theta_{T_\varepsilon}$ (i.e. waiting up to T_ε and then stopping the first time X is below s^2 in the continuation game). Applying Lemma 11-(iv) with $i = 2$, $a = s^2$, $b = q_n^1$ and $u = \bar{J}^2$, we deduce that for sufficiently small ε, $f(s^2) > \bar{J}^2(s^2)$, which contradicts the fact that (μ^2, S^2) is a PBR to (μ^1, S^1). Thus $\bar{J}^2$ satisfies (50).

That $\bar{J}^2$ satisfies (51) follows from Lemma 13 below.

Lemma 13 *If* $((\mu^1, S^1), (\mu^2, S^2)) = \left(\left(\sum_{i=1}^n a_i \delta_{q_i^1}, \emptyset\right), \left(\sum_{i=1}^{n-1} b_i \delta_{q_i^2}, (\alpha, s^2]\right)\right)$ *is a mixed strategy MPE then, for every stopping time* $\tau \in \mathcal{T}$ *and every* $x \in \mathcal{I}$,

$$\begin{aligned}
\bar{J}^2(x) &= \mathbf{E}_x\Big[\sum_{i=1}^n \int_{[0, \tau \wedge \tau_*^2)} e^{-rs} G^2(q_i^1) \Lambda_s^1 a_i dL_s^{q_i^1} + e^{-r\tau_*^2} \Lambda_{\tau_*^2}^1 R^2(X_{\tau_*^2}) 1_{\tau_*^2 < \tau} \\
&\quad + e^{-r\tau} \Lambda_\tau^1 \bar{J}^2(X_\tau) 1_{\tau \leq \tau_*^2}\Big] \qquad (C.10)\\
&= \mathbf{E}_x\Big[\sum_{i=1}^n \int_{[0, \tau \wedge \tau_*^2)} ((\bar{J}^2(q_i^1) a_i - \frac{1}{2}\Delta(\bar{J}^2)'(q_i^1)) e^{-rs} \Lambda_s^1 dL_s^{q_i^1} + e^{-r\tau_*^2} \Lambda_{\tau_*^2}^1 R^2(X_{\tau_*^2}) 1_{\tau_*^2 < \tau} \\
&\quad + e^{-r\tau} \Lambda_\tau^1 \bar{J}^2(X_\tau) 1_{\tau \leq \tau_*^2}\Big] \qquad (C.11)
\end{aligned}$$

As the proof will show, Equation (C.10) follows from the strong Markov property and Equation (C.11) follows from an application of the Itô-Meyer-Tanaka formula. Let us admit for a while Lemma 13. It follows that, for every stopping time τ,

$$\mathbf{E}_x\left[\sum_{i=1}^n \int_{[0,\tau)} 1_{s < \tau_*^2} e^{-rs} G^2(q_i^1) a_i \Lambda_s^1 dL_s^{q_i^1}\right] = \mathbf{E}_x\left[\sum_{i=1}^n \int_{[0,\tau)} 1_{s < \tau_*^2} e^{-rs} (\bar{J}^2(q_i^1) a_i - \frac{1}{2}\Delta \bar{J}^{2\prime}(q_i^1)) \Lambda_s^1 dL_s^{q_i^1}\right].$$

Equivalently, for every stopping time τ, $\mathbf{E}_x[M_\tau] = \mathbf{E}_x[M_0] = 0$ where

$$M_t \equiv \sum_{i=1}^n \int_{[0,\tau)} 1_{s < \tau_*^2} e^{-rs} (a_i [G^2(q_i^1) - \bar{J}^2(q_i^1)] + \frac{1}{2}\Delta \bar{J}^{2\prime}(q_i^1)) \Lambda_s^1 dL_s^{q_i^1}.$$

It follows that the process $(M_t)_{t\geq 0}$ is a martingale.[9] Noting that $(M_t)_{t\geq 0}$ is a continuous bounded variation process, we deduce that, for every stopping time τ,

$$M_\tau = M_0 = 0. \tag{C.12}$$

Let us assume that for some i with $1 \leq i \leq n$,

$$a_i[G^2(q_i^1) - \bar{J}^2(q_i^1)] + \frac{1}{2}\Delta \bar{J}^{2\prime}(q_i^1) \neq 0. \tag{C.13}$$

Let us consider $X_0 = q_i^1$ and $T_\varepsilon := \inf\{t \geq 0 \,|\, X_t \notin (q_i^1 - \varepsilon, q_i^1 + \varepsilon)\}$ where $\varepsilon > 0$ is such that $q_{i+1}^1 < q_i^1 - \varepsilon < q_i^1 + \varepsilon < q_{i-1}^1$ (with the convention that $q_{-1}^1 = \beta$ and $q_{n+1}^1 = s^2$). From the properties of the local time process we have that, for all $t > 0$, $L_t^{q_i^1} > 0$ $\mathbf{P}_{q_i^1} - a.s.$[10] It then follows from (C.13) that $M_{T_\varepsilon} \neq 0$, which contradicts (C.12). We conclude that $\bar{J}^2$ satisfies (51) and that $\bar{J}^2$ is a solution to $\mathcal{VS}^2$.

An analogous reasoning shows that if $\bar{J}^1$ is the best reply value function to strategy (μ^2, S^2) then $\bar{J}^1$ is a solution to $\mathcal{VS}^1$.

Let us now prove Lemma 13.

Proof of Lemma 13 From (v) of Proposition 1, if $((\mu^1, S^1), (\mu^2, S^2))$ is a MPE, then $(0, (\alpha, s^2])$ is a best reply to (μ^1, S^1). Applying the strong Markov property (A.7) to the expected payoff of player 2 associated to the pair of Markov strategies $((\mu^1, S^1), (0, (\alpha, s^2]))$ yields for every stopping time τ and every $x \in \mathcal{I}$

$$\begin{aligned}\bar{J}^2(x) \;&= \mathbf{E}_x\Big[\;\sum_{i=1}^{n}\int_{[0,\tau\wedge\tau_*^2)} e^{-rs} G^2(q_i^1)\Lambda_s^1 a_i dL_s^{q_i^1} + e^{-r\tau_{s^2}}\Lambda^1_{\tau_{s^2}} R^2(X_{\tau_*^2}) 1_{\tau_*^2<\tau}\\ &\qquad + e^{-r\tau}\Lambda_\tau^1 \bar{J}^2(X_\tau) 1_{\tau\leq\tau_*^2}\Big]\end{aligned}$$

which shows (C.10).

To prove (C.11), we apply the Ito-Meyer-Tanaka formula to the process $e^{-r(\tau\wedge\tau_*^2\wedge\tau_k)}\Lambda^1_{\tau\wedge\tau_*^2\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_*^2\wedge\tau_k})$ where, for each $k \in \mathbb{N}$, $\tau_k := \inf\{t \geq 0 \,:\, X_t \neq [\alpha_k, \beta_k]\}$ with $([\alpha_k, \beta_k])_{k\in\mathbb{N}}$ an increasing sequence of compacts intervals of $\mathcal{I}$ such that $\cup_{k\in\mathbb{N}}[\alpha_k, \beta_k] = (\alpha, \beta)$. Observe that $\tau_k < \infty$ and that $X_t \in [\alpha_k, \beta_k]$ over $\{t \leq \tau_k\}$, $\mathbf{P}_x$-almost surely for $x \in [\alpha_k, \beta_k]$. Also, because X is a regular diffusion, $\lim_{k\longrightarrow\infty}\tau_k = \infty$ and, hence, $\lim_{k\longrightarrow\infty}\tau\wedge\tau_k = \tau$ with $\tau \in \mathcal{T}$.

[9]See for instance Proposition 3.5 page 70 in Revuz and Yor (1999).

[10]See for instance the proof of Proposition 2.5 page 241 in Revuz and Yor (1999).

We thus obtain from the Ito-Meyer-Tanaka formula

$$\begin{aligned}\bar{J}^2(x) &= e^{-r(\tau\wedge\tau_*^2\wedge\tau_k)}\Lambda^1_{\tau\wedge\tau_*^2\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_*^2\wedge\tau_k}) - \int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\bar{J}^2(X_s)d\Lambda^1_s \\ &\quad + \int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\Lambda^1_s(r\bar{J}^2(X_s) - \mathcal{L}\bar{J}^2(X_s))\Pi_{i=1}^n 1_{\{X_s\neq q_i^1\}}\,ds \\ &\quad - \int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\Lambda^1_s\sigma(X_s)\bar{J}^{2\prime}(X_s)\Pi_{i=1}^n 1_{\{X_s\neq q_i^1\}}dW_s \qquad \text{(C.14)} \\ &\quad - \frac{1}{2}\sum_{i=1}^n \Delta\bar{J}^{2\prime}(q_i^1)\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\Lambda^2_s dL_s^{q_i^1}. \qquad \text{(C.15)}\end{aligned}$$

After taking expectations, we get,

$$\begin{aligned}\bar{J}^2(x) = \mathbf{E}_x[e^{-r\tau\wedge\tau_*^2\wedge\tau_k}\Lambda^1_{\tau\wedge\tau_*^2\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_*^2\wedge\tau_k})] - \mathbf{E}_x\Big[\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\bar{J}^2(X_s)d\Lambda^1_s\Big] \\ -\mathbf{E}_x\Big[\frac{1}{2}\sum_{i=1}^n \Delta(\bar{J}^2)'(q_i^1)\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\Lambda^1_s dL_s^{q_i^1}\Big],\end{aligned}$$

where we have used the fact that $\bar{J}^2$ satisfies (47) and that,

$$\mathbf{E}_x\Big[\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\Lambda^1_s\sigma(X_s)\bar{J}^{2\prime}(X_s)\Pi_{i=1}^n 1_{\{X_s\neq q_i^1\}}dW_s\Big] = 0. \qquad \text{(C.16)}$$

To justify this latter equality, note that σ is continuous on I, and $\bar{J}^2 \in C^1(I \setminus \{(q_i^1)_{1\leq i\leq n}\})$ with $|(\bar{J}^2)'(x+)| < \infty$ and $|(\bar{J}^2)'(x-)| < \infty$ for $x \in \{(q_i^1)_{1\leq i\leq n}\}$. Thus, there exists $C_k > 0$ such that $|\sigma(X_t)(\bar{J}^2)'(X_t)| \leq C_k$ over $\{t \leq \tau_*^2 \wedge \tau_k\}$, $\mathbf{P}_x$-almost surely, which implies (C.16). It follows that,

$$\begin{aligned}\bar{J}^2(x) &= \mathbf{E}_x[e^{-r\tau\wedge\tau_k}\Lambda^1_{(\tau\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_k})1_{\tau\wedge\tau_k\leq\tau_*^2}] + \mathbf{E}_x[e^{-r\tau_*^2}\Lambda^1_{\tau_*^2}R^2(X_{\tau_*^2})1_{\tau_*^2<\tau\wedge\tau_k}] \\ &\quad + \mathbf{E}_x\Big[\sum_{i=1}^n\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\bar{J}^2(X_s)\Lambda^1_s a_i dL_s^{q_i^1}\Big] - \mathbf{E}_x\Big[\frac{1}{2}\sum_{i=1}^n \Delta\bar{J}^{2\prime}(q_i^1)\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)} e^{-rs}\Lambda^1_s dL_s^{q_i^1}\Big].\end{aligned}$$

That is,

$$\begin{aligned}\bar{J}^2(x) = \mathbf{E}_x[e^{-r\tau\wedge\tau_k}\Lambda^1_{(\tau\wedge\tau_k)^-}\bar{J}^2(X_{\tau\wedge\tau_k})1_{\tau\wedge\tau_k\leq\tau_*^2}] + \mathbf{E}_x[e^{-r\tau_{q_n^1}}\Lambda^1_{\tau_{q_n^1}}R^2(X_{\tau_*^2})1_{\tau_*^2<\tau\wedge\tau_k}] \\ + \mathbf{E}_x\Big[\sum_{i=1}^n\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)}((\bar{J}^2(q_i^1)a_i - \frac{1}{2}\Delta\bar{J}^{2\prime}(q_i^1))e^{-rs}\Lambda^1_s dL_s^{q_i^1}\Big] \qquad \text{(C.17)}\end{aligned}$$

We have that

$$\begin{aligned}\lim_{k\longrightarrow\infty}\mathbf{E}_x\Big[\int_{[0,\tau\wedge\tau_*^2\wedge\tau_k)}((\bar{J}^2(q_i^1)a_i - \frac{1}{2}\Delta\bar{J}^{2\prime}(q_i^1))e^{-rs}\Lambda^1_s dL_s^{q_i^1}\Big] \\ = \mathbf{E}_x\Big[\int_{[0,\tau\wedge\tau_*^2)}((\bar{J}^2(q_i^1)a_i - \frac{1}{2}\Delta\bar{J}^{2\prime}(q_i^1))e^{-rs}\Lambda^1_s dL_s^{q_i^1}\Big]\end{aligned}$$

and

$$\lim_{k\longrightarrow\infty} \mathbf{E}_x[e^{-r\tau_{s^2}}\Lambda^1_{\tau_{s^2}} R^2(X_{\tau^2_*})1_{\tau^2_*<\tau\wedge\tau_k}] = \mathbf{E}_x[e^{-r\tau_{s^2}}\Lambda^1_{\tau_{s^2}} R^2(X_{\tau^2_*})1_{\tau^2_*<\tau}]$$

by the monotone convergence theorem. Because from Proposition 1 $\bar{J}^2 \leq G^2$, it follows from assumption **A4** that the sequence $(e^{-r\tau\wedge\tau_k}\bar{J}^2(X_{\tau^-\wedge\tau_k})1_{\tau\wedge\tau_k\leq\tau^2_*}1_{\tau<\infty})_{k\in\mathbb{N}}$ is uniformly integrable. We have

$$\lim_{k\to\infty} \mathbf{E}_x[e^{-r(\tau\wedge\tau_k)}\Lambda^1_{(\tau\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_k})1_{\tau\wedge\tau_k\leq\tau^2_*}1_{\tau<\infty}] = \mathbf{E}_x[e^{-r\tau}\Lambda^1_\tau\bar{J}^2(X_\tau)1_{\tau\leq\tau^2_*}1_{\tau<\infty}]$$

by Vitali's convergence theorem.

Finally, over $\{\tau=\infty\}$, we have $e^{-r(\tau\wedge\tau_k)}\Lambda^1_{(\tau\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_k})1_{\tau\wedge\tau_k\leq\tau^2_*} = e^{-r\tau_k}\Lambda^1_{\tau_k}\bar{J}^2(X_{\tau_k})1_{\tau_k\leq\tau^2_*}$. For k large enough, $x\in(\alpha_k,\beta_k)$, thus

$$\begin{aligned}
\mathbf{E}_x[e^{-r\tau_k}\Lambda^1_{\tau_k}\bar{J}^2(X_{\tau_k})1_{\tau_k\leq\tau^2_*}] &\leq \mathbf{E}_x[e^{-r\tau_k}\Lambda^1_{\tau_k}\bar{J}^2(X_{\tau_k})1_{\{X_{\tau_k}=\alpha_k\}}] + E_x[e^{-r\tau_k}\Lambda^1_{\tau_k}\bar{J}^2(X_{\tau_k})1_{\{X_{\tau_k}=\beta_k\}}] \\
&= \mathbf{E}_x[e^{-r\tau_{\alpha_k}}\Lambda^1_{\tau_{\alpha_k}}\bar{J}^2(\alpha_k)] + E_x[e^{-r\tau_{\beta_k}}\Lambda^1_{\tau_{\beta_k}}\bar{J}^2(\beta_k)] \\
&\leq \frac{\phi(x)}{\phi(\alpha_k)}G^2(\alpha_k) + \frac{\psi(x)}{\psi(\beta_k)}G^2(\beta_k),
\end{aligned}$$

We deduce from the growth properties (11) that

$$\lim_{k\longrightarrow\infty} \mathbf{E}_x[e^{-r\tau_k}\Lambda^1_{\tau_k}\bar{J}^2(X_{\tau_k})1_{\tau_k\leq\tau^2_*}] = 0.$$

Thus, letting k go to ∞ in (C.17) yields

$$\begin{aligned}
\bar{J}^2(x) &= \mathbf{E}_x\Big[\sum_{i=1}^n \int_{[0,\tau\wedge\tau^2_*)} ((\bar{J}^2(q^1_i)a_i - \frac{1}{2}\Delta(\bar{J}^2)'(q^1_i))e^{-rs}\Lambda^1_s dL^{q^1_i}_s + e^{-r\tau^2_*}\Lambda^1_{\tau^2_*}R^2(X_{\tau^2_*})1_{\tau^2_*<\tau} \\
&\quad + e^{-r\tau}\Lambda^1_\tau\bar{J}^2(X_\tau)1_{\tau\leq\tau^2_*}\Big],
\end{aligned}$$

which shows (C.11).

To close the proof of Theorem 3 it simply remains to establish a verification Lemma that shows that, if a pair (w^1,w^2) are solutions to the systems $\mathcal{VS}^1$ and $\mathcal{VS}^2$ then,

$$((\mu^1,S^1),(\mu^2,S^2)) = \left(\left(\sum_{i=1}^n a_i\delta_{q^1_i},\emptyset\right),\left(\sum_{i=1}^{n-1} b_i\delta_{q^2_i},(\alpha,s^2]\right)\right)$$

is a mixed strategy MPE. Specifically, Lemma 14 below shows that the strategy $(\sum_{i=1}^n a_i\delta_{q^1_i},\emptyset)$ is a PBR to player's 2 strategy $(\sum_{i=1}^{n-1} b_i\delta_{q^2_i},(\alpha,s^2])$ and that the strategy $(\sum_{i=1}^{n-1} b_i\delta_{q^2_i},(\alpha,s^2])$ is a PBR to player's 1 strategy $(\sum_{i=1}^n a_i\delta_{q^1_i},\emptyset)$. This ends the proof of Theorem 3.

Lemma 14 *For $i = 1, 2$, let w^i be a solution to $\mathcal{VS}^i$ then, with $j \neq i$*

$$
\begin{aligned}
w^i(x) &\geq \sup_\tau J^i(x, \tau, (\mu^j, S^j)), && \text{(C.18)}\\
w^i(x) &= J^i(x, (\mu^1, S^1), (\mu^2, S^2)). && \text{(C.19)}
\end{aligned}
$$

Proof of Lemma 14. We prove Lemma 14 for $i = 2$ and $j = 1$.

At first, let us observe that (A.7) leads to

$$
J^2(x, (\mu^1, S^1), \tau) = \mathbf{E}_x[e^{-r\tau} R^2(X_\tau)\Lambda^1_\tau \;+\; \sum_{i=1}^n \int_{[0,\tau)} e^{-rs} G^2(X_s)\Lambda^1_s a_i \, dL^{q^1_i}_s].
$$

Let w^2 be a solution to $\mathcal{VS}^2$. We apply the Itô-Meyer-Tanaka formula to the process $(e^{-r(\tau\wedge\tau_k)}\Lambda^1_{\tau\wedge\tau_k} w^2(X_{\tau\wedge\tau_k}))_{t\geq 0}$ where $\tau_k := \inf\{t \geq 0 \,:\, X_t \neq [\alpha_k, \beta_k]\}$ and where $([\alpha_k, \beta_k])_{k\in\mathbb{N}}$ be an increasing sequence of compact intervals of $\mathcal{I}$ such that $\cup_{k\in\mathbb{N}}[\alpha_k, \beta_k] = \mathcal{I}$ as in the proof of Lemma 13. We get

$$
\begin{aligned}
w^2(x) &= e^{-r(\tau\wedge\tau_k)}\Lambda^1_{\tau\wedge\tau_k} w^2(X_{\tau\wedge\tau_k}) - \int_{[0,\tau\wedge\tau_k)} e^{-rs} w^2(X_s) d\Lambda^1_s \\
&\quad + \int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s (r w^2(X_s) - \mathcal{L}w^2(X_s)) \Pi^n_{i=1} 1_{\{X_s \neq q^1_i\}} \, ds \\
&\quad - \int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s \sigma(X_s) w^{2\prime}(X_s) \Pi^n_{i=1} 1_{\{X_s \neq q^1_i\}} dW_s && \text{(C.20)}\\
&\quad - \frac{1}{2}\sum_{i=1}^n \Delta w^{2\prime}(q^1_i) \int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s dL^{q^1_i}_s. && \text{(C.21)}
\end{aligned}
$$

From assumption **A3** and (48), we have $\mathcal{L}w^2 - rw^2 = \mathcal{L}R^2 - rR^2 \leq 0$ on (α, s^2). It then follows from (47) that

$$
\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s (r w^2(X_s) - \mathcal{L}w^2(X_s)) \Pi^n_{i=1} 1_{\{X_s \neq q^1_i\}} \, ds\right] \geq 0. \tag{C.22}
$$

From (51) and from the properties of the local time process $(L^{q^1_i}_t)_{t\geq 0}$, we have

$$
\begin{aligned}
-\frac{1}{2}\sum_{i=1}^n \Delta w^{2\prime}(q^1_i) \int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s dL^{q^1_i}_s &= \sum_{i=1}^n a_i (G^2(q^1_i) - w^2(q^1_i)) \mathbf{E}_x[\int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s dL^{q^1_i}_s] \\
&= \mathbf{E}_x[\sum_{i=1}^n \int_{[0,\tau\wedge\tau_k)} e^{-rs} G^2(X_s)\Lambda^1_s a_i dL^{q^1_i}_s] - \mathbf{E}_x[\sum_{i=1}^n \int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s w^2(X_s) a_i dL^{q^1_i}_s] \\
&= \mathbf{E}_x[\sum_{i=1}^n \int_{[0,\tau\wedge\tau_k)} e^{-rs} G^2(X_s)\Lambda^1_s a_i dL^{q^1_i}_s] + \mathbf{E}_x[\int_{[0,\tau\wedge\tau_k)} e^{-rs} w^2(X_s) d\Lambda^1_s]
\end{aligned}
\tag{C.23}
$$

We obtain from (C.21), (C.22), (C.23)

$$\begin{aligned} w^2(x) &\geq \mathbf{E}_x[e^{-r(\tau\wedge\tau_k)}\Lambda^1_{\tau\wedge\tau_k} w^2(X_{\tau\wedge\tau_k})] + \mathbf{E}_x[\sum_{i=1}^n \int_{[0,\tau\wedge\tau_k)} e^{-rs}G^2(X_s)\Lambda^1_s a_i dL_s^{q_i^1}] \\ &\geq \mathbf{E}_x[e^{-r(\tau\wedge\tau_k)}\Lambda^1_{\tau\wedge\tau_k} R^2(X_{\tau\wedge\tau_k})] + \mathbf{E}_x[\sum_{i=1}^n \int_{[0,\tau\wedge\tau_k)} e^{-rs}G^2(X_s)\Lambda^1_s a_i dL_s^{q_i^1}] \end{aligned}$$

where we have used that the stochastic integral in (C.20) is a centered square integrable martingale as shown in Lemma 13 and that $w^2(x) \geq R^2(x)$ on $\mathcal{I}$. Using again the same arguments than in Lemma 13, letting k go to ∞ yields

$$w^2(x) \geq \mathbf{E}_x[e^{-r(\tau)}\Lambda^1_\tau R^2(X_\tau)] + \mathbf{E}_x[\sum_{i=1}^n \int_{[0,\tau)} e^{-rs}G^2(X_s)\Lambda^1_s a_i dL_s^{q_i^1}],$$

where we note that, from (16), the left-hand-side of the above equation corresponds to $J^2(x, (\mu^1, S^1), \tau)$. Then, taking the supremum over τ yields (C.18).

To establish (C.19), we apply the Ito-Meyer-Tanaka formula to $e^{-r\tau_k}\Lambda^1_{\tau_k}\Lambda^2_{\tau_k} w^2(X_{\tau_k})$ where w^2 is a solution to $\mathcal{VS}^2$. We obtain that

$$\begin{aligned} w^2(x) = \mathbf{E}_x[e^{-r\tau_k}\Lambda^1_{\tau_k}\Lambda^2_{\tau_k} w^2(X_{\tau_k})] - \mathbf{E}_x[\int_{[0,\tau_k)} e^{-rs}w^2(X_s)\Lambda^2_s d\Lambda^1_s] - \mathbf{E}_x[\int_{[0,\tau_k)} e^{-rs}w^2(X_s)\Lambda^1_s d\Lambda^2_s] \\ -\mathbf{E}_x[\frac{1}{2}\sum_{i=1}^n \Delta w^{2\prime}(q_i^1) \int_{[0,\tau_k)} e^{-rs}\Lambda^1_s\Lambda^2_s dL_s^{q_i^1}]. \end{aligned} \tag{C.24}$$

where, as in the proof of Lemma 13, we have used that

$$\mathbf{E}_x\Big[\int_{[0,\tau_k)} e^{-rs}\Lambda^1_s\Lambda^2_s\sigma(X_s)w^{2\prime}(X_s)\Pi_{i=1}^n 1_{\{X_s\neq q_i^1\}}dW_s\Big] = 0.$$

and

$$\mathbf{E}_x[\int_{[0,\tau\wedge\tau_k)} e^{-rs}\Lambda^1_s\Lambda^2_s(rw^2(X_s) - \mathcal{L}\bar{w}^2(X_s))\Pi_{i=1}^n 1_{\{X_s\neq q_i^1\}}\, ds] = 0.$$

This latter equality follows from (47) and from the definition of Λ^2_s which contains the indicator $1_{s<\tau_*^2}$. Next, we have

$$\begin{aligned} -\mathbf{E}_x[\int_{[0,\tau_k)} e^{-rs}w^2(X_s)\Lambda^1_s d\Lambda^2_s] &= \mathbf{E}_x[\sum_{i=1}^{n-1}\int_{[0,\tau_k)} e^{-rs}w^2(X_s)\Lambda^1_s\Lambda^2_s b_i dL_s^{q_i^2}] \\ &\quad + \mathbf{E}_x[e^{-r\tau_*^2}e^{-rs}w^2(X_{\tau_{s^2}})\Lambda^1_{\tau_*^2}\Lambda^2_{\tau_*^2}1_{\tau_*^2\leq\tau_k}] \\ &= \mathbf{E}_x[\sum_{i=1}^{n-1}\int_{[0,\tau_k)} e^{-rs}R^2(X_s)\Lambda^1_s\Lambda^2_s b_i dL_s^{q_i^2}] \\ &\quad + \mathbf{E}_x[e^{-r\tau_*^2}e^{-rs}R^2(X_{\tau_{s^2}})\Lambda^1_{\tau_*^2}\Lambda^2_{\tau_*^2}1_{\tau_*^2\leq\tau_k}]. \end{aligned} \tag{C.25}$$

Using (45), we get

$$
\begin{aligned}
-\mathbf{E}_x[\frac{1}{2}\sum_{i=1}^{n}\Delta w^{2\prime}(q_i^1)\int_{[0,\tau_k)} e^{-rs}\Lambda_s^1\Lambda_s^2 dL_s^{q_i^1}] &= \mathbf{E}_x[\sum_{i=1}^{n}\int_{[0,\tau_k)} e^{-rs}G^2(q_i^1)\Lambda_s^1\Lambda_s^2 a_i dL_s^{q_i^1}] \\
&\quad -\mathbf{E}_x[\sum_{i=1}^{n}\int_{[0,\tau_k)} e^{-rs}w^2(q_i^1)\Lambda_s^1\Lambda_s^2 a_i dL_s^{q_i^1}] \\
&= \mathbf{E}_x[\sum_{i=1}^{n}\int_{[0,\tau_k)} e^{-rs}G^2(X_s)\Lambda_s^1\Lambda_s^2 a_i dL_s^{q_i^1}] \\
&\quad +\mathbf{E}_x[\int_{[0,\tau_k)} e^{-rs}w^2(X_s)\Lambda_s^2 d\Lambda_s^1]. \qquad \text{(C.26)}
\end{aligned}
$$

We get from (C.24), (C.25), (C.26) that,

$$
\begin{aligned}
w(x) = \mathbf{E}_x[e^{-r\tau_k}\Lambda_{\tau_k}^1\Lambda_{\tau_k}^2 w(X_{\tau_k})] + \mathbf{E}_x[\sum_{i=1}^{n-1}\int_{[0,\tau_k)} e^{-rs}R^2(X_s)\Lambda_s^1\Lambda_s^2 b_i dL_s^{q_i^2}] \\
+\mathbf{E}_x[e^{-r\tau_*^2}R^2(X_{s^2})\Lambda_{\tau_*^2}^1\Lambda_{\tau_*^2}^2 1_{\tau_*^2\le\tau_k}] + \mathbf{E}_x[\sum_{i=1}^{n}\int_{[0,\tau_k)} e^{-rs}G^2(X_s)\Lambda_s^1\Lambda_s^2 a_i dL_s^{q_i^1}]. \qquad \text{(C.27)}
\end{aligned}
$$

Letting k go to ∞ as in Lemma 13, yields

$$
\begin{aligned}
w^2(x) = \mathbf{E}_x[\sum_{i=1}^{n-1}\int_{[0,\infty)} e^{-rs}R^2(X_s)\Lambda_s^1\Lambda_s^2 b_i\, dL_s^{q_i^2} \\
+ e^{-r\tau_*^2}R^2(X_{\tau_{s^2}})\Lambda_{\tau_{s^2}}^1\Lambda_{\tau_{s^2}}^2 + \sum_{i=1}^{n}\int_{[0,\infty)} e^{-rs}G^2(X_s)\Lambda_s^1\Lambda_s^2 a_i\, dL_s^{q_i^1}].
\end{aligned}
$$

Finally, we observe from (16) that the right hand side of the latter equation corresponds to $J^2(x,(\mu^1,S^1),(\mu^2,S^2))$, which ends the proof.

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