Online Appendix for *Optimally Stubborn*

Rather than artificially separating the proofs of Propositions 4 and 5, I show convergence and existence jointly. Given that the proofs are mostly constructive, the statements in the appendix are usually stronger than what is stated in the main text. The proof is structured as follows. First, I consider pooling equilibria. Then I consider semi-separating equilibria and finally separating equilibria.

Propositions 7 to 9 (together with Proposition 3) establish that any pooling equilibrium converges to the limits stated in Proposition 4. Moreover, they show such sequences of pooling equilibria exist as stated in Proposition 5 (a) and (b).

Pooling Equilibria: Convergence and Existence

Pooling Equilibria with $K > 2$

Before proving the existence of pooling equilibria with $K > 2$, it is helpful to state the following supplementary lemma and its proof:

Lemma 1. *There exists a pooling equilibrium with support $\{\alpha_1, \ldots, \alpha_K\}$ only if the demands α_1 through α_K along with probabilities q_1 through to q_K, and positive numbers μ_1 through to μ_K solve (5)–(8).*

Proof. Fix $z > 0$, and an equilibrium, specifying $\{\alpha_1, \ldots, \alpha_K\}$, $\mu_1, \ldots, \mu_K > 0$, and $q_1, \ldots, q_K > 0$. For any $k \leq K$, define

$$\begin{aligned} v_k^r = & \sum_{\substack{i \text{ s.t.} \\ \alpha_i \leq 1-\alpha_k}} q_i \left(\frac{\alpha_k + 1 - \alpha_i}{2} \right) \\ & + \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1-\alpha_k}} q_i \left(\alpha_k \min\left\{ 0, 1 - \left(\frac{\mu_i}{\mu_k}\right)^{1-\alpha_i} \right\} + (1-\alpha_i) \min\left\{ 1, \left(\frac{\mu_i}{\mu_k}\right)^{1-\alpha_i} \right\} \right), \end{aligned} \tag{1}$$

$$v_k^s = v_k^r - \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1-\alpha_k}} q_i (1-\alpha_i) \max\left\{ \mu_i^{\alpha_k}, \left(\frac{\mu_i}{\mu_k}\right)^{1-\alpha_i} \mu_k^{\alpha_k} \right\}. \tag{2}$$

For a detailed derivation of these payoffs see the supplementary material on my website. For any $k, k' \leq K$, define

$$\Delta^r_{k,k'} = v^r_k - v^r_{k'}, \tag{3}$$

$$\Delta^s_{k,k'} = v^s_k - v^s_{k'}. \tag{4}$$

Given z and $\{\alpha_1, \ldots, \alpha_K\}$, define the following system in (q_i, μ_i), $i = 1, \ldots, K$:

$$\Delta^r_{k,k+1} = 0, \; \forall k < K, \tag{5}$$

$$\Delta^r_{k,k+1} - \Delta^s_{k,k+1} = 0, \; \forall k < K \tag{6}$$

$$\sum_{i=1}^{K} q_i \mu_i^{1-\alpha_i} = z, \text{ and} \tag{7}$$

$$\sum_{i=1}^{K} q_i = 1. \tag{8}$$

Note that there are $2K$ equations (and as many variables). For a candidate equilibrium with support $\{\alpha_1, \ldots, \alpha_K\}$, both types need to be indifferent over all demands α_1 through to α_K, with probabilities $q_i > 0$, given an ex ante probability of a player being stubborn, z. Equation (5) shows the difference in payoff for a rational type between making a demand of α_k and making a demand of α_{k+1}, conditional on the opponent mixing over the offers α_1 through to α_K. Hence, equation (5) ensures indifference of the rational type between any two offers, α_k and α_{k+1}. In the same manner, equation (6) ensures indifference of the stubborn type between any two offers, simplified using the indifference of the rational type. Equation (8) ensures that the probabilities of being faced with a given offer add up to 1; and equation (7) ensures that the conditional probabilities of stubbornness, $\mu_i^{1-\alpha_i}$, are consistent with the ex ante probability of a player being stubborn, z.

Fix K demands (satisfying Lemmas 1 and 2). Suppose that for all $\bar{z} > 0$, there exists $z < \bar{z}$, such that there exist $q_i > 0$, and $\mu_i > 0$ for $i = 1, 2, \ldots, K$ such that (z, α, q, μ) satisfies (5) to (8). Then there exists a sequence $(z^n, \alpha^n, q^n, \mu^n)_{n \in \mathbb{N}}$, with $\lim_{n \to \infty} z^n \to 0$, solving (5)–(8), such that it is *not* the case that $\alpha^n_i - \alpha^n_{i+1} \to 0$

for all i, $i+1 \leq \lceil K/2 \rceil - 1$ and all $i, i+1 \geq \lceil K/2 \rceil$ with $i+1 < K$. Recall, that $\alpha^n, q^n, \mu^n \in [0,1]$. Hence, without loss, assume that α^n, q^n and μ^n converge. By continuity, $(z=0, \lim_{z\to 0}\alpha, \lim_{z\to 0} q, \lim_{z\to 0}\mu)$ also solves (5)–(8). In the following, I drop the subscript n; limits are indicated explicitly by $\lim_{z\to 0}$ throughout.

In other words, if the system has a solution for small enough z, then for at least one $i \notin \{\lceil K/2 \rceil - 1, K\}$, $\alpha_i \neq \alpha_{i+1}$. □

Proposition 1 (Proposition 7)**.** *(a) Fix demands $\{\alpha_1, \ldots, \alpha_K\}$ (satisfying Lemmas 1 and 2), with $K > 3$. Then there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, there exist no $q_i > 0$, and $\mu_i > 0$ for $i = 1, 2, \ldots, K$ such that (z, α, q, μ) satisfies (5)–(8).*

(b) Fix a sequence $z^n \to 0$, and a corresponding convergent sequence of equilibria (α^n, q^n, μ^n), where $\alpha^n = (\alpha_i^n)_{i=1}^K$ with $K > 3$. Then there exists $a_1 \in (0, 1/2]$ and $a_K \in (1 - a_1, 1]$ such that

$$\lim_{n\to\infty} (\alpha_1^n, \ldots, \alpha_{k-1}, \alpha_k, \ldots, \alpha_K^n) = (\underbrace{a_1, \ldots, a_1}_{\lceil K/2 \rceil - 1 \text{ terms}}, \underbrace{1 - a_1, \ldots, 1 - a_1, a_K}_{K - \lceil K/2 \rceil + 1 \text{ terms}}),$$

where $k = \lceil K/2 \rceil$. Moreover, along any such sequence,

$$\lim_{n\to\infty} q^n = (\underbrace{0, \ldots, 0}_{K-2 \text{ terms}}, 1, 0).$$

Proof of Proposition 7. The proof that follows is divided into the following steps. First, I show that in any sequence of equilibria, $\mu_i \to 0$ for any $i > 1$ (Claims 1 and 2). I then show that there must be more than one immediate concession in the limit (Claim 3)[1] , i.e., there exist i, j with $i > j > 1$ such that

$$\frac{\mu_i}{\mu_j} \to 0.$$

[1]Recall from Lemma **??** that $\mu_i \leq \mu_j$ for any $i > j$. Hence, if there exist i, j with $i > j > 1$ such that

$$\frac{\mu_i}{\mu_j} \to 0,$$

then

$$\frac{\mu_i}{\mu_1} \to 0.$$

Finally, I show that if there is more than one immediate concession in the limit, an equilibrium with support $\{\alpha_1, \ldots, \alpha_K\}$ does not exist for z small enough (Claim 4). Together these claims establish that fixing any K demands satisfying Lemmas 1 and 2, an equilibrium with support $\{\alpha_1, \ldots, \alpha_K\}$ does not exist for z small enough.

Claim 1. *For (5)–(8) to be satisfied, it is necessary that*

$$\lim_{z \to 0} \mu_K = 0.$$

Proof. Recall that by Lemma 1, in order for (5) to be satisfied it must be that $\mu_{k+1} \leq \mu_k$, $\forall k$, $\forall z > 0$, and hence also $\mu_K \leq \mu_{K-k}$, $\forall z > 0$. Note that (8) implies that

$$q_i \geq \frac{1}{K},$$

for at least one i. Therefore, by (7), $\lim_{z \to 0} \mu_i = 0$ for some i. It then follows that $\lim_{z \to 0} \mu_K = 0$. □

Claim 2. *If* $\lim_{z \to 0} \mu_K = 0$*, then for (5)–(8) to be satisfied it is necessary that for any* $i > 1$*,*

$$\lim_{z \to 0} \mu_i = 0.$$

Proof. Suppose $\lim_{z \to 0} \mu_{k+1} = 0$ and $\lim_{z \to 0} \mu_k \neq 0$ for some $k > 1$. Then for every $i < k$, $\lim_{z \to 0} \mu_i \neq 0$, and hence, it follows from (7) that $\lim_{z \to 0} \sum_{i=1}^{k} q_i = 0$.

Case 1: $\alpha_k > 1/2$. Then I can write (5) for $k' = k - 1$ (i.e., $\Delta^r_{k-1,k}|_{\alpha_k > 1/2}$) as:

$$\begin{aligned}
&\sum_{\substack{i \text{ s.t.} \\ \alpha_i < 1 - \alpha_k}} \underbrace{q_i \frac{1}{2} (\alpha_{k-1} - \alpha_k)}_{\to 0} + \sum_{\substack{i \text{ s.t.} \\ 1 - \alpha_k < \alpha_i \leq \min\{\alpha_k, 1 - \alpha_{k-1}\}}} \underbrace{q_i \frac{1}{2} (\alpha_{k-1} + \alpha_i - 1)}_{\to 0} \\
&+ \sum_{\substack{i \text{ s.t.} \\ \alpha_k \geq \alpha_i > \max\{\alpha_{k-1}, 1 - \alpha_{k-1}\}}} \underbrace{q_i \left(\alpha_{k-1} + l_{i,k-1}^{1-\alpha_i} (1 - \alpha_i - \alpha_{k-1})\right)}_{\to 0} \\
&+ \sum_{i > k}^{K} q_i \left(\alpha_{k-1} + \underbrace{l_{i,k-1}^{1-\alpha_i}}_{\to 0} (1 - \alpha_i - \alpha_{k-1}) - \alpha_k - \underbrace{l_{i,k}^{1-\alpha_i}}_{\to 0} (1 - \alpha_i - \alpha_k) \right) = 0.
\end{aligned} \tag{9}$$

Since $\lim_{z\to 0} q_i = 0$ for any $i \leq k$, the first three terms go to 0. Note that there must exist $\lim_{z\to 0} q_i > 0$ for some $i > k$. Moreover, $\alpha_{k-1} \neq \alpha_k$. For the last term to go to 0, it is necessary that $\lim_{z\to 0} l_{i,k} \neq 0$ for some i. By assumption $\lim_{z\to 0} \mu_k \neq 0$, yet $\lim_{z\to 0} \mu_i = 0$ for any $i > k$. Hence, if $\lim_{z\to 0} \mu_k \neq 0$ with $\alpha_k > 1/2$, (9) cannot be satisfied. Therefore, it is necessary that $\forall \alpha_k > 1/2$,

$$\lim_{z\to 0} \mu_k = 0.$$

Case 2: $\alpha_k \leq 1/2$. Then I can write (5) for $k' = k-1$ (i.e., $\Delta^r_{k-1,k}|_{\alpha_k \leq 1/2}$) as:

$$\begin{aligned}
&\sum_{i\leq k} q_i \underbrace{\frac{1}{2}(\alpha_{k-1}-\alpha_k)}_{\to 0} + \sum_{\substack{i \text{ s.t.}\\ \alpha_k<\alpha_i\leq 1-\alpha_k}} \underbrace{q_i \frac{1}{2}(\alpha_{k-1}+\alpha_k-1)}_{\leq 0} \\
&+ \sum_{\substack{i \text{ s.t.}\\ 1-\alpha_k<\alpha_i\leq 1-\alpha_{k-1}}} q_i \left(\underbrace{\frac{\alpha_{k-1}+1-\alpha_i}{2} - \alpha_k}_{<0} + \underbrace{l^{1-\alpha_i}_{i,k}}_{\to 0}(\alpha_i+\alpha_k-1) \right) \\
&+ \sum_{\substack{i \text{ s.t.}\\ \alpha_i>1-\alpha_{k-1}}} q_i \left(\alpha_{k-1} + \underbrace{l^{1-\alpha_i}_{i,k-1}}_{\to 0}(1-\alpha_i-\alpha_{k-1}) - \alpha_k - \underbrace{l^{1-\alpha_i}_{i,k}}_{\to 0}(1-\alpha_i-\alpha_k) \right) = 0.
\end{aligned} \tag{10}$$

As in Case 1, since $\lim_{z\to 0} q_i = 0$ for any $i \leq k$, the first term goes to 0. The second term is strictly negative if $\lim_{z\to 0} q_i > 0$. Suppose the second term is strictly negative. Then either the third or fourth term need to be strictly positive for (10) to be satisfied. Regarding the third term, note that

$$\frac{\alpha_{k-1}+1-\alpha_i}{2} - \alpha_k = \frac{1}{2}\underbrace{(\alpha_{k-1}-\alpha_k)}_{<0} + \frac{1}{2}\underbrace{(1-\alpha_i-\alpha_k)}_{<0} < 0.$$

Hence, for either the third or fourth term to be weakly positive, it is necessary that $\lim_{z\to 0} l_{i,k} \neq 0$. Since $\lim_{z\to 0} \mu_i = 0$ for any $i > k$, this would require $\lim_{z\to 0} \mu_k = 0$. However, by assumption this does not hold. Recall there must be at least one i for which $\lim_{z\to 0} q_i > 0$. Therefore, at least one of the remaining three terms is strictly negative, and no term is strictly positive. Hence, if $\lim_{z\to 0} \mu_k \neq$ with $\alpha_k \leq \frac{1}{2}$, $\Delta_{k-1} = 0$

cannot be satisfied. Therefore, it is necessary that for any $\frac{1}{2} \geq \alpha_k > \alpha_1$,

$$\lim_{z \to 0} \mu_k = 0.$$

Therefore, if $\lim_{z \to 0} \mu_K = 0$, then for (5)–(8) to be satisfied it is necessary that for any $i > 1$,

$$\lim_{z \to 0} \mu_i = 0.$$

□

Note that the system (5)–(8) is linear in the probability of being faced with an offer α_i, q_i. Hence, without loss, I normalize $q_{K-1} = 1$ for the remaining part of the proof.

Claim 3. *There exist i, j with $i > j > 1$ such that*

$$\lim_{z \to 0} l_{i,j} = 0.$$

Proof. Suppose not; i.e., suppose that for all i, j with $i > j > 1$, there exist $1 \geq \lim_{z \to 0} l_{i,j} > 0$, such that

$$\frac{\mu_i}{\mu_j} = l_{i,j}.$$

Evaluating (6) for $k = 1$ and rearranging, I get:

$$q_{K-1} \left(1 - \alpha_{K-1}\right) \mu_2^{\alpha_2 + \alpha_{K-1} - 1} \mu_{K-1}^{1-\alpha_{K-1}} = q_K \left(1 - \alpha_K\right) \mu_K^{1-\alpha_K} \left(\mu_1^{\alpha_1 + \alpha_K - 1} - \mu_2^{\alpha_2 + \alpha_K - 1}\right) \tag{11}$$

Suppose $\lim_{z \to 0} l_{K,1} > 0$. Then I can write (11) as:

$$\begin{aligned} q_{K-1} \left(1 - \alpha_{K-1}\right) \mu_2^{\alpha_2} l_{K-1,2}^{1-\alpha_{K-1}} &= q_K \left(1 - \alpha_K\right) \left(\mu_1^{\alpha_1} l_{K,1}^{1-\alpha_K} - \mu_2^{\alpha_2} l_{K,2}^{1-\alpha_K}\right), \text{ or} \\ q_{K-1} \left(1 - \alpha_{K-1}\right) \underbrace{\mu_2^{\alpha_2 - \alpha_1}}_{\to 0} \underbrace{l_{2,1}^{\alpha_1} l_{K-1,2}^{1-\alpha_{K-1}}}_{>0} &= q_K \left(1 - \alpha_K\right) \underbrace{\left(l_{K,1}^{1-\alpha_K} - \mu_2^{\alpha_2 - \alpha_1} l_{2,1}^{\alpha_1} l_{K,2}^{1-\alpha_K}\right)}_{>0} \end{aligned} \tag{12}$$

Since $q_{K-1} = 1$, it must be that $\lim_{z \to 0} q_K = 0$.

Suppose instead $\lim_{z\to 0} l_{K,1} = 0$. Since $\lim_{z\to 0} l_{K,2} > 0$, this implies that $\lim_{z\to 0} l_{2,1} = 0$. Hence, I can write (11) as:

$$q_{K-1}\left(1-\alpha_{K-1}\right) \underbrace{l_{2,1}^{\alpha_1+\alpha_K-1}\mu_2^{\alpha_2-\alpha_1}}_{\to 0} \underbrace{l_{K-1,2}^{\alpha_K-\alpha_{K-1}}}_{>0} = q_K\left(1-\alpha_K\right) l_{K,K-1}^{1-\alpha_K}\left(1-\underbrace{l_{2,1}^{\alpha_1+\alpha_K-1}\mu_2^{\alpha_2-\alpha_1}}_{\to 0}\right) \tag{13}$$

Since $q_{K-1} = 1$, it must be that $\lim_{z\to 0} q_K = 0$.

Evaluating (5) for $k = 1$ gives:

$$\begin{aligned} &q_K\left(\alpha_1\left(1-l_{K,1}^{1-\alpha_K}\right)-\alpha_2\left(1-l_{K,2}^{1-\alpha_K}\right)\right)-q_K(1-\alpha_K)\left(l_{K,2}^{1-\alpha_K}-l_{K,1}^{1-\alpha_K}\right)\\ &-q_{K-1}\left(\frac{1-\alpha_1-\alpha_{K-1}}{2}\left(\alpha_2+\alpha_{K-1}-1\right)\left(1-l_{K-1,2}^{1-\alpha_{K-1}}\right)\right)-\sum_{i=1}^{K-2} q_i\left(\frac{\alpha_2-\alpha_1}{2}\right)=0. \end{aligned} \tag{14}$$

Note that the last three terms in (14) are negative. However, $\lim_{z\to 0} q_K = 0$, and $q_{K-1} = 1$. Hence, (14) cannot be satisfied. Hence, it is necessary for there to exist i, j with $i > j > 1$ such that

$$\lim_{z\to 0} l_{i,j} = 0.$$

□

Claim 4. *Suppose that there exist i, j with $i > j > 1$ such that*

$$\lim_{z\to 0} l_{i,j} = 0.$$

Then for all $(z = 0, \lim_{z\to 0}\alpha, \lim_{z\to 0} q, \lim_{z\to 0}\mu)$, solving (5)–(8), it must be that

$$\lim_{z\to 0} \alpha_i - \alpha_{i+1} = 0$$

for all i, $i+1 \leq \lceil K/2\rceil - 1$ and all $i, i+1 \geq \lceil K/2\rceil$ with $i+1 < K$.

Proof. Suppose there exist i, j with $i > j > 1$ such that

$$\lim_{z\to 0} l_{i,j} = 0.$$

Recall that by Lemma 1, $\mu_i \le \mu_j$ for any $i > j$. Hence, without loss, there exists i such that

$$\lim_{z\to 0} l_{K,i} = 0.$$

It follows that

$$\lim_{z\to 0} l_{K,i} = 0$$

for $i = 1, 2$. Note that this implies that:

$$\lim_{z\to 0} q_K (1-\alpha_K) \left(l_{K,2}^{1-\alpha_K} - l_{K,1}^{1-\alpha_K} \right) = 0.$$

Therefore, the unique candidate solution to (14) is:

$$\lim_{z\to 0} q_i = 0, \text{ for any } i \ne K-1 \tag{15}$$

$$\lim_{z\to 0} l_{K-1,2} = 1, \text{ and} \tag{16}$$

$$\lim_{z\to 0} \alpha_1 + \alpha_{K-1} = 1. \tag{17}$$

Note that in this case, the rational type is indifferent between any two demands – α_{K-1} never concedes to a lower demand, and a player is faced with a demand of α_{K-1} with probability 1. Hence, the rational receives $1 - \alpha_{K-1}$ regardless of his demand.

Evaluating (6) at $k = 2$ and rearranging gives

$$\begin{aligned}
q_{K-2}\left(1-\alpha_{K-2}\right)\mu_3^{\alpha_3+\alpha_{K-2}-1}\mu_{K-2}^{1-\alpha_{K-2}} &= q_K\left(1-\alpha_K\right)\mu_K^{1-\alpha_K}\left(\mu_2^{\alpha_2+\alpha_K-1}-\mu_3^{\alpha_3+\alpha_K-1}\right)\\
&\quad + q_{K-1}\left(1-\alpha_{K-1}\right)\mu_{K-1}^{1-\alpha_{K-1}}\left(\mu_2^{\alpha_2+\alpha_{K-1}-1}-\mu_3^{\alpha_3+\alpha_{K-1}-1}\right),\\
&\iff\\
q_{K-2}\left(1-\alpha_{K-2}\right)\mu_3^{\alpha_3} l_{K-2,3}^{1-\alpha_{K-2}} &= q_K\left(1-\alpha_K\right)\left(l_{K,2}^{1-\alpha_K}\mu_2^{\alpha_2}-l_{K,3}^{1-\alpha_K}\mu_3^{\alpha_3}\right)\\
&\quad + q_{K-1}\left(1-\alpha_{K-1}\right)\left(l_{K-1,2}^{1-\alpha_{K-1}}\mu_2^{\alpha_2}-l_{K-1,3}^{1-\alpha_{K-1}}\mu_3^{\alpha_3}\right),\\
&\iff\\
q_{K-2}\left(1-\alpha_{K-2}\right) l_{3,2}^{\alpha_3}\mu_2^{\alpha_3-\alpha_2} l_{K-2,3}^{1-\alpha_{K-2}} &= q_K\left(1-\alpha_K\right)\left(l_{K,2}^{1-\alpha_K}-l_{K,3}^{1-\alpha_K} l_{3,2}^{\alpha_3}\mu_2^{\alpha_3-\alpha_2}\right)\\
&\quad + q_{K-1}\left(1-\alpha_{K-1}\right)\left(l_{K-1,2}^{1-\alpha_{K-1}}-l_{K-1,3}^{1-\alpha_{K-1}} l_{3,2}^{\alpha_3}\mu_2^{\alpha_3-\alpha_2}\right).
\end{aligned} \tag{18}$$

Recall that $\lim_{z\to 0} q_i = 0$, for any $i \neq K-1$. Hence, the LHS and the first term on the RHS of (18) go to 0. Further, recall that $\lim_{z\to 0} l_{K-1,2} = 1$ and $\lim_{z\to 0} \mu_2 = 0$. Hence, it follows from (18) that $\lim_{z\to 0} q_{K-1} = 0$. A contradiction since by (15),

$$\lim_{z\to 0} q_{K-1} = 1.$$

Hence, if there exist i, j with $i > j > 1$ such that

$$\lim_{z\to 0} l_{i,j} = 0,$$

then for all $(z = 0, \lim_{z\to 0} \alpha, \lim_{z\to 0} q, \lim_{z\to 0} \mu)$, solving (5)–(8), it must be that

$$\lim_{z\to 0} \alpha_i - \alpha_{i+1} = 0$$

for all i, $i+1 \leq \lceil K/2 \rceil - 1$ and all $i, i+1 \geq \lceil K/2 \rceil$ with $i+1 < K$. But by Claim 3 there exist i, j with $i > j > 1$ such that

$$\lim_{z\to 0} l_{i,j} = 0.$$

Hence, I have established that for all $(z = 0, \lim_{z\to 0} \alpha, \lim_{z\to 0} q, \lim_{z\to 0} \mu)$ solving (5)–(8), it must be that

$$\lim_{z\to 0} \alpha_i - \alpha_{i+1} = 0$$

for all i, $i+1 \leq \lceil K/2 \rceil - 1$ and all $i, i+1 \geq \lceil K/2 \rceil$ with $i+1 < K$. □

□

Define

$$(s_1, s_2, s_3) = \begin{cases} \left(\frac{1-a_3}{2-a_1-a_3}, 0, \frac{1-a_1}{2-a_1-a_3}\right) & \text{if } a_1 > 1 - \frac{a_3}{4} - \sqrt{a_3(8-7a_3)}, \\ (0, 1, 0) & \text{otherwise.} \end{cases}$$

Proposition 2 (Proposition 8.). *(a) Let (z^n, r^n, s^n) be a convergent sequence of pooling equilibria with $|\operatorname{supp} r^n \cup \operatorname{supp} s^n| = 3$ and $\lim_{n\to\infty} z^n = 0$. Then there exist $a_1 \in (0, 1/2]$ and $a_3 \in (1 - a_1, 1]$ such that*

$$\lim_{n\to\infty}\left(\alpha_1^n,\alpha_2^n,\alpha_3^n\right)=\left(a_1,1-a_1,a_3\right). \tag{19}$$

Moreover, along any such sequence,

$$\lim_{n\to\infty} r^n=(0,1,0),\quad \lim_{n\to\infty} s^n=\left(s_1,s_2,s_3\right). \tag{20}$$

(b) Fix a sequence $z^n\to 0$ and fix $a_1\in(0,1/2]$ and $a_3\in(1-a_1,1]$. Then there exists N such that for any $n>N$, a corresponding sequence of pooling equilibria (z^n,r^n,s^n) satisfying (19) and (20) exists.

Proof of Proposition 8. When $K=3$, I can write (5) for $k=1,2$ as:

$$\begin{aligned}&-q_1\frac{\alpha_2-\alpha_1}{2}-q_2\frac{1-\alpha_1-\alpha_2}{2}\\&+q_3\left(\left(\alpha_1+\alpha_3-1\right)\left(1-l_{3,1}^{1-\alpha_3}\right)-\left(\alpha_2+\alpha_3-1\right)\left(1-l_{3,2}^{1-\alpha_3}\right)\right)=0,\end{aligned} \tag{21}$$

$$-q_1\frac{1-\alpha_1-\alpha_2}{2}+q_3\left(\alpha_2+\alpha_3-1\right)\left(1-l_{3,2}^{1-\alpha_3}\right)=0, \tag{22}$$

and respectively, I can write (6) for $k=1,2$ as:

$$q_2\left(1-\alpha_2\right)\mu_2^{\alpha_2}-q_3\left(1-\alpha_3\right)\mu_3^{1-\alpha_3}\left(\mu_1^{\alpha_1+\alpha_3-1}-\mu_2^{\alpha_2+\alpha_3-1}\right)=0, \tag{23}$$

$$q_1\left(1-\alpha_1\right)\mu_1^{\alpha_3}-q_2\left(1-\alpha_2\right)\left(\mu_2^{\alpha_2}-\mu_2^{\alpha_3}\right)-q_3\left(1-\alpha_3\right)\left(l_{3,2}^{1-\alpha_3}\mu_2^{\alpha_2}-\mu_3^{\alpha_3}\right)=0. \tag{24}$$

The proof that follows is divided into the following steps. I first show that in any sequence of equilibria, $\mu_i\to 0$ for any $i>1$ (Claim 5). I then show that if $\alpha_2+\alpha_1<1$, an equilibrium with support $\{\alpha_1,\alpha_2,\alpha_3\}$ does not exist in the limit (Claim 6). Next, I show that if $\alpha_2+\alpha_1=1$, an equilibrium with support $\{\alpha_1,\alpha_2,\alpha_3\}$ does exist in the limit (Claim 7). Finally, I show that if $\alpha_2+\alpha_1=1$ and $K=3$, the system (5)–(8) can be solved locally around $z=0$, with $q_i\in(0,1)$, and $\mu_i\in(0,1)$ for $i=1,2,3$ (Claim 8). Together these claims establish that fixing any 3 demands satisfying Lemmas 1 and 2 with $\alpha_2=1-\alpha_1$, an equilibrium with support $\{\alpha_1,\alpha_2,\alpha_3\}$ does exist for z small enough.

Claim 5. *For (5)–(8) to be satisfied when $K = 3$, $\lim_{z\to 0}\mu_i = 0$ for $i = 2, 3$.*

Proof. By (7), either $\lim_{z\to 0} q_i = 0$ or $\lim_{z\to 0}\mu_i = 0$. By (8) and (7), there must exist $\lim_{z\to 0}\mu_i = 0$. Recall that by Lemma 1, $\mu_3 \le \mu_2 < \mu_1$. Hence, $\lim_{z\to 0}\mu_3 = 0$. Suppose $\lim_{z\to 0}\mu_2 \neq 0$. Then $\lim_{z\to 0} q_i = 0$, for $i = 1, 2$. But then (21) cannot be satisfied – since the last term is non-zero. Hence, it must be that $\lim_{z\to 0}\mu_2 = 0$. □

Claim 6. *If $\alpha_2 + \alpha_1 < 1$ and $K = 3$, the system (5)–(8) cannot be solved in the limit.*

Proof. Using (8), I can replace q_3 by $1 - q_1 - q_2$ in (21) and (22). I can then solve (21) and (22) for q_1 and q_2 as a function of μ_i, $i = 1, 2, 3$, only. I can then replace q_i, $i = 1, 2, 3$ in (23). I can then write (23) as a function of μ_i, $i = 1, 2, 3$ only, which allows me to solve for μ_3:

$$\mu_3^{1-\alpha_3} = -\frac{2\left(1-\alpha_2\right)\left(\alpha_2-\alpha_1\right)\left(\alpha_3-\alpha_1\right) l_{2,1}^{\alpha_2}\mu_1^{1-\alpha_3+\alpha_2-\alpha_1}}{\left(1-\alpha_1-\alpha_2\right)^2\left(1-\alpha_3\right)+k_0\mu_2^{\alpha_2-\alpha_1}}, \tag{25}$$

where

$$\begin{aligned} k_0 = &\left(-\alpha_1^2\left(1-\alpha_3\right)+\left(1-\alpha_2\right)\left(2\alpha_1\left(2\alpha_2+\alpha_3-1\right)-\left(\alpha_2+\alpha_3+\alpha_2\alpha_3-1\right)\right)\right) l_{2,1}^{\alpha_1+\alpha_3-1} \\ &+ 2\left(1-\alpha_2\right)\left(1-\alpha_1-\alpha_2\right)\left(\alpha_1+\alpha_3-1\right) l_{2,1}^{\alpha_1}. \end{aligned}$$

Note first that k_0 is only a function of the demands and $l_{2,1}$, all of which are bounded, and hence, k_0 is bounded. Since $\lim_{z\to 0}\mu_2^{\alpha_2-\alpha_1} = 0$, the denominator of (25) is positive for n large enough. Moreover, all terms in the numerator of (25) are positive. This implies there exists N (finite) such that for any $n > N$, $\mu_3 < 0$. But by definition this cannot be. Hence, if $\alpha_2 \neq 1 - \alpha_1$ and $K = 3$, (5)–(8) cannot be satisfied. □

Claim 7. *If $\alpha_2 = 1 - \alpha_1$ and $K = 3$, the system (5)–(8) has a solution in the limit.*

I first simplify (7)–(8) and (21)–(24) by $\alpha_2 = 1 - \alpha_1$. In particular, simplifying (22) to:

$$q_3\left(\alpha_2+\alpha_3-1\right)\left(1-\left(\frac{\mu_3}{\mu_2}\right)^{1-\alpha_3}\right) = 0. \tag{26}$$

It follows immediately that $\mu_3 = \mu_2$. I can solve the simplified versions of (21) and (23) for q_1 and q_2:

$$q_1 = \frac{2\alpha_1 (\alpha_1 + \alpha_3 - 1) \left(\mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3 - \alpha_1} - \mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1}\right)}{(1-2\alpha_1)(1-\alpha_3) - (\alpha_3 + \alpha_1 - 1)\left(2\alpha_1 \mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1} - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)} \tag{27}$$

$$q_2 = \frac{(1-2\alpha_1)(1-\alpha_3)\left(1 - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)}{(1-2\alpha_1)(1-\alpha_3) - (\alpha_3 + \alpha_1 - 1)\left(2\alpha_1 \mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1} - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)}. \tag{28}$$

Note that it follows immediately from this that,

$$\lim_{z\to 0} q_1 = 0, \quad \lim_{z\to 0} q_2 = 1, \quad \text{and} \quad \lim_{z\to 0} q_3 = 0.$$

Replacing q_1 through q_3 in (24), dividing by $\alpha_1 \mu_2^{1-\alpha_1}$ and simplifying, I get:

$$\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)\left(\mu_1^{\alpha_3-\alpha_1} - \frac{\mu_1^{1-\alpha_1}}{\mu_2^{1-\alpha_3}}\right) + (1-2\alpha_1)(1-\alpha_3)\left(1 - \mu_2^{\alpha_1+\alpha_3-1}\right)}{-(1-2\alpha_1)(1-\alpha_3) + (\alpha_1+\alpha_3-1)\left(2\alpha_1 \mu_2^{1-2\alpha_1} l_{2,1}^{\alpha_1} - \mu_1^{1-2\alpha_1} l_{2,1}^{\alpha_3-\alpha_1}\right)} = 0, \tag{29}$$

Note the second term of the denominator of (29) is 0 in the limit, and hence, the limit of the denominator is a constant. For (29) to be satisfied, it must then be that

$$\lim_{z\to 0} \frac{\mu_1^{1-\alpha_1}}{\mu_2^{1-\alpha_3}} = \frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}.$$

Put differently,

$$\mu_1 = k_2 \mu_2^{\frac{1-\alpha_3}{1-\alpha_1}} + O(\mu_2^{x_0}), \tag{30}$$

where $x_0 = \min\left\{\alpha_1 + \alpha_3 - 1 + \frac{1-\alpha_3}{1-\alpha_1}, \frac{1-\alpha_3}{1-\alpha_1}(1 + \alpha_3 - \alpha_1)\right\}$ and

$$k_2 = \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{1}{1-\alpha_1}}.$$

Using (30), I can rewrite (27) and (28):

$$q_1 = \frac{2\alpha_1(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)} k_2^{1-\alpha_3-\alpha_1} \mu_2^{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}} + O(\mu_2^{x_2}), \tag{31}$$

$$q_2 = 1 - k_2^{1-\alpha_3-\alpha_1} \frac{(2\alpha_3 - 1)\alpha_1}{(1-2\alpha_1)(1-\alpha_3)} \mu_2^{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}} + O(\mu_2^{x_2}), \text{ and similarly,} \tag{32}$$

$$q_3 = \frac{\alpha_1}{1-\alpha_3} k_2^{1-\alpha_1-\alpha_3} \mu_2^{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}} + O(\mu_2^{x_1}), \tag{33}$$

where

$$x_1 = x_0 + \frac{(1-\alpha_1)^2 - (1-\alpha_3^2)}{1-\alpha_1},$$

and

$$x_2 = \min\left\{x_1, 2\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}\right\}.$$

Equivalently,

$$x_1 = \min\left\{\frac{\alpha_3(\alpha_3-\alpha_1)}{1-\alpha_1}, \frac{1-\alpha_1(3-\alpha_1-\alpha_3)}{1-\alpha_1}\right\}, \text{ and}$$

$$x_2 = \min\left\{\frac{\alpha_3(\alpha_3-\alpha_1)}{1-\alpha_1}, \frac{1-\alpha_1(3-\alpha_1-\alpha_3)}{1-\alpha_1}, 2\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}\right\}.$$

Recall that using $\mu_2 = \mu_3$ and $\mu_1 = k_2\mu_2^{\frac{1-\alpha_3}{1-\alpha_1}} + O(\mu_2^{x_0})$. With some abuse of notation, I can write (7) as:

$$q_1\left(k_2\mu_2^{\frac{1-\alpha_3}{1-\alpha_1}} + O(\mu_2^{x_0})\right)^{1-\alpha_1} + q_2\mu_2^{\alpha_1} + q_3\mu_2^{1-\alpha_3} = z, \tag{34}$$

with q_1 through q_3 defined in (31) through (33). This simplifies to:

$$\frac{\alpha_1}{1-\alpha_3}\left(\frac{2(\alpha_1+\alpha_3-1)}{1-2\alpha_1}k_2^{1-\alpha_1}+1\right)k_2^{1-\alpha_1-\alpha_3}\mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}+1-\alpha_3} + \mu_2^{\alpha_1} + \mathcal{O}\left(\mu_2^{x_3}\right) = z, \tag{35}$$

where

$$x_3 = \min\left\{\frac{1-\alpha_1-\alpha_3+\alpha_3^2}{1-\alpha_1}, \frac{2-\alpha_3-\alpha_1(4-\alpha_1-2\alpha_3)}{1-\alpha_1}\right\}.$$

If

$$\alpha_1 < \frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3,$$

then

$$\mu_2^{\alpha_1} + \mathcal{O}\left(\mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}+1-\alpha_3}\right) = z.$$

Similarly, if

$$\alpha_1 > \frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3,$$

then

$$\frac{\alpha_1}{1-\alpha_3}\left(\frac{2(\alpha_1+\alpha_3-1)}{1-2\alpha_1}k_2^{1-\alpha_1}+1\right)k_2^{1-\alpha_1-\alpha_3}\mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}+1-\alpha_3} + \mathcal{O}\left(\mu_2^{\alpha_1}\right) = z.$$

Recall that $s_i = \frac{\mu_i^{1-\alpha_i}q_i}{z}$.

Case 1: $\alpha_1 < \frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3$. Taylor approximation as before yields:

$$s_1 = \frac{\alpha_1}{1-\alpha_1} k_2^{-\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}} \mu_2^{\frac{2(1-\alpha_1)^2-\alpha_3(2-\alpha_1)+\alpha_3^2}{1-\alpha_1}} + O(\mu_2^{x_4}), \tag{36}$$

$$s_2 = 1 - \mathcal{O}\left(\mu_2^{\frac{2(1-\alpha_1)^2-\alpha_3(2-\alpha_1)+\alpha_3^2}{1-\alpha_1}}\right) \tag{37}$$

$$s_3 = \frac{\alpha_1}{1-\alpha_3} k_2^{1-\alpha_3-\alpha_1} \mu_2^{\frac{2(1-\alpha_1)^2-\alpha_3(2-\alpha_1)+\alpha_3^2}{1-\alpha_1}} + \mathcal{O}\left(\mu_2^{x_4}\right) \tag{38}$$

where

$$x_4 = \min\left\{\frac{(1-\alpha_1)^2 - \alpha_3 + \alpha_3^2}{1-\alpha_1}, \frac{(1-2\alpha_1)(2-\alpha_1-\alpha_3)}{1-\alpha_1}, 2\frac{2(1-\alpha_1)^2 - \alpha_3(2-\alpha_1) + \alpha_3^2}{1-\alpha_1}\right\}.$$

Case 2: $\alpha_1 > \frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3$. Then we get,

$$s_1 = \frac{1-\alpha_3}{2-\alpha_1-\alpha_3} + O(\mu_2^{x_5}), \tag{39}$$

$$s_2 = \frac{(1-\alpha_1)(2\alpha_3-1)}{(1-2\alpha_1)(2-\alpha_1-\alpha_3)} \mu_2^{\alpha_1+\alpha_3-1} + O(\mu_2^{x_6}), \tag{40}$$

$$s_3 = \frac{1-\alpha_1}{2-\alpha_1-\alpha_3} + \mathcal{O}\left(\mu_2^{\alpha_1+\alpha_3-1}\right), \tag{41}$$

where

$$x_5 = \min\left\{\frac{(1-\alpha_1)^2 - \alpha_3(1-\alpha_3)}{1-\alpha_1}, \frac{\alpha_3(2-\alpha_1) - 2(1-\alpha_1)^2 - \alpha_3^2}{1-\alpha_1}\right\},$$
$$x_6 = \min\left\{\frac{\alpha_3(\alpha_3-\alpha_1)}{1-\alpha_1}, \frac{\alpha_3(3-\alpha_3-2\alpha_1) - 3(1-\alpha_1)^2}{1-\alpha_1}\right\}.$$

To summarize:

Case 1 $\alpha_1 < \frac{1}{4}\left(4 - \alpha_3 - \sqrt{(8-7\alpha_3)\alpha_3}\right)$. Then:

$$\lim_{z\to 0} s_1 = 0, \quad \lim_{z\to 0} s_2 = 1, \quad \lim_{z\to 0} s_3 = 0,$$
$$\lim_{z\to 0} r_1 = 0, \quad \lim_{z\to 0} r_2 = 1, \quad \lim_{z\to 0} r_3 = 0. \tag{42}$$

Case 2 $\alpha_1 > \frac{1}{4}\left(4-\alpha_3-\sqrt{(8-7\alpha_3)\alpha_3}\right)$. Then in the same fashion, I derive:

$$\begin{aligned}&\lim_{z\to 0} s_1 = \frac{1-\alpha_3}{2-\alpha_1-\alpha_3}, \quad \lim_{z\to 0} s_2 = 0, \quad \lim_{z\to 0} s_3 = \frac{1-\alpha_1}{2-\alpha_1-\alpha_3},\\ &\lim_{z\to 0} r_1 = 0, \qquad\qquad\qquad \lim_{z\to 0} r_2 = 1, \quad \lim_{z\to 0} r_3 = 0.\end{aligned} \tag{43}$$

When $\alpha_1 = \frac{1}{4}\left(4-\alpha_3-\sqrt{(8-7\alpha_3)\alpha_3}\right)$, then $\lim_{z\to 0} s_2 = \frac{1-\alpha_3}{\alpha_1}$. Hence, in this case all three probabilities of the stubborn player are strictly interior (everything else unchanged).

Claim 8. *If $\alpha_2 = 1-\alpha_1$ and $K=3$, the system (5)–(8) can be solved locally around $z=0$, with $q_i \in (0,1)$, and $\mu_i \in (0,1)$ for $i=1,2,3$.*

Proof. Define

$$\begin{aligned}A(\mu_1,\mu_2) =& 2\left(1-\alpha_1\right)\left(\alpha_1+\alpha_3-1\right)\left(\mu_1^{\alpha_3-\alpha_1} - \mu_1^{1-\alpha_1}\mu_2^{-(1-\alpha_3)}\right)\\ &+\left(1-2\alpha_1\right)\left(1-\alpha_3\right)\left(1-\mu_2^{\alpha_1+\alpha_3-1}\right),\end{aligned} \tag{44}$$

$$\begin{aligned}B(\mu_1,\mu_2,z) =& \left(1-2\alpha_1\right)\left(\left(1-\alpha_3\right)\mu_2^{\alpha_1} + \mu_1^{1-\alpha_1-\alpha_3}\mu_2^{\alpha_3-\alpha_1}\left(\alpha_1\mu_2^{1-\alpha_3} - \left(1-\alpha_3\right)\mu_2^{\alpha_1}\right)\right)\\ &+2\alpha_1\left(\alpha_1+\alpha_3-1\right)\mu_1^{1-2\alpha_1}\left(\mu_1^{1-\alpha_3}\mu_2^{\alpha_3-\alpha_1} - \mu_2^{1-2\alpha_1}\right) - dz,\end{aligned} \tag{45}$$

where

$$d = \left(1-2\alpha_1\right)\left(1-\alpha_3\right) - \left(\alpha_1+\alpha_3-1\right)\left(2\alpha_1\mu_2^{1-\alpha_1}\mu_1^{-\alpha_1} - \mu_1^{1-\alpha_1-\alpha_3}\mu_2^{\alpha_3-\alpha_1}\right). \tag{46}$$

Following identical steps to the first part of the proof, I can reduce the system (5)–(8) to $A(\mu_1,\mu_2)=0$ and $B(\mu_1,\mu_2,z)=0$.

Case 1 Throughout Case 1, $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} > \alpha_1+\alpha_3-1$. Let me introduce two auxiliary variables, y_b and n_b, where

$$y_b = \mu_1^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{1-\alpha_3}}, \text{ and} \tag{47}$$

$$n_b = \left(\mu_2\mu_1^{-\frac{1-\alpha_1}{1-\alpha_3}} - \left(\frac{2\left(1-\alpha_1\right)\left(\alpha_1+\alpha_3-1\right)}{\left(1-2\alpha_1\right)\left(1-\alpha_3\right)}\right)^{\frac{1}{1-\alpha_3}}\right)^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}}. \tag{48}$$

Using the IFT, I will explicitly derive the derivatives:

$$\left.\frac{dy_b}{dz}\right|_{y_b=n_b=z=0} = \frac{(1-\alpha_3)}{\alpha_1}\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{\alpha_1}{1-\alpha_3}} > 0, \text{ and} \tag{49}$$

$$\begin{aligned}\left.\frac{dn_b}{dz}\right|_{y_b=n_b=z=0} &= \left(\frac{1-\alpha_3}{\alpha_1}\right)^{1-\frac{2+\alpha_1^2-\alpha_1(3-\alpha_3)-\alpha_3(2-\alpha_3)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}} \\ &\cdot \left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{1-\alpha_1}{1-\alpha_3}+\frac{1-\alpha_1}{\alpha_1+\alpha_3-1}-\frac{1-2\alpha_1-\alpha_3}{1-\alpha_1}} > 0.\end{aligned} \tag{50}$$

In order to compute those derivatives, it is useful to further introduce y_a and n_a:

$$y_a = \mu_1^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{1-\alpha_3}}, \text{ and} \tag{51}$$

$$n_a = \mu_2\mu_1^{-\frac{1-\alpha_1}{1-\alpha_3}} - \left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{1}{1-\alpha_3}}. \tag{52}$$

Note that

$$\begin{aligned}\left.\frac{dy_a}{dn_a}\right|_{y_b=n_b=z=0} &= \left.\frac{d\left(y_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}}\right)}{d\left(n_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}}\right)}\right|_{y_b=n_b=z=0} \\ &= \left.\left(\frac{y_b}{n_b}\right)^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}-1}\right|_{y_b=n_b=z=0} \left.\frac{dy_b}{dn_b}\right|_{y_b=n_b=z=0}\end{aligned} \tag{53}$$

Hence,

$$\begin{aligned}\left.\frac{dy_b}{dn_b}\right|_{y_b=n_b=z=0} &= \left.\left(\frac{y_a}{n_a}\right)^{\frac{(1-\alpha_1)^2-(2-\alpha_1-\alpha_3)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}}\right|_{y_b=n_b=z=0} \left.\frac{dy_a}{dn_a}\right|_{y_b=n_b=z=0} \\ &= \left.\left(\frac{dy_a}{dn_a}\right)^{\frac{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)(\alpha_1+\alpha_3-1)}}\right|_{y_b=n_b=z=0}.\end{aligned} \tag{54}$$

Using the IFT on (44), I show that:

$$\left.\frac{dy_a}{dn_a}\right|_{y_a=n_a=o} = -\left.\frac{\frac{\partial A_1}{\partial n_a}}{\frac{\partial A_1}{\partial y_a}}\right|_{y_a=n_a=0} > 0. \tag{55}$$

I can rewrite (44) as a function of y_a and n_a, using (51) and (52). Denote this new function $A_1(y_a, n_a)$. Note that y_a is simply constructed such that the smallest exponent on y_a in A_1 is 1. Taking derivatives of A_1 w.r.t. y_a and n_a, and evaluating the derivative at $y_a = n_a = 0$, I get

$$\left.\frac{\partial A_1}{\partial y_a}\right|_{y_a=n_a=0} = (1-2\alpha_1)(1-\alpha_3)\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_3}} > 0, \tag{56}$$

$$\left.\frac{\partial A_1}{\partial n_a}\right|_{y_a=n_a=0} = -\frac{(1-2\alpha_1)(1-\alpha_3)^2}{\alpha_1}\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{-\alpha_1}{1-\alpha_3}} < 0. \tag{57}$$

Hence,

$$\left.\frac{dy_a}{dn_a}\right|_{y_a=n_a=o} = -\left.\frac{\frac{\partial A_1}{\partial n_a}}{\frac{\partial A_1}{\partial y_a}}\right|_{y_a=n_a=0} \tag{58}$$

$$= \frac{1-\alpha_3}{\alpha_1}\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{-\frac{2\alpha_1+\alpha_3-1}{1-\alpha_3}} > 0. \tag{59}$$

Similarly, I can rewrite (45) and (46) using (47) and (48). Denote these new functions $B_1(y_b, n_b, z)$, and d_1, where

$$\begin{aligned} d_1 = & (1-2\alpha_1)(1-\alpha_3) \\ & - 2\alpha_1(\alpha_1+\alpha_3-1)\left(n_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}} + k_2^{-\frac{1-\alpha_1}{1-\alpha_3}}\right)^{1-\alpha_1} y_b^{\frac{(1-\alpha_1)^2-\alpha_1(1-\alpha_3)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} \\ & + (\alpha_1+\alpha_3-1)\left(n_b^{\frac{(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}} + k_2^{-\frac{1-\alpha_1}{1-\alpha_3}}\right)^{\alpha_3-\alpha_1} y_b^{1-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}}. \end{aligned}$$

Note that, y_b is simply constructed such that the smallest exponent on y_b in $B_1 - d_1 z$. Taking derivatives of B_1 with respect to y_b, n_b and z, I get:

$$\left.\frac{\partial B_1}{\partial y_b}\right|_{y_b=n_b=z=0} = \alpha_1(1-2\alpha_1)\left(\frac{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}{(1-2\alpha_1)(1-\alpha_3)}\right)^{\frac{1-\alpha_1}{1-\alpha_3}} > 0, \tag{60}$$

$$\left.\frac{\partial B_1}{\partial n_b}\right|_{y_b=n_b=z=0} = 0, \tag{61}$$

$$\left.\frac{\partial B_1}{\partial z}\right|_{y_b=n_b=z=0} = -(1-2\alpha_1)(1-\alpha_3) < 0. \tag{62}$$

Since

$$\left.\frac{dy_b}{dz}\right|_{y_b=n_b=z=0} = -\left.\frac{\frac{\partial B_1}{\partial z}}{\frac{\partial B_1}{\partial y_b}}\right|_{y_b=n_b=z=0}, \text{ and} \tag{63}$$

$$\left.\frac{dn_b}{dz}\right|_{y_b=n_b=z=0} = -\left.\frac{\frac{\partial B_1}{\partial z}}{\frac{\partial B_1}{\partial n_b} + \frac{\partial B_1}{\partial y_b}\frac{\partial y_b}{n_b}}\right|_{y_b=n_b=z=0}, \tag{64}$$

(49) and (50) follow immediately. Note that the exponents on y_b in d_1 are less than 1. It is easy to verify that the derivative of d_1 w.r.t. y_b, evaluated at $y_b = n_b = z = 0$ is 0. I simply need to take into account the rate at which y_b goes to 0 relative to z. Recall that $\lim_{z\to 0} \frac{\mu_1^{1-\alpha_1}}{\mu_2^{1-\alpha_3}} = k_2^{1-\alpha_1}$. Moreover, (i) if $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 < \alpha_1$, then $\lim_{z\to 0} \frac{\mu_2^{\alpha_1}}{z} = 1$. Therefore,

$$\lim_{z\to 0} \frac{y_b^{\frac{(1-\alpha_1)\alpha_1}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}}}{z} = k_3,$$

where k_3 is some positive constant. It follows that[2]

$$\lim_{z\to 0} y_b^{\frac{(1-\alpha_1)^2-\alpha_1(1-\alpha_3)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}-1} z = 0, \text{ and}$$
$$\lim_{z\to 0} y_b^{-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} z = 0.$$

Similarly, (ii) if $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 < \alpha_1$, then

$$\lim_{z\to 0} \frac{\mu_2^{\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1}+1-\alpha_3}}{z} = k_4,$$

where k_4 is some positive constant. Therefore, $\lim_{z\to 0} \frac{y_b}{z} = k_5$, where k_5 is some positive constant. Hence,

$$\lim_{z\to 0} y_b^{\frac{(1-\alpha_1)^2-\alpha_1(1-\alpha_3)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}-1} z = 0, \text{ and}$$
$$\lim_{z\to 0} y_b^{-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2-(1-\alpha_3)(\alpha_1+\alpha_3-1)}} z = 0.$$

Therefore, if $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} > \alpha_1 + \alpha_3 - 1$, $\alpha_2 = 1 - \alpha_1$ and $K = 3$, the system (5)–(8) can be solved locally around $z = 0$, with $q_i \in (0,1)$, and $\mu_i \in (0,1)$ for $i = 1, 2, 3$.

[2] Note that

$$0 > \left(\frac{(1-\alpha_1)\alpha_1}{(1-\alpha_1)^2 - (1-\alpha_3)(\alpha_1+\alpha_3-1)}\right)^{-1}\left(-\frac{(1-\alpha_3)(1-\alpha_1)}{(1-\alpha_1)^2 - (1-\alpha_3)(\alpha_1+\alpha_3-1)}\right) + 1.$$

Case 2 Throughout Case 2, $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} < \alpha_1 + \alpha_3 - 1$. Case 2 has two subcases, where I will introduce a pair of auxiliary variables for each subcase. Let me first state the auxiliary variables, and the derivatives for each case before giving a sketch of the proof.

Case 2.1 Throughout Case 2.1, $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 < \alpha_1$. Let me introduce two auxiliary variables, y_b and m_b, where

$$y_b = \mu_2^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{1-\alpha_1}}, \text{ and} \tag{65}$$

$$n_b = \left(\mu_1 \mu_2^{-\frac{1-\alpha_3}{1-\alpha_1}} - \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{1}{1-\alpha_1}}\right)^{\frac{(1-\alpha_1)^2+(1-\alpha_3)(1-\alpha_1-\alpha_3)}{(1-\alpha_1)}}. \tag{66}$$

Using the IFT, I will explicitly derive the derivatives:

$$\left.\frac{dy_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_3}{\alpha_1}\left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}} > 0, \text{ and} \tag{67}$$

$$\left.\frac{dm_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_3}{\alpha_1}\left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{1-\alpha_1}{1-\alpha_3}} > 0. \tag{68}$$

Case 2.2 Throughout Case 2.2, $\frac{(1-\alpha_1)^2-\alpha_3(1-\alpha_3)}{1-\alpha_1} + 1 - \alpha_3 > \alpha_1$. Again, let me introduce two auxiliary variables, y_b and m_b, where

$$y_b = \mu_2^{\frac{(1-\alpha_1)(\alpha_3-\alpha_1)+(1-\alpha_3)(2-2\alpha_1-\alpha_3)}{1-\alpha_1}}, \text{ and} \tag{69}$$

$$m_b = \left(\mu_1 \mu_2^{-\frac{1-\alpha_3}{1-\alpha_1}} - \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{1}{1-\alpha_1}}\right)^{\frac{(1-\alpha_1)(\alpha_3-\alpha_1)+(1-\alpha_3)(2-2\alpha_1-\alpha_3)}{(1-\alpha_1)(\alpha_3-\alpha_1)}} \tag{70}$$

Using the IFT, one can again explicitly derive the derivatives:

$$\left.\frac{dy_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_1}{\alpha_1}\left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}} > 0, \text{ and} \tag{71}$$

$$\left.\frac{dm_b}{dz}\right|_{y_b=m_b=z=0} = \frac{1-\alpha_1}{\alpha_1}\left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)}\right)^{\frac{1-\alpha_1}{1-\alpha_3}} > 0. \tag{72}$$

For both Case 2.1 and Case 2.2, in order to compute the above derivatives, it is useful to further introduce y_a and m_a, where where

$$y_a = \mu_2^{\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1}} \tag{73}$$

$$m_a = \mu_1 \mu_2^{-\frac{1-\alpha_3}{1-\alpha_1}} - \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{\frac{1}{1-\alpha_1}} \tag{74}$$

As set out in great detail for Case 1, I can rewrite (44) as a function of y_a and m_a, using (73) and (74). As with Case 1, taking derivatives and using the IFT, this gives me

$$\left.\frac{dy_a}{dm_a}\right|_{y_a=m_a=0} = \left(\frac{(1-2\alpha_1)(1-\alpha_3)}{2(1-\alpha_1)(\alpha_1+\alpha_3-1)} \right)^{-\frac{\alpha_3-\alpha_1}{1-\alpha_1}} > 0. \tag{75}$$

Following the identical steps of Case 1, we get the derivatives given in (67), (68), (71), and (72). Therefore, if $\frac{(1-\alpha_3)(\alpha_3-\alpha_1)}{1-\alpha_1} < \alpha_1 + \alpha_3 - 1$, and $\alpha_2 = 1 - \alpha_1$ and $K = 3$, the system (5)–(8) can be solved locally around $z = 0$, with $q_i \in (0, 1)$, and $\mu_i \in (0, 1)$ for $i = 1, 2, 3$. □

□

Pooling Equilibria with $K = 2$

See Proposition 3.

Pooling Equilibria with $K = 1$

Proposition 3 (Proposition 9)**.** *Pooling equilibria where players make a demand α with probability 1 exist. In any such equilibrium, there is either*

(a) infinitely long delay and $\alpha = 1$, or

(b) immediate agreement and $\alpha = 1/2$.

Proof of Proposition 9. Suppose players choose a demand $\alpha < 1/2$. Then both types of players have an incentive to deviate to $1 - \alpha$. Suppose instead players choose a demand $1 > \alpha > 1/2$. The expected payoff for a rational player in this candidate equilibrium is

$1-\alpha$. The expected payoff for a stubborn player from demanding α is $(1-\alpha)(1-z^{\frac{\alpha}{1-\alpha}})$. However, a stubborn player could receive $1-\alpha$, by demanding $1-\alpha$. If players demand $1/2$, then $\alpha = 1-\alpha$, and hence, there is no such deviation. Suppose $\alpha = 1/2$. Then if any deviation is believed to come from a rational type, neither player type wants to deviate. If players demand $\alpha = 1$, then similarly there is no such deviation. □

Semi-Separating Equilibria: Convergence and Existence

Recall that

(a) for any $z > 0$, there is at most one separating demand by the stubborn type (Lemma 3.3).

(b) the lowest and highest pooling demands are incompatible (Lemma 2.1).

Note also that if there are multiple pooling demands, then the highest pooling demand is compatible with the separating demand of the stubborn type (a straightforward consequence of Lemma 2).

Lemma 2. *If there are multiple pooling demands, then either both types separate or neither.*

Proof. Recall that if there are multiple pooling demands, then it must be that the lowest pooling demand must be incompatible with the highest pooling demand. If the highest pooling demand is incompatible with the lowest pooling demand and there is no separating demand by the stubborn type, then the rational type cannot benefit from making a separating demand: he would receive the same payoff from facing any pooling demand and would receive a lower payoff conditional on facing a separating demand by the rational type. Hence, if there are multiple pooling demands, then either there are no separating demands by either type or there are separating demands by both types. □

Lemma 3 (Lemma 6)**.** *There exists a semi-separating equilibrium with* $\operatorname{supp} s \subseteq \{\alpha_0, \alpha_1, \dots, \alpha_K\}$ *and* $\operatorname{supp} r \subseteq \{\alpha_1, \dots, \alpha_K, \dots, \alpha_{K+L}\}$ *for some* $K, L \geq 1$ *only if the demands* α_0 *through* α_{K+L} *along with probabilities* q_0 *through to* q_{K+L} *and positive numbers* μ_0 *through* μ_{K+L} *solve (76) – (79).*

Proof. Given z and $\{\alpha_0, \alpha_1, \dots, \alpha_K, \dots, \alpha_{K+L}\}$, define the following system in (q_i, μ_i), $i = 0, 1, \dots, K, \dots, K+L$:

$$\Delta^r_{k,k+1} = 0, \ \forall \alpha_k, \alpha_{k+1} \in \operatorname{supp} r, \tag{76}$$

$$\Delta^s_{k,k+1} = 0, \ \forall \alpha_k, \alpha_{k+1} \in \operatorname{supp} s, \tag{77}$$

$$\sum_{i=0}^{K+L} q_i \mu_i^{1-\alpha_i} = z, \text{ and} \tag{78}$$

$$\sum_{i=0}^{K+L} q_i = 1. \tag{79}$$

The proof of Lemma 6 follows the same lines as the proof of Lemma 4 (with the obvious adjustments for separating demands). □

Proposition 4 (Proposition 10)**.** *(a) Fix* $\operatorname{supp} s = \{\alpha_0, \alpha_1, \dots, \alpha_K\}$ *and* $\operatorname{supp} r = \{\alpha_1, \dots, \alpha_K, \dots, \alpha_{K+L}\}$ *for some* $K \geq 2$ *and* $L \geq 1$*. Then there exists* $\bar{z} > 0$ *such that for any* $z < \bar{z}$*, there exist no* $q_i > 0$*, and* $\mu_i > 0$ *for* $i = 0, 1, 2, \dots, K, \dots K+L$ *such that* (z, α, q, μ) *satisfies (76) – (79).*

(b) Fix a sequence $z^n \to 0$*, and a corresponding convergent sequence of semi-separating equilibria* (α^n, q^n, μ^n)*, where* $\alpha^n = (\alpha_i^n)_{i=0}^K + L$ *with* $K \geq 2$ *and* $L \geq 1$*. Then there exist* $a_0 < a_1$*,* $a_1 \in (0, 1/2]$*,* $a_K \in (1 - a_1, 1]$ *and* $a_{K+\ell} > a_K$ *for al* $\ell = 1, \dots, L$ *such that*

$$\lim_{n\to\infty} \left(\alpha_0^n, \alpha_1^n, \dots, \alpha_{k-1}, \alpha_k, \dots, \alpha_K^n, \alpha_{K+1}^n, \dots, \alpha_{K+L}^n\right)$$
$$= (a_0, \underbrace{a_1, \dots, a_1,}_{\lceil K/2\rceil - 1 \text{ terms}} \underbrace{1 - a_1, \dots, 1 - a_1, a_K}_{K - \lceil K/2\rceil + 1 \text{ terms}}, \underbrace{a_{K+1}, \dots, a_{K+L}}_{L \text{ terms}}),$$

where $k = \lceil K/2 \rceil$*. Moreover, along any such sequence,*

$$\lim_{n\to\infty} q^n = (\underbrace{0, \dots, 0}_{K-2 \text{ terms}}, 1, \underbrace{0, \dots, 0}_{1+L \text{ terms}}).$$

Proof of Proposition 10. The proof of Proposition 7 does not rely on the equilibrium being a pooling equilibrium. Consider instead a potential semi-separating equilibrium with $\operatorname{supp} s = \{\alpha_0, \alpha_1, \ldots, \alpha_K\}$ and $\operatorname{supp} r = \{\alpha_1, \ldots, \alpha_K, \ldots, \alpha_{K+L}\}$ for some $K \geq 2$ and $L \geq 1$. Claims 1 and 2 and the proofs thereof require no changes (other than adjusting the sum over probabilities in equations (7) and (8) to include the separating demands - see (78) and (79)). Claim 3 holds, but the proof requires cosmetic changes. In particular, when (5) is evaluated at $k = 1$, additional terms for the separating demands are needed - however, conditional on facing a separating demand by either type, the payoff from making higher pooling demand (α_2) is strictly higher than the payoff from making the lower pooling demand (α_1). Hence, the argument goes through as before. Claim 4 also holds, but convergence in the value of demands is only established for pooling demands. Therefore, if there exist at least two pooling demands, $\lim_{n\to\infty} q^n = \lim_{n\to\infty} r^n = \mathbb{1}_{1-a}$.

□

Lemma 4. *For any $z > 0$, and any $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_i = \alpha_j$ for $i, j = 1, 2, 3$, there exists no equilibrium with* $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$ *and* $\operatorname{supp} r = \{\alpha_2, \alpha_3\}$.

Proof. Suppose there exists exactly one pooling demand, and both types separate. Then it must be that the separating demand by the stubborn type and the pooling demand are incompatible - otherwise the stubborn type is not willing to separate. Suppose then that the separating demand by the stubborn type and the pooling demand are incompatible. It then follows that the lowest separating demand by the rational type does just as well as the pooling demand when facing either the separating demand by the stubborn type or the pooling demand. However, when facing a separating demand by the rational type, the separating demand by the rational type does strictly worse than the pooling demand. Hence, the rational type does not want to separate. Hence, there exists no equilibrium with one pooling demand where both types make separating demands.

□

Lemma 5. *Fix any α_1, α_2 and any $z > 0$. Then there exists an equilibrium with* $\operatorname{supp} r = \{\alpha_2\}$ *and* $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$ *if and only if* $\alpha_1 + \alpha_2 = 1$ *and* $\alpha_2 > \frac{1}{2}$.

Proof.

Claim 9. *Fix any $\{\alpha_1, \alpha_2\}$, with $\alpha_1 + \alpha_2 \neq 1$. Then there exists no symmetric equilibrium with* $\operatorname{supp} r = \{\alpha_2\}$ *and* $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$.

Proof. Since strength is decreasing in any equilibrium and $\mu_1 = 1$, it follows that $\alpha_1 < \alpha_2$.

Case 1: $\alpha_1 > 1/2$. In this case, the rational type has an incentive to deviate to α_1:

$$v_1^r = q_1 (1 - \alpha_1) + q_2 \left(\left(1 - \mu_2^{1-\alpha_2}\right) \alpha_1 + \mu_2^{1-\alpha_2} (1 - \alpha_2) \right), \tag{80}$$

$$v_2^r = q_1 (1 - \alpha_1) + q_2 (1 - \beta). \tag{81}$$

Hence, $v_1^r > v_2^r$.

Case 2: $\alpha_2 \leq 1/2$. In this case, the rational type has an incentive to deviate to $1 - \alpha_1$: Conditional on facing a demand of α_1, demanding α_2 gives $1/2 (\alpha_2 + 1 - \alpha_1)$, while demanding $1 - \alpha_1$ gives $1 - \alpha_1 > \alpha_2$. Conditional on facing a demand of α_2, demanding α_2 gives $1/2$, while demanding $1 - \alpha_1$ gives at least $1 - \alpha_2$. Hence, the rational type has a strictly higher payoff from demanding $1 - \alpha_1$ than from demanding α_2.

Case 3: $\alpha_1 \leq 1/2 < \alpha_2$. If $\alpha_1 + \alpha_2 < 1$, then the stubborn type has an incentive to deviate from α_1 to $1 - \alpha_2$: Conditional on facing a demand of α_1, a stubborn type receives $1/2$ from demanding α_1 and receives $1/2 (1 - \alpha_1 + \alpha_2)$ from demanding $1 - \alpha_2$. Note that $\alpha > \alpha_1$. Conditional on facing a demand of α_2, a stubborn type receives $1/2 (\alpha_1 + 1 - \alpha_2)$ from demanding α_1, while demanding $1 - \alpha_2$ would give the stubborn type $1 - \alpha_2$. Hence, the stubborn type has a strictly higher payoff from demanding $1 - \alpha_2$ than from demanding α_1.

Hence, it must be that $\alpha_1 + \alpha_2 > 1$. A stubborn type's payoff from demanding α_2 and $\alpha_3 = 1 - \alpha_2$ is:

$$v_2^s = q_2 (1 - \alpha_2)(1 - \mu_2^{\alpha_2}), \tag{82}$$

$$v_3^s = q_1 \left(\frac{1 - \alpha_2 + 1 - \alpha_1}{2} \right) + q_2 (1 - \alpha_2). \tag{83}$$

Hence, $v_2^s < v_3^s$. □

Claim 10. *Fix any* α_1, α_2, *with* $\alpha_2 = 1 - \alpha_1 > 1/2$. *Then there exists an equilibrium with* $\operatorname{supp} r = \{\alpha_2\}$ *and* $\operatorname{supp} s = \{\alpha_1, \alpha_2\}$.

Proof. Evaluating (**??**) for $k = 1$ gives:

$$\Delta_{1,2}^r = -(1 - q_2)\left(\alpha_2 - \frac{1}{2}\right) < 0. \tag{84}$$

Hence, the rational type has no incentive to deviate to α_1. Evaluating (**??**) for $k = 1$, i.e. $\Delta_{1,2}^s$, gives:

$$q_2 (1 - \alpha_2) \mu_2^{\alpha_2} - (1 - q_2)\left(\alpha_2 - \frac{1}{2}\right) = 0. \tag{85}$$

Solving (85) for q_2 I get:

$$q_2 = \frac{2\alpha_2 - 1}{2\alpha_2 - 1 + 2\mu_2^{\alpha_2}(1 - \alpha_2)} \in (0, 1]. \tag{86}$$

Plugging (89) into (**??**) and simplifying, I get:

$$z = \frac{2\mu_2^{\alpha_2}(1 - \alpha_2) + \mu_2^{1-\alpha_2}(2\alpha_2 - 1)}{2\mu_2^{\alpha_2}(1 - \alpha_2) + 2\alpha_2 - 1}. \tag{87}$$

Note that $\mu_2^{\alpha_2} < \mu_2^{1-\alpha_2}$, and hence,

$$\mu_2^{1-\alpha_2}/z \to 1. \tag{88}$$

□

Note that $1 - q_2 = zs_1$ and recall that $z = 1 - q_2 + q_2\mu_2^{1-\alpha_2}$. Hence,

$$s_1 = \frac{1 - q_2}{1 - q_2 + q_2\mu_2^{1-\alpha_2}}.$$

Given that

$$q_2 = \frac{2\alpha_2 - 1}{2\alpha_2 - 1 + 2\mu_2^{\alpha_2}(1-\alpha_2)} \in (0,1]. \tag{89}$$

we can write

$$s_1 = \frac{2(1-\alpha_2)\mu_2}{2(1-\alpha_2)\mu_2 + \mu_2^{2(1-\alpha_2)}(2\alpha_2 - 1)} = \frac{2(1-\alpha_2)\mu_2^{\alpha_2}}{2(1-\alpha_2)\mu_2^{2\alpha_2-1} + 2\alpha_2 - 1}.$$

Hence, $\lim_{n\to\infty} s_1^n = 0$.

Hence,

$$\lim_{n\to\infty} s^n = (0,1),$$

and

$$\lim_{n\to\infty} r^n = (0,1).$$

□

Lemma 6. *(a) Let (z^n, r^n, s^n) be a convergent sequence of equilibrium triples with $\lim_{n\to\infty} = 0$ with $\operatorname{supp} s = \{\alpha_1\}$ and $\operatorname{supp} r = \{\alpha_1, \dots, \alpha_K\}$. Then there exists $a_1 < \frac{1}{2}$ and $a_i > 1 - a_1$ for $i = 2, \dots, K$ such that*

$$\lim_{n\to\infty} r^n = \mathbb{1}_{a_1} 2\left(a_1 - \sum_{i=2}^{K} r_i(1-a_i)\right) + \sum_{i\neq 1} \mathbb{1}_{a_i} r_i,$$

with $\sum_{i\neq 1} r_i = 1 - r_1$.

(b) Fix any $\alpha_1 < \frac{1}{2}$ and any $\alpha_2 > 1 - \alpha_1$. Then for any $z < \frac{2(\alpha_1+\alpha_2-1)}{2\alpha_2-1}$, there exists an equilibrium with $s = \mathbb{1}_{\alpha_1}$ and

$$r = \mathbb{1}_{\alpha_1}\left(1 - \frac{1-2\alpha_1}{(2\alpha_2-1)(1-z)}\right) + \mathbb{1}_{\alpha_2}\left(\frac{1-2\alpha_1}{(2\alpha_2-1)(1-z)}\right).$$

Proof. Suppose the rational type separates over multiple demands. Then the lowest such demand, call it α_2, cannot be strictly compatible with the pooling demand α_1: we require $\alpha_2 + \alpha_1 \geq 1$. Otherwise, the rational type has no incentive to separate to a demand that is strictly compatible. Hence, all rational demands are incompatible with the pooling demand. As a result, it must be that fixing a demand of the opponent, the payoff from making any of the rational separating demands is identical. Since the lowest demand is a pooling demand, and the higher demands are separating demands incompatible with other demands made, the stubborn type has no incentive to deviate. The rational type is willing to randomize over the demands if the payoff from the pooling demand is identical to the payoff from any of the rational separating demands:

$$q_1 \frac{1}{2} + (1 - q_1)\alpha_1 = \sum_{i=1}^{K} q_i(1 - \alpha_i).$$

Hence,

$$q_1 = 2\left(\alpha_1 - \sum_{i=2}^{K} q_i(1 - \alpha_i)\right).$$

Given $q_1 = z + (1 - z)r_1$ and $q_i = (1 - z)r_i$ for all $i > 1$, the first part of the lemma follows. The second part of the lemma follows when we set $K = 2$. In particular, $r_1 > 0$ iff $z < \left(1 - \frac{1-2\alpha_1}{(2\alpha_2-1)(1-z)}\right)$.

□

Separating Equilibria: Convergence and Existence

Proposition 5. *(a) Fix any set of demands $\{\alpha_1, \ldots, \alpha_K\}$ with $K \geq 2$ and fix any probability distribution over demands $\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$: $\tilde{r} = (0, \tilde{r}_2, \ldots, \tilde{r}_K)$. Then there exists a fully separating equilibrium with $s = (1, 0, \ldots, 0)$ and $r = \tilde{r}$ if and only if $\alpha_1 < 1/2$, $\alpha_1 + \alpha_2 > 1$ and*

$$z \geq \bar{z} = \frac{2\left(\alpha_1 - \sum_{i=2}^{K} \tilde{r}_i(1 - \alpha_i)\right)}{1 - 2\sum_{i=2}^{K} \tilde{r}_i(1 - \alpha_i)}.$$

(b) Let (z^n, r^n, s^n) be a convergent sequence of separating equilibrium triples such that $\lim_{n\to\infty} z^n = 0$. Then there exists $a \in (0, 1/2)$ such that

$$\lim_{n\to\infty} r^n = \mathbb{1}_{1-a},$$

and

$$\lim_{n\to\infty} s^n = \mathbb{1}_a.$$

Proof of Proposition 14. Recall there exists at most one separating demand by the stubborn type. Moreover, note that the rational type's separating demands must be incompatible with the stubborn type's separating demand - otherwise, the stubborn type has an incentive to deviate to the lowest separating demand by the rational type. Hence, $\alpha_1 + \alpha_2 > 1$. Suppose the rational type randomizes over demands $\alpha_2, \ldots, \alpha_K$, i.e., $r = (0, \tilde{r}_2, \ldots, \tilde{r}_K)$. Then the rational type has no incentive to deviate to the stubborn type's demand iff

$$z\frac{1}{2} + (1-z)\alpha_1 \leq z(1-\alpha_1) + (1-z)\sum_{i=2}^{K} \tilde{r}_i(1-\alpha_i).$$

Hence, a separating equilibrium with $s = (1, 0, \ldots, 0)$ and $r = \tilde{r}$ exists iff:

$$z \geq \bar{z} = \frac{2\left(\alpha_1 - \sum_{i=2}^{K} \tilde{r}_i(1-\alpha_i)\right)}{1 - 2\sum_{i=2}^{K} \tilde{r}_i(1-\alpha_i)}. \tag{90}$$

Note further that as $z \to 0$, it is clear that $z \geq \bar{z}$ iff Moreover, take a sequence of separating triples such that $\lim_{n\to\infty} z^n = 0$. Then it follows from (90) that there exists $a \in (0, 1/2)$ such that

$$\lim_{n\to\infty} r^n = \mathbb{1}_{1-a},$$

and

$$\lim_{n\to\infty} s^n = \mathbb{1}_a.$$

In other words, in the limit the two types make demands with probability 1 which are exactly compatible.

□

Refinement: D1

Proof of Proposition 6. Denote a potential deviation by d, and the associated strength by μ_d. Moreover, denote the strength that makes the rational type indifferent between his equilibrium demand and d by μ_d^r. Consider a pooling equilibrium, where both players randomize over exactly three demands. Moreover, suppose $\alpha_2 = 1 - \alpha_1$. Clearly, neither type wants to deviate to $d < 1-\alpha_3$. Moreover, the rational type is always willing to deviate to $d > \alpha_3$ regardless of the opponent's belief.

If $d \in (1-\alpha_3, \alpha_1)$, then the stubborn type's payoff difference between α_1 and d evaluated at μ_d^r is given by:

$$v_1^s - v_d^s|_{\mu_d = \mu_d^r} = -q_3\left(1-\alpha_3\right)\mu_3^{1-\alpha_3}\left(\mu_1^{\alpha_1+\alpha_3-1} - \left(\mu_d^r\right)^{d+\alpha_3-1}\right). \tag{91}$$

Since $\mu_d > \mu_1$ and $\alpha_1 > d$, $v_1^s - v_d^s|_{\mu_d=\mu_d^r} > 0$. The stubborn type prefers his equilibrium demand to d at μ_d^r.

If $d \in (\alpha_1, 1-\alpha_1)$, then the stubborn type's payoff difference between α_2 and d evaluated at μ_d^r is given by:

$$\begin{aligned} v_2^s - v_d^s|_{\mu_d=\mu_d^r} =& - q_2\left(1-\alpha_2\right)\left(\mu_2^{\alpha_2} - \left(\mu_d^r\right)^{\alpha_2+d-1}\mu_2^{1-\alpha_2}\right) \\ & - q_3\left(1-\alpha_3\right)\left(\mu_2^{\alpha_2} - \left(\mu_d^r\right)^{d+\alpha_3-1}\mu_3^{1-\alpha_3}\right). \end{aligned} \tag{92}$$

Since $\alpha_2 > d$ and $\mu_d^r > \mu_2 = \mu_3$, $v_2^s - v_d^s|_{\mu_d=\mu_d^r} > 0$.

If $d \in (1-\alpha_1, \alpha_3)$, then the rational type is willing to deviate regardless of the belief of the opponent.

Hence, any three offer pooling equilibrium with $\alpha_2 = 1-\alpha_1$ satisfies D1. □