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"Large mechanism design with Moment-based allocation externality"

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Large mechanism design with moment-based allocation externality*

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Abstract

In many mechanism design problems in practice, often allocation externality exists (e.g., peer effects in student allocation, and post-license competition in oligopoly). Despite the practical importance, mechanism design with allocation externality has not been much explored in the literature, perhaps due to the tractability issue of the problem. In this paper, we propose a simple and tractable model of mechanism design with allocation externality. We characterize the optimal mechanism, which has a very simple form in the sense that it is identified by only a few parameters. This simplicity of the optimal mechanism is also useful to obtain comparative statics results.

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1 Introduction

In many mechanism design problems in practice, often *allocation externality* exists. For example, in a student-allocation problem to schools (or to classes, groups, etc.), which studies how to divide a pool of students into different schools, often one of the important concerns is peer effects. In a license-allocation problem in oligopoly, studying which firms should be given special licenses to operate in certain markets, each firm's willingness-to-pay for a license crucially depends on which other firms would get licensed and hence be rivals. Another example is an optimal intervention problem in an adverse selection market, where the market price depends on which types of sellers are certified.

Despite the practical importance, in the literature, mechanism design with allocation externality has not been much explored, perhaps due to the tractability issue of the problem: For example, Jehiel, Moldovanu, and Stacchetti (1996, 1999); Jehiel and Moldovanu (2001) consider a very general model of externality, where agent i's externality on agent j can be different from i's externality on another agent k, corresponding to different parameters. This multi-dimensional characteristics of agents naturally makes the problem complicated, as is well-known in the multi-dimentional screening literature. Figueroa and Skreta (2009, 2011) consider one-dimensional type, summarizing this agent's payoff-type and his externality on all the other agents (as in this paper). On the other hand, their main interest is in a general form of type-dependent outside options (including externality as a special case), again making characterization of the optimal mechanism difficult. In the optimal taxation literature, several papers consider externalities such as those due to an occupational choice and its resulting wage changes (e.g., Rothschild and

Scheuer (2013, 2016)), and price changes in a product market (e.g., Kushnir and Zubrickas (2019)). Here again, they consider very general environments (such as multi-dimensional types and multi-dimensional externality channels). It is certainly important to allow for a general externality structure and study properties of desirable mechanisms in such general environments, even if fully characterizing optimal mechanisms are prohibitive. However, as its complement, it is also important to provide a simpler model which enables us to characterize optimal mechanisms, conduct comparative statics, and obtain key economic insights more straightforwardly.

This paper's goal is to provide a simple and tractable model of mechanism design with externality. With mild technical conditions, we can fully characterize the optimal mechanism, and moreover, the optimal mechanism has a very simple form in the sense that it is identified by only a few parameters. Furthermore, thanks to this simplicity of the optimal mechanism, some clean comparative statics results are provided.

As the cost of tractability, the model is admittedly restrictive in several dimensions, and hence, would not cover all possible applications of mechanism design with externality. Nevertheless, some applications may well be studied in this model, and for those applications, our approach could be useful.

1.1 Related papers

As discussed above, this paper basically lies in the literature of mechanism design with allocation externality. The main contribution within this literature is to propose a tractable model with characterization of optimal mechanisms and comparative statics.

The moment-based externality makes this paper also very related to the literature of information design / Bayesian persuasion. Indeed, our problem of dividing a set of heterogeneous agents into two groups can be interpreted as a special kind of information design problem where the principal ("sender") designs a signaling device of generating a binary signal (e.g., "good school" signal or "bad school" signal) about the payoff-relevant state (e.g., "student's ability"), where the expected value of a function of the state is payoff-relevant. In the standard approach in this literature, the sender is not only restricted to a binary signaling structure, and in this sense, our problem is different from the standard approach. As the other key difference, our mechanism or "signaling device" must satisfy the agents' incentive compatibility, while it is not usually required in this literature (except for some papers which look at monotone signaling devices: e.g., Mensch (2019) and Arieli, Babichenko, Smorodinsky, and Yamashita (2021)). Despite those differences, it is useful to understand our results in relation to the basic insights in this literature. For example, if the principal's objective exhibits certain convexity or concavity property, then as in Kamenica and Gentzkow (2011), maximum "diversification" or "concentration" of treated agent types would be optimal.

2 Baseline model

In this section, we describe the baseline model. To obtain tractability, the model has a number of special features.

First, we assume that there exists a continuum of agents (e.g., students or firms). Each agent is identified by his *type* $\theta \in [0, 1]$, and the density of type θ

in the population is given by $f(\theta)$. Let $F(\theta) = \int_{\tilde{\theta}\leq\theta} f(\tilde{\theta})d\tilde{\theta}$. This f is common knowledge among the agents, and also known to the mechanism designer (the "principal").[1]

An *allocation* is denoted by $(q, p) \in [0, 1] \times \mathbb{R}$, where q is interpreted as a probability of a *treatment*, and p as a (monetary or non-monetary) *payment*. In the school allocation, the treatment to agent i can mean that this agent is admitted to a better school, perhaps with a higher tuition or costly effort of passing an exam (captured by p); in license allocation, the treatment to firm i can mean that this firm obtains a license to operate in a certain market, possibly with a fee or costly rent-seeking behavior (captured by p).

The principal commits to a *(menu) mechanism* given by $(q(\theta), p(\theta))_{\theta\in[0,1]}$. An agent reports θ to the principal, and the principal assigns $(q(\theta), p(\theta))$.[2] It is without loss to focus on the class of mechanisms where every agent has an incentive to report his type truthfully.

The second key component of the model is that externality is given in a "moment-based" manner. More specifically, there exists an increasing function

[1]The case with aggregate uncertainty (i.e., with unknown f) would be an interesting generalization.

[2]We implicitly assume that an agent's allocation only depends on his report, and not on the other agents' reports. This assumption may be interpreted as a (quasi-)anonymity assumption on mechanisms, as follows. A general (direct) mechanism determines an agent's allocation as a function of his report and the other agents' reports. Assume that we only consider a class of (quasi-)anonymous mechanism in the sense that an agent's allocation depends on his report and only anonymously on the other reports, that is, permuting the other reports does not change this agent's allocation. Then, because of continuously many agents, these anonymous reports are fully summarized by f. However, because f is already known to the principal, we can omit the dependence of the allocation on f without loss of generality.

$\chi : [0,1] \to \mathbb{R}$ and we define the *externality index* $x \in \mathbb{R}$ by:

$$\begin{aligned} x &= E[\chi(\theta)|\text{treated}] \\ &= \frac{\int \chi(\theta) q(\theta) dF}{\int q(\theta) dF}. \end{aligned}$$

In what follows, without loss of generality, we focus on the case where χ is an identify function, which implies $x = E[\theta|\text{treated}]$.[3]

Agent θ's payoff is given by:

$$U(\theta, x, q, p) = qu_1(\theta, x) + (1-q)u_0(\theta, x) - p,$$

where (q, p) denotes the assigned allocation and x is the externality index, u_1 is the payoff when he is treated, and u_0 is the payoff when not treated. We assume that U exhibits single-crossing in (θ, q): for each x,

$$\text{sgn}\left[\frac{d}{d\theta}(u_1(\theta, x) - u_0(\theta, x))\right]$$

is constant in θ. This implies that one of the extreme types, the highest-θ or lowest-θ type, is the most willing type for the treatment (and which one is can depend on x). In what follows, for simplicity, we focus on the case where $\frac{d}{d\theta}(u_1(\theta, x) - u_0(\theta, x)) > 0$, but the other case can be treated similarly.

As in the standard argument, the agent's truth-telling condition of a mechanism $(q(\theta), p(\theta))_\theta$ becomes equivalent to monotonicity of $q(\cdot)$ (and the corresponding $p(\cdot)$ is fully determined by the envelope theorem up to a constant).

[3] In applications where θ has some natural meaning, it may be more useful to work with χ. For example, imagine a school allocation problem where the peer-effect index is given by a weighted sum of the mean and variance of treated students' abilities.

The principal's objective is:

$$\int q(\theta)v_1(\theta,x)+(1-q(\theta))v_0(\theta,x)dF.$$

For example, if the principal cares about the utilitarian (non-monetary) surplus, we would have $v_k(\theta,x)=u_k(\theta,x)$ for $k=0,1$; if the principal cares about the total monetary transfer from the agents ("revenue"), then we would have $v_1(\theta,x)-v_0(\theta,x)$ as the agent's virtual valuation.

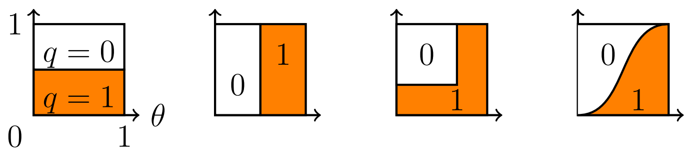
Finally, we only consider a class of mechanisms such that type-0 agent's payoff coincides with some exogenous value $\underline{u}$. The result does not depend on the exact value of $\underline{u}$ in the sense that, if $(q(\theta),p(\theta))_\theta$ is the optimal mechanism with some $\underline{u}$, $(q(\theta),p(\theta)+\delta)_\theta$ is the optimal mechanism with $\underline{u}-\delta$.

Therefore, the optimal mechanism design problem is given by:

$$\begin{aligned}
\max_{(q(\cdot),p(\cdot))} \quad & \int q(\theta)v_1(\theta,x)+(1-q(\theta))v_0(\theta,x)dF \\
\text{sub. to} \quad & q(\theta)u_1(\theta,x)+(1-q(\theta))u_0(\theta,x)-p(\theta) \\
& \geq q(\theta')u_1(\theta,x)+(1-q(\theta'))u_0(\theta,x)-p(\theta'),\ \forall\theta,\theta', \\
& q(0)u_1(0,x)+(1-q(0))u_0(0,x)-p(0)=\underline{u}, \\
& Q=\int q(\theta)dF\in\mathbf{Q},\ x=\frac{\int\theta q(\theta)dF}{Q},
\end{aligned}$$

where $\mathbf{Q}$ captures a possible restriction of the total size of the treatment. For example, if we are interested in the unrestricted case, we would set $\mathbf{Q}=[0,1]$; if we are interested in the fixed-capacity problem, then we would set $\mathbf{Q}$ as a singleton. Because our main focus is on the role of the externality index in the optimal mechanism, in what follows, we focus on the case where $\mathbf{Q}$ is any fixed singleton (i.e., $\mathbf{Q} = \{Q\}$ for some $Q \in (0,1)$). However, it is possible to extend our analysis to the cases with more flexible Q.

At this point, it may be instructive to present a few instances of potential treatment profile $q(\cdot)$.



The first figure exhibits "full mixing", because the treatment decision does not depend on θ, and the second figure exhibits "full separation" in the sense that the treatment decision differs completely before and after a cutoff value of θ. The third figure exhibits both separation and mixing in a particular way, and the last one exhibits more smooth boundary.

As these figures suggest, the principal's problem is to divide a unit-square into two regions of fixed measures (Q and $1-Q$), appreciating the agents' incentive compatibility, and importantly, the fact that x changes endogenously with $q(\cdot)$.

Despite this infinite-dimensional nature of the problem and endogenous externality, the optimal mechanism has a simple form, as explained in the following sections.

3 Examples

Here, we describe three applications of the model.

3.1 School allocation

The following is based on Arnott and Rowse (1987). There exist a continuum of students, each with ability θ, and two schools (0 and 1) of equal capacity $\frac{1}{2}(=Q)$.

Let q denote the probability of being admitted to school 1, and $\frac{p}{q}$ denote the payment in case of admission (which might be tuition or effort cost of passing an exam). A *peer effect* index in school $s = 0, 1$ is given by x_s, where

$$\begin{aligned} x = x_1 &= \frac{\int \theta q(\theta) dF}{1/2}, \\ x_0 &= \frac{\int \theta (1 - q(\theta)) dF}{1/2} = 2\mathbb{E}[\theta] - x. \end{aligned}$$

Assume that student θ's payoff in school s is give by $u_s = \beta\theta^\alpha x_s + (1-\beta)\sqrt{x_s}$ for some $\alpha \in (0, 1]$, a convex combination (with some exogenous weight β) of a product θx_s and a concave function of the peer effect index $\sqrt{x_s}$. The principal's objective is pure welfare maximization:

$$v_s(\theta, x) = \beta\theta^\alpha x_s + (1-\beta)\sqrt{x_s}.$$

Intuitively, the first term exhibits supermodularity in type and the peer effect index, which makes the principal prefer more separation. Indeed, it is well-known that, with $\beta = 1$, full separation is optimal (i.e., $q(\theta) = 1\{\theta > \theta^*\}$ where $F(\theta^*) = \frac{1}{2}$). On the other hand, the second term is concave in the peer effect index, which makes the principal prefer more mixing. Indeed, it is easy to show that, with $\beta = 0$, full mixing is optimal (i.e., $q(\theta) = \frac{1}{2}$ for all θ).

A question is when β is more intermediate. It is perhaps easy to show that neither full separation nor full mixing is optimal, but *what is* optimal is more challenging. Would it look like the third figure at the end of the previous section, where high-θ types are surely admitted and low-θ types are admitted only with some constant probability? Or would it look like the fourth figure where the boundary is smooth? Or different from them?

As a related point, if β gradually increases, it seems quite natural to expect that the optimal mechanism exhibits "more separation". However, with two mechanisms at hand for low β and high β, how to judge whether one exhibits more separation than the other?

Theorem 1 in the next section shows that the optimal mechanism belongs to a simple class identified by only a few parameters (and in some cases, it has a form as in the third figure). This simple structure allows clean comparative statics: with higher β, in a quite natural sense, the optimal mechanism exhibits more separation. For example, in the case where the optimal mechanism has a form as in the third figure, with higher β, the optimal mechanism exhibits higher cutoff type below which the agent is never admitted (and accordingly, the probability of admission above the cutoff decreases).

3.2 License allocation

The following is based on Melitz (2003). There exist a representative consumer with a CES utility function with elasticity of substitution $\rho \in (0,1)$ across a variety of goods, and continuum of firms, each with different marginal production cost $\frac{1}{\psi}$, where $\psi \in [0,1]$ with density $f_0(\psi)$. Let $\theta = \psi^{\frac{\rho}{1-\rho}}$, and in what follows, we interpret this θ (which is strictly increasing in ψ) as the firm's cost parameter. The density of θ is $f(\theta)$ where $f(\theta) = f_0(\theta^{\frac{1-\rho}{\rho}})\frac{1-\rho}{\rho}\theta^{\frac{1-2\rho}{\rho}}$. The firms play the following game: In the first period, each firm, without knowing θ, must decide whether to pay an (exogenously given) cost ϕ to be informed of θ or not. Assume measure M firms pay ϕ to be informed of their types.[4] After learning, in the second period, each firm reports θ to the principal to get licensed with prob $q(\theta)$ by paying $p(\theta)$. We interpret $\frac{p(\theta)}{q(\theta)}$ as a license fee (in case of getting licensed), a monetary transfer to the principal. Finally, those who get licensed operate in a monopolistic competition market. This last part involves a standard but additional argument about the consumer's purchase decision and the firms' production decisions, and hence at this moment, we skip it and only provide a "solved" or "reduced-form" description.

More specifically, firm θ's payoff is 0 if he does not learn his type θ, $-\phi$ if he learns θ but does not get licensed, and

$$\frac{C}{Mx}\theta q(\theta) - p(\theta) - \phi$$

if he learns θ and gets licensed, where C is a constant and $x = \frac{\int \theta q(\theta) dF}{\int q(\theta) dF}$. Naturally, the firm's payoff increases with its competitiveness θ, but notice that the payoff decreases with Mx: with higher M, the market is more congested, and hence each firm's payoff becomes smaller; and with higher x, each firm competes with more cost-efficient firms, again implying a lower payoff.

The principal's objective is a weighted sum of the consumers' surplus, a monotone transformation of Mx, and the fee revenue $M \int p(\theta) dF$. The producers' surplus is zero by free entry, and hence does not appear in the principal's objective. However, if ϕ is interpreted as inefficient value burning, one may want to

[4] In the equilibrium, M is determined so that the firms' ex ante expected payoff is zero.

make the principal's surplus decreasing with respect to $M\phi$. The principal's payoff is therefore $Mx + \gamma M \int p(\theta)dF - \delta M\phi$, where $\gamma, \delta \geq 0$ are exogenously given weights. In fact, a further examination of Melitz's model shows that

$$Mx = \frac{\beta(1-\rho)}{\phi} \int \frac{1-F(\theta)}{f(\theta)} q(\theta)dF,$$

where $\beta > 0$ is a constant, and hence, one can further rewrite the principal's payoff so that his ex ante objective becomes proportional to the following (and hence fits our framework):

$$v_1(\theta, x) - v_0(\theta, x) = \int \left(\frac{\beta}{\phi} \frac{1-F(\theta)}{f(\theta)} + \frac{\gamma}{x} \left(\theta - \frac{1-F(\theta)}{f(\theta)} \right) - \frac{\delta\phi}{x} \right) q(\theta)dF.$$

The first term corresponds to the consumer's surplus. The second term is a standard "virtual value" expression for the principal's revenue, except the coefficient $\frac{1}{x}$ in front. The third term corresponds to the sunk cost of learning. Intuitively, the second (revenue) part makes the principal prefer more separation, because given any size of the licenses, more separation enables the principal to charge a higher license fee. However, notice that the marginal return of higher license fee would be rapidly decreasing because of the coefficient $\frac{1}{x}$ in front of this revenue expression: if the rival firms become "too competitive", each firm's willingness-to-pay decreases. A similar effect exists for the third (sunk cost) part. More separation implies higher x, which decreases M, implying smaller total sunk cost. However, again, its marginal effect is decreasing. Those two effects balance with the negative effect of inviting less firms with higher x.

As in the first example, unless the parameters are extreme, the optimal mechanism exhibits both separation and mixing. A question is how the optimal mechanism looks like, and how the changes in the parameters change this balance of separation and mixing.

3.3 Intervention in adverse selection market

The following is partly motivated by Tirole (2012) and Philippon and Skreta (2012). Consider a bilateral-trade setting in which a mass of risk-neutral sellers, each endowed with a single unit of an indivisible good, engages in exchanges with a larger mass of risk-neutral buyers.[5] The quality of the good, $\theta \sim F$, is private information of the seller. The buyers' valuation for a good of quality θ is given by θ; while the (opportunity) cost for the seller is $c(\theta)$, strictly increasing and convex function of θ. The trade surplus $\theta - c(\theta)$ is positive for all θ and increasing in θ. However, the market suffers from severe adverse selection in the sense that there is no mutually acceptable trading price π except for $\pi = 0$:

$$\mathbb{E}[\theta | \pi - c(\theta) \geq 0] - \pi \geq 0 \; \Rightarrow \pi = 0.$$

In order to realize some surplus-generating trade, the government (principal) intervenes the market through certification with subsidization (e.g., buybacks of legacy assets by the government, in the setting of Tirole (2012)). Let q denote the probability of being certified, and p denote the expected transfer from the seller (agent) to the government (hence, negative p means subsidy from the government

[5] Hence, the trading price in the market is given by the buyer's break-even condition.

to the seller). The externality parameter is the resulted market price:

$$x = \frac{\int_0^1 \theta q(\theta) d\theta}{\int_0^1 q(\theta) d\theta}.$$

The seller with θ earns payoff $q(\theta)(x - c(\theta)) + (1 - q(\theta))u_0(\theta, x) - p(\theta)$, where $x - c(\theta)$ is his ex post pre-subsidy payoff if certified, and $u_0(\theta, x)$ otherwise. There may be several possible specifications of u_0. The simplest one (and the one we adopt here) is $u_0 \equiv 0$, corresponding to the case where non-certified sellers cannot trade. An interesting extension may be when even non-certified sellers can trade in the market, possibly with some friction relative to the certified case:

$$u_0(\theta, x) = \phi(x^* - c(\theta)),$$

where ϕ is some function representing the friction and x^* denotes the market price for non-certified sellers:

$$\begin{aligned} x^* &= \frac{\int_0^1 \theta(1 - q(\theta)) d\theta}{\int_0^1 (1 - q(\theta)) d\theta} \\ &= \frac{1}{1 - Q}\left(\frac{1}{2} - Qx\right) \end{aligned}$$

for $Q = \int_0^1 q(\theta) d\theta$. The case we consider here can be interpreted as the extreme case with $\phi \equiv 0$.

The government's problem is to optimally allocate certificates among the sellers, taking into account the total trade surplus and the total subsidy, both dependent on the endogenous market price x. More specifically, the government's payoff satisfies:

$$v_1(\theta) - v_0(\theta) = \omega q(\theta)\left(x - c(\theta) - c'(\theta)\frac{F(\theta)}{f(\theta)}\right) + (1-\omega)q(\theta)(\theta - c(\theta)).$$

Given any fixed size of the certificates $Q = \int_0^1 q(\theta)d\theta$, it is natural to guess that more separation would be preferable for the government who puts more weight on the subsidy, by focusing on less costly seller types; conversely, more mixing would be preferable for the surplus-oriented government, because the higher seller types induce higher trade surplus. A non-trivial question is how the optimal balance is achieved in the intermediate case, and its comparative statics.

4 Optimal mechanism

This section characterizes the optimal mechanism. A potential challenge of this problem is the following circularity. Imagine some x is exogenously fixed. Then, the standard mechanism design technique is applicable, enabling us to obtain the optimal mechanism in a well-known manner. With x as the endogenously-determined externality index, however, a change in a mechanism can change the externality index, changing the optimal mechanism given the new externality index, changing the optimal mechanism, and so on.

To circumvent this potential circularity, in what follows, we solve the problem in the following steps. To begin with, for each $(Q, x) \in \mathcal{Q} \times \mathbb{R}$, we say that (Q, x)

is feasible if there exists non-decreasing $q(\cdot)$ with which:

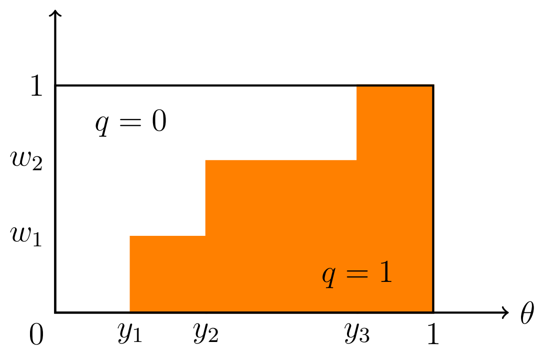
$$\begin{aligned} Q &= \int q(\theta) dF \\ Qx &= \int \theta q(\theta) dF. \end{aligned}$$

Fix any feasible (Q, x). It is possible that, in general, there are multiple non-decreasing $q(\cdot)$ which induces (Q, x). The first step of the solution is to maximize the principal's ex ante payoff among all $q(\cdot)$ that induces this (Q, x). Let $V^*(Q, x)$ denote the maximized objective in this first step. Then, in the second step, we maximize $V^*(Q, x)$ among all feasible (Q, x). The advantage of this procedure is two-fold. First, although the first-step problem is infinite-dimensional, the problem is relatively standard: given (Q, x) fixed, the problem is close to the standard mechanism design problem without externality. It is still not fully standard because of the "auxiliary feasibility constraint" that $q(\cdot)$ induces the prefixed (Q, x). However, thanks to the moment-based form of the externality index, it can tractably be accommodated. Second, once the first-step problem is solved, the remaining problem is just a finite-dimensional problem. In particular, in case $\mathcal{Q}$ is singleton, it is just a one-variable problem.

The solution to the first-step problem for any feasible (Q, x) is given as follows (see also the figure below). Given any (Q, x), let $\mathcal{Q}(Q, x)$ denote the set of all q

such that, for some $\exists 0 \leq y_1 \leq y_2 \leq y_3 \leq 1$, $\exists 0 < w_1 < w_2 < 1$, we have:

$$\begin{aligned}
q(\theta) &= \begin{cases} 0 & \text{if} \quad \theta < y_1 \\ w_1 & \text{if} \quad \theta \in [y_1, y_2) \\ w_2 & \text{if} \quad \theta \in [y_2, y_3) \\ 1 & \text{if} \quad \theta \geq y_3, \end{cases} \\
Q &= w_1(F(y_2) - F(y_1)) + w_2(F(y_3) - F(y_2)) + 1 - F(y_3), \\
Qx &= w_1 \int_{y_1}^{y_2} \theta dF + w_2 \int_{y_2}^{y_3} \theta dF + \int_{y_3}^{1} \theta dF.
\end{aligned}$$



Theorem 1. Given any (Q, x), an optimal q is in $\mathcal{Q}(Q, x)$.

The solution simply comprises the four regions. If $\theta < y_1$, then the agent is never treated; if $\theta \in [y_1, y_2)$, the agent is treated with some fixed probability w_1; if $\theta \in [y_2, y_3)$, the agent is treated with a higher fixed probability $w_2 (> w_1)$; and if $\theta \geq y_3$, then the agent is surely treated. Of course, y, w satisfy the two feasibility conditions in the first-step problem, corresponding to the fixed total size of the treatment and the fixed externality index. In this sense, out of five parameters, only three degrees of freedom exist.

The proof is relegated to the appendix.

4.1 Simpler cases

Even simpler mechanisms are optimal with additional conditions on v. Let $V^{cav-vex}$ denote the set of all v with a strictly increasing second derivative; $V^{vex-cav}$ denote the set of all v with a strictly decreasing second derivative; V^{vex} denote the set of all v with a strictly positive second derivative; and let V^{cav} denote the set of all v with a strictly negative second derivative.

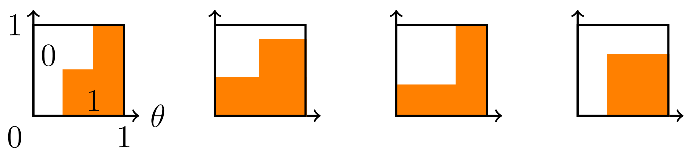
Proposition 1. 1. If $v \in V^{cav-vex}$, then an optimal mechanism is in $\mathcal{Q}(Q,x)$ identified by $(y_1, y_2, y_3; w_1, w_2)$ with $y_2 \in \{y_1, y_3\}$.

2. If $v \in V^{vex-cav}$, then an optimal mechanism is in $\mathcal{Q}(Q,x)$ identified by $(y_1, y_2, y_3; w_1, w_2)$ with $y_1 \in \{0, y_2\}$ and $y_3 \in \{y_2, 1\}$.

3. If $v \in V^{vex}$ then an optimal mechanism is in $\mathcal{Q}(Q,x)$ identified by $(y_1, y_2, y_3; w_1, w_2)$ with either $y_1 = y_2 = 0$ or $y_1 = y_2 = y_3$.

4. If $v \in V^{cav}$ then an optimal mechanism is in $\mathcal{Q}(Q,x)$ identified by $(y_1, y_2, y_3; w_1, w_2)$ with either $y_2 = y_3 = 1$ or $y_1 = y_2 = y_3$.

The first case with $v \in V^{cav-vex}$ corresponds to the first figure below (and as special cases, the last two figures are also possible).[6] The second case with $v \in V^{vex-cav}$ corresponds to the second figure and the last two figures. The third case with $v \in V^{vex}$ corresponds to the third figure. The last case with $v \in V^{cav}$ corresponds to the last figure.



[6]The full-separation and full-mixing cases are also possible as special cases.

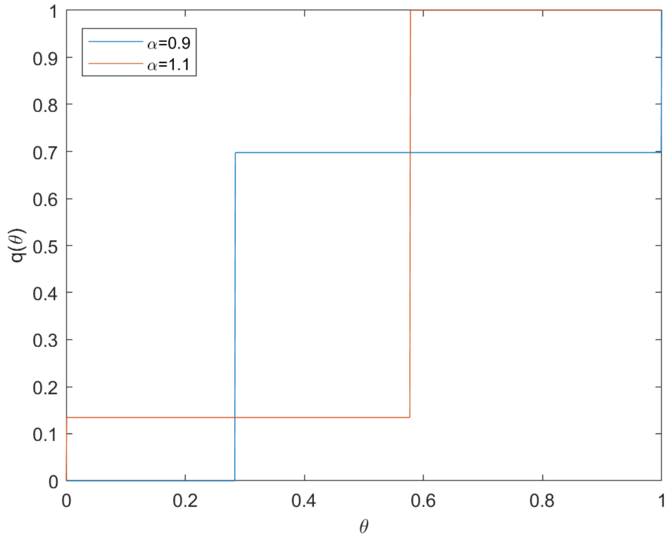


Figure 1: Optimal allocation function for $\beta = 0.28$ and different values of α.

Example 1. We revisit the school allocation example introduced in Section 3. Recall that $v_s(\theta, x) = \beta\theta^\alpha x_s + (1-\beta)\sqrt{x_s}$ for each s.

Figure 1 plot the optimal q for $(\alpha, \beta) = (1.1, 0.28)$ and for $(\alpha, \beta) = (0.9, 0.28)$. The first case with $\alpha = 1.1$ is covered in Proposition 1.3, where the optimal policy exhibits sure treatment of high-enough θ; while the second case with $\alpha = 0.9$ is covered in Proposition 1.4, where the optimal policy exhibits sure non-treatment of low-enough θ. As suggested in this example, the optimal policy may change drastically even if model parameters change only slightly.

□

5 Comparative Statics

5.1 Separation or Mixing

Here, we investigate the conditions under which the optimal mechanism exhibits more separation (and hence higher x) or more mixing (and hence lower x).

We say that q *exhibits more separation (less mixing) than* q' if the distribution of treated types given q (with density $\frac{q(\theta)f(\theta)}{Q}$) first-order stochastically dominates that given q' (with density $\frac{q'(\theta)f(\theta)}{Q}$). Intuitively, q treats more of higher types than q' does. Accordingly, given any F, q implies higher externality index than q' does.

In the following, let q denote the optimal mechanism given v_1, v_0; and q' denote the optimal mechanism given v_1', v_0'. By Theorem 1, both q and q' are (at-most-)four-step functions:

$$q(\theta) = \begin{cases} 0 & \text{if} \quad \theta < y_1 \\ w_1 & \text{if} \quad \theta \in (y_1, y_2) \\ w_2 & \text{if} \quad \theta \in (y_2, y_3) \\ 1 & \text{if} \quad \theta > y_3. \end{cases}, q'(\theta) = \begin{cases} 0 & \text{if} \quad \theta < y_1' \\ w_1' & \text{if} \quad \theta \in (y_1', y_2') \\ w_2' & \text{if} \quad \theta \in (y_2', y_3') \\ 1 & \text{if} \quad \theta > y_3'. \end{cases}$$

We establish the following comparative statics result. Let $v(\theta, x) = v_1(\theta, x) - v_0(\theta, x)$, and similarly, $v'(\theta, x) = v_1'(\theta, x) - v_0'(\theta, x)$.

Proposition 2. Assume that, for all θ, x, $\frac{\partial}{\partial \theta}(v'(\theta, x) - v(\theta, x)) \geq 0$ and $\frac{\partial^2}{\partial x \partial \theta}(v'(\theta, x) - v(\theta, x)) \geq 0$. Then, q does not exhibits more separation than q'.

Although the above result establishes a condition with which q does *not* exhibit more separation than q', one might expect a stronger claim that "q' exhibits more separation than q" (under the stated condition). However, that claim is not generally true. For example, recall Example 1, where the optimal policies given different values of α cannot be ordered according to the first-order stochastic dominance, even though the change in α satisfies the conditions in Proposition 2. Intuitively, the change from v to v' does not only make the principal prefer more separation (in the sense of the conditions in Proposition 2), but also make him prefer more *diversity* of treated types, or put it differently, prefer more *concentration* of non-treated types. Thus, the stronger comparative statics result requires more assumptions on the environment.

Proposition 3. Assume that either $v, v' \in V^{vex}$ or $v, v' \in V^{cav}$. Assume also $\frac{\partial^2}{\partial x \partial \theta}(v'(\theta, x) - v(\theta, x)) \geq 0$ and $\frac{\partial}{\partial \theta}(v'(\theta, x) - v(\theta, x)) \geq 0$ for all θ, x. Then, the optimal mechanism given v' exhibits more separation than that given v.

Example 2. We revisit the school allocation example introduced in Section 3. Recall that $v_s(\theta, x) = \beta\theta^\alpha x_s + (1-\beta)\sqrt{x_s}$ for each s. Thus, letting $x = x_1$ (which implies $x_0 = 1 - x$) with $x \in \left[\frac{1}{2}, \frac{3}{4}\right]$ without loss of generality, we have:

$$\begin{aligned} v(\theta, x) &= v_1(\theta, x) - v_0(\theta, 1-x) \\ &= \beta\theta^\alpha(2x-1) + (1-\beta)(\sqrt{x} - \sqrt{1-x}). \end{aligned}$$

Thus, for $\alpha, \beta \in (0, 1)$,

$$\begin{aligned} \frac{\partial v(\theta, x)}{\partial \theta} &= \alpha\theta^{\alpha-1}\beta(2x-1) > 0 \\ \frac{\partial^2 v(\theta, x)}{\partial \theta^2} &= -\alpha(1-\alpha)\theta^{\alpha-2}\beta(2x-1) < 0 \end{aligned}$$

for $x \in \left(\frac{1}{2}, \frac{3}{4}\right)$, that is, $v \in V^{cav}$. For example, the optimal policy given $(\alpha, \beta) = (0.9, 0.28)$ and given $(\alpha, \beta) = (0.9, 0.285)$ are plotted as follows:

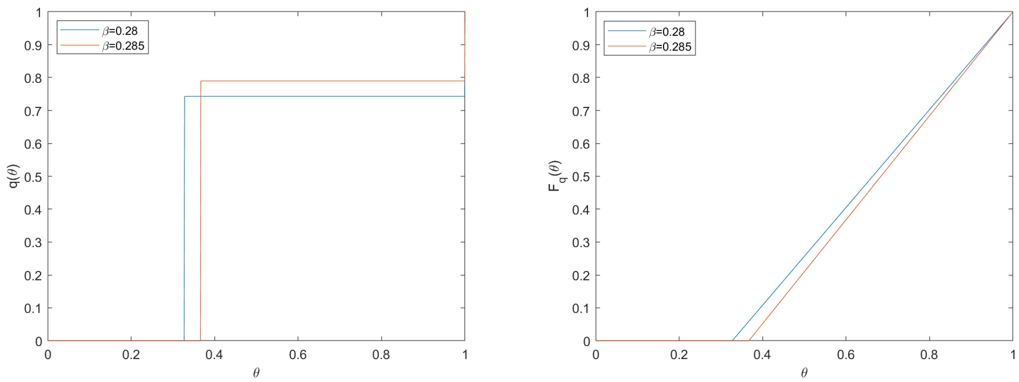


Figure 2: Optimal allocation functions (left) and its FOSD relationship (right) with $\alpha = 0.9$ and different values of β.

As in the figure, the distribution of the treated types in the optimal q with $(\alpha, \beta) = (0.9, 0.285)$ first-order stochastically dominates that with $(\alpha, \beta) = (0.9, 0.28)$.

□

5.2 Diversity or Concentration

Two policies may differ not only in terms of separation/mixing, but also in other dimensions. For example, as illustrated in the previous example, the principal may prefer to have more or less variety of treated/non-treated types.

To obtain cleaner comparative statics results regarding the variety of treated/non-treated types, in the following, we focus on the first-step problem of finding the optimal mechanism given (Q, x).

Intuitively, more diversity of treated types keeping (Q, x) unchanged corresponds to mean-preserving spread of treated types. We say that q *exhibits more diversity (less concentration) of treated types than* q' if the distribution of treated types given q (with density $\frac{q(\theta)f(\theta)}{Q}$) is second-order stochastically dominated by that given q' (with density $\frac{q'(\theta)f(\theta)}{Q}$).

Let q denote the optimal mechanism given v_1, v_0; and q' denote the optimal mechanism given v_1', v_0', as follows:

$$q(\theta) = \begin{cases} 0 & \text{if} \quad \theta < y_1 \\ w_1 & \text{if} \quad \theta \in (y_1, y_2) \\ w_2 & \text{if} \quad \theta \in (y_2, y_3) \\ 1 & \text{if} \quad \theta > y_3. \end{cases}, q'(\theta) = \begin{cases} 0 & \text{if} \quad \theta < y_1' \\ w_1' & \text{if} \quad \theta \in (y_1', y_2') \\ w_2' & \text{if} \quad \theta \in (y_2', y_3') \\ 1 & \text{if} \quad \theta > y_3'. \end{cases}$$

Let $v(\theta, x) = v_1(\theta, x) - v_0(\theta, x)$, and similarly, $v'(\theta, x) = v_1'(\theta, x) - v_0'(\theta, x)$.

Proposition 4. Assume that, for all θ, x, $\frac{\partial^2}{\partial \theta^2}(v'(\theta, x) - v(\theta, x)) \geq 0$. Then, in the problem with any fixed (Q, x), q does not exhibit more diversity of treated types than q'.

As in the previous comparative statics result, one cannot expect that the optimal mechanism given v' exhibits more concentration than that given v'. However, such a stronger claim is possible with additional assumptions on the environment.

Proposition 5. Assume that either $v, v' \in V^{cav-vex}$ or $v, v' \in V^{vex-cav}$. Assume also that $\frac{\partial^2}{\partial \theta^2}(v'(\theta, x) - v(\theta, x)) \geq 0$ for all θ, x. Then, in the problem with any fixed (Q, x), the optimal mechanism given v' exhibits more concentration of treated types than the optimal mechanism given v.

Example 3. We revisit the license allocation example introduced in Section 3. Recall the principal's objective:

$$v(\theta, x) = \frac{\beta}{\phi} \frac{1 - F(\theta)}{f(\theta)} + \frac{\gamma}{x} \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) - \frac{\delta \phi}{x}.$$

Let $H(\theta)=\frac{1-F(\theta)}{f(\theta)}$ denote the inverse hazard rate. Notice that

$$\frac{\partial^2}{\partial\theta^2}v(\theta,x)=H''(\theta)\left(\frac{\beta}{\phi}-\frac{\gamma}{x}\right),$$

and in this sense, the curvature of the inverse hazard rate is crucial for the curvature of v. We assume that

$$F(\theta)=\theta^2$$

so that the inverse hazard rate is convex.

In order to illustrate the comparative statics regarding diversification and concentration, let us fix $Q=0.5$ and $x=0.75$, and we only consider policies which achieve this (Q,x) pair (and hence the "optimal" policy below refers to the best one among those attaining this pair).

Let two objectives $\tilde{v}$ and v be such that:

$$\widetilde{v}(\theta,x)-v(\theta,x)=\left(\frac{\widetilde{\beta}}{\widetilde{\phi}}-\frac{\beta}{\phi}\right)H(\theta)+\frac{(\widetilde{\gamma}-\gamma)}{x}(\theta-H(\theta))-\frac{\left(\widetilde{\delta}\widetilde{\phi}-\delta\phi\right)}{x},$$

where we assume $(\tilde{\phi},\tilde{\gamma},\tilde{\delta})=(\phi,\gamma,\delta)=(1,1,1)$, but $\tilde{\beta}=5\neq 1=\beta$. Then we have:

$$\frac{\partial^2}{\partial\theta^2}[\widetilde{v}(\theta,x)-v(\theta,x)]\ =H''(\theta)\left(\tilde{\beta}-\beta\right).$$

The optimal policy given $\beta=1$ exhibits maximal concentration, while it exhibits maximal diversity given $\tilde{\beta}=5$ (the left panel of Figure 3). Indeed, the distributions of treated types are ordered according to second-order stochastic dominance (the right panel of Figure 3).

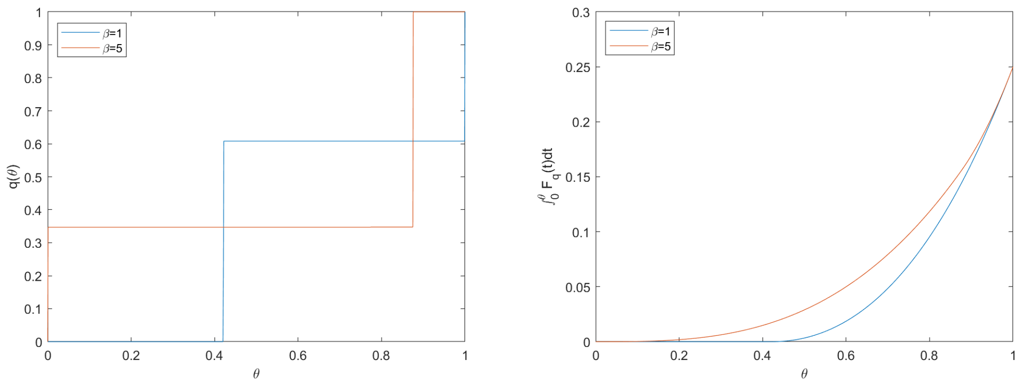


Figure 3: Optimal allocation rules (left) and their SOSD relationship (right) for fixed externality index ($x = 0.75$) given $\beta = 1$ and $\beta' = 5$.

Intuitively, a more consumer welfare-oriented principal (i.e., with higher β) favors more diversity of firms to congest the market and thus transfer surplus from the monopolistic firms to the consumers, whilst a principal with lower β concentrates its license allocation to firms with high virtual values, maximizing separation of types (given the capacity Q and externality x) and thus revenue obtained by means of higher fees.

□

6 Conclusion

This paper proposes a tractable model of mechanism design with allocation externality. The key simplifying assumptions include a large number of agents and moment-based allocation externality. We characterize the optimal mechanism, which has a very simple form in the sense that it is identified by only a few parameters. This simplicity of the optimal mechanism is also useful to obtain comparative statics results, in terms of separation/mixing (based on a first-order stochastic dominance idea) and concentration/diversity of optimal treatment groups (based on a second-order stochastic dominance idea).

Admittedly, the model is restrictive in many respects, as the cost of tractability. Future researches seeking less restrictive but still useful and insightful models would be important.

A Omitted proofs

A.1 Proof of Theorem 1

Fix any (Q, x). Let q_* be optimal given v.

Lemma 1. There exists a sequence of non-decreasing, finite-step functions $\{q_k\}_{k=1}^{\infty}$, such that (i) each q_k achieves the same (Q, x), and (ii) q_k converges uniformly to q_*.

Proof. For each k, divide $[0, 1]$ into (connected) subsets of the form $I_i = [\frac{i-1}{2^k}, \frac{i}{2^k})$ for $i = 1, \ldots, 2^k - 1$ and $I_{2^k} = [\frac{2^k-1}{2^k}, 1]$. Let $\Theta_i = \{\theta | q_*(\theta) \in I_i\}$.

Define $\underline{q}_k$ as follows. For each i and $\theta \in \Theta_i$, let $\underline{q}_k(\theta) = q_*(\theta)$ if Θ_i is singleton; otherwise:

$$\underline{q}_k(\theta) = \frac{\int_{\theta \in \Theta_i} q_*(\theta) dF}{\int_{\theta \in \Theta_i} dF}.$$

Because $\underline{q}_k(\theta) \in [\inf\{q_*(\theta)|\theta \in \Theta_i\}, \sup\{q_*(\theta)|\theta \in \Theta_i\}]$ for $\theta \in \Theta_i$, this $\underline{q}_k$ is non-decreasing. It is also a finite-step function. It also satisfies:

$$\int_0^1 \underline{q}_k(\theta) dF = Q.$$

However, in general, we have

$$\int_0^1 \theta \underline{q}_k(\theta) dF \leq Qx,$$

because the distribution of treated types given q_* (whose density is $\frac{q_*(\theta) f(\theta)}{Q}$) first-order stochastically dominates that of $\underline{q}_k$ (whose density is $\frac{\underline{q}_k(\theta) f(\theta)}{Q}$).

Next, define $\overline{q}_k$ as follows. For each i and $\theta \in \Theta_i$, let $\overline{q}_k(\theta) = q_*(\theta)$ if Θ_i is singleton; otherwise, for some $\theta_i^* \in \Theta_i^*$:

$$\overline{q}_k(\theta) = \begin{cases} \inf\{q_*(\theta)|\theta \in \Theta_i\} & \text{if} \quad \theta \leq \theta_i^* \\ \sup\{q_*(\theta)|\theta \in \Theta_i\} & \text{if} \quad \theta > \theta_i^* \end{cases}$$

This $\overline{q}_k$ is a non-decreasing, finite-step function, which satisfies:

$$\begin{aligned} \int_0^1 \underline{q}_k(\theta)dF &= Q \\ \int_0^1 \theta\overline{q}_k(\theta)dF &\geq Qx, \end{aligned}$$

because the distribution of treated types given q_* is first-order stochastically dominated by that of $\overline{q}_k$. Therefore, an appropriate convex combination of $\underline{q}_k$ and $\overline{q}_k$ can be used as q_k, which itself is a non-decreasing, finite-step function, and satisfies:

$$\begin{aligned} \int_0^1 q_k(\theta)dF &= Q \\ \int_0^1 \theta q_k(\theta)dF &= Qx. \end{aligned}$$

Its uniform convergence to q_* is immediate from the construction. □

Thus, we have $\lim_k \int_0^1 v(\theta, x)(q_*(\theta) - q_k(\theta))dF = 0$.

Next, we show that, for each k, there exists $q^k \in \mathcal{Q}(Q, x)$ such that $\int_0^1 v(\theta, x)(q^k(\theta) - q_k(\theta))dF \geq 0$. This implies that $\lim_k \int_0^1 v(\theta, x)(q_*(\theta) - q^k(\theta))dF = 0$.

Lemma 2. Within the class of finite-step function q that achieves (Q, x), an optimal policy is in $\mathcal{Q}(Q, x)$.

Proof. Take any feasible and finite-step q that achieves (Q,x) but that is not in $\mathcal{Q}(Q,x)$. Then, there exist intervals $\Theta_k \subseteq [0,1]$ for $k=1,2,3$ and $0<w_1<w_2<w_3<1$ such that:

$$q(\theta) = w_k \text{ iff } \theta \in \Theta_k,$$

for each $k=1,2,3$.

Consider another finite-step $\tilde{q}$ defined as follows: for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$,

$$\begin{aligned}
\tilde{q}(\theta) &= w_1 + \varepsilon_1 \quad \text{iff} \quad \theta \in \Theta_1 \\
\tilde{q}(\theta) &= w_2 - \varepsilon_2 \quad \text{iff} \quad \theta \in \Theta_2 \\
\tilde{q}(\theta) &= w_3 + \varepsilon_3 \quad \text{iff} \quad \theta \in \Theta_3,
\end{aligned}$$

and $\tilde{q}(\theta) = q(\theta)$ otherwise, where

$$\begin{aligned}
\varepsilon_1 \int_{\theta\in\Theta_1} dF + \varepsilon_2 \int_{\theta\in\Theta_2} dF + \varepsilon_3 \int_{\theta\in\Theta_3} dF &= 0 \\
\varepsilon_1 \int_{\theta\in\Theta_1} \theta dF + \varepsilon_2 \int_{\theta\in\Theta_2} \theta dF + \varepsilon_3 \int_{\theta\in\Theta_3} \theta dF &= 0.
\end{aligned}$$

This $\tilde{q}$ is non-decreasing if $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are sufficiently small, and achieves (Q,x) by construction. The objective increases by:

$$\varepsilon_1 \int_{\theta\in\Theta_1} v(\theta,x) dF + \varepsilon_2 \int_{\theta\in\Theta_2} v(\theta,x) dF + \varepsilon_3 \int_{\theta\in\Theta_3} v(\theta,x) dF,$$

which must be non-positive because q is optimal. Applying the same logic but with $\varepsilon_1, \varepsilon_2, \varepsilon_3 < 0$ implies that it must be non-negative too. Therefore:

$$\varepsilon_1 \int_{\theta \in \Theta_1} v(\theta, x) dF + \varepsilon_2 \int_{\theta \in \Theta_2} v(\theta, x) dF + \varepsilon_3 \int_{\theta \in \Theta_3} v(\theta, x) dF = 0.$$

However, then, we can take ε so that the number of steps in $\tilde{q}$ is that in q less one. If $\tilde{q} \in \mathcal{Q}(Q, x)$, then we complete the proof. If not, then we can apply the same procedure as here in order to obtain $\tilde{\tilde{q}}$ whose number of steps is that in q less two. Because q has finitely many steps, this procedure leads to an optimal policy in $\mathcal{Q}(Q, x)$. □

Lemma 3. There exists $q^* \in \mathcal{Q}(Q, x)$ such that $\lim_k \int_0^1 v(\theta, x)(q^k(\theta) - q^*(\theta)) dF = 0$.

This lemma implies $\int_0^1 v(\theta, x)(q_*(\theta) - q^*(\theta)) dF = 0$, and therefore, q^* is an optimal mechanism.

Proof. First, given that each $q \in \mathcal{Q}(Q, x)$ is identified by a finite-dimensional vector (y, w), define a metric for $\mathcal{Q}(Q, x)$ as a Euclidean metric for (y, w). Note that the principal's expected payoff is continuous with respect to this metric.

Consider a slightly larger space than $\mathcal{Q}(Q, x)$, denoted by $\overline{\mathcal{Q}}(Q, x)$, defined as the set of all q such that, for some $0 \le y_1 \le y_2 \le y_3 \le 1$ and $0 \le w_1 \le w_2 \le 1$:

$$\begin{aligned} q(\theta) &= \begin{cases} 0 & \text{if} \quad \theta < y_1 \\ w_1 & \text{if} \quad \theta \in [y_1, y_2) \\ w_2 & \text{if} \quad \theta \in [y_2, y_3) \\ 1 & \text{if} \quad \theta \ge y_3, \end{cases} \\ Q &= w_1(F(y_2) - F(y_1)) + w_2(F(y_3) - F(y_2)) + 1 - F(y_3), \\ Qx &= w_1 \int_{y_1}^{y_2} \theta dF + w_2 \int_{y_2}^{y_3} \theta dF + \int_{y_3}^{1} \theta dF. \end{aligned}$$

Notice the difference: $0 \le w_1 \le w_2 \le 1$ for the definition of $\overline{\mathcal{Q}}(Q, x)$, and $0 < w_1 < w_2 < 1$ for that of $\mathcal{Q}(Q, x)$. With these inequality constraints, $\overline{\mathcal{Q}}(Q, x)$ is compact.

Consider the sequence $\{q^k\}_k$. Because $\mathcal{Q}(Q, x) \subseteq \overline{\mathcal{Q}}(Q, x)$ and $\overline{\mathcal{Q}}(Q, x)$ is compact, q^k has a limit point $\overline{q}^* \in \overline{\mathcal{Q}}(Q, x)$. By the continuity of the principal's objective with respect to the Euclidean metric for (y, w), we have:

$$\lim_k \int_0^1 v(\theta, x)(q^k(\theta) - \overline{q}^*(\theta)) dF = 0.$$

If $\overline{q}^* \in \mathcal{Q}(Q, x)$, then we are done, so suppose not. For example, suppose that $\overline{q}^*$ is identified by (y, w) with $0 = w_1 < w_2 < 1$ (the other cases can be treated similarly, and hence omitted). As a mechanism, this is essentially equivalent to another $q' \in \mathcal{Q}(Q, x)$ identified by (y', w') where $y'_1 = y'_2 = y_2$, $y'_3 = y_3$, $w'_1 \in (0, w_2)$, and $w'_2 = w_2$. In particular, we have:

$$\int_0^1 v(\theta, x)(q'(\theta) - \overline{q}^*(\theta)) dF = 0.$$

Thus, we complete the proof by setting $q^* = q'$. □

A.2 Proof of Proposition 1.1

Let θ^* be such that $v''(\theta) < 0$ if $\theta < \theta^*$, and $v''(\theta) > 0$ if $\theta > \theta^*$.

Suppose contrarily that an optimal mechanism satisfies $y_1 < y_2 < y_3$. Consider an alternative mechanism in $\mathcal{Q}(Q, x)$ identified by $(y_1 + \varepsilon_1, y_2 + \varepsilon_2, y_3; w_1 + \delta_1, w_2)$. We choose $\varepsilon_1, \varepsilon_2, \delta_1 > 0$ so that the distribution of treated types in the optimal mechanism is a mean-preserving spread of that in the alternative mechanism:

$$\begin{aligned} -w_1 \int_{y_1}^{y_1+\varepsilon_1} dF - (w_2 - w_1 - \delta_1) \int_{y_2}^{y_2+\varepsilon_2} dF + \delta_1 \int_{y_1+\varepsilon_1}^{y_2} dF &= 0, \\ -w_1 \int_{y_1}^{y_1+\varepsilon_1} \theta dF - (w_2 - w_1 - \delta_1) \int_{y_2}^{y_2+\varepsilon_2} \theta dF + \delta_1 \int_{y_1+\varepsilon_1}^{y_2} \theta dF &= 0. \end{aligned}$$

By optimality, we must have $\theta^* < y_2$: Suppose contrarily that $\theta^* \geq y_2$. Then the principal's expected payoff given the alternative mechanism is higher than that given the optimal mechanism by:

$$\Delta = -w_1 \int_{y_1}^{y_1+\varepsilon_1} v(\theta, x) dF - (w_2 - w_1 - \delta_1) \int_{y_2}^{y_2+\varepsilon_2} v(\theta, x) dF + \delta_1 \int_{y_1+\varepsilon_1}^{y_2} v(\theta, x) dF.$$

Letting $\gamma = \frac{v(y_2+\varepsilon_2)-v(y_1)}{y_2+\varepsilon_2-y_1}$, we have:

$$\begin{aligned} \int_{y_1}^{y_1+\varepsilon_1} v(\theta, x) dF &= \int_{y_1}^{y_1+\varepsilon_1} v(y_1) + (\theta - y_1)\gamma + o(\varepsilon_1) dF, \\ \int_{y_2}^{y_2+\varepsilon_2} v(\theta, x) dF &= \int_{y_2}^{y_2+\varepsilon_2} v(y_1) + (\theta - y_1)\gamma + o(\varepsilon_2) dF, \\ \int_{y_1+\varepsilon_1}^{y_2} v(\theta, x) dF &= \int_{y_1+\varepsilon_1}^{y_2} v(y_1) + (\theta - y_1)\gamma + H dF, \end{aligned}$$

where $H > 0$ can be taken independently of $\varepsilon_1, \varepsilon_2, \delta_1$ as long as they are small enough, because of concavity of v on this region. Therefore,

$$\begin{aligned}\Delta &= -w_1 \int_{y_1}^{y_1+\varepsilon_1} (o(\varepsilon_1) - H) dF - (w_2 - w_1 - \delta_1) \int_{y_2}^{y_2+\varepsilon_2} (o(\varepsilon_2) - H) dF \\ &> 0,\end{aligned}$$

for sufficiently small $\varepsilon_1, \varepsilon_2, \delta_1$.

Similarly, consider an alternative mechanism in $\mathcal{Q}(Q, x)$ identified by $(y_1, y_2 - \varepsilon_2, y_3 - \varepsilon_3; w_1, w_2 - \delta_2)$ with $\varepsilon_2, \varepsilon_3, \delta_2 > 0$. By the same logic above, we must have $\theta^* > y_2$. This is a contradiction.

A.3 Proof of Proposition 1.2

Let θ^* be such that $v''(\theta) > 0$ if $\theta < \theta^*$, and $v''(\theta) < 0$ if $\theta > \theta^*$.

Suppose contrarily that an optimal mechanism q satisfies $y_2 < y_3 < 1$. The other case with $0 < y_1 < y_2$ is similar, and hence is omitted.

If $y_2 = 0$, then q is equivalent to another mechanism in $\mathcal{Q}(Q, x)$ identified by (y', w') with $y'_1 = 0$ and $y'_2 = y'_3 = y_3$ and $w'_1 = w_2$. This alternative mechanism satisfies the conditions in the statement. Thus, in what follows, we consider the case with $y_2 \neq 0$.

First, consider an alternative mechanism q' such that, for small $\varepsilon, \delta_1, \delta_2 > 0$, we have $q'(\theta) = w_1 + \delta_1$ for $\theta < y_2 + \varepsilon$; $q'(\theta) = w_2 + \delta_2$ for $\theta \in (y_2 + \varepsilon, y_3)$; and $q'(\theta) = q(\theta)$ otherwise. We choose $\varepsilon, \delta_1, \delta_2$ so that the distribution of treated types in the alternative mechanism is a mean-preserving spread of that in the optimal mechanism:

$$\begin{aligned}\delta_1 \int_{y_1}^{y_2+\varepsilon} dF - \delta_2 \int_{y_2}^{y_3} dF + (w_2 - \delta - 2 - w_1) \int_{y_2}^{y_2+\varepsilon} dF &= 0, \\ \delta_1 \int_{y_1}^{y_2+\varepsilon} \theta dF - \delta_2 \int_{y_2}^{y_3} \theta dF + (w_2 - \delta - 2 - w_1) \int_{y_2}^{y_2+\varepsilon} \theta dF &= 0.\end{aligned}$$

Consider first the case with $v(y_3, x) \geq v(y_2, x) + v'(y_2, x)(y_3 - y_2)$. This implies $\theta^* > y_2$, and moreover, $v(\theta) > v(y_2) + v'(y_2)(\theta - y_2)$ for $\theta \in (y_2, y_3)$. Therefore, the principal's expected payoff given the alternative mechanism is higher than that given the optimal mechanism by:

$$\begin{aligned}\Delta &= \delta_1 \int_{y_1}^{y_2+\varepsilon} v(\theta, x) dF - \delta_2 \int_{y_2}^{y_3} v(\theta, x) dF + (w_2 - \delta - 2 - w_1) \int_{y_2}^{y_2+\varepsilon} v(\theta, x) dF \\ &> \delta_1 \int_{y_1}^{y_2+\varepsilon} v(\theta, x) dF - \delta_2 \int_{y_2}^{y_3} v(y_2) + v'(y_2)(\theta - y_2) dF \\ &\quad + (w_2 - \delta - 2 - w_1) \int_{y_2}^{y_2+\varepsilon} v(\theta, x) dF.\end{aligned}$$

Let $\tilde{v}(\theta) = v(\theta, x)$ for $\theta \in (y_1, y_2]$ and $\tilde{v}(\theta) = v(y_2) + v'(y_2)y_3 - y_2(\theta - y_2)$ for $\theta \in (y_2, y_3)$. Note that $\tilde{v}$ is convex on (y_1, y_3). Thus, we have:

$$\begin{aligned}\Delta &= \delta_1 \int_{y_1}^{y_2+\varepsilon} \tilde{v}(\theta) dF - \delta_2 \int_{y_2}^{y_3} \tilde{v}(\theta) dF + (w_2 - \delta - 2 - w_1) \int_{y_2}^{y_2+\varepsilon} \tilde{v}(\theta) dF \\ &> 0,\end{aligned}$$

where the inequality is because of convexity of $\tilde{v}$. This contradicts that q is optimal. Thus, we must have the other case with $v(y_3, x) < v(y_2, x) + v'(y_2, x)(y_3 - y_2)$.

$$
\begin{aligned}
\Delta &= \delta_1 \int_{y_1}^{y_2+\varepsilon} v(\theta,x)dF - \delta_2 \int_{y_2}^{y_3} v(\theta,x)dF + (w_2-\delta-2-w_1)\int_{y_2}^{y_2+\varepsilon} v(\theta,x)dF \\
&> \delta_1 \int_{y_1}^{y_2+\varepsilon} v(\theta,x)dF - \delta_2 \int_{y_2}^{y_3} v(y_2) + \frac{v(y_3)-v(y_2)}{y_3-y_2}(\theta-y_2)dF \\
&\quad +(w_2-\delta-2-w_1)\int_{y_2}^{y_2+\varepsilon} v(\theta,x)dF.
\end{aligned}
$$

Let $\tilde{v}(\theta) = v(\theta,x)$ for $\theta \in (y_1,y_2]$ and $\tilde{v}(\theta) = v(y_2) + \frac{v(y_3)-v(y_2)}{y_3-y_2}(\theta-y_2)$ for $\theta \in (y_2,y_3)$. Obviously:

$$
\Delta = \delta_1 \int_{y_1}^{y_2+\varepsilon} \tilde{v}(\theta)dF - \delta_2 \int_{y_2}^{y_3} \tilde{v}(\theta)dF + (w_2-\delta-2-w_1)\int_{y_2}^{y_2+\varepsilon} \tilde{v}(\theta)dF.
$$

If $\theta^* \geq y_2$ and $\frac{v(y_3)-v(y_2)}{y_3-y_2} \geq v'(y_2)$, then $\tilde{v}$ is convex on (y_1,y_3). Then we obtain $\Delta > 0$, which is a contradiction. Therefore, we must have either $\theta^* < y_2$ or $\frac{v(y_3)-v(y_2)}{y_3-y_2} < v'(y_2)$. Note that $\frac{v(y_3)-v(y_2)}{y_3-y_2} < v'(y_2)$ implies $\theta^* < y_3$.

Next, consider an alternative mechanism q' such that, for small $\varepsilon, \delta_2, \delta_3 > 0$, we have $q'(\theta) = w_2 - \delta_2$ for $\theta \in (y_2, y_3-\varepsilon)$; $q'(\theta) = 1-\delta_3$ for $\theta \in (y_3-\varepsilon, 1)$; and $q'(\theta) = q(\theta)$ otherwise. We choose $\varepsilon, \delta_2, \delta_3$ so that the distribution of treated types in the optimal mechanism is a mean-preserving spread of that in the alternative mechanism. By the same logic, for this alternative mechanism not to be strictly better than the optimal mechanism, we must have $\frac{v(y_3)-v(y_2)}{y_3-y_2} < v'(y_3)$.

Combining the two sets of conditions, we must have $\frac{v(y_3)-v(y_2)}{y_3-y_2} < \min\{v'(y_2), v'(y_3)\}$. Consider a linear function $\hat{v}$ such that $\hat{v}(\theta) = v(\theta,x)$ for $\theta \in \{y_2,y_3\}$. Hence, the slope of $\hat{v}$ is $\frac{v(y_3)-v(y_2)}{y_3-y_2}$. By the intermediate value theorem, there must exist $y^* \in (y_2,y_3)$ such that $v'(y^*) = \tilde{v}'(y^*)$. However, by assumption, $v'(\theta)$ is strictly increasing for $\theta < \theta^*$ and strictly decreasing for $\theta > \theta^*$. This, together with $\frac{v(y_3)-v(y_2)}{y_3-y_2} < \min\{v'(y_2), v'(y_3)\}$, implies $v'(\theta) > \frac{v(y_3)-v(y_2)}{y_3-y_2}$ for all $\theta \in [y_2, y_3]$. This is a contradiction.

A.4 Proof of Proposition 1.3

First, consider the case with $y_1 > 0$. Because v is a limit case of $V^{vex-cav}$, the above result applies so that $y_1 = y_2$ and $y_3 \in \{y_2, 1\}$. If $y_2 = y_3$, the mechanism satisfies the condition in the statement, so consider the case with $y_3 = 1$.

However, it is possible to construct another mechanism whose distribution of treated types is a mean-preserving spread of that of the optimal mechanism. This is a contradiction, as the optimal mechanism can be further improved.

Next, consider the case with $y_1 = 0$. If $y_2 = 0$, the mechanism satisfies the condition in the statement, so consider the case with $y_2 > 0$. Because v is a limit case of $V^{vex-cav}$, the above result applies so that $y_3 \in \{y_2, 1\}$. In either case, it is possible to construct another mechanism whose distribution of treated types is a mean-preserving spread of that of the optimal mechanism. This is a contradiction, as the optimal mechanism can be further improved.

A.5 Proof of Proposition 1.4

We omit the proof, as it is similar to that of the previous result.

A.6 Proof of Proposition 2

Let $z_1 = \min\{0, y_1, y_2, y_3, y_1', y_2', y_3', 1\}$, $z_2 = \min(\{0, y_1, y_2, y_3, y_1', y_2', y_3', 1\} \setminus \{z_1\})$, and so on, and let $z_K = \max\{0, y_1, y_2, y_3, y_1', y_2', y_3', 1\}$.

Suppose that q exhibits more separation than q', in order to obtain a contradiction. Then, there exist $\alpha_{kk'} \geq 0$ and $\tau_{kk'} : [0,1] \to \mathbb{R}$ for each k, k' with $1 \leq k < k' \leq K$ such that (i):

$$\begin{aligned} \tau_{kk'}(\theta) = \tau_{kk'}(\theta') < 0 \qquad & \text{if } \theta, \theta' \in (z_k, z_{k+1}), \\ \tau_{kk'}(\theta) = \tau_{kk'}(\theta') > 0 \qquad & \text{if } \theta, \theta' \in (z_{k'}, z_{k'+1}), \\ \tau_{kk'}(\theta) = \tau_{kk'}(\theta') = 0 \qquad & \text{otherwise;} \end{aligned}$$

(ii):

$$\int_0^1 \tau_{kk'}(\theta) dF = 0;$$

and (iii):

$$q(\theta) = q'(\theta) + \sum_{1 \leq k < k' \leq K} \alpha_{kk'} \tau_{kk'}(\theta), \ \forall \theta.$$

That is, q is obtained from q' by shifting the treatment pattern according to each $\tau_{kk'}(\theta)$. By (i), this shift is "upward", that is, more of higher types and less of lower types are treated. By (ii), this preserves the total size of treated types.

Thus, in what follows, it suffices to show the following: Fixing any treatment policy $q^* : [0,1] \to \mathbb{R}$ and any shift $\tau_{kk'}(\theta)$ satisfying (i) and (ii), if the principal is better off by this shift given v, then he must also be better off by this shift given v'. Let x be the externality index induced by q^*, and $x'(\geq x)$ be the externality index induced by $q^* + \tau_{kk'}$. Suppose that the principal is better off by this shift given v, that is:

$$\int_0^1 v(\theta,x')(q^*(\theta)+\tau_{kk'}(\theta))dF - \int_0^1 v(\theta,x)q^*(\theta)dF \geq 0.$$

It suffices to show that

$$\int_0^1 (v'(\theta,x')-v(\theta,x'))(q^*(\theta)+\tau_{kk'}(\theta))dF - \int_0^1 (v'(\theta,x)-v(\theta,x))q^*(\theta)dF \geq 0,$$

because it then implies

$$\int_0^1 v'(\theta,x')(q^*(\theta)+\tau_{kk'}(\theta))dF - \int_0^1 v'(\theta,x)q^*(\theta)dF \geq 0$$

and hence, the desired contradiction.

Let $\varepsilon = -\tau_{kk'}(\theta) > 0$ for $\theta \in (z_k, z_{k+1})$ and $\varepsilon' = \tau_{kk'}(\theta) > 0$ for $\theta \in (z_{k'}, z_{k'+1})$. By (ii), $\varepsilon \int_{z_k}^{z_{k+1}} dF = \varepsilon' \int_{z_{k'}}^{z_{k'+1}} dF$. Thus, we have:

$$\begin{aligned}
& \int_0^1 (v'(\theta,x')-v(\theta,x'))(q^*(\theta)+\tau_{kk'}(\theta))dF - \int_0^1 (v'(\theta,x)-v(\theta,x))q^*(\theta)dF \\
= \; & \int_0^1 (v'(\theta,x')-v(\theta,x')-(v'(\theta,x)-v(\theta,x)))q^*(\theta)dF + \int_0^1 (v'(\theta,x')-v(\theta,x'))\tau_{kk'}(\theta)dF \\
\geq \; & -\varepsilon \int_{z_k}^{z_{k+1}} (v'(\theta,x')-v(\theta,x'))dF + \varepsilon' \int_{z_{k'}}^{z_{k'+1}} (v'(\theta,x')-v(\theta,x'))dF \\
\propto \; & -\frac{\int_{z_k}^{z_{k+1}} (v'(\theta,x')-v(\theta,x'))dF}{\int_{z_k}^{z_{k+1}} dF} + \frac{\int_{z_{k'}}^{z_{k'+1}} (v'(\theta,x')-v(\theta,x'))dF}{\int_{z_{k'}}^{z_{k'+1}} dF} \\
\geq \; & 0,
\end{aligned}$$

where the first inequality is because $\frac{\partial^2}{\partial x \partial \theta}(v'(\theta,x)-v(\theta,x)) \geq 0$ for all θ, x; and the last inequality is because $\frac{\partial}{\partial \theta}(v'(\theta,x)-v(\theta,x)) \geq 0$ for all θ, x.

A.7 Proof of Proposition 3

Under the stated conditions, Proposition 2 implies that the optimal mechanism given v' does not exhibit more separation than that given v. If $v, v' \in V^{vex}$ or $v, v' \in V^{cav}$, then by Proposition 1.3 or 1.4, the optimal mechanisms given v and given v' can always be ordered according to first-order stochastic dominance of treated types. Therefore, the optimal mechanism given v' exhibits more separation than that given v.

A.8 Proof of Proposition 4

Let $z_1 = \min\{0, y_1, y_2, y_3, y_1', y_2', y_3', 1\}$, $z_2 = \min(\{0, y_1, y_2, y_3, y_1', y_2', y_3', 1\} \setminus \{z_1\})$, and so on, and let $z_K = \max\{0, y_1, y_2, y_3, y_1', y_2', y_3', 1\}$.

Suppose that q exhibits more diversity of treated types than q', in order to obtain a contradiction. Then, there exist $\alpha_{ijk} \geq 0$ and $\tau_{ijk} : [0,1] \to \mathbb{R}$ for each i, j, k with $1 \leq i < j < k \leq K$ such that (i):

$$\begin{aligned}
\tau_{ijk}(\theta) &= \tau_{ijk}(\theta') > 0 && \text{if } \theta, \theta' \in (z_i, z_{i+1}), \\
\tau_{ijk}(\theta) &= \tau_{ijk}(\theta') < 0 && \text{if } \theta, \theta' \in (z_j, z_{j+1}), \\
\tau_{ijk}(\theta) &= \tau_{ijk}(\theta') > 0 && \text{if } \theta, \theta' \in (z_k, z_{k+1}), \\
\tau_{ijk}(\theta) &= \tau_{ijk}(\theta') = 0 && \text{otherwise;}
\end{aligned}$$

(ii):

$$\int_0^1 \tau_{ijk}(\theta) dF = 0, \ \int_0^1 \theta \tau_{ijk}(\theta) dF = 0;$$

and (iii):

$$q(\theta) = q'(\theta) + \sum_{1 \le i < j < k \le K} \alpha_{ijk}\tau_{ijk}(\theta),\ \forall \theta.$$

That is, q is obtained from q' by a series of mean-preserving spreads.

Thus, in what follows, it suffices to show the following: Fixing any treatment policy $q^* : [0,1] \to \mathbb{R}$ and any shift $\tau_{ijk}(\theta)$ satisfying (i) and (ii), if the principal is better off by this shift given v, then he must also be better off by this shift given v'. Suppose that the principal is better off by this shift given v, that is:

$$\int_0^1 v(\theta,x)(q^*(\theta) + \tau_{ijk}(\theta))dF - \int_0^1 v(\theta,x)q^*(\theta)dF \ge 0,$$

or equivalently,

$$\int_0^1 v(\theta,x)\tau_{ijk}(\theta)dF \ge 0.$$

Because $v' - v$ is convex in θ and τ_{ijk} is a mean-preserving spread, we obtain:

$$\int_0^1 (v'(\theta,x') - v(\theta,x'))\tau_{ijk}(\theta)dF \ge 0,$$

implying

$$\int_0^1 v'(\theta,x')\tau_{ijk}(\theta)dF \ge 0,$$

and hence, the desired contradiction.

A.9 Proof of Proposition 5

Under the stated conditions, Proposition 4 implies that the optimal mechanism given v' does not exhibit more concentration of treated types than that given v. If $v, v' \in V^{cav-vex}$ or $v, v' \in V^{vex-cav}$, then by Proposition 1.1 or 1.2, the optimal mechanisms given v and given v' (with any fixed (Q, x)) can always be ordered according to second-order stochastic dominance of treated types. Therefore, the optimal mechanism given v' exhibits more concentration of treated types than that given v.

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