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Abstract

In this paper we examine a game-theoretical generalization of the landscape theory introduced by Axelrod and Bennett (1993). In their two-bloc setting each player ranks the blocs on the basis of the sum of her individual evaluations of members of the group. We extend the Axelrod-Bennett setting by allowing an arbitrary number of blocs and expanding the set of possible deviations to include multi-country gradual deviations. We show that a Pareto optimal landscape equilibrium which is immune to profitable gradual deviations always exists. We also indicate that while a landscape equilibrium is a stronger concept than Nash equilibrium in pure strategies, it is weaker than strong Nash equilibrium.

JEL numbers: C72, D74.

Keywords: Landscape theory, landscape equilibrium, blocs, gradual deviation, potential functions, hedonic games.

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1 Introduction

This paper examines the class of strategic environments covered by the "landscape theory"introduced by Axelrod and Bennett (1993) (AB — henceforth). In the landscape setting the actors (countries) are partitioned into two mutually exclusive blocs on the basis of their propensity to work together with other bloc members on the bilateral basis. All players rank groups according to the sum of her individual evaluations of all members of the group. In this sense the AB approach to international interactions is related to Bueno de Mesquita (1975, 1981) who constructed a proximity matrix for every pair of nations based on history of their defense cooperation.

Each actor i is characterized by the value of her size/strength/influence parameter s_i. For each pair of countries i and j there is a parameter p_{ij} (positive or negative), the value of which represents the propensity for collaboration between i and j. Thus, the data of the model consists of an n-dimensional vector of countries' strength parameters and an $n \times n$ matrix $\mathcal{P}$ of pairwise proximity coefficients.

For an arbitrary partition of all countries into two blocs, AB define the *frustration* of a country i as the sum of the proximity coefficients p_{ij} for all members outside her bloc weighted by their strength parameter s_j. Obviously, the country frustration will be reduced if it avoids countries with whom it has a strong negative propensity to align. The *energy* of any two-bloc partition is then determined as the sum of individual frustrations of all countries weighted by their size. The objective of the theory is to identify the configurations that yield, as AB call it, a local and *global* minimum of energy. To attain these outcomes, AB used the *incremental* or *gradual* approach by allowing single countries to switch their membership, one at a time, to generate a new configuration with the reduced energy level. Assuming the symmetry of the proximity matrix $\mathcal{P}$, i.e. $p_{ij} = p_{ji}$ for all pairs of players i, j, AB showed that for any initial bloc structure, the sequential gradual reduction of energy does not contain cycles and is terminated when a stable configuration is attained. Note that the symmetry of the proximity matrix $\mathcal{P}$ is essential to obtain stable configurations. Indeed, consider a game with two players, where player 1 prefers to join player 2, i.e., $p_{12} > 0$, whereas player 2 would like to avoid being together with 1, i.e., $p_{21} < 0$. The game obviously does not admit a stable partition, as the partition in two groups would be challenged by player 1, while the creation of a two-country bloc would be rejected by player 2.

AB provide a spectacular application of the landscape theory to European alliances prior to World War II. By using the Correlates of War data and estimating the propensity for cooperation based on ethnic and border conflicts, history, etc., AB calibrate a matrix $\mathcal{P}$ to conclude that that there were two stable configurations. One is the expected partition to the Axis and Allies of World War II, while the other separates USSR, Yugoslavia and Greece from the rest of Europe! Axelrod et al. (1995) also illustrate and test the landscape theory by estimating the choices of nine computer companies to join one of two alliances sponsoring competing UNIX operating system standards in 1988.

Even though AB have not done that explicitly, it is natural to present their setting in game-theoretical terms. Each actor i is a player with two available pure strategies corresponding to two blocs, X and Y, and her payoff function is represented by her frustration level derived from the two-bloc partition. Thus, after minor adjustments, proper reformulation, and clarifications, AB in fact show the existence of a pure strategies Nash equilibrium in landscape games. As Bennett (2000, p. 51) points out: "A local optimum is defined as a configuration for which every adjacent configuration has higher (worse) energy. When the system reaches one of those points, no further improvement in energy is possible given a single step (change of coalition by one actor). This optimum is akin to a Nash equilibrium in game theory, wherein no single actor can improve its own payoff by choosing a different move." Interestingly enough, the AB energy function E could be viewed as a potential, so that symmetric landscape games belong to the class of potential games examined by Monderer and Shapley (1996).

Notice that landscape games belong to the class of hedonic games pioneered by Banerjee, Konishi and Sönmez (2001) and Bogomolnaia and Jackson (2002). Hedonic games are coalition formation games, where the payoff of any player depends solely upon the composition of the coalition to which she belongs, and a strategic choice made by the coalition does not impact its members' payoffs. This is the case for landscape games, where each player possesses a precise evaluation of every potential partner and then ranks groups according to the sum of her individual evaluations of all members of the group she may join. In the case of equal values of the strength parameter s_i for all countries, this model belongs to the class of *additively separable* games in Banerjee, Konishi and Sönmez (2001). By constructing a potential function, as in AB, Bogomolnaia and Jackson (2002) show that the symmetric additively separable hedonic games, including the landscape games, admit a Nash stable configuration.

In this paper, we consider the class of landscape games that expands the AB framework in two aspects. First, we allow for an arbitrary number of blocs to form, without limiting ourselves to two-bloc configurations.[1] The configurations with more than two blocs have a place in various environments. In fact, during the Cold War between East and West that followed the end of the World War II, an important role has been played by the third bloc of non-aligned countries. And nowdays, when the world is often described as a multi-polar environment, the study of multi-bloc settings becomes even more relevant. Another distinction with regard to the AB model is that we expand the notion of incremental or gradual deviations in AB, in which only one country at a time was allowed to switch its bloc membership. In our framework we take the gradual approach further by allowing several countries to switch their blocs

[1]See Florian and Galam (2000) for a discussion on a three-bloc extension of the landscape theory.

at the same time. However, the cost of absorption of new members from different blocs could be quite high. Thus, a switch will be allowed only for a subgroup from one bloc to another. We call such a deviation gradual and define a *landscape equilibrium* as a configuration immune to gradual deviations. Note that individual deviations are obviously allowed under the umbrella of gradualism.

Our main result shows that, under the symmetry assumption, there is a landscape equilibrium. Our proof relies on the *potential* approach. While our game itself is not a potential game, we show that it is strategically equivalent to a potential game. That is our game admits a weighted potential in Monderer and Shapley's (1996) terminology. The application of the potential approach allows us to obtain even a stronger result, namely the existence of a Pareto optimal landscape equilibrium. It is important pointing out that the existence of a Pareto optimal landscape equilibrium rules out an emergence of prisoner's dilemma type of situation, where countries acting in their own self-interest generate a suboptimal outcome. Interestingly, some aspects of Pareto optimality have been discussed by AB, who searched for the global optimum as the lowest energy level of any configuration. Since the concept of landscape equilibrium is stronger than Nash equilibrium, our result reconfirms the existence of a Nash equilibrium in landscape games. On the other hand, we also consider a more demanding notion of strong Nash equilibrium introduced by Aumann (1959), which requites immunity against any deviation by any group of players. However, as is implied by a result in Banerjee, Konishi and Sönmez (2001), a strong Nash equilibrium in landscape games may fail to exist. Thus, the unrestricted extension of the set of feasible deviations not only violates the concept of gradualism, but also diminishes the likelihood of obtaining a meaningful existence result.

The paper is organized as follows. In the next section we offer a brief review of the literature. In Section 3 we present a model and the necessary definitions. In Section 4 we state and prove our result on existence of a Pareto optimal landscape equilibrium. In Section 5 we discuss the links of our equilibrium concept with other modifications of Nash equilibrium.

2 Related Literature

The main feature of landscape games examined in this paper is that the strategic decisions made by players are driven by their pairwise evaluations of other players. The environments with such a property were widely studied in the literature. In the international relations setting, Bueno de Mesquita (1975, 1981) constructed a matrix that captures the proximity between pairs of nations according to their alliances on defense issues and defined "indicators of tightness" which are used as a key determinant to evaluate the war proneness of the international system. Le Breton and Weber

(1994) consider such a setting in the case where only a two-player coalitions can be formed. Desmet et al. (2011) consider a nation formation game where pairwise hedonic heterogeneity is described by the matrix of genetic distances between nations as calculated by scholars in population genetics.

The family of landscape games belongs to the class of *hedonic games* studied by Banerjee, Konishi and Sönmez (2001) and Bogomonalia and Jackson (2002), where the payoff of every player solely depends on the composition of the coalition she belongs to. In fact, landscape games are strategically equivalent to the additively separable hedonic games. In their study of hedonic games, both Banerjee, Konishi and Sönmez (2001) and Bogomonalia and Jackson (2002) have shown that various requirements of stability may yield the nonexistence of stable profiles. Banerjee, Konishi and Sönmez (2001) also derive sufficient conditions for a hedonic game to be *core stable.* Bogomolnaia and Jackson (2001) derive the existence results with regard to two other notions of coalitional stability. Both notions they examine, that of individual stability and contractual stability, are weaker than Nash stability.[2] Milchtaich and Winter (2002) demonstrate that a coalition formation game in which each player is identified by a one dimensional characteristic (status) and player's payoff is a decreasing function of the average distance between her status and that of other players in the group, may fail to admit a strongly stable configuration. A possibility of the nonexistence of a strong Nash equilibrium has been reinforced by Kukushkin (2019) for a different variant of the status game.

Nash equilibria and its refinements have also been investigated for various families of *congestion games*. When all non-diagonal entries p_{ij} are negative, our game is a non-anonymous congestion game a la Milchtaich (1996), where all individuals are adversly impacted by the presence of others in their coalition[3]. Any such congestion game belongs to the class of games considered by Quint and Shubik (1994), Milchtaich (1996) and Konishi, Le Breton and Weber (1997a,b), who prove the existence of a Nash equilibrium in pure strategies for anonymous congestion games. For the latter class of games Konishi, Le Breton and Weber (1997a), in fact, show the existence of a strong Nash equilibrium. Our game is also stragically equivalent to a congestion game a la Rosenthal (1973)[4], where players do not possess an *a priori* defined set of strategies (actually, a strategy is a path on network). The payoff of a player is defined

[2]In this setting, Nash and core stability are logically independent. An equivalent notion of strong Nash equilibrium in the context of hedonic games, called strong stability in Milchtaich and Winter (2002), allows deviating players to join existing blocs. At the same time, the core stability requires that all deviating players select an *empty slot*, i.e., a previously unused alternative.

[3]Obviously, if all off-diagonal entries of the matrix $\mathcal{P}$ are positive, the grand coalition represents a unique Pareto efficient Nash equilibrium, as well as a strong Nash equilibrium of our game. We are grateful to an anonymous referee for pointing out the correct statement in this regard.

[4]The congestion games considered in Rosenthal (1973) differ from those considered by Milchtaich (1996).

as the sum of her payoffs on each segment of the path that she ultimately selects. Assuming that on any given segment the payoffs of the players who have access to that segment are identical, the seminal contribution of Rosenthal (1973) shows that any game in his class is a potential game, and, thus, admits a Nash equilibrium in pure strategies. This linkage was later systematically explored by Monderer and Shapley (1996).

Note that Nash equilibria in these types of games do not need to be strong. To address this point, Holzman and Law-Yone (1997) introduce the notion of strong potential and obtain conditions on the network that guarantee the existence of a strong Nash equilibria. This topic is further explored in Voorneveld et al. (1999) and Harks, Klimm and Möhring (2013). Finally, we would like to point out that strong Nash equilibria has recently received a lot of attention in algorithmic game theory.[5]

3 The Model and Definitions

The landscape game Γ^0 is defined as follows. Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. Each player i is associated with the positive value s_i which represents her influence, or in the case of countries, the population size or military power. For every pair of players i and j from N there is a value p^0_{ij} (positive or negative) that represents the strength of ties between i and j and their benefit of being members of the same coalition. It is assumed that this value is symmetric for every pair i and j, i.e., $p^0_{ij} = p^0_{ji}$ with $p^0_{ii} = 0$ for every player i. The data on pairwise propensities is therefore represented by the symmetric $n \times n$ matrix $\mathcal{P}$.

The set of alternatives $X = \{x^1, \ldots, x^m\}$ is common for all players. Each player $i \in N$ chooses an alternative $x_i \in X$.[6] Two players i and j belong to the same bloc if $x_i = x_j$. A vector of players' choices $\mathbf{x} = (x_1, \ldots, x_n)$ generates the partition $\pi(\mathbf{x})$ of the set N, which consists of no more than m non-empty blocs. We denote $G^i(\mathbf{x}) \in \pi(\mathbf{x})$ the bloc that contains player i.

The payoff $U^0_i(\mathbf{x})$, $i \in N$, of each player solely depends upon the bloc to which she belongs[7]. More specifically we assume that

$$U^0_i(\mathbf{x}) = \sum_{j \in G^i(\mathbf{x})} p^0_{ij} s_j. \tag{1}$$

We shall analyze the stability of emerging bloc formations. To do so, we need to examine a threat of feasible deviations. For a given strategy profile $\mathbf{x}$, a group of

[5] See, e.g., Andelman, Feldman and Mansour (2009), Chien and Sinclair (2009), Epstein, Feldman and Mansour (2009) who compute strong versions of the price of anarchy for various classes of games.

[6] The lower and upper indices indicate players and alternatives, respectively. The expression $x_i \in X$ means that there is $x^k \in X$ such that $x_i = x^k$.

[7] Note that in our paper, we use a more common notion of *utility* rather than frustation. The utility maximization and the frustration minimization are, obviously, equivalent objectives.

players S would deviate if each member i of S would switch to another bloc, while raising her utility. Formally,

Definition 1: Deviation: Let a strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ be given. A *(feasible)* deviation from $\mathbf{x}$ by a group of players S is a profile $\mathbf{x}' = \left(\{x_i'\}_{i \in S}, \{x_i\}_{i \notin S}\right)$ consisting of "new" choices for players in S and unaltered choices for the rest of the players. It is *profitable* if:

$$U_i^0(\mathbf{x}') > U_i^0(\mathbf{x}) \quad \text{for all } i \in S.$$

However, in our setting, as in many others, one needs to impose some restrictions on feasible deviations. The coordination challenges, switching costs and other factors may limit the size and the composition of deviating groups. In fact, AB argued for need for *incrementalism* and allowed only for one single actor to switch bloc. Thus, we first consider the case where the only feasible deviating coalitions are singletones. It immediately yields the notion of Nash equilibrium.

Definition 2: Nash equilibrium: A strategic profile $\mathbf{x} = (x_1, \ldots, x_n)$ is a Nash equilibrium if for all $i \in N$, there is no profitable deviation from $\mathbf{x}$ by an individual i in N.

Following AB, we limit the range of feasible deviations while allowing for a wider set of deviations than mere singletons. In our view, the main bulk of bloc formation costs in the landscape theory boils down to bloc costs incurred by the absorption of new members. To mitigate these costs, any bloc A is allowed to absorb members from only one other bloc. Thus, we rule out the situation where members from two other different blocs switch to A. It is important to point out that no country should be prevented from switching to a bloc of its choice. What we examine here is the first stage of a possible realignment process.

Definition 3: Gradualism and Landscape Equilibrium: Let strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ be given. Assume that the strategy profile $\mathbf{x}'$ represents a deviation from $\mathbf{x}$ by a group of players S. The deviation is called *gradual* if the following condition is satisfied for every of two players i and j in S:

$$(GR) \;\; \text{If } i, j \in S \text{ and } x_i' = x_j' \quad \text{then} \;\; x_i = x_j.$$

A profile $\mathbf{x}$ is called a *landscape equilibrium* if it does not allow a profitable gradual deviation.

It would be useful to recall the notion of a strong Nash equilibrium (Aumann (1959)), which is immune against the unrestricted set of coalitional deviations.

Definition 4: Strong Nash Equilibrium: A strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ is a *strong Nash equilibrium* if there is no $S \subseteq N$ that profitably deviates from $\mathbf{x}$.

Finally, we use the standard notion of strong Pareto optimality:

Definition 5: Strong Pareto Optimality: A profile $\mathbf{x}$ is *strongly Pareto optimal* if there is no other strategy profile $\mathbf{x}'$ such that $U_i^0(\mathbf{x}') \geq U_i^0(\mathbf{x})$ for all $i \in N$ with a strict inequality for at least one i.

4 The Main Result

Our main result is:

Theorem: The game Γ^0 admits a strongly Pareto optimal landscape equilibrium.

Proof of Theorem: We proceed in three steps.

First, we modify the game Γ^0 by multiplying the utility function U_i^0 for each player $i \in N$ by i's influence parameter s_i:

$$s_i U_i^0(\mathbf{x}) = \sum_{j \in G^i(\mathbf{x})} p_{ij}^0 s_i s_j.$$

While this modification does not alter the equilibrium structure of the game[8], it makes the summation term $p_{ij}^0 s_i s_j$ symmetric in i and j: $p_{ij}^0 s_i s_j = p_{ji}^0 s_j s_i$. We therefore define the game Γ that differs from Γ^0 only with respect to individual utility functions where for each $i \in N$

$$U_i(\mathbf{x}) = \sum_{j \in G^i(\mathbf{x})} p_{ij}, \tag{2}$$

with

$$p_{ij} = p_{ij}^0 s_i s_j$$

for every pair of players i and j.

Second, we demonstrate that if a profile $\mathbf{y}$ is subject to a profitable gradual coalitional deviation by coalition C, satisfying (GR), then there exists a coalition $C' \subset C$ (possibly, C itself), that profitably deviates from $\mathbf{y}$ via profile $\mathbf{y}'$, and in the same time, in addition to (GR), also satisfies the following condition (G):

$$(G) \quad \text{for any } i, j \in C' \quad \text{the equality} \quad y_i = y_j \quad \text{implies } y_i' = y_j'.$$

[8] As pointed out by an anonymous referee and indicated in the introduction, the game Γ^0 is not necessarily a potential game, while Γ is. Thus, the game Γ^0 admits a weighted potential (Monderer and Shapley (1996)).

(G) means that deviating members of the same bloc should stay in the same bloc after the deviation as well. Formally,

Lemma: Let strategy profile $\mathbf{y}$ be given. Assume that there is a profitable gradual deviation by coalition C from $\mathbf{y}$, that satisfies (GR). Then there is a coalition $C' \subset C$ that allows a profitable deviation from $\mathbf{y}$ that satisfies both (GR) and (G).

Proof of Lemma: Take a strategy profile $\mathbf{y}$. Assume that there is a profitable gradual deviation by coalition C from $\mathbf{y}$ via profile $\mathbf{y}' = \left(\{y_i'\}_{i \in C}, \{y_j\}_{j \notin C}\right)$ that satisfies (GR).

For every alternative $j = 1, \ldots, m$, denote by C_j the set of players who chose alternative x^j at $\mathbf{y}$ and switched to a different alternative (bloc) at $\mathbf{y}'$. Denote by $J \subset M$ the list of blocs j for which C_j is nonempty. (There is, at least, one such bloc). For every $j \in J$ denote by $K(j)$ the list of blocs which are joined at $\mathbf{y}'$ by, at least, one nember of C_j. There are two cases to consider.

Case 1. $\bigcup_{j \in J} K_j \subset J$. If for some $j \in J$, K_j contains more than one element, then according to the pigeonhole principle, players from two different blocs at $\mathbf{y}$ join the same bloc at $\mathbf{y}'$, contradicting (GR). Thus, every $K(j)$ consists of one bloc for every $j \in J$. That is, the deviating coalition C satisfies (G).

Case 2. For at least one $j \in J$, K_j contains a bloc k, which is not in J. Thus, the set $C_j' = \{i \in C_j : y_i' = x^k\}$ is nonempty. Consider the shift of all members of C_j' from x^j to x^k. The condition (GR) guarantees that no other players join x^k at $\mathbf{y}'$. Note that bloc k at $\mathbf{y}$ is either empty or consists of players who stayed put under the original deviation of coalition C. In both cases the payoffs of members of C_j' are the same under $\mathbf{y}'$ and $\mathbf{y}''$, where $\mathbf{y}'' = \left(\{y_i'\}_{i \in C_j'}, \{y_j\}_{j \notin C_j'}\right)$. Thus, the deviation of C_j' via $\mathbf{y}''$ is profitable and obviously satisfies (GR) and (G). **Q.E.D.**

Third, we introduce the utilitarian social welfare function

$$\mathbf{P}(\mathbf{x}) \equiv \sum_{i \in N} U_i(\mathbf{x}),$$

and show that it is a strong landscape potential in the following sense: If there is a profitable deviation of group S from $\mathbf{x}$ via $\mathbf{x}'$ satisfying (GR) and (G) in game Γ, then $\mathbf{P}(x') > \mathbf{P}(x)$**.**

Let $\tilde{\mathbf{x}}$ be the maximum of the function $\mathbf{P}(\mathbf{x})$ over the finite set of all strategy profiles. Obviously, $\tilde{\mathbf{x}}$ is strongly Pareto optimal. We shall now demonstrate that $\mathbf{x}$ is immune against coalitional deviations satisfying (GR) and (G). This will complete the proof of our theorem, as by the lemma, any profitable deviation from $\mathbf{x}$ satisfying (GR) induces a profitable deviation satisfying both (GR) and (G).

Suppose, in negation, that there is a coalition S that profitably deviates from $\tilde{\mathbf{x}}$ via $\mathbf{x}'$, while satisfying (GR) and (G). We shall show this deviation results in the change in the potential, which is equal to the total change in the utilities of the deviated players multiplied by two:

$$\mathbf{P}(\tilde{\mathbf{x}}) - \mathbf{P}(\mathbf{x}') = 2\sum_{i\in S}(U_i(\tilde{\mathbf{x}}) - U_i(\mathbf{x}')). \tag{3}$$

In fact, equation (3) holds for arbitrary $\tilde{\mathbf{x}}$ and $\mathbf{x}'$ such that $\mathbf{x}'$ is obtained from $\tilde{\mathbf{x}}$ by deviation of coalition S, which satisfies both (GR) and (G). However, to prove the Theorem, we utilize (3) only for the maximum of the potential $\tilde{\mathbf{x}}$ from which the coalition S profitably deviates via the strategy profile $\mathbf{x}'$. Indeed, the gain for all $i \in S$ implies that the right hand side of (3) is negative. The latter would contradict the fact that $\tilde{\mathbf{x}}$ is the maximum of the potential. The rest of the proof is focused on deriving equation (3).

Players' choices at the maximum $\tilde{\mathbf{x}}$ induce the partition of the set N onto m (not necessarily nonempty) coalitions. Denote by $H^j(\tilde{\mathbf{x}})$, $j = 1,\dots,m$ the set of players $i \in N$ who choose the strategy $\tilde{x}_i = x^j$ in X.

For every pair of alternatives $x^k \neq x^l$ in X let $T^{kl} \subset S$ be the set of players who changed their choice from x^k to x^l. Then there are (at most) $m(m-1)$ such groups. For every $x^k \in X$ denote three sets of players :

Q^k — those who left the bloc that chose x^k at $\tilde{\mathbf{x}}$: $Q^k = \cup_{l\neq k} T^{kl}$,

R^k — those who choose x^k at $\mathbf{x}'$ but not at $\tilde{\mathbf{x}}$: $R^k = \cup_{l\neq k} T^{lk}$,

Ψ^k — those who choose x^k at both $\mathbf{x}'$ and $\tilde{\mathbf{x}}$.

Note that $\Psi^k = G^k(\tilde{\mathbf{x}}) \setminus Q^k$. (Recall that $G^k(\tilde{\mathbf{x}})$ is the set of players whose choice coincides with that of player k at $\tilde{\mathbf{x}}$). Since each player from $T^{kl} \subset S$ increases her payoff by switching from x^k to x^l, it follows that

$$U_i(\mathbf{x}') - U_i(\tilde{\mathbf{x}}) > 0 \tag{4}$$

for all $i \in T^{kl}$.

To simplify the notation, we introduce the mapping $\sigma(\cdot,\cdot) : N \times N \longrightarrow \Re$ which assigns a real number to any two subsets N_1 and N_2 of N as follows:

$$\sigma(N_1, N_2) = \sum_{i\in N_1}\sum_{j\in N_2} p_{ij}.$$

In particular, for every $i \in N$ and a strategy profile $\mathbf{x}$ we have

$$\sigma(\{i\}, G^i(\mathbf{x})) = \sum_{j\in G^i(\mathbf{x})} p_{ij} = U_i(\mathbf{x}). \tag{5}$$

Moreover, the symmetry of $\mathcal{P}$ induces the symmetry of σ:

$$\sigma(N_1, N_2) = \sigma(N_2, N_1). \tag{6}$$

In addition, for every triple $N_1, N_2, N_3 \subset N$, we have

$$\sigma(N_1, N_2 \cup N_3) = \sigma(N_1, N_2) + \sigma(N_1, N_3) \tag{7}$$

We extend the notation $\sigma(N_1, N_2)$ to the case of $N_1 = \emptyset$ or/and $N_2 = \emptyset$ by assigning $\sigma(\emptyset, N_2) = \sigma(N_1, \emptyset) = \sigma(\emptyset, \emptyset) = 0$.

Suppose that player i switches from x^k to x^l. Then combining observation (5) with the decompositions

$$G^k(\tilde{\mathbf{x}}) = \Psi^k \cup_{r=1, r\neq k}^m T^{kr} \quad \text{and} \quad G^l(\mathbf{x}') = \Psi^l \cup_{q=1, q\neq l}^m T^{ql},$$

we rewrite inequality (4) in the following way:

$$\sigma(\{i\}, \Psi^l \cup_{q=1, q\neq l}^m T^{ql}) - \sigma(\{i\}, \Psi^k \cup_{r=1, r\neq k}^m T^{kr}) > 0 \quad \text{for all } i \in T^{kl}. \tag{8}$$

Summing up inequalities (8) over all $i \in T^{kl}$, we have

$$\sigma(T^{kl}, \Psi^l \cup_{q=1, q\neq l}^m T^{ql}) > \sigma(T^{kl}, \Psi^k \cup_{r=1, r\neq k}^m T^{kr}). \tag{9}$$

Let us use the symmetry condition (6) and the decomposition property (7) of σ and apply them to equation (9):

$$\sigma(T^{kl}, \Psi^l) + \sum_{q=1, q\neq l}^{m} \sigma(T^{kl}, T^{ql}) > \sigma(T^{kl}, \Psi^k) + \sum_{r=1, r\neq k}^{m} \sigma(T^{kl}, T^{kr}). \tag{10}$$

Condition (GR) does not allow players from two different blocs at $\tilde{\mathbf{x}}$ to join the same bloc at $\mathbf{x}'$, i. e., either T^{kl} or T^{ql} is empty set for each pair (T^{kl}, T^{ql}), $q = 1, \ldots, m$, $q \neq l$, $q \neq k$. The corresponding terms $\sigma(T^{kl}, T^{ql})$ are equal to zero. In the same way, according to (G), $\sigma(T^{kl}, T^{kr}) = 0$, $r = 1, \ldots, m$, $r \neq l$, $r \neq k$. Therefore, (10) is simplified to

$$\sigma(T^{kl}, \Psi^l) > \sigma(T^{kl}, \Psi^k). \tag{11}$$

There are $m(m-1) - 1$ (not necessarily nonempty) other groups of players from S that alter their strategies and raise their payoff by shifting from $\tilde{\mathbf{x}}$ to $\mathbf{x}'$. Summing up all inequalities (11) obtained for different pairs (k, l) we end up with

$$\sum_{i \in S} \big(U_i(\mathbf{x}') - U_i(\tilde{\mathbf{x}})\big) = \sum_{q=1}^{m} \big(\sigma(R^q, \Psi^q) - \sigma(Q^q, \Psi^q)\big) > 0. \tag{12}$$

Now we evaluate the difference of the potential at the two points that represent the players' choices before and after the deviation:

$$\mathbf{P}(\tilde{\mathbf{x}}) - \mathbf{P}(\mathbf{x}') = \sum_{q=1}^{m} \sigma(\Psi^q \cup Q^q, \Psi^q \cup Q^q) - \sum_{q=1}^{m} \sigma(\Psi^q \cup R^q, \Psi^q \cup R^q).$$

Once again, the symmetry property (6) is applied and the terms $\sigma(\Psi^q, \Psi^q)$ are cancelled out. Then

$$\mathbf{P}(\tilde{\mathbf{x}}) - \mathbf{P}(\mathbf{x}') = \sum_{q=1}^{m} \big(2\sigma(Q^q, \Psi^q) + \sigma(Q^q, Q^q)\big) - \sum_{q=1}^{m} \big(2\sigma(R^q, \Psi^q) + \sigma(R^q, R^q)\big). \quad (13)$$

The conditions (GR) and (G) allow us to conclude that

$$\sum_{q=1}^{m} \sigma(Q^q, Q^q) - \sum_{q=1}^{m} \sigma(R^q, R^q) = 0. \quad (14)$$

Indeed, each (non-empty) R^q consists of a single sub-group T^{kq}: for any $q = 1, \dots, m$ there is $k = k(q)$: $R^q = T^{kq}$ (the existence of the second sub-group would violate condition (GR)). According to (G), each Q^q also consists of a single sub-group T^{ql}: for any $q = 1, \dots, m$ there is $l = l(q)$: $Q^q = T^{ql}$. Thus, the both sums in (14) consist of the same terms. From (13) and (14) we have

$$\mathbf{P}(\tilde{\mathbf{x}}) - \mathbf{P}(\mathbf{x}') = 2\sum_{q=1}^{m} \big(\sigma(Q^q, \Psi^q) - \sigma(R^q, \Psi^q)\big). \quad (15)$$

Combining (12) and (15), we obtain (3) to complete the proof of Theorem. **Q.E.D.**

5 Comments

Let us first offer some comments on the connection between landscape equilibria and the two other equilibrium notions introduced above. First, the notion of gradual deviation in Definition 3 is stronger than individual deviations, and, therefore, the set of landscape equilibria is smaller than the set of Nash equilibria. Thus, Theorem yields the existence of a Nash equilibrium in pure strategies as well. The following example shows that, in general, the sets of landscape and Nash equilibria do not coincide, and a Nash equilibrium does not necessarily constitute a landscape equilibrium.

Example: Consider the game Γ^0 with four players and two alternatives ($n = 4, m = 2$). The influence parameter s_i is assumed to be equal to 1 for all players, and the propensity matrix $\mathcal{P}$ is given by:

$$\mathcal{P} = \begin{pmatrix} 0 & 1000 & -100 & -50 \\ 1000 & 0 & -100 & -50 \\ -100 & -100 & 0 & -400 \\ -50 & -50 & -400 & 0 \end{pmatrix}$$

It is easy to see that the strategy profile (a, a, a, b) is a Nash equilibrium. However, players 1 and 2 would benefit by switching to b and joining player 4. By Definition 3, it is a profitable gradual deviation, and this profile is not a landscape equilibrium. Notice that in this example the profile (a, a, b, a) is a landscape equilibrium.

Similarly, the notion of gradual deviation is weaker than the unrestricted notion of deviation that yields strong Nash equilibrium. Thus, the set of landscape equilibria is larger than the set of strong Nash equilibria, and natural question is whether our assumptions yield the existence of strong Nash equilibria. A negative answer to this question follows from the result in Banerjee, by Konishi and Sönmez (2001) which asserts that a symmetric additively separable hedonic game may fail to admit a core stable configuration. Indeed, if the number of alternatives m exceeds the number of players n, a strong Nash equilibrium is necessarily core stable.[9] Thus, their example of nonexistence of a core stable configuration implies that the set of strong Nash equilibria in our model could be empty. This observation reinforces the importance of examination of a weaker concept of landscape equilibria.

Finally, let us point out that in the case of two blocs ($m = 2$) considered by AB, all profitable deviations are gradual. Thus, the sets of landscape equilibria and strong Nash equilibria coincide and our Theorem can be used to show the existence of a strong Nash equilibria in the case of two blocs (see Dower at al. (2020)).

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[9] Note that Banerjee, by Konishi and Sönmez (2001) do not explicitly include a set of alternatives in the description of their setting.

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