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Spatial competition with unit-demand functions*

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Abstract

This paper studies a spatial competition game between two firms that sell a homogeneous good at some pre-determined fixed price. A population of consumers is spread out over the real line, and the two firms simultaneously choose location in this same space. When buying from one of the firms, consumers incur the fixed price plus some transportation costs, which are increasing with their distance to the firm. Under the assumption that each consumer is ready to buy one unit of the good whatever the locations of the firms, firms converge to the median location: there is “minimal differentiation”. In this article, we relax this assumption and assume that there is an upper limit to the distance a consumer is ready to cover to buy the good. We show that the game always has at least one Nash equilibrium in pure strategy. Under this more general assumption, the “minimal differentiation principle” no longer holds in general. At equilibrium, firms choose “minimal”, “intermediate” or “full” differentiation, depending on this critical distance a consumer is ready to cover and on the shape of the distribution of consumers’ locations.

Keywords: Spatial competition games, horizontal differentiation, willingness to pay

1 Introduction

The choice of product characteristics - and strategic product differentiation in particular - is a central issue in Industrial Organization.

A large number of studies on this topic build on Hotelling’s seminal model of firm location (Hotelling [1929]). In Hotelling’s model, consumers are uniformly distributed on a line. Two firms selling an homogeneous good simultaneously and non-cooperatively choose a location on this line (stage 1). Once locations are observed, firms simultaneously choose a price at which they sell the good (stage 2). Consumers are ready to buy exactly one unit of the good (whatever the prices and the locations). They incur linear transportation costs when traveling on the line to purchase the good. Hotelling claims that this two-stage game has a unique Nash equilibrium, where both firms choose the same location at stage 1 - hence the name of “minimal differentiation principle” given to Hotelling’s result.

Note that the game is framed here in geographical terms, but there is an immediate analogy with a situation where firms, instead of a geographical location, choose some characteristics of

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their products in some space product à la Lancaster, and consumers differ in their preferences for product characteristics. In this interpretation, the counter-part of the transportation cost is the utility loss suffered by a consumer who consumes a product whose characteristics do not exactly match her preferred ones. In the paper, we will use the geography terminology, and talk about firms' "positions" or "locations", but all the results can also be equally interpreted in terms of more general product characteristics.

d'Aspremont et al. [1979] challenge Hotelling's convergence result, demonstrating that there is a flaw in the resolution of the price subgame stage: the price subgame has no equilibrium in pure strategies when firms are too close one from this other. Assuming that consumers have quadratic transportation costs (instead of linear as in the original article), they show that the price subgame always has a pure strategy equilibrium. They show that in that case, a "maximum differentiation" principle holds, firms locating at the two ends of the line. The powerful intuition behind this result is that firms differentiate to avoid too fierce a price competition at the second stage.¹ As summarized by Downs [1957], this two-stage model where firms first choose product characteristics and then choose prices offers the standard explanation in industrial organization as to why "(...) *firms generally do not want to locate at the same place in the product space. The reason is simply the Bertrand paradox: Two firms producing perfect substitutes face unbridled price competition (at least in a static framework). In contrast, product differentiation establishes clienteles ("market niches", in the business terminology) and allows firms to enjoy some market power over these clienteles. Thus, firms usually wish to differentiate themselves from other firms*" (Downs [1957], page 278).²

In the present paper, Hotelling's convergence result is also challenged, but on completely different grounds. We argue that softening the price competition is not the only force which may drive firms apart. We do so by relaxing Hotelling's assumption that the market is always covered, whatever the locations and prices of the firms. Instead of assuming perfectly inelastic demand, we assume unit demand functions: a consumer buys the good only if her valuation for the good is higher than the total cost, where the total cost is the price of the good augmented by the transportation cost. If for both firms, the total cost is lower than her valuation, the consumer buys from the firm with the lowest total cost (and randomizes equally between the two firms in case of equality). Under this more general assumption, if both firms are too far away from her location, a consumer might prefer not to buy the good. We will show that introducing this option to abstain/stay out of the market, can be a powerful force in favor of differentiation.

In order to make this argument as transparent as possible, we study a one-stage location game, where firms are assumed to sell the good at some pre-determined fixed price. This will allow us to clearly distinguish our effect from the one driven by the price competition.

Note that this fixed-price situation is interesting *per se*, since there are many situations where, for legal or technical reasons, price is not a free parameter in the competition. As noted by Downs [1957], "*There may exist legal or technical reasons why the scope of price competition is limited. For instance, the prices of airline tickets in the United States (before deregulation) were determined exogenously, as the price of gas and books in France once were*" (page 287). Some shops sell products whose price is exogenously determined, for instance newsstands, pharmacies, or franchises of brand

¹See also Economides [1986], Osborne and Pitchik [1987] and Bester et al. [1996] for further discussion on Hotelling's result.

²A few papers have also analyzed how uncertainty about demand impacts this incentive to differentiate, such as De Palma et al. [1985] and Meagher and Zauner [2004].

clothes for example.³

In this fixed-price spatial competition, if the demand is assumed to be perfectly inelastic, both firms choose the “median consumer” location, i.e. the location such that one half of the consumers lay on its left-hand side, and the other half lay on its right-hand side. This result holds whatever the form of the transportation costs and the distribution of consumers.⁴ The intuition behind this convergence result is quite powerful. Consider any situation where the firms choose different locations. Then both firms could increase their profit by moving closer to their opponent. Indeed, with such a move, each firm would win additional consumers (among those initially located between the two firms), without losing any consumers on the other side. This shows that at any equilibrium in pure strategies, the firms should converge.⁵

If instead there exists a maximum distance that consumers are ready to travel to buy the good, we show that the convergence result may not hold anymore. One may observe some “intermediate” or even “full differentiation”. To be more precise about what we mean by partial or full differentiation, we define the “potential attraction zone” of a firm as the set of consumers who prefer buying from this firm rather than not buying the good at all. We say that there is “partial differentiation” when firms choose different locations but their potential attraction zones intersect; and that there is “full differentiation” when the two potential attraction zones do not intersect (or intersect over a set of consumers of measure 0). We characterize all pure strategy equilibria, and discuss their properties under quite general assumptions about the transportation cost functions and the distribution of consumers. Assuming mild assumptions on the distribution of consumers⁶, our results are the following. If the maximum traveling distance is high enough, both firms converge to the median/modal location⁷ (the standard convergence result). Now, if this distance is small enough, firms diverge at equilibrium. To understand the main intuition behind this result, suppose that a firm has chosen the median/modal position. In that case, if its opponent also selects this central position, the two firms will have exactly the same potential attraction zones, and thus each

³In the survey by Gabszewicz and Thisse (1992) of the early literature on spatial competition, Section 4 (pages 298-302) is devoted to this fixed-price situation.

⁴This result has been known in political economy under the name of “Median Voter theorem”(see Black [1948], Tirole [1988]). In this interpretation, two political parties compete to attract voters, who are located along an “ideological”left-right axis. If parties seek to attract as many voters as possible, they will at equilibrium both choose the median location. Note that this analogy was already noted by Hotelling in his seminal article, where he writes: “*So general is this [agglomerative] tendency that it appears in the most diverse fields of competitive activity, even quite apart from what is called economic life. In politics it is strikingly exemplified. The competition for votes between the Republican and Democratic parties does not lead to a clear drawing of issues, an adoption of two strongly contrasted positions between which the voter may choose. Instead, each party strives to make its platform as much like the other’s as possible.*”(page 54)

⁵Note that the result crucially depends on the duopoly assumption: with more than two firms, the situation in which all firms converge at the median is no longer an equilibrium. Indeed, consider a pure location game with $n \geq 2$ firms. If all firms locate at the median, each gets a share $1/n$ of the consumers. By moving slightly to left or the right, a deviating firm could attract almost $1/2$ of the consumers. This simple argument shows that convergence of all firms at the median cannot be an equilibrium at soon at $n \geq 3$. In such a game, it has been shown that an equilibrium in pure strategy exists only under very restricted assumptions about the distribution of consumers, and when it does, firms do not all converge, see for example Eaton and Lipsey [1975] and Fournier [2019]. Peters et al. [2018] study a pure location model with congestion where consumers are uniformly distributed and each consumer selects one of the firms based on distances as well as the number of consumers visiting each firm. They provide conditions for the existence of subgame perfect Nash equilibrium, and show that firms do not converge when the number of firms is larger than two.

⁶We assume that the distribution is single-peaked, symmetric and that it has a continuous log-concave density. Most standard distributions satisfy these assumptions.

⁷Since we assume that f is single-peaked and symmetric, the median and modal location coincide.

will attract one half of the consumers who are located within acceptable distance of the central position. The latter firm may fare better in that case by avoiding this frontal competition, and moving somewhat to the left or the right. In doing so, it might win new consumers located “at the periphery”, although it will come at the cost of losing some “central” consumers. We expect the incentives to move away to be greater when the distribution is flatter (less concentration at the modal position) and when the width of the attraction zone is larger. It will be shown to be indeed the case. Depending of the width of the attraction zone compared to some indicator of the flatness of the distribution of consumers’ location, we can observe full convergence to the central position, intermediate differentiation, or complete differentiation (in the sense that no consumer is located at equilibrium within acceptable distance of both firms). In particular, some necessary and sufficient conditions on this ratio are provided for the convergence result to hold.

We are not the first to revisit Hotelling’s assumption of perfectly inelastic demand. Early contributions by Lerner and Singer [1937] and Smithies [1941] note the centrifugal forces that a more elastic demand may generate. Economides [1984] study a two-stage location-then-price Hotelling game where consumers have a finite valuation for the good. In that case, even with linear costs of transportation, a price equilibrium may exist at the second stage, and in the first stage, firms may differentiate.⁸ Imperfectly inelastic demand in a pure location game has been studied by Feldman et al. [2016] and Shen and Wang [2017], who consider a model where each seller has an interval of attraction, as it is the case in our model, but they suppose that consumers randomly select where to buy among attractive sellers. Contrary to our assumption, buyers do not necessarily buy from the closest place. Feldman et al. [2016] study the case of uniformly distributed consumers; whereas Shen and Wang [2017] study more general distributions. They prove the existence of pure Nash equilibrium, but do not describe it. In the political science literature (see Footnote 4 for the analogy between a fixed-price location game between firms and an electoral competition game between parties), Tirole [1988], Hinich and Ordeshook [1970] or more recently Xefteris et al. [2017] have also noted that if voters prefer to abstain when neither party is close enough to their ideal policy, differentiation may result at equilibrium. Xefteris et al. [2017] study a more general abstention function, and show that the game admits an equilibrium in mixed strategies (existence result). They characterize equilibria in pure strategies only under the assumption that voters are uniformly distributed and for a special case of the abstention function. Hinich and Ordeshook [1970] mostly focus on the comparison of parties’ objectives: plurality maximization versus vote maximization. In the latter case, which is the one we study in this paper, they show that differentiation can occur in equilibria. We provide a more complete characterization of all equilibria.⁹

The paper is organized as follows. The model is presented in more detail in section 2. Section 3 characterizes all Nash equilibria in which firms play pure strategies. Section 4 comments the results and discusses some of the assumptions. Section 5 contains the proofs.

2 The model

We study a (fixed-price) spatial competition game between two firms facing consumers with unit demand functions.

⁸However, because a price equilibrium does not exist for every pair of locations, a complete analysis of the spatial competition is impossible. The paper focuses on local firms’ deviations.

⁹In particular, because in their paper they mostly focus on first order conditions, they fail to notice that a continuum of asymmetric equilibria may exist under some configurations of the parameters, and that first order conditions are not necessarily sufficient.

- The two firms ($i = 1, 2$) produce the same homogeneous good. Firm $i \in \{1, 2\}$ produces quantity q at cost $\gamma_i(q)$. Firms sell the good at some identical pre-determined fixed price $p > 0$. Before selling the good, they simultaneously select locations x_1 and x_2 on the real line \mathbb{R} .
- A mass 1 of potential consumers is distributed on $X = \mathbb{R}$ according to a probability distribution that is absolutely continuous with respect to the Lebesgue measure. We denote f its density, and F its cumulative distribution. We focus our analysis on the set \mathcal{D} of distributions that have continuous log-concave densities (i.e. such that f can be written $f(x) = e^{g(x)}$ where g is a concave function), and such that f is symmetric around 0 and strictly decreasing on \mathbb{R}^+ . The analysis would be identical if the distribution was symmetric around a mode different from 0. The above hypotheses describe a very large class \mathcal{D} of distributions that contains for example the normal (centered) distributions, the Laplace distributions, the symmetric exponential distributions, the logistic distributions, the symmetric gamma distributions, the symmetric extreme value distributions, etc.¹⁰
- All consumers have the same valuation for the good $v > 0$. They also incur transportation costs: they have a utility loss of traveling a distance $d \geq 0$ that is denoted $c(d)$. We suppose that $c(0) = 0$ and that c is strictly increasing and continuous. If a consumer travels a distance d to buy the good at the pre-determined price p , she gets the total utility:

$$u = v - p - c(d).$$

If she doesn't buy the good, her utility is normalized to 0. Assuming that $v - p > 0$, u is positive whenever the distance d is smaller than δ , where:

$$\delta := \begin{cases} c^{-1}(v - p) > 0 & \text{if } v - p \leq \lim_{d \rightarrow \infty} c(d), \\ +\infty & \text{otherwise.} \end{cases} \quad (1)$$

Parameter $\delta > 0$ denotes the maximal distance that a consumer is ready to travel to buy the good. It is strictly increasing in the valuation of the good (v) and decreasing in its price (p). Under these assumptions, a consumer buys from the closest firm if her distance to the firm is smaller than δ (randomly choosing a firm if both firms are equidistant from her own position), and she doesn't buy otherwise.

- We assume that firms serve all the demand they face at price p , and that they maximize their profit. When a quantity q of consumers buy from firm i , it makes a profit equal to $p \times q - \gamma_i(q)$. We assume $\gamma'_i \geq 0$, $\gamma''_i \geq 0$ and $\gamma'_i(1) < p$, which imply that this profit function is strictly increasing with respect to q . Under these assumptions, maximizing its profit is equivalent for the firm to maximizing the quantity it sells.
- We can now formally define the 2-player game $\mathcal{H}(f, \delta)$ associated to distribution f and parameter δ . The firms simultaneously select locations x_1 and x_2 in \mathbb{R} . We denote by $q_i(x_1, x_2)$ the quantity of consumers who buy from firm i when players choose locations x_1 and $x_2 \in \mathbb{R}$. Since for a firm, maximizing its profit is equivalent to maximizing the quantity of consumers who buy from this firm, we define the payoff of firm i as being $q_i(x_1, x_2)$. Given our assump-

¹⁰In Section 4.3, we also study the case of a uniform distribution of consumers.

tions about consumer behavior, the payoffs of the players are defined by:

$$q_i(x_1, x_2) := \begin{cases} \int_{\{t : \{|x_i - t| \leq \delta \text{ and } |x_i - t| = \min\{|x_1 - t|, |x_2 - t|\}\}} f(t) dt & \text{if } x_1 \neq x_2, \\ \frac{1}{2} \int_{\{t : |x_i - t| \leq \delta\}} f(t) dt & \text{if } x_1 = x_2. \end{cases}$$

We now introduce a number of definitions that will be useful to present our main results.

Definition “Potential attraction zones” : We call *potential attraction zone* of a firm the set of locations such that consumers at these locations prefer buying from this firm rather than not buying the good at all. Formally, the potential attraction zone of firm i when locations are (x_1, x_2) , denoted by $A_i(x_1, x_2)$, is $A_i(x_1, x_2) := \{t : |x_i - t| \leq \delta\}$.

Definition “No differentiation”/ “Full convergence” : We say that at profile of locations (x_1, x_2) , there is *no differentiation* (or *full convergence*) if the potential attraction zones of the two firms exactly coincide. Formally, this is the case if $A_1(x_1, x_2) = A_2(x_1, x_2)$. Note that this happens if and only if $x_1 = x_2$.

Definition “Partial differentiation” : We say that at profile of locations (x_1, x_2) , there is *partial differentiation* if the potential attraction zones of the two firms partially overlap. Formally, this means that the following two conditions are simultaneously satisfied: (i) the two potential attraction zones $A_1(x_1, x_2)$ and $A_2(x_1, x_2)$ intersect over a set of consumers of positive measure, (ii) $x_1 \neq x_2$.

Definition “Full differentiation” : We say that at profile of locations (x_1, x_2) , there is *full differentiation* if the potential attraction zones of the two firms do not intersect, or intersect over a set of consumers of measure 0.

Note that if $\delta = +\infty$, then for any distribution in \mathcal{D} we have that $(0, 0)$ is the unique equilibrium (Median Voter theorem). In the following we focus on the case where $\delta \in]0, +\infty[$.

In the next section, we characterize all Nash equilibria in pure strategies of the games $\mathcal{H}(f, \delta)$ for $\delta \in]0, +\infty[$ and $f \in \mathcal{D}$.

3 Equilibria

In this section, we characterize all Nash equilibria in pure strategies of the games $\mathcal{H}(f, \delta)$. We will show that for any $\delta \in]0, +\infty[$ and $f \in \mathcal{D}$, the game $\mathcal{H}(f, \delta)$ always has at least one equilibrium in pure strategy.¹¹ We present these equilibria according to the level of differentiation they entail. The main results of the paper are Proposition 1, Proposition 2 and Proposition 3, which show that a pure strategy equilibrium always exists, and characterize the set of equilibria. They deal respectively with equilibria inducing no, partial and full differentiation.

¹¹We know from Xefteris et al. [2017](Proposition 1) that the game admits a symmetric Nash equilibrium (possibly) in mixed strategies.

We will show that the necessary and sufficient conditions for the existence of these different types of equilibria only depend on parameter δ (the maximum distance a consumer is ready to travel to buy the good) and on a parameter κ that is defined as the positive solution of the equation:

$$\frac{1}{2}f(0) = f(\kappa). \quad (2)$$

Note that Equation (2) admits exactly one positive solution. Indeed, f is continuous, strictly decreasing on \mathbb{R}^+ with $f(0) > 0$ and $\lim_{+\infty} f(x) = 0$ (because f is a decreasing probability density). Parameter κ is the time it takes for f to decrease to half its modal value. It measures how 'flat' the consumer distribution is.

Proposition 1 (No differentiation)

- (i) A (pure strategy) equilibrium with no differentiation exists if and only if $\delta \geq \kappa$.
- (ii) In this case, the unique equilibrium of the game is $(0, 0)$: both firms converge at the median/modal position.

Proposition 2 (Partial differentiation)

- (i) A (pure strategy) equilibrium with partial differentiation exists if and only if $\frac{\kappa}{2} < \delta < \kappa$.
- (ii) In this case, the unique equilibrium of the game is $(\delta - \kappa, \kappa - \delta)$ (up to a permutation of the players).

Proposition 3 (Full differentiation)

- (i) A (pure strategy) equilibrium with full differentiation exists if and only if $\delta \leq \frac{\kappa}{2}$.
- (ii) In this case, there is a unique symmetric equilibrium $(-\delta, \delta)$. Besides, as soon as $\delta < \frac{1}{2}\kappa$, there is also a continuum of asymmetric Nash equilibria, where firms are located at distance exactly 2δ one from the other.
More specifically, (up to a permutation of the players) the whole set of equilibria is $(m - \delta, m + \delta)$ for $m \in [-\alpha, \alpha]$, where $\alpha \in [0, \delta]$ is uniquely defined by:

$$\alpha := \max\{t \in [0, \delta] : \frac{1}{2}f(t) \leq f(t + 2\delta)\}. \quad (3)$$

The proof of these three propositions is provided in the Appendix (Section 5.1). Before we give in the next section an economic intuition for these main results, note that parameters κ and α are easy to derive from the distribution f of consumers, as illustrated in the following examples.

Example 4 (Normal distribution)

Suppose that consumers are distributed according to a normal distribution $\mathcal{N}(0, \sigma^2)$, i.e. $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$ with $\sigma > 0$.

Then: $\kappa = \sigma\sqrt{2\ln(2)}$ and $\alpha = \min\left\{\delta, \frac{\sigma^2\ln(2)}{2\delta} - \delta\right\}$.

Example 5 (Laplace distribution)

Suppose that consumers are distributed according to a Laplace distribution $\mathcal{L}(0, \beta)$, i.e. $f(x) = \frac{1}{2\beta}e^{-\frac{|x|}{\beta}}$ with $\beta > 0$.

Then: $\kappa = \beta\ln(2)$ and $\alpha = \delta$.

4 Comments and discussion

In this section, we first comment upon our main results, and give the main economic intuition. We then propose some efficiency considerations. Last, we discuss how our results should be adapted in the case of a uniform distribution of consumers.

4.1 Comments

As explained in the introduction, we explain differentiation by a direct demand-driven effect, stemming from the fact that there is a maximal distance consumers are ready to cover to buy the good. To understand how this effect operates, consider again the powerful argument leading to convergence in the case of a perfectly inelastic demand ($\delta = +\infty$). Suppose that the firms choose different locations. Then each of them can unambiguously increase its profit by moving closer to its competitor. Indeed, consider the firm initially located on the right hand side, say Firm 2. By moving closer to its opponent (that is, moving to the left),

1. Firm 2 does not lose any consumers on its right-hand side (by assumption, these consumers will still buy from Firm 2), and
2. Firm 2 attracts a larger quantity of consumers located between the two firms.

When we relax the assumption that consumers are ready to buy the good whatever the locations of the firms, this latter argument is still active: There is still a force towards convergence, due to the willingness to compete for the "central" consumers. But the former argument according to which Firm 2 does not lose any consumers on its right-hand side is no longer valid. In that case, by moving closer to its competitor, the firm may lose the "peripheral" consumers who were indifferent between buying from Firm 2 and not buying the good. There are now two types of relevant marginal consumers: those who are located between the two firms and could potentially buy from both, and those who are located at the border of the domains of attraction and who are indifferent between buying from the closest firms and not buying the good. Depending on how this trade-off is solved, there can be no, partial or full differentiation at equilibrium.

Propositions 1, 2 and 3 taken together show that the regime regarding the firm differentiation depends on the ratio $\frac{\delta}{\kappa}$. Interestingly, the characterization of the equilibria does not depend on the details of the transport cost function. The only thing that matters is the maximal distance the consumer is ready to cover to buy the good (δ) and the shape of the distribution of consumers, as summarized by parameter κ , where κ is the time it takes for the density f to decrease to half its initial value (see Equation (2)).

Case $\frac{\delta}{\kappa} \geq 1$: No differentiation. In that case, there is a unique Nash equilibrium, at which both firms choose to locate at the median position. As noticed in the introduction, the intuition suggests that firms will converge at the center if the distance a consumer is ready to cover to buy the good is large enough (δ large) or if consumers are sufficiently numerous around the center. Proposition 1 provides a precise quantification for these conditions: The situation where both firms converge is an equilibrium if and only if $\delta \geq \kappa$.

To understand the intuition behind this condition, assume that one firm, say firm 1, chooses the modal median location (0). If its opponent also selects this central position, both firms will have exactly the same potential attraction zones: $A_1(0, 0) = A_2(0, 0) = [-\delta, \delta]$, and each will attract one half of the consumers who are located within acceptable distance of 0. Therefore, the payoff for firm 2 is $q_2(0, 0) = \frac{F(\delta) - F(-\delta)}{2}$. Since f is assumed to be symmetric, note that $q_2(0, 0) = F(\delta) - F(0)$.

If firm 2 moves slightly to the right, say by some small $\varepsilon > 0$, its potential attraction zone will now be $A_2(0, \varepsilon) = [\delta - \varepsilon, \delta + \varepsilon]$. By doing so, it will attract all consumers located between $\frac{\varepsilon}{2}$ and $\delta + \varepsilon$, and $q_2(0, \varepsilon) = F(\delta + \varepsilon) - F(\frac{\varepsilon}{2})$. The move is beneficial if the mass of consumers located at δ is larger than half the mass of the consumers located at 0. This condition is $\frac{1}{2}f(0) < f(\delta)$, which is exactly the condition $\delta < \kappa$ (remember that κ is the time it takes for the density f to decrease to half its initial value). The assumptions about the logconcavity of f are sufficient to guarantee that the examination of first order conditions are sufficient to characterize equilibrium. We also show in the appendix that there is no equilibrium with convergence at another location than 0.

Consider Example 4. In the case of a normal distribution with variance σ^2 , the condition $\delta \geq \kappa$ can be written as $\delta \geq \sigma \times \sqrt{2 \ln(2)}$. Note that $\sqrt{2 \ln(2)}$ is approximately equal to 1.18. This shows that, for the no differentiation principle to hold, the total length of a firm's potential attraction zone (2δ) has to be approximately at least as large as 2.35 time the standard deviation of the distribution of consumer locations. This figure is quite high. By instance, one may check that when $\delta/\sigma = \sqrt{2 \ln(2)}$, the potential attraction zone of a firm located at the center covers over 75% of the population.

Case $\frac{1}{2} < \frac{\delta}{\kappa} < 1$: Partial differentiation. In that case, for each (δ, κ) , there is a unique Nash equilibrium, in which the two firms engage in "partial differentiation". The unique equilibrium is symmetric, with firms choosing locations $(\delta - \kappa, \kappa - \delta)$, where $0 < \kappa - \delta < \delta$. The distance between the two firms is $2(\kappa - \delta) < 2\delta$: a positive mass of consumers, in particular the median consumer, are located within acceptable distance of both firms. Note that the distance between the two firms is decreasing in δ .

To understand the intuition behind this result, remember that, as discussed in the case of full convergence, whenever the potential attraction zones of the two firms intersect on a set of positive mass, a firm faces a trade-off. Indeed, by moving away from its opponent, it could attract new "peripheral" consumers, who were not buying the good at the initial locations. But this move would imply losing the "central" consumers who were initially indifferent between the two firms, a share 1/2 of which were buying from this firm in the initial situation. The equilibria described in that case are characterized by the fact that these two effects exactly offset one another. Besides, one can show (see Appendix) that only symmetric equilibria exist in that case: firms choose symmetric locations, say $(-x, +x)$, $x \geq 0$. Consider the firm at location x . For this firm, the "peripheral" consumers that it could attract by moving further to the right are those located around $x + \delta$, whereas the "central" consumers who are initially indifferent between the two firms are those located around 0. The condition stating that the two opposite effects exactly counter-balance is therefore $\frac{1}{2}f(0) = f(x + \delta)$, which yields $x = \kappa - \delta$. Note that this equilibrium only exists when the resulting distance is strictly lower than 2δ , that is, when $2\kappa - 2\delta < 2\delta$ ($\frac{1}{2} < \frac{\delta}{\kappa}$).

When the ratio $\frac{\delta}{\kappa}$ is small enough so that this condition is no longer satisfied, we move to a situation of full differentiation.

Case $\frac{\delta}{\kappa} \leq \frac{1}{2}$: Full differentiation. In that case, for each pair (δ, κ) , there is a unique symmetric equilibrium $(-\delta, \delta)$; besides, as soon as $\frac{\delta}{\kappa} < \frac{1}{2}$, there is also a continuum of asymmetric Nash equilibria. In all these equilibria, the two firms are located at distance 2δ one from the other: the potential attraction zones of the two firms do not intersect (more precisely, a mass zero of consumers simultaneously belong to both potential attraction zones).

In the case of a normal distribution, the condition $\delta \leq \frac{1}{2}\kappa$ states that the domain of attraction of a firm located at the center has to cover at most 45% of the population.

4.2 Efficiency of equilibria

In this subsection, we compare equilibrium locations to these which would be optimal either from the consumers' point of view (consumer surplus maximizing locations) or from the firms' perspective (aggregate profit maximizing locations). Aggregate profit maximizing locations and consumer surplus maximizing locations are described in the following proposition.

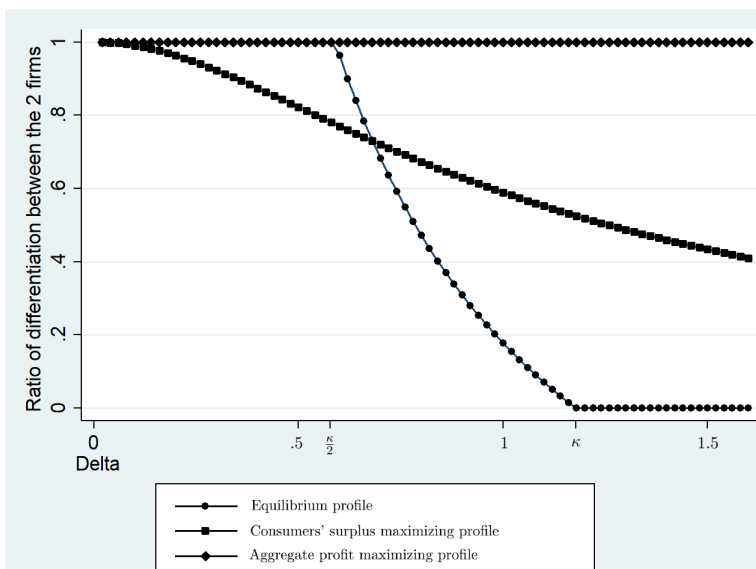
Proposition 6 (Efficiency)

(1) (*Aggregate profit maximizing locations*) Assume that both firms have the same production functions ($\gamma_1 = \gamma_2$). Then the location profile maximizing the sum of the firms' profits is $(-\delta, \delta)$, which entails full differentiation.

(2) (*Consumer surplus maximizing locations*) The location profile maximizing consumer surplus entails partial differentiation. The detail of the location profile maximizing consumer surplus depends on the transportation cost function $c(\cdot)$.

The proof of Proposition 6 is provided in the appendix, section 5.2.

Figure 1 provides an illustration of Proposition 6 when consumers are distributed according to a standard normal distribution. It plots the level of differentiation as a function of δ : (i) in the Nash equilibria profile, (ii) in the aggregate profit maximizing profile, and (iii) in the consumers' surplus maximizing profile. More precisely, on the vertical axis, it shows the ratio of the resulting distance between the two firms to the minimal distance between the firms that guarantees full differentiation (2δ). A ratio of 1 means full differentiation and a ratio of 0 means no differentiation. As emphasized in Proposition 6, the profile of locations that maximizes consumer surplus depends on the transportation cost function; in this example, we choose linear transportation costs $c(d) = d$. As noted earlier (see Example 4), when consumers are distributed according to $\mathcal{N}(1, \kappa = \sqrt{2 \ln(2)})$, which is approximately equal to 1.18.



This figure illustrates that there exists a unique value of $\delta \in]0, +\infty[$ such that the equilibrium and the consumers surplus maximizing profile coincide. For smaller value of δ , the distance between firms at equilibrium is strictly larger than it would be in a consumer surplus maximizing profile, for larger value of δ , it is strictly smaller.

4.3 Uniform distribution of consumers

So far, we have supposed that the distribution of consumers is symmetric around 0, log-concave and strictly decreasing on \mathbb{R}^+ . These assumptions are weak as this case includes most of standard distributions (Normal, Laplace, Logistic, etc.). However, a large part of the literature on horizontal differentiation has studied the particular case of consumers uniformly distributed on an interval. In this subsection we discuss this case, which is not included in our general model as in the uniform case, the function f is no longer strictly decreasing on \mathbb{R}^+ .

We consider the case where consumers are distributed uniformly in the interval $X = [-\kappa, \kappa]$, for some $\kappa > 0$. We choose this notation to be consistent with our previous notation (see Definition (2)). Indeed, consider the following alternative (more general) definition for κ :

$$\kappa = \inf \left\{ t \in \mathbb{R}^+ : \frac{1}{2}f(0) > f(t) \right\}.$$

It coincides with Definition (2) when f is continuous and strictly decreasing on \mathbb{R}^+ , but can also be used in the uniform case. We still assume that the firms can choose any location on the real line \mathbb{R} .

Proposition 7 shows that most of the results stated in Propositions 1, 2 and 3 extend to the uniform case, the only adaption to be made being the characterization of equilibria with full differentiation.

Proposition 7 (Uniform distribution)

Assume that consumers are uniformly distributed on $[-\kappa, \kappa]$.

Proposition 1, Proposition 2 and Part (i) of Proposition 3 extend to this uniform case.

The only difference lays with the characterization of equilibria with full differentiation (Part (ii) of Proposition 3): In the uniform case, if $\delta \leq \frac{\kappa}{2}$, there exists a continuum of equilibria where the two firms locate at distance at least 2δ one from the other. More specifically, supposing without loss of generality that $x_1 \leq x_2$, the whole set of equilibria is (x_1, x_2) for $-\kappa + \delta \leq x_1 \leq x_1 + 2\delta \leq x_2 \leq \kappa - \delta$.

The proof of this proposition is provided in section 5.3 in the appendix. Proposition 5.3 shows that in the case of a uniform distribution, the three regimes of no, partial, and full differentiation still exist. The main difference is that now, in the case of full differentiation, the firms can locate at a distance strictly larger than 2δ one from the other at equilibrium.

5 Appendix

5.1 Proof of Proposition 1, Proposition 2, and Proposition 3 (Characterization of equilibria)

The proof of the propositions rely on the following lemmas.

Lemma 8

If the density f is symmetric around 0 and strictly decreasing on \mathbb{R}_+ , then the cumulative function F satisfies the following properties:

For any $\delta > 0$,

- (1) if $x < 0$, then $F(x + \delta) - F(x) > F(x) - F(x - \delta)$*
- (2) if $x > 0$, then $F(x + \delta) - F(x) < F(x) - F(x - \delta)$.*

Proof. of Lemma 8

(1) Note that

$$\begin{aligned} F(x + \delta) - F(x) &= \int_{t=x}^{t=x+\delta} f(t)dt = \int_{t=x}^{t=-x} f(t)dt + \int_{t=-x}^{t=x+\delta} f(t)dt \\ F(x) - F(x - \delta) &= \int_{t=x-\delta}^{t=x} f(t)dt = \int_{t=x-\delta}^{t=-x-\delta} f(t)dt + \int_{t=-x-\delta}^{t=x} f(t)dt \end{aligned}$$

Assume $x < 0$.

Consider first the case $x + \delta \leq -x$. Then for all $t \in [x, x + \delta]$, $f(t) \geq f(x)$ with a strict inequality if $x < t < x + \delta$. Besides, for all $t \in [x - \delta, x]$, $f(t) \leq f(x)$ with a strict inequality if $t < x$. This shows that in that case $F(x + \delta) - F(x) > F(x) - F(x - \delta)$.

Consider now the case $x + \delta \geq -x$. By symmetry of f , $\int_{t=-x}^{t=x+\delta} f(t)dt = \int_{t=-x-\delta}^{t=x} f(t)dt$. For all $t \in [x - \delta, -x - \delta]$, $f(t) < f(x)$. And for all $t \in [x, -x]$, $f(t) \geq f(x)$ with a strict inequality if $x < t < -x$. This shows that in that case too $F(x + \delta) - F(x) > F(x) - F(x - \delta)$.

(2) By symmetry, the proof is the same as for claim (1). ■

Lemma 9 provides a few useful remarks about the structure of equilibria and best responses.

Lemma 9

- (1) $x_2 = 0$ is the unique best response to any x_1 such that $|x_1| \geq 2\delta$.
- (2) Any best response to $x_1 \in [-2\delta, 0[$ belongs to the interval $]x_1, x_1 + 2\delta]$.
- (3) Any best response to $x_1 \in]0, 2\delta]$ belongs to the interval $[x_1 - 2\delta, x_1[$.
- (4) Any best response to $x_1 = 0$ belongs to the interval $[-2\delta, 2\delta]$.
- (5) Any equilibrium (x_1, x_2) such that $x_1 \leq x_2$ satisfies $x_1 \in [-2\delta, 0]$ and $x_2 \in [0, 2\delta]$.

Proof. of Lemma 9

(1) If $|x_1| \geq 2\delta$ then $q_2(x_1, 0) = F(\delta) - F(-\delta)$, which is the strictly maximal feasible payoff since f is symmetric around 0 and strictly decreasing on \mathbb{R}^+ .

(2) Let $x_1 \in [-2\delta, 0[$.

First note that because f is strictly decreasing on \mathbb{R}^+ , if $x_2 > x_1 + 2\delta$, then $q_2(x_1, x_2) < q_2(x_1, x_1 + 2\delta)$, and if $x_2 < x_1 - 2\delta$ then $q_2(x_1, x_2) < q_2(x_1, x_1 - 2\delta)$. Therefore the best response belongs to the interval $[x_1 - 2\delta, x_1 + 2\delta]$.

But, because f is symmetric, if $x_2 \in [x_1 - 2\delta, x_1[$ then $q_2(x_1, x_2) < q_2(x_1, -x_2)$, which shows that the best response belongs to the interval $]x_1, x_1 + 2\delta]$.

It remains to show that $x_2 = x_1$ cannot be a best response against $x_1 \in [-2\delta, 0[$. Note that for $\varepsilon > 0$ small enough,

$$\begin{aligned} q_2(x_1, x_1) &= \frac{F(x_1 + \delta) - F(x_1 - \delta)}{2} \\ q_2(x_1, x_1 + \varepsilon) &= F(x_1 + \varepsilon + \delta) - F(x_1 + \frac{\varepsilon}{2}) \end{aligned}$$

Therefore

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} q_2(x_1, x_1 + \varepsilon) - q_2(x_1, x_1) = \frac{F(x_1 + \delta) + F(x_1 - \delta)}{2} - F(x_1),$$

which by Lemma 8 is positive since by assumption $x_1 < 0$. This concludes the proof of claim (2).

(3) By symmetry, the proof is the same as for claim (2).

(4) Let $x_1 = 0$. If $x_2 > 2\delta$, then $q_2(0, x_2) = F(x_2 + \delta) - F(x_2 - \delta)$, which is strictly decreasing in x_2 for $x_2 \in]2\delta, +\infty[$. Therefore, $x_2 > 2\delta$ cannot be a best response against $x_1 = 0$. By symmetry,

$x_2 < -2\delta$ cannot be a best response against $x_1 = 0$.

(5) Let (x_1, x_2) be an equilibrium with $x_1 \leq x_2$.

Suppose first that $|x_1| > 2\delta$. Then, $x_2 = 0$ according to claim (1), and claim (4) contradicts the fact that x_1 is a best response to x_2 , so it must be the case that $|x_1| \leq 2\delta$.

Suppose now that $0 < x_1 \leq 2\delta$. Then, by claim (3), $x_2 < x_1$. Since by assumption, $x_1 \leq x_2$, it implies a contradiction. Therefore $x_1 \in [-2\delta, 0]$.

Similar arguments show that $x_2 \in [0, 2\delta]$, which concludes the proof of claim (5). ■

According to Lemma 9, at any equilibrium (x_1, x_2) such that $x_1 \leq x_2$, we have that $x_1 \in [-2\delta, 0]$ (claim (5)) and Player 2's best response against x_1 belongs to the interval $]x_1, x_1 + 2\delta]$ (claim (2)). Player 2's payoff when it selects x_2 in this interval is:

$$q_2(x_1, x_2) = F(x_2 + \delta) - F\left(\frac{x_2 + x_1}{2}\right),$$

and

$$\frac{\partial q_2}{\partial x_2}(x_1, x_2) = f(x_2 + \delta) - \frac{1}{2}f\left(\frac{x_2 + x_1}{2}\right). \quad (4)$$

Note that when $x_2 = x_1 + 2\delta$, we only compute a left derivative.

Remark 10 Before turning to the proofs of the propositions, let us introduce the function Ψ_z , where for $t, z \in \mathbb{R}$, Ψ_z is defined as follows:

$$\Psi_z(t) := \frac{f(t)}{f(t+z)}. \quad (5)$$

Because of the log-concavity of f , we have that:

- For any $z > 0$, $t \mapsto \Psi_z(t)$ is increasing in $t \in \mathbb{R}$,
- For any $z < 0$, $t \mapsto \Psi_z(t)$ is decreasing in $t \in \mathbb{R}$.

Indeed, the log-concave function f can be written e^g , where g is a concave function. Therefore $\Psi'_z(t) = (g'(t) - g'(t+z))e^{g(t)-g(t+z)}$ has the same sign than $g'(t) - g'(t+z)$. The monotony of Ψ_z follows from the fact that g' is decreasing.

We are now ready to complete the proofs of propositions 1, 2 and 3.

Proof of Propostion 1

Assume that (x_1, x_2) is an equilibrium such that $x_1 = x_2$. Lemma 9 implies that $x_1 = x_2 = 0$. Indeed, Claim (5) states that $x_1 \in [-2\delta, 0]$, and Claim (2) states that any best response to $x_1 \in [-2\delta, 0[$ is strictly larger than x_1 . Therefore, there exists an equilibrium with no differentiation if and only if $(0, 0)$ is a Nash equilibrium. In that case, it is the unique equilibrium with no differentiation.

It remains to show $(0, 0)$ is a Nash equilibrium if and only if $\kappa \leq \delta$. A necessary condition for $(0, 0)$ to be a Nash equilibrium is that

$$\lim_{\substack{x_2 \rightarrow 0 \\ x_2 > 0}} \frac{\partial q_2}{\partial x_2}(0, x_2) \leq 0,$$

which given equation (4) can be written as $f(\delta) \leq \frac{1}{2}f(0)$. This is exactly condition $\kappa \leq \delta$. Although the function $x_2 \mapsto g_2(x_1, x_2)$ is in general discontinuous in $x_2 = x_1$, it is continuous in the particular case where $x_1 = 0$ (because f is symmetric), so that it's enough to consider the derivative in $x_2 > 0$, $x_2 \rightarrow 0$.

Last, let us show that when condition $\kappa \leq \delta$ holds, $\frac{\partial q_2}{\partial x_2}(0, x_2) < 0$ for all $x_2 \in]0, 2\delta]$, which will guarantee that $x_2 = 0$ is a best response against $x_1 = 0$. Given equation (4), for $x_2 \in]0, 2\delta]$:

$$\frac{\partial q_2}{\partial x_2}(0, x_2) = f(x_2 + \delta) - \frac{1}{2}f\left(\frac{x_2}{2}\right).$$

Note that when $x_2 = 2\delta$, we only compute a left derivative. We have:

$$\frac{f\left(\frac{x_2}{2}\right)}{f(x_2 + \delta)} = \Psi_{\frac{x_2}{2} + \delta}\left(\frac{x_2}{2}\right) \geq \Psi_{\frac{x_2}{2} + \delta}(0) = \frac{f(0)}{f\left(\frac{x_2}{2} + \delta\right)} > \frac{f(0)}{f(\delta)},$$

where the first inequality follows from the observation that $\frac{x_2}{2} + \delta > 0$ and $\frac{x_2}{2} > 0$, and the second inequality follows from the fact that $0 < \delta < \frac{x_2}{2} + \delta$ and f is strictly decreasing on \mathbb{R}^+ . Since by assumption $f(\delta) \leq \frac{1}{2}f(0)$, this proves that $\frac{\partial q_2}{\partial x_2}(0, x_2) < 0$ for all $x_2 \in]0, 2\delta]$, and $x_2 = 0$ is a best response against $x_1 = 0$.

The same argument shows that when $f(\delta) \leq \frac{1}{2}f(0)$, $x_1 = 0$ is a best response against $x_2 = 0$.

Therefore, condition $\kappa \leq \delta$ is a necessary and sufficient condition for $(0, 0)$ to be a Nash equilibrium. This concludes the proof of Proposition 1.

Proof of Proposition 2. Assume that x_1 *leq* x_2 and that (x_1, x_2) is an equilibrium with partial differentiation, meaning that $0 < x_2 - x_1 < 2\delta$.

According to Lemma 9, at any equilibrium (x_1, x_2) such that $x_1 \leq x_2$, it must be the case that $x_1 \in [-2\delta, 0] \cap [x_2 - 2\delta, x_2[$ and $x_2 \in [-2\delta, 0] \cap]x_1, x_1 + 2\delta]$. Since $x_2 - 2\delta < x_1 < x_2 < x_1 + 2\delta$, the first-order conditions imply that $\frac{\partial q_1}{\partial x_1}(x_1, x_2) = \frac{\partial q_2}{\partial x_2}(x_1, x_2) = 0$, and therefore that:

$$f(x_2 + \delta) = f(x_1 - \delta),$$

which, when $x_2 - x_1 < 2\delta$, is possible only if $x_2 + x_1 = 0$: An equilibrium with partial differentiation is necessarily symmetric.

Assume that $(-x_2, x_2)$ is a symmetric equilibrium with $x_2 > 0$. From equation (4), it must be the case that

$$\frac{\partial q_2}{\partial x_2}(-x_2, x_2) = 0 \Leftrightarrow f(x_2 + \delta) = \frac{1}{2}f(0).$$

If $(-x_2, x_2)$ is an equilibrium partial differentiation, it must be the case that $x_2 < \delta$. By definition of κ (see (2)), equation $f(x_2 + \delta) = \frac{1}{2}f(0)$ has a solution in $]0, \delta[$ if and only if $\frac{\kappa}{2} < \delta < \kappa$. In that case, the solution is unique and is $x_2 = \kappa - \delta$.

Assume that $\frac{\kappa}{2} < \delta < \kappa$. It remains to show $(\delta - \kappa, \kappa - \delta)$ is a Nash equilibrium.

Let us first show that $x_2 = \kappa - \delta$ is a best response against $x_1 = \delta - \kappa$. It is sufficient to prove that:

$$\begin{aligned} \frac{\partial q_2}{\partial x_2}(\delta - \kappa, x_2) &> 0 \text{ if } \delta - \kappa < x_2 < \kappa - \delta \\ \frac{\partial q_2}{\partial x_2}(\delta - \kappa, x_2) &< 0 \text{ if } \kappa - \delta < x_2 < 3\delta - \kappa \end{aligned}$$

Given (4), for $x_2 \in]\delta - \kappa, 3\delta - \kappa]$:

$$\frac{\partial q_2}{\partial x_2}(\delta - \kappa, x_2) = f(x_2 + \delta) - \frac{1}{2}f\left(\frac{\delta - \kappa + x_2}{2}\right).$$

Consider first the case $\delta - \kappa < x_2 < \kappa - \delta$. Note that

$$\frac{f\left(\frac{\delta - \kappa + x_2}{2}\right)}{f(x_2 + \delta)} = \Psi_{\frac{\delta + \kappa + x_2}{2}}\left(\frac{\delta - \kappa + x_2}{2}\right) \leq \Psi_{\frac{\delta + \kappa + x_2}{2}}(0) = \frac{f(0)}{f\left(\frac{\delta + \kappa + x_2}{2}\right)} < \frac{f(0)}{f(\kappa)},$$

where the first inequality follows from the observation that $\frac{\delta + \kappa + x_2}{2} > 0$ and $\frac{\delta - \kappa + x_2}{2} < 0$, and the second inequality follows from the fact that $0 < \frac{\delta + \kappa + x_2}{2} < \kappa$ and f is strictly decreasing on \mathbb{R}^+ . Since by assumption $f(\kappa) = \frac{1}{2}f(0)$, this proves that $\frac{\partial q_2}{\partial x_2}(0, x_2) > 0$ for all x_2 such that $\delta - \kappa < x_2 < \kappa - \delta$.

Consider now the case $\kappa - \delta < x_2 < 3\delta - \kappa$. Note that

$$\frac{f\left(\frac{\delta - \kappa + x_2}{2}\right)}{f(x_2 + \delta)} = \Psi_{\frac{\delta + \kappa + x_2}{2}}\left(\frac{\delta - \kappa + x_2}{2}\right) \geq \Psi_{\frac{\delta + \kappa + x_2}{2}}(0) = \frac{f(0)}{f\left(\frac{\delta + \kappa + x_2}{2}\right)} > \frac{f(0)}{f(\kappa)},$$

where the first inequality follows from the observation that $\frac{\delta + \kappa + x_2}{2} > 0$ and $\frac{\delta - \kappa + x_2}{2} > 0$, and the second inequality follows from the fact that $0 < \kappa < \frac{\delta + \kappa + x_2}{2}$ and f is strictly decreasing on \mathbb{R}^+ . Since by assumption $f(\kappa) = \frac{1}{2}f(0)$, this proves that $\frac{\partial q_2}{\partial x_2}(0, x_2) < 0$ for all x_2 such that $\kappa - \delta < x_2 < 3\delta - \kappa$.

This shows that $x_2 = \kappa - \delta$ is a best response against $x_1 = \delta - \kappa$.

A symmetric argument shows that $x_1 = \delta - \kappa$ is a best response against $x_2 = \kappa - \delta$.

Proposition 2 is proved.

Proof of proposition 3. It follows from Lemma 9 that any equilibrium with full differentiation is necessarily of the form $(a - \delta, a + \delta)$ for some $a \in [-\delta, \delta]$. Besides, $(a - \delta, a + \delta)$ is an equilibrium only if:

$$\begin{aligned} \lim_{\substack{x_1 \rightarrow a - \delta \\ x_1 > a - \delta}} \frac{\partial q_1}{\partial x_1}(x_1, a + \delta) &= \frac{1}{2}f(a) - f(a - 2\delta) \leq 0 \Leftrightarrow \Psi_{-2\delta}(a) \leq 2, \\ \lim_{\substack{x_2 \rightarrow a + \delta \\ x_2 < a + \delta}} \frac{\partial q_2}{\partial x_2}(a - \delta, x_2) &= f(a + 2\delta) - \frac{1}{2}f(a) \geq 0 \Leftrightarrow \Psi_{2\delta}(a) \leq 2, \end{aligned}$$

where $\Psi_z(\cdot)$ is defined by (5). Therefore, a necessary condition for $(a - \delta, a + \delta)$ to be an equilibrium is that $\max(\Psi_{-2\delta}(a), \Psi_{2\delta}(a)) \leq 2$.

Note that because of the logconcavity of f , $t \mapsto \Psi_{-2\delta}(t)$ is a decreasing function on \mathbb{R} and $t \mapsto \Psi_{2\delta}(t)$ is an increasing function on \mathbb{R} . Besides, $\Psi_{-2\delta}(0) = \Psi_{2\delta}(0)$. Therefore

$$\max(\Psi_{-2\delta}(a), \Psi_{2\delta}(a)) = \begin{cases} \Psi_{2\delta}(a) & \text{if } a \geq 0 \\ \Psi_{-2\delta}(a) & \text{if } a \leq 0 \end{cases}$$

Since $\Psi_{-2\delta}(a) = \Psi_{2\delta}(-a)$, then $\max(\Psi_{-2\delta}(a), \Psi_{2\delta}(a)) = \Psi_{2\delta}(|a|)$.

Note that there exists $a \in [-\delta, \delta]$ such that $\Psi_{2\delta}(|a|) \leq 2$ if and only if $\Psi_{2\delta}(0) \leq 2 \Leftrightarrow 2\delta \leq \kappa$.

If this condition holds, $\alpha(f, \delta)$ defined in 3 exists and is uniquely defined, and $(-\delta + a, +\delta + a)$ is an equilibrium only if $a \in [-\alpha(f, \delta), \alpha(f, \delta)]$.

Note that the case $a = 0$ is the unique symmetric equilibria in this class.

Assume that $2\delta \leq \kappa$ and consider $x_1 \in [-\delta - \alpha(f, \delta), -\delta + \alpha(f, \delta)]$. Let us show that $\frac{\partial q_2}{\partial x_2}(x_1, x_2) > 0$ for all $x_2 \in]x_1, x_1 + 2\delta]$, which will guarantee that $x_2 = x_1 + 2\delta$ is a best response against x_1 . Given (4), for $x_2 \in]x_1, x_1 + 2\delta[$:

$$\frac{\partial q_2}{\partial x_2}(x_1, x_2) = f(x_2 + \delta) - \frac{1}{2}f\left(\frac{x_1 + x_2}{2}\right).^{12}$$

Note that

$$\frac{f\left(\frac{x_1 + x_2}{2}\right)}{f(x_2 + \delta)} = \Psi_{\delta + \frac{x_2 - x_1}{2}}\left(\frac{x_1 + x_2}{2}\right) \leq \Psi_{\delta + \frac{x_2 - x_1}{2}}(x_1 + \delta) = \frac{f(x_1 + \delta)}{f\left(\frac{x_1 + x_2}{2} + 2\delta\right)} < \frac{f(x_1 + \delta)}{f(x_1 + \delta + 2\delta)},$$

where the first inequality follows from the observation that $\delta + \frac{x_2 - x_1}{2} > 0$ and $\frac{x_1 + x_2}{2} < x_1 + \delta$, the second inequality follows from the fact that $0 \leq \frac{x_1 + x_2}{2} + 2\delta < (x_1 + \delta) + 2\delta$ and f is strictly decreasing on \mathbb{R}^+ .

Note also that

$$\frac{f(x_1 + \delta)}{f(x_1 + \delta + 2\delta)} = \Psi_{2\delta}(x_1 + \delta) \leq \Psi_{2\delta}(\alpha(f, \delta)) = \frac{f(\alpha(f, \delta))}{f(\alpha(f, \delta) + 2\delta)} \leq 2$$

where the first inequality follows from the observation that $2\delta > 0$ and $x_1 + \delta \leq \alpha(f, \delta)$, and the second inequality follows from the definition of $\alpha(f, \delta)$ (see (3)).

This proves that $\frac{\partial q_2}{\partial x_2}(x_1, x_2) < 0$ for all $x_2 \in]x_1, x_1 + 2\delta]$, and $x_2 = x_1 + 2\delta$ is a best response against x_1 .

This concludes the proof of Proposition 3.

5.2 Proof of Proposition 6 (Efficiency)

Claim (1): Consumers' surplus.

Suppose that firms' locations are x_1 and x_2 . When $x_1 \leq x_2$, the consumers' surplus is:

$$CS(x_1, x_2) = \begin{cases} \int_{x_1 - \delta}^{\frac{x_1 + x_2}{2}} (v - p - c(|x_1 - t|))f(t)dt + \int_{\frac{x_1 + x_2}{2}}^{x_2 + \delta} (v - p - c(|x_2 - t|))f(t)dt & \text{if } |x_2 - x_1| \leq 2\delta, \\ \int_{x_1 - \delta}^{x_1 + \delta} (v - p - c(|x_1 - t|))f(t)dt + \int_{x_2 - \delta}^{x_2 + \delta} (v - p - c(|x_2 - t|))f(t)dt & \text{if } |x_2 - x_1| \geq 2\delta. \end{cases}$$

Because players are anonymous, we have $CS(x_1, x_2) = CS(x_2, x_1)$. Therefore, the previous expression also holds for $x_1 \geq x_2$.

Note first that there exists a profile that maximizes the consumers' surplus. Indeed, the surplus is a continuous function and because $f(x)$ goes to zero as $|x|$ goes to $+\infty$, we can restrict the analysis of $CS(x_1, x_2)$ on a compact subset of $\mathbb{R} \times \mathbb{R}$.

Remark that a situation with full convergence cannot be optimum. Indeed, for any $x, y \in \mathbb{R}$, $x \neq y$, $CS(x, y) > CS(x, x)$.

It remains to show that a situation where $|x_2 - x_1| \geq 2\delta$ cannot be an optimum. Straightforward computations show that when firms' locations are x_1 and x_2 , with $x_1 \leq x_2$ and $|x_2 - x_1| \geq 2\delta$,

then:

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{CS(x_1 + \varepsilon, x_2) - CS(x_1, x_2)}{\varepsilon} = \int_0^\delta c'(s) [f(x_1 + s) - f(x_1 - s)] ds,$$

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{CS(x_1, x_2 - \varepsilon) - CS(x_1, x_2)}{\varepsilon} = \int_0^\delta c'(s) [f(x_2 - s) - f(x_2 + s)] ds.$$

Note that if $x_1 < 0$, for any s such that $0 \leq s \leq \delta$, $f(x_1 + s) - f(x_1 - s) > 0$. Indeed, if $x_1 + s \leq 0$, it is true since f is increasing on \mathbb{R}_- . And if $x_1 + s \geq 0$: $f(x_1 + s) - f(x_1 - s) > 0 \Leftrightarrow x_1 + s < |x_1 - s| \Leftrightarrow x_1 < 0$. Therefore if $x_1 < 0$, then $f(x_1 + s) - f(x_1 - s) > 0$ and $\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{CS(x_1 + \varepsilon, x_2) - CS(x_1, x_2)}{\varepsilon} > 0$: the consumer surplus would increase if Firm 1 were to move closer to Firm 2.

Consider now the case $x_1 \geq 0$. Since $|x_2 - x_1| \geq 2\delta$, it implies that $x_2 > 0$. A similar argument shows that if $x_2 > 0$, for any s such that $0 \leq s \leq \delta$, $f(x_2 + s) - f(x_2 - s) < 0$ and $\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{CS(x_1, x_2 - \varepsilon) - CS(x_1, x_2)}{\varepsilon} > 0$: the consumer surplus would increase if Firm 2 were to move closer to Firm 1.

This completes the proof of part 1.

Claim (2): Aggregate profit.

Assume the following conditions hold: (i) $\gamma_1 = \gamma_2 = \gamma$; (ii) $\gamma'' \geq 0$; (iii) $\gamma'(1) < p$. For $q \in [0, 1]$ and $\alpha \in [0, 1]$, denote by $\Pi(q, \alpha)$ the aggregate profit when total production is q and Firm 1 realizes a share α of the total production (the remaining share being produce by Firm 2):

$$\Pi(q, \alpha) = pq - \gamma(\alpha q) - \gamma((1 - \alpha)q).$$

Then

$$\frac{\partial \Pi}{\partial q}(q, \alpha) = p - \alpha \gamma'(\alpha q) - (1 - \alpha) \gamma'((1 - \alpha)q).$$

Since $\gamma'(x) < p$ for all $x \in [0, 1]$, $\frac{\partial \Pi}{\partial q}(q, \alpha) > 0$: aggregate profit is increasing with the aggregate output.

Note also that:

$$\frac{\partial \Pi}{\partial \alpha}(q, \alpha) = -q \gamma'(\alpha q) + q \gamma'((1 - \alpha)q)$$

$$\frac{\partial^2 \Pi}{\partial \alpha^2}(q, \alpha) = -q^2 \gamma''(\alpha q) - q^2 \gamma''((1 - \alpha)q) \leq 0,$$

therefore for all $\alpha \in [0, 1]$,

$$\Pi(q, \alpha) \leq \Pi\left(q, \frac{1}{2}\right),$$

which means that fixing the total output q , an equal sharing of the production is efficient (there is no way to make costs strictly lower).

Since the profile of location $(-\delta, +\delta)$ is the unique profile which maximizes total sales, and since it is symmetric, it is the unique solution of the aggregate profit maximization program. Which concludes the proof of Part 2.

5.3 Proof of Proposition 7 (Uniform case).

Assume that consumers are uniformly distributed on the $[-\kappa, \kappa]$ interval. Firms can choose any location on the real line.

The proof straightforwardly follows from the following five lemmas.

Lemma 11

If (x_1, x_2) is an equilibrium, then, necessarily, $x_1, x_2 \in [-\kappa, \kappa]$.

Proof. of Lemma 11

Note first that at equilibrium, both firms receive a positive payoff. Indeed, a firm could secure a positive payoff by moving to the center 0. This remark proves that at equilibrium, $x_1, x_2 \in]-\kappa - \delta, \kappa + \delta[$.

Assume that (x_1, x_2) is an equilibrium, with $x_1 \leq x_2$.

Assume that $-\kappa - \delta \leq x_1 < -\kappa$. If $x_1 = x_2$, Firm 2 could strictly increase its payoff by moving to position $-x_1$, which contradicts the fact that (x_1, x_2) is an equilibrium. If $x_1 < x_2$, Firm 1 could strictly increase its payoff by moving slightly closer to Firm 2, which again contradicts the fact that (x_1, x_2) is an equilibrium.

By symmetry, there can be no equilibria where $\kappa < x_2 \leq \kappa + \delta$.

This concludes the proof of Lemma 11. ■

Lemma 12

There exists an equilibrium where the firms locate at distance at least 2δ one from the other if and only if $\delta \leq \frac{\kappa}{2}$. In that case, there exists a continuum of equilibria. More specifically, supposing without loss of generality that $x_1 \leq x_2$, the whole set of equilibria is (x_1, x_2) such that $-\kappa + \delta \leq x_1 \leq x_1 + 2\delta \leq x_2 \leq \kappa - \delta$.

Proof. of Lemma 12

By Lemma 11, one can restrict attention to (x_1, x_2) such that $-\kappa \leq x_1 \leq x_1 + 2\delta \leq x_2 \leq \kappa$. Note that it must be the case that $x_2 \leq \kappa - \delta$. Otherwise, Firm 2 could strictly increase its payoff by moving slightly to the left. Similarly, it must be the case that $x_1 \geq -\kappa + \delta$. These two conditions, together with the fact that $x_1 + 2\delta \leq x_2$ implies that $2\delta \leq \kappa$.

Last, note that if $2\delta \leq \kappa$, any (x_1, x_2) such that $-\kappa + \delta \leq x_1 \leq x_1 + 2\delta \leq x_2 \leq \kappa - \delta$ gives both firms the maximal possible payoff $(\frac{\delta}{\kappa})$, and is thus an equilibrium. ■

Lemma 13

There exists an equilibrium where the firms locate at distance less than 2δ one from the other without converging if and only if $\frac{1}{2} < \frac{\delta}{\kappa} < 1$. If this condition holds, the unique equilibrium is $(\delta - \kappa, \kappa - \delta)$ (up to a permutation of the players).

Proof. of Lemma 13

By Lemma 11, one can restrict attention to (x_1, x_2) such that $-\kappa \leq x_1 < x_2 < x_1 + 2\delta$ and $x_2 \leq \kappa$. In that case, $q_2(x_1, x_2) = \frac{1}{2\kappa} [\min(\kappa, x_2 + \delta) - \frac{x_1 + x_2}{2}]$. The fact that firm 2 cannot increase its payoff by deviating slightly to the left implies that $x_2 + \delta \geq \kappa$. The fact that firm 2 cannot increase its payoff by deviating slightly to the right implies that $x_2 + \delta \leq \kappa$. It must therefore be the case that $x_2 = \kappa - \delta$. Similarly, it must be the case that $x_1 = -\kappa + \delta$. The distance between firm 1 and 2 has to be $2\kappa - 2\delta$. Since we have imposed that this distance should be positive and less than 2δ , one gets the following necessary condition: $\frac{1}{2} < \frac{\delta}{\kappa} < 1$.

One may easily check that if this condition holds, $(\delta - \kappa, \kappa - \delta)$ is an equilibrium. ■

Lemma 14

$(0, 0)$ is an equilibrium if and only if $\delta \geq \kappa$.

Proof. of Lemma 14

Let us first show that if $\delta \geq \kappa$, $(0, 0)$ is an equilibrium. Assume that $\delta \geq \kappa$. Note that $q_1(0, 0) = q_2(0, 0) = \frac{1}{2}$. And note that

$$\begin{aligned} q_2(0, x_2) &= \frac{1}{2\kappa} \left[\kappa - \frac{x_2}{2} \right] = \frac{1}{2} - \frac{x_2}{4\kappa} \text{ if } x_2 \leq 2\kappa \\ &= 0 \text{ otherwise.} \end{aligned}$$

This proves that $\forall x_2 > 0$, $q_2(0, x_2) < q_2(0, 0)$. Therefore, $(0, 0)$ is an equilibrium.

Let us now complete the proof by showing that if $\delta < \kappa$, $(0, 0)$ is not an equilibrium. If $\delta < \kappa$

$$q_2(0, 0) = \frac{\delta}{2\kappa}$$

and for $0 < \varepsilon < \kappa - \delta$,

$$q_2(0, \varepsilon) = \frac{1}{2\kappa} \left[\varepsilon + \delta - \frac{\varepsilon}{2} \right] = \frac{\delta}{2\kappa} + \frac{\varepsilon}{4\kappa} > q_2(0, 0).$$

This completes the proof. ■

Lemma 15

There is no equilibrium where the firms choose the same location if this location is different from 0.

Proof. of Lemma 15

Assume that (x_1, x_2) is an equilibrium such that $x_1 = x_2 = x > 0$.

By Lemma 11, we know that necessarily, $0 < x \leq \kappa$.

Then:

$$q_1(x, x) = \frac{1}{2} \times \frac{1}{2\kappa} [\min(x + \delta, \kappa) - \max(x - \delta, -\kappa)]$$

and for $\varepsilon > 0$ small enough:

$$q_1(x - \varepsilon, x) = \frac{1}{2\kappa} \left[x - \frac{\varepsilon}{2} - \max(x - \varepsilon - \delta, -\kappa) \right].$$

For (x, x) to be an equilibrium, it is therefore necessary that

$$x - \max(x - \delta, -\kappa) \leq \frac{1}{2} \times [\min(x + \delta, \kappa) - x + x - \max(x - \delta, -\kappa)],$$

which is equivalent to

$$\begin{aligned} x - \max(x - \delta, -\kappa) &\leq \min(x + \delta, \kappa) - x \\ x + \min(-x + \delta, \kappa) &\leq \min(x + \delta, \kappa) - x \\ \min(\delta, x + \kappa) &\leq \min(\delta, \kappa - x) \end{aligned}$$

If $\min(\delta, x + \kappa) = x + \kappa$, this implies $x + \kappa \leq \kappa - x$, which is impossible since by assumption $x > 0$.

Therefore $\min(\delta, x + \kappa) = \delta$ and $x \leq \kappa - \delta$. Note that since $x > 0$, this implies that $\delta < \kappa$.

If $x \leq \kappa - \delta$, then $\min(x + \delta, \kappa) = x + \delta$ and $\max(x - \delta, -\kappa) = x - \delta$, therefore:

$$q_1(x, x) = \frac{1}{2} \times \frac{\delta}{\kappa}.$$

But then note that

$$\begin{aligned}q_1(-x, x) &= \frac{1}{2\kappa} [\min(-x + \delta, 0) - (-x + \delta)] \\ &= \frac{1}{2} \times \frac{\delta}{\kappa} + \frac{1}{2\kappa}x + \min(-x + \delta, 0).\end{aligned}$$

Since $x > 0$, $q_1(-x, x) > q_1(x, x)$ and (x, x) is not an equilibrium. ■

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