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“Identification and Estimation in a Third-Price Auction Model”

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Identification and Estimation in a Third-Price Auction Model *

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Abstract

The first novelty of this paper is that we show global identification of the private values distribution in a sealed-bid third-price auction model using a fully nonparametric methodology. The second novelty of the paper comes from the study of the identification and estimation of the model using a quantile approach. We consider an i.i.d. private values environment with risk-averse bidders. In the first place, we consider the case where the risk-aversion parameter is known. We show that the speed of convergence in process of our nonparametric estimator produces at the root-n parametric rate and we explain the intuition behind this apparently surprising result. Next, we consider that the risk-aversion parameter is unknown and we locally identify it using exogenous variation in the number of participants. We extend our procedure to the case where we observe only the bids corresponding to the transaction prices, and we generalize the model so as to account for the presence of exogenous variables. The methodological toolbox used to analyse identification of the third-price auction model can be employed in the study of other games of incomplete information. Our results are interesting also from a policy perspective,

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as some authors recommend the use of the third-price auction format for certain Internet auctions. Moreover, we contribute to the econometric literature on auctions using a quantile approach.

Keywords: structural nonparametric estimation, nonlinear inverse problems, global identification, functional convergence of estimators, third-price auction model

JEL Classification: $C73, C40$

1 Introduction

In recent years there has been a growing interest in the structural analysis of auction data. Its success is motivated by a coherent body of theory and by its usefulness in providing insights into many practical policy issues. The structural approach allows the econometrician to estimate the data generating process, i.e. the distribution of bidders' private values. Thus, this approach is extremely useful for the normative aspect of designing an optimal selling mechanism. It is well known that the design of optimal auctions has been criticized for relying upon the distributions of bidders' valuations, which are unknown to the auctioneer. Therefore, the structural econometrics can compensate for this drawback by estimating the law of probability governing these valuations. Next, the estimated distribution can be used to determine the revenue-maximizing selling mechanism, the optimal reserve price or to conduct counterfactuals.

The novelty of our paper comes from the fact that we are using an inverse problem approach to assess the identification of the distribution of private values in a third-price auction model. This means that the econometrician observes the data (in our case the bids) and, starting from these observations, he/she tries to uncover the primitives that lead to the observables. We show that in the case of the third-price auction we are actually dealing with a well-posed inverse problem. This aspect is new in the econometric models of games of incomplete information, where very often the inverse problems are ill-posed. Moreover the well-posedness of our problem is associated with a parametric rate of convergence for our nonparametric estimator. Although the third-price auction is not very common in practice, the study of identification and asymptotic properties of the estimator of the density of private values sheds some light on the econometric properties of analogous estimators in more common auction formats, such as the first-price auction model and the second-price auction model.

The econometric literature of auction data is quite rich and some of the first seminal works in this field are due to Paarsch (1992), Donald and Paarsch (1993), Donald and Paarsch (1996), who worked in a parametric environment using the method of maximum likelihood to estimate the distribution of private values. In order to address some of the criticisms of the maximum likelihood approach, Laffont et al. (1995) proposed the method of simulated nonlinear least-squares based on the Revenue Equivalence Principle.

When speaking about estimation in a structural model, a strongly related concept is the notion of identification. In the field of econometrics for auctions, the paper by Guerre et al. (2000) is a milestone concerning the identification of the private values distribution. The authors proved that the distribution of unobserved valuations in first-price sealed-bid auctions, within the independent private values paradigm, is nonparametrically identified from the observables whether all bids are observed or only the winning bid. Another contribution to the identification issue is the article by Florens et al. (1998) where the authors studied the general conditions of identification for empirical games of incomplete information. An extension of this work has been conducted by Florens and Sbaï (2010), who introduced additional parameters to be identified and also studied several theoretical situations (e.g. partial observations of the exogenous variables, randomized strategies).

Our paper uses a structural approach to study the nonparametric identification and estimation in the case of a third-price sealed-bid auction model. Moreover, using a quantile approach, we obtain results of constructive nonparametric identification following Matzkin (2013) that allow us to provide a straightforward estimation of the model.[1] The quantile approach has previously been used in an independent private values auction environment by Marmer and Shneyerov (2012) who nonparametrically estimated the density of latent variables in a first-price sealed-bid auction. Their paper was influenced by the paper of Haile et al. (2003) where the authors used nonparametric quantile-based tests for common values in first-price sealed-bid auction models. Marmer et al. (2013) employed quantile estimation to test for selective entry in first-price auctions used in highway procurement. Guerre et al. (2009) and Zincenko (2013) developed a quantile methodology for the study of the first-price

[1]Although the quantiles contain the same type of information as the distribution functions, rewriting the model in terms of quantiles allows us to obtain an explicit solution at the identification stage. If one applies our methodology to the first-price auction (see Enache and Florens (2017)), one can obtain a global identification result as in Guerre et al. (2000). Nevertheless, by contrast with the Guerre et al. (2000) paper, our quantile approach leads to a closed-form solution of the model linking the primitives to the observables.

auction model while accounting for the presence of risk-averse bidders. Liu et al. (2014) used quantiles to perform nonparametric testing for exogenous participation in first-price auctions. Other related works that study games of incomplete information using quantile regressions are Enache and Florens (2018a), Enache and Florens (2018b), Enache and Florens (2017), Gimenes et al. (2016), Gimenes (2017) and Guerre and Sabbah (2012).

The third-price auction is considered to be a theoretical type of auction, as for the moment there is no known instance (to our knowledge) of such a mechanism in practice. The rule of this game is such that the highest bidder wins the object (efficient mechanism), but he pays a price equal to the third-highest bid. Kagel and Levin (1993) is the seminal paper that formalizes the body of theory underlying the third-price auction under the assumption of uniformly distributed private values and risk-neutrality or constant absolute risk aversion. Thus, the authors show the following peculiar features of the third-price auction: the bids exceed the private values, the marginal effect on bids of an increase in private values is greater than one, and increasing the number of bidders reduces the bids. Besides this, the paper by Kagel and Levin (1993) is a valuable source of experimental data as it also contains a section where the authors describe a laboratory experiment using the third-price auction format.

In a more recent paper, Monderer and Tennenholtz (2000a) extend the results obtained by Kagel and Levin (1993) for any distribution function of private values and any attitude towards the risk. Moreover, they prove that there exists at most one equilibrium and that the bidding strategy implies overbidding.[2] Therefore, the issue of multiplicity of equilibria does not arise in the third-price auction. In our econometric model, we suppose that the equilibrium exists[3] and we do not have to face the estimation challenges generated by multiple equilibria. The authors show that, in the presence of risk-seeking bidders, the revenue of the seller is higher than in the second-price auction (the latter type of auction outperforms the first-price auction in case of risk-seeking). Thus a third-price auction is appropriate for situations where bidders are risk-seeking, as the seller could better exploit the love for risk of the agents.

[2]For more details about the existence of at most one equilibrium in third-price auction model, see Section 3 of Monderer and Tennenholtz (2000a).

[3]For a discussion of sufficient conditions for equilibrium existence, see Section 5 of Monderer and Tennenholtz (2000a).

Moreover, Monderer and Tennenholtz (2000a,b) provide recommendations for the use of such a design in the case of Internet auctions for relatively cheap items. For more arguments along these lines, see mainly Monderer and Tennenholtz (1998) and also Monderer and Tennenholtz (2004) where the authors consider that buyers of cheap items on the Internet are fun-seeking, somehow like players in a casino.[4]

Our paper proceeds as follows: in the next part we present the model of a third-price auction, in the third section we proceed to the global identification approach, assuming the risk-aversion parameter as being given. Section 4 provides the corresponding estimation methodology along with a Monte Carlo experiment. Section 5 provides the asymptotic statistics for our estimator. Sections 6 to 8 illustrate some extensions, namely the case where the risk-averse parameter is unknown, the case where only the bids corresponding to the transaction prices are observed, and the case in which exogenous variables are present. Section 9 concludes.

2 Model

A third-price auction is a game with at least 3 players ($N \geq 3$), where each player, $j = 1, ...N$, has private information (his type) denoted by v_j. This private information is unknown to the other players and does not influence the valuation of the good that the other players have (a situation which describes a game of incomplete information in a private value environment). The equilibrium concept adequate for such situations is *Bayesian Nash Equilibrium*.

We assume that the private information is independently and identically distributed, drawn from a common cumulative distribution function $v_j \sim F$, which is common knowledge for the players, but unobservable to the statistician. Without loss of generality, we choose the support of the private values as being $[0, 1]$.

At equilibrium each player plays an action x_j, which is influenced by his private value v_j

[4] Other works on the third-price auction include Goeree et al. (2005), Tauman (2002) and Wolfstetter (1996). The latter article claims that, in the presence of corruption, bidders might behave according to a third-price auction, although the auction was designed as a second-price auction.

and by the distribution function, F. The latter aspect constitutes the strategic component of the game. Otherwise said, $x_j = \sigma_j(v_j, F)$, where the function σ_j is the solution of a Bayesian Nash Equilibrium and is known by the statistician. For symmetric bidders $\sigma_j = \sigma$.

Following Monderer and Tennenholtz (2000a), we consider symmetric bidders with a constant absolute risk-aversion (CARA) attitude[5] and therefore the equilibrium bid in the case of a third-price auction with risk-aversion bidders is:

$$x = v + \frac{1}{\lambda} \ln \left\{ 1 + \frac{\lambda}{M} \frac{F(v)}{f(v)} \right\}, \tag{1}$$

where f is the probability density function (hereafter p.d.f.) of F, $\lambda > 0$ is the CARA parameter and, $M = N - 2$. From *Theorem AT* of Monderer and Tennenholtz (2000a) we have that the equilibrium bid is increasing on $[0,1]$.

We make the following assumptions on the c.d.f. F:

assumption 1. *$F \in \mathscr{C}^3[0,1]$, i.e. F is three times continuously differentiable on $[0,1]$.*

assumption 2. *$f(v) = F'(v) > 0$ for every $v \in (0,1]$.*

assumption 3. *$\frac{F}{f}$ is a strictly increasing function and $\lim_{v \to 0} \frac{F(v)}{f(v)} = 0$.*

Many of these assumptions can be found in the theoretical model of Monderer and Tennenholtz (2000a). The only exception is the condition on the smoothness of the distribution function of the valuations. Indeed, Monderer and Tennenholtz (2000a) suppose that $F \in \mathscr{C}^2[0,1]$, but we need a stronger assumption in order to insure that $G \in \mathscr{C}^2$, where G denotes the cumulative distribution function of the bids. This latter property will be useful in the derivation of the asymptotic properties of our estimators. The strictly increasing $\frac{F}{f}$ is a sufficient condition for the existence of the equilibrium (see Section 4 of Monderer and Tennenholtz (2000a)), but we need it for our identification strategy as well.

[5] For buyers with constant absolute risk-aversion, the utility function has the following shape $u(x) = a\left(1 - e^{-\lambda x}\right)$, with $\lambda > 0$ and $a > 0$. The risk neutrality corresponds to $\lambda = 0$ and in this case the first-order condition writes as $x = v + \frac{F(v)}{Mf(v)}$.

Let us comment below on the assumptions on the F function.

Remark 1. *First we may note that there is a one to one relation between functions F satisfying Assumptions 1 to 3 and functions π defined on $[0,1]$ that verify:*

(i) $\pi \in \mathscr{C}^2[0,1]$.

(ii) $\pi(0) = 0$ *and* π *is increasing.*

(iii) $\displaystyle\int_0^1 \frac{1}{\pi(x)} \mathrm{d}x = +\infty$.

Indeed given π, one may define F by:

$$F(v) = e^{-\int_0^v \frac{1}{\pi(x)} \mathrm{d}x}$$

and $\pi = \dfrac{F}{f}$. This relation is similar to the usual relation between the survivor function and the hazard-rate in duration models. Note that if property (iii) is not satisfied, the function F has a probability mass at 0, but we do not consider that case in our approach. An elementary class of models is characterized by $\pi(x) = ax^b$, $0 < b < 1$.

Remark 2. *Secondly, we may note that assumptions 1 to 3 are common in the literature on games of incomplete information and in principal-agent models (see Laffont and Tirole (1993), for example) and we may consider these hypotheses as identification restrictions. Indeed, if we do not assume some properties on F it would be impossible to identify F from G. Moreover, we may imagine that there exists a continuum of possible bidders characterized by their type v and the "order" of these bidders is arbitrary. Our assumptions mean that the bidders are "ranked" such that π is increasing and that by convention $\pi(0) = 0$.*

For identification reasons, we assume that the risk-aversion parameter is known. We will relax this assumption later on, assuming a more complex observation scheme. From the relation $x = \sigma(v, F)$ specified in (1) we may easily deduce that

$$G = F \circ \sigma^{-1}, \tag{2}$$

where G is the cdf of the bids and σ^{-1} means the reciprocal function of σ as a function of v, F fixed. This implicit form can be easily obtained from:

$$G(x) = \Pr(x_j \leq x) = \Pr(\sigma^{-1}(x_j) \leq \sigma^{-1}(x)) = \Pr(v_j \leq \sigma^{-1}(x)) = F\left(\sigma^{-1}(x)\right).$$

This expression is equivalent to $G \circ \sigma = F$ or to the quantile expression:

$$G^{-1}(t) = (\sigma \circ F^{-1})(t),\ t \in [0,1], \tag{3}$$

where G^{-1} and F^{-1} are the quantile functions of bids and of private values. This equation becomes, in our particular case:

$$G^{-1}(t) = F^{-1}(t) + \frac{1}{\lambda} \ln\left\{1 + \frac{\lambda}{M} t F^{-1'}(t)\right\}. \tag{4}$$

Third-price auctions belong to a class of games where the dependence between the distribution of bids and the distribution of types occurs through the "hazard-rate" $\frac{F}{f}$ (F and f are the cdf and the density of private values). We already pointed out that in this case the link between the distribution of types and the distribution of bids is easy to analyze in terms of quantile functions and leads to global analysis (identification and estimation) of the problem. Equation (4) will be the basis of our analysis of identification and of our estimation approach.

3 Identification

In structural econometrics, the models contain the economic information underlying the process that is being studied. Before estimating the model one has to ensure that there is an injective mapping between the distribution of data (the observables) and the elements to be estimated (the structural parameters). In other words, identification trivially means that there is only one structure that can explain the process under observation. We are dealing here with an inverse problem in the sense that the bids are an indirect observation of the private values. The task of the econometrician consists in checking that the mapping from the data to the primitives is one to one. Identification is also one of the conditions for an inverse problem to be well-posed along with the stability condition [6].

At this stage we assume that we observe all the bids of several games. Then the actual data is x_{jl}, i.e. the bid of player j in game l. The number of players is assumed to be identical for each game, but the players may be different for different games. In this section, the distribution F is identical for different games and we actually observe iid sample of x_{jl} of the c.d.f. G. For simplicity we denote by i the couple (j,l) and $i = 1,...,n$ ($n = NL$, where N is the number of players in each game and L is the total number of games, with $l = 1,...,L$). Then the data identify the c.d.f. G. Recall that we assume that the strategy function σ is known and that λ is also given. We will relax this assumption later on.

We analyze globally the identification using equation (4) which links the reduced form quantile function G^{-1} to the structural parameter F^{-1}.

The following proposition illustrates the first important result of the paper:

Proposition 1. *In a third-price auction model with symmetric risk-averse bidders and equilibrium strategy given by* (1)*, the cumulative distribution function of the private values, $F(v)$, is globally nonparametrically identified without any restrictions and the quantile function corresponding to it, $F^{-1}(t)$, is described by the following relation:*

$$F^{-1}(t) = \frac{1}{\lambda} \ln \frac{1}{t^M} \int_0^t M s^{M-1} e^{\lambda G^{-1}(s)} \, \mathrm{d}s, \tag{5}$$

[6] Following the conditions proposed by the French mathematician Jacques Hadamard, an inverse problem is well-posed if there exists a solution to that problem, there is at most one solution, and a small change in the data is associated with small changes in the solution. The existence of a solution is equivalent to supposing that the model is correctly specified, i.e. we observe third-price auction bids and the players play Nash. The uniqueness of the solution is equivalent to the model being identified, i.e. from the reduced form, we uniquely retrieve the structural parameters (see Carrasco et al. (2007)).The last of the three conditions is known as a stability condition.

where $t \in [0,1]$, $F^{-1}(t)$ is the quantile function of the private values and $G^{-1}(t)$ is the quantile function of the bids.

As Guerre et al. (2009) show in their paper, one cannot identify at the same time the distribution function of latent variables and the parameter of risk aversion. Indeed, G^{-1} contains all the information coined by the data and we need some supplementary variation in the data in order to identify this extra parameter. This variation can be obtained by the use of exogenous variables (as we show in Section 6). This identification strategy is comparable to the one of Guerre et al. (2009), but applied to the third-price auction model. Moreover, from equations (A.2) and (A.1) of the Appendix it follows that there exists a one to one relation between λG^{-1} and λF^{-1}. Then, given the identified function G^{-1} and, for any λ, we may recover F^{-1}. So the model is not identified if λ is unknown.

4 Estimation

In the previous step we have provided a constructive identification strategy (see Matzkin (2013)), in the sense that we have obtained a closed-form solution to our inverse problem. The usefulness of the constructive identification is that it provides a straightforward estimation method. Basically we will replace the true distribution of data by its empirical counterpart in the identification result. Thus we use the empirical quantile function as a natural estimator for $G^{-1}(t)$ and then we plug it in (5):

$$\widehat{F^{-1}}(t) = \frac{1}{\lambda} \ln \frac{1}{t^M} \int_0^t M s^{M-1} e^{\lambda \widehat{G^{-1}}(s)} \, \mathrm{d}s, \tag{6}$$

where $\widehat{G^{-1}}(t)$ is the sample quantile function for the observables (data on bids) and is given by $\widehat{G^{-1}}(t) = \inf \left\{ x : \widehat{G}(x) \geq t \right\}$.

Before analyzing the asymptotic properties of our estimator, we will briefly discuss some practical computation issues. We will also perform a Monte Carlo experiment to see the performance of our estimator in finite samples. We used a Beta distribution of parameters

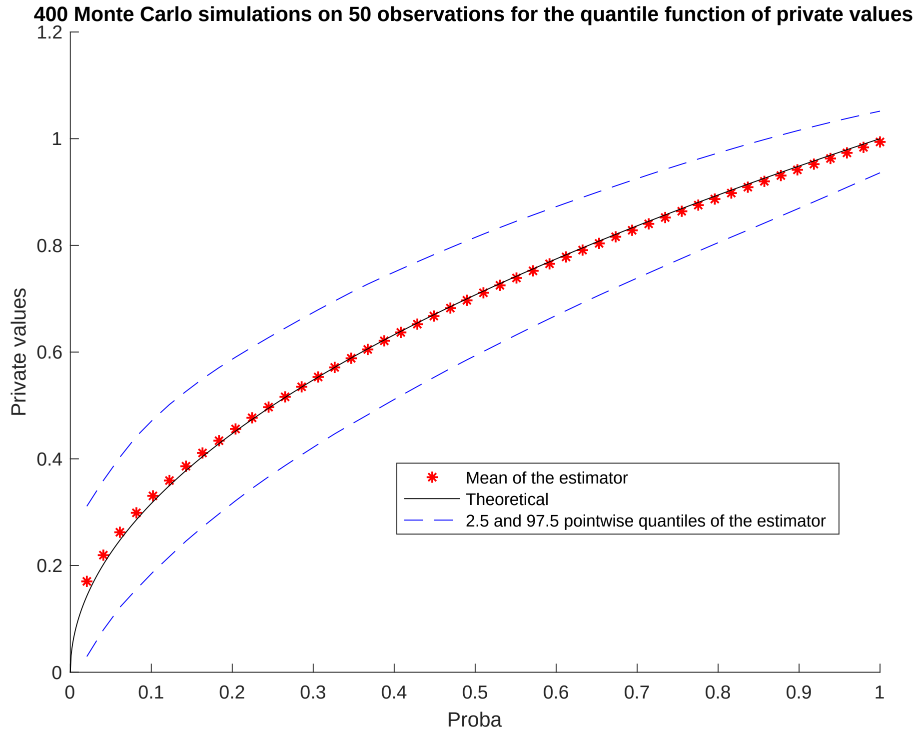

Figure 1: The true quantile function and the mean of the estimator for 400 simulations with a sample of 50 observations.

2 and 1 from which we drew i.i.d. the private values. The coefficient of risk-aversion was fixed to 1. We conducted two configurations: one with 50 observations and another one with 100 observations. The results are illustrated in Figure 1 and Figure 2. One can see that the estimator performs well in small samples (given the common scarcity of the data in practice) and we highlight the fact that, usually in the literature for auctions, simulations are conducted with larger samples compared with those chosen by us.

The quantile density of private values, i.e. $F^{-1'}$, and the density function, f, can be deduced from:

$$F^{-1'}(t) = -\frac{M}{\lambda t} + \frac{1}{\lambda} \frac{t^{M-1} e^{\lambda G^{-1}(t)}}{\int_0^t s^{M-1} e^{\lambda G^{-1}(s)} \mathrm{d}s} \quad \text{and} \tag{7}$$

$$f(v) = \frac{1}{F^{-1'}\left(F(v)\right)}. \tag{8}$$

From the estimation of F^{-1} we may immediately obtain an estimation of F by symmetry.

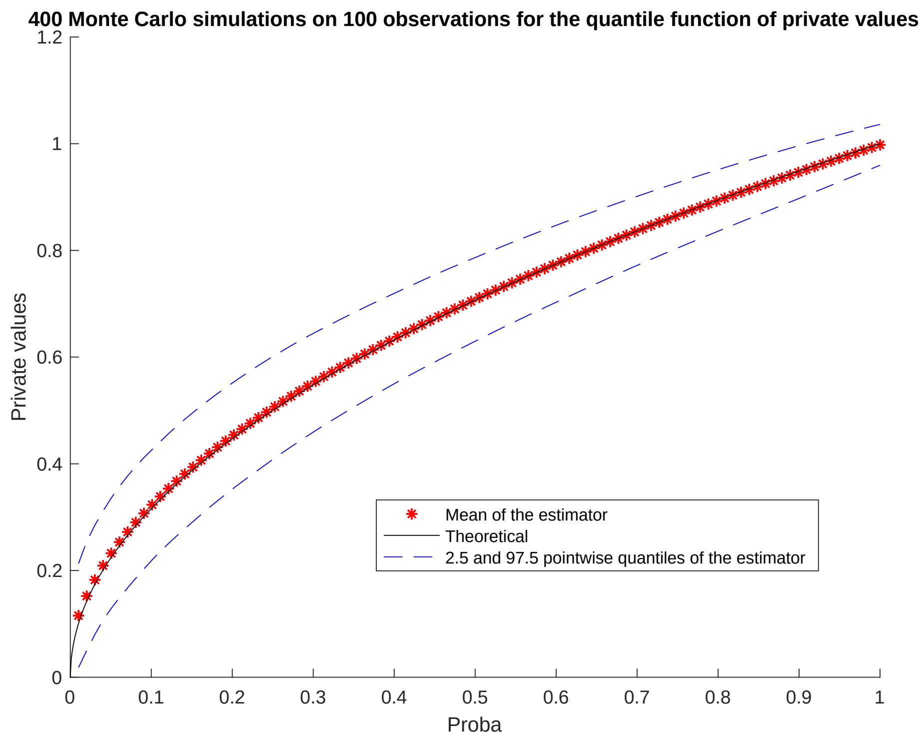

Figure 2: The true quantile function and the mean of the estimator for 400 simulations with a sample of 100 observations.

The estimation of $F^{-1'}$ can be done by replacing G^{-1} by its nonparametric estimation in equation (7). In a similar manner, the estimation of f can be conducted by plugging-in the estimators of $F^{-1'}$ and F in equation (8). Note that the estimation of $F^{-1'}$ and of f does not require any smoothing procedure. In Figure 3 and Figure 4 we report the Monte Carlo simulations for the density function constructed as described above.

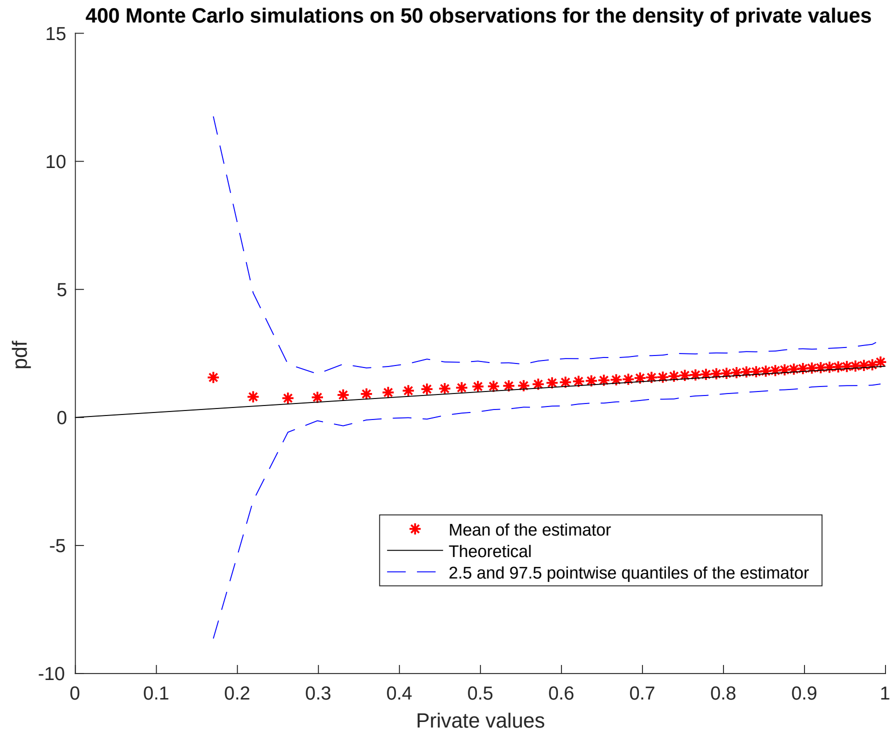

Figure 3: The true density function and the mean of the estimator for 400 simulations with a samp

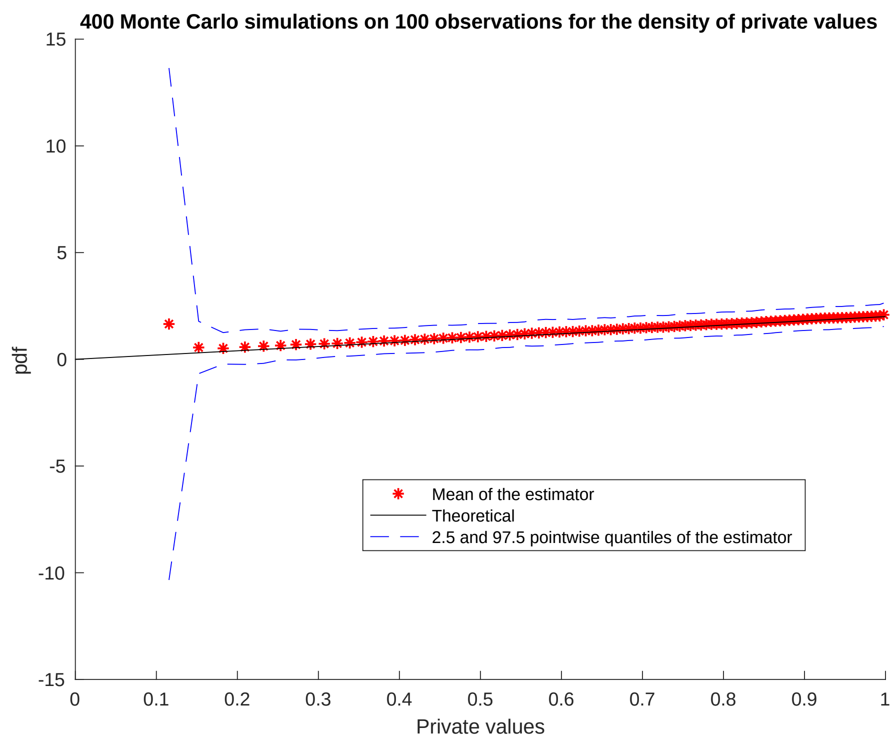

Figure 4: The true density function and the mean of the estimator for 400 simulations with a sample of 100 observations.

5 Asymptotic statistics

We first consider the asymptotic properties of the estimation of F^{-1}. These properties are deduced from the relation (6), where $\widehat{F^{-1}}$ is a nonlinear integral transform of $\widehat{G}^{-1}$. The intuition is the following: we may consider a linear approximation of this integral and $\sqrt{n}\left(\widehat{F^{-1}}-F^{-1}\right)$ will be expressed as a linear integral transform of $\sqrt{n}\left(\widehat{G^{-1}}-G^{-1}\right)$. This last expression is known to be asymptotically Gaussian and we may conclude that $\sqrt{n}\left(\widehat{F^{-1}}-F^{-1}\right)$ will be asymptotically Gaussian. The precise proof of that result needs to control the residuals of the different approximations and to take care of the difficulties at the boundary of $[0,1]$.

Proposition 2. *Let $L^{\infty}[a,b]$ define the collection of all bounded functions on the interval [a,b], for any a, b such that $0<a<b<1$. Under Assumptions 1 to 3, the processes $\sqrt{n}\left(\widehat{F^{-1}}-F^{-1}\right)$ and $\sqrt{n}\left(\widehat{F^{-1\prime}}-F^{-1\prime}\right)$ converge in distribution*[7] *in the space $L^{\infty}[a,b]$ to Gaussian processes with zero mean and covariance kernels Υ and Φ (given in the equations* (B.7) *and* (B.8) *of Appendix B). The process $\sqrt{n}\left(\widehat{f}-f\right)$ converges in distribution in the space $L^{\infty}[F^{-1}(a),F^{-1}(b)]$ to a Gaussian process with zero mean and variance Γ given in the equation* (B.11) *of Appendix B.*

The $\sqrt{n}$ convergence of f might seem paradoxical. However it is not in contradiction with the usual statistical results. It is well known that the min-max rate for the nonparametric estimation of the density of data is not $\sqrt{n}$, but it is slower and depends on the regularity of the problem. In our case, the density f is not the density of the actual data, but it is a structural parameter. The paradox comes from the fact we would not do better if we could observe the latent variable v. The $\sqrt{n}$ speed of convergence comes from the fact that our estimator is obtained as integrals of distributions of observed data.[8] Moreover, the Taylor

[7] In the sense of Van der Vaart (1998), section 18.2.

[8] In the case of a first-price auction, the speed of convergence for an estimator $\hat{F}$ is less than the speed of convergence for $\hat{G}$ (in fact it equals the speed of convergence for $\hat{g}$). The two speeds are equal in the case of a second-price auction, whereas in the case of a third-price auction both $\hat{F}$ and $\hat{f}$ converge at the speed of $\hat{G}$. We interpret that this result is due to the presence of the hazard-rate in the equilibrium bid of the third-price auction. Basically, in the first-price auction model, the Bayesian Nash Equilibrium gives rise to an integral equation, whereas in our model the hazard-rate leads to a differential equation. When solving for the inverse problem

expansion in (B.10) shows that the empirical process of the density of private values depends continuously on the empirical process of the quantile and the quantile density of private values. These two latter processes converge in distribution to a Gaussian process at a $\sqrt{n}$ speed of convergence, and thus the same result applies to the density of private values. Moreover, as all our estimators converge as stochastic processes to Gaussian processes, nonparametric bootstrap procedure would be justified for the treatment of real data (see Chapter 1 from Mammen (2012) for more details).

The stochastic process convergence of these estimators is very important because we may immediately deduce many results from this property. We may construct confidence intervals or tests (for example $F^{-1} = F_0^{-1}$) using Kolmogorov-Smirnov or Cramer Von Mises methods.

6 Estimation of the risk-aversion parameter

We have assumed in the previous paragraphs that we observe bids generated by identical auctions with the same number of players and the same distribution F. We have seen that in that case, the parameter λ should be known. Let us now consider another observation scheme: we observe K types of games which differ by number of bidders N_k (k=1,..., K), but with the same distribution F for all the participants of all the games. In order to obtain identification we need the number of bidders to be exogenous.[9] If all games are observed L_k times, the number of observations is $N_k L_k$ and we assume that $L_k \to \infty$ for any k. The distribution of the bids G_k depends on N_k and if $N_{k_1} \neq N_{k_2}$, then $G_{k_1} \neq G_{k_2}$. In that case, the parameter λ is identified under minor regularity conditions.

Consider for simplicity the case where $K = 2$ and $N_1 \neq N_2$. The computation of the same F in the two auctions gives the equality:

in the first-price auction, one applies the differential operator in order to retrieve the distribution of primitives. The differentiation operation has as a consequence a reduction in the speed of convergence. In contrast to the previously described situation, in the case of the third-price auction, the inverse operator used to back-out the distribution of types is an integral operator and, hence, does not reduce the speed of convergence (if the speed of convergence is less than root-n, then it has an improving effect). Finally, in the second-price auction model the inverse operator is the identity operator so we retrieve in a natural way the same speed of convergence for both the cdfs and the densities of the data and of latent variables.

[9] This approach is similar to Guerre et al. (2009).

$$\frac{1}{t^{M_1}}\int_0^t M_1 s^{M_1-1} e^{\lambda G_1^{-1}(s)}\,\mathrm{d}s - \frac{1}{t^{M_2}}\int_0^t M_2 s^{M_2-1} e^{\lambda G_2^{-1}(s)}\,\mathrm{d}s = 0 \quad \forall t \in [0,1], \tag{9}$$

where $M_1 = N_1 - 2$ and $M_2 = N_2 - 2$.

The parameter λ is identified if equation (9) (for given G_1^{-1}, G_2^{-1}, M_1 and M_2) has only one solution. The nonlinearity of this equation makes the characterization of the uniqueness condition difficult. The solution of this equation is locally unique if the derivative of the LHS term of (9) with respect to λ is not 0 for the true value. This condition can be written as:

$$\frac{M_1}{t^{M_1}}\int_0^t s^{M_1-1} G_1^{-1}(s) e^{\lambda G_1^{-1}(s)}\,\mathrm{d}s - \frac{M_2}{t^{M_2}}\int_0^t s^{M_2-1} G_2^{-1}(s) e^{\lambda G_2^{-1}(s)}\,\mathrm{d}s \neq 0,$$

for at least one value of t. A sufficient condition of local identification can be obtained for $t = 1$:

$$M_1\int_0^1 s^{M_1-1} G_1^{-1}(s) e^{\lambda G_1^{-1}(s)}\,\mathrm{d}s - \int_0^1 s^{M_2-1} G_2^{-1}(s) e^{\lambda G_2^{-1}(s)}\,\mathrm{d}s \neq 0,$$

The estimation can be done by replacing G_1^{-1} and G_2^{-1} with their estimators and by minimizing:

$$\int_a^b \left\{ \frac{1}{t^{M_1}}\int_a^t M_1 s^{M_1-1} e^{\lambda \widehat{G_1^{-1}}(s)}\mathrm{d}s - \frac{1}{t^{M_2}}\int_a^t M_2 s^{M_2-1} e^{\lambda \widehat{G_2^{-1}}(s)}\mathrm{d}s \right\}^2 \mathrm{d}t, \tag{10}$$

for any $0 < a \leq t \leq b \leq 1$.

The asymptotic normality of the estimator at a $\sqrt{n}$ rate may be computed using standard techniques, because $\sqrt{n}\left(\widehat{G_1^{-1}} - G_1^{-1}\right)$ and $\sqrt{n}\left(\widehat{G_2^{-1}} - G_2^{-1}\right)$ are asymptotically Gaussian on $[a,b]$.

This analysis may also be extended to the case where K different auctions with $N_1, \ldots, N_k$ participants are observed. Identification just requires that there exist two different values of N.

7 Transaction prices

Let us suppose now that we observe L auctions, but that for each of these auctions we only observe the transaction price, that is the bid corresponding to the third highest bid proposed. In this case, the distribution of v corresponding to the observed price is the distribution of the third-order statistics of an i.i.d. sample of L values. Thus the number of observations is $n = L \times 1$. The distribution of a third-order statistic is given by:

$$F_{N,3}(v) = \sum_{k=3}^{N} C_N^k F^k(v)(1-F(v))^{N-k}, \tag{11}$$

and its density:

$$f_{N,3}(v) = \frac{N!}{2(N-3)!} F^2(v)(1-F(v))^{N-3} f(v).$$

Therefore the proposed strategy is the following: using the distribution of prices paid by the winner, G, we estimate $F_{N,3}(v)$ using (5). This estimator converges in distribution to a Gaussian process as shown previously. Next, we compute F using (11). This transformation has been studied by Arnold et al. (2008), who showed that the relation between F and $F_{N,3}$ is bijective (see also Athey and Haile (2002) who showed nonparametric identification in ascending auctions from transaction prices). Otherwise said, using the estimation of $F_{N,3}$, we can compute F as the unique solution of the equation:

$$F_{N,3}(v) = \int_0^{F(v)} \frac{N!}{2(N-3)!} t^2 (1-t)^{N-3} \, \mathrm{d}t.$$

Several ways to resolve this problem have been proposed in the literature (see Arnold et al. (2008)). The speed of estimation of F and its asymptotic properties are deduced from this equation. Briefly, we can see this problem as a nonlinear inverse problem $T(F) = F_{N,3}$ whose properties are characterized by the Fréchet differential of T :

$$T'_F(\widetilde{F}) = \frac{N!}{2(N-3)!} F^2 (1-F)^{N-3} \widetilde{F}.$$

This differential shows that the problem is well-posed and, using a Taylor expansion, we obtain:

$$\sqrt{n}\left(\widehat{F}-F\right) = \frac{2(N-3)!}{N!F^2(1-F)^{N-3}}\sqrt{n}\left(\widehat{F}_{N,3}-F_{N,3}\right)+o_p(1).$$

We have thus a convergence in $\sqrt{n}$ of $\widehat{F}-F$ and the fact that $\sqrt{n}\left(\widehat{F}-F\right)$ converges to a centered Gaussian process, whose variance can be inferred from the above expression. We can conduct a similar analysis for f using a linear approximation of:

$$f_{N,3} = \frac{N!}{2(N-3)!}F^2(1-F)^{N-3}f.$$

We have that:

$$\sqrt{n}\left(\widehat{f}-f\right) = \left[\frac{N!}{(N-3)!}F\left(1-F\right)^{N-3}f + \frac{N!}{2(N-4)!}F^2(1-F)^{N-4}f + \right.$$
$$\left. + \frac{N!}{2(N-3)!}F^2(1-F)^{N-3}\right]\sqrt{n}\left(\widehat{f}_{N,3}-f_{N,3}\right)+o_p(1).$$

We have therefore proved the convergence of $\sqrt{n}\left(\widehat{f}-f\right)$ to a Gaussian process.

8 Exogenous variables

Let us suppose that we observe L games with N players and for which all bids are observed. Thus we have $n = L\times N$ observations. Moreover, let us suppose that for each game the distribution of types is identical for all players, but it varies between the games conditionally on the observed exogenous variables. The data allows us therefore to identify the conditional distribution $G(x|Z=z_j)$ and the conditional quantiles, $G^{-1}(t|z)$. Hence, we can identify $F(v|z)$ using the generalization of (5):

$$F^{-1}(t|z) = \frac{1}{\lambda}\ln\frac{1}{t^M}\int_0^t Ms^{M-1}e^{\lambda G^{-1}(s|z)}\,\mathrm{d}s. \tag{12}$$

The next step is to generalize the estimation procedure presented in Section 4. The conditional quantile function $\widehat{G^{-1}}(t|z)$ may be estimated by several techniques and $F^{-1}(t|z)$ is then estimated by replacing G^{-1} by its estimator in equation (12).[10] We do not present the details of the asymptotic properties of this estimator. Briefly speaking, $\sqrt{nh_n^q}\left(\widehat{G^{-1}}(t|z) - G^{-1}(t|z)\right)$ converges almost surely to a Gaussian process, where h_n is a bandwidth of the estimation of $G^{-1}(t|z)$ (if kernel smoothing is used or h_n is any regularization parameter) and q is the dimension of z (see Stute (1986) and Van Keilegom (1998)). Then the results of Proposition 2 apply using a comparable argument where $\sqrt{n}$ is replaced by $\sqrt{nh_n^q}$ and where the covariance operators of the Gaussian processes become functions of z. Thus for a fixed z and a fixed M, $G^{-1}(t|z)$ still converges in process but at a nonparametric speed. If $N \to \infty$, $\widehat{G^{-1}}(t|z)$ converges in process to $G^{-1}(t|z)$ at a speed of $\sqrt{N}$. If M varies between the games we could estimate also λ using the arguments of this section and of Section 6. In case λ is not a constant and depends on the exogenous variables of the game, we would need to have exclusion conditions to be able to identify $\lambda(z_l)$ and $F(v|z_l)$. We leave this last case for further research.

9 Conclusions

This paper solves both for identification and estimation in a sealed-bid third-price auction model using a fully nonparametric quantile approach. We consider an i.i.d. private values environment with risk-averse bidders and we treat the case where the parameter of risk-aversion is known. Thus, using a global approach, we obtain nonparametric identification of the distribution of valuations by linking the quantiles of bids to the quantiles of valuations. Moreover, the speed of convergence of our nonparametric estimator is a parametric one. We have addressed three other issues: auctions with an unknown risk-aversion parameter, auctions where only the transaction prices are observed, and auctions with exogenous variables. To our knowledge, the identification issues concerning this type of auction have not been studied before. Although the third-price auction is an "exotic" type of game, solving for its identification helps our understanding of more common auction formats.

[10] Given that our approach is fully non-parametric we do not have to deal with the quantile crossing problem.

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Appendix

A Proof of Proposition 1

We rewrite equation (4) as:

$$G^{-1}(t) = F^{-1}(t) + \frac{1}{\lambda} \ln \left\{ 1 + \frac{\lambda}{M} \frac{F\left(F^{-1}(t)\right)}{f\left(F^{-1}(t)\right)} \right\}. \tag{A.1}$$

Equation (A.1) can be written as the following differential equation:

$$\beta(t) + \frac{t}{M}\beta'(t) = e^{\lambda G^{-1}(t)},$$

where $\beta(t) = e^{\lambda F^{-1}(t)}$.

Hence we managed to reduce our complex equation to a simple first-order differential equation. To solve this equation note that:

$$\frac{(t^M \beta(t))'}{M t^{M-1}} = \beta(t) + \frac{t}{M}\beta'(t)$$

and

$$t^M \beta(t) = \int_0^t M s^{M-1} e^{\lambda G^{-1}(s)} \, \mathrm{d}s + C.$$

Since $\beta(0) = 1$ implies that $C = 0$, the constant is identified. Moreover, using l'Hôpital's rule we have that:

$$\lim_{t \to 0^+} \frac{1}{t^M} \int_0^t M s^{M-1} e^{\lambda G^{-1}(s)} \, \mathrm{d}s = 1.$$

Thus the solution of (A.1) is given by:

$$F^{-1}(t) = \frac{1}{\lambda} \ln \frac{1}{t^M} \int_0^t M s^{M-1} e^{\lambda G^{-1}(s)} \, \mathrm{d}s, \tag{A.2}$$

where $F^{-1}(1) = 1$. The previous equation highlights a link between the quantile function of the private values and the quantile function of the bids. Therefore we obtain an explicit form for the solution of our structural equation and we can state the global identification for the sealed-bid third-price auction model (in the spirit of Guerre et al. (2000)). G^{-1} is identified by the data and therefore everything in equation (A.2) is identified. $\square$

B Proof of Proposition 2

We first recall some results about the estimation of G^{-1} in B.1. Then in B2 we analyze $\sqrt{n}\left(\widehat{F^{-1}} - F^{-1}\right)$ and in B3 we extend this result to $F^{-1\prime}$ and f.

B.1 Some results on the estimation of G^{-1}

Property 1. *G has a compact support $[0, x_{max}]$ with*

$$x_{max} = 1 + \frac{1}{\lambda} \ln\left\{1 + \frac{\lambda}{M}\frac{1}{f(1)}\right\} < \infty.$$

Property 2. *$G \in \mathscr{C}^2$ on $[0, x_{max}]$ and $g > 0$ on $(0, x_{max}]$.*

Thus G satisfies the necessary conditions in order to apply the following property:

Property 3. *(Lemma 21.4, Van der Vaart (1998)) G has compact support $[0, x_{max}]$ and is continuously differentiable on its support with strictly positive derivative g. Then, $\forall\ 0 < a < b < 1$, $\sqrt{n}\left(\widehat{G^{-1}} - G^{-1}\right)$ converges in distribution in $L^{\infty}[a,b]$ to a process $\dfrac{BB}{g\left(G^{-1}(t)\right)}$, where BB is the usual Brownian Bridge on $[0,1]$.*

The covariance of this process is:

$$\Sigma(s,t) = \frac{s(1-t)}{g\left(G^{-1}(s)\right) g\left(G^{-1}(t)\right)}, \tag{B.1}$$

where $0 < s \leq t \leq 1$.

Property 4. *Furthermore, G satisfies the conditions of Lemma 1.4.1 from Csörgö (1983) on the interval* $[a, x_{max}],\ a > 0$, *as* $G(x)(1-G(x))\frac{|g'(x)|}{g^2(x)}$ *is continuous on the compact* $[a, x_{max}]$ *and it is hence bounded by a* $\gamma > 0$.

Property 5. *This property refers to an extension of Dvoretzky-Kiefer-Wolfowitz inequality (see Dvoretzky et al. (1956)) to the sample quantile process. Given that G satisfies the Property 4, we have that (see Theorem 1.4.3 from Csörgö (1983)):*

$$\sqrt{n} \sup_{t \in [a,b]} |\widehat{G^{-1}}(t) - G^{-1}(t)| \sim O(1), \forall\ 0 < a < b < 1.$$

B.2 Asymptotic behavior of $\sqrt{n}\left(\widehat{F^{-1}} - F^{-1}\right)$

B.2.1 Step 1:

$$\sqrt{n}\left(\widehat{F^{-1}}(t) - F^{-1}(t)\right) = \frac{\int\limits_0^t s^{M-1} e^{\lambda G^{-1}(s)} \sqrt{n}\left(\widehat{G^{-1}} - G^{-1}\right)(s)\, \mathrm{d}s}{\int\limits_0^t s^{M-1} e^{\lambda G^{-1}(s)}\, \mathrm{d}s} + o_p(1).$$

Indeed if we consider F^{-1} as a function of G^{-1}, its Gâteaux derivative in G^{-1} as a function of a $\widetilde{G^{-1}}$, defined by $\frac{\partial}{\partial \alpha} F^{-1}\left(G^{-1} + \alpha\left(\widetilde{G^{-1}} - G^{-1}\right)\right)\Big|_{\alpha=0}$, verifies:

$$\mathrm{d}F^{-1}_{G^{-1}}\left(\widetilde{G^{-1}}\right) = \frac{\int\limits_0^t s^{M-1} e^{\lambda G^{-1}(s)}\, \widetilde{G^{-1}}(s) \mathrm{d}s}{\int\limits_0^t s^{M-1} e^{\lambda G^{-1}(s)}\, \mathrm{d}s}.$$

This expression is actually the Frechet derivative (for the L^∞ norm), because it is a continuous linear operator as a function of $\widetilde{G^{-1}}$ and it is continuous as a function of the two arguments G^{-1} and $\widetilde{G^{-1}}$ (see Nashed (1971)). A first order Taylor expansion gives the above expression. The behavior of the residual follows from Serfling (1980), Lemma B, page 218 and Property 5.

B.2.2 Step 2:

Let us consider a sequence $c_n = \frac{1}{n}\ln n$. If $t > c_n$, we have:

$$\sqrt{n}\left(\widehat{F^{-1}}(t) - F^{-1}(t)\right) = \frac{\int\limits_{c_n}^{t} s^{M-1} e^{\lambda G^{-1}(s)} \sqrt{n}\left(\widehat{G^{-1}} - G^{-1}\right)(s)\,\mathrm{d}s}{\int\limits_{0}^{t} s^{M-1} e^{\lambda G^{-1}(s)}\,\mathrm{d}s} + o_p(1).$$

Indeed, $\widehat{G^{-1}} - G^{-1}$ is bounded because $\widehat{G^{-1}}$ and G^{-1} are smaller or equal than x_{max} and

$$\int\limits_{0}^{c_n} s^{M-1} e^{\lambda G^{-1}(s)} \sqrt{n}\left(\widehat{G^{-1}}(s) - G^{-1}(s)\right)\mathrm{d}s = o_p(\sqrt{n}c_n^M) \Rightarrow o_p(1). \tag{B.2}$$

B.2.3 Step 3:

Let us consider the process $\delta = \frac{BB}{g(G^{-1}(t))}$, where BB is the Brownian Bridge for $t > c_n$ and $t < 1 - c_n$. Using Csörgö (1983) we have:

$$\sup_{c_n < s < 1 - c_n} \left|\sqrt{n}\left(\widehat{G}^{-1}(s) - G^{-1}(s) - \delta(s)\right)\right| = o_p(1). \tag{B.3}$$

Then for $c_n < t < 1 - c_n$:

$$\sqrt{n}\left(\widehat{F^{-1}}(t) - F^{-1}(t)\right) = \frac{\int\limits_{c_n}^{t} s^{M-1} e^{\lambda G^{-1}(s)} \delta(s)\,\mathrm{d}s}{\int\limits_{0}^{t} s^{M-1} e^{\lambda G^{-1}(s)}\,\mathrm{d}s} + o_p(1). \tag{B.4}$$

B.2.4 Step 4:

As a bounded linear transformation of the Gaussian process δ, the first part of the right-hand side of (B.4) (denoted ξ_n) is distributed as a Gaussian process with 0 mean and covariance kernel operator characterized by:

$$\Upsilon_n(s,t) = \frac{\int\limits_{c_n}^{t}\int\limits_{c_n}^{s} a(u)a(v)\sigma(u,v)\mathrm{d}u\mathrm{d}v}{b(s)b(t)}, \tag{B.5}$$

where $c_n < s < 1-c_n$, $c_n < t < 1-c_n$, $a(s) = s^{M-1}e^{\lambda G^{-1}(s)}$ and $b(t) = \int\limits_0^t s^{M-1}e^{\lambda G^{-1}(s)}\mathrm{d}s$ (see computation below).

B.2.5 Step 5:

If $0 < a \leq t \leq b < 1$ take $c_n < a$ and $b < 1-c_n$ for n large. Then the process ξ_n converges on $[a,b]$ to the zero mean Gaussian process with covariance kernel operator Υ, computed in (B.7). Indeed if $B_n\delta = \frac{\int\limits_{c_n}^{t} s^{M-1}e^{\lambda G^{-1}(s)}\delta(s)\,\mathrm{d}s}{\int\limits_0^t s^{M-1}e^{\lambda G^{-1}(s)}\,\mathrm{d}s}$ and $B\delta$ has the same expression with c_n replaced by 0, we have:

$$\sqrt{n}\sup_{t\in[a,b]}\left|B_n\delta - B\delta\right| \to 0 \text{ if } n\to\infty. \tag{B.6}$$

B.2.6 Step 6: Computation of the asymptotic covariance

We have to compute the covariance kernel operator of $\frac{\int\limits_0^t a(s)\delta(s)\mathrm{d}s}{b(t)}$, where $a(s) = s^{M-1}e^{\lambda G^{-1}(s)}$, $b(t) = \int\limits_0^t s^{M-1}e^{\lambda G^{-1}(s)}\,\mathrm{d}s$ and $\delta(t) = \sqrt{n}\left(\widehat{G^{-1}}(t) - G^{-1}(t)\right)$.

We compute the covariance operator as follows:

$$\begin{aligned}
\Upsilon(s,t) &= \mathrm{Cov}\left[\frac{1}{b(s)}\int_0^s a(v)\delta(v)\,\mathrm{d}v, \frac{1}{b(t)}\int_0^t a(u)\delta(u)\,\mathrm{d}u\right] \\
&= \frac{1}{b(s)b(t)}\mathbb{E}\left[\int_0^1 \mathbb{1}_{[v\leq s]}a(v)\delta(v)\mathrm{d}v \times \int_0^1 \mathbb{1}_{[u\leq t]}a(u)\delta(u)\,\mathrm{d}u\right] \\
&= \frac{1}{b(s)b(t)}\int_0^1\int_0^1 \mathbb{1}_{[v\leq s]}\mathbb{1}_{[u\leq t]}a(v)a(u)\mathbb{E}\left[\delta(v)\delta(u)\right]\mathrm{d}v\mathrm{d}u \\
&= \frac{1}{b(s)b(t)}\int_0^1\int_0^1 \mathbb{1}_{[v\leq s]}\mathbb{1}_{[u\leq t]}a(v)a(u)\Sigma(v,u)\,\mathrm{d}v\mathrm{d}u.
\end{aligned} \tag{B.7}$$

B.3 Asymptotic distribution of $\sqrt{n}\left(\widehat{F^{-1'}} - F^{-1'}\right)$

The proof of the asymptotic behavior of $\sqrt{n}\left(\widehat{F^{-1'}} - F^{-1'}\right)$ is very similar to the proof of the asymptotic distribution of $\sqrt{n}\left(\widehat{F^{-1}} - F^{-1}\right)$ and we just give the main elements of this analysis. The first step is to compute a Taylor expansion of our object:

$$\sqrt{n}\left(\widehat{F^{-1'}}(t) - F^{-1'}(t)\right) = \frac{t^{M-1}e^{\lambda G^{-1}(t)}\sqrt{n}\left(\widehat{G^{-1}}(t) - G^{-1}(t)\right)}{\int_0^t s^{M-1}e^{\lambda G^{-1}(s)}\,\mathrm{d}s} -$$

$$- \frac{t^{M-1}e^{\lambda G^{-1}(t)}\int_0^t s^{M-1}e^{\lambda G^{-1}(s)}\sqrt{n}\left(\widehat{G^{-1}}(s) - G^{-1}(s)\right)\mathrm{d}s}{\left(\int_0^t s^{M-1}e^{\lambda G^{-1}(s)}\,\mathrm{d}s\right)^2} + o_p(1).$$

The two components of the above expression are asymptotically Gaussian for $t \in [a,b]$ (0<a<b<1). The proof for the first component is a direct application of Property 3 and the second part should be treated in the same way as in (B.2). The final step of this analysis is to compute the asymptotic covariance kernel operator of the Gaussian process:

$$\Phi(s,t) = \mathrm{Cov}\left[\frac{a(s)\delta(s)}{b(s)} - \frac{a(s)\int\limits_0^s a(v)\delta(v)\,\mathrm{d}v}{(b(s))^2}, \frac{a(t)\delta(t)}{b(t)} - \frac{a(t)\int\limits_0^t a(u)\delta(u)\,\mathrm{d}u}{(b(t))^2}\right]$$

$$= \mathrm{Cov}\left[\frac{a(s)\delta(s)}{b(s)}, \frac{a(t)\delta(t)}{b(t)}\right] + 2\mathrm{Cov}\left[\frac{a(s)\delta(s)}{b(s)}, -\frac{a(t)\int\limits_0^t a(u)\delta(u)\,\mathrm{d}u}{(b(t))^2}\right]$$

$$+ \mathrm{Cov}\left[\frac{a(s)\int\limits_0^s a(v)\delta(v)\,\mathrm{d}v}{(b(s))^2}, \frac{a(t)\int\limits_0^t a(u)\delta(u)\,\mathrm{d}u}{(b(t))^2}\right].$$

Equivalently:

$$\Phi(s,t) = \frac{a(s)a(t)}{b(s)b(t)}\Sigma(s,t) - 2\frac{a(s)a(t)}{b(s)(b(t))^2}\mathbb{E}\left[\delta(s)\int\limits_0^t a(u)\delta(u)\,\mathrm{d}u\right]$$

$$+ \frac{a(s)a(t)}{(b(s))^2(b(t))^2}\mathbb{E}\left[\int\limits_0^1 \mathbb{1}_{[v\leq s]}a(v)\delta(v)\mathrm{d}v \times \int\limits_0^1 \mathbb{1}_{[u\leq t]}a(u)\delta(u)\,\mathrm{d}u\right].$$

The last term of the above sum can be written as:

$$\frac{a(s)a(t)}{(b(s))^2(b(t))^2}\int_0^1\int_0^1 \mathbb{1}_{[v\leq s]}\mathbb{1}_{[u\leq t]}a(v)a(u)\mathbb{E}\left[\delta(v)\delta(u)\right]\mathrm{d}v\mathrm{d}u$$

$$= \frac{a(s)a(t)}{(b(s))^2(b(t))^2}\int_0^1\int_0^1 \mathbb{1}_{[v\leq s]}\mathbb{1}_{[u\leq t]}a(v)a(u)\Sigma(v,u)\,\mathrm{d}v\mathrm{d}u$$

Thus, we finally obtain:

$$\begin{aligned}\Phi(s,t) &= \frac{a(s)a(t)}{b(s)b(t)}\Sigma(s,t) - 2\frac{a(s)a(t)}{b(s)(b(t))^2}\mathbb{E}\left[\delta(s)\int_0^t a(u)\delta(u)\,\mathrm{d}u\right] \\ &+ \frac{a(s)a(t)}{(b(s))^2(b(t))^2}\int_0^1\int_0^1 \mathbb{1}_{[v\leq s]}\mathbb{1}_{[u\leq t]}a(v)a(u)\Sigma(v,u)\,\mathrm{d}v\mathrm{d}u.\end{aligned} \tag{B.8}$$

B.4 Asymptotic covariance between the empirical processes of the quantile function and the quantile density of private values

$$\begin{aligned}\Delta(s,t) &= \mathrm{Cov}\left[\sqrt{n}\left(\widehat{F^{-1}}'(s) - F^{-1'}(s)\right), \sqrt{n}\left(\widehat{F^{-1}}(t) - F^{-1}(t)\right)\right] \\ &= \mathrm{Cov}\left[\frac{a(s)\delta(s)}{b(s)} - \frac{a(s)\int_0^s a(v)\delta(v)\,\mathrm{d}v}{(b(s))^2}, \frac{\int_0^t a(u)\delta(u)\,\mathrm{d}u}{b(t)}\right] \\ &= \frac{a(s)}{b(s)(b(t)}\mathbb{E}\left[\delta(s)\int_0^t a(u)\delta(u)\,\mathrm{d}u\right] - \frac{a(s)}{(b(s))^2 b(t)}\int_0^1\int_0^1 \mathbb{1}_{[v\leq s]}\mathbb{1}_{[u\leq t]}a(v)a(u)\Sigma(v,u)\,\mathrm{d}v\mathrm{d}u.\end{aligned} \tag{B.9}$$

B.5 Weak convergence for $\sqrt{n}\left(\widehat{f}-f\right)$

We just give the main steps of the proof as the technical details are identical to those given in Section B.2.

We start by the relation $f(\nu)=\dfrac{1}{F^{-1'}\left(F(\nu)\right)}$ and we compute a first-order Taylor approximation:

$$\sqrt{n}\left(\widehat{f(\nu)}-f(\nu)\right)\approx\frac{-\sqrt{n}\left(\widehat{F^{-1'}}-F^{-1'}\right)(F)+F^{-1''}(F)\times\dfrac{\sqrt{n}\left(\widehat{F^{-1}}-F^{-1}\right)(F)}{F^{-1'}(F)}}{\left[F^{-1'}(F)\right]^{2}}. \tag{B.10}$$

Using Von Mises calculus, we have the following result of weak convergence in the space $L^{\infty}[F^{-1}(a),F^{-1}(b)]$:

$$\sqrt{n}\left(\widehat{f}-f\right)\rightsquigarrow N(0,\Gamma)$$

The covariance kernel operator Γ satisfies:

$$\begin{aligned}\Gamma(s,t)=&\frac{1}{\left[F^{-1'}(F(s))\right]^{2}}\Phi(s,t)\frac{1}{\left[F^{-1'}(F(t))\right]^{2}}+\frac{F^{-1''}(F(s))}{\left[F^{-1'}(F(s))\right]^{3}}\Upsilon(s,t)\frac{F^{-1''}(F(t))}{\left[F^{-1'}(F(t))\right]^{3}}\\&-2\frac{1}{\left[F^{-1'}(F(s))\right]^{2}}\Delta(s,t)\frac{F^{-1''}(F(t))}{\left[F^{-1'}(F(t))\right]^{3}},\end{aligned} \tag{B.11}$$

where $\Delta(s,t)$ is the covariance kernel computed in equation (B.9).