

Dynamics of cash holdings, learning about profitability, and access to the market*

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Abstract: We study a dynamic model of a firm whose shareholders learn about its profitability, face costs of external financing and costs of holding cash. The shareholders' problem involves a notoriously challenging singular stochastic control problem with a two-dimensional degenerate diffusion process. We solve it by means of an explicit construction of its value function, and derive a corporate life-cycle with two stages: a “probation stage” where the firm has no access to capital markets, and a “mature stage” where the firm does have access to capital markets. The cash target level is nonmonotonic in the belief about the profitability and reaches its highest value at the access threshold to the market. It follows new insights on the firm's volatility and its payout ratio which depend on the firm's stage in its life cycle.

JEL classification: C02; C11; C61; G32; G33; G35

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1 Introduction

In this paper, we explain how learning about profitability impacts corporate cash management. While the trade-off between the costs and benefits of holding cash has received much attention in recent literature, little is known on the dynamics of this trade-off in a setting where profitability is difficult to ascertain. Nevertheless, this issue is key for corporations that are uncertain of the profitability of their project and face important external financing constraints. The dynamic interaction between learning about profitability and cash management is most relevant for young firms developing new technologies, conducting intensive research and development (R&D) and innovative activities.¹ For those firms, information problems and lack of collateral value of intangible assets make external capital very costly. This leads them to finance their activities with internal cash flow and to issue stock only when cash flow is exhausted.²

With few exceptions that we comment below, existing studies develop corporate cash models in a complete information setting and typically remain silent on the intertwining between holding cash and learning about profitability. Our paper contributes to close this gap. We develop a stylized continuous-time model of an all-equity firm confronted with three frictions: imperfect information about profitability, external financing costs and costs of holding cash. Cash constrained shareholders observe at any time realized earnings and update their beliefs about the firm’s profitability. They control the dynamics of cash through issuance and payment policies, thereby weighing the costs and benefits of holding cash. In such a framework, shareholders must cope with both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of having to liquidate a profitable project).

Formally, we solve a new two-dimensional singular control problem where the state variables are the controlled cash reserves and the profitability prospects resulting from Bayesian learning about the actual profitability. The problem is highly nontrivial. Intuitively, a positive shock to earnings increases the profitability prospects. This should facilitate external financing and induce the firm’s management to lower cash target levels. Nevertheless, a firm has more to lose from liquidity constraints when profitability prospects are high than it does when they are low. This may induce the firm’s management to accumulate more cash when the profitability prospects increase. We spell out these interactions and show that they result in a two-stage life-cycle for the firm: a “probation stage” in which the firm has no access to capital markets and a “mature stage” in which the firm does have access to capital markets. Thus, our model generates a corporate life-cycle dynamics of cash holdings that stems from

¹A striking textbook case is the funding crisis Intel faced in the early 1980s. At that time, the large potential of microprocessors was difficult to realize, which is a main reason external financing was extremely costly for Intel, as for instance documented in Passov (2003). We refer to Hall and Lerner (2010) and Kerr and Nanda (2015) for surveys on the financing of innovation and the role of learning in running innovative projects.

²Brown, Fazzari and Petersen (2009) point out this feature and find evidence that young, high-tech firms almost entirely financed by cash flow or public share issues explain most of the 1994 to 2004 aggregate R&D cycle. More recently, Graham and Leary (2018) find evidence that a large number of new public Nasdaq firms from 1980 to 2000 were holding large amounts of cash. They find that this effect was most pronounced among unprofitable, largely debt-free, high-growth, and high-volatility firms operating in the health care or high-tech industries.

the optimal equity issuance, payout and liquidation policies that we derive analytically. A rich set of implications follows. We highlight our primary findings here.

Issuance occurs when cash reserves are depleted if and only if the cost of issuance is not too high and profitability prospects are larger than an endogenous threshold required to access the capital markets. Above that threshold, the firm is in the mature stage and can issue new shares. Below that threshold, the firm is in the probation stage, cannot tap the market and is liquidated when cash is exhausted. Because of liquidity shocks, the firm can go back and forth between the two stages. Therefore, the terms “probation stage” and “mature stage” refer to the firm’s ability to finance its activity with new share issues. If the cost of issuance is high, the firm never issues new shares and is therefore limited to the probation stage.³

The uncertainty about the firm’s actual profitability impacts the corporate cash policy, which, in terms of cash target levels, changes as the firm learns about its profitability. We establish that a continuous function of the profitability prospects, *the dividend boundary*, characterizes the cash target levels. We show that when cash reserves reach the dividend boundary, shareholders pay out as dividends a fraction of the cash above the dividend boundary and reinvest the complement into the firm. The fraction that shareholders receive corresponds to the payout ratio of the firm and is also a function of the profitability prospects. Our theoretical analysis yields dynamics of cash holdings that are drastically different in the two different regimes of the corporate life cycle. Salient results are as follows.

- In the probation stage, the precautionary motive for holding cash is strong because the firm has no access to the market. The dividend boundary is increasing in the profitability prospects. The payout ratio is slightly increasing in the profitability prospects and takes very low value, which means that the firm pays little in dividends. We establish that the firm reaches its maximum cash target level on the edge between the probation stage and the mature stage.
- Shareholders of a firm entering into the mature stage have built a large amount of cash reserves and have increased profitability prospects. They optimally decide to initiate dividend payments. This causes a discontinuity in the payout ratio, the firm dis-saves and uses its reserves to pay more dividends than its last profit. Then, the payout ratio decreases as profitability prospects increase and tends to the payout ratio of the complete information benchmark of our model. In the mature stage, the dividend boundary is decreasing in the profitability prospects and tends toward the cash target level of the complete information benchmark.
- The profitability prospects required to access to the market, as well as the cash target levels, increase with the cost of external financing. Furthermore, the model predicts that a high-cost firm dis-saves more aggressively when its profitability is proven.

Therefore, our model generates life cycle dynamics and provides in a unified setting a theoretical ground for empirical evidences that, dividend policy, seasoned equity offerings, cash flow patterns are related to a firm’s life-cycle stage.⁴

³Our analysis characterizes high issuance costs.

⁴See for instance, DeAngelo, DeAngelo and Stulz (2006, 2010), Dickinson (2011). We discuss in section 5, the relation between our results and the empirical literature.

Additional economic insights follow from the model analysis. We find that a single indicator, defined as the firm’s performance, characterizes the two stages of the corporate life cycle. The firm’s performance reflects payments to shareholders, the issuance proceeds and the ex ante prior beliefs on the firm’s profitability. The firm’s performance remains unchanged as long as the firm neither pays dividends nor issues new securities. It increases whenever the firm reaches a cash target level and decreases whenever the firm issues new shares. The initial assessment of the firm’s profitability (for example, by specialized intermediaries for the financing of innovation) has lasting effects and matters for evaluating the firm’s performance at any date. We find that the firm has access to the capital markets if and only if its performance is above an endogenous threshold, hereafter *the market access threshold*. We show that for a fixed level of performance, the firm value can be described as a function of its cash reserves. That function is increasing and convex for a fixed performance level sufficiently below the market access threshold, while it is increasing and concave for a fixed performance level sufficiently above the market access threshold. The value of the firm as a function of its cash reserves is neither convex nor concave for fixed levels of performance close to the market access threshold. These properties are also unique to our model. Intuitively, in the probation stage, profitability is an important issue. One more unit of profitability prospects has a larger impact on the value of the firm than one more unit of cash. The opposite occurs in the mature stage. This yields the above concavity/convexity properties. A series of novel implications follow. The model predicts that the relationship between the volatility of the firm and its value depends on the firm’s stage in its life cycle. We get a positive relationship between the volatility of the firm and its value when the firm is far from accessing to the market and a negative relationship between firm value and volatility when the firm has proven access to the market. Close to the market access threshold, the volatility of the firm is an inverted U-shaped function of the value of the firm. Overall, our model suggests that most changes in the features of a firm’s key indicators (volatility, cash target levels, payout ratios) occur at transition phases between life-cycle stages.

All these results are grounded on a new contribution to the literature on stochastic control. We solve a two-dimensional Bayesian adaptive control problem that combines singular control and stopping.⁵ In stochastic control theory, the route to obtaining value functions and optimal control policies involves two steps: (i) derive a Hamilton-Jacobi-Bellman (HJB) equation whose solution gives a candidate value function, and, as a by-product, a candidate optimal policy, if any, (ii) leaning on regularity properties of the candidate value function, apply a verification lemma based on the Itô’s formula which asserts that the candidates are actually the value function and the optimal control. This procedure is anything but trivial. Academics have provided a lot of efforts to get sets of simple conditions that guarantee the regularity of the value function and the existence of optimal strategies for one-dimensional stopping/control problems.⁶ Difficulties are increasing in a two-dimensional framework: a closed form solution to the HJB equation cannot generally be provided and, deriving required regularity properties to apply the verification lemma and to prove the existence of an optimal control is very challenging. These technical difficulties reduce the scope of two-dimensional

⁵The literature on two-dimensional explicitly solvable control and stopping problems relies mainly on the study of leading examples. See e.g Benes, Shepp and Witsenhausen (1980), Peskir (1998), Karatzas, Ocone, Wang and Zervos (2000), Guo and Zervos (2010).

⁶See for instance Strulovici and Szydlowski (2015).

corporate cash models, which, for the largest majority, rely on numerical approximations that unfortunately do not characterize optimal policies. In contrast, we solve our two-dimensional control problem by means of an explicit construction of its value function and prove all the required regularity properties. It is worth noting that, in our problem the associated HJB equation has an uncountable set of solutions that satisfy the so called *super contact condition*.⁷ Among these solutions, only one satisfies the integrability conditions required to apply the verification procedure. A remarkable feature of our model is that this solution corresponds to the unique solution to the HJB equation that converges to the solution of the shareholders' problem when information about the profitability is complete. We fully characterize the optimal issuance and dividend policies which involve the analysis of three related highly non linear equations. A first equation defines the threshold about profitability prospects above which the firm optimally issues new shares. The condition of convergence towards the value function associated to the complete information benchmark dictates this threshold. Two other equations (one associated to the probation stage, one associated to the mature stage) define two functions that characterize the dividend boundary. In particular, an ordinary differential equation, which terminal condition accounts for the issuance policy, characterizes the dividend boundary in the probation stage. As we explained, the dividend boundary is continuous and nonmonotonic. It reaches its maximum at a point where it is not differentiable. This non differentiability drives the discontinuity of the payout ratio at the edge between the two stages. To the best of our knowledge, these results are unique to our study. They allow us to provide a rigorous model analysis and to obtain the rich set of results that we summarized above.

Relationship to the literature. Jeanblanc and Shiryaev (1995), Radner and Shepp (1996) have set the benchmark case for the analysis of corporate cash management in continuous time. The cumulative net cash flow generated by the firm follows an arithmetic Brownian motion. The constant drift represents the firm's profitability per unit of time, and the Brownian shock is interpreted as a liquidity shock. External financing is costly, which creates a precautionary demand for cash. Agency costs of free cash flow create a cost of carrying cash. This results in a unique optimal payout policy that requires paying shareholders 100% of earnings beyond an endogenous constant cash target level.⁸

These contributions have been extended in a number of directions. Lokka and Zervos (2008) introduce issuance costs into the analysis. Décamps, Mariotti, Rochet and Villeneuve (DMRV) (2011) study the interaction between cash management, agency costs, issuance costs and stock price; Bolton, Chen and Wang (2011) extend the model to the case of flexible firm size in order to study the dynamic patterns of corporate investment. Bolton, Chen and Wang (2013) and Hugonnier, Malamud and Morellec (2014) introduce capital supply uncertainty and the necessary time needed to secure outside funds into the analysis. Décamps, Gryglewicz, Morellec and Villeneuve (2017) assume that the firm's operating cash flow is proportional to profitability, the dynamics of which are governed by a geometric Brownian

⁷The super contact condition requires that the solution of the HJB Equation is twice continuously differentiable at the free boundary function. In standard one dimensional corporate cash models, the super contact condition characterizes the optimal policy. We refer to Dumas (1991) for an insightful discussion of the super contact condition as an optimality condition for singular control problems.

⁸Influential empirical papers driving the theory are Opler, Pinkowitz and Stulz (1999) and Bates, Kahle and Stulz (2009).

motion. This leads to a dividend boundary that is linear and increasing in profitability. Malamud and Zucchi (2019) develop a model of endogenous growth where firms face costly access to financial markets and hoard cash reserves to sustain innovation. All these contributions develop one-dimensional models (or can be reduced to one-dimensional models thanks to scaling properties).

Anderson and Caverhill (2012), Murto and Tervio (2014), Bolton, Wang and Yang (2019), Reppen, Rochet and Soner (2020) develop in different settings two dimensional corporate cash models with random profitability. As in our study, the firm’s decision depends upon two state variables, the current profitability and the current level of liquid assets. In contrast to our study, information about the parameters of the model is complete. In Anderson and Caverhill (2012) and Reppen, Rochet and Soner (2020) the long-term profitability of the firm’s project corresponds to the mean parameter of the drift of an Ornstein Uhlenbeck process. In Murto and Tervio (2014) and Bolton, Wang and Yang (2019), the long-term profitability corresponds to the drift of a Geometric Brownian motion that models the earnings fundamentals. These models share important common features. First, technical difficulties do not allow them to characterize the dividend boundary that stems from the trade-off between the costs and benefits of holding cash. Its study follows from a numerical treatment of the HJB equation that underlies the shareholders’ problem.⁹ Second, the models generate high cash target levels at the profitability threshold that triggers liquidation. The dividend boundary is decreasing in the profitability prospects, in absence of growth opportunities.¹⁰ The intuition is simple. The long-term profitability is known and positive. To avoid costly liquidation or issuance, shareholders increase cash target levels when the current profitability decreases. In contrast, in our learning model, the cash target level takes its minimum (which is equal to zero) at the belief threshold about the profitability below which the project is liquidated. Here also the intuition is simple. Shareholders decrease cash target levels when they become more and more convinced that the firm’s project is not profitable. Under the optimal policy, there is no more cash inside the firm when it is liquidated. Furthermore, we provide an analytical characterization of the dividend boundary. This function is single peaked when issuance costs are low and reaches its maximum at the market access threshold. It is increasing when the issuance costs are high because of the fear of having to liquidate a profitable project. Therefore, the two approaches correspond to different economic issues and generate different results. We take the view of a firm whose profitability is difficult to

⁹Interestingly, Reppen, Rochet and Soner (2020) use the theory of viscosity solutions to be the first to prove the continuity of the value function of the shareholders’ problem in this type of model. However, the concept of viscosity solutions is of no help to characterize the optimal dividend policy. Bolton, Wang and Yang (2019) proves regularity properties of the value function and characterize optimal policies when the firm accumulates cash at no cost. In this particular case the issue of cash target levels is irrelevant. The general case is tackled with numerical simulations. Introducing cash-carry costs into two dimensional corporate cash models raises major mathematical difficulties regarding the existence and the characterization of optimal policies.

¹⁰The situation is more intricate when a growth option increases profitability and volatility of earnings fundamentals as studied in Bolton, Wang and Yang (2019). In their model, the firm goes from a growth stage to a mature stage when it invests in the growth option. Their numerical simulations indicate that the dividend boundary in the mature stage is decreasing except in a small right neighborhood of the profitability threshold that triggers liquidation. The dividend boundary in the growth stage has a more complex shape. In both stages numerical simulations yield high cash target levels near liquidation.

ascertain. Shareholders estimate it from cash flow realizations.

Our study is naturally related to the corporate finance literature that emphasizes the role of learning about profitability and its importance for corporate decision-making. Pastor and Veronesi (2003) study stock prices in a model in which shareholders of an all-equity firm learn about profitability over time. Their model avoids both the liquidity and liquidation issues, assuming a peculiar dividend strategy that maintains a positive book value of the firm's equity at any time. In Moyen and Platinakov (2013), shareholders update their beliefs about firms' quality (high or low) in a dynamic Tobin's q framework. They find evidence that firms with unclear quality are more sensitive to earnings in their investment decisions than are well-established firms. In these models, cash-constrained firms become well established as time passes by learning about their profitability. Unlike in our model, cash management plays no role in these models. DeMarzo and Sannikov (2017) study a dynamic contracting model with learning about the profitability of the firm. In their model, asymmetric information arises endogenously because by shirking, an entrepreneur can distort the beliefs of investors about the project's profitability. The paper studies the relationship between incentives and learning.¹¹ Our focus is different. We do not model hidden actions. In our model, information is incomplete but symmetric between shareholders. We study the interplay between the evolution of profitability prospects and the evolution of the trade-off between the cost and benefit of holding cash.

Our problem is very simple in its formulation and is a natural extension of pioneering models. In our paper the drift parameter of the cumulative cash flow process is a random variable that can only take two opposite values over which shareholders learn over time. Our study is closely related to Gryglewicz (2011) and De Angelis (2020) who consider a similar setting. In Gryglewicz (2011) there are no frictions inside the firm, so holding cash is not costly. It follows that any dividend policy that maintains the reserves above a critical level is optimal. Gryglewicz (2011) studies how this framework impacts the optimal capital structure that results from the trade-off between tax shields and bankruptcy costs when equity financing is either costless or infinitely costly. De Angelis (2020) considers cash-carry costs and infinite issuance costs. De Angelis (2020) establishes for the first time a link between optimal dividends with partial information and the so called problems of optimal stopping with creation. This connection allows to establish regularity properties of the value function and to prove existence and uniqueness of a dividend boundary which is increasing in the profitability prospects. However, there is no explicit construction of the value function nor analytical characterization of the dividend boundary, which limits the economic analysis. Furthermore, applying his methodology to the case of equity issuance remains an open research question.¹² In our study, holding cash is costly and issuing new shares has a finite cost. We provide an explicit construction of the value function and of the optimal issuance and payment policies, which is unique to our study. We manage to solve the problem at the cost of a simplifying hypothesis with respect to De Angelis (2020). We consider that the possible values for the drift parameter are opposite (and not simply of opposite signs). We show that when the firm cannot issue new shares, an ordinary differential equation

¹¹See also the related studies Prat and Jovanovic (2014) and He, Wei, Yu and Gao (2017).

¹²When equity issuance is not allowed, the cash reserve process is absorbed at zero. This is a crucial feature for establishing the link between the shareholders' control problem and a stopping problem with creation.

characterizes the dividend boundary which is increasing in the profitability prospects.¹³

The paper is organized as follows. We lay out the model in Section 2. Section 3 studies benchmarks in which shareholders face profitability and liquidity concerns separately. Section 4 solves the model in a closed form and presents the optimal corporate policies. Section 5 develops the model implications. Section 6 concludes. All the proofs are in the Appendix.

2 The model

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a one-dimensional Brownian motion $W = \{W_t; t \geq 0\}$. On the same probability space, we also have a random variable Y which is independent of W and takes either of the two values $-\mu < 0 < \mu$.

2.1 Learning

A firm has a single investment project that generates random cash flows over time. The cumulative cash flow process $\{R_t; t \geq 0\}$ follows an arithmetic Brownian motion with unknown profitability Y and known variance σ

$$dR_t = Y dt + \sigma dW_t, \quad t \geq 0.$$

The firm is held by risk-neutral shareholders who do not fully know the project's profitability Y but observe the cumulative cash flow process $R = \{R_t; t \geq 0\}$. We denote by (\mathcal{F}_t^R) the filtration generated by the cash-flow process to model the flow of information available to shareholders.¹⁴ The conditional expectation

$$Y_t := \mathbb{E}[Y | \mathcal{F}_t^R], \quad (1)$$

defines the profitability prospects at time t . It is well-known that the process $\{Y_t; t \geq 0\}$ satisfies the filtering equation

$$dY_t = \frac{1}{\sigma}(\mu^2 - Y_t^2)dB_t, \quad (2)$$

where the so-called innovation process $\{B_t; t \geq 0\}$ is a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t^R; t \geq 0\}$ and satisfies

$$dB_t = \frac{1}{\sigma}(dR_t - Y_t dt). \quad (3)$$

The cumulative cash flow process is a sufficient statistic for Bayesian updating. Specifically, a direct application of Itô's formula yields the relation

$$dR_t = d\phi(Y_t), \quad (4)$$

¹³We first obtained this result in Décamps and Villeneuve (2015).

¹⁴Two filtrations on $(\Omega, \mathcal{F}, \mathbb{P})$ are worth emphasizing: one is the filtration generated by Y and W which corresponds to the information available to the modeller. The other is the filtration generated by R . The process W is not a Brownian motion under this filtration. See, for instance, Liptser and Shiryaev (1977).

where the function $\phi(y) = \frac{\sigma^2}{2\mu} \ln \left(\frac{\mu + y}{\mu - y} \right)$ is increasing and defined on $(-\mu, \mu)$. Finally, we obtain from (3) that

$$\mathbb{E} \left[\int_0^\infty e^{-rs} dR_s \right] = \mathbb{E} \left[\int_0^\infty e^{-rs} Y_s ds \right] \leq \frac{\mu}{r}.$$

Therefore, the present value of the future cash flows under partial information is lower than the present value of future cash flows of a project with observed profitability μ .

2.2 The shareholders' problem

Risk-neutral shareholders discount future payments at the risk-free interest rate $r > 0$ and must keep positive liquid reserves at all times if they want to avoid liquidation. The model builds on the standard *cost versus benefit trade-off of holding cash*. The firm accumulates cash for precautionary motives in a costly external financing environment. We allow the firm to increase its cash holdings or cover operating losses by raising funds in the capital markets. External financing involves a proportional cost $p > 1$: for each dollar of new shares issued, the firm only receives $1/p$ dollars in cash, where $p - 1 > 0$ represents the marginal issuance cost. Carrying cash is costly. Because of internal frictions, the rate of return of cash inside the firm is lower than the cost of capital. To capture in a simple and tractable way the agency costs of free cash flow, we consider that the cost per unit of time of holding one dollar of cash inside the firm corresponds to the cost of capital $r dt$. Shareholders can reduce these costs by deciding to distribute cash.¹⁵ In addition to this trade-off, shareholders are not aware of the profitability, positive or negative, of the firm's project. Thus, shareholders face both a profitability concern (the risk of running a project that is not profitable) and a liquidity concern (the risk of having to liquidate a profitable project).

Formally, at each date, shareholders decide whether to continue the project, whether to distribute dividends and whether to issue new shares. For simplicity, we assume that the liquidation value of the project is equal to 0. Let $D = \{D_t; t \geq 0\}$ and $I = \{I_t; t \geq 0\}$ be two $(\mathcal{F}_t^R)_{t \geq 0}$ -adapted, nondecreasing, right-continuous processes with $D_{0-} = I_{0-} = 0$. The process $D = \{D_t; t \geq 0\}$ represents the cumulative amount of dividends paid by the firm up to time t , while $I = \{I_t; t \geq 0\}$ represents the cumulative amount of equities issued by the firm up to time t . Under the policy (D, I) , the cumulative cash reserve process $X = \{X_t; t \geq 0\}$ evolves according to the dynamics

$$dX_t = dR_t + \frac{dI_t}{p} - dD_t. \quad (5)$$

Because D is nondecreasing, shareholders' limited liability is satisfied. In our setting, dividends can be distributed at no cost, and we do not need to allow share repurchases; thus, I is nondecreasing. Thus, dX_t , the cash reserves at time t , corresponds to the operating cash

¹⁵These modeling assumptions are standard in corporate cash models. Agency costs of free cash flow can be notably important in the innovation sector due, for instance, to moral hazard between the inventor and financiers. See, e.g., Allen and Michaely (2003), DeAngelo, DeAngelo and Skinner (2008), and Hall and Lerner (2010) for insightful surveys of the literature.

flow dR_t plus the cash flow from financing activities $\frac{dI_t}{p} - dD_t$, that is, the cash received from issuing securities minus the cash paid as dividends.

Using (4), we rewrite (5) in the form

$$X_t = \phi(Y_t) - \phi(Y_0) + X_0 + \frac{I_t}{p} - D_t. \quad (6)$$

Equation (6) is an accounting identity that specifies the relationships between cash reserves, profitability prospects, cumulative issuance and cumulative dividends. Observe that since ϕ is an increasing function, by holding the cash reserves X_t fixed and the cumulative issuance I_t fixed, the higher the cumulative dividend D_t is, the larger the profitability prospects Y_t . Thus, Equation (5) shows a positive relationship between cumulative dividend and profitability prospects, all else being equal. Similarly, there is a negative relationship between the cumulative issuance and the profitability prospects, all else being equal. Additionally, by holding the cumulative dividend and the cumulative issuance fixed, there is a one-to-one relationship between the cash reserves and the profitability prospects.

The firm ceases its activity for two possible reasons: (i) the firm cannot meet its short-term operating costs by issuing new shares or by drawing cash from its reserves, and/or (ii) shareholders strategically decide to liquidate because the profitability prospects are not high enough. Thus, given a policy (I, D) , equation (5) represents the dynamics of the cash reserves up to the liquidation time τ_0 defined as

$$\tau_0 = \inf\{t \geq 0 \mid X_t = 0\}.$$

For current cash reserves $x \in [0, \infty)$ and current profitability prospects $y \in (-\mu, \mu)$, the value of the firm corresponds to the expected present value of all future dividends minus the expected present value of all future gross issuance proceeds

$$\mathcal{V}(x, y; I, D) = \mathbb{E} \left[\int_0^{\tau_0} e^{-rt} (dD_t - dI_t) \right], \quad (7)$$

with $D_{\tau_0} - D_{\tau_0^-} = \max(X_{\tau_0^-}, 0)$. The case $D_{\tau_0} - D_{\tau_0^-} = X_{\tau_0^-} > 0$ corresponds to a strategic liquidation.¹⁶ The shareholders' problem is to find the optimal value function defined as the supremum of (7) over all admissible issuance and dividend policies

$$V^*(x, y) = \sup_{(I, D) \in \mathcal{A}} \mathcal{V}(x, y; I, D). \quad (8)$$

where \mathcal{A} is the set of policies (D, I) such that the associated cash reserves $\{X_t, \mathcal{F}_t^R; t \geq 0\}$ satisfy $X_t \geq 0$, $e^{-rt} X_t$ integrable and, $\lim_{t \rightarrow \infty} \mathbb{E}[e^{-rt} X_t] = 0$.

3 Benchmarks

Two polar cases are worth mentioning: (i) the case where shareholders face only a profitability concern and, (ii) the case where the shareholders face only a liquidity concern. These benchmarks provide natural bounds to the value of the firm V^* .

¹⁶In that case, shareholders take the remaining cash reserves $X_{\tau_0^-} > 0$ as dividends.

3.1 First-best benchmark

Let us assume that shareholders can issue and repurchase shares at no cost whenever they want. That is, we assume that $p = 1$ in (5). In this framework, shareholders face only a profitability concern. Accumulating cash does not bring any benefit, while carrying cash is costly. Intuitively, an optimal policy is to distribute all of the firm's initial cash reserves x as a special payment at date 0, to hold no cash beyond that time, and to liquidate when the profitability prospects are too low. In this first-best environment, the firm's value \hat{V} is equal to the sum of current cash reserves plus the option value to liquidate the firm¹⁷:

$$\hat{V}(x, y) \equiv x + \sup_{\tau \in \mathcal{T}^R} \mathbb{E} \left[\int_0^\tau e^{-rs} Y_s ds \right]. \quad (9)$$

The firm is optimally liquidated when the firm's profitability prospects hit the so-called first best liquidation threshold \hat{y} , which can be explicitly computed. Formally, the stopping time $\hat{\tau} = \inf\{t \geq 0 \mid Y_t = \hat{y}\}$ is optimal for problem (9). The mapping $y \rightarrow \hat{V}(x, y) - x$ is an increasing and convex function over $(-\mu, \mu)$. It corresponds to the option value to liquidate the firm.

The logic of Miller and Modigliani (1961) applies, and there is a large degree of freedom in designing the cumulative issuance and dividend processes that deliver the value (9). To see this, consider the payout policy $D_t = (x \mathbb{1}_{t=0} + lt) \mathbb{1}_{t \leq \hat{\tau}}$. That is, the firm distributes all its cash reserves at time 0 and pays an exogenous constant dividend flow $l > 0$ up to the stopping time $\hat{\tau}$, which corresponds to the date at which shareholders strategically decide to liquidate. To keep cash reserves constant and equal to zero after time 0, shareholders issue or repurchase new shares to offset earnings up to the liquidation date $\hat{\tau}$, that is, $I_t = ((Y_t - l)dt + \sigma dB_t) \mathbb{1}_{t \leq \hat{\tau}}$. Thus, we have

$$\hat{V}(x, y) = \mathbb{E} \left[\int_0^{\hat{\tau}} e^{-rt} (dD_t - dI_t) \right] = x + \sup_{\tau \in \mathcal{T}^R} \mathbb{E} \left[\int_0^\tau e^{-rs} Y_s ds \right].$$

To summarize,

Proposition 3.1 *Suppose that shareholders face only a profitability concern. Then, the following holds:*

- (i) *The value of the firm, $\hat{V}(x, y)$, is an increasing and convex function of the level y of the profitability prospects.*
- (ii) *Distributing all initial cash reserves x at time 0, holding no cash beyond that time, and liquidating at the stopping time $\hat{\tau} = \inf\{t \geq 0 : Y_t = \hat{y}\}$ are an optimal policy. There are many degrees of freedom in designing the dividend policy.*

Finally, the shareholders' value function V^* in (8) is bounded above by the benchmark \hat{V} .

Corollary 3.2 *The value functions V^* and \hat{V} satisfy $V^*(x, y) \leq \hat{V}(x, y)$ for any $(x, y) \in [0, \infty) \times (-\mu, \mu)$.*

¹⁷In (9), we denote by \mathcal{T}^R the set of \mathcal{F}^R -stopping times.

In particular, if the profitability prospects are lower than the first-best liquidation threshold \hat{y} , then the firm is liquidated regardless of the amount of cash within the firm. Formally, $V^*(x, y) = \hat{V}(x, y) = x$ for all $x \geq 0$ and $y \leq \hat{y}$.

3.2 Complete information benchmark

Another useful benchmark is the complete information setting in which shareholders face only a liquidity concern. This corresponds to the case where $y = -\mu$ or $y = \mu$ in the main problem (8).¹⁸ We shall denote by $V_{-\mu}(x)$ and $V_{\mu}(x)$ the associated values of the firm.

When $y = -\mu$, the firm's profitability is negative, and it is optimal for shareholders to take the initial cash reserves and to liquidate the firm at time $t = 0$. We have that $V_{-\mu}(x) = x, \forall x \geq 0$.

When $y = \mu$, the dynamics of the cash reserve process take the form

$$dX_t = \mu dt + \sigma dB_t + \frac{dI_t}{p} - dD_t.$$

and we revert to a classic case studied by many authors.¹⁹

Specifically, the value of the firm, $V_{\mu}(x)$, is an increasing and concave function of the level x of its cash reserves. The marginal value of cash, $V'_{\mu}(x)$, is strictly greater than one up to $x_{\mu} = \inf\{x > 0 \mid V'_{\mu}(x) = 1\}$. The threshold x_{μ} corresponds to the firm's cash target level at which dividends are paid. If cash holdings x exceed x_{μ} , the firm places no premium on internal funds, and it is optimal to make a lump sum payment $x - x_{\mu}$ to shareholders. Accordingly, $V_{\mu}(x) = x - x_{\mu} + V_{\mu}(x_{\mu})$ for any $x \geq x_{\mu}$.

Because external financing is costly, it is optimal to postpone the issuance of new shares for as long as possible: equity issuance only takes place whenever cash reserves are depleted and occurs if and only if the cost of issuance is not too high. Specifically, there exists a threshold \bar{p} such that there is equity issuance every time $X_t = 0$ if and only if $p < \bar{p}$.²⁰ In that case, the marginal benefit, $V'_{\mu}(0)$, is equal to the proportional issuance cost, p . Given that the value of the firm is concave in x , one obtains $V'_{\mu}(x) \leq p$ for $x \geq 0$. This means that it is indeed never optimal to issue new shares before cash reserves are depleted. Finally, the optimal issuance strategy induces a reflection at level zero of the cash reserve process so that infinitesimal amounts of new equity are issued every time $X_t = 0$.

The following proposition summarizes these standard results, first established in Lokka and Zervos (2008) and then used and generalized in several studies, especially in DMRV (2011) and Bolton, Chen and Wang (2011). We provide for completeness a rigorous statement in the Appendix.

Proposition 3.3 *Suppose that shareholders face only a liquidity concern, so $y = \mu$. Then, the value of the firm, $V_{\mu}(x)$, is an increasing and concave function of the level x of its cash reserves. Any excess of cash over the dividend boundary x_{μ} is paid out to shareholders, so the firm's payout ratio is 100%. Furthermore,*

¹⁸Technically, $-\mu$ and μ are absorbing barriers for the process Y : if $y = \mu$ (resp. $y = -\mu$), then $Y_t = \mu$ a.s. (resp. $Y_t = -\mu$ a.s.).

¹⁹See, for instance, the textbook by Moreno-Bromberg and Rochet (2018).

²⁰The analysis yields that $\bar{p} = \bar{V}'_{\mu}(0)$, where \bar{V}_{μ} corresponds to the value of the firm if issuance of new shares is not allowed (see the Appendix).

- (i) If issuance costs are high such that $p \geq \bar{p}$, then it is never optimal to issue new equities, and the firm is liquidated as soon as it runs out of cash.
- (ii) If issuance costs are low such that $p < \bar{p}$, then equity issuance takes place whenever the firm runs out of cash, so that the cash reserve process is reflected back whenever it hits 0, and the firm is never liquidated.

The complete information case also provides a useful upper-bound to the shareholders' value function V^* .

Corollary 3.4 *The value functions V^* and V_μ satisfy $V^*(x, y) \leq V_\mu(x)$ for any $(x, y) \in [0, \infty) \times (-\mu, \mu)$.*

As we will see now, an imperfect information about profitability dramatically impacts Proposition 3.3.

4 Model solution

The next section is heuristic and leads to a variational system that should satisfy the value of the firm V^* solution to (8).

4.1 Heuristic discussion

Taking the cash reserve and liquidating is an admissible policy, thus the value function V^* satisfies the inequality $V^*(x, y) \geq x$ for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$. From Corollary 3.2, $V^*(x, y) = x$ for all $y \leq \hat{y}$, where \hat{y} is the first-best liquidation threshold. To proceed further, we assume in this section that V^* is as smooth as necessary, and we derive some properties that V^* should satisfy.

Dynamic programming. Let us fix some pair $(x, y) \in (0, \infty) \times (-\mu, \mu)$. Let us consider the policy that consists of abstaining from issuing new shares and paying dividends for $t \wedge \tau_0$ units of time and, then, in applying the optimal policy associated with the resulting couple $(x + \int_0^{t \wedge \tau_0} Y_s ds + \sigma dB_s, y + \int_0^{t \wedge \tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2)dB_s)$, implied by dynamics (2) and (3). This policy must yield no more than the optimal policy:

$$\begin{aligned}
0 &\geq \mathbb{E} \left[e^{-r(t \wedge \tau_0)} V^* \left(x + \int_0^{t \wedge \tau_0} Y_s ds + \sigma dB_s, y + \int_0^{t \wedge \tau_0} \frac{1}{\sigma}(\mu^2 - Y_s^2)dB_s \right) \right] \\
&\quad - V^*(x, y) \\
&= \mathbb{E} \left[\int_0^{t \wedge \tau_0} e^{-rs} (\mathcal{L}V^*(X_s, Y_s) - rV^*(X_s, Y_s)) ds \right]. \tag{10}
\end{aligned}$$

The last equality follows from Itô's formula, where \mathcal{L} denotes the partial differential operator defined by

$$\mathcal{L}V(x, y) = \frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V_{yy} + \frac{1}{2}\sigma^2 V_{xx} + (\mu^2 - y^2)V_{xy} + yV_x.$$

Letting t go to zero in (10) yields

$$\mathcal{L}V^*(x, y) - rV^*(x, y) \leq 0$$

for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$.

Dividend boundary. The intuition that underlies the complete information benchmark applies: fix some $(x, y) \in (0, \infty) \times (-\mu, \mu)$; the policy that consists of making a payment $\varepsilon \in (0, x)$, and then immediately executing the optimal policy associated with cash reserves $x - \varepsilon$ must yield no more than the optimal policy. That is,

$$V^*(x, y) \geq V^*(x - \varepsilon, y) + \varepsilon.$$

Subtracting $V^*(x - \varepsilon, y)$ from both sides of this inequality, dividing through by ε and letting ε approach 0 yield

$$V_x^*(x, y) \geq 1$$

for all $(x, y) \in (0, \infty) \times (-\mu, \mu)$. It is expected that the inequality $V_x^*(x, y) > 1$ holds for any $x \in (0, b^*(y))$, where $b^*(y) = \inf\{x, V_x^*(x, y) = 1\} > 0$. Intuitively, for any fixed profitability prospects y , any excess of cash above $b^*(y)$ should be paid out. Therefore, the optimal cash policy should not be characterized by a constant threshold, as in the complete information benchmark, but rather by a dividend boundary $y \rightarrow b^*(y)$.

Issuance policy. If it is never optimal to issue new shares when the firm's profitability is known and equal to μ , then it should also never be optimal to issue new shares in the incomplete information setting. Thus, if $p \geq \bar{p}$, we expect that there are no equity issuances at all, so the firm is liquidated when it runs out of cash. If $1 < p < \bar{p}$, the logic of the complete information benchmark applies again: if there is any issuance activity, this must be when cash reserves drop down to zero to avoid liquidation. In such a situation, the marginal value of cash should be equal to the proportional issuance cost p , formally, $V_x^*(0, y) = p$. Intuitively, this latter equality should require that the profitability prospects when the cash reserves are depleted are sufficiently high. Accordingly, we conjecture the existence of an (endogenous) threshold y_i^* such that $V_x^*(0, y) = p$ for any $y \geq y_i^*$, whereas $V^*(0, y) = 0$ for any $y \leq y_i^*$. In this latter case, the profitability prospects are too low with regard to the cost of external financing, and the firm defaults when the cash reserves are depleted.

Convergence toward the complete information benchmark. Finally, we expect that when shareholders are increasingly confident that the profitability of the firm is μ , the value of the firm tends to the one derived in the complete information benchmark. We should have for all $x \geq 0$

$$\lim_{y \rightarrow \mu} V^*(x, y) = V_\mu(x). \quad (11)$$

One thus expects that the value function of the shareholders' problem (8) satisfies on $[0, \infty) \times (-\mu, \mu)$ the HJB Equation

$$\max(\mathcal{L}V - rV, 1 - V_x, V_x - p) \leq 0,$$

the condition $\max(-V(0, y), V_x(0, y) - p) = 0$ on $y \in (-\mu, \mu)$, and the limit condition (11). We rephrase this guess in terms of the following variational system: find a smooth function

V , a constant $y_i \in (-\mu, \mu)$ and a positive function b continuously differentiable almost everywhere over $(-\mu, \mu)$ that solve

$$\mathcal{L}V(x, y) - rV(x, y) = 0 \quad \text{on the domain } \{(x, y), 0 < x < b(y), -\mu < y < \mu\}, \quad (12)$$

$$V(0, y) = 0 \quad \forall y \in (-\mu, y_i], \quad (13)$$

$$V_x(0, y) = p \quad \forall y \in [y_i, \mu), \quad (14)$$

$$V_x(x, y) = 1, \text{ for } x \geq b(y), \quad (15)$$

$$V_{xy}(b(y), y) = 0, \quad (16)$$

$$\lim_{y \rightarrow \mu} V(x, y) = V_\mu(x) \quad \forall x \geq 0. \quad (17)$$

Equation (16) ensures that the mapping $x \rightarrow V(x, y)$ is twice continuously differentiable at the boundary function b so that the so-called super contact condition holds in our two-dimensional model. Specifically, differentiating equation $V_x(b(y), y) = 1$ with respect to y and using $V_{xy}(b(y), y) = 0$ lead to $V_{xx}(b(y), y) = 0$. The super contact condition characterizes optimality in standard one-dimensional singular control problems.²¹ In contrast, we will see that the system (12), (13), (14), (15) has an uncountable set of solutions that satisfy (16).²² It turns out that, the limit condition (17) pins down the only solution of system (12)-(16) that has bounded first derivatives. This result allows us to develop a verification procedure and to show that the unique solution (V, y_i^*, b^*) to the system (12)-(17) coincides with the value function V^* of the shareholders' problem, and that y_i^* and b^* characterize the optimal issuance and dividend policies. We obtain a closed form expression for V given y_i^* and b^* , and show that three highly non linear equations characterize the threshold y_i^* and the function b^* . A rich set of implications follows.

4.2 Solution to the shareholders' problem

In this section, we focus on the case $1 < p < \bar{p}$ and explain how we solve the system (12)-(17).²³ Then, we state our main result. Our analysis relies on a simple change of variable, which will be proved very useful for both the mathematical treatment and the economic analysis of the model.

We remark from equation (6) that, as long as the controls I and D are not activated, the process $Z = \{Z_t; t \geq 0\}$ with

$$Z_t \equiv \phi(Y_t) - X_t \quad (18)$$

remains constant. That is, keeping unchanged issuance and dividend policies, there is a one-to-one mapping between the cash reserves and the profitability prospects. The change of variable (18) allows us to restate problem (12)-(17) in the (z, y) -space and to solve it analytically. The change of variable (18) also provides new economic insights that we comment on below.

²¹This is notably the case for the complete information benchmark problem.

²²It is interesting to remark the analogy with two-dimensional stopping problems involving a running maximum process in which the associated variational inequality has an uncountable set of solutions that satisfy the so-called smooth-fit condition, see for instance Peskir (1998) and Guo and Zervos (2010).

²³We refer the reader to the Appendix for the complete analysis.

To develop the intuition, let us consider some admissible issuance and dividend policies I and D leading to cash reserve process

$$X_t = \phi(Y_t) - \phi(Y_0) + X_0 - D_t + \frac{I_t}{p}.$$

It follows that

$$Z_t = \phi(Y_t) - X_t = D_t - \frac{I_t}{p} + (\phi(Y_0) - X_0).$$

The process Z_t corresponds to the cash outflows from financing activities, $D_t - \frac{I_t}{p}$, corrected for the initial amount $\phi(Y_0) - X_0$. The process Z_t increases whenever the firm reaches a cash target level and decreases whenever the firm issues new shares. It measures the performance record of the firm at time t and defines a one-to-one mapping between profitability prospects and cash reserves that holds true as long as the firm neither pays dividends nor issues new securities.

Therefore, in our two-dimensional setting, the determinants of the shareholders' policy at a given date t are the cash position X_t and the profitability prospects Y_t ; or, equivalently, the firm's performance Z_t and the profitability prospects Y_t ; or, also equivalently, the cash position X_t and the firm's performance Z_t . We will use these three points of view to develop the economic implications of our model.

It is worth noting that the current cash outflow from financing activities, $D_t - \frac{I_t}{p}$, is not a sufficient statistic to define the performance of the firm at date t . The firm's performance at date t also depends on the initial profitability prospects Y_0 through the relation $\phi(Y_0) - X_0$. The initial profitability prospects Y_0 are not directly observable and follow, for instance, from a specific analysis by financial analysts of the relevance of the firm's project at the early stage of the firm's life. Thus, the performance of the firm is defined in light of the initial assessment of the profitability prospects. We will see that the firm's performance process Z indicates whether the firm can issue new shares if needed.

Using the change of variable (18), we define

$$U(z, y) \equiv V(\phi(y) - z, y),$$

and we restate problem (12)-(17) in the (z, y) -space.

In the (z, y) -space, the equation

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 V_{yy} + \frac{1}{2}\sigma^2 V_{xx} + (\mu^2 - y^2)V_{xy} + yV_x - rV(x, y) = 0$$

becomes

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y) = 0. \quad (19)$$

The solution to (19) is explicit and can be written in the form

$$U(z, y) = A(z)h_1(y) + B(z)h_2(y), \quad (20)$$

where $h_1(y) = (\mu + y)^\gamma(\mu - y)^{1-\gamma}$ and $h_2(y) = (y + \mu)^{1-\gamma}(\mu - y)^\gamma$, with γ being the negative root of the equation $x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0$. The functions A and B are to be determined.

The condition $V(0, y) = 0$ for all $y \leq y_i$ becomes

$$U(\phi(y), y) = U(z, \psi(z)) = 0, \text{ for all } z \leq z_i,$$

where $z_i \equiv \phi(y_i)$ and where $\psi(z) \equiv \phi^{-1}(z) = \mu \frac{e^{\frac{2\mu}{\sigma^2}z} - 1}{e^{\frac{2\mu}{\sigma^2}z} + 1}$. Thus, for a given performance z , the real number $\psi(z) \in (-\mu, +\mu)$ corresponds to the profitability prospects when cash reserves are depleted, that is, when $x = 0$.

The condition $V_x(0, y) = p$ becomes

$$U_z(z, \psi(z)) = -p,$$

which holds for any $z \geq z_i$. In other words, the firm issues new shares when cash reserves are depleted only if its performance is higher than $z_i \equiv \phi(y_i)$ (equivalently, if the profitability prospects when the cash reserves are depleted are higher than y_i). We call z_i , the market access threshold.

The condition $V_x(b(y), y) = 1$ becomes

$$U_z((\phi - b)(y), y) = U_z(z, k(z)) = -1,$$

where $k(z) \equiv \inf\{y \mid (\phi - b)(y) = z\}$. Thus, according to the change of variable (18), $k(z)$ corresponds to the profitability prospects at the cash target level $b(k(z)) = \phi(k(z)) - z$. Therefore, in our two-dimensional setting, each performance z defines a cash target level $b(k(z))$, the set of which forms the dividend boundary. We will see that the function $\phi - b$ is invertible, so $k(z) = (\phi - b)^{-1}(z)$. We will prove that k is an increasing function. Intuitively, the higher the firm's performance is, the higher the profitability prospects at the cash target level.

The super contact condition $V_{xy}(b(y), y) = 0$ becomes

$$U_{zy}((\phi - b)(y), y) = U_{zy}(z, k(z)) = 0.$$

The convergence condition to the complete information benchmark, $\lim_{y \rightarrow \mu} V(x, y) = V_\mu(x)$, becomes

$$\lim_{z \rightarrow \infty} U(z, \psi(x + z)) = V_\mu(x).$$

Indeed, the change of variable (18) leads to $y = \psi(x + z)$ and, in turn, $V(x, \psi(x + z)) = U(z, \psi(x + z))$.

Overall, the free boundary problem (12)-(17) writes in the (z, y) -space in the following form: find a function U , a constant z_i , and a function k that solve the variational system

$$\frac{1}{2\sigma^2}(\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y) = 0 \quad \text{on the domain} \\ \{(z, y), z \in \mathbb{R}, \psi(z) < y < k(z)\}, \quad (21)$$

$$U(z, \psi(z)) = 0 \quad \forall z \leq z_i, \quad (22)$$

$$U_z(z, \psi(z)) = -p \quad \forall z \geq z_i, \quad (23)$$

$$U_z(z, y) = -1, \quad \text{for } k(z) \leq y, \quad (24)$$

$$U_{zy}(z, k(z)) = 0. \quad (25)$$

$$\lim_{z \rightarrow \infty} U(z, \psi(x+z)) = V_\mu(x). \quad (26)$$

We find the unique solution (U, z_i^*, k^*) to the system (21)-(26). We obtain the solution (V, y_i^*, b^*) to the system (12)-(17) through the relations $U(z, y) = V(\phi(y) - z, y)$, $k^*(z) = (\phi - b^*)^{-1}(z)$ and, $z_i^* = \phi(y_i^*)$. We show that this solution coincides with the value function of the shareholder's problem and we characterize the optimal strategies. We establish these results in a series of Propositions in the Appendix. To ease its reading, we present informally below the main ideas that underly the proofs.

Let us consider a solution (U, z_i, k) , if any, to the system (21)-(25). Note that we do not take into account for the moment the condition (26). We obtain from (20), (23), (24), (25) that the function k satisfies for $z \geq z_i$

$$-h'_1(k(z))h_2(\psi(z)) + h'_2(k(z))h_1(\psi(z)) + \frac{2\mu^2}{\hat{y}}p = 0. \quad (27)$$

Proposition 8.11, assertion (i) establishes that (27) defines over $[z_i, +\infty)$ a unique continuously differentiable increasing function k . Proposition 8.11, assertion (ii) characterizes the unique z_i^* that satisfies (26). We obtain from (20), (22), (24), (25) that, for $z \leq z_i^*$, the function k satisfies an ordinary differential equation, which terminal condition ensures the continuity of k at z_i^* . This uniquely defines a function k^* associated to the system (21)-(26). Getting this ordinary differential equation is quite involved and requires the analysis of the case where shareholders are not allowed to issue new shares, which we develop as a preliminary step in section 8.3.1. Equation (99) in Proposition 8.11, provides the full description of the function k^* which therefore involves three highly non linear equations.²⁴ Proposition 8.12 establishes that the above procedure determines a unique solution (V, y_i^*, b^*) to the system (12)-(17). The function V has a closed form expression given $z_i^* = \phi(y_i^*)$ and $k^* = (\phi - b^*)^{-1}$. The function b^* is not differentiable at $k^*(\phi(y_i^*))$. We check that the value function V solution to the system (12)-(17) coincides with the value function V^* solution to the shareholder's problem. To this end, Proposition 8.13 shows that the solution to (12)-(17) satisfies the verification Lemma 8.9. Notably, Proposition 8.13 shows that a solution V to the system (12)-(16) has bounded first derivatives if and only if V converges to the value function of the complete information benchmark, that is, if and only if condition (17) holds.

²⁴Precisely, Equation (27), Equation (98) in Proposition 8.11 that characterizes z_i^* and, the ordinary differential equation studied in Lemma 8.5 with the terminal terminal condition that ensures the continuity of k^* at z_i^* as stated in assertion (iii) of Proposition 8.11.

Thus, a remarkable feature of our two-dimensional control problem is that the condition (17) pins down the unique solution to the system (12)-(16) which coincides with the value function of the shareholder's problem. Proposition 8.14 describes the optimal issuance and dividend policies. Finally, Proposition 8.15 establishes that the dividend boundary function b^* is increasing in the profitability prospects when issuance costs are large (in which case $y_i^* = \mu$), and nonmonotonic in the profitability prospects when issuance costs are low ($p < \bar{p}$). It reaches its maximum at $k^*(z_i^*) = k^*(\phi(y_i^*))$

The next Theorem summarizes our results. Its statement is in mirror of proposition 3.3.

Theorem 4.1 *The value of the firm V^* solving (8) coincides with the unique solution (V, y_i^*, b^*) to the system (12)-(17). The threshold y_i^* corresponds to the required profitability prospects above which the firm issues new shares when the cash reserves are depleted and below which the firm is liquidated when it runs out of cash. Payments are made whenever cash reserves hit the dividend boundary function b^* . The function b^* is continuous over $[-\mu, \mu]$ and satisfies $b^*(y) = 0$ for $y \leq y^*$ and $b^*(\mu) = x_\mu$. The threshold x_μ is the constant dividend boundary of the complete information benchmark. The threshold $y^* = \max\{y \in (-\mu, \mu) \mid y = b^{*-1}(0)\} > \hat{y}$ corresponds to the minimum profitability prospects required by shareholders to run the project. The optimal cash reserve process is reflected along the function b^* in a horizontal direction on the (x, y) -plane. Furthermore:*

(i) *If the proportional issuance cost p satisfies $p \geq \bar{p}$,*

- $y_i^* = \mu$, so it is never optimal to recapitalize the firm. The firm is liquidated when it runs out of cash, $V^*(0, y) = 0$ for all $y \in (-\mu, \mu)$.
- The dividend $f b^*$ is continuously increasing and differentiable.

(ii) *If the proportional issuance cost p satisfies $1 < p < \bar{p}$,*

- $y_i^* \in (y^*, \mu)$, so that equity issuance takes place whenever the firm runs out of cash if and only if the profitability prospects are greater than the threshold y_i^* . At issuance dates, the optimal cash reserve process is reflected in a horizontal direction on the (x, y) -plane, the marginal value of cash $V_x^*(0, y)$ is equal to the issuance cost p , and $V^*(0, y) > 0$ for all $y > y_i^*$.
- The dividend boundary b^* is increasing for $y \leq k^*(z_i^*)$ and decreasing for $y \geq k^*(z_i^*)$, where $z_i^* = \phi(y_i^*)$ and $k^* = (\phi - b^*)^{-1}$. The maximum cash target level, $b^*(k^*(z_i^*))$, satisfies $b^*(k^*(z_i^*)) > x_\mu$. The dividend boundary is not differentiable at $k^*(z_i^*)$.

Theorem 4.1 delivers several results. In our model, uncertainty about the firm's profitability impacts the corporate cash policy, which, in terms of cash target levels, changes as the firm learns about its profitability. Two opposite effects are at work. First, a positive shock to earnings increases profitability prospects and may induce the firm's management to lower cash target levels because of the cost of accumulating cash. Second, a firm has more to lose from liquidity constraints when profitability prospects are high than when they are low. This may induce the firm's management to accumulate more cash when profitability prospects increase. Theorem 4.1 shows that the second effect dominates when the firm has

no access to capital markets, while the first effect dominates when the firm does have access to capital markets. As a consequence, the dividend boundary b^* is increasing in the profitability prospects for high issuance costs (when $p \geq \bar{p}$) and is nonmonotonic in the profitability prospects for low issuance costs (when $p < \bar{p}$). In the latter case, shareholders increase the cash target levels after a new performance record as long as the firm cannot tap the market, that is, as long as the firm's performance is lower than the market access threshold z_i^* (equivalently, as long as the profitability prospects are lower than y_i^* when the cash reserves are depleted). When the cash reserves reach the level $b^*(k^*(z_i^*))$, shareholders decide to lower the cash target levels because the firm now has access to the capital markets and has an important amount of cash reserves that is costly to bear. Note that the dividend boundary converges to the complete information dividend threshold x_μ when the profitability prospects tend to μ . This implies the inequality $b^*(k^*(z_i^*)) > x_\mu$. Thus, the corporate cash target level is at its highest value at the market access threshold z_i^* . At that moment, the firm has a higher cash target level than what would have been optimal in a complete information setting. Another salient feature that we comment on in the next section is the nondifferentiability of b^* at $k^*(z_i^*)$.

When $p < \bar{p}$, in contrast to the complete information benchmark, firms either optimally issue new equity or are liquidated. That is, if the profitability prospects when cash reserves are depleted are above y_i^* , then the firm issues new shares, whereas the firm is liquidated if profitability prospects are lower than y_i^* . This result echoes the empirical findings of DeAngelo, DeAngelo, and Stulz (2010) that a SEO reflects the corporate life cycle and that, without the offer proceeds, most firms would run out of cash the year after the SEO. Thus, in our model, profitability issues create liquidity issues, and ultimately, the firm is liquidated for liquidity reasons. This latter feature is specific to our model in which random profitability is linked to bayesian learning about the project's profitability.

Finally, let us note that the profitability prospects y^* required to run the project are strictly larger than the first-best liquidation threshold \hat{y} . In particular, a negative exogenous shock that leads to profitability prospects below y^* triggers liquidation even if cash reserves are abundant.

In the next section, we alleviate the notations and we write V , y_i , z_i , k and b instead of V^* , y_i^* , z_i^* , k^* and b^* . Theorem 4.1 allows this shortcut since it establishes that the solution to the system (12)-(17) coincides with the value function of the shareholders problem and characterizes optimal policies.

5 Model Analysis

Despite its two-dimensional nature, our model leads to analytical formulae that allow for a rigorous analysis of the interplay between dynamics of cash holdings and dynamics of profitability prospects. Our analytical formulae also allow simple numerical illustrations that yield additional insights.

Figure 1 plots on the (x, y) -plane the dividend boundary b and the curves $z = \phi(y) - x$ that link cash reserves and profitability prospects for a firm's different performance levels,

z . It illustrates the joint dynamics of cash and profitability prospects.²⁵

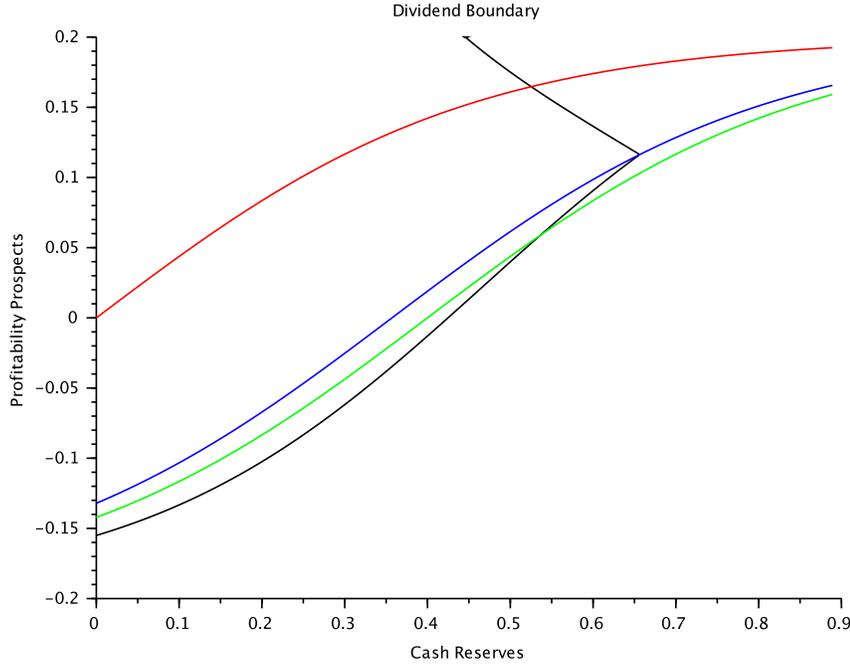


Figure 1: The dividend boundary b (black curve) and three performance level curves $z = \phi(y) - x$ with, from bottom to top, $z = -0.40$, $z = z_i$, $z = 0$. The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.5$. For those parameters, $y_i = -0.1321$, $z_i = -0.3571$, $k(z_i) = k(\phi(y_i)) = 0.1163$, $b(k(z_i)) = 0.6563$ and $x_\mu = 0.44$. Accordingly, $z_i = \phi(y_i)$ corresponds to the minimum level of performance that allows the firm to issue new shares when cash reserves are depleted, and $k(z_i) = (\phi - b)^{-1}(z_i)$ corresponds to the profitability prospects at the cash target level when the performance is z_i .

The joint dynamics of cash and profitability prospects: an illustration based on Figure 1 . Assume that, at date $t = 0$, the cash reserves X_0 and the profitability prospects Y_0 satisfy the equation $Z_0 = \phi(Y_0) - X_0$ with $Z_0 = -0.40$ such that the pair (X_0, Y_0) is on the green curve in Figure 1. The amount $Z_0 = -0.40$ corresponds to the initial value of the performance process $Z_t = \phi(Y_t) - X_t = Z_0 + D_t - \frac{I_t}{p}$. As long as there are neither payments nor issuances, $D_t = I_t = 0$, the performance process Z remains constant. Therefore, the two-dimensional process (X_t, Y_t) satisfies $\phi(Y_t) - X_t = Z_0$ and thus evolves on the green curve. If the cash reserves increase to the point of exceeding the dividend boundary b , cash is paid out, the cash reserve process is reflected back in the horizontal direction on the (x, y) -plane, and the performance process increases. If performance records accumulate, the process Z_t will

²⁵To ease comparisons, we use in our simulations the baseline parameters considered in studies in which shareholders face only a liquidity concern; see, for instance, Bolton, Chen and Wang (2011) or DMRV (2011). The parameters r , μ and σ are annualized; σ and μ are expressed in millions of dollars.

eventually increase to the value $z = 0$, so that the process (X_t, Y_t) will satisfy $\phi(Y_t) - X_t = 0$ and thus will evolve on the red curve.

Consider now that the cash reserves decrease after a series of negative shocks on cash flows to hit zero. Shareholders then decide whether to issue new shares. They do so whenever the profitability prospects are larger than y_i or, equivalently, whenever the performance of the firm is above z_i . When the firm issues new shares, the cash reserve process is reflected back in the horizontal direction on the (x, y) -plane, and the performance process decreases accordingly. If, as time passes, cumulative issuances become too large, the process (X_t, Y_t) will eventually evolve on a constant-performance curve below the blue curve $z_i = \phi(y) - x$. The firm then runs the risk of being liquidated because, for such a level of performance, the profitability prospects are too low compared to the financing cost p to allow shareholders to issue new shares when cash reserves are depleted.

Our model results in a two-stage life cycle for the firm: a “probation” stage in which the firm has no access to capital markets and a “mature” stage in which the firm does have access to capital markets. In the probation stage, cash target levels are increasing in the profitability prospects. In the mature stage, the precautionary motive for holding cash weakens, and cash target levels decline when the profitability prospects increase. Therefore, the dividend boundary b attains its maximum when the firm is at the market access threshold as Figure 1 illustrates. Figure 1 also illustrates that the function b attains its maximum at a kink, meaning that it is non differentiable at its maximum as stated in Theorem 4.1. We shall comment further these results and the relationship with the firm’s life cycle theory in the next paragraph where we analyse our model’s implications on the firm’s payout ratio.

Finally, it is worth noting the importance of the initial value of profitability prospects. Two identical firms with the same cash outflow from financing activities, $D_t - \frac{I_t}{p} > 0$, but with different initial profitability prospects and thus different values for Z_0 , for example, $Z_0^1 < z_i < Z_0^2$, can be in drastically different situations when cash reserves are depleted: it can happen that firm 1 is liquidated because $Z_t^1 = Z_0^1 + D_t - \frac{I_t}{p} < z_i$, whereas firm 2 issues new shares because $Z_t^2 = Z_0^2 + D_t - \frac{I_t}{p} \geq z_i$. Thus, our model predicts that the initial profitability prospects, which may result, for instance, from financial analysts, have lasting effects on the corporate cash policy. This feature is consistent with the important role of specialized intermediaires in the financing of innovation as, for instance, pointed in Kerr and Nanda (2015).

Payout ratios. What does a firm do with a marginal \$1 when it is at its target level of cash? In models with only liquidity issues where the cash target level is a constant threshold, the firm pays out \$1 as a dividend. This generates the prediction that the payout ratio of a firm is constant and equal to 100%. In a two dimensional model with random profitability, one expects that the payout ratio is a function of the profitability prospects. As we emphasized, existing studies rely on numerical schemes which makes difficult to push further the analysis. In contrast, we provide an analytical characterization of the dividend boundary and are able to study the payout ratio that our learning model induces. This is a unique feature of our paper.

Specifically, suppose that the firm is at its cash target level $x = b(y)$ and consider what happens after a positive shock to the cash flow. To account for the sign of the change in cash flow over a small period of time h , we consider a \sqrt{h} Euler approximation of the model,

$$X_h = x + \sigma\sqrt{h}B_1,$$

and

$$Y_h = y + \frac{\mu^2 - y^2}{\sigma}\sqrt{h}B_1,$$

where B_1 is a standard Gaussian variable. Therefore, $X_h - x$, the amount of cash available for distribution at time h , and $X_h - b(Y_h)$, the amount paid out to shareholders at time h , satisfy

$$\begin{aligned} X_h - b(Y_h) &= X_h - (b(y) + b'(y)(Y_h - y)) \\ &= X_h - x - b'(y)\frac{\mu^2 - y^2}{\sigma}\sqrt{h}B_1 \\ &= P(y)(X_h - x), \end{aligned}$$

with

$$P(y) = 1 - b'(y)\frac{\mu^2 - y^2}{\sigma^2}. \quad (28)$$

The function $P(y)$ indicates, for profitability prospects y , the percentage of each dollar earned above the dividend boundary that is distributed to shareholders in the form of cash. It corresponds to the firm's payout ratio. Therefore, in our two-dimensional model, a dividend decision is made when cash reserves reach the cash target level $b(y)$. Shareholders then receive $P(y)$ percent of the cash above $b(y)$ and reinvest the complement into the firm.

Formula (28) relates the payout ratio to the derivative of the dividend boundary b and yields the following result. In the probation stage, after a new performance record the firm increases its cash target level, (the derivative of the dividend boundary is positive). It follows from (28) that the payout ratio is lower than 100%. In the mature stage, the firm has access to the capital market. It decreases its cash target level after a performance record (the derivative of the dividend boundary is negative). The firm has a payout ratio larger than 100% meaning that the firm dis-saves and uses its reserves to pay more dividends than its last profit. The decision to decrease cash target levels yields a discontinuity in the payout ratio at $k(z_i)$ that originates from the nondifferentiability of b at $k(z_i)$. This break reflects the change in the cash management policy when changing between corporate life-cycle stages. When its performance level allows the firm to enter the mature stage, the cost of holding cash becomes prominent, leading to a change in the payout policy. It is worth noting that the jump in the payout ratio is not related to a jump in firm value, which continuously evolves as a function of cash reserves and profitability prospects. All our findings result from the (endogenous) dynamics of the interaction between learning about profitability and the cost and benefit of holding cash. Let us also observe that because of the nonmonotonicity of the dividend boundary, the cash target level is not a sufficient statistic to infer the payout ratio. We must know what stage the firm is in to deduce the payout ratio from cash target levels. Finally, the payout ratio tends toward 100% when y tends toward μ , reflecting the fact that the model converges to the complete information benchmark.

Numerical simulations provide additional insights. Figure 2 is representative of our numerical simulations and plots the payout ratio for proportional issuance costs $p = 1.5$ and $p = 1.05$ (with other baseline parameters remaining the same). Due to the single peaked form of the dividend boundary, our model suggests overall a positive relationship between cash target levels and the payout ratio. Precisely, in the probation stage, the payout ratio is increasing in the profitability prospects. In the mature stage, it is decreasing for large issuance costs (left). For low issuance costs (right), the payout ratio is slightly decreasing when the firm enters in its mature phase, and then it is increasing. From a quantitative point of view, our model suggests that a cash-constrained firm which profitability is difficult to ascertain pays very little in dividends as long as it has no access to the market and initiates dividend when it enters into the mature stage. This effect is even more true when the cost of external financing is high. In particular a firm in a costly issuance environment has a larger payout ratio when it enters into the mature stage than a firm in a cheap issuance environment. The reason is that both the firm's optimal level of cash and required profitability prospects to access the mature stage increase with the cost of external financing. It follows that a high-cost firm has more slack with which to respond to positive shocks to earnings and initiates high payments (equivalently dis-saves aggressively) in the mature stage to counteract the cost of holding cash. Specifically, Figure 2 shows for $p = 1.5$ (left) payout ratios lower than 10% in the probation stage. When entering the mature stage, for profitability prospects just above the threshold $k(z_i) = 0,1163$, the payout ratio is around 180%. The maximum cash target level, $b(k(z_i)) = 0,6563$, is far above the cash target level $x_\mu = 0,4457$ that prevails in the complete information case. For $p = 1.05$, payout ratios in the probation stage are lower than 30%. When entering the mature phase, for profitability prospects just above the threshold $k(z_i) = -0,1025$, the payout ratio is around 105%. Here, the maximum cash target level $0,2358$ is slightly higher than the constant cash target level of the complete information case $x_\mu = 0,1829$.

These findings provide new insights on the firm life cycle theory, which develops the idea that the trade-off between the advantages and costs of cash retention evolves over time as profits accumulate and investment opportunities decline.²⁶ In our model the corporate life cycle does not rely on the firm's ability to find new profitable investment opportunities, but on its ability to process information about the profitability of its ongoing project. Within our simple learning model, dynamics of cash target levels (increasing in the probation stage, decreasing in the mature stage) characterize the firm's life cycle. At the market access threshold, the cash target level is at its highest level and the firm has sufficiently high profitability prospects (both of these levels are all the higher because external financing costs are high). This provides the firm with the ability to generate and sustain free cash flows. It is then optimal to initiate dividend payments. This causes a jump in the payout ratio function as already highlighted. In line with the firm life cycle theory of dividends, the optimal dividend policy in our model depends on the firm's stage in its life cycle. Furthermore, because of the nonmonotonicity of the dividend boundary, we can have large cash holdings with both low or high payout ratios.

These results are in line with several recent documented features. Drobetz, Halling and

²⁶See the seminal paper Mueller (1972), and Bulan and Subramanian (2009) for a survey on the firm life cycle theory of dividends.

Schroder (2015) use the Dickinson (2011) methodology and relate the corporate life cycle to the dynamics and valuation of cash holdings. They notably find evidence of increases in cash holdings in stages of the life cycle where access to the market is more constrained and of decreases in cash holdings when the firms move toward maturity. Faff, Kwok, Podolski and Wong (2016) point out the nonmonotonicity of the firm’s cash holdings across the corporate life-cycles and question the relevance of cash holdings as a good proxy of the stages of the life-cycle. Bulan, Subramanian and Tanlu (2007) document that firms that initiate dividends have increased their profitability and have greater cash reserves. Our results are also in line with DeAngelo, DeAngelo and Stulz (2006), whose analysis suggests that, if well-established firms had not paid dividends as observed, their cash balances would be enormous, thus granting extreme discretion to managers of these firms. DeAngelo, DeAngelo and Stulz (2006) find also that the relation between cash holdings and propensity to pay dividends is ambiguous. Their proposed explanation for this is that cash holdings “could indicate a build-up of excess funds (best suited for distribution) or of resources to fund an abundance of new investments (best suited for retention)”. Interestingly, the nonmonotonicity of cash target levels that we derive in our model delivers another explanation based on shareholders bayesian learning about the project’s profitability.

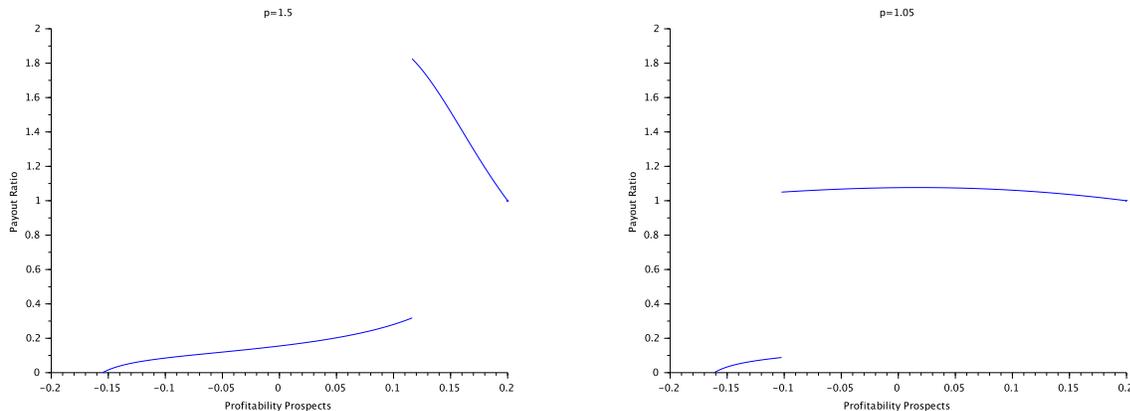


Figure 2: The payout ratio as a function of profitability prospects. (Left) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.5$. (Right) The parameters are $r = 0.1$, $\sigma = 0.3$, $\mu = 0.2$ and $p = 1.05$

Firm dynamics. Our model also delivers new insights into the dynamics of firm value across life stages. Let us first consider the firm value process between two consecutive dates of issuance decision and payment decision. We saw that for a given performance $z \in [\phi(y^*), \infty)$, payment occurs at time $\tau_z \equiv \{t \geq 0 \mid X_t = b(k(z))\}$, and issuance (or liquidation) at time $\tau_0 = \{t \geq 0 \mid X_{t-} = 0\}$. Thus, the firm value process can be written on time interval $[0, \tau_0 \wedge \tau_z]$ as a function of the cash reserve process $\{X_t, t \geq 0\}$. We use the change of variable (18) to obtain that $V(x, y) = V(x, \psi(x + z))$, and we denote $W(x, z) \equiv V(x, \psi(x + z))$. Then, by applying Itô’s formula to the process $\{W(X_t, z), t \geq 0\}$, we easily obtain the following

proposition.

Proposition 5.1 *For a given performance, $z \geq \phi(y^*)$, the mapping $x \rightarrow W(x, z)$ is increasing on $[0, b(k(z))]$. The firm value process $\{W(X_t, z), t \geq 0\}$ satisfies, for any $t \in [0, \tau_0 \wedge \tau_z]$, the dynamics*

$$dW(X_t, z) = rW(X_t, z) dt + \sigma W_x(X_t, z) dB_t,$$

with

$$\sigma W_x(x, z) = \sigma V_x(x, \psi(x + z)) + \sigma \psi'(x + z) V_y(x, \psi(x + z)), \quad (29)$$

so that the volatility of the firm at cash target $b(k(z))$ satisfies

$$\sigma W_x(b(k(z)), z) = \sigma + \frac{\mu^2 - k(z)^2}{\sigma} V_y(b(k(z)), k(z)). \quad (30)$$

The mapping $x \rightarrow W(x, z)$ represents the value of the firm as a function of the cash reserves for a fixed level of performance z .²⁷ It corresponds to the value of the firm between two consecutive dates of issuance decision and payment decision. Formulae (29) and (30) give the volatility of the firm as a function of the cash reserves.

For a fixed performance z , an increase in cash also increases profitability prospects so that the relation $x - \phi(y) = z$ holds true between two consecutive dates of issuance decision and payment decision. Then, two effects are at work. First, an increase in cash diminishes the precautionary motive for holding cash. The marginal value of cash decreases in cash levels. Accordingly, one more unit of cash has a larger impact on the value of the firm when the latter is initially low than when it is high. Formally, the mapping $x \rightarrow V(x, y)$ is concave, as proven in the Appendix.²⁸ Second, an increase in cash increases profitability prospects. The ongoing project is more likely to have a positive profitability. Accordingly, one more unit of profitability prospects has a larger impact on the value of the firm when the latter is initially high than when it is low. Formally, the mapping $y \rightarrow V(x, y)$ is convex.

Our numerical study shows that the first effect dominates when the firm has proven access to the market, in the sense that its performance z is sufficiently above the market access threshold z_i . The second effect, which we call the learning effect, dominates when the firm is fully in the probation stage in the sense that its performance z is sufficiently below z_i . Close to the market access threshold (when z is around z_i), both effects are present. Specifically, Figure 3 illustrates that the mapping $x \rightarrow W(x, z)$ is concave for z values sufficiently above z_i , convex for z values sufficiently below z_i , and convex then concave for z values around z_i .

Additionally, Figure 3 illustrates that the mapping $z \rightarrow W(x, z)$ is increasing, as we show in the Appendix. A direct implication follows: Because the firm's performance process is increasing in cumulative payments and decreasing in the cumulative issuances, our two-dimensional model predicts that, by holding liquid assets constant, there is a positive relationship between payments and firm value and a negative relationship between issuances and firm value.

Now, we turn to the volatility of the firm. Figure 4 depicts, for different issuance costs, the volatility of the firm as a function of the cash reserves with different levels of performance z

²⁷The amount $\phi(y^*)$ is the minimum level of performance required by shareholders to run the firm.

²⁸See the proof of Proposition 8.13.

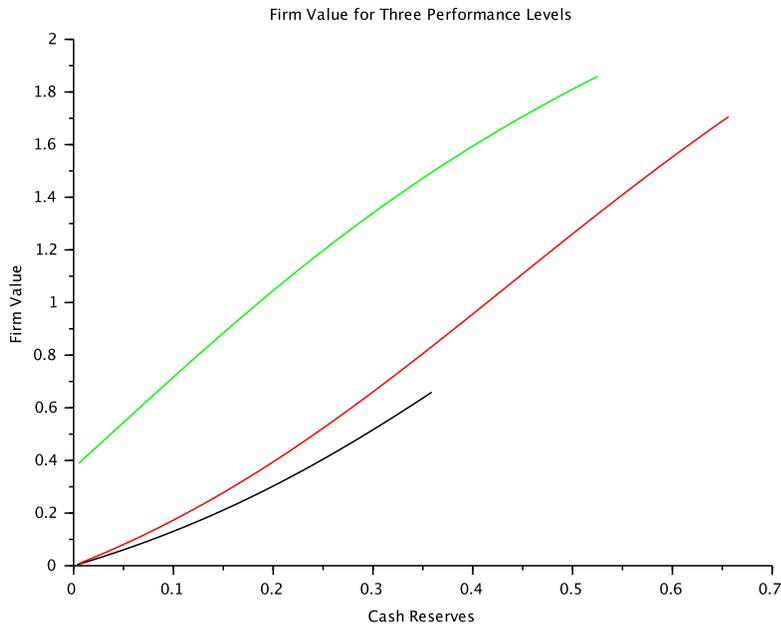


Figure 3: The value of the firm as a function of cash reserves for three levels of performance, $z = -0.4$ (black curve), $z = z_i$ (red curve) and $z = 0$ (green curve). The parameters are $r = 0.1$, $\mu = 0.2$, $\sigma = 0.3$ and $p = 1.5$.

(Equation (29)). In line with the previous observations, the volatility of the firm is decreasing in the cash reserves for z sufficiently above z_i and increasing in the cash reserves for z sufficiently below z_i . Close to the market access threshold (when z is around z_i), the volatility of the firm is increasing and then decreasing in the cash reserves.

Noting that the mapping $x \rightarrow W(x, z)$ is increasing, we obtain new testable results on the relationship between the value of the firm and its volatility across firm life-cycle stages. Our model predicts that, for firms with proven access to the market, we should observe a negative relationship between firm value and volatility. For firms still far from accessing the market, we should observe a positive relationship between firm value and volatility. The negative relationship between the firm's value and its volatility is a standard feature of corporate cash models with complete information.²⁹ The positive relationship between the volatility of the firm and its value is a well-known feature of corporate models with deep-pocketed shareholders³⁰ and holds true in our first-best benchmark. Our study shows that the result still holds for cash-constrained shareholders who learn about profitability and have no access to the financial markets. Close to the market access threshold, the volatility of the firm is an inverted U-shaped function of the value of the firm, reflecting the two effects. Therefore, our model suggests that the relationships between the value of the firm and its volatility can drastically change in transition phases between life-cycle stages.

²⁹See Bolton, Chen, and Wang (2011) and DMRV (2011), among many others.

³⁰We refer to the literature initiated by Merton (1973) and Leland (1994).

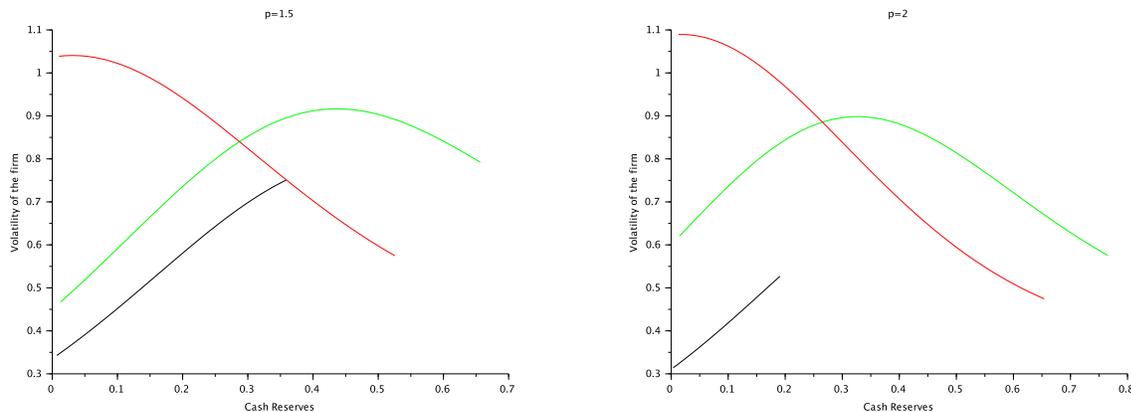


Figure 4: The volatility of the firm as a function of cash reserves for two different levels of issuance costs. (Left) The parameters are $r = 0.1$, $\mu = 0.2$, $\sigma = 0.3$, $z = -0.4$ (black curve), $z = z_i$ (green curve) and $z = 0$ (red curve) and $p = 1.5$. (Right) The parameters are $r = 0.1$, $\mu = 0.2$, $\sigma = 0.3$, $z = -0.4$ (black curve), $z = z_i$ (green curve) and $z = 0$ (red curve) and $p = 2$.

In the complete information benchmark, the volatility of the firm corresponds to the volatility of the cash flows times the marginal value of cash (that is, $\sigma V'_\mu(x)$ with the notations of Proposition 2). The volatility of the firm is more involved in our study. The novel insight is the learning effect that links the volatility of the firm and the marginal value of profitability prospects (that is, the second term of the right-hand side of (29)). Equation (30) highlights that the volatility of the firm at cash target levels is larger than the volatility of cash flows, whereas these volatilities coincide in the complete information benchmark.³¹

Two additional insights on the firm's volatility follow from our model. First, we observe in Figure 4 that the volatility of the firm when cash reserves are depleted is increasing in the performance level z . Thus, the model predicts that the higher the profitability prospects, the higher the volatility of the firm is at issuance dates. Second, at cash target levels, the firm reaches its higher level of volatility when it is at the threshold of access to capital markets, that is, for $z = z_i$. This is consistent with the prediction that the dynamics of cash holdings change drastically when the firm's performance crosses the threshold z_i .

6 Conclusion

We study the corporate cash management of an all-equity firm that is conducting a project whose profitability is unknown and that faces external financing costs and costs of holding cash. We show that the trade-off between the benefits and the costs of holding cash evolves

³¹Technically, at the limit, for $z = \infty$, the model coincides with the complete information benchmark. We show in the Appendix (see Propositions 8.7 and 8.13) that $\lim_{z \rightarrow \infty} W(x, z) = V_\mu(x)$ and that $\lim_{z \rightarrow \infty} W_x(x, z) = V'_\mu(x)$. In particular, we have the equality $\sigma \lim_{z \rightarrow \infty} W_x(b(k(z)), z) = \sigma$.

as shareholders learn about profitability and yields a two-stage life-cycle for the firm: a probation stage and a mature stage. In the mature stage, the firm issues new shares when cash is exhausted. In the probation stage, the firm has no access to capital markets and is liquidated when cash reserves are depleted. As a consequence, issuing new shares requires a sufficiently high level of profitability prospects. A payment decision is made when the total flow of cash into the firm reaches a target level, which depends on the profitability prospects. The cash target level is nonmonotonic in the profitability prospects. It increases in the probation stage and decreases in the mature stage. Thus, the firm reaches its maximum level of cash reserves at the market access threshold. The firm's payout ratio is a function of the profitability prospects and reflects the firm's ability to raise cash on the market. Payouts are low in the probation stage and high in the mature stage. The payout ratio function is not continuous at the market access threshold. This reflects the firm's decision to initiate dividends. When the costs of external financing increase, both the profitability prospects necessary to access to the market and the cash target levels increase. This implies higher payout ratios when the firm access to the market. The model offers a rationale for several stylized facts stemming from the life-cycle theory of the firm. It also delivers new predictions that call for additional theoretical and empirical studies. In particular, the model suggests that the firm's volatility depends on the firm's stage in its life cycle. In the probation stage, the model predicts a positive relationship between the firm's value and its volatility, while the relationship is negative for firms with proven access to the market. Close to the market access threshold, the volatility of the firm is an inverted U-shaped function of the value of the firm. Overall, our model suggests that corporate cash policy and the dynamics of firm value drastically change in transition phases between life-cycle stages. Additionally, the model suggests that the initial assessment of the firm's profitability has long lasting consequences on its corporate cash management and matters for evaluating its performance at any future date.

These results rely on two unique features of our model: First, random profitability follows from bayesian learning. This notably yields a dividend boundary that has different properties than those obtained in models with complete information about the parameters of the model. Second, we solve the shareholders' two-dimensional control problem by means of an explicit construction of its value function. This allows a complete and rigorous analysis of the model based on analytical characterizations of optimal policies.

Our concern to develop a simple and natural model that allows a precise analysis of the economic forces led us to introduce several simplifications. For example, we have assumed proportional issuance costs. In this setting, equity issues occur in infinitesimal amounts. Introducing fixed issuance costs require the study of a difficult mixed singular/impulse control problem with stopping which is beyond the scope of this paper.³² A combination of proportional and fixed issuance costs would clearly imply lumpy equity issuance. Also, the property that the dividend boundary is increasing in the profitability prospects in the probation stage should be robust to the introduction of fixed issuance costs.

³²Even in a one dimensional model, considering both proportional and fixed issuance costs is challenging and most of the studies rely on numerical approximations. An exception is DMRV (2011) who provide a complete mathematical analysis. In two dimensional models with random profitability, numerical approximations suggest that the presence of fixed costs on top of proportional issuance costs does not impact the general decreasing shape of the dividend boundary, see for instance Reppen, Rochet and Soner (2020).

Another natural extension of our model is to introduce capital stock adjustment over time. This can be done in the line of Bolton Chen and Wang (2011) with assumptions guaranteeing that the cash to capital ratio and the belief about the firm's profitability would be the relevant two state variables for the model analysis. This approach should lead to an interesting dynamic Tobin's q framework with bayesian learning and financing constraints, allowing a richer corporate life-cycle analysis.

Finally, as explained in the introduction, our model addresses the case of all-equity young firms that have little collateral to offer. Well-established firms with more elaborated financial structure face also corporate cash management issues, and learn about their profitability when they launch new projects or engage in major restructurings. Clearly, future learning models should integrate into the analysis a wider range of financial tools, especially debt issuance and the use of credit lines. We lack of studies on the joint evolution of cash holdings, debt, and profitability prospects. In addition, note that other learning issues arise in corporate finance. For instance, some studies focus on learning about the state of the economy, or learning about a rival's characteristics, nevertheless avoiding cash management considerations.³³ Here again, we lack of studies that integrate these learning issues in a setting of constrained financing. These and related questions must await for future work.

³³See for instance, Grenadier and Malenko (2010), Décamps and Mariotti (2004).

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8 Appendix

8.1 First-best benchmark

Proposition 3.1 relies on the following Lemma.

Lemma 8.1 *The value of the firm V^* can be written in the form*

$$V^*(x, y) = x + \sup_{(I, D) \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} (Y_s - rX_s) ds \right] \leq x + \frac{\mu}{r}. \quad (31)$$

Proof of Lemma 8.1. For all $t > 0$ and all admissible controls I and D , we have

$$e^{-r(t \wedge \tau_0)} X_{t \wedge \tau_0} = x + \int_0^{t \wedge \tau_0} e^{-rs} dX_s - r \int_0^{t \wedge \tau_0} e^{-rs} X_s ds.$$

Because X_t is nonnegative for $t \leq \tau_0$ and Y is bounded above by μ , we obtain

$$\begin{aligned} e^{-r(t \wedge \tau_0)} X_{t \wedge \tau_0} + \int_0^{t \wedge \tau_0} e^{-rs} (dD_s - dI_s) &= x + \int_0^{t \wedge \tau_0} e^{-rs} dR_s - r \int_0^{t \wedge \tau_0} e^{-rs} X_s ds \quad (32) \\ &\leq x + \frac{\mu}{r} + \sigma \int_0^{t \wedge \tau_0} e^{-rs} dB_s. \end{aligned}$$

Applying the optional sampling theorem to the uniformly \mathcal{F}^R -martingale $\left(\int_0^t e^{-rs} dB_s \right)_{t \geq 0}$, we obtain after taking expectations,

$$\mathbb{E} \left[e^{-r(t \wedge \tau_0)} X_{t \wedge \tau_0} + \int_0^{t \wedge \tau_0} e^{-rs} (dD_s - dI_s) \right] \leq x + \frac{\mu}{r}.$$

Letting t goes to $+\infty$, we obtain for all $(x, y) \in [0, \infty) \times (-\mu, \mu)$

$$\mathcal{V}(x, y; I, D) \leq x + \frac{\mu}{r} < \infty.$$

Thus, the value function V^* is finite. We obtain (31) from equations (3) and (32). \square

Proof of Proposition 3.1. Let us define

$$\Gamma(y) = \sup_{\tau \in \mathcal{T}^R} \mathbb{E} \left[\int_0^{\tau} e^{-rs} Y_s ds \right].$$

Standard results in optimal stopping theory³⁴ yield that the optimal value function Γ is C^1 on $[0, \infty)$ and that a threshold strategy $\hat{\tau} = \inf\{t \geq 0 \mid Y_t = \hat{y}\}$ is optimal. The value function Γ can be written in terms of the free boundary problem:

$$\begin{cases} \frac{1}{2\sigma^2}(y + \mu)^2(\mu - y)^2\Gamma''(y) - r\Gamma(y) + y = 0, & y \geq \hat{y}, \\ \Gamma(\hat{y}) = 0, & \Gamma'(\hat{y}) = 0. \end{cases}$$

³⁴See for instance Peskir and Shiryaev (2006).

Standard computations yield

$$\begin{cases} \hat{V}(x, y) = x, & -\mu < y \leq \hat{y}, \\ \hat{V}(x, y) = x + \frac{y}{r} - \frac{h_1(y)}{h_1(\hat{y})} \frac{\hat{y}}{r} & \hat{y} \leq y < \mu, \end{cases} \quad (33)$$

where

$$h_1(y) = (\mu + y)^\gamma (\mu - y)^{1-\gamma}, \quad \hat{y} = \frac{-\mu}{1 - 2\gamma} < 0, \quad (34)$$

$$\text{and where } \gamma \text{ is the negative root of the equation } x^2 - x - \frac{r\sigma^2}{2\mu^2} = 0. \quad (35)$$

It follows from (33), (34) that $\hat{V}(x, y)$ is increasing and convex in y .

Proof of Corollary 3.2. Because the cash reserves are positive for all admissible controls, it follows from (31) that,

$$x + \sup_{(I, D) \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} Y_s ds \right]$$

is an upper bound for V^* . From Equation (3), any admissible control (I, D) acts on $\mathbb{E} \left[\int_0^{\tau_0} e^{-rs} Y_s ds \right]$ by modifying only the \mathcal{F}^R -stopping time τ_0 . Thus,

$$\sup_{(I, D) \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} Y_s ds \right] \leq \sup_{\tau \in \mathcal{T}^R} \mathbb{E} \left[\int_0^{\tau} e^{-rs} Y_s ds \right],$$

which implies that the function V^* is bounded above by \hat{V} . \square

8.2 Complete information benchmark.

In this section, we consider that the firm's profitability is known and is equal to μ . We develop the mathematical formulation of Proposition 3.3. This formulation yields useful formulae for the proof of Theorem 4.1. We prove also in this section the Corollary 3.4.

8.2.1 No equity issuance

We start the analysis with the case where security issuances are not allowed. The dynamics of the cash reserves satisfy

$$dX_t = \mu dt + \sigma dB_t - dD_t,$$

and the shareholders' problem writes

$$\bar{V}_\mu(x) = \sup_{D \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} dD_s \right], \quad (36)$$

where $\tau_0 = \inf\{t \geq 0 \mid X_t = 0\}$. The following result is due to Jeanblanc and Shiryaev (1995).

Proposition 8.2 *The value function \bar{V}_μ of problem (36) is concave, twice continuously differentiable and it satisfies the following HJB equation on $(0, +\infty)$:*

$$\max \left\{ \frac{\sigma^2}{2} \bar{V}_\mu'' + \mu \bar{V}_\mu' - r \bar{V}_\mu, 1 - \bar{V}_\mu' \right\} = 0.$$

Moreover, we have

$$\begin{cases} \bar{V}_\mu(x) = \frac{e^{-\beta\gamma x} - e^{\beta(\gamma-1)x}}{-\beta\gamma e^{-\beta\gamma\bar{x}_\mu} + (1-\gamma)\beta e^{\beta(\gamma-1)\bar{x}_\mu}} & 0 \leq x \leq \bar{x}_\mu, \\ \bar{V}_\mu(x) = x - \bar{x}_\mu + \frac{\mu}{r}, & x \geq \bar{x}_\mu, \end{cases} \quad (37)$$

with

$$\bar{x}_\mu = \frac{1}{\beta(1-2\gamma)} \ln\left(\frac{1-\gamma}{\gamma}\right)^2 = 2\frac{\hat{y}}{\mu}\phi(\hat{y}), \quad (38)$$

where (34) and (35) define \hat{y} and γ and where $\beta = \frac{2\mu}{\sigma^2}$. Any excess of cash over the dividend boundary \bar{x}_μ is paid out to shareholders, such that the cash reserve process is reflected back each time it reaches \bar{x}_μ . The process $D = \{D_t; t \geq 0\}$ with

$$D_t = (x - \bar{x}_\mu)^+ \mathbb{1}_{t=0} + L_t^{\bar{x}_\mu} \mathbb{1}_{t>0} \quad (39)$$

is the optimal dividend payment process. In equation (39), $L_t^{\bar{x}_\mu}$ denotes the so-called local time process solution to the Skorohod problem³⁵ at \bar{x}_μ for the drifted Brownian motion $\mu t + B_t$.

8.2.2 Equity issuance

When security issuances are allowed at a proportional issuance cost $p > 1$, the dynamics of the cash reserves satisfy

$$dX_t = \mu dt + \sigma dB_t - dD_t + \frac{dI_t}{p},$$

and shareholders' problem writes

$$V_\mu(x) = \sup_{I, D \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} (dD_s - dI_s) \right], \quad (40)$$

where $\tau_0 = \inf\{t \geq 0 \mid X_t = 0\}$. The following proposition is due to Lokka and Zervos (2008) and provides a rigorous formulation of Proposition 3.3 in the main text.

Proposition 8.3 *The value function defined in (40) is concave, twice continuously differentiable and it satisfies the following HJB equation on $(0, +\infty)$:*

$$\max \left\{ \frac{\sigma^2}{2} V_\mu'' + \mu V_\mu' - r V_\mu, 1 - V_\mu', V_\mu' - p \right\} = 0.$$

Moreover, we have

- If $p \geq \bar{V}'_\mu(0)$ then, $V_\mu(x) = \bar{V}_\mu(x)$ for all $x \geq 0$.

³⁵See Karatzas and Shreve (1991) page 210.

- If $p < \bar{V}'_\mu(0)$ then,

$$\begin{cases} V_\mu(x) = \frac{1-\gamma}{\beta\gamma} \frac{\hat{y}}{\mu} e^{-\beta\gamma(x-x_\mu)} + \frac{\gamma}{\beta(\gamma-1)} \frac{\hat{y}}{\mu} e^{\beta(\gamma-1)(x-x_\mu)} & 0 \leq x \leq x_\mu, \\ V_\mu(x) = x - x_\mu + \frac{\mu}{r}, & x \geq x_\mu, \end{cases} \quad (41)$$

where x_μ is defined as the unique solution to the equation

$$p = -\frac{\hat{y}}{\mu}(1-\gamma)e^{\gamma\beta x_\mu} + \frac{\hat{y}}{\mu}\gamma e^{(1-\gamma)\beta x_\mu}. \quad (42)$$

Any excess of cash over the dividend boundary x_μ is paid out to shareholders, so that the cash reserve process is reflected back each time it reaches x_μ . There is equity issuance whenever the firm runs out of cash, so that the cash reserve process is reflected back each time it reaches 0. The processes $D = \{D_t; t \geq 0\}$ and $I = \{I_t; t \geq 0\}$ with

$$D_t = (x - x_\mu)^+ \mathbb{1}_{t=0} + L_t^{x_\mu} \mathbb{1}_{t>0} \quad \text{and} \quad I_t = L_t^0 \mathbb{1}_{t>0} \quad (43)$$

are the optimal dividend payment and equity issuance processes. In equation (43), L^{x_μ} and L^0 denote the solution to the Skorohod problem at x_μ and at 0 for the drifted Brownian motion $\mu t + B_t$.

Hereafter, we will note $\bar{p} = \bar{V}'_\mu(0)$. The thresholds \bar{x}_μ and x_μ defined in (38) and (42) satisfy $\bar{x}_\mu > x_\mu$. Moreover, equation (37) yields that $\bar{p} = \frac{1-\gamma}{-\gamma} e^{\gamma\beta\bar{x}_\mu}$. We deduce that for $1 < p < \bar{p}$, we have

$$p < \frac{1-\gamma}{-\gamma} e^{\beta\gamma x_\mu}. \quad (44)$$

We will use later this inequality.

Proof of Corollary 3.4. We use equation (31) and the fact that the process Y_t is bounded by μ to get

$$V^*(x, y) \leq x + \sup_{(I, D) \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} (\mu - rX_s) ds \right].$$

Proceeding analogously as in the proof of Lemma 8.1, we have

$$V_\mu(x) = x + \sup_{(I, D) \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} (\mu - rX_s) ds \right],$$

which leads to $V^*(x, y) \leq V_\mu(x)$. □

8.3 Model Solution

We devote this section to the proof of Theorem 4.1. We use the following notations and relationships throughout the proof.

- The function $h_1(y) = (y + \mu)^\gamma(\mu - y)^{1-\gamma}$ is decreasing and convex over $(-\mu, \mu)$ and

$$h_1'(y) = h_1(y)\left(\frac{\mu^2}{\hat{y}} - y\right)\frac{1}{\mu^2 - y^2}, \quad h_1''(y) = \frac{2r\sigma^2}{(\mu^2 - y^2)^2}h_1(y). \quad (45)$$

- The function $h_2(y) = (y + \mu)^{1-\gamma}(\mu - y)^\gamma$ is increasing and convex over $(-\mu, \mu)$ and

$$h_2'(y) = h_2(y)\left(-\frac{\mu^2}{\hat{y}} - y\right)\frac{1}{\mu^2 - y^2}, \quad h_2''(y) = \frac{2r\sigma^2}{(\mu^2 - y^2)^2}h_2(y). \quad (46)$$

- For all $y \in (-\mu, \mu)$

$$h_1(y)h_2(y) = \mu^2 - y^2, \quad \text{and} \quad h_1'(y)h_2(y) - h_2'(y)h_1(y) = \frac{2\mu^2}{\hat{y}}. \quad (47)$$

- The function $\psi(z) = \phi^{-1}(z) = \mu \frac{e^{\beta z} - 1}{e^{\beta z} + 1}$ with $\beta = \frac{2\mu}{\sigma^2}$, is increasing over \mathbb{R} and

$$\psi'(z) = \frac{\mu^2 - \psi(z)^2}{\sigma^2}. \quad (48)$$

- For all $z \in \mathbb{R}$,

$$h_1(\psi(z)) = 2\mu e^{(\gamma-1)\beta z} \frac{1}{e^{-\beta z} + 1}, \quad h_2(\psi(z)) = 2\mu e^{-\gamma\beta z} \frac{1}{e^{-\beta z} + 1}. \quad (49)$$

As for the complete information benchmark, it is useful to start the analysis under the assumption that the shareholders are not allowed to issue new shares.

8.3.1 No equity issuance

Thus, in this subsection we solve the problem

$$\bar{V}(x, y) = \sup_{D \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rt} dD_t \right], \quad (50)$$

where $\tau_0 = \inf\{t \geq 0 \mid X_t = 0\}$ with $X_t = \phi(Y_t) - \phi(y) + x - D_t$.

As a preliminary but essential step, we establish a standard verification Lemma that specifies conditions under which a function V defined on $[0, \infty) \times (-\mu, \mu)$ is a majorant of the value function \bar{V} of the problem (50).

Lemma 8.4 (*Verification Lemma*) *Assume there exists a function V defined on $[0, \infty) \times (-\mu, \mu)$ that satisfies*

1. V is twice differentiable,
2. V has bounded first derivatives,

3. $V(0, y) = 0$ for all $y \in (-\mu, \mu)$ and

$$\max(\mathcal{L}V - rV, 1 - V_x) \leq 0 \text{ on } [0, \infty) \times (-\mu, \mu),$$

then V is a majorant of \bar{V} .

Proof of Lemma 8.4. See the online appendix. \square

Second, we explicitly build such a majorant. To this end, we prove that the following variational problem

$$\mathcal{L}V(x, y) - rV(x, y) = 0 \text{ on the domain } \{(x, y), 0 < x < b(y), -\mu < y < \mu\}, \quad (51)$$

$$V(0, y) = 0 \quad \forall y \in (-\mu, \mu), \quad (52)$$

$$V_x(x, y) = 1, \text{ for } x \geq b(y), \quad (53)$$

$$V_{xy}(b(y), y) = 0, \quad (54)$$

$$\lim_{y \rightarrow \mu} V(x, y) = \bar{V}_\mu(x) \quad \forall x \geq 0, \quad (55)$$

has a unique solution (V, b) (see Proposition 8.6), such that V satisfies Lemma 8.4 (see Proposition 8.7) and thus dominates \bar{V} . Finally, we show in Proposition 8.8 that V can be reached by an admissible policy and thus coincides with the solution \bar{V} to the problem (50). This last step also provides the optimal dividend policy and concludes the study of problem (50).

We start with a technical lemma.

Lemma 8.5 *The ordinary differential equation*

$$g'(y) = f(g(y), y), \quad (56)$$

$$g(\mu) = \bar{x}_\mu, \quad (57)$$

with

$$f(x, y) = \frac{\sigma^2}{\mu^2 - y^2} \frac{y\hat{y} + \left(\mu - r\sigma^2 \left(\frac{\hat{y}}{\mu}\right)^2 \frac{1}{\mu}\right) \phi^{-1}\left(-\frac{\mu}{\hat{y}}x\right)}{y\hat{y} + \mu\phi^{-1}\left(-\frac{\mu}{\hat{y}}x\right)} \quad (58)$$

defined on the domain $\{(x, y) \in [0, \infty) \times (-\mu, \mu) \mid x > \frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu})\}$ has a unique solution. The solution g is C^1 and increasing over $[\bar{y}^*, \mu]$ where the threshold $\bar{y}^* \equiv g^{-1}(0)$ is well defined and strictly larger than \hat{y} .

Moreover, if we define $\bar{b} = \max(g, 0)$, then $\bar{k} = (\phi - \bar{b})^{-1} : [\phi(\bar{y}^*), \infty) \rightarrow [\bar{y}^*, \mu)$ is a well defined C^1 increasing function. The function \bar{k} is the unique solution to the ordinary differential equation

$$\bar{k}'(z) = \Theta(z, \bar{k}(z)), \quad (59)$$

$$\lim_{z \rightarrow \infty} \phi(\bar{k}(z)) - z = \bar{x}_\mu, \quad (60)$$

with

$$\Theta(z, y) = \frac{\mu^3}{\hat{y}r\sigma^2} \frac{\mu^2 - y^2}{\sigma^2} \frac{\psi\left(\frac{\mu}{\hat{y}}(\phi(y) - z)\right) \mu - y\hat{y}}{\hat{y}\psi\left(\frac{\mu}{\hat{y}}(\phi(y) - z)\right)} \quad (61)$$

defined on the domain $\{(z, y) \in \mathbb{R} \times (-\mu, \mu) \mid \phi(y) - z > \max(0, \frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu}))\}$.

Proof of Lemma 8.5. See the online appendix □

We are now in a position to solve the system (51)-(55).

Proposition 8.6 *Let us consider the functions \bar{b} and \bar{k} defined in Lemma 8.5 and the function $(x, y) \rightarrow V(x, y)$ defined on $[0, \infty) \times (-\mu, \mu)$ by the relations*

$$\begin{cases} V(0, y) = 0, & \text{for } y \in (-\mu, \mu), \\ V(x, y) = A(\phi(y) - x) \left(h_1(y) - e^{\frac{2}{\sigma^2} \frac{\mu^2}{\hat{y}} (\phi(y) - x)} h_2(y) \right), & \text{for } 0 \leq x \leq \bar{b}(y), y \in (-\mu, \mu), \\ V(x, y) = x - \bar{b}(y) + V(\bar{b}(y), y), & \text{for } x \geq \bar{b}(y), y \in (-\mu, \mu), \end{cases} \quad (62)$$

where

$$A(z) = \frac{\sigma^2}{4} \left(\frac{\hat{y}}{\mu} \right)^2 \left(\frac{1}{\mu} \right)^2 \left(h_1'(\bar{k}(z)) e^{-\frac{2}{\sigma^2} \frac{\mu^2}{\hat{y}} z} - h_2'(\bar{k}(z)) \right). \quad (63)$$

Then, the couple (V, \bar{b}) is the unique solution to the system (51)-(55). Furthermore, the function $\bar{b} : [\bar{y}^*, \mu] \rightarrow [0, \bar{x}_\mu]$ is C^1 and increasing.

Proof of Proposition 8.6. Having in mind the change of variable (18), we are looking for a smooth function U defined on $[0, \infty) \times (-\mu, \mu)$ and a C^1 function $k : \mathbb{R} \rightarrow (-\mu, \mu)$ that solve the variational system

$$\frac{1}{2\sigma^2} (\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y) = 0 \text{ on } \{(z, y), z \in \mathbb{R}, \psi(z) < y < k(z)\}, \quad (64)$$

$$U(z, \psi(z)) = 0, \text{ for } z \in \mathbb{R}, \quad (65)$$

$$U_z(z, y) = -1, \text{ for } k(z) \leq y, \quad (66)$$

$$U_{xy}(z, k(z)) = 0, \quad (67)$$

$$\lim_{z \rightarrow \infty} U(z, \psi(x+z)) = \bar{V}_\mu(x). \quad (68)$$

First, we establish a set of necessary conditions for the existence of such a pair (U, k) by observing that any solution to the o.d.e. (64) can be written in the form

$$U(z, y) = A(z)h_1(y) + B(z)h_2(y), \quad (69)$$

where (45) and (46) define h_1 and h_2 . Using (69), we obtain from (66) and (67) that

$$A'(z) = h_2'(k(z)) \frac{\hat{y}}{2\mu^2}, \text{ and } B'(z) = -h_1'(k(z)) \frac{\hat{y}}{2\mu^2}. \quad (70)$$

Using again (69), we rewrite (65) in the form

$$A(z) = -B(z) \frac{h_2(\psi(z))}{h_1(\psi(z))} = -B(z) e^{-\frac{2}{\sigma^2} \frac{\mu^2}{\hat{y}} z}. \quad (71)$$

Taking the derivative of (71), we obtain

$$A'(z) = -B'(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z} + B(z)\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}$$

which yields using again (71),

$$A(z) = -\frac{\sigma^2\hat{y}}{2\mu^2}\left(A'(z) + B'(z)e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}\right). \quad (72)$$

Using (69) and (71), and plugging (70) into (72) yield that,

$$U(z, y) = A(z)\left(h_1(y) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}h_2(y)\right), \quad (73)$$

where

$$A(z) = \frac{\sigma^2}{4}\left(\frac{\hat{y}}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2\left(h_1'(k(z))e^{-\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z} - h_2'(k(z))\right). \quad (74)$$

Taking the derivative of (72) and using (70), we obtain that

$$k'(z) = \frac{2\mu^2 h_1'(k(z))h_2(\psi(z)) + h_2'(k(z))h_1(\psi(z))}{\sigma^2\hat{y} h_1''(k(z))h_2(\psi(z)) - h_2''(k(z))h_1(\psi(z))}. \quad (75)$$

A computation based on formulae (45)-(49) yields that

$$k'(z) = \frac{\mu^3}{\hat{y}r\sigma^2}\frac{\mu^2 - k(z)^2}{\sigma^2}\frac{\psi\left(\frac{\mu}{\hat{y}}(\phi(k(z)) - z)\right)\mu - k(z)\hat{y}}{\hat{y}\psi\left(\frac{\mu}{\hat{y}}(\phi(k(z)) - z)\right)}, \quad (76)$$

which is positive on the domain $\phi(k(z)) - z > \max(0, \frac{\hat{y}}{\mu}\phi(\frac{\hat{y}}{\mu}k(z)))$.

Thus, (73), (74) and (76) is a set of necessary conditions for the existence of a smooth solution (U, k) to (64), (65), (66), (67). It remains to find a necessary condition for a solution to satisfy (68). Below we prove that U satisfies (68) if and only if $\lim_{z \rightarrow \infty} \phi(k(z)) - z = \bar{x}_\mu$.

From Lemma 8.5, it will imply that $k = \bar{k}$. To do this, we use (49) and (74) to write

$$U(z, \psi(x+z)) = A(z)\left[h_1(\psi(x+z)) - e^{\frac{2}{\sigma^2}\frac{\mu^2}{\hat{y}}z}h_2(\psi(x+z))\right],$$

in the form

$$U(z, \psi(x+z)) = \frac{\sigma^2}{4}\left(\frac{\hat{y}}{\mu}\right)^2\left(\frac{1}{\mu}\right)^2 f(x)\Delta(z)\frac{1 + e^{-\beta z}}{1 + e^{-\beta(x+z)}}, \quad (77)$$

with

$$\begin{aligned} f(x) &= e^{(\gamma-1)\beta x}(1 - e^{(1-2\gamma)\beta x}), \\ \Delta(z) &= h_1'(k(z))h_2(\psi(z)) - h_2'(k(z))h_1(\psi(z)). \end{aligned}$$

Observing that $k(z) = \psi((\phi(k(z)) - z) + z)$ and using relations (45), (46), (49), we obtain the following asymptotics

$$\begin{aligned} h'_1(k(z))h_2(\psi(z)) &\sim_{z=\infty} 2\mu(\gamma - 1)e^{\gamma\beta(\phi(k(z))-z)}, \\ h'_2(k(z))h_1(\psi(z)) &\sim_{z=\infty} -2\mu\gamma e^{(1-\gamma)\beta(\phi(k(z))-z)}, \end{aligned}$$

yielding

$$\lim_{z \rightarrow \infty} \Delta(z) = \lim_{z \rightarrow \infty} 2\mu \left((\gamma - 1)e^{\gamma\beta(\phi(k(z))-z)} - \gamma e^{(1-\gamma)\beta(\phi(k(z))-z)} \right). \quad (78)$$

Using (37), we observe that

$$\bar{V}_\mu(x) = f(x) \frac{e^{\beta\gamma\bar{x}_\mu}}{\beta(\gamma + (\gamma - 1)e^{(2\gamma-1)\beta\bar{x}_\mu})}. \quad (79)$$

It then follows from (77), (78), (79) that $\lim_{z \rightarrow \infty} U(z, \psi(x + z)) = \bar{V}_\mu(x)$ is equivalent to $\lim_{z \rightarrow \infty} \phi(k(z)) - z = \bar{x}_\mu$.

Thus, a smooth solution (U, k) to (64)-(68), if it exists, must satisfy (73), (74) where $k = \bar{k}$ is uniquely defined in Lemma 8.5. Conversely, a direct computation shows that the function defined by (73), (74) with $k = \bar{k}$ is a smooth solution to (64)-(68). Finally, posing $z = \phi(y) - x$ in (73) and (74) leads to (62) and (63) where \bar{b} is uniquely defined in Lemma 8.5. Thus, the couple (V, \bar{b}) defined in Proposition 8.6 is the unique solution to the system (51)-(55). Observe that the uniqueness of the function V comes from the uniqueness of the function \bar{b} which follows from condition (68). \square

To prove that $V = \bar{V}$, we proceed in two steps. First, we show in Proposition 8.7 that the function V solution to (64)-(68) satisfies the assumptions of the verification Lemma 8.4, which implies that $\bar{V} \leq V$. Second, we construct an admissible policy for problem (50), the value of which coincides with V . This latter result implies that $V \leq \bar{V}$.

Proposition 8.7 *The function V defined in Proposition 8.6 satisfies the assumptions of Lemma 8.4.*

Proof of Proposition 8.7. It is clear from (62) that V is twice continuously differentiable on any open set in $(0, \infty) \times (-\mu, \mu)$ away from the set $\{(x, y), x = \bar{b}(y)\}$. By construction V_x and V_{xx} are continuous across the boundary \bar{b} . Therefore, to prove that V is twice differentiable on $(0, \infty) \times (-\mu, \mu)$, we only have to show that the functions V_y and V_{yy} are continuous across the boundary \bar{b} , that is

$$\begin{aligned} \lim_{x \rightarrow \bar{b}(y)^-} V_y(x, y) &= -\bar{b}'(y) + \nu'(y), \\ \lim_{x \rightarrow \bar{b}(y)^-} V_{yy}(x, y) &= -\bar{b}''(y) + \nu''(y), \end{aligned} \quad (80)$$

where the function ν is defined on $(-\mu, \mu)$ by the relation

$$\nu(y) = A(\phi(y) - \bar{b}(y)) \left(h_1(y) - e^{\frac{2}{\sigma^2} \frac{\mu^2}{y} (\phi(y) - \bar{b}(y))} h_2(y) \right).$$

Let us define

$$H(y) \equiv e^{\frac{2}{\sigma^2} \frac{\mu^2}{y} (\phi(y) - \bar{b}(y))} = \frac{h_1(\psi(\phi(y) - \bar{b}(y)))}{h_2(\psi(\phi(y) - \bar{b}(y)))}.$$

Using the relations (45),(46),(47),(48), we obtain

$$H'(y) = \frac{2\mu^2}{\sigma^2 \hat{y}} (\phi'(y) - \bar{b}'(y)) H(y). \quad (81)$$

Remembering the relations (70) and (72) and the definition of \bar{k} , we observe that

$$A(z) = -\frac{\sigma^2 \hat{y}}{2\mu^2} \left(A'(\phi(y) - \bar{b}(y)) + B'(\phi(y) - \bar{b}(y)) \frac{1}{H(y)} \right).$$

We are in a position to compute the derivative of ν . We have,

$$\begin{aligned} \nu'(y) &= A(\phi(y) - \bar{b}(y))(h_1'(y) - H(y)h_2'(y)) \\ &+ (\phi'(y) - \bar{b}'(y))A'(\phi(y) - \bar{b}(y))(h_1(y) - H(y)h_2(y)) - A(\phi(y) - \bar{b}(y))H'(y)h_2(y). \end{aligned}$$

Using the relations (81), (72), (70) and the definition of \bar{k} , the second term of the right-hand side is equal to

$$(\phi'(y) - \bar{b}'(y)) \frac{\hat{y}}{2\mu^2} (h_2'(y)h_1(y) - h_1'(y)h_2(y)).$$

We note that the change of variable $V(x, y) = U(\phi(y) - x, y)$ leads to the relations

$$V_y(x, y) = \phi'(y)U_z(\phi(y) - x, y) + U_y(\phi(y) - x, y), \quad (82)$$

$$\begin{aligned} V_{yy}(x, y) &= \phi''(y)U_z(\phi(y) - x, y) + \phi'(y)^2 U_{zz}(\phi(y) - x, y) \\ &+ 2\phi'(y)U_{zy}(\phi(y) - x, y) + U_{yy}(\phi(y) - x, y). \end{aligned} \quad (83)$$

As a consequence,

$$\begin{aligned} \nu'(y) &= (\phi'(y) - \bar{b}'(y)) \frac{\hat{y}}{2\mu^2} ((h_2'(y)h_1(y) - h_1'(y)h_2(y)) \\ &+ A(\phi(y) - \bar{b}(y))(h_1'(y) - e^{\frac{2}{\sigma^2} \frac{\mu^2}{y} (\phi(y) - \bar{b}(y))} h_2'(y))) \\ &= -\phi'(y) + \bar{b}'(y) + U_y(\phi(y) - \bar{b}(y), y) \\ &= \bar{b}'(y) + \lim_{x \rightarrow \bar{b}(y)^-} V_y(x, y), \end{aligned}$$

where the first equality comes from (70) and the last equality comes from (82) and from the relation $U_z(\phi(y) - \bar{b}(y), y) = -V_x(\bar{b}(y), y) = -1$. Thus (80) is satisfied. Moreover,

$$\begin{aligned} \nu''(y) &= -\phi''(y) + \bar{b}''(y) + A((\phi(y) - \bar{b}(y))h_1''(y) + B((\phi(y) - \bar{b}(y))h_2''(y) \\ &+ (\phi'(y) - \bar{b}'(y))A'(\phi(y) - \bar{b}(y))(h_1'(y) - e^{\frac{2}{\sigma^2} \frac{\mu^2}{y} (\phi(y) - \bar{b}(y))} h_2'(y))) \\ &= -\phi''(y) + \bar{b}''(y) + \frac{2r\sigma^2}{(\mu^2 - y^2)^2} U((\phi(y) - \bar{b}(y), y) \\ &+ (\phi'(y) - \bar{b}'(y))U_{zy}((\phi(y) - \bar{b}(y), y) \\ &= -\phi''(y) + \bar{b}''(y) + \frac{2r\sigma^2}{(\mu^2 - y^2)^2} U((\phi(y) - \bar{b}(y), y) \\ &= \bar{b}''(y) + \lim_{x \rightarrow \bar{b}(y)^-} V_{yy}(x, y), \end{aligned}$$

where the last equality comes from (83) and from the relations $U_{zz}(\phi(y) - \bar{b}(y), y) = U_{zy}(\phi(y) - \bar{b}(y), y) = 0$, $U_z(\phi(y) - \bar{b}(y), y) = -1$ and $U_{yy}(\phi(y) - \bar{b}(y), y) = \frac{2r\sigma^2}{(\mu^2 - y^2)^2} U((\phi(y) - \bar{b}(y), y))$. Therefore, V is twice differentiable on $(0, \infty) \times (-\mu, \mu)$.

We show below that the function V has bounded first derivatives V_x and V_y . This amounts to show that $\lim_{x \rightarrow \infty} V_x(x, y) < \infty$ and $\lim_{y \rightarrow \mu} V_y(x, y) < \infty$. We write the change of variable $x = \phi(y) - z$ in the form $y = \psi(x + z)$ and define

$$W(x, z) \equiv V(x, \psi(x + z)) = U(z, \psi(x + z)).$$

We have

$$\begin{aligned} V_x(x, y) &= W_x(x, \phi(y) - x) - W_z(x, \phi(y) - x), \\ V_y(x, y) &= \phi'(y)W_z(x, \phi(y) - x) = \frac{\sigma^2}{\mu^2 - y^2}W_z(x, \phi(y) - x). \end{aligned}$$

We then deduce that V_x is bounded if and only if

$$\lim_{z \rightarrow \infty} W_x(x, z) - W_z(x, z) < \infty. \quad (84)$$

From (48), we deduce that V_y is bounded if and only if $\lim_{z \rightarrow \infty} \frac{1}{\psi'(x + z)}W_z(x, z) < \infty$, or, equivalently if and only if

$$\lim_{z \rightarrow \infty} \frac{1}{\psi'(x + z)}\Delta'(z) < \infty, \quad (85)$$

where the latter expression follows from a computation that uses (77). Using (75) we obtain that (85) is equivalent to

$$\begin{aligned} \lim_{z \rightarrow \infty} \frac{1}{\psi'(x + z)} \frac{2\mu^2}{\sigma^2 \hat{y}} (h'_1(\bar{k}(z))h_2(\psi(z)) + h'_2(\bar{k}(z))h_1(\psi(z))) \\ + \frac{\psi'(z)}{\psi'(x + z)} (h'_1(\bar{k}(z))h'_2(\psi(z)) - h'_2(\bar{k}(z))h'_1(\psi(z))) < \infty. \end{aligned} \quad (86)$$

Recalling that $\lim_{z \rightarrow \infty} \phi(\bar{k}(z)) - z = \bar{x}_\mu$ and observing that $\bar{k}(z) = \psi((\phi(\bar{k}(z)) - z))$, formulae (45)-(49) lead to the relations

$$\begin{aligned} \frac{1}{\psi'(x + z)} \frac{2\mu^2}{\sigma^2 \hat{y}} h'_1(\bar{k}(z))h_2(\psi(z)) &\sim_{z=\infty} \frac{\mu}{\hat{y}} (\gamma - 1) e^{\beta x} e^{\gamma \beta \bar{x}_\mu} e^{\beta z}, \\ \frac{1}{\psi'(x + z)} \frac{2\mu^2}{\sigma^2 \hat{y}} h'_2(\bar{k}(z))h_1(\psi(z)) &\sim_{z=\infty} -\frac{\mu}{\hat{y}} \gamma e^{\beta x} e^{(1-\gamma)\beta \bar{x}_\mu} e^{\beta z}, \\ \frac{\psi'(z)}{\psi'(x + z)} h'_1(\bar{k}(z))h'_2(\psi(z)) &\sim_{z=\infty} -\gamma(\gamma - 1) e^{\beta x} e^{\gamma \beta \bar{x}_\mu} e^{\beta z}, \\ \frac{\psi'(z)}{\psi'(x + z)} h'_2(\bar{k}(z))h'_1(\psi(z)) &\sim_{z=\infty} -\gamma(\gamma - 1) e^{\beta x} e^{(1-\gamma)\beta \bar{x}_\mu} e^{\beta z}. \end{aligned}$$

Aggregating these relations in (86), we obtain that

$$\lim_{z \rightarrow \infty} \frac{1}{\psi'(x + z)} W_z(x, z) = \lim_{z \rightarrow \infty} e^{\beta x} e^{\beta z} ((\gamma - 1)^2 e^{\gamma \beta \bar{x}_\mu} - \gamma^2 e^{(1-\gamma)\beta \bar{x}_\mu}) = 0, \quad (87)$$

where the last equality follows from (38). Thus, V_y is bounded. We deduce from (87) that $\lim_{z \rightarrow \infty} W_z(x, z) = 0$ and from (77) that

$$W_x(x, z) \sim_{z=\infty} \frac{\sigma^2}{4} \left(\frac{\hat{y}}{\mu} \right)^2 \left(\frac{1}{\mu} \right)^2 f'(x) \Delta(z).$$

Since $\lim_{z \rightarrow \infty} \phi(\bar{k}(z)) - z = \bar{x}_\mu$, we obtain from (78) that (84) is satisfied. Thus V_x is bounded. Finally, we prove below that

$$\max(\mathcal{L}V - rV, 1 - V_x) \leq 0 \text{ on } [0, \infty) \times (-\mu, \mu).$$

Note that, by construction, $V(0, y) = 0$ for all $y \in (-\mu, \mu)$ and that $\mathcal{L}V - rV = 0$ on the set $\{x \leq \bar{b}(y)\}$. We first show that the mapping $x \rightarrow V(x, y)$ is concave on the set $\{x < \bar{b}(y)\}$. The change of variable (18) and relation (64) yield that, on the set $\{x \leq \bar{b}(y)\}$,

$$V(x, y) = A(\phi(y) - x)h_1(y) + B(\phi(y) - x)h_2(y).$$

Using (70), we obtain that

$$V_{xx}(x, y) = \bar{k}'(\phi(y) - x) \frac{\hat{y}}{2\mu^2} (h_2''(\bar{k}(\phi(y) - x)h_1(y) - h_1''(\bar{k}(\phi(y) - x)h_2(y))). \quad (88)$$

From (46), the right hand side of (88) has the same sign than

$$\bar{k}'(\phi(y) - x) \frac{\hat{y}}{2\mu^2} (h_2(\bar{k}(\phi(y) - x)h_1(y) - h_1(\bar{k}(\phi(y) - x)h_2(y))). \quad (89)$$

Since h_1 is positive decreasing and h_2 is positive increasing, the function $(h_2(\bar{k}(\phi(y) - x)h_1(y) - h_1(\bar{k}(\phi(y) - x)h_2(y)))$ is positive if and only if $\bar{k}(\phi(y) - x) > y$. Since \bar{k} is increasing, this latter inequality is equivalent to $\phi(y) - x > \bar{k}^{-1}(y)$, that is $x < \bar{b}(y)$. Thus, (89) is negative since $\hat{y} < 0$. Therefore, the mapping $x \rightarrow V(x, y)$ is concave on the set $\{x < \bar{b}(y)\}$. Because $V_x(x, y) = 1$ for all $x \geq \bar{b}(y)$, we conclude that $x \rightarrow V(x, y)$ is concave over $[0, \infty)$, and in turn that $V_x \geq 1$ on $[0, \infty)$.

It remains to show that $\mathcal{L}V - rV < 0$ on the set $\{x > \bar{b}(y)\}$. On the set $\{x > \bar{b}(y)\}$, we have that $V(x, y) = x - \bar{b}(y) + \nu(y)$ and $V_{xy}(\bar{b}(y), y) = 0$. We deduce the equalities $V_y(x, y) = V_y(\bar{b}(y), y)$ and $V_{yy}(x, y) = V_{yy}(\bar{b}(y), y)$. Therefore, using the fact that V is twice differentiable across \bar{b} , we obtain that, on the set $\{x > \bar{b}(y)\}$,

$$\begin{aligned} (\mathcal{L}V - rV)(x, y) &= \frac{1}{2\sigma^2} (\mu^2 - y^2)^2 V_{yy}(\bar{b}(y), y) + y - rV(x, y) \\ &= -r(x - \bar{b}(y)) \\ &< 0. \end{aligned}$$

The proof of Proposition 8.7 is complete and thus $\bar{V} \leq V$. \square

Finally, we show that the solution V can be reached by an admissible policy. Our guess is that the optimal cash reserve process is reflected along the free boundary function \bar{b} on a horizontal direction in the (x, y) -plane. We formalize this using a 2-dimensional version to

Skorohod's lemma established by Burdzy and Toby (1995). Specifically, there exists a unique continuous process $\{L = (L_t)_t; t \geq 0\}$ defined on $(\Omega, \mathcal{F}^R, \mathbb{P})$ such that, for \mathbb{P} -a.e. $\omega \in \Omega$,

- $(\phi(Y_t(\omega)) - \phi(y) + x - L_t(\omega), Y_t(\omega)) \in [0, \bar{b}(y)] \times [\bar{y}^*, \mu], \quad \forall t \in [0, \tau_0], \quad (90)$

where $\tau_0 = \inf\{t \geq 0 \mid \phi(Y_t) - \phi(y) + x - L_t = 0\}$,

- $L_0(\omega) = 0$, and $t \rightarrow L_t(\omega)$ is nondecreasing on $\{t \geq 0 : \phi(Y_t) - \phi(y) + x - L_t = \bar{b}(Y_t)\}$, (91)

- $t \rightarrow L_t(\omega)$ is constant on $\{t \geq 0 : (\phi(Y_t(\omega)) - \phi(y) + x - L_t(\omega), Y_t(\omega)) \in (0, \bar{b}(y)) \times (\bar{y}^*, \mu)\}$. (92)

Conditions (90)-(92) ensure that the policy L is admissible and that the process $\phi(Y_t) - \phi(y) + x - L_t$ is reflected in a horizontal direction whenever it hits $\bar{b}(Y_t)$.

Proposition 8.8 *The function V can be attained by an admissible policy and thus $V \leq \bar{V}$.*

Proof of Proposition 8.8 Let us consider the process $\bar{D} = \{\bar{D}_t; t \geq 0\}$ with

$$\bar{D}_t = ((x - \bar{b}(y))^+ \mathbb{1}_{y \geq \bar{y}^*} + x \mathbb{1}_{y \leq \bar{y}^*}) \mathbb{1}_{t=0} + L_t \mathbb{1}_{t>0}, \quad (93)$$

where L is defined by (90)-(92) and let us consider the continuous process

$$X_t \equiv \phi(Y_t) - \phi(y) + x - \bar{D}_t.$$

A computation based on Itô's formula yields that, for all $t \geq 0$,

$$\begin{aligned} \mathbb{E} [e^{-rt \wedge \tau_0} V(X_{t \wedge \tau_0}, Y_{t \wedge \tau_0})] &= V(x, y) - \mathbb{E} \left[\int_0^{t \wedge \tau_0} e^{-rs} V_x(X_s, Y_s) d\bar{D}_s \right] \\ &= V(x, y) - \mathbb{E} \left[\int_0^{t \wedge \tau_0} e^{-rs} d\bar{D}_s \right] \end{aligned} \quad (94)$$

where the second equality comes from (91) and (92) along with the fact that $V_x(\bar{b}(y), y) = 1$. Letting t go to ∞ in (94) yields

$$V(x, y) = \mathbb{E} \left[\int_0^{t \wedge \tau_0} e^{-rs} d\bar{D}_s \right] \leq \bar{V}.$$

□

Thus, from Propositions 8.7 and 8.8, the function V defined in Proposition 8.6 coincides with the value function \bar{V} of problem (50). Equation (93) provides the optimal dividend policy: The function \bar{b} corresponds to the dividend boundary of the shareholders' problem (50). The optimal cash reserve process is reflected along the function \bar{b} on a horizontal direction in the (x, y) -plane.

8.3.2 Equity issuance

We are now ready to prove Theorem 4.1 and to solve problem (8):

$$V^*(x, y) = \sup_{(I, D) \in \mathcal{A}} \mathbb{E} \left[\int_0^{\tau_0} e^{-rt} (dD_t - dI_t) \right].$$

We proceed as in the previous section where equity issuance was not allowed. The verification Lemma 8.4 has to be adapted in the following way.

Lemma 8.9 (*Verification Lemma*) *Assume there exists a function V defined on $[0, \infty) \times (-\mu, \mu)$ that satisfies*

1. V is twice differentiable almost everywhere,
2. V has bounded first derivatives,
3. $\max(-V(0, y), V_x(0, y) - p) = 0$ for all $y \in (-\mu, \mu)$ and,

$$\max(\mathcal{L}V - rV, 1 - V_x, V_x - p) \leq 0 \text{ almost everywhere on } [0, \infty) \times (-\mu, \mu),$$

then $V \geq V^*$.

Proof of Lemma 8.9. See the online appendix.

We first assume that the proportional issuance costs p satisfies $p \geq \bar{p} = \bar{V}'_{\mu}(0)$. We prove that the firm value V^* solution to (8) coincides with \bar{V} solution to (50), so that also the functions b^* and \bar{b} coincide. This proves Theorem 4.1 when $p \geq \bar{p}$. The result is a direct consequence of the following Lemma.

Lemma 8.10 *Let us assume that $p \geq \bar{p}$, then $\bar{p} > \bar{V}_x(x, y)$ for all $(x, y) \in [0, \infty) \times (-\mu, \mu)$.*

Proof of Lemma 8.10. See the online appendix.

From Lemma 8.10, \bar{V} satisfies the assumptions of Lemma 8.9. It follows that $V^*(x, y) \leq \bar{V}(x, y)$. On the other hand, considering the policies $I^* = 0$ and D^* defined in (93) lead to the inequality $V^*(x, y) \geq \bar{V}(x, y)$, thus the result.

The case $p \leq \bar{p}$ is much more involved. The analysis relies on the following technical Proposition.

Proposition 8.11 *The following holds.*

(i) *Fix $z_i > \hat{z} = \phi(\hat{y})$, the relation*

$$-h'_1(k(z))h_2(\psi(z)) + h'_2(k(z))h_1(\psi(z)) + \frac{2\mu^2}{\hat{y}}p = 0 \tag{95}$$

uniquely defines over $[z_i, \infty)$ a continuously differentiable increasing function $k > \psi$ that satisfies

$$\lim_{z \rightarrow \infty} \phi(k(z)) - z = x_\mu, \quad (96)$$

$$k(z) \underset{+\infty}{\sim} \psi(z + x_\mu). \quad (97)$$

(ii) The equation

$$\frac{p}{\psi'(z)} h_1(\psi(z)) + \int_z^\infty h_1'(k(u)) du = 0 \quad (98)$$

has a unique solution $z_i^* \in (\hat{z}, \infty)$.

(iii) Let us denote $y_i^* = \psi(z_i^*)$ and let us consider the function k^* with

$$k^*(z) = k_1(z) \mathbb{1}_{z \leq z_i^*} + k_2(z) \mathbb{1}_{z > z_i^*}, \quad (99)$$

where (95) characterizes k_2 , and where k_1 is the solution to the ordinary differential equation $k_1'(z) = \Theta(k_1(z), z)$ with terminal condition $k_1(z_i^*) = k_2(z_i^*)$ where Θ is defined by (61). Then, k^* is a well defined continuous increasing function over (\hat{z}, ∞) and continuously differentiable over $\mathbb{R} \setminus \{z_i^*\}$.

Proof of Proposition 8.11. See the online appendix. □

We state now the main result of this section.

Proposition 8.12 *Let us consider the function k^* defined by (99) and the function b^* defined by the relation*

$$b^*(y) = \max(0, b_1(y)) \mathbb{1}_{y \leq y_i^*} + b_2(y) \mathbb{1}_{y > y_i^*}, \quad (100)$$

with $b_2(y) = (\phi - k_2^{-1})(y)$ for any $y \geq y_i^*$ and, $b_1(y) = (\phi - k_1^{-1})(y)$ for any $y \leq y_i^*$. The function b^* is well defined and positive on $[y^*, \mu)$ with $y^* = b^{-1}(0)$. It is differentiable on $(y^*, \mu) \setminus \{y_i^*\}$ where $y_i^* \equiv k_1(z_i^*) = k_2(z_i^*)$. Moreover, let us consider the function $(x, y) \rightarrow V(x, y)$ defined on $\mathbb{R}^+ \times (-\mu, \mu)$ by the relations

- For $y \in [y_i^*, \mu)$,

$$\begin{cases} V(x, y) = A(\phi(y) - x) h_1(y) + B(\phi(y) - x) h_2(y), & \forall 0 \leq x \leq b^*(y), \\ V(x, y) = x - b^*(y) + V(b^*(y), y), & \forall x \geq b^*(y), \end{cases}$$

where for $z > z_i^*$:

$$\begin{cases} A(z) = A(z_i^*) + \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_2'(k^*(u)) du, \\ B(z) = B(z_i^*) - \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_1'(k^*(u)) du \end{cases} \quad (101)$$

and,

$$\begin{cases} A(z_i^*) = \frac{p}{\psi'(z_i^*)} \frac{\hat{y}}{2\mu^2} h_2(\psi(z_i^*)), \\ B(z_i^*) = -\frac{p}{\psi'(z_i^*)} \frac{\hat{y}}{2\mu^2} h_1(\psi(z_i^*)). \end{cases} \quad (102)$$

- For $y \in (-\mu, y_i^*]$,

$$\begin{cases} V(0, y) = 0, & \forall -\mu < y < y, \\ V(x, y) = A(\phi(y) - x) \left(h_1(y) - e^{\frac{2}{\sigma^2} \frac{\mu^2}{y} (\phi(y) - x)} h_2(y) \right), & \forall 0 \leq x \leq b^*(y), \\ V(x, y) = x - b^*(y) + V(b^*(y), y), & \forall x \geq b^*(y) \end{cases}$$

where for $z \leq z_i$,

$$A(z) = \frac{\sigma^2}{4} \left(\frac{\hat{y}}{\mu} \right)^2 \left(\frac{1}{\mu} \right)^2 \left(h_1'(k^*(z)) e^{-\frac{2}{\sigma^2} \frac{\mu^2}{\hat{y}} z} - h_2'(k^*(z)) \right). \quad (103)$$

Then, the triple (V, y_i^*, b^*) is the unique solution to the system (12)-(17).

Proof of Proposition 8.12. The proof follows the same route than the proof of Proposition 8.6. We first consider a solution (U, z_i, k) to the system

$$\frac{1}{2\sigma^2} (\mu^2 - y^2)^2 U_{yy}(z, y) - rU(z, y) = 0 \quad \text{on } \{(z, y), z \in \mathbb{R}, \psi(z) < y < k(z)\}, \quad (104)$$

$$U(z, \psi(z)) = 0 \quad \forall z \leq z_i, \quad (105)$$

$$U_z(z, \psi(z)) = -p \quad \forall z \geq z_i, \quad (106)$$

$$U_z(z, y) = -1, \quad \text{for } k(z) \leq y, \quad (107)$$

$$U_{xy}(z, k(z)) = 0, \quad (108)$$

$$\lim_{z \rightarrow \infty} U(z, \psi(x+z)) = V_\mu(x). \quad (109)$$

The relations derived in the proof of Proposition 8.6 hold true for $z \leq z_i$, so that the solution (U, z_i, k) satisfies (73), (74) and (76) for $z \leq z_i$.

Note that (69) and (70) hold true for any $(z, y) \in \{(z, y), z \in \mathbb{R}, \psi(z) < y < k(z)\}$. We then deduce from (106) that (95) characterizes the function k on $[z_i, \infty)$.

Consider (105) and take the derivative with respect to z of $U(z, \psi(z))$. One get

$$U_z(z, \psi(z)) + \psi'(z) U_y(z, \psi(z)) = 0. \quad (110)$$

Then, using (69) and (106), Equations (105) and (110) evaluated at z_i yield

$$\begin{cases} A(z_i) = \frac{p}{\psi'(z_i)} \frac{\hat{y}}{2\mu^2} h_2(\psi(z_i)), \\ B(z_i) = -\frac{p}{\psi'(z_i)} \frac{\hat{y}}{2\mu^2} h_1(\psi(z_i)). \end{cases} \quad (111)$$

We then obtain from (70) that

$$\begin{cases} A(z) = A(z_i) + \int_{z_i}^z \frac{\hat{y}}{2\mu^2} h_2'(k(u)) du, \\ B(z) = B(z_i) - \int_{z_i}^z \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du. \end{cases} \quad (112)$$

Thus, a smooth solution (U, z_i, k) to (104)-(108) satisfies $U(z, y) = A(z)h_1(y) + B(z)h_2(y)$ on $\{(z, y), z \in \mathbb{R}, \psi(z) < y < k(z)\}$ and the relations (74), (76), (111), (112). We prove

below that such a smooth solution satisfies (109) if and only if $z_i = z_i^*$. This will imply that $k = k^*$. We have

$$U(z, \psi(x+z)) = A(z)h_1(\psi(x+z)) + B(z)h_2(\psi(x+z)) \quad (113)$$

where (71) and (74) define A and B for $z \leq z_i$ and (111) and (112) define A and B for $z \geq z_i$. We deduce from (46), (49) and (97) that

$$h_1'(k(u)) \underset{+\infty}{\sim} (\gamma-1)e^{\gamma\beta(u+x_\mu)} \quad \text{and} \quad h_2'(k(u)) \underset{+\infty}{\sim} -\gamma e^{(1-\gamma)\beta(u+x_\mu)}. \quad (114)$$

It follows that

$$\int_{z_i}^{\infty} \frac{\hat{y}}{2\mu^2} h_2'(k(u)) du = -\infty, \quad \int_{z_i}^{\infty} \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du < \infty, \quad (115)$$

and also that,

$$\int_{z_i}^z h_2'(k(u)) du \underset{+\infty}{\sim} \int_{z_i}^z -\gamma e^{(1-\gamma)\beta(u+x_\mu)} du, \quad (116)$$

yielding

$$\begin{aligned} \int_{z_i}^z -\gamma e^{(1-\gamma)\beta(u+x_\mu)} du &= \frac{\gamma}{\gamma-1} \frac{1}{\beta} e^{-(\gamma-1)\beta(z+x_\mu)} \\ &\quad - \frac{\gamma}{\gamma-1} \frac{1}{\beta} e^{-(\gamma-1)\beta(z_i+x_\mu)}. \end{aligned}$$

Using (49), we deduce that

$$\begin{aligned} &\lim_{z \rightarrow \infty} \int_{z_i}^z \frac{\hat{y}}{2\mu^2} h_2'(k(u)) du h_1(\psi(x+z)) \\ &= \frac{\gamma}{\beta(\gamma-1)} \frac{\hat{y}}{\mu^2} e^{-(\gamma-1)\beta x_\mu} e^{(\gamma-1)\beta x} - \lim_{z \rightarrow \infty} \frac{\gamma}{\beta(\gamma-1)} \frac{\hat{y}}{\mu} e^{-(\gamma-1)\beta(z_i+x_\mu)} e^{(\gamma-1)\beta(x+z)} \\ &= \frac{\gamma}{\beta(\gamma-1)} \frac{\hat{y}}{\mu^2} e^{-(\gamma-1)\beta x_\mu} e^{(\gamma-1)\beta x}, \end{aligned} \quad (117)$$

where the last equality comes from the fact that $\gamma < 0$. We also have,

$$\int_{z_i}^z \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du = \int_{z_i}^{\infty} \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du - \int_z^{\infty} \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du$$

from which we deduce that,

$$\begin{aligned} \lim_{z \rightarrow \infty} \int_{z_i}^z \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du h_2(\psi(x+z)) &= \left(\lim_{z \rightarrow \infty} \int_{z_i}^{\infty} \frac{\hat{y}}{\mu} h_1'(k(u)) du e^{-\beta\gamma(x+z)} \right. \\ &\quad \left. - \int_z^{\infty} \frac{\hat{y}}{\mu} e^{\beta\gamma(u+x_\mu)} (\gamma-1) du e^{-\beta\gamma(x+z)} \right). \end{aligned}$$

Thus,

$$\lim_{z \rightarrow \infty} \int_{z_i}^z \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du h_2(\psi(x+z)) = -\frac{1-\gamma}{\beta\gamma} \frac{\hat{y}}{\mu} e^{\beta\gamma x_\mu} + \lim_{z \rightarrow \infty} e^{-\beta\gamma(x+z)} \int_{z_i}^{\infty} \frac{\hat{y}}{\mu} h_1'(k(u)) du \quad (118)$$

Using (111), (112) and (113) together with (41), (117), (118) we obtain

$$\begin{aligned}
\lim_{z \rightarrow \infty} U(z, \psi(x+z)) &= \lim_{z \rightarrow \infty} \left(\int_{z_i}^z \frac{\hat{y}}{2\mu^2} h'_2(k(u)) du h_1(\psi(x+z)) + B(z_i) h_2(\psi(x+z)) \right. \\
&\quad \left. - \int_{z_i}^z \frac{\hat{y}}{2\mu^2} h'_1(k(u)) du h_2(\psi(x+z)) \right) \\
&= V_\mu(x) + \lim_{z \rightarrow \infty} \left(B(z_i) h_2(\psi(x+z)) - \int_{z_i}^\infty \frac{\hat{y}}{2\mu^2} h'_1(k(u)) du 2\mu e^{-\beta\gamma(x+z)} \right) \\
&= V_\mu(x) - \lim_{z \rightarrow \infty} \left(\frac{P}{\psi'(z_i)} h_1(\psi(z_i)) + \int_{z_i}^\infty h'_1(k(u)) du \right) \frac{\hat{y}}{\mu} e^{-\beta\gamma(x+z)}.
\end{aligned}$$

Since $\gamma < 0$, the function U satisfies (109) if and only if z_i satisfies (98), or equivalently if $z_i = z_i^*$. Therefore, a smooth solution (U, z_i, k) to (104)-(109) satisfies $U(z, y) = A(z)h_1(y) + B(z)h_2(y)$ on $\{(z, y), z \in \mathbb{R}, \psi(z) < y < k(z)\}$ and the relations (74), (76), (111), (112) with $k = k^*$ (and thus $z_i = z_i^*$). Conversely, a computation shows that $U(z, y) = A(z)h_1(y) + B(z)h_2(y)$ on $\{(z, y), z \in \mathbb{R}, \psi(z) < y < k(z)\}$ and the relations (74), (76), (111), (112) with $k = k^*$ is a solution to the system (104)-(109).

Finally, from Proposition 8.11, k_1 and k_2 in (99) are increasing so that, b^* in (100) is indeed a well defined function which is not differentiable at y^{i*} from assertion (iii) of Proposition 8.11. Then, using the change of variable $z = \phi(y) - x$, we obtain that (V, y_i^*, b^*) is a solution to the system (12)-(17). Observe that the uniqueness of the function V comes from the uniqueness of the function b^* that follows from condition (109). \square

Proceeding as in the previous section, we show the following

Proposition 8.13 *The function V defined in Proposition 8.12 satisfies the assumptions of Lemma 8.9 and thus $V \geq V^*$.*

Proof of Proposition 8.13. The proof relies on arguments developed in Proposition 8.7. We first show that the function V is \mathcal{C}^1 on the domain $(0, \infty) \times (-\mu, \mu)$ and is \mathcal{C}^2 on the domain $(0, \infty) \times (-\mu, \mu) \setminus \{(b^*(y^{i*}), y^{i*})\}$. By construction, V is twice continuously differentiable on any open set in $(0, \infty) \times (-\mu, \mu)$ away from the set $\{(x, y), x = b^*(y)\}$. Since b^* is differentiable on $(y^*, \mu) \setminus \{y^{i*}\}$, we can proceed as in the proof of Proposition 8.7 to prove that V is of class \mathcal{C}^2 on $[0, \infty) \times (-\mu, \mu) \setminus \{(b^*(y^{i*}), y^{i*})\}$. Also by construction, the study of the \mathcal{C}^1 -differentiability of V^* does not involve the derivative of b^* : to establish that V is \mathcal{C}^1 on $(0, \infty) \times (-\mu, \mu)$, it is sufficient to check that A and B are continuously differentiable at z_i^* . For the continuity of A (or equivalently of B) at z_i^* , observe that (74) and (95) evaluated at z_i^* , lead to (102). The differentiability of A and B at z_i^* comes from (70) and (101). Thus, V is twice differentiable almost everywhere.

Second, let us fix any $y \in (y^*, \mu)$. We deduce from the proof of Proposition 8.7 that the mapping $x \rightarrow V(x, y)$ is concave on $[0, \infty) \setminus \{\phi(y) - z_i^*\}$ if and only if

$$(k^*)'(\phi(y) - x)(h_2(k^*(\phi(y) - x))h_1(y) - h_1(k^*(\phi(y) - x))h_2(y)) < 0. \quad (119)$$

Note that (119) is well defined since $\phi(y) - x \neq z_i^*$ on the considered domain, such that the derivative of k^* is well defined. The reasoning developed in the proof of Proposition 8.7

shows that (119) holds true. Now, because V is linear in x outside $\{x < b^*(y)\}$ and that $V_x(b^*(y), y) = 1$ for any $y \in (-\mu, \mu)$, we deduce from the concavity of $x \rightarrow V(x, y)$ that $V_x(x, y) \geq 1$ on $[0, \infty) \setminus \{\phi(y) - z_i^*\}$. Then, because V is \mathcal{C}^1 on $(0, \infty) \times (-\mu, \mu)$, one obtains that $V_x(x, y) \geq 1$ on $(0, \infty) \times (-\mu, \mu)$.

Finally, the concavity and \mathcal{C}^1 -differentiability properties together with the fact that $V_x(0, y) \leq p$ for any $y \in (-\mu, \mu)$ lead to $V_x(x, y) \leq p$ on $(0, \infty) \times (-\mu, \mu)$.

The arguments developed in the proof of Proposition 8.7 show that $\mathcal{L}V - rV \leq 0$ on $(0, \infty) \times (-\mu, \mu) \setminus \{(b^*(y^{i*}), y^{i*})\}$. Aggregating all these results, we obtain that, almost everywhere on $(0, \infty) \times (-\mu, \mu)$,

$$\max(\mathcal{L}V - rV, 1 - V_x, V_x - p) \leq 0.$$

Observe also that V satisfies by construction $\max(-V(0, y), V_x(0, y) - p) = 0$ for all $y \in (-\mu, \mu)$.

To conclude, it remains to show that the function V has bounded first derivatives. We have $V_x(x, y) \leq p$ on $(0, \infty) \times (-\mu, \mu)$ such that $(x, y) \rightarrow V_x(x, y)$ is bounded over $[0, \infty) \times (-\mu, \mu)$. From the expression of V in Proposition 8.12, we deduce that V_y is bounded if and only if

$$\lim_{y \rightarrow \mu} V_y(x, y) < \infty.$$

That is, as shown in Proposition 8.7, if and only if

$$\lim_{z \rightarrow \infty} \frac{1}{\psi'(x+z)} W_z(x, z) < \infty,$$

where $W(x, z) = V(x, \psi(x+z)) = U(z, \psi(x+z))$.

We have that

$$\begin{aligned} W(x, z) &= A(z)h_1(\psi(x+z)) + B(z)h_2(\psi(x+z)) \\ &= A(z_i^*)h_1(\psi(x+z)) + B(z_i^*)h_2(\psi(x+z)) \\ &\quad + \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_2'(k(u)) du h_1(\psi(x+z)) \\ &\quad - \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du h_2(\psi(x+z)). \end{aligned}$$

This leads to

$$\begin{aligned} \frac{1}{\psi'(x+z)} W_z(x, z) &= A(z_i^*)h_1'(\psi(x+z)) + B(z_i^*)h_2'(\psi(x+z)) \\ &\quad + \frac{\hat{y}}{2\mu^2} h_2'(k(z)) \frac{h_1(\psi(x+z))}{\psi'(x+z)} - \frac{\hat{y}}{2\mu^2} h_1'(k(z)) \frac{h_2(\psi(x+z))}{\psi'(x+z)} \\ &\quad + \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_2'(k(u)) du h_1'(\psi(x+z)) \\ &\quad - \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h_1'(k(u)) du h_2'(\psi(x+z)), \end{aligned}$$

or, equivalently using (47), (48),

$$\begin{aligned}
\frac{1}{\psi'(x+z)} W_z(x, z) &= A(z_i^*) h'_1(\psi(x+z)) + B(z_i^*) h'_2(\psi(x+z)) \\
&+ \frac{\hat{y} \sigma^2}{2\mu^2} \frac{h'_2(k(z))}{h_2(\psi(x+z))} - \frac{\hat{y} \sigma^2}{2\mu^2} \frac{h'_1(k(z))}{h_1(\psi(x+z))} \\
&+ \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h'_2(k(u)) du h'_1(\psi(x+z)) \\
&- \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h'_1(k(u)) du h'_2(\psi(x+z)). \tag{120}
\end{aligned}$$

Using (114), (115), (116) and (49), we have that

$$h'_1(\psi(x+z)) \underset{+\infty}{\sim} (\gamma-1)e^{\gamma\beta(x+z)}, \quad h'_2(\psi(x+z)) \underset{+\infty}{\sim} -\gamma e^{(1-\gamma)\beta(x+z)},$$

$$\begin{aligned}
\frac{\hat{y} \sigma^2}{2\mu^2} \frac{h'_2(k(z))}{h_2(\psi(x+z))} \underset{+\infty}{\sim} -\frac{\hat{y}}{2\mu^2} \frac{1}{\beta} \gamma e^{(1-\gamma)\beta x_\mu} e^{\gamma\beta x} e^{\beta z}, \quad \frac{\hat{y} \sigma^2}{2\mu^2} \frac{h'_1(k(z))}{h_1(\psi(x+z))} \underset{+\infty}{\sim} -\frac{\hat{y}}{2\mu^2} \frac{1}{\beta} (1-\gamma) e^{\gamma\beta x_\mu} e^{(1-\gamma)\beta x} e^{\beta z}, \\
\int_{z_i^*}^z h'_2(k(u)) \frac{\hat{y}}{2\mu^2} du h'_1(\psi(x+z)) \underset{+\infty}{\sim} \frac{\hat{y}}{2\mu^2} \gamma \frac{1}{\beta} e^{\gamma\beta x} e^{-(\gamma-1)\beta x_\mu} e^{\beta z}.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{z \rightarrow \infty} \int_{z_i^*}^z \frac{\hat{y}}{2\mu^2} h'_1(k(u)) du h'_2(\psi(x+z)) &= \lim_{z \rightarrow \infty} \left(\int_{z_i^*}^{\infty} \frac{\hat{y}}{2\mu^2} h'_1(k(u)) du h'_2(\psi(x+z)) \right. \\
&\left. - \int_z^{\infty} \frac{\hat{y}}{2\mu^2} h'_1(k(u)) du h'_2(\psi(x+z)) \right).
\end{aligned}$$

Aggregating all these relations in (120), a last computation shows that proving $\lim_{y \rightarrow \infty} V_y(x, y) < \infty$ is equivalent to prove

$$\lim_{z \rightarrow \infty} \left(-B(z_i^*) + \frac{\hat{y}}{2\mu^2} \int_{z_i^*}^z h'_1(k(u)) du \right) \gamma e^{(1-\gamma)\beta(x+z)} < \infty.$$

Noting that $\beta > 0$ and using (102), we obtain that $\lim_{y \rightarrow \infty} V_y(x, y) < \infty$ if and only if z_i^* satisfies (98), which indeed holds true by definition of z_i^* and concludes the proof of Proposition 8.13, from which it follows that $V^* \leq V$. \square

The next proposition establishes the converse inequality

Proposition 8.14 *The function V can be attained by an admissible policy and thus $V \leq \bar{V}$.*

Proof of Proposition 8.14. The proof follows exactly the same arguments than those developed in Proposition 8.8. Let L^{b^*} and L^0 positive continuous increasing processes such that the process

$$\phi(Y_t) - \phi(y) + x - L_t^{b^*} + L_t^0$$

is reflected in a horizontal direction whenever $X_t = b^*(Y_t)$ and whenever $X_t = 0$. Following the results of Burdzy and Toby (1995), the processes $L_t^{b^*}$ and L_t^0 are well-defined. Then, the policies

$$D_t^* = ((x - b^*(y))^+ \mathbb{1}_{y \geq y^*} + x \mathbb{1}_{y \leq y^*}) \mathbb{1}_{t=0} + L_t^{b^*} \mathbb{1}_{t>0},$$

$$I_t^* = L_t^0 \mathbb{1}_{Y_{\tau_0} > y_i^*} \mathbb{1}_{t>0},$$

are admissible and a computation based on Itô's formula yields

$$V(x, y) = \mathbb{E} \left[\int_0^{\tau_0} e^{-rt} (dD_t^* - dI_t^*) \right]$$

which concludes the proof of Proposition 8.14. \square

Thus, from Propositions 8.13 and 8.14, the function V defined in Proposition 8.12 coincides with the value function V^* of problem (8). The function b^* corresponds to the dividend boundary of the shareholders' problem (8). The optimal cash reserve process is reflected along the function b^* on a horizontal direction in the (x, y) -plane. The threshold y_i^* corresponds to the level of the profitability prospects above which new shares are issued when cash reserves are depleted.

To conclude the proof of our main Theorem 4.1, it remains to show the properties of the dividend boundary b^* . The next proposition yields these results.

Proposition 8.15 *The function b^* is increasing for $y \leq y^{i^*}$ and decreasing for $y \geq y^{i^*}$, where $y^{i^*} = k^*(z_i^*)$. The function b^* attains its maximum at y^{i^*} , $b^*(y^{i^*}) > x_\mu$, and $\bar{y}^* > y^* > \hat{y}$.*

Proof of Proposition 8.15. Given the previous results, we only need to show that b^* is increasing over $[y^*, y^{i^*}]$, decreasing over $[y^{i^*}, \mu)$ and that $\bar{y}^* > y^* > \hat{y}$. First, we show that b^* is decreasing over $[y^{i^*}, \mu)$. Let us introduce the notation $\tilde{b}(z) \equiv b^*(k^*(z)) = \phi(k^*(z)) - z$ with $z > z_i^*$. Because $(k^*)^{-1}$ is increasing, we deduce from the relation $b^*(y) = \tilde{b}((k^*)^{-1}(y))$ that, for any $z > z_i^*$, $\tilde{b}'(z) = (k^*)'(z)\phi'(k^*(z)) - 1$ has the same sign than $b^{*'}(y)$ for any $y > y^{i^*}$. In addition, the relation $k^*(z) = \psi(\tilde{b}(z) + z)$ leads to $(k^*)'(z) = (1 + \tilde{b}'(z))\psi'(z + \tilde{b}(z))$, that is $\tilde{b}'(z) = \frac{k^{*'}(z)}{\psi'(z + \tilde{b}(z))} - 1$. Therefore, to show that b^* is decreasing over $[y^{i^*}, \mu)$, we prove that $\frac{k^{*'}(z)}{\psi'(z + \tilde{b}(z))} - 1 > 0$ for any $z > z_i^*$. This latter inequality follows from a computation developed in Lemma 8.17 in the additional appendix. Then, since b^* is decreasing on $[y^{i^*}, \mu)$ and $b^*(\mu) = x_\mu$, we have that $b^*(y^{i^*}) > x_\mu$.

It remains to show that b^* is increasing over $[y^*, y^{i^*}]$ and that $\bar{y}^* > y^* > \hat{y}$. We obtain from our previous results that

$$V(x, y; b^*) = V^*(x, y) \geq \mathbb{E} \left[\int_0^{\tau_0} e^{-rt} d\bar{D}_t \right] = V(x, y; \bar{b}),$$

where $V(x, y; b^*)$ is defined in Proposition 8.12, and $V(x, y; \bar{b})$ is defined in Proposition 8.6. Equation (93) defines the process \bar{D} . In particular we have that

$$V(b^*(\bar{y}^*), \bar{y}^*; b^*) \geq V(\bar{b}(\bar{y}^*), \bar{y}^*; \bar{b}) = V(0, \bar{y}^*; \bar{b}),$$

which implies $b^*(\bar{y}^*) \geq \bar{b}(\bar{y}^*)$. Let us recall that, for $y \leq y^{i^*}$ the functions b^* and \bar{b} satisfy (56) and (58). Then, the non crossing property of ordinary differential equations implies $b^*(y) \geq \bar{b}(y)$, for $y \leq y^{i^*}$. It then follows from the proof of Lemma 8.5 that the function b^* is increasing for $y \leq y^{i^*}$ and that $\bar{y}^* \geq y^*$. The same reasoning than in the proof of Lemma 8.5 shows that $y^* > \hat{y}$. Finally, a computation shows that y^{i^*} is increasing in the proportional issuance cost p . Recalling that $V(x, y; b^*) = V(x, y; \bar{b})$ for $p = \bar{p}$, we then deduce that $\bar{y}^* > y^*$ for $1 < p < \bar{p}$. This concludes the proof of Proposition 8.15. The proof of Theorem 4.1 is complete. \square

ONLINE APPENDIX

Proofs of Lemma 8.9 and Lemma 8.4. The proof of Lemma 8.4 follows from a straightforward adaptation of the proof of Lemma 8.9 that we show below.

Let us consider a pair of admissible policies D and I and let us write $D_t = D_t^c + D_t^d$, $I_t = I_t^c + I_t^d$, where D_t^c (resp. I_t^c) is the continuous part of D_t (resp. I_t) and D_t^d (resp. I_t^d) is the pure discontinuous part of D_t (resp. I_t). We recall the dynamics of the cash reserve process and of the profitability prospects process:

$$\begin{aligned} dX_t &= Y_t dt + \sigma dB_t - dD_t + \frac{dI_t}{p}, \\ dY_t &= \frac{\mu^2 - Y_t^2}{\sigma} dB_t. \end{aligned}$$

Applying the generalized Itô's formula to a function V that satisfies the assumptions of Lemma 8.9, we can write for $\tau_0 = \inf\{t \geq 0, X_t = 0\}$,

$$\begin{aligned} e^{-r(t \wedge \tau_0)} V(X_{t \wedge \tau_0}, Y_{t \wedge \tau_0}) &= V(x, y) + \int_0^{t \wedge \tau_0} e^{-rs} (\mathcal{L}V(X_{s-}, Y_s) - rV(X_{s-}, Y_s)) ds \\ &+ \int_0^{t \wedge \tau_0} e^{-rs} V_x(X_{s-}, Y_s) \sigma dB_s \\ &+ \int_0^{t \wedge \tau_0} e^{-rs} V_y(X_{s-}, Y_s) \frac{\mu^2 - Y_s^2}{\sigma} dW_s \\ &- \int_0^{(t \wedge \tau_0)} e^{-rs} V_x(X_s, Y_s) dD_s^c \\ &- \int_0^{(t \wedge \tau_0)} e^{-rs} V_x(X_s, Y_s) \frac{dI_s^c}{p} \\ &+ \sum_{s \leq t \wedge \tau_0} e^{-rs} (V(X_s, Y_s) - V(X_{s-}, Y_s)) (\mathbb{1}_{(\Delta X)_s > 0} + \mathbb{1}_{(\Delta X)_s < 0}), \end{aligned}$$

where $(\Delta X)_s = X_s - X_{s-}$. By assumption, the second term of the right-hand side is negative and, because V has bounded first derivatives, the two stochastic integrals are centered square integrable martingales. Taking expectations and using $1 \leq V_x \leq p$, we obtain

$$\begin{aligned} \mathbb{E} [e^{-r(t \wedge \tau_0)} V(X_{t \wedge \tau_0}, Y_{t \wedge \tau_0})] &\leq V(x, y) - \mathbb{E} \left[\int_0^{t \wedge \tau_0} e^{-rs} dD_s^c \right] + \mathbb{E} \left[\int_0^{t \wedge \tau_0} e^{-rs} dI_s^c \right] \\ &+ \mathbb{E} \left[\sum_{s \leq t \wedge \tau_0} e^{-rs} (V(X_s, Y_s) - V(X_{s-}, Y_s)) (\mathbb{1}_{(\Delta X)_s > 0} + \mathbb{1}_{(\Delta X)_s < 0}) \right]. \end{aligned}$$

Observe that there are two types of jumps for the cash reserve process $(X_t)_{t \geq 0}$. When there is a dividend distribution, $(\Delta X)_s < 0$ so that $X_{s-} - X_s = D_s - D_{s-} > 0$. When there is an issue of shares, $(\Delta X)_s > 0$ and $X_s - X_{s-} = I_s - I_{s-} > 0$. Therefore, the Mean-Value theorem gives the existence of a random variable $\theta \in [X_s, X_{s-}]$ when $X_{s-} - X_s = D_s - D_{s-} > 0$ and a random variable $\eta \in [X_{s-}, X_s]$ when $X_s - X_{s-} = \frac{I_s - I_{s-}}{p} > 0$ such that:

- on the set $\{X_{s-} - X_s = D_s - D_{s-}\}$, $V(X_s, Y_s) - V(X_{s-}, Y_s) = -V_x(\theta, Y_s)(X_{s-} - X_s)$,
- on the set $\{X_s - X_{s-} = \frac{I_s - I_{s-}}{p}\}$, $V(X_s, Y_s) - V(X_{s-}, Y_s) = V_x(\eta, Y_s)(X_s - X_{s-})$.

Therefore,

$$\begin{aligned}
V(X_s, Y_s) - V(X_{s-}, Y_s) &= (V(X_s, Y_s) - V(X_{s-}, Y_s))(\mathbb{1}_{(\Delta X)_s > 0} + \mathbb{1}_{(\Delta X)_s < 0}) \\
&= V_x(\eta, Y_s)(X_s - X_{s-})\mathbb{1}_{(\Delta X)_s > 0} - V_x(\theta, Y_s)(X_{s-} - X_s)\mathbb{1}_{(\Delta X)_s < 0} \\
&= V_x(\eta, Y_s)\left(\frac{I_s - I_{s-}}{p}\right)\mathbb{1}_{(\Delta X)_s > 0} - V_x(\theta, Y_s)(D_s - D_{s-})\mathbb{1}_{(\Delta X)_s < 0} \\
&\leq (I_s - I_{s-})\mathbb{1}_{(\Delta X)_s > 0} - (D_s - D_{s-})\mathbb{1}_{(\Delta X)_s < 0}.
\end{aligned}$$

Finally, we obtain

$$V(x, y) \geq \mathbb{E} \left[e^{-r(t \wedge \tau_0)} V(X_{(t \wedge \tau_0)}, Y_{(t \wedge \tau_0)}) \right] + \mathbb{E} \left[\int_0^{t \wedge \tau_0} e^{-rs} (dD_s - dI_s) \right].$$

In order to get rid of the first term of the right-hand side, we observe that under the assumptions of Lemma 8.9, we have $V(x, y) \leq V(0, y) + px$ that implies

$$\mathbb{E} \left[e^{-r(t \wedge \tau_0)} V(X_{(t \wedge \tau_0)}, Y_{(t \wedge \tau_0)}) \right] \leq e^{-rt} V_\mu(0) + p \mathbb{E}[e^{-rt} X_t].$$

Letting t go to ∞ yields

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[e^{-r(t \wedge \tau_0)} V(X_{(t \wedge \tau_0)}, Y_{(t \wedge \tau_0)}) \right] = 0,$$

and thus

$$V(x, y) \geq \mathbb{E} \left[\int_0^{\tau_0} e^{-rs} (dD_s - dI_s) \right],$$

which ends the proof of Lemma 8.9. \square

Proof of Lemma 8.5. Observe that the function f defined in (58) does not satisfy the Lipschitz condition on an open domain containing (x, μ) with $x \geq 0$, so that the existence and uniqueness of a solution to (56), (57) require a specific analysis.

We remark that, the denominator of (58) is strictly positive if and only if $x > l_1(y)$ where

$$l_1(y) = \frac{\hat{y}}{\mu} \phi\left(y \frac{\hat{y}}{\mu}\right).$$

Thus, f satisfies a local Lipschitz condition with respect to x in \mathcal{D} , where $\mathcal{D} = \{(x, y) \in \mathbb{R} \times (-\mu, \mu) \mid x > l_1(y)\}$. Thus, for any (x, y) , there exists a unique solution $g_{x,y}$ to (56) defined on a maximal interval $I \subset (-\mu, \mu)$ passing through (x, y) .

Second, the numerator of (58) is strictly positive if and only if³⁶ $x > l_2(y)$ where,

$$l_2(y) = \frac{\hat{y}}{\mu} \phi\left(\frac{y \hat{y}}{\mu - r \sigma^2 \left(\frac{\hat{y}}{\mu}\right)^2 \frac{1}{\mu}}\right).$$

³⁶Note that, the definition of \hat{y} in (34) implies that $\mu - r \sigma^2 \left(\frac{\hat{y}}{\mu}\right)^2 \frac{1}{\mu} = \mu \left(1 - r \sigma^2 \frac{1}{\mu^2 + 2r \sigma^2}\right) > 0$.

The function l_2 is continuously increasing on $[-\mu, \mu]$ and satisfies the inequality $l_2(y) > l_1(y)$ for any $y \in (0, \mu]$. Furthermore, $l_1(0) = l_2(0) = 0$ and $l_1(\mu) < l_2(\mu) = \bar{x}_\mu$. To see the last equality, use (38) and remark that $2\phi(\hat{y}) = \phi(\frac{\hat{y}}{1-r\sigma^2(\frac{\hat{y}}{\mu})^2 \frac{1}{\mu^2}})$. This leads to $l_2(\mu) = \bar{x}_\mu$.

We deduce from the above observations, that any solution g to (56) entering in the domain $\{(x, y) \in \mathcal{D} \mid l_1(y) < x < l_2(y)\}$ remains in this domain. Since l_2 is bounded above by \bar{x}_μ on $[-\mu, \mu]$, it follows also that any solution g to (56) defined on a maximal interval I and passing through $(x_0, y_0) \in \{(x, y) \in \mathcal{D} \mid x \geq \bar{x}_\mu\}$ is strictly increasing and satisfies $g(y) > l_2(y)$ for all $y \in I$.

Now, let $(y_n)_{n \geq 0}$ an increasing sequence converging to μ . For each $n \in \mathbb{N}$ there exists a unique solution $g_{\bar{x}_\mu, y_n}$ to (56) satisfying $g_n(\bar{x}_\mu) = y_n$. Let us consider the sequence of functions $(g_n)_{n \geq 0}$ defined by the relations

$$\begin{cases} g_n(y) = g_{\bar{x}_\mu, y_n}(y) & \forall y \in (0, y_n], \\ g_n(y) = \bar{x}_\mu & \forall y \in [y_n, \mu]. \end{cases}$$

Our previous remarks on the solutions to (56) together with a standard non crossing property yield that, $(g_n)_{n \geq 0}$ is a decreasing sequence of increasing functions defined on $(0, \mu]$ and bounded above by \bar{x}_μ . Thus, it admits a pointwise limit g defined on $(0, \mu]$. The function g is bounded above by \bar{x}_μ and satisfies $g(\mu) = \bar{x}_\mu$. We show below that g satisfies (56).

By construction, for each $n \in \mathbb{N}$, for any $y \in (0, \mu)$ one has

$$g_n(y) = \bar{x}_\mu - \int_y^\mu f(g_n(s), s) \mathbb{1}_{s \leq y_n} ds.$$

A direct computation shows that, for any fixed $y > 0$, the mapping $x \rightarrow f(x, y)$ is continuously increasing over $\{x \mid x \geq l_2(y)\}$. We deduce that, for any $y \in (0, \mu)$,

$$\int_y^\mu \lim_{n \rightarrow \infty} f(g_n(s), s) \mathbb{1}_{s \leq y_n} ds = \int_y^\mu f(g(s), s) ds \leq \int_y^\mu f(\bar{x}_\mu, s) ds < \infty,$$

where the last inequality comes from the fact that the mapping $s \rightarrow f(\bar{x}_\mu, s)$ is continuous over $(0, \mu)$ with $\lim_{s \rightarrow \mu} f(\bar{x}_\mu, s) = \frac{1}{2} \frac{1-r\sigma^2(\frac{\hat{y}}{\mu})^2 \frac{1}{\mu^2}}{r(\frac{\hat{y}}{\mu})^2} < \infty$. It results from the dominated convergence Theorem that,

$$g(y) = \bar{x}_\mu - \int_y^\mu f(g(s), s) ds. \tag{121}$$

Thus, g is defined and increasing on $(0, \mu]$, satisfies the ode (56)-(57). A standard extension argument ensures that g is defined on a maximal interval $I \subset (-\mu, \mu)$ as well.

We show that $\bar{y}^* \equiv g^{-1}(0)$ is well defined and satisfies $\bar{y}^* > \hat{y}$. Take the solution $g_{0, \hat{y}}$ to (56) defined on a maximal interval $I \subset (-\mu, \mu)$ passing through $(0, \hat{y})$. A computation shows that the function $v_1(y) = \frac{\hat{y}}{\mu}(\phi(\hat{y}) - \phi(y))$ defined on $(-\mu, \mu)$ satisfies $v_1(\hat{y}) = g_{0, \hat{y}}(\hat{y}) = 0$ together with the inequality $v_1'(y) < f(v_1(y), y)$ for any $y \in (-\mu, 0]$. We deduce that $g_{0, \hat{y}}(y) >$

$v_1(y)$ for all $y \in (\hat{y}, 0]$. From the Cauchy-Lipschitz Theorem it follows that $g_{0,\hat{y}} > g_{v_1(0),0}$ on a maximal interval, where $g_{v_1(0),0}$ is the solution to (56) passing through $(v_1(0), 0)$.

Now, let us consider the function $v_2(y) = \frac{\hat{y}}{\mu}(\phi(y\frac{\hat{y}}{\mu}) + \phi(\hat{y})) \geq l_2(y)$ on $[0, \mu]$. Computations shows that $v_2(0) = v_1(0)$, $v_2(\mu) = \bar{x}_\mu$ and $v_2'(y) \leq f(v_2(y), y)$ for any $y \in [0, \mu]$. We deduce that $g_{0,\hat{y}}(y) \geq v_2(y)$ for any $y \in [0, \mu]$. It follows that $g_{0,\hat{y}} > g_{v_1(0),0} \geq g$ which implies that $\bar{y}^* \equiv g^{-1}(0) > \hat{y}$.

Finally, we show that g is the unique solution to $g'(y) = f(g(y), y)$ satisfying the boundary condition $g(\mu) = \bar{x}_\mu$. Suppose the contrary, let g and \tilde{g} be two solutions to (56) with $g(\mu) = \tilde{g}(\mu) = \bar{x}_\mu$ and $\tilde{g}(y) > g(y)$ over $(0, \mu)$. The functions g and \tilde{g} satisfy (121). It follows that,

$$\tilde{g}(y) - g(y) = \int_y^\mu f(g(s), s) - f(\tilde{g}(s), s) ds. \quad (122)$$

The right hand side of (122) is strictly positive whereas its left hand side is negative because the mapping $x \rightarrow f(x, y)$ is increasing for any fixed $y > 0$, thus, a contradiction.

We now turn to the study of the function $\bar{k} = (\phi - \bar{b})^{-1} : [\phi(\bar{y}^*), \infty) \rightarrow [\bar{y}^*, \mu)$. We observe that

$$\begin{aligned} \phi'(y) - g'(y) &= \frac{\sigma^2}{\mu^2 - y^2} \left(1 - \frac{y\hat{y} - \left(\mu - r\sigma^2 \left(\frac{\hat{y}}{\mu} \right)^2 \frac{1}{\mu} \right) \psi \left(\frac{\mu}{\hat{y}} g(y) \right)}{y\hat{y} - \mu \psi \left(\frac{\mu}{\hat{y}} g(y) \right)} \right) \\ &= \frac{\sigma^2}{\mu^2 - y^2} \frac{\frac{1}{\mu} r\sigma^2 \left(\frac{\hat{y}}{\mu} \right)^2 \psi \left(\frac{\mu}{\hat{y}} g(y) \right)}{-y\hat{y} + \mu \psi \left(\frac{\mu}{\hat{y}} x \right)} \end{aligned}$$

is positive for $g(y) > \max(0, \frac{\hat{y}}{\mu} \phi(g(y)\frac{\hat{y}}{\mu}))$, so that $\phi - \bar{b}$ is strictly increasing over $[\bar{y}^*, \mu)$ where $\bar{b} = \max(0, g)$ with g satisfying (56) and (58). Thus, $\bar{k} = (\phi - \bar{b})^{-1} : [\phi(\bar{y}^*), \infty) \rightarrow [\bar{y}^*, \mu)$ is well defined, increasing and satisfies

$$\begin{aligned} \bar{k}'(z) &= \frac{1}{(\phi - \bar{b})'(\bar{k}(z))} \\ &= \frac{\mu^3}{\hat{y}r\sigma^2} \frac{\mu^2 - k(z)^2}{\sigma^2} \frac{\psi \left(\frac{\mu}{\hat{y}} (\phi(\bar{k}(z)) - z) \right) \mu - y\hat{y}}{\hat{y} \psi \left(\frac{\mu}{\hat{y}} (\phi(\bar{k}(z)) - z) \right)}. \end{aligned}$$

Because $\bar{b} > 0$ and $\bar{k} = (\phi - \bar{b})^{-1}$ we have that $\phi(\bar{k}(z)) - z > 0$. Because $\bar{b}(y) > \frac{\hat{y}}{\mu} \phi(\bar{b}(y)\frac{\hat{y}}{\mu})$ we have that $\psi \left(\frac{\mu}{\hat{y}} (\phi(\bar{k}(z)) - z) \right) \mu - y\hat{y} > 0$, so that $(z, \bar{k}(z)) \in \tilde{\mathcal{D}} = \{(z, y) \in \mathbb{R} \times (-\mu, \mu) \mid \phi(y) - z > \max(0, \frac{\hat{y}}{\mu} \phi(y\frac{\hat{y}}{\mu}))\}$. Finally, the relation $\bar{k} = (\phi - \bar{b})^{-1}$ together with $\lim_{y \rightarrow \mu} \bar{b}(y) = \bar{b}(\mu) = \bar{x}_\mu < \infty$ implies that $\lim_{y \rightarrow \mu} \bar{k}^{-1}(y) = \infty$ and in turn that $\lim_{z \rightarrow \infty} \bar{k}(z) = \mu$ since \bar{k} is increasing. Then, posing $y = \bar{k}(z)$, the relation $\bar{b}(y) = \phi(y) - \bar{k}^{-1}(y)$ leads to $\lim_{z \rightarrow \infty} \bar{b}(\bar{k}(z)) = \phi(\bar{k}(z)) - z = \bar{x}_\mu$.

Conversely, let us consider a C^1 -function k solution to the ordinary differential equation (59), (60) on the domain $\tilde{\mathcal{D}}$. The function k is increasing so that, $b = \phi - k^{-1}$ is well defined over (\bar{y}^*, μ) . It is positive since by assumption $(z, k(z)) \in \tilde{\mathcal{D}}$. A direct computation shows that b satisfies the ordinary differential equation (59), (60) on the domain \mathcal{D} . Thus, we obtain that $b = \bar{b}$ and in turn $k = \bar{k}$. The proof of Lemma 8.5 is complete. \square

Proof of Lemma 8.10. Since for any y fixed in $(-\mu, \mu)$, the mapping $x \rightarrow \bar{V}_x(x, y)$ is concave on $[0, \infty)$, we only have to check that $\bar{V}_x(0, y) < \bar{p}$ for any $y \in (-\mu, \mu)$. Noting that $\bar{V}'_\mu(0) = \bar{p}$ and that $\lim_{y \rightarrow \mu} \bar{V}_x(0, y) = \bar{V}'_\mu(0)$, the result follows from the fact that the mapping $y \rightarrow \bar{V}_x(0, y)$ is increasing.³⁷ To see that latter point, note that $\bar{V}(x, y) = U(\phi(y) - x, y)$ where the function U is defined in the proof of Proposition 8.6. We obtain that $\bar{V}_x(0, y) = -U_z(0, \phi(y))$ and thus that $\frac{d}{dy} \bar{V}_x(0, y) = -\phi'(y)U_{zz}(0, \phi(y))$. The result follows since ϕ is increasing and $U_{zz}(z, y) = \frac{\hat{y}}{2\mu^2} \bar{k}'(z)(h_2''(\bar{k}(z))h_1(y) - h_1''(\bar{k}(z))h_2(y))$ which we know to be negative from the proof of the concavity of the mapping $x \rightarrow \bar{V}(x, y)$ in Proposition 8.7. \square

Proof of Proposition 8.11.

Proof of Assertion (i). From (45), (46) and (47) the mapping

$$y \rightarrow G(z, y) = -h_1'(y)h_2(\psi(z)) + h_2'(y)h_1(\psi(z)) + \frac{2\mu^2}{\hat{y}}p$$

is well defined over $[\psi(z), \infty)$, increasing and satisfies the equality $G(\psi(z), z) = \frac{2\mu^2}{\hat{y}}(p-1) < 0$ and $\lim_{y \rightarrow \mu} G(z, y) = +\infty$. Thus, the function k is well defined over $[z_i, \infty)$ and satisfies the inequality $k > \psi$. Applying the implicit function Theorem, we deduce from the relation $G(z, k(z)) = 0$ that, the function k is differentiable and satisfies for any $z > z_i$

$$k'(z) = \psi'(z) \frac{h_2'(k(z))h_1'(\psi(z)) - h_1'(k(z))h_2'(\psi(z))}{h_1''(k(z))h_2(\psi(z)) - h_2''(k(z))h_1(\psi(z))}. \quad (123)$$

We saw in the proof of Proposition 8.7 that the denominator of the right hand side of (123) is negative. The numerator of k' in (123) is also negative. To see this point, remark that $x \rightarrow h_2'(x)h_1'(y) - h_1'(x)h_2'(y)$ is decreasing over $[y, \mu)$ since it takes the value 0 at $x = y$, and its derivative has the same sign as $h_2(x)h_1'(y) - h_1(x)h_2'(y) < 0$. Thus, k is increasing over $[z_i, \infty)$. Then, assertion (i) of Propostion 8.11 follows from the next lemma.

Lemma 8.16 *the following holds*

(i) $\psi(z + x_\mu) < k(z) \quad \forall z,$

(ii) $\psi(z + x_\mu + \epsilon) > k(z) \quad \forall \epsilon > 0$ for z sufficiently large.

³⁷A computation based on (77) and (79) yields $\lim_{y \rightarrow \mu} \bar{V}_x(0, y) = \bar{V}'_\mu(0)$.

Proof of Lemma 8.16. We show that, for any z , $G(z, \psi(x_\mu + z)) < 0$ and that, for any $\epsilon > 0$ and any z sufficiently large, $G(z, \psi(x_\mu + z + \epsilon)) > 0$. Using (45), (46), (49), computations yield, for $x \geq 0$

$$G(z, \psi(x + z)) = g_1(x) + e^{-\beta z} g_2(x),$$

with

$$\begin{aligned} g_1(x) &= \left(-\frac{\mu^2}{\hat{y}} + \mu\right)e^{\beta\gamma x} - \left(\frac{\mu^2}{\hat{y}} + \mu\right)e^{-(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}}, \\ &= 2(1-\gamma)\mu e^{\beta\gamma x} - 2\gamma\mu e^{-(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}}, \\ g_2(x) &= \left(-\frac{\mu^2}{\hat{y}} + \mu\right)e^{-\beta\gamma x} - \left(\frac{\mu^2}{\hat{y}} + \mu\right)e^{(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}}, \\ &= 2(1-\gamma)\mu e^{-\beta\gamma x} - 2\gamma\mu e^{(\gamma-1)\beta x} + p\frac{2\mu^2}{\hat{y}}. \end{aligned} \quad (124)$$

To prove (i) and (ii), we show that the functions g_1 and g_2 are increasing and satisfy $g_2(x_\mu) < g_1(x_\mu) = 0$. A computation leads to

$$g_1'(x) > 0 \Leftrightarrow e^{(1-2\gamma)\beta x} > 1, \quad \text{and} \quad g_2'(x) > 0 \Leftrightarrow e^{-(1-2\gamma)\beta x} < 1.$$

Both inequalities hold true since $(1-2\gamma)\beta > 0$. The relation $g_1(x_\mu) = 0$ follows from the definition of x_μ . Finally, using (124) and (42) and rearranging terms yield that

$$g_2(x_\mu) < 0 \Leftrightarrow \bar{g}(x_\mu) > \underline{g}(x_\mu) \quad (125)$$

with $\bar{g}(x) = (1-\gamma)(e^{\gamma\beta x} - e^{-\beta\gamma x})$ and $\underline{g}(x) = \gamma(e^{(1-\gamma)\beta x} - e^{(\gamma-1)\beta x})$. It is easy to see that $\bar{g}(0) = \underline{g}(0) = 0$ and that \bar{g} and \underline{g} are decreasing. To prove (125), we remark that, for any $x > 0$, $\bar{g}'(x) > \underline{g}'(x)$. Indeed, this latter inequality is equivalent to

$$e^{\gamma\beta x} + e^{-\beta\gamma x} < e^{\beta(\gamma-1)x} + e^{\beta(1-\gamma)x},$$

which, given that $-\beta\gamma < \beta(1-\gamma)$, follows the properties of the function cosh. \square

Thus, from (i) and (ii) we have that,

$$\forall \epsilon > 0, \exists \bar{z}, \forall z \geq \bar{z}, \quad \psi(z + x_\mu) < k(z) < \psi(z + x_\mu + \epsilon).$$

Assertions (96) and (97) follow by noting that, $k(z) = \psi((\phi(k(z)) - z) + z)$ and that ψ is a bounded continuous and increasing function. The proof of assertion (i) is complete.

Proof of Assertion (ii). We start with the existence of a solution z_i^* to equation (98). Let us consider

$$f(z) = \frac{p}{\psi'(z)} h_1(\psi(z)) + \int_z^\infty h_1'(k(u)) du. \quad (126)$$

To begin, we show that $f(z_i^*) > 0$ and that $\lim_{z \rightarrow \infty} f(z) < 0$. Since function h_1' is negative and increasing, we deduce from the inequality $\psi < k$ that

$$\int_{z_i^*}^\infty h_1'(\psi(u)) du < \int_{z_i^*}^\infty h_1'(k(u)) du. \quad (127)$$

Thus, to show that $f(z_i^*) > 0$, we show that

$$\frac{p}{\psi'(z_i^*)} h_1(\psi(\hat{z})) + \int_{z_i^*}^{\infty} h_1'(\psi(u)) du > 0.$$

Computations are explicit and yield that (127) is equivalent to

$$p(1 + e^{-\beta z_i^*}) - \frac{\gamma}{\gamma - 1} e^{-\beta z_i^*} + \frac{1 - \gamma}{\gamma} > 0. \quad (128)$$

An easy computation shows that the left hand side of (128) is equal to zero when $p = 1$. This implies that $f(z_i^*) > 0$ for $p > 1$. We already know that $k = \psi$ when $p = 1$. We thus obtained as a by product result that $z_i^* = \hat{z}$ when $p = 1$.

We show that $\lim_{z \rightarrow \infty} f(z) = 0^-$. From (49) and (97), it is sufficient to show that,

$$\begin{aligned} \frac{p}{\beta} e^{\beta \gamma z} (1 + e^{-\beta z}) + \int_z^{\infty} h_1'(\psi(u + x_\mu)) du &< 0 \\ \Leftrightarrow p(1 + e^{-\beta z}) &< \frac{1 - \gamma}{-\gamma} e^{\beta \gamma x_\mu}. \end{aligned}$$

This latter inequality follows from (44), thus the result. Therefore, there exists z_i^* such that $f(z_i^*) = 0$.

Uniqueness of z_i^ .* A direct computation shows that $f'(z) < 0$ for $z < 0$. Therefore, given that $\lim_{z \rightarrow \infty} f(z) < 0$, if f has more than one zero, there exists z_1 and z_2 such that $0 < z_1 < z_2$, $f(z_1) = f(z_2) = 0$ and $f'(z_1) > 0$ and $f'(z_2) < 0$. We reason by way of contradiction and prove that if there are z_1 and z_2 such that $0 < z_1 < z_2$, $f(z_1) = f(z_2) = 0$ and $f'(z_1) > 0$ then $f'(z_2) > 0$ which contradicts $\lim_{z \rightarrow \infty} f(z) < 0$.

We consider below $g(z) = f(z)h_2(\psi(z))$ that as the same zeros and the same sign than f . From (47) and (48), we have that

$$g(z) = p\sigma^2 + \int_z^{\infty} h_1'(k(u)) du h_2(\psi(u)).$$

Thus,

$$\begin{aligned} g'(z) &= \psi'(z)h_2'(\psi(z)) \int_z^{\infty} h_1'(k(u)) du - h_2(\psi(z))h_1'(k(z)) \\ &= \psi'(z)h_2'(\psi(z)) \int_z^{\infty} h_1'(k(u)) du - p \frac{2\mu^2}{\hat{y}} - h_1(\psi(z))h_2'(k(z)), \end{aligned}$$

where the last equality follows from (95).

Now, z_1 satisfies by assumption $g(z_1) = 0$ and $g'(z_1) > 0$, thus

$$\begin{aligned} g'(z_1) &= \psi'(z_1)h_2'(\psi(z_1)) \left(-\frac{p}{\psi'(z_1)} h_1(\psi(z_1)) - p \frac{2\mu^2}{\hat{y}} - h_1(\psi(z_1))h_2'(k(z_1)) \right) \quad (129) \\ &= -p h_2'(\psi(z_1))h_1(\psi(z_1)) - p \frac{2\mu^2}{\hat{y}} - h_1(\psi(z_1))h_2'(k(z_1)) > 0, \end{aligned}$$

where the first term of the right hand side of (129) follows from (126). Any zero z of g satisfies

$$g'(z) = -ph_2'(\psi(z))h_1(\psi(z)) - p\frac{2\mu^2}{\hat{y}} - h_1(\psi(z))h_2'(k(z)).$$

We show that

$$q_1 : z \longrightarrow -h_2'(\psi(z))h_1(\psi(z)),$$

and

$$q_2 : z \longrightarrow -h_1(\psi(z))h_2'(k(z))$$

are increasing functions which will imply that $g'(z_2) > 0$. We have

$$\text{sign}\{q_1'(z)\} = \text{sign}\{-2r\sigma^2 - (\frac{\mu^2}{\hat{y}} - \psi(z))(-\frac{\mu^2}{\hat{y}} - \psi(z))\},$$

and

$$-2r\sigma^2 - (\frac{\mu^2}{\hat{y}} - \psi(z))(-\frac{\mu^2}{\hat{y}} - \psi(z)) = \mu^2 - \psi^2(z) > 0.$$

Also, we have

$$q_2'(z) = -k'(z)h_2''(k(z))h_1(\psi(z)) - h_1'(\psi(z))\psi'(z)h_2'(k(z)).$$

It follows that

$$q_2'(z) \geq 0 \Leftrightarrow k'(z)\frac{2r\sigma^2}{\mu^2 - k(z)^2} < \psi'(z)\frac{h_1'(\psi(z))h_2(k(z))(-\frac{\mu^2}{\hat{y}} - k(z))}{-h_2(k(z))h_1(\psi(z))}. \quad (130)$$

Using (123), a computation yields that

$$k'(z)\frac{2r\sigma^2}{\mu^2 - k(z)^2} = \psi'(z)\frac{h_1'(\psi(z))h_2(k(z))(-\frac{\mu^2}{\hat{y}} - k(z)) - h_1(k(z))h_2'(\psi(z))(\frac{\mu^2}{\hat{y}} - k(z))}{h_1(k(z))h_2(\psi(z)) - h_2(k(z))h_1(\psi(z))}. \quad (131)$$

Using (130) and (131), a computation shows that $q_2'(z) \geq 0$ is equivalent to $\psi(z) < k(z)$, which we know to be true, thus the result.

Proof of Assertion (iii). Given assertion (i), we only need to prove that the solution k_1 to the ordinary differential equation

$$\begin{aligned} k_1'(z) &= \Theta(z, k_1(z)), \\ k_1(z_i^*) &= k_2(z_i^*), \end{aligned}$$

with

$$\Theta(z, y) = \frac{\mu^3}{\hat{y}r\sigma^2} \frac{\mu^2 - y^2}{\sigma^2} \frac{\psi\left(\frac{\mu}{\hat{y}}(\phi(y) - z)\right) \mu - y\hat{y}}{\hat{y}\psi\left(\frac{\mu}{\hat{y}}(\phi(y) - z)\right)}$$

defined on the domain $\tilde{\mathcal{D}} = \{(z, y) \in \mathbb{R} \times (-\mu, \mu) \mid \phi(y) - z > \max(0, \frac{\hat{y}}{\mu}\phi(y\frac{\hat{y}}{\mu}))\}$, is a well defined continuously differentiable and increasing function over $(-\infty, z_i^*)$.

We deduce from the proof of Lemma 8.5 that the ode

$$\begin{aligned} k'(z) &= \Theta(z, k(z)), \\ k(z_i) &= y_i, \end{aligned}$$

where the couple (z_i, y_i) satisfies

$$\phi(y_i) - z_i > \max\left(0, \frac{\hat{y}}{\mu} \phi\left(y_i \frac{\hat{y}}{\mu}\right)\right) \quad (132)$$

has a unique solution that is continuously differentiable and increasing over $(-\infty, z_i)$. To establish Assertion (iii), it thus remain to show that the couple $(z_i^*, k_2(z_i^*))$ satisfies (132). We deduce from (75) and (76) that this requirement is equivalent to

$$h'_1(k_2(z_i^*))h_2(\psi(z_i^*)) + h'_2(k_2(z_i^*))h_1(\psi(z_i^*)) > 0. \quad (133)$$

Consider first the case $z_i^* \leq 0$. Let us recall that the mapping $y \rightarrow h'_1(y)h_2(\psi(z)) + h'_2(y)h_1(\psi(z))$ is increasing and that $k_2 > \psi$. It follows that

$$\begin{aligned} &h'_1(k_2(z_i^*))h_2(\psi(z_i^*)) + h'_2(k_2(z_i^*))h_1(\psi(z_i^*)) \\ &> h'_1(\psi(z_i^*))h_2(\psi(z_i^*)) + h'_2(\psi(z_i^*))h_1(\psi(z_i^*)) = -2\psi(z_i^*) \geq 0. \end{aligned}$$

Thus, (133) is satisfied. Next, consider that $z_i^* > 0$. The function k_2 satisfies (95). It follows that (133) is equivalent to $h'_2(k_2(z_i^*))h_1(\psi(z_i^*)) + \frac{\mu^2}{\hat{y}}p > 0$. From the proof of assertion (ii) we know that z_i^* satisfies³⁸

$$-ph'_2(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{2\mu^2}{\hat{y}} - h_1(\psi(z_i^*))h'_2(k(z_i^*)) < 0.$$

that is

$$p\frac{\mu^2}{\hat{y}} + h_1(\psi(z_i^*))h'_2(k(z_i^*)) > -ph'_2(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{\mu^2}{\hat{y}}.$$

Then, a direct computation shows that

$$-ph'_2(\psi(z_i^*))h_1(\psi(z_i^*)) - p\frac{\mu^2}{\hat{y}} > 0$$

if and only if $\psi(z_i^*) > 0$ which is true since we consider $z_i^* > 0$. As a final remark, an easy computation based on (75) and (123) yields that $k'_1(z) \neq k'_2(z)$ so that k^* is not differentiable at z_i^* . The proof of assertion (iii) is complete. \square

The next Lemma completes the proof of Proposition 8.15.

Lemma 8.17 *The following holds*

$$\frac{k^*(z)}{\psi'(z + \tilde{b}(z))} - 1 > 0 \text{ for any } z > z_i^*.$$

³⁸See the argument underlying Equation (129).

Proof of Lemma 8.17. From Proposition 8.11 and Proposition 8.14, we deduce that k^* satisfies (123). Therefore, to prove Lemma 8.17, we show that

$$\frac{\psi'(z)}{\psi'(z+x)} \frac{h_2'(\psi(z+x))h_1'(\psi(z)) - h_1'(\psi(z+x))h_2'(\psi(z))}{h_1''(\psi(z+x))h_2(\psi(z)) - h_2''(\psi(z+x))h_1(\psi(z))} - 1 < 0.$$

To prove the latter inequality, it is enough to prove

$$\begin{aligned} & \psi'(z)(h_2'(\psi(z+x))h_1'(\psi(z)) - h_1'(\psi(z+x))h_2'(\psi(z))) \\ & - \psi'(z+x)(h_1''(\psi(z+x))h_2(\psi(z)) - h_2''(\psi(z+x))h_1(\psi(z))) > 0 \end{aligned}$$

or equivalently,

$$\begin{aligned} & \psi'(z)(h_1(\psi(z+x))h_2(\psi(z))(-\frac{\mu^2}{\hat{y}} - \psi(x+z))(\frac{\mu^2}{\hat{y}} - \psi(z))) \\ & - h_2(\psi(z+x))h_1(\psi(z))(\frac{\mu^2}{\hat{y}} - \psi(x+z))(-\frac{\mu^2}{\hat{y}} - \psi(z))) \\ & - \frac{\psi'(z+x)2\sigma^2r}{(\mu^2 - \psi^2(x+z))^2}(h_1(\psi(z+x))h_2(\psi(z)) - h_2(\psi(z+x))h_1(\psi(z))) > 0. \end{aligned}$$

This latter expression is equivalent to

$$\begin{aligned} & h_1(\psi(z+x))h_2(\psi(z)) \left(\psi'(z)(-\frac{\mu^2}{\hat{y}} - \psi(x+z))(\frac{\mu^2}{\hat{y}} - \psi(z)) - \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))} \right) + \\ & h_2(\psi(z+x))h_1(\psi(z)) \left(\psi'(z)(-\frac{\mu^2}{\hat{y}} + \psi(x+z))(-\frac{\mu^2}{\hat{y}} - \psi(z)) + \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))} \right) > 0. \end{aligned} \quad (134)$$

We remark that

$$\left(\psi'(z)(-\frac{\mu^2}{\hat{y}} + \psi(x+z))(-\frac{\mu^2}{\hat{y}} - \psi(z)) + \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))} \right) > 0, \quad (135)$$

and

$$h_2(\psi(z+x))h_1(\psi(z)) > h_1(\psi(z+x))h_2(\psi(z)). \quad (136)$$

A straightforward computation shows that the sign of

$$\begin{aligned} & \psi'(z)(-\frac{\mu^2}{\hat{y}} + \psi(x+z))(-\frac{\mu^2}{\hat{y}} - \psi(z)) + \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))} \\ & + \psi'(z)(-\frac{\mu^2}{\hat{y}} - \psi(x+z))(\frac{\mu^2}{\hat{y}} - \psi(z)) - \frac{2r}{h_1(\psi(x+z))h_2(\psi(x+z))} \end{aligned}$$

is the same as the sign of $-\psi(z) + \psi(x+z)$. But clearly, $-\psi(z) + \psi(x+z)$ is positive. It then follows from (134), (135) and (136) that the mapping $y \rightarrow b^*(y)$ is decreasing over $[y^{i*}, \mu)$.