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Competing Mechanisms and Folk Theorems: Two Examples

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Abstract

We study competing-mechanism games under exclusive competition: principals first simultaneously post mechanisms, after which agents simultaneously choose to participate and communicate with at most one principal. In this setting, which is common to competing-auction and competitive-search applications, we develop two complete-information examples that question the relevance of the folk theorems for competing-mechanism games documented in the literature. The first example shows that there can exist pure-strategy equilibria in which some principal obtains a payoff below her min-max payoff, computed over all principals' decisions. Thus folk-theorem-like results may have to involve a bound on principals' payoffs that depends on the spaces of messages available to the agents, and not only on the players' actions. The second example shows that even this nonintrinsic approach is misleading when agents' participation decisions are strategic: there can exist incentive-feasible allocations in which principals obtain payoffs above their min-max payoffs, computed over arbitrary spaces of mechanisms, but which cannot be supported in equilibrium.

Keywords: Competing Mechanisms, Folk Theorems, Exclusive Competition.
JEL Classification: D82.

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1 Introduction

Competition in financial, labor, and monetary markets is often modeled by assuming that sellers noncooperatively design trade mechanisms. Final allocations are then determined by buyers' strategic participation and communication decisions in these mechanisms. In competing auctions (McAfee (1993), Peters (1997), Peters and Severinov (1997), Virag (2010)), privately informed buyers observe the posted mechanisms, choose the auction they want to participate to, and then bid according to their valuations. In competitive search (Moen (1997), Eeckhout and Kircher (2010), Wright, Kircher, Julien and Guerrieri (2017), Auster and Gottardi (2019)), buyers apply to their preferred trade mechanism, meet a seller according to some meeting technology, and, once in a meeting, communicate with the seller they are matched with, possibly revealing private information to her.

These interactions are instances of competing-mechanism games in which principals first simultaneously commit to mechanisms and agents then simultaneously participate and communicate with principals. Mechanisms are public and are not contingent on one another. That is, when designing a mechanism, a principal cannot directly condition her decisions on the *market information* generated by her competitors' mechanisms, and held by the agents. However, she can, in principle, indirectly condition her decisions on reports by the agents about their market information. Since the seminal work of Epstein and Peters (1999), the literature on competing mechanisms has emphasized that providing agents with the opportunity to report both their exogenous private information and their endogenous market information can spectacularly enlarge the set of equilibrium allocations. Following Yamashita (2010), several contributions (Peters and Troncoso-Valverde (2013), Xiong (2013), Ghosh and Han (2018)) have indeed offered different versions of a folk theorem: in a nutshell, letting principals' mechanisms be sufficiently reactive to agents' reports about their market information allows one to support in equilibrium any incentive-compatible allocation that yields each principal a payoff above a well-specified min-max bound. These results are established under fairly general conditions on the primitives of the game, which questions the relevance of the equilibrium analyses provided in the applied literature.

The present paper further elaborates on this issue. We focus on situations in which agents' participation decisions are strategic, in line with the intended economic applications of the competing-mechanism paradigm, and we provide two examples that fundamentally challenge the logic of folk theorems in this context.

As a contrast to our results, it is useful to review the arguments leading to folk theorems for competing-mechanism games, as first developed by Yamashita (2010). Let each principal commit to decisions to punish a unilateral deviation by any of her competitors, when reported by a majority of agents through appropriate messages, and assume that the agents' message sets are rich enough to allow them to select a specific punishment for each of these deviations. Then, if there are at least three agents, any such punishment can be selected by majority voting in the message game played by the agents. When unilaterally deviating, each principal hence anticipates that her competitors will react by min-maximizing her: as a result, any incentive-compatible allocation yielding her a payoff above the corresponding bound can be supported in equilibrium by such mechanisms. The lower bound of a principal's equilibrium payoffs turns out to coincide with her min-max payoff computed over all mechanisms available to principals, which establishes the folk theorem. This equality, however, obtains by making agents able to select the worst punishment against every principal in every subgame, which effectively requires that *each* agent participates and communicates with *each* principal for *any* profile of posted mechanisms. We find this assumption hard to justify in the light of economic applications, in which agents' communication decisions are closely tied to their participation decisions. That is, an agent can communicate the information he possesses to a given principal only if he chooses to participate with her.

We show that taking into account agents' participation decisions in competing-mechanism games has dramatic implications for the possibility of deriving folk-theorem results. We focus on the situation in which agents participate and communicate with at most one principal, as is assumed both in competitive-auction and in competitive-search models. In this exclusive-competition scenario, we construct two examples for the complete-information case in which agents' types are degenerate, so that they can only take participation decisions and can only report about their market information.

Our first example exhibits equilibria of competing-mechanism games in which some principal obtains a payoff below her min-max payoff, computed over the set of principals' decisions. In this example, the explicit consideration of agents' participation decisions leads to discontinuities and nonconvexities that prevent from applying the standard min-max logic, despite the fact that there are only two principals who are allowed to randomize over their decisions. The min-max payoff of the principal in question is strictly higher than her max-min payoff, and hence cannot be a relevant bound for equilibrium characterization. The result suggests that to establish a folk theorem in complete-information games, one may need to specify a nonintrinsic bound that depends on the agents' message sets, which in turn limits the predictive power of the approach.

Our second example establishes that even this approach is unsatisfactory when agents' participation decisions are strategic. In this example, the min-max payoff for each principal can be straightforwardly computed over arbitrary mechanisms, and it coincides with the corresponding max-min payoff. Yet, the fact that each agent can communicate with at most one principal makes it impossible to construct sophisticated equilibrium threats. As a consequence, although there exist many incentive-feasible allocations in which principals obtain payoffs above their min-max payoffs, none of them can be supported in equilibrium. Each principal's equilibrium payoff thus coincides with its min-max payoff, even if the analysis is extended so as to allow for mixed-strategy equilibria.

Related Literature

This note is closely related to the recent literature on folk theorems in competing-mechanism games initiated by Yamashita (2010). The main contribution of Yamashita (2010) is to highlight the role of *recommendation mechanisms*. By offering such a mechanism, a principal commits to a direct mechanism if all but one agent recommend her to do so. Letting principals post recommendation mechanisms makes it possible to reproduce the equilibrium allocations associated to the *universal* space of mechanisms identified by Epstein and Peters (1999). Yamashita (2010) further assumes that each agent participates and communicates with all principals for any profile of mechanisms, which allows him to rely on recommendation mechanisms to derive an equilibrium characterization in terms of principals' min-max payoffs. His analysis has been extended in two important directions.

First, the bound for principals' equilibrium payoffs proposed by Yamashita (2010) is sensitive to the mechanisms available in the game. This makes it difficult to evaluate his contribution in the light of standard folk theorems. The recent work of Peters and Troncoso-Valverde (2013) provides an abstract framework for incomplete-information games and formulates the corresponding bounds through agents' incentive constraints in the spirit of Myerson (1979). The bounds are therefore defined in terms of the primitives of the model and, in contrast to Yamashita (2010), do not depend on the set of available (indirect) mechanisms. Ghosh and Han (2018) extend Yamashita (2010) to repeated interactions and reformulate the bounds on principals' equilibrium payoffs in these settings.

Yet, neither Peters and Troncoso-Valverde (2013) nor Ghosh and Han (2018) allow players to take any action after mechanisms are posted. Our first example shows that, under complete information, the bounds identified by Peters and Troncoso-Valverde (2013) and Ghosh and Han (2018) are no longer relevant for principals' equilibrium payoffs if agents' participation decisions are taken into account.

Second, Yamashita (2010) restricts principals to deterministic mechanisms and only considers pure-strategy equilibria of the agents' game. Szentes (2009) shows that this restriction is critical by constructing a simple complete-information game in which the equilibrium allocations supported by deterministic mechanisms yield a principal a payoff below Yamashita's (2010) min-max bound. Xiong (2013) provides a generalized version of Yamashita (2010), in which random mechanisms are allowed and a folk theorem is established. Crucially, he also assumes that agents always communicate with all principals. We share with Szentes (2009) the focus on complete information, but we allow principals to post random mechanisms, and we do not restrict agents to play pure strategies. In contrast with these approaches, we explicitly model agents' participation decisions. Our second example then shows that recommendation mechanisms may not guarantee a system of punishments allowing each principal to min-maximize his opponents.

An alternative route to folk theorems in competing mechanism games is based on the notion of *contractible contracts*. Following Tennenholz (2004) and Kalai, Kalai, Lehrer, and Samet (2010), Peters and Szentes (2012), Peters (2015) and Szentes (2015) let principals design mechanisms that depend on the mechanisms of their opponents. This allows them to *directly* punish a deviator in a way that depends on the specific deviation she chooses, which can yield a folk theorem even in the absence of a strategic role for the agents. The observability requirements underlying this approach, however, are too demanding in the light of the economic applications we consider.

The paper is organized as follows. Section 2 introduces a general model of exclusive competition under complete information. Sections 3 and 4 present our examples. Section 5 concludes. Proofs not given in the text can be found in the Appendix.

2 The Model

We consider a setting in which several principals, indexed by $j \in \mathcal{J} \equiv \{1, \ldots, J\}$, contract with several agents, indexed by $i \in \mathcal{I} \equiv \{1, \ldots, I\}$. Agents have no private information, and we denote each agent's single type by t.

Actions and Payoffs Agents only take participation decisions, and we denote by $a^i_j \in A^i_j \equiv \{Y, N\}$ agent i's decision to participate (Y) or not (N) with principal j. Such decisions are exclusive, in that each agent i can participate with at most one principal j. Hence, overall, each agent i takes an action a^i in the set $A^i \equiv \{(a^i_1, \ldots, a^i_J) : a^i_j = Y \text{ for at most one } j\}$. Each principal j in turn takes an action x_j in a finite set X_j. We let $v_j : A \times X \to \mathbb{R}$ and

$u^i : A \times X \to \mathbb{R}$ be the payoff functions of principal j and of agent i, respectively, where $A \equiv A^1 \times \ldots \times A^I$ and $X \equiv X_1 \times \ldots \times X_J$.

Communication Communication takes place through the public mechanisms posted by the principals and the messages sent by the agents in these mechanisms. Formally, agent i sends a private message m^i_j to principal j in some Polish space M^i_j.[1] Each message space M^i_j includes the empty message $\emptyset$, which corresponds to agent i not communicating with principal j, as well as the trivial message t. Communication is tied to participation, in the sense that agent i sends a nonempty message to principal j if and only if he decides to participate with her. Hence, overall, each agent i sends messages m^i in the space $M^i \equiv \{(m^i_1, \ldots, m^i_J) : m^i_j \neq \emptyset \text{ for at most one } j\}$, and we say that a profile $(m^i, a^i) \in M^i \times A^i$ is *consistent for agent* i whenever $m^i_j \neq \emptyset$ if and only if $a^i_j = Y$ for all j. We denote by C^i the space of such consistent communication and participation profiles for agent i.

Mechanisms Each principal j can take an action contingent on the messages $m_j \in M_j$ she receives and the agents' decisions $a_j \in A_j$ to participate with her, where by definition $M_j \equiv M^1_j \times \ldots \times M^I_j$ and $A_j \equiv A^1_j \times \ldots \times A^I_j$. We say that a profile $(m_j, a_j) \in M_j \times A_j$ is *consistent for principal* j if $m^i_j \neq \emptyset$ if and only if $a^i_j = Y$ for all i. We denote by C_j the space of such consistent communication and participation profiles for principal j. Notice that, because M_j is Polish and A_j is finite, C_j is Polish. A mechanism for principal j is a Borel-measurable mapping $\gamma_j : C_j \to \Delta(X_j)$ that associates to every consistent communication and participation profile for principal j a lottery over her actions.

Admissibility Whereas most of our analysis focuses on situations in which principals play pure strategies in equilibrium, a general requirement for defining expected payoffs in our and related games is that the evaluation mapping $(\gamma_j, c_j) \mapsto \gamma_j(c_j)$ describing how the distribution of principal j's action varies with her mechanism and the consistent communication and participation profile she observes be measurable. Thus, at a minimum, we must define a measurable structure on the set $\Gamma^{M_j}_j$ of mechanisms for principal j. Two cases can arise. If M_j and, hence, C_j, is countable, we can take $\Gamma^{M_j}_j$ to be the set of all Borel-measurable mappings $\gamma_j : C_j \to \Delta(X_j)$; a natural measurable structure on $\Gamma^{M_j}_j$ is then the product Borel σ-field on the product of at most countably infinitely many copies of $\Delta(X_j)$. If M_j is uncountable, however, there is no measurable structure on the set of all Borel-measurable mappings $\gamma_j : C_j \to \Delta(X_j)$ such that the evaluation mapping for principal j is measurable

[1] Our first example only allows for minimal communication, and thus finite message sets. Our second example, by contrast, allows for rich communication, and thus uncountable message spaces. Requiring these spaces to be Polish entails no loss of generality.

(Aumann (1961)); in that case, there is no other choice than to restrict the set of admissible mechanisms $\Gamma_j^{M_j}$. Admissibility can be shown to coincide with the requirement that $\Gamma_j^{M_j}$ be of bounded Borel class (Aumann (1961), Rao (1971)), allowing for a rich class of mechanisms for our analysis. With this caveat in mind, we hereafter fix an admissible space $\Gamma_j^{M_j}$ of mechanisms for every principal j, with associated σ-field $\mathcal{G}_j^{M_j}$.

Strategies and Timing The competing-mechanism game G^M induced by M unfolds in three stages:

1. The principals simultaneously post mechanisms.

2. The agents simultaneously take consistent communication and participation decisions.

3. The principals' mechanisms are implemented, lotteries realize, and all payoffs accrue.

A strategy for principal j is a probability measure $\mu_j \in \Delta(\Gamma_j^{M_j})$ over the σ-field $\mathcal{G}_j^{M_j}$. A strategy for agent i is a measurable mapping $\lambda^i : \Gamma^M \to \Delta(C^i)$ that associates to every profile of mechanisms a probability measure over consistent communication and participation profiles for agent i, where $\Gamma^M \equiv \Gamma_1^{M_1} \times \ldots \times \Gamma_J^{M_J}$ for $M \equiv M_1 \times \ldots \times M_J$ is endowed with the product σ-field $\mathcal{G}_1^{M_1} \otimes \ldots \otimes \mathcal{G}_J^{M_J}$ and $\Delta(C^i)$ with the Borel σ-field. The allocation $z(\gamma, \lambda)$ induced by the mechanisms $\gamma \equiv (\gamma_1, \ldots, \gamma_J)$ and the strategies $\lambda \equiv (\lambda^1, \ldots, \lambda^I)$ is the probability measure over outcomes in $A \times X$ uniquely defined by the marginal of $\lambda^1(\gamma) \otimes \ldots \otimes \lambda^I(\gamma)$ over A and the probability transitions γ_j from C_j to X_j. Notice that the mapping $z(\cdot, \lambda) : \Gamma^M \to \Delta(A \times X) : \gamma \mapsto z(\gamma, \lambda)$ is measurable. Hence we can define the allocation $z(\mu, \lambda)$ induced by the strategies $\mu \equiv (\mu_1, \ldots, \mu_J)$ and $\lambda \equiv (\lambda^1, \ldots, \lambda^I)$ by

$$z(\mu, \lambda)(a, x) \equiv \int_{\Gamma^M} z(\gamma, \lambda)(a, x)\, \mu_1(\mathrm{d}\gamma_1) \otimes \ldots \otimes \mu_J(\mathrm{d}\gamma_J)$$

for all $(a, x) \in A \times X$.

Equilibrium The strategy profile (μ, λ) is a subgame-perfect equilibrium (SPNE) of G^M whenever:

(i) For each $\gamma \in \Gamma^M$, $(\lambda^1(\gamma), \ldots, \lambda^I(\gamma))$ is a Nash equilibrium in the subgame γ played by the agents.

(ii) Given the continuation equilibrium λ, μ is a Nash equilibrium of the game played by the principals.

We denote by $\Lambda^*(\gamma)$ the set of Nash equilibria of the subgame γ. Following Epstein and Peters (1999) and Han (2007), we will mostly focus on SPNEs of G^M in which principals play pure strategies (SPNE-PSP). That is, each principal deterministically posts a mechanism; notice, however, that this mechanism may involve randomization over her final actions.

Direct Mechanisms In our complete-information setting, a *direct* mechanism for principal j is a mechanism such that the message space M_j is restricted to the trivial message t and the empty message $\emptyset$. Because in this case communication decisions are redundant relative to participation decisions, such a mechanism can be identified to a mapping $\tilde{\gamma}_j : A_j \to \Delta(X_j)$ that associates to every participation profile for principal j a lottery over her actions. We denote by $\tilde{\Gamma}_j$ the set of direct mechanisms for principal j, and we let $\tilde{\Gamma} \equiv \tilde{\Gamma}_1 \times \ldots \times \tilde{\Gamma}_J$ and $\tilde{G}$ be the competing-mechanism game in which principals are restricted to direct mechanisms. Notice that, because A_j and X_j are finite, $\tilde{\Gamma}_j$ is a compact and convex subset of a Euclidean space. A strategy for agent i in $\tilde{G}$ can be identified to a Borel-measurable mapping $\tilde{\lambda}^i : \tilde{\Gamma} \to \Delta(A^i)$ that associates to every profile of direct mechanisms a lottery over participation profiles for agent i. We say that the allocation $z(\tilde{\gamma}, \tilde{\lambda})$ is *incentive-feasible* if $\tilde{\lambda} \in \Lambda^*(\tilde{\gamma})$.

3 The First Example

Let $I = J \equiv 2$ and let $X_1 \equiv \{x_{11}, x_{12}\}$ and $X_2 \equiv \{x_{21}, x_{22}\}$ be the sets of actions of principal 1 (P1) and principal 2 (P2), respectively. Let $A^1 = A^2 \equiv \{YN, NY, NN\}$ be agent 1's (A1) and agent 2's (A2) sets of actions, where YN, for instance, refers to the agent participating with P1 but not with P2. Payoffs are represented in the matrix in Table 1 below, in which agents choose in the external box and principals choose in the internal 2×2 cells. Each array represents the payoffs to P1, P2, A1, and A2, respectively.

		YN			NY	
		x_{21}	x_{22}		x_{21}	x_{22}
YN	x_{11}	$(0, \zeta, 5, \frac{25}{2})$	$(0, \zeta, 5, \frac{25}{2})$	x_{11}	$(0, 0, 5, 12)$	$(0, 10, 5, 8)$
	x_{12}	$(0, \zeta, 10, \frac{9}{2})$	$(0, \zeta, 10, \frac{9}{2})$	x_{12}	$(0, 0, 10, 12)$	$(0, 10, 10, 8)$
		x_{21}	x_{22}		x_{21}	x_{22}
NY	x_{11}	$(0, 10, 8, 12)$	$(0, 8, 9, 12)$	x_{11}	$(0, \zeta, 40, 7)$	$(0, \zeta, 4, 13)$
	x_{12}	$(0, 10, 8, 8)$	$(0, 8, 9, 8)$	x_{12}	$(0, \zeta, 40, 7)$	$(0, \zeta, 4, 13)$

Table 1: The payoff matrix.

Observe that P1's payoff is constantly equal to 0, and that, for any profile of participation decisions by the agents, P1's choice of action has no impact on P2's payoff. Hence there are no direct payoff externalities between the principals, and P1 can affect P2's payoff only insofar as she can influence the agents' participation decisions through her choice of a mechanism. We assume, in addition, that the no-participation decision NN is strictly dominated for every agent, and we let $\zeta < 0$ be an arbitrarily large loss for P2 if neither A1 nor A2 participate with her, or if both A1 and A2 participate with her. Notice also that there exists at least one incentive-feasible allocation yielding P2 her maximal payoff of 10. To see this, consider the simple direct mechanisms in which P1 chooses x_{11} and P2 chooses x_{21} for any participation decisions of the agents. The resulting subgame played by the agents only admits the Nash equilibrium (NY, YN), which yields P2 a payoff of 10.

3.1 The Game $\tilde{G}$

We first consider the game $\tilde{G}$ in which each principal does not ask for private messages and associates to every profile of agents' decisions to participate with her a lottery over her actions. Let us first observe that a direct mechanism for P1, say $\tilde{\gamma}_1$, is represented by the following list of participation-contingent probability distributions over X_1:

$$\begin{aligned} \tilde{\gamma}_1(Y,Y) &= (\delta_{Y,Y}, 1-\delta_{Y,Y}), & \tilde{\gamma}_1(Y,N) &= (\delta_{Y,N}, 1-\delta_{Y,N}), \\ \tilde{\gamma}_1(N,Y) &= (\delta_{N,Y}, 1-\delta_{N,Y}), & \tilde{\gamma}_1(N,N) &= (\delta_{N,N}, 1-\delta_{N,N}), \end{aligned}$$

where $\delta_{a_1^1,a_1^2}$ denotes the probability of P1 choosing action x_{11} given a participation profile $(a_1^1, a_1^2) \in A_1$ for P1. Thus, for instance, $\delta_{Y,N}$ is the probability that P1 chooses x_{11} if only A1 chooses to participate with her. Similarly, a direct mechanism for P2, say $\tilde{\gamma}_2$, is represented by participation-contingent probability distributions over X_2, and we let $\sigma_{a_2^1,a_2^2}$ denote the probability that P2 chooses x_{21} given a participation profile $(a_2^1, a_2^2) \in A_2$ for P2. Our first result is that the principals' payoffs are uniquely pinned down in any SPNE-PSP of $\tilde{G}$.

Proposition 1 *The principals obtain payoffs* $(0, 10)$ *in any SPNE-PSP of* $\tilde{G}$.

Proof. Recall that P1's payoff is constantly equal to 0. We thus only need to show that for every direct mechanism posted by P1, that is, for every family of transition probabilities $\delta_{a_1^1,a_1^2}$, there exists a direct mechanism for P2, that is, a family of transition probabilities $\sigma_{a_2^1,a_2^2}$, inducing a unique Nash equilibrium in the subgame played by the agents, in which P2 achieves her maximal payoff of 10.

Case 1 Suppose first that $\delta_{Y,Y} > \frac{2}{5}$. In this case, let P2 post a mechanism such that $\sigma_{Y,Y} = \sigma_{N,Y} = \sigma_{Y,N} = 1$, which induces the subgame in Table 2.

Because $\delta_{Y,Y} > \frac{2}{5}$, we have $10 - 5\delta_{Y,Y} < 8$. Thus NY is a strictly dominant strategy for A1 in this subgame, which guarantees that (NY, YN) is the unique Nash equilibrium. Because $\sigma_{Y,N} = 1$, P2 obtains a payoff of 10.

	YN	NY
YN	$(10 - 5\delta_{Y,Y}, \frac{9}{2} + 8\delta_{Y,Y})$	$(10 - 5\delta_{Y,N}, 12)$
NY	$(8, 8 + 4\delta_{N,Y})$	$(40, 7)$

Table 2: The subgame of $\tilde{G}$ induced by $\delta_{Y,Y} > \frac{2}{5}$ and $\sigma_{Y,Y} = \sigma_{Y,N} = \sigma_{N,Y} = 1$.

Case 2 Suppose next that $\delta_{Y,Y} \leq \frac{2}{5}$. In this case, let P2 post a mechanism such that $\sigma_{Y,Y} = \sigma_{N,Y} = \sigma_{Y,N} = 0$, which induces the subgame in Table 3.

	YN	NY
YN	$(10 - 5\delta_{Y,Y}, \frac{9}{2} + 8\delta_{Y,Y})$	$(10 - 5\delta_{Y,N}, 8)$
NY	$(9, 8 + 4\delta_{N,Y})$	$(4, 13)$

Table 3: The subgame of $\tilde{G}$ induced by $\delta_{Y,Y} \leq \frac{2}{5}$ and $\sigma_{Y,Y} = \sigma_{Y,N} = \sigma_{N,Y} = 0$.

Because $\delta_{Y,Y} \leq \frac{2}{5}$, we have $\frac{9}{2} + 8\delta_{Y,Y} < 8$. Thus NY is a strictly dominant strategy for A2 in this subgame, which, as $10 - 5\delta_{Y,N} \geq 4$, guarantees that (YN, NY) is the unique Nash equilibrium. Because $\sigma_{N,Y} = 0$, P2 obtains a payoff of 10. Hence the result. ■

The proof of Proposition 1 actually shows the stronger result that, given any direct mechanism $\tilde{\gamma}_1$ posted by P1, P2 can defend his maximal payoff of 10 by posting a direct mechanism $\tilde{\gamma}_2$ that induces a unique Nash equilibrium in the subgame $\tilde{\gamma} \equiv (\tilde{\gamma}_1, \tilde{\gamma}_2)$. That is, the following min-max-min payoff for P2:

$$\overline{V}_2 \equiv \min_{\tilde{\gamma}_1 \in \tilde{\Gamma}_1} \max_{\tilde{\gamma}_2 \in \tilde{\Gamma}_2} \min_{\tilde{\lambda} \in \Lambda^*(\tilde{\gamma})} \mathbf{E}_{z(\tilde{\gamma}, \tilde{\lambda})}[v_2(a, x)], \tag{1}$$

is equal to 10 in the game $\tilde{G}$. Notice that the definition of $\overline{V}_2$ only allows principals to offer direct mechanisms; as such, it is specified only in terms of the primitives of the model, that is, the actions available to the players and the resulting payoffs. In this respect, it is analogous to the min-max bounds introduced by Peters and Troncoso-Valverde (2013), Peters (2014), and Ghosh and Han (2018) in competing-mechanism games with complete information, the only difference being that agents can take real participation decisions. Conversely, $\overline{V}_2$ differs from the min-max-min bound introduced by Yamashita (2010), in which the set of admissible mechanisms for each principal includes a recommendation mechanism, that is, a specific indirect mechanism committing her to asking agents to recommend her a direct mechanism and to following the majority recommendation.

3.2 Indirect Mechanisms with Minimal Private Communication

In the game $\tilde{G}$, private communication between the agents and the principals can only take place through the trivial message t, which an agent sends to a principal if he decides to participate with her, and the empty message $\emptyset$, which she sends to her otherwise. Such messages are redundant relative to participation decisions and, hence, essentially trivial. We now consider a game G^M in which a minimal degree of meaningful private communication is allowed for. Specifically, we allow A1 to send *one* additional message m to P1. That is, $M_1^1 \equiv \{t, m, \emptyset\}$, while $M_1^2 = M_2^1 = M_2^2 \equiv \{t, \emptyset\}$ as in the game $\tilde{G}$.

In the game G^M, P2 can only offer direct mechanisms $\tilde{\gamma}_2$ described as above by transition probabilities $\sigma_{a_2^1, a_2^2}$. By contrast, P1 can also offer indirect mechanisms γ_1 contingent on the message m sent by A1, allowing her to generate additional threats. Extending our previous notation, such a mechanism is represented by the following list of message- and participation-contingent probability distributions over X_1:[2]

$$\begin{aligned} \gamma_1((t,Y),Y) &= (\delta_{(t,Y),Y}, 1-\delta_{(t,Y),Y}), & \gamma_1((t,Y),N) &= (\delta_{(t,Y),N}, 1-\delta_{(t,Y),N}), \\ \gamma_1((m,Y),Y) &= (\delta_{(m,Y),Y}, 1-\delta_{(m,Y),Y}), & \gamma_1((m,Y),N) &= (\delta_{(m,Y),N}, 1-\delta_{(m,Y),N}), \\ \gamma_1(N,Y) &= (\delta_{N,Y}, 1-\delta_{N,Y}), & \gamma_1(N,N) &= (\delta_{N,N}, 1-\delta_{N,N}). \end{aligned}$$

The subgame $(\gamma_1, \tilde{\gamma}_2)$ is represented in Table 4 below.

	YN	NY
$(t,Y)N$	$(10 - 5\delta_{(t,Y),Y}, \frac{9}{2} + 8\delta_{(t,Y),Y})$	$(10 - 5\delta_{(t,Y),N}, 8 + 4\sigma_{N,Y})$
$(m,Y)N$	$(10 - 5\delta_{(m,Y),Y}, \frac{9}{2} + 8\delta_{(m,Y),Y})$	$(10 - 5\delta_{(m,Y),N}, 8 + 4\sigma_{N,Y})$
NY	$(9 - \sigma_{Y,N}, 8 + 4\delta_{N,Y})$	$(4 + 36\sigma_{Y,Y}, 13 - 6\sigma_{Y,Y})$

Table 4: The subgame $(\gamma_1, \tilde{\gamma}_2)$ of G^M.

The following result shows that this minimal enlargement of a single agent's message space compared to the game $\tilde{G}$ has a dramatic impact on P2's SPNE-PSP payoff set.

Proposition 2 *If the loss ζ incurred by P1 when both A1 and A2 participate with her is large enough, the principals can obtain any payoffs in $\{(0, \pi) : \pi \in [0, 10]\}$ in an SPNE-PSP of G^M.*

Proof. For the sake of clarity, all equilibrium objects will be indexed by a $*$. We prove that, for each $\sigma^* \in [0, 1]$, there exists an SPNE-PSP of G^M in which P1 posts a mechanism γ_1^* and P2 posts a direct mechanism $\tilde{\gamma}_2^*$ such that $\sigma_{N,Y}^* = \sigma^*$ and, on the equilibrium path,

[2] To alleviate the notation, and when no confusion can arise, we hereafter only indicate the nonempty messages t and m sent by A1 to P1.

A1 participates with P1 and A2 participates with P2 with probability 1, which yields P2 a payoff $10(1-\sigma^*)$.

Thus fix some $\sigma^* \in [0,1]$. To construct an SPNE-PSP in which P2 posts a direct mechanism $\tilde{\gamma}_2^*$ such that $\sigma^*_{N,Y} = \sigma^*$, we proceed as follows. First, let P1 post a mechanism γ_1^* in which $\delta^*_{(t,Y),Y}$ is such that $\frac{9}{2} + 8\delta^*_{(t,Y),Y} = 8 + 4\sigma^*$, that is, $\delta^*_{(t,Y),Y} = \frac{7+8\sigma^*}{16} \in (0,1)$. In addition, let $\delta^*_{(t,Y),N} = 0$, $\delta^*_{(m,Y),Y} = 0$, $\delta^*_{(m,Y),N} = 1$, and $\delta^*_{N,Y} = 1$. Second, let P2 post a mechanism $\tilde{\gamma}_2^*$ in which $\sigma^*_{Y,Y} = \frac{1}{6}$ and $\sigma^*_{N,Y} = \sigma^*$. One can check that the subgame $(\gamma_1^*, \tilde{\gamma}_2^*)$ has an equilibrium in which A1 and A2 play $((t,Y)N, NY)$. This yields P2 a payoff $10(1-\sigma^*)$.

Suppose next that P2 deviates to some direct mechanism $\tilde{\gamma}_2$. The agents play the game in Table 5.

	YN	NY
$(t,Y)N$	$(10 - \frac{5}{16}(7+8\sigma^*), 8+4\sigma^*)$	$(10, 8+4\sigma_{N,Y})$
$(m,Y)N$	$(10, \frac{9}{2})$	$(5, 8+4\sigma_{N,Y})$
NY	$(9-\sigma_{Y,N}, 12)$	$(4+36\sigma_{Y,Y}, 13-6\sigma_{Y,Y})$

Table 5: The subgame $(\gamma_1^*, \tilde{\gamma}_2)$ of G^M.

The analysis of such subgames consists of three steps.

Step 1 Consider first the subgames $(\gamma_1^*, \tilde{\gamma}_2)$ for $\sigma_{Y,Y} \leq \frac{1}{6}$ and $\sigma_{N,Y} \geq \sigma^*$. Our candidate for an SPNE-PSP of G^M has A1 and A2 playing $((t,Y)N, NY)$ in any such subgame, which is indeed a Nash equilibrium if $\sigma_{Y,Y} \leq \frac{1}{6}$ and $\sigma_{N,Y} \geq \sigma^*$, because NY is then weakly dominant for A2 and $(t,Y)N$ is then a best response of A1 to A2 playing NY. The corresponding payoff for P2 is $10(1-\sigma_{N,Y})$, which is strictly decreasing in $\sigma_{N,Y}$. By construction, P2 does not want to deviate to a mechanism $\tilde{\gamma}_2$ such that $\sigma_{Y,Y} \leq \frac{1}{6}$ and $\sigma_{N,Y} > \sigma^*$, which would make her strictly worse off, or to a mechanism $\tilde{\gamma}_2$ such that $\sigma_{Y,Y} < \frac{1}{6}$ and $\sigma_{N,Y} = \sigma^*$, which would leave her indifferent.

Step 2 We next show that P2 does not want to deviate to any $\tilde{\gamma}_2$ such that $\sigma_{Y,Y} > \frac{1}{6}$. To see why, observe first that, if $\sigma_{Y,Y} > \frac{1}{6}$, then $(t,Y)N$ is strictly dominated in $(\gamma_1^*, \tilde{\gamma}_2)$ for A1 and $(\gamma_1^*, \tilde{\gamma}_2)$ has no pure strategy Nash equilibrium. It is easy to check that $(\gamma_1^*, \tilde{\gamma}_2)$ has a unique mixed-strategy Nash equilibrium in which A1 plays $(m,Y)N$ with probability p and NY with probability $1-p$, where

$$p \equiv \frac{6\sigma_{Y,Y} - 1}{6\sigma_{Y,Y} + 4\sigma_{N,Y} + \frac{5}{2}},$$

and A2 plays YN with probability q and NY with probability $1-q$, where

$$q \equiv \frac{36\sigma_{Y,Y} - 1}{36\sigma_{Y,Y} + \sigma_{Y,N}}.$$

Then P2's payoff is

$$pq\zeta + p(1-q)10(1-\sigma_{N,Y}) + (1-p)q(8+2\sigma_{Y,N}) + (1-p)(1-q)\zeta,$$

which is negative when the loss ζ is large enough as p is bounded away from 1 and q is bounded away from 0 and 1 no matter the values of $\sigma_{Y,Y} > \frac{1}{6}$ and $\sigma_{Y,N}$.

Step 3 We finally show that P2 does not want to deviate to a mechanism $\tilde{\gamma}_2$ such that $\sigma_{Y,Y} \leq \frac{1}{6}$ and $\sigma_{N,Y} < \sigma^*$. Consider first the subgames such that $\sigma_{Y,Y} = \frac{1}{6}$ and $\sigma_{N,Y} < \sigma^*$. Our candidate for an SPNE-PSP of G^M has A1 and A2 playing (NY, NY) in any such subgame, which is indeed a Nash equilibrium. The corresponding payoff for P2 is ζ, so that she has no incentive to deviate from her postulated mechanism. Consider next the subgames such that $\sigma_{Y,Y} < \frac{1}{6}$ and $\sigma_{N,Y} < \sigma^*$. Observe that none of the resulting subgames $(\gamma_1^*, \tilde{\gamma}_2)$ has a Nash equilibrium in which A2 plays a pure strategy. Thus, A1 must play $(t, Y)N$ with positive probability, for, otherwise, the unique best response of A2 would be NY. Moreover, A1 must play $(m, Y)N$ or NY with positive probability, for, otherwise, the unique best response of A2 would be YN. We distinguish three cases.

Case 1 Suppose first that A1 randomizes over $(t, Y)N$ and $(m, Y)N$. For A2 to be indifferent between YN and NY, it must be that A1 plays $(t, Y)N$ with probability p' and $(m, Y)N$ with probability $1-p'$, where

$$p' \equiv \frac{\frac{7}{2} + 4\sigma_{N,Y}}{\frac{7}{2} + 4\sigma^*}.$$

Similarly, for A1 to be indifferent between $(t, Y)N$ and $(m, Y)N$, it must be that A2 plays YN with probability q' and NY with probability $1-q'$, where

$$q' \equiv \frac{1}{1 + \frac{5}{16}(7 + 8\sigma^*)}.$$

These strategies form a Nash equilibrium if A1 is not tempted to deviate to NY, which is the case if and only if $q' \geq q$, with q as defined in Step 2. Then P2's payoff is

$$q'\zeta + p'(1-q')10(1-\sigma_{N,Y}) + (1-p')(1-q')\zeta,$$

which is negative when the loss ζ is large enough as q' is bounded away from 0 no matter the value of σ^*.

Case 2 Suppose next that A2 randomizes over $(t, Y)N$ and NY. For A2 to be indifferent between YN and NY, it must be that A1 plays $(t, Y)N$ with probability p'' and NY with probability $1 - p''$, where

$$p'' \equiv \frac{1 - 6\sigma_{Y,Y}}{1 - 6\sigma_{Y,Y} + 4(\sigma^* - \sigma_{N,Y})}.$$

Similarly, for A1 to be indifferent between $(t, Y)N$ and NY, it must be that A2 plays YN with probability q'' and NY with probability $1 - q''$, where

$$q'' \equiv \frac{6(1 - 6\sigma_{Y,Y})}{6(1 - 6\sigma_{Y,Y}) + \frac{1}{16}(19 + 40\sigma^*) - \sigma_{Y,N}}.$$

These strategies form a Nash equilibrium if A1 is not tempted to deviate to $(m, N)Y$, which is the case if and only if $q \geq q'$, with q as defined in Step 2. Then P2's payoff is

$$p''q''\zeta + p''(1 - q'')10(1 - \sigma_{N,Y}) + (1 - p'')q''(8 + 2\sigma_{Y,N}) + (1 - p'')(1 - q'')\zeta,$$

and we must show that, when the loss ζ is large enough, this does not exceed the candidate equilibrium payoff $10(1 - \sigma^*)$ uniformly in $\sigma_{Y,Y} < \frac{1}{6}$, $\sigma_{N,Y} < \sigma^*$, and $\sigma_{Y,N}$, given the above expressions for p'' and q''. For conciseness, let us write

$$\begin{aligned}
\varepsilon &\equiv 1 - 6\sigma_{Y,Y}, \\
\eta &\equiv \sigma^* - \sigma_{N,Y}, \\
\xi &\equiv \frac{1}{16}(19 + 40\sigma^*) - \sigma_{Y,N},
\end{aligned}$$

and notice that ξ, unlike ε and η, is bounded away from 0. An upper bound for P2's payoff from deviating is

$$B \equiv [p''q'' + (1 - p'')(1 - q'')]\zeta + p''(1 - q'')10(1 - \sigma_{N,Y}) + 10(1 - p'')q'',$$

and, with the above notation, we have

$$\begin{aligned}
B - 10(1 - \sigma^*) &\leq [p''q'' + (1 - p'')(1 - q'')]\zeta + p''(1 - q'')10(\sigma^* - \sigma_{N,Y}) + 10(1 - p'')q'' \\
&\propto (6\varepsilon^2 + 4\eta\xi)\zeta + \varepsilon\xi\eta + 240\eta\varepsilon \\
&< \eta(4\xi\zeta + \varepsilon\xi + 240\varepsilon),
\end{aligned}$$

which, as ξ is bounded away from 0 and $\varepsilon \in [0, 1]$, is negative when the loss ζ is large enough, uniformly in η and ε.

Case 3 Finally, if $q = q' = q''$, A1 is ready to randomize over $(t, Y)N$, $(m, Y)N$, and NY. Instead of considering a completely mixed Nash equilibrium of $(\gamma_1^*, \tilde{\gamma}_2)$, we can however select either of the equilibria constructed in Cases 1 and 2, which ensures that P2 has no incentive to deviate. Hence the result. ■

3.3 Discussion: Min-Max versus Max-Min

Our example suggests that, in settings in which agents can take real participation decisions, instead of mere reporting decisions, the min-max-min payoff (1) is not relevant for describing principals' equilibrium payoffs of competing-mechanism games with communication: indeed, Proposition 2 shows that even minimal communication allows one to support equilibrium payoffs for P2 below this value, which is equal to 10 in our example.

It is interesting, in that respect, to observe that the max-min-min payoff for P2,

$$\underline{V}_2 \equiv \max_{\tilde{\gamma}_2 \in \tilde{\Gamma}_2} \min_{\tilde{\gamma}_1 \in \tilde{\Gamma}_1} \min_{\tilde{\lambda} \in \Lambda^*(\tilde{\gamma})} \mathbf{E}_{z(\tilde{\gamma},\tilde{\lambda})}[v_2(a,x)], \tag{2}$$

is at most equal to 0. To see this, notice that, by Proposition 2, we certainly have

$$\max_{\tilde{\gamma}_2 \in \tilde{\Gamma}_2} \min_{\gamma_1 \in \Gamma_1^{M_1}} \min_{\lambda \in \Lambda^*(\gamma_1,\tilde{\gamma}_2)} \mathbf{E}_{z(\gamma_1,\tilde{\gamma}_2,\lambda)}[v_2(a,x)] \leq 0. \tag{3}$$

Thus, to show that $\underline{V}_2 \leq 0$, we only need to establish the following lemma.

Lemma 1 *The two max-min-min payoffs in (2)–(3) coincide.*

The upshot of this discussion is that the max-min-min and the min-max-min payoffs (1)–(2) for P2 are ordered as follows:

$$\underline{V}_2 \leq 0 < 10 = \overline{V}_2.$$

This wedge between the payoffs $\underline{V}_2$ and $\overline{V}_2$ reflects the fact that, in our two-principals example, agents take strategic participation decisions. This contrasts with the min-max theorem for one-shot complete-information games established by Ghosh and Han (2018, Theorem 3) in line with the general discussions of Peters and Troncoso-Valverde (2013) and Peters (2014).[3] There are indeed two reasons why Sion's (1958) min-max theorem does not apply in the present context. The first is that the Nash correspondence $\Lambda^* : \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 \twoheadrightarrow \Delta(A^1) \times \Delta(A^2)$ in our example is not single-valued and is only upper hemicontinuous. This implies that the mapping $(\tilde{\gamma}_1, \tilde{\gamma}_2) \mapsto \min_{\tilde{\lambda} \in \Lambda^*(\tilde{\gamma})} \mathbf{E}_{z(\tilde{\gamma},\tilde{\lambda})}[v_2(a,x)]$ is not upper semicontinuous in $\tilde{\gamma}_2$;[4] for instance, it is easy to check that it exhibits a downward discontinuity at $\sigma_{Y,Y} = \frac{1}{6}$.[5]

[3] In two-principal settings, Theorem 3 in Ghosh and Han (2018) follows directly from Sion's (1958) min-max theorem.

[4] By contrast, it follows from the second half of Berge's maximum theorem that this mapping is lower semicontinuous in $\tilde{\gamma}_1$ (Aliprantis and Border (2006, Lemma 17.30)).

[5] Indeed, if $\sigma_{Y,Y} = \frac{1}{6}$, (NY, NY) is a pure-strategy Nash equilibrium in the game played by the agents that yields P2 a payoff of ζ. By contrast, if $\sigma_{Y,Y} < \frac{1}{6}$ and, for instance, $\delta_{Y,Y} = 1$ and $\delta_{Y,N} = \delta_{N,Y} = 0$, then the game between the agents admits only a completely mixed Nash equilibrium in which the probability p''' with which A1 plays YN is bounded away from 0, and the probability q''' with which A2 plays YN tends to 0 as $\sigma_{Y,Y}$ tends to $\frac{1}{6}$. The limit payoff for P2 is then $(1-p''')\zeta + p'''10(1-\sigma_{N,Y}) > \zeta$.

The second is that, whereas the expectation $\mathbf{E}_{z(\tilde{\gamma},\tilde{\lambda})}[v_2(a,x)]$ is linear in $(\tilde{\gamma},\tilde{\lambda})$, there is no reason for its minimum with respect to $\tilde{\lambda} \in \Lambda^*(\tilde{\gamma})$ to be quasiconvex in $\tilde{\gamma}_1$ and quasiconcave in $\tilde{\gamma}_2$. Thus letting the agents take real participation decisions naturally leads to discontinuities and nonconvexities that prevent the usual min-max logic from applying, even if we allow principals to randomize over their actions. Given the importance of participation decisions in the applied literature on competing mechanisms, this casts serious doubt on the general relevance of this logic.

Admittedly, Proposition 2 does not provide a full characterization of equilibrium payoffs for principals; specifically, we have not checked whether $\underline{V}_2 < 0$ and whether equilibria with payoff for P2 down to that level can be sustained. Our second example will show that there is no hope for obtaining a general result along these lines: neither the min-max-max payoff (1) nor the max-min-min payoff (2) are relevant bounds for the characterization of equilibrium payoffs of competing-mechanism games with communication.

4 The Second Example

Let $J \equiv 2$ and $I \equiv 3$, and let $X_1 \equiv \{x_{11}, x_{12}, x_{13}\}$ and $X_2 \equiv \{x_{21}, x_{22}, x_{23}\}$ be the sets of actions of principal 1 (P1) and principal 2 (P2), respectively. Let $A^1 = A^2 = A^3 \equiv \{YN, NY, NN\}$ be agent 1's (A1), agent 2's (A2), and agent 3's (A3) sets of actions, with the same interpretation as in the first example. To make the example as simple as possible, we assume that the principals' payoffs are not affected by agents' participation decisions. These payoffs are represented in the matrix below. Each array represents the payoffs to P1 and P2, respectively.

	x_{21}	x_{22}	x_{23}
x_{11}	$(1,1)$	$(-1,2)$	$(0,0)$
x_{12}	$(1,-1)$	$(-1,0)$	$(0,0)$
x_{13}	$(0,0)$	$(0,0)$	$(0,0)$

Table 6: The payoff matrix for principals.

The agents' payoffs are as follows. Each agent's payoff is independent of the other agents' participation decisions. Moreover, A1 strictly prefers to participate with P1, while A2 and A3 strictly prefer to participate with P2. With these assumptions in mind, we conventionally suppress the agents' participation decisions in the expressions of their payoffs. In addition, A1 payoffs are such that

$$u^1(x_{11}, x_{22}) = 2 > -2 = u^1(x_{12}, x_{22}) = u^1(x_{13}, x_{22}), \tag{4}$$

so that, if P2 chooses x_{22}, A1 would strictly prefer P1 to choose x_{11} rather than x_{12} or x_{13}. Unlike in our first example, we assume from the outset that the agents can communicate with the principals through rich message spaces. The following lemma is an easy consequence of the fact that agents' actions have no impact on principals's payoffs.

Lemma 2 *For each $j = 1, 2$, and for arbitrary message spaces M_j^i,*

$$V_j \equiv \min_{\gamma_{-j} \in \Gamma_{-j}^{M_{-j}}} \max_{\gamma_j \in \Gamma_j^{M_j}} \min_{\lambda \in \Lambda^*(\gamma)} \mathbf{E}_{z(\gamma,\lambda)}[v_j(a, x)] = \max_{\gamma_j \in \Gamma_j^{M_j}} \min_{\gamma_{-j} \in \Gamma_{-j}^{M_{-j}}} \min_{\lambda \in \Lambda^*(\gamma)} \mathbf{E}_{z(\gamma,\lambda)}[v_j(a, x)] = 0. \tag{5}$$

Proof. Because principals' payoffs do not depend on agents' participation decisions, we can disregard the $\min_{\lambda \in \Lambda^*(\gamma)}$ operator in the min-max-min and max-min-min payoffs in (5). To show that they are equal to 0, observe that, for any mechanism $\gamma_{-j} \in \Gamma_{-j}^{M_{-j}}$, P$j$ can defend a 0 payoff by committing to choose x_{j3} with probability 1, regardless of the agents' participation and communication decisions. Likewise, P$-j$ can bring down Pj's payoff to 0 by committing to choose x_{-j3} with probability 1, regardless of the agents' participation and communication decisions. The result follows. ■

An alternative, if less direct, proof of this result consists to notice that, because the sets of direct mechanisms $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are compact and convex subsets of a Euclidean space and the payoff function $\mathbf{E}_{z(\tilde{\gamma},\tilde{\lambda})}[v_j(a, x)]$ is linear in $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, it follows from Sion's (1958) min-max theorem that

$$\min_{\tilde{\gamma}_{-j} \in \tilde{\Gamma}_{-j}} \max_{\tilde{\gamma}_j \in \tilde{\Gamma}_j} \min_{\tilde{\lambda} \in \Lambda^*(\tilde{\gamma})} \mathbf{E}_{z(\tilde{\gamma},\tilde{\lambda})}[v_j(a, x)] = \max_{\tilde{\gamma}_j \in \tilde{\Gamma}_j} \min_{\tilde{\gamma}_{-j} \in \tilde{\Gamma}_{-j}} \min_{\tilde{\lambda} \in \Lambda^*(\tilde{\gamma})} \mathbf{E}_{z(\tilde{\gamma},\tilde{\lambda})}[v_j(a, x)], \tag{6}$$

and then to apply Ghosh and Han (2018, Lemma 1). This expresses the fact that, if the payoffs (1)–(2)—which are specified only in terms of the primitives of the model—coincide, no harsher punishment can be forced upon any principal by letting the other principal post an indirect mechanism.

In contrast with our first example, this seems an a priori more promising scenario for the min-max logic to apply. Indeed, a mechanical application of Yamashita's (2010) and Peters and Troncoso-Valverde's (2013) results would suggest that any incentive-feasible allocation yielding each principal a payoff above 0 can be supported in an SPNE-PSP of a game G^M with sufficiently rich message spaces; for instance, the message spaces may be large enough as to incorporate all direct mechanisms as in Yamashita (2010), allowing principals to use recommendation mechanisms. Yet this approach is mislead, as we shall now see.

To first illustrate this point in an intuitive manner, notice that the only way P1 and P2 can reach payoffs $(1, 1)$ is by deterministically choosing the actions (x_{11}, x_{21}). Suppose then, by way of contradiction, that there exists an SPNE-PSP (γ^*, λ^*) of some game G^M in which P1 and P2 choose these actions, and only these actions, on the equilibrium path. For this to be the case, there must exist a message $m_1^1 \in M_1^1$ such that P1 chooses x_{11} following the consistent profile $(m_1^1, \emptyset, \emptyset, Y, N, N)$—recall that, by assumption, A1 strictly prefers to participate with P1, while A2 and A3 strictly prefer to participate with P2. But P2 could then deviate to a mechanism γ_2 committing her to choose x_{22} regardless of the agents' communication and participation decisions. Because of (4), A1 would then strictly prefer that P1 chooses x_{11}. In the subgame (γ_1^*, γ_2), he can ensure this by participating with P1 and sending her the message m_1^1. This yields P2 a payoff of $2 > 1$, and hence γ_2 is a profitable deviation for P2. Thus there is no SPNE-PNP of G^M supporting the payoffs $(1, 1)$ for P1 and P2. The following result generalizes this insight to all payoffs above $(0, 0)$ for P1 and P2, even if we allow them to play mixed strategies in equilibrium.

Proposition 3 *The principals obtain payoffs* $(0, 0)$ *in any SPNE of any game* G^M.

It should be noted that Proposition 3 does not depend on the size of the message spaces. In particular, it holds true if these are large enough to incorporate the infinite-dimensional space of direct mechanisms, allowing principals to use recommendation mechanisms. This finding contrasts with the folk theorems of Yamashita (2010) and Peters and Troncoso-Valverde (2013) in which every agent participates and communicates with all the principals.

5 Concluding Remarks

The upshot of the two examples discussed in this paper is threefold.

Antifolk Theorems The examples lead us to question the logic of the folk theorems for competing-mechanism games. Our first example illustrates a situation in which principals' payoffs below the min-max-min bounds identified by Peters and Troncoso-Valverde (2013), Peters (2014), and Ghosh and Han (2018) can be supported in equilibrium. Our second example illustrates a situation in which no payoff for the principals above the min-max-min bound identified by Yamashita (2010) can be supported in equilibrium even if agents can choose messages from arbitrarily rich message spaces. The examples highlights the two key reasons why the folk-theorem logic fails to hold. In the first example, agents' participation decisions are payoff-relevant. In the second example, an agent can communicate with a principal only if he chooses to participate with her. Both features are prominent in economic applications of competing-mechanism games.[6]

Exclusive Competition and Direct Mechanisms In line with competing-auctions and competitive-search models in the applied literature, our examples postulate that competition is exclusive: each agent can participate and communicate with at most one principal. Yet, a main feature of these applications is the restriction to direct mechanisms. That is, each seller only requires the buyers who participate with her to submit their exogenous private information. Under complete information, this amounts to each seller posting take-it-or-leave-it offers for the buyers participating with her. The difficulties in establishing a general folk theorem that we highlight may lead one to ask whether, under exclusive competition, direct mechanisms achieve a full characterization of equilibrium. Our first example provides a clear negative answer. Indeed, an immediate implication of Propositions 1–2 is the existence of many equilibrium outcomes that cannot be supported by direct mechanisms. In this respect, we document a failure of the revelation principle in competing-mechanism games of exclusive competition: direct mechanisms are not flexible enough to reproduce all the threats that P1 can implement using the payoff-irrelevant message m.[7] This failure is dramatic: while the game $\tilde{G}$ has a unique equilibrium payoff vector, a continuum of Pareto-ranked equilibria can be sustained in the game G^M as soon as a single agent has the opportunity to send a single additional message to a single principal.

To interpret this failure of the revelation principle under exclusive competition, it is helpful to contrast our example with related results in the literature. First, the example is cast in a complete-information framework; this contrasts with Martimort (1996), Peck (1997), and Attar, Campioni, and Piaser (2018), who crucially exploit the agents' private information. Second, the example does not rely on direct externalities between the principals, but instead exploits the strategic role of the agents' joint participation and communication decisions; hence it does not rely on the intuitions developed by Martimort (1996) in the

[6]Clearly, our results do not contradict the possibility of multiple equilibrium allocations arising in a specific class of applications. In this respect, Han (2016) shows how indirect mechanisms can be used to sustain multiple symmetric equilibria in a simple competitive-search setting.

[7]Thus Propositions 1–2 extend to exclusive-competition environments the insights developed by Peters (2001) and Martimort and Stole (2002) in single-agent, nonexclusive-competition environments. The example suggests that, to implement all relevant threats, there is no need for each agent to communicate with all principals; this ex-post vindicates our assumption that communication is tied to participation, in the sense that an agent who chooses not to participate with a principal does not get to meaningfully communicate with her.

context of competing hierarchies. Third, principals in the example play pure strategies in equilibrium; hence we do not rely on the limited power of direct mechanisms to extract the agents' information on the realization of principals' mixed strategies, in contrast with Peck (1997). In a sense, thus, we have put ourselves in the worst possible scenario for communication to play a role: agents have no private information of their own and each principal's equilibrium mechanism is deterministic so that there is no need to use the agents to reveal it to the other principal. Despite these drastic features, our example documents the potentially destabilizing role of communication even in complete-information environments. This questions the prevalent use of direct mechanisms in the applied literature.

Equilibrium Characterization: New Ideas Taken together, our examples emphasize that disregarding agents' participation decisions in competing-mechanism games entails a severe loss of generality. Observe, in this respect, that our results do not depend on the specific assumption of exclusive competition. In particular, the second example can be reformulated in a nonexclusive scenario where each agent can participate with more than one principal at a time, provided that communication is tied to participation. The explicit consideration of agents' participation decisions introduces a fundamental constraint on principals' design of mechanisms. When designing a mechanism, a principal anticipates that her payoff will be affected by agents' participation decisions, which in turn depend on the entire profile of posted mechanisms. Dealing with this additional moral-hazard dimension may require a more sophisticated class of mechanisms, in which a principal can send private signals to agents. In recent work, Attar, Campioni and Piaser (2019) provide an example of a complete-information competing-mechanism game in which principals use such private communication to correlate their decisions with those of the agents in equilibrium. The resulting set of equilibrium allocations and the set of equilibrium allocations that can be supported with standard recommendation mechanisms turn out to be disjoint. All these insights point towards the need to develop new ideas and devices to approach equilibrium characterization in competing-mechanism games.

Appendix

Proof of Lemma 1. This is in fact an instance of a general result in the spirit of Myerson's (1982) revelation principle. The intuition is that, for any direct mechanism $\tilde{\gamma}_2$ of P2 and any mechanism γ_1 of P1, and for any Nash equilibrium λ of the subgame $(\gamma_1, \tilde{\gamma}_2)$ of G^M, there exist a direct mechanism $\tilde{\gamma}_1$ of P1 and a Nash equilibrium $\tilde{\lambda}$ of the subgame $\tilde{\gamma} \equiv (\tilde{\gamma}_1, \tilde{\gamma}_2)$ of $\tilde{G}$ such that the two resulting allocations coincide, $z(\tilde{\gamma}, \tilde{\lambda}) = z(\gamma_1, \tilde{\gamma}_2, \lambda)$. Indeed, P1 can reproduce the randomizations over messages performed by A1 in $\lambda^1(\gamma_1, \tilde{\gamma}_2)$ in case he decides to participate with P1 by offering a direct mechanism $\tilde{\gamma}_1$ such that

$$\tilde{\gamma}_1(Y, a_1^2) \equiv \frac{\lambda^1(\gamma_1, \tilde{\gamma}_2)((t, Y)N)\gamma_1((t, Y), a_1^2) + \lambda^1(\gamma_1, \tilde{\gamma}_2)((m, Y)N)\gamma_1((m, Y), a_1^2)}{\lambda^1(\gamma_1, \tilde{\gamma}_2)((t, Y)N) + \lambda^1(\gamma_1, \tilde{\gamma}_2)((m, Y)N)}$$

for all $a_1^2 \in A_1^2$. That is, if A1 chooses to participate with P1, then P1 first draws a lottery with outcomes (t, Y) and (m, Y), with probabilities $\frac{\lambda^1(\gamma_1, \tilde{\gamma}_2)((t,Y)N)}{\lambda^1(\gamma_1, \tilde{\gamma}_2)((t,Y)N) + \lambda^1(\gamma_1, \tilde{\gamma}_2)((m,Y)N)}$ and $\frac{\lambda^1(\tilde{\gamma}_1, \tilde{\gamma}_2)((m,Y)N)}{\lambda^1(\tilde{\gamma}_1, \tilde{\gamma}_2)((t,Y)N) + \lambda^1(\tilde{\gamma}_1, \tilde{\gamma}_2)((m,Y)N)}$, respectively, and then, depending on the outcome of this lottery, chooses x_{11} with probability $\gamma_1((t, Y), a_1^2)$ or $\gamma_1((m, Y), a_1^2)$. If A1 chooses not to participate with P1, then we set

$$\tilde{\gamma}_1(N, a_1^2) \equiv \gamma_1((\emptyset, N), a_1^2)$$

for all $a_1^2 \in A_1^2$. Turning to the agent's strategies, we define

$$\begin{aligned}
\tilde{\lambda}^1(\tilde{\gamma}_1, \tilde{\gamma}_2)(YN) &\equiv \lambda^1(\gamma_1, \tilde{\gamma}_2)((t, Y)N) + \lambda^1(\gamma_1, \tilde{\gamma}_2)((m, Y)N), \\
\tilde{\lambda}^1(\tilde{\gamma}_1, \tilde{\gamma}_2)(NY) &\equiv \lambda^1(\gamma_1, \tilde{\gamma}_2)((\emptyset, N)Y), \\
\tilde{\lambda}^2(\tilde{\gamma}_1, \tilde{\gamma}_2)(YN) &\equiv \lambda^2(\gamma_1, \tilde{\gamma}_2)(YN).
\end{aligned}$$

By construction, $z(\tilde{\gamma}, \tilde{\lambda}) = z(\gamma_1, \tilde{\gamma}_2, \lambda)$. Moreover, because P1 reproduces the randomizations of A1 in case she decides to participate with her, the incentives of the agents are unchanged. Hence $\tilde{\lambda}$ is a Nash equilibrium of the subgame $\tilde{\gamma}$, as required. ■

Proof of Proposition 3. Suppose, by way of contradiction, that there exists an SPNE of some game G^M in which at least one of the principals obtains a positive payoff. The proof consists of two steps.

Step 1 We first claim that, on the candidate SPNE equilibrium path, P1 chooses x_{11} with positive probability. Suppose, indeed, that P1 never chooses x_{11} in the SPNE under consideration. Then P2 must obtain a 0 payoff, which she can guarantee by committing to choose x_{23} with probability 1, regardless of the agents' communication and participation decisions. By assumption, P1 must, therefore, obtain a positive payoff. If P1 never chooses x_{11}, this can occur if and only if P1 and P2 choose (x_{12}, x_{21}) with positive probability. But P2 would then obtain a negative profit, a contradiction. The claim follows.

Step 2 We next claim that P2's unique best response in the candidate SPNE consists in committing to play x_{22} with probability 1, regardless of the agents' communication and participation decisions. We distinguish two types of subgames, depending on the mechanism posted by P1 according to her—possibly mixed—equilibrium strategy $\mu_1^* \in \Delta(\Gamma_1^{M_1})$.

Case 1 Consider first the mechanisms γ_1 in the support of μ_1^* such that there is no message $m_1^1 \in M_1^1$ such that P1 chooses x_{11} with positive probability following the consistent profile $(m_1^1, \emptyset, \emptyset, Y, N, N)$; call C_1 the corresponding set of mechanisms. Committing herself to choose x_{22} with probability 1, regardless of the agents' communication and participation decisions, ensures P2 to obtain her maximal payoff of 0 in any subgame in which P1 posts a mechanism in C_1.

Case 2 Consider next the mechanisms γ_1 in the support of μ_1^* such that there is a message $m_1^1 \in M_1^1$ such that P1 chooses x_{11} with positive probability following the consistent profile $(m_1^1, \emptyset, \emptyset, Y, N, N)$; that is, $\gamma_1 \in \operatorname{supp} \mu_1^* \setminus C_1$. It follows from (4) that A1 will, among these messages, choose one that maximizes the probability $\gamma_1(m_1^1, \emptyset, \emptyset, Y, N, N)(x_{11})$ of P1 choosing x_{11}. Committing herself to choose x_{22} with probability 1, regardless of the agents' communication and participation decisions, ensures P2 to obtain her maximal payoff of $2 \max_{m_1^1 \in M_1^1} \{\gamma_1(m_1^1, \emptyset, \emptyset, Y, N, N)(x_{11})\}$ in any subgame in which P1 posts a mechanism in $\operatorname{supp} \mu_1^* \setminus C_1$.

According to Step 1, $\mu_1^*(\operatorname{supp} \mu_1^* \setminus C_1) > 0$, so that Case 2 arises with positive probability. The claim follows.

Step 3 According to Steps 1 and 2, in the candidate SPNE, P1 chooses x_{11} with positive probability while P2 chooses x_{22} with probability 1. Hence P1 must earn a negative payoff. This, however, is a contradiction, as she can guarantee herself a 0 payoff by committing to choose x_{13} with probability 1. Hence the result. ■

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