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“Repeated Games with Incomplete Information”

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THE LARGE SPACE OF INFORMATION STRUCTURES

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ABSTRACT. We revisit the question of modeling incomplete information among 2 Bayesian players, following an ex-ante approach based on values of zero-sum games. K being the finite set of possible parameters, an information structure is defined as a probability distribution u with finite support over $K \times \mathbb{N} \times \mathbb{N}$ with the interpretation that: u is publicly known by the players, (k, c, d) is selected according to u , then c (resp. d) is announced to player 1 (resp. player 2). Given a payoff structure g , composed of matrix games indexed by the state, the value of the incomplete information game defined by u and g is denoted $\text{val}(u, g)$. We evaluate the pseudo-distance $d(u, v)$ between 2 information structures u and v by the supremum of $|\text{val}(u, g) - \text{val}(v, g)|$ for all g with payoffs in $[-1, 1]$, and study the metric space Z^* of equivalent information structures.

We first provide a tractable characterization of $d(u, v)$, as the minimal distance between 2 polytopes, and recover the characterization of Peski (2008) for $u \succeq v$, generalizing to 2 players Blackwell's comparison of experiments via garblings. We then show that Z^* , endowed with a weak distance d_W , is homeomorphic to the set of consistent probabilities with finite support over the universal belief space of Mertens and Zamir. Finally we show the existence of a sequence of information structures, where players acquire more and more information, and of $\varepsilon > 0$ such that any two elements of the sequence have distance at least ε : having more and more information may lead nowhere. As a consequence, the completion of (Z^*, d) is not compact, hence not homeomorphic to the set of consistent probabilities over the states of the world *à la* Mertens and Zamir. This example answers by the negative the second (and last unsolved) of the three problems posed by J.F. Mertens in his paper "Repeated Games", ICM 1986.

1. INTRODUCTION

Given a countable set S , we denote by $\Delta(S) = \{x = (x(s))_{s \in S} \in \mathbb{R}_+^S, \sum_{s \in S} x(s) = 1\}$ the set of probability distributions over S , and by $\Delta_f(S)$ the set of probability distributions with finite support over S . The Dirac measure on an element s will be denoted δ_s . More generally if S is a compact metric space, $\Delta(S)$ is the set of Borel probability distributions on S , and is endowed with the weak topology.

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2. THE SPACE OF INFORMATION STRUCTURES

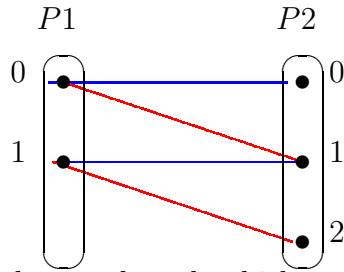
Throughout the paper, K is a fixed finite set of parameters or states of nature, e.g. $K = \{0, 1\}$ or $K = \{Blue, Red\}$. There is a true state k in K , which is imperfectly known by two Bayesian players. The general question is : What is the set of possible situations ?

Definition 1. *An information structure is a probability with finite support over $K \times \mathbb{N} \times \mathbb{N}$. The set of information structures is denoted by $\mathcal{U} = \Delta_f(K \times \mathbb{N} \times \mathbb{N})$.*

The interpretation of an information structure u is the following : u is publicly known by the players, a triple (k, c, d) is selected according to u , then the state is k , player 1 learns c and player 2 learns d . So an information structure represents an ex-ante situation, before the players have received their signals.

Unless otherwise specified, in our examples K will have two elements and u will be uniform over a finite subset of $K \times \mathbb{N} \times \mathbb{N}$.

Example 1. $K = \{Blue, Red\}$, and u is represented by:



Here with probability $1/4$, the top blue edge is selected, which means that the state is blue, player 1 receives the signal 0 and player 2 receives the signal 0. With probability $1/4$, the top red edge is selected : the state is red, player 1 receives the signal 0 and player 2 receives the signal 1. Etc...

After receiving signal 0, player 1 believes that both states in K are equally likely. It is the same after receiving signal 1. However the two signals of player 1 convey distinct information for him: after receiving signal 0, player 1 knows that if the state is blue then player 2 knows it, whereas after receiving signal 1, player 1 knows that if the state is blue then player 2 has a uniform belief on K . ■

The central idea is to evaluate an information structure via the values of associated zero-sum Bayesian games. We first define payoff structures, which are given by a matrix game with payoff in $[-1, 1]$ for each state in K . Since we don't want to fix a priori the size of the matrices, we will formally consider infinite matrices with only finitely many relevant rows and columns.

Definition 2. *Given $L \geq 1$, a payoff structure of size L is a map $g : K \times \mathbb{N} \times \mathbb{N} \rightarrow [-1, 1]$, such that for all (k, i, j) : $g(k, i, j) = -1$ if $i \geq L > j$ and $g(k, i, j) = 1$ if $j \geq L > i$.*

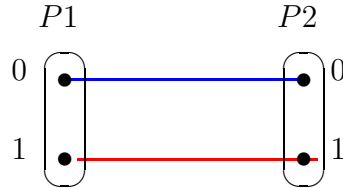
The set of payoff structures of size L is denoted by $\mathcal{G}(L)$, and the set of payoff structures is $\mathcal{G} = \bigcup_{L \geq 1} \mathcal{G}(L)$.

Example 2. $K = \{Blue, Red\}$. To represent a payoff structure g of size 2, it is enough to give a blue and a red matrix such as $\left\{ \left(\begin{array}{cc} 0 & 0 \\ -\frac{3}{5} & 1 \end{array} \right), \left(\begin{array}{cc} 1 & -\frac{3}{5} \\ 0 & 0 \end{array} \right) \right\}$. ■

Definition 3. An information structure u and a payoff structure g together define a zero-sum Bayesian game $\Gamma(u, g)$ played as follows: First, (k, c, d) is selected according to u , player 1 learns c and player 2 learns d . Then simultaneously player 1 chooses i in \mathbb{N} and player 2 chooses j in \mathbb{N} , and finally the payoff of player 1 is $g(k, i, j)$. $\Gamma(u, g)$ can be seen as a finite zero-sum game, and we denote its value by $\text{val}(u, g)$.

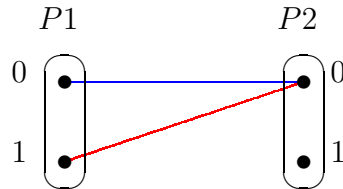
Example 3. Consider the payoff structure g of example 2.

1) The information structure is u_1 , where players have complete information on the state:



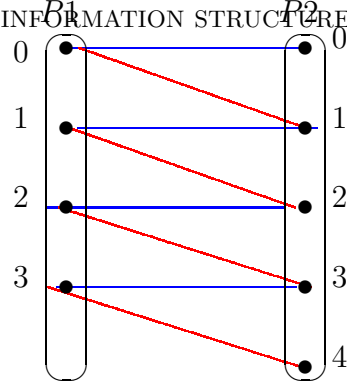
Here the unique optimal strategies are, for player 2, to play the left column after signal 0 and to play the right column after signal 1 and, for player 1 to play the top row after signal 0 and the bottom row after signal 1. $\text{val}(u_1, g) = 0$.

2) The information structure is u_2 , with lack of information on the side of player 2 :



Here the unique optimal strategy for player 1 is to play bottom after 0 and top after 1, whereas any strategy of player 2 which plays after signal 0 both left and right with probability at least $3/8$ is optimal. And $\text{val}(u_2, g) = 1/5$. Comparing with u_1 , we recover that optimal strategies of player 1 do not only depend on his belief on K .

3) Here the information structure u_3 is given by :



Here $\text{val}(u_3, g) = 1/10$, and the unique optimal strategy of player 1 is to play top after signals 0 and 2, and bottom after signal 1 and 3. Note that player 1 should play very differently after receiving signal 1 and 2, whereas in both cases : player 1 believes that both states on K are equally likely, player 1 believes that player 2 believes that both states are equally likely, and player 1 believes that player 2 believes that player 1 believes that both states are equally likely. ■

Given u, v in \mathcal{U} , a natural distance between u and v is given by the L^1 -norm:

$$\|u - v\| = \sum_{k \in K, (c,d) \in \mathbb{N}^2} |u(k, c, d) - v(k, c, d)|.$$

If g is a payoff structure in \mathcal{G} , since all payoffs are in $[-1, 1]$ it is easy to see that $|\text{val}(u, g) - \text{val}(v, g)| \leq \|u - v\|$.

We now order and compare information structures.

Definition 4. Given u, v in \mathcal{U} , say that $u \succeq v$ if for all g in \mathcal{G} , $\text{val}(u, g) \geq \text{val}(v, g)$.

Definition 5. Given u, v in \mathcal{U} , define :

$$d(u, v) = \sup_{g \in \mathcal{G}} |\text{val}(u, g) - \text{val}(v, g)|.$$

Clearly $d(u, v) \leq \|u - v\| \leq 2$. $d(u, v) = d(v, u) \in [0, 1]$ and d satisfies the triangular inequality but we may have $d(u, v) = 0$ for $u \neq v$, so d is a pseudo-distance on \mathcal{U} . Similarly \preceq is reflexive and transitive but one may have $u \succeq v$ and $v \succeq u$ for $u \neq v$. If we start from an information structure u and relabel the signals of the players, we obtain an information structure u' which is formally different from u , but “equivalent” to u .

Definition 6. Say that u and v are equivalent, and write $u \sim v$, if for all game structures g in \mathcal{G} , $\text{val}(u, g) = \text{val}(v, g)$. We let $\mathcal{U}^* = \mathcal{U} / \sim$ be the set of equivalence classes.

d and \succeq are naturally defined on \mathcal{U}^* , and by construction d is a distance and \succeq is a partial order on \mathcal{U}^* . We will study the metric d , and focus on three main questions:

- 1) How to compute $d(u, v)$?
- 2) What is the link between \mathcal{U}^* and the Mertens-Zamir space ?
- 3) How large is the space of information structures ? Given $\varepsilon > 0$, can we cover \mathcal{U}^* with finitely many balls of radius ε ?

Whereas previous papers in the literature restrict attention¹ to a particular subset of \mathcal{U} (independent information, lack of information on one side, fixed support...), we will study the general case of information structures in \mathcal{U} and \mathcal{U}^* .

3. COMPUTING $d(u, v)$

We give here a tractable characterization of $d(u, v)$, based on duality between signals and actions. We start with the notion of garbling, used by Blackwell to compare statistical experiments [1].

Definition 7. A garbling is an element $q : \mathbb{N} \rightarrow \Delta_f(\mathbb{N})$, and the set of all garblings is denoted by \mathcal{Q} . Given a garbling q in \mathcal{Q} and an information structure u in \mathcal{U} , we define the information structures $q.u$ and $u.q$ in \mathcal{U} by: $\forall k \in K, \forall c, c', d, d' \in \mathbb{N}$,

$$q.u(k, c', d) = \sum_{c \in \mathbb{N}} u(k, c, d)q(c)(c') \quad \text{and} \quad u.q(k, c, d') = \sum_{d \in \mathbb{N}} u(k, c, d)q(d)(d').$$

The interpretation of $q.u$ is as follows: first (k, c, d) is selected according to u , the state is k and player 2 learns d . c' is selected according to the probability $q(c)$, and player 1 learns c' . Here, the signal received by player 1 has been deteriorated through the garbling q . And $u.q$ corresponds to the dual situation where the signal of player 2 has been deteriorated. Since in a zero-sum game the value is monotonic in the information of the players, regardless of the payoffs player 1 always weakly prefers u to $q.u$, and $u.q$ to u :

Lemma 1. For all u in \mathcal{U} and q in \mathcal{Q} , $q.u \preceq u \preceq u.q$

To compute $d(u, v)$, we will use here a second and new interpretation. A garbling q in \mathcal{Q} will also be seen as a behavior strategy of a player in a Bayesian game $\Gamma(u, g)$: if the signal received is c , play the mixed action $q(c)$.

Notations: Given $L \geq 1$, we denote by $\mathcal{U}(L)$ the set of information structures u with support in $K \times \{0, \dots, L-1\}^2$: only the first L signals of each player matter. We also denote by $\mathcal{Q}(L)$ the set of garblings $q : \mathbb{N} \rightarrow \Delta_f(\mathbb{N})$, with range in $\Delta(\{0, \dots, L-1\})$.

¹For instance, one can read in [2] “We leave open the question of what happens when the components of the state on which the players have some information fail to be independent.... In this situation the notion of monotonicity is unclear, and the duality method is not well understood.”

$\mathcal{U}(L)$ is a convex compact subset of a finite dimensional vector space. Notice that for u in \mathcal{U} and $L \geq 1$, the sets $\mathcal{Q}(L).u = \{q.u, q \in \mathcal{Q}(L)\}$ and $u.\mathcal{Q}(L) = \{u.q, q \in \mathcal{Q}(L)\}$ are also convex compacta in Euclidean spaces.

Consider now u and v in \mathcal{U} . Since u and v have finite support, we can find L such that both u and v are in $\mathcal{U}(L)$. Our first theorem shows that $\sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g))$ can be simply computed as the minimal distance, measured by the norm $\|\cdot\|$, between the convex compact subsets $\mathcal{Q}(L).u$ and $v.\mathcal{Q}(L)$ of $\mathcal{U}(L)$. Moreover, the supremum is achieved by a payoff structure of size L .

Theorem 1. *For u, v in $\mathcal{U}(L)$,*

$$\begin{aligned} \sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g)) &= \max_{g \in \mathcal{G}(L)} (\text{val}(v, g) - \text{val}(u, g)), \\ &= \min_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \|q_1.u - v.q_2\|, \\ &= \min_{q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}} \|q_1.u - v.q_2\|. \end{aligned}$$

Since $d(u, v) = \max\{\sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g)), \sup_{g \in \mathcal{G}} (\text{val}(u, g) - \text{val}(v, g))\}$, the following corollary is immediate, and explains how to compute $d(u, v)$.

Corollary 1. *For u, v in \mathcal{U} ,*

$$d(u, v) = \max_{g \in \mathcal{G}} |\text{val}(u, g) - \text{val}(v, g)| = \max \left\{ \min_{q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}} \|q_1.u - v.q_2\|, \min_{q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}} \|u.q_1 - q_2.v\| \right\}.$$

We can also recover from theorem 1 that : $u \succeq v \iff \exists q_1, q_2 \in \mathcal{Q}, q_1.u = v.q_2$, as obtained by Peski [9], generalizing the Blackwell characterization of more informative experiment to the multi-player setting. And we get a simple characterization of the equivalence relation:

$$u \sim v \iff \exists q_1, q_2, q_3, q_4 \in \mathcal{Q}, q_1.u = v.q_2, u.q_3 = q_4.v.$$

The proof of Theorem 1 relies on two main aspects : the two interpretations of a garbling (deterioration of signals, and strategy), and the use of a minmax theorem due to the fact that we consider information structures with finitely many signals.

Proof of Theorem 1.

1) We start with general considerations. For u in \mathcal{U} and $g \in \mathcal{G}$, we denote by $\gamma_{u,g}(q_1, q_2)$ the payoff of player 1 in the zero-sum game $\Gamma(u, g)$ when player 1 plays $q_1 \in \mathcal{Q}$ and player 2 plays $q_2 \in \mathcal{Q}$. Extending as usual g to mixed actions, we have: $\gamma_{u,g}(q_1, q_2) = \sum_{k,c,d} u(k, c, d)g(k, q_1(c), q_2(d))$. Notice that in $\Gamma(u, g)$, both players can play the identity

strategy Id in \mathcal{Q} which plays with probability one the signal received. And for u in \mathcal{U} and g in \mathcal{G} , the scalar product $\langle g, u \rangle = \sum_{k \in K, (c,d) \in \mathbb{N}^2} g(k, c, d) u(k, c, d)$ is well defined, and corresponds to the expectation of g with respect to u , and to the payoff $\gamma_{u,g}(Id, Id)$.

Let us now compute the payoff $\gamma_{u,g}(q_1, q_2)$, for any q_1 and q_2 in \mathcal{Q} :

$$\begin{aligned}
\gamma_{u,g}(q_1, q_2) &= \sum_{k,c,d} u(k, c, d) g(k, q_1(c), q_2(d)) \\
&= \sum_{k,c,d} u(k, c, d) \sum_{(c',d') \in \mathbb{N}^2} q_1(c)(c') q_2(d)(d') g(k, c', d') \\
&= \sum_{k,c',d'} g(k, c', d') \sum_{c,d} u(k, c, d) q_1(c)(c') q_2(d)(d') \\
&= \sum_{k,c',d'} g(k, c', d') q_1 \cdot u \cdot q_2(k, c', d') \\
&= \langle g, q_1 \cdot u \cdot q_2 \rangle.
\end{aligned}$$

Consequently, $\text{val}(u, g) = \max_{q_1 \in \mathcal{Q}} \min_{q_2 \in \mathcal{Q}} \langle g, q_1 \cdot u \cdot q_2 \rangle = \min_{q_2 \in \mathcal{Q}} \max_{q_1 \in \mathcal{Q}} \langle g, q_1 \cdot u \cdot q_2 \rangle$. Since both players can play the Id strategy in $\Gamma_{u,g}$, we obtain for all $u \in \mathcal{U}(L)$ and $g \in \mathcal{G}(L)$:

$$\inf_{q_2 \in \mathcal{Q}} \langle g, u \cdot q_2 \rangle \leq \inf_{q_2 \in \mathcal{Q}(L)} \langle g, u \cdot q_2 \rangle \leq \text{val}(u, g) \leq \sup_{q_1 \in \mathcal{Q}(L)} \langle g, q_1 \cdot u \rangle \leq \sup_{q_1 \in \mathcal{Q}} \langle g, q_1 \cdot u \rangle.$$

Notice also that for all u, v in \mathcal{U} , $\|u - v\| = \sup_{g \in \mathcal{G}} \langle g, u - v \rangle$.

2) We now prove Theorem 1.

Consider g in \mathcal{G} , q_1 and q_2 in \mathcal{Q} . $\text{val}(v \cdot q_2, g) \geq \text{val}(v, g)$ and $\text{val}(u, g) \geq \text{val}(q_1 \cdot u, g)$, so: $\text{val}(v, g) - \text{val}(u, g) \leq \text{val}(v \cdot q_2, g) - \text{val}(q_1 \cdot u, g) \leq \|q_1 \cdot u - v \cdot q_2\|$. We first obtain:

$$\sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g)) \leq \inf_{q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}} \|q_1 \cdot u - v \cdot q_2\|.$$

Clearly, $\sup_{g \in \mathcal{G}(L)} (\text{val}(v, g) - \text{val}(u, g)) \leq \sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g))$ and $\inf_{q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}} \|q_1 \cdot u - v \cdot q_2\| \leq \inf_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \|q_1 \cdot u - v \cdot q_2\|$. So it will be enough to prove that

$$\inf_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \|q_1 \cdot u - v \cdot q_2\| \leq \sup_{g \in \mathcal{G}(L)} (\text{val}(v, g) - \text{val}(u, g)). \quad (3.1)$$

We have $\inf_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \|q_1 \cdot u - v \cdot q_2\| = \inf_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \sup_{g \in \mathcal{G}(L)} \langle g, v \cdot q_2 - q_1 \cdot u \rangle$. The sets $\mathcal{Q}(L)$ and $\mathcal{G}(L)$ are compact, and by Sion's theorem :

$$\inf_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \sup_{g \in \mathcal{G}(L)} \langle g, v \cdot q_2 - q_1 \cdot u \rangle = \sup_{g \in \mathcal{G}(L)} \inf_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \langle g, v \cdot q_2 - q_1 \cdot u \rangle.$$

Inequality (3.1) now follows from :

$$\begin{aligned} \sup_{g \in \mathcal{G}(L)} \inf_{q_1 \in \mathcal{Q}(L), q_2 \in \mathcal{Q}(L)} \langle g, v.q_2 - q_1.u \rangle &= \sup_{g \in \mathcal{G}(L)} \left(\inf_{q_2 \in \mathcal{Q}(L)} \langle g, v.q_2 \rangle - \sup_{q_1 \in \mathcal{Q}(L)} \langle g, q_1.u \rangle \right) \\ &\leq \sup_{g \in \mathcal{G}(L)} (\text{val}(v, g) - \text{val}(u, g)). \end{aligned}$$

Finally notice that the compactness of $\mathcal{Q}(L)$ and $\mathcal{G}(L)$ also give that the above infima and suprema are achieved. \blacksquare

Remark 1. Theorem 1 and its proof also imply the followings.

1) For u, v in $\mathcal{U}(L)$, the sets $A = \{q_1.u - v.q_2, q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}\}$ and $B = \{q_1.v - u.q_2, q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}\}$ are polytopes in $\mathbb{R}^{K \times \{0, \dots, L-1\}^2}$, and to compute $d(u, v)$ it is enough to compute $\alpha = \text{Min}\{\|x\|_1, x \in A\}$ and $\beta = \text{Min}\{\|x\|_1, x \in B\}$. Then $d(u, v) = \max\{\alpha, \beta\}$.

2) Relationship between d , $\|\cdot\|$ and \succeq : We have for all u, v in \mathcal{U} ,

$$\sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g)) = \min_{u' \preceq u, v' \succeq v} \|u' - v'\|.$$

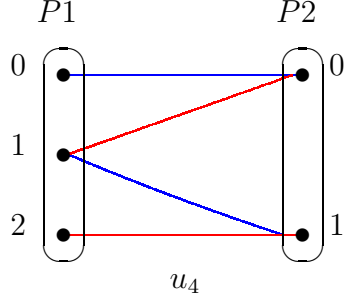
3) Optimal payoff structure : If u, v are in $\mathcal{U}(L)$, $\sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g))$ is achieved for $g \in \mathcal{G}(L)$ maximizing $\min_{q_1, q_2 \in \mathcal{Q}(L)} \langle g, v.q_2 - q_1.u \rangle$. This shows how to find g such that $d(u, v) = |\text{val}(u, g) - \text{val}(v, g)|$.

4) Optimal strategies : Consider u, v in $\mathcal{U}(L)$, and let q_1 and q_2 achieving the minimum in $\min_{q'_1 \in \mathcal{Q}(L), q'_2 \in \mathcal{Q}(L)} \|q'_1.u - v.q'_2\|$. We have $\|q_1.u - v.q_2\| = \sup_{g \in \mathcal{G}} (\text{val}(v, g) - \text{val}(u, g)) \leq d(u, v)$. Let g be a payoff structure in \mathcal{G} , there is a canonical way to transform optimal strategies in the Bayesian game $\Gamma(v, g)$ into $2d(u, v)$ -optimal strategies in $\Gamma(u, g)$. Indeed let σ in \mathcal{Q} be optimal for player 1 in $\Gamma(v, g)$, and define $\sigma.q_1$ in \mathcal{Q} by $\sigma.q_1(c) = \sum_{c'} q_1(c)(c')\sigma(c')$ for each signal c : player 1 receives signal c , then selects c' according to $q_1(c)$ and plays $\sigma(c')$. Using the notations of the proof of theorem 1, we have for every strategy τ of player 2 in \mathcal{Q} :

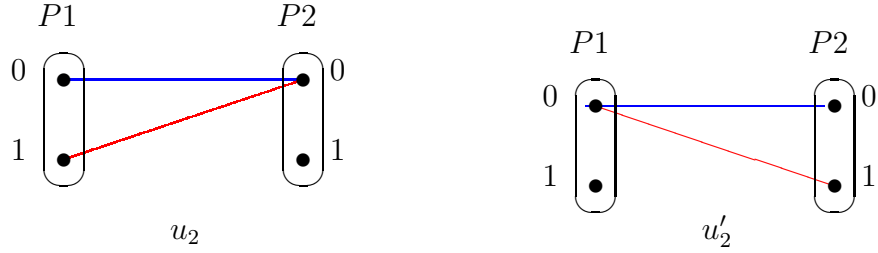
$$\begin{aligned} \gamma_{u, g}(\sigma.q_1, \tau) &= \langle g, (\sigma.q_1).u.\tau \rangle \\ &= \langle g, \sigma.(q_1.u).\tau \rangle \\ &\geq \langle g, \sigma.(v.q_2).\tau \rangle - \|q_1.u - v.q_2\| \\ &\geq \langle g, \sigma.v.(\tau.q_2) \rangle - d(u, v) \\ &\geq \text{val}(v, g) - d(u, v) \\ &\geq \text{val}(u, g) - 2d(u, v), \end{aligned}$$

so $\sigma.q_1$ is $2d(u, v)$ optimal in $\Gamma(u, g)$. Similarly if τ is optimal for player 2 in $\Gamma(u, g)$, then $\tau.q_2$ is $2d(u, v)$ optimal for player 2 in $\Gamma(v, g)$. ■

Example 4. Consider for instance the following information structure u_4 .



How valuable is u_4 to player 1, in which sense it is profitable for player 1? What are $d(u_2, u_4)$ and $d(u'_2, u_4)$?



We first have $\|u_2 - u_4\| = 1$, so $d(u_2, u_4) \leq 1$. We have $u_2 \succeq u_4$, hence $d(u_2, u_4) = \min_{q_1 \in \mathcal{Q}, q_2 \in \mathcal{Q}} \|q_1.u_4 - u_2.q_2\|$. Define q_1 in \mathcal{Q} such that $q_1(0) = \delta_0$, $q_1(1) = q_1(2) = \delta_1$, and q_2 in \mathcal{Q} satisfying $q_2(0) = 1/2 \delta_0 + 1/2 \delta_1$. The information structures $q_1.u_4$ and $u_2.q_2$ can be represented as follows:



Notice that $u_2.q_2 \sim u_2$, whereas $q_1.u_4 \preceq u_4$. $\|q_1.u_4 - u_2.q_2\| = 1/2$, hence $d(u_2, u_4) \leq 1/2$.

Consider now the payoff structure g given by $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$. In the game (u_2, g) , it is optimal for player 1 to play Top if 0 and Bottom if 1, and $\text{val}(u_2, g) = 1/2$. In the game (u_4, g) it is optimal for player 2 to play Left if 0 and Right if 1, and $\text{val}(u_4, g) = 0$. Consequently, $d(u_2, u_4) \geq 1/2$, and we obtain $d(u_2, u_4) = 1/2$.

Notice that $u'_2 \sim u''_2$, with u''_2 obtained from u_2 by exchanging the signals 0 and 1 for each player, and $\|u_4 - u''_2\| = 1$. Considering the payoff structure given by $\left\{ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \right\}$ gives $d(u'_2, u_4) = 1$, so u_4 is closer to u_2 than to u'_2 .

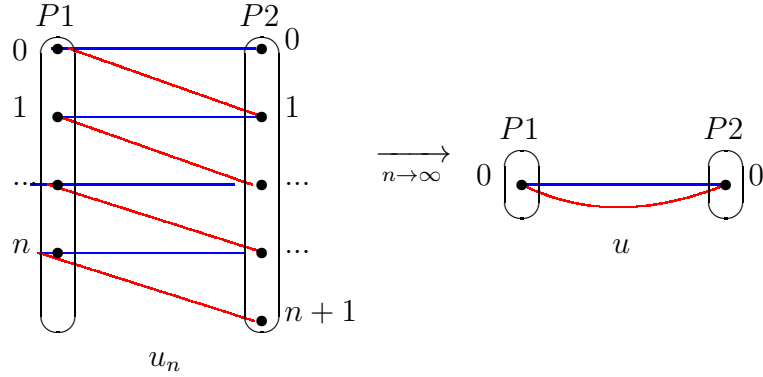
Example 5. Maximal distance with a given marginal on K . Consider $p = (p_k)_{k \in K}$ in $\Delta(K)$.

$$\max_{\Delta(K)} \{d(u, v), \text{marg}(u) = \text{marg}(v) = p\} = 2(1 - \max_k p_k).$$

Proof: Assume w.l.o.g. that $p_1 = \max_k p_k$. Define u_{max} and u_{min} in \mathcal{U} such that $u_{max}(k, c, d) = p_k \mathbf{1}_{c=k} \mathbf{1}_{d=0}$ (complete information for player 1, trivial information for player 2) and $u_{min}(k, c, d) = p_k \mathbf{1}_{c=0} \mathbf{1}_{d=k}$ for all (k, c, d) (trivial information for player 1, complete information for player 2). Since the value of a zero-sum game is weakly increasing with player 1's information and weakly decreasing with player 2's information, we have $u_{min} \preceq u \preceq u_{max}$ and $u_{min} \preceq v \preceq u_{max}$. It implies that $d(u, v) \leq \|u_{max} - u_{min}\| = 2(1 - p_1)$.

Define now the payoff structure g such that $g(k, c, d) = \mathbf{1}_{k=c} - \mathbf{1}_{k \neq c}$. Clearly, $\text{val}(u_{max}, g) = 1$. In the game $\Gamma(u_{min}, g)$, it is optimal for player 1 to play $c = 0$, and $\text{val}(u_{min}, g) = p_1 - (1 - p_1) = 2p_1 - 1$. Hence $\text{val}(u_{max}, g) - \text{val}(u_{min}, g) = 2(1 - p_1)$, and $d(u_{max}, u_{min}) = 2(1 - p_1)$.

Example 6. An example of convergence in the metric space (\mathcal{U}^*, d) :



The idea is that when n is large, with high probability the players will receive signals far from 0 and n . These signals convey very little information to the players and only differ for very high-order beliefs. Optimal strategies of Bayesian games may differ after receiving one signal or another (as for u_3 in Example 3), but if we restrict attention to the values of the Bayesian games, u_n is close to the trivial information structure u .

We now prove the convergence. Consider garblings q_1, q_2 , such that $q_1(0)$ is uniform on $\{0, \dots, n\}$, and $q_2(c) = \delta_0$ for each c . Then $q_1 \cdot u = u_n \cdot q_2$. We obtain $u \succeq u_n$, and $d(u, u_n) = \min_{q'_1, q'_2 \in \mathcal{Q}} \|q'_1 \cdot u_n - u \cdot q'_2\|$. Consider now $q'_1 = q_2$ and q'_2 such that $q'_2(0)$ is uniform on $\{0, \dots, n+1\}$. We get $\|q'_1 \cdot u_n - u_n \cdot q'_2\| \leq 1/(n+1) \xrightarrow{n \rightarrow \infty} 0$.

Remark 2. Decision problems. Our approach can also be used for 1-player games or decision problems, with $\mathcal{U}_0 = \Delta_f(K \times \mathbb{N})$, $\mathcal{G}_0 = \{g : K \times \mathbb{N} \rightarrow [-1, 1], \exists L \text{ s.t. } \forall i \geq L, g(k, i) = -1\}$, and $d_0(u, v) = \sup_{g \in CG_0} |\text{val}(v, g) - \text{val}(u, g)|$. We obtain for u, v in \mathcal{U}_0 , that $d_0(u, v) = \max\{\min_{q \in \mathcal{Q}} \|q.u - v\|, \min_{q \in \mathcal{Q}} \|q.v - u\|\}$ and the Blackwell characterization : $u \succeq v \Leftrightarrow \exists q \in \mathcal{Q}, q.u = v$.

Notice that what matters here for an information structure u in \mathcal{U}_0 is the induced law \tilde{u} of the a posteriori of the player after receiving his signal. We also have, if D is the set of suprema of affine functions from $\Delta(K)$ to $[-1, 1]$ and E_1 is the set of 1-Lipchitz functions on $\Delta(K)$: $d_0(u, v) = \sup_{f \in D} \left| \int_{p \in \Delta(K)} f(p) d\tilde{u}(p) - \int_{p \in \Delta(K)} f(p) d\tilde{v}(p) \right|$,

$$\begin{aligned} \text{and } u_n \xrightarrow[n \rightarrow \infty]{} u &\iff \forall f \in E_1, \int_{p \in \Delta(K)} f(p) du_n(p) \xrightarrow[n \rightarrow \infty]{} \int_{p \in \Delta(K)} f(p) du(p) \\ &\iff \left(\sup_{f \in E_1} \left(\int_{p \in \Delta(K)} f(p) du_n(p) - \int_{p \in \Delta(K)} f(p) du(p) \right) \xrightarrow[n \rightarrow \infty]{} 0 \right) \quad \blacksquare \end{aligned}$$

4. LINKS WITH THE UNIVERSAL BELIEF SPACE

In the standard approach (Harsanyi, Mertens-Zamir), a situation of incomplete information is described by a *state of the world*. A state of the world specifies the true state k , the belief of each player on k , the belief of each player on the belief of each player on k , etc... The set of states of the world is the *universal belief space* :

$$\Omega = K \times \Theta_1 \times \Theta_2,$$

where for $i = 1, 2$, Θ_i is the *universal type space* of player i , containing all the coherent belief hierarchies of this player. The type space of a player is always endowed with the weak topology, and a crucial property is that Θ_i is compact and homeomorphic to the set of Borel probabilities over $K \times \Theta_{-i}$.

Any information structure in \mathcal{U} naturally induces a Borel probability distribution over the universal belief space, which is consistent since we have a common prior and beliefs are derived by Bayes's rule. We denote by Π the set of consistent (Borel) probabilities over the universal belief space, and by Π_f the set of elements of Π with finite support. We use the weak topology on Π and Π_f , the space Π is then compact and Π_f is dense in Π (see corollary III.2.3 and theorem III.3.1 in [7]). All elements of Π_f are induced by some information structure in \mathcal{U} , since given P in Π_f we can associate an information structure u in \mathcal{U} selecting (k, θ_1, θ_2) according to P (formally, $(k, f_1(\theta_1), f_2(\theta_2))$ in $K \times \mathbb{N} \times \mathbb{N}$, with f_1 and f_2 being one-to-one).

Given P in Π_f and g in \mathcal{G} , we can define $\text{val}(P, g)$ as the value of the zero-sum Bayesian game where first: (k, θ_1, θ_2) is selected according to P , then the players simultaneously select i and j in \mathbb{N} , and the payoff to player 1 is $g(k, i, j)$. By Proposition III.4.4 in [7], $\text{val}(u, g) = \text{val}(\Phi(u), g)$ and an optimal strategy in the game defined by P and g induces an optimal strategy in the zero-sum game $\Gamma(u, g)$. Now, it is known that the value functions of finite games separate the elements of Π (lemma 41 in Gossner Mertens [4]), so equivalent information structures in \mathcal{U} induce the same element of Π_f , and we can associate to each equivalence class in \mathcal{U}^* an element of Π_f . We obtain a natural bijection from \mathcal{U}^* to Π_f , that we denote by Φ , and one can ask how similar the topological spaces \mathcal{U}^* and Π_f are.

In this section only, we will not consider the distance d , but the weak topology of pointwise convergence on \mathcal{U} and \mathcal{U}^* .

Definition 8. *A sequence of information structures $(u_n)_{n \geq 1}$ weakly converges to u if for all payoff structures g in \mathcal{G} , $\text{val}(u_n, g) \xrightarrow[n \rightarrow \infty]{} \text{val}(u, g)$.*

Since the set of payoff structures can be seen as a countable union of sets of payoff matrices of a given size, one can find a sequence g_1, \dots, g_n, \dots of elements of \mathcal{G} such that for each g in \mathcal{G} and $\varepsilon > 0$, there exists n with $\max_{k \in K, (i, j) \in \mathbb{N}^2} |g(k, i, j) - g_n(k, i, j)| \leq \varepsilon$. The sequence (g_n) is dense in \mathcal{G} for the sup norm, and the weak convergence is metrizable by the metric:

$$d_W(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\text{val}(u, g_n) - \text{val}(v, g_n)|.$$

(\mathcal{U}^*, d_W) is now another metric space, a priori different from (\mathcal{U}^*, d) since we have changed the metric. It can not be compact, since we have only considered information structures with finite support.

Theorem 2.

1) *The metric space (\mathcal{U}^*, d_W) is homeomorphic to the space Π_f of consistent probabilities with finite support over the universal belief space.*

2) *Its completion is homeomorphic to the compact space Π of consistent probabilities over the universal belief space.*

Proof of Theorem 2.

Define, for P and Q in Π ,

$$d_W^*(P, Q) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\text{val}(P, g_n) - \text{val}(Q, g_n)|.$$

If for each n , $\text{val}(P, g_n) - \text{val}(Q, g_n) = 0$ then for all g in \mathcal{G} , $\text{val}(P, g) - \text{val}(Q, g) = 0$, and $P = Q$ by lemma 41 of [4] again. d_W^* is a metric on Π .

For each payoff structure g , the mapping $(P \mapsto \text{val}(P, g))$ is continuous for the weak topology on Π (see Lemma 2 in [6] or Proposition III.4.3. in [7]). So if a sequence $(P_t)_t$ of elements of Π weakly converges to some limit P , we have $d_W^*(P_t, P) \xrightarrow[t \rightarrow \infty]{} 0$.

Conversely, consider a sequence $(P_t)_t$ of elements of Π converging for d^* to some limit P , we have for all n : $\text{val}(P_t, g_n) \xrightarrow[t \rightarrow \infty]{} \text{val}(P, g_n)$. For any converging subsequence $(P_{\phi_t})_t$, for the weak topology, with limit Q , we have by the previous paragraph, that for all n , $\text{val}(P_{\phi_t}, g_n) \xrightarrow[t \rightarrow \infty]{} \text{val}(Q, g_n)$. So $d_W^*(P, Q) = 0$ for each limit point Q , and since Π is compact the sequence $(P_t)_t$ converges to P .

We obtain that d_W^* induces the weak topology on Π . By construction, the bijection Φ is isometric from (\mathcal{U}^*, d_W) to (Π_f, d_W^*) , hence an homeomorphism.

Finally, the completion of (\mathcal{U}^*, d_W) is homeomorphic to the completion of (Π_f, d_W^*) . Since d_W^* induces the weak topology on Π , the completion of (Π_f, d_W^*) is the closure of Π_f . Since Π is compact and Π_f is dense in Π , this completion is Π . ■

Theorem 2 suggests a possible alternative construction of the set Π of consistent probability over the universal belief space. The alternative construction is simply based on the values of finite zero-sum Bayesian games.

In the remainder of the paper we come back to the distance d on \mathcal{U}^* .

5. HOW LARGE IS THE SPACE OF INFORMATION STRUCTURES ?

We consider the metric space (\mathcal{U}^*, d) (or simply \mathcal{U}^*). As \mathcal{U} only contains information structures with finite support, \mathcal{U}^* can not be compact, and we denote by $\overline{\mathcal{U}}$ its completion. We focus here on a major property : is $\overline{\mathcal{U}}$ compact ? Equivalently, is \mathcal{U}^* totally bounded, i.e. given $\varepsilon > 0$ can we cover \mathcal{U}^* with finitely many balls of radius ε ? Can we see \mathcal{U}^* as a subset of a compact metric space ?

One can show that this question is equivalent to any of the following ones:

- A) Is $\overline{\mathcal{U}}$ homeomorphic to the set Π of consistent probabilities over the universal belief space ?
- B) Are the distances d and the weak distance d_W uniformly equivalent on \mathcal{U}^* ?
- C) Is the family $(P \mapsto \text{val}(P, g))_{g \in CG}$ an equicontinuous family of mappings from Π to \mathbb{R} ?

Question C) corresponds to the second of the three problems² posed by J.F. Mertens in his Repeated Games survey from ICM 1986 [6] : “This equicontinuity or Lipschitz property character is crucial in many papers...”.

Remark 3. Repeated Games. Consider a general zero-sum repeated game (stochastic game, with incomplete information and signals), given by a transition $q : K \times I \times J \rightarrow \Delta(K \times C \times D)$, a payoff function $g : K \times I \times J \rightarrow [-1, 1]$ and an initial probability u_0 in $\Delta(K \times A \times B)$, where K, I, J, A and B are finite subsets of \mathbb{N} . Before stage 1, an initial state k_1 in K and initial private signals a_1 in A for player 1, and b_1 in B for player 2, are selected according to u_0 . Then at each stage t , simultaneously player 1 chooses an action i_t in I and player 2 chooses an action j_t in J , and the stage payoff is $g(k_t, i_t, j_t)$, an element $(k_{t+1}, a_{t+1}, b_{t+1})$ is selected according to $q(k_t, i_t, j_t)$, the new state is k_{t+1} , player 1 receives the signal a_{t+1} , player 2 the signal b_{t+1} , and the play proceeds to stage $t + 1$.

An appropriate state variable is here u in \mathcal{U} , representing the current state in K and the finite sequence of signals previously received by each player. As a consequence, a recursive formula can be explicitly written as follows: for all discount λ in $(0, 1]$ and all u in \mathcal{U} ,

$$\begin{aligned} v_\lambda(u) &= \max_{q_1 \in \mathcal{Q}} \min_{q_2 \in \mathcal{Q}} \lambda G(u, q_1, q_2) + (1 - \lambda)v_\lambda(F(u, q_1, q_2)), \\ &= \min_{q_2 \in \mathcal{Q}} \max_{q_1 \in \mathcal{Q}} \lambda G(u, q_1, q_2) + (1 - \lambda)v_\lambda(F(u, q_1, q_2)), \end{aligned}$$

with $G(u, q_1, q_2) = \sum_{k,c,d} u(k, c, d)g(k, q_1(c), q_2(d)) \in [-1, 1]$, and $F(u, q_1, q_2) \in \mathcal{U}$ is defined, for all (k, i, a, j, b) in $K \times I \times A \times J \times B$, by $F(u, q_1, q_2)(k', f_1(c, i, a), f_2(d, j, b)) = \sum_k u(k, c, d)q_1(c)(i)q_2(d)(j)q(k, i, j)(k', a, b)$ (where f_1 and f_2 are fixed one-to-one mappings from \mathbb{N}^3 to \mathbb{N}).

The value function v_λ can be approximated by the value functions of finite games. Since such value functions are, by construction, 1-Lipschitz from (\mathcal{U}, d) to $[-1, 1]$, so is v_λ . Hence the family $(v_\lambda)_\lambda$ is equicontinuous, and if it happens that the set of information structures that can be reached during the game is totally bounded, by Ascoli’s theorem this family has a uniform limit point when $\lambda \rightarrow 0$. ■

Compactness of $\overline{\mathcal{U}}$ is then strongly related to the equivalence between the strong distance d and the weak distance d_W . Notice that in the 1-player case of Remark 2, weak and strong convergence are equivalent, and \mathcal{U}_0 is homeomorphic to $\Delta_f(\Delta(K))$, which is dense in the

²Problem 1 asked for the convergence of the value functions $(v_\lambda)_\lambda$ and $(v_n)_n$ in a general zero-sum repeated game with finitely many states, actions and signals, and was disproved during the PhD thesis of B. Ziliotto [11]. Problem 3 asks if the existence of a uniform value follows from the uniform convergence of (v_λ) , and was disproved by Lehrer and Monderer [5] for 1-player games, see also [8].

compact set $\Delta(\Delta(K))$. For 2 players, compactness has been obtained in every particular case tackled so far. If \mathcal{U}' is a subset of \mathcal{U}^* , we denote by $\overline{\mathcal{U}'}$ the closure of \mathcal{U}' in $\overline{\mathcal{U}}$:

- Set \mathcal{U}_1 of information structures where both players receive the same signal: $\overline{\mathcal{U}_1}$ is compact, and homeomorphic to $\Delta(\Delta(K))$. Here given u in \mathcal{U}_1 , what matters is the induced law \tilde{u} on the common a posteriori of the players on K . Another characterization of $d(u, v)$ has been obtained in [10]. let D_1 be the subset of 1-Lipschitz functions from $\Delta(K)$ to \mathbb{R} satisfying $\forall p, q \in \Delta(K), \forall a, b \geq 0, af(p) - bf(q) \leq \|ap - bq\|_1$. We have :

$$\forall u, v \in \mathcal{U}_1, d(u, v) = \sup_{f \in D_1} \left(\int_{p \in \Delta(K)} f(p) d\tilde{u}(p) - \int_{p \in \Delta(K)} f(p) d\tilde{v}(p) \right).$$

- Set \mathcal{U}_2 of information structures where player 1 knows the signal of player 2: $\overline{\mathcal{U}_2}$ is compact, and homeomorphic to $\Delta(\Delta(\Delta(K)))$ (see [6], [3]).

- Set \mathcal{U}_3 of independent information structures : \mathcal{U}_3 is the set of u in \mathcal{U} such that $u(c, d|k) = u(c|k)u(d|k)$ (the signals c and d are conditionally independent given k). Here $\overline{\mathcal{U}_3}$ is homeomorphic to $\Delta(\Delta(K) \times \Delta(L))$.

We now present our main counterexample, where it is assumed that there are at least 2 states in K .

Theorem 3. *There exists $\varepsilon > 0$ and a sequence $(\mu^l)_{l \geq 1}$ of information structures in \mathcal{U} satisfying :*

- 1) $d(\mu^l, \mu^p) > \varepsilon$ for all $l \neq p$,
- 2) for each l the conditional law of μ^{l+1} on the support of μ^l is μ^l , and
- 3) for all $l > p$, the distribution on states and $2p$ -order beliefs induced by μ^l does not depend on l .

Remarks :

Condition 1) implies that (\mathcal{U}^*, d) is not totally bounded, and $\overline{\mathcal{U}}$ is not compact. The space of information structures \mathcal{U}^* is very large, in the sense that it is *not* a subset of a compact metric space, one cannot approximate the space with finite sets. All questions A), B), C) above have a negative answer, in particular $\overline{\mathcal{U}}$ is not homeomorphic to Π .

Condition 2) means that to go from μ^l to μ^{l+1} , each player gets an extra signal. So having more and more information may lead... nowhere. This has to be contrasted with the 1-player case, where the sequence of beliefs of a player receiving more and more signals is a martingale, which converges in law. We don't have a "strategic martingale" convergence theorem here.

Condition 3) implies there exists no n such that knowing the joint distribution of n -order beliefs is enough to determine, up to ε , the value of every finite game with payoffs in $[-1, 1]$.

Computing the largest ε such that a sequence satisfying condition 1) exists seems very difficult, but we believe it is very small. Rough estimates of our proof only gives $\varepsilon \geq 3.10^{-17}$.

6. PROOF OF THEOREM 3.

Without loss of generality we assume that there are two states: $K = \{0, 1\}$. For convenience we will consider information structures u in $\Delta(K \times \mathcal{C} \times \mathcal{D})$ where \mathcal{C} and \mathcal{D} are arbitrary finite sets (which can be easily identified with subsets of \mathbb{N}). Similarly, we will consider game structures $g : K \times \mathcal{C} \times \mathcal{D} \rightarrow [-1, 1]$, where \mathcal{C} and \mathcal{D} are the respective finite sets of actions of player 1 and player 2.

N is a very large even integer to be fixed later, and we write $A = C = D = \{1, \dots, N\}$, with the idea of using C while speaking of actions or signals of player 1, and using D while speaking of actions and signals of player 2. We fix ε and α , to be used later, such that

$$0 < \varepsilon < \frac{1}{10(N+1)^2}, \text{ and } \alpha = \frac{1}{25}.$$

We will consider a Markov chain with law ν on A , satisfying:

- the law of the first state of the Markov chain is uniform on A ,
- for each a in A , there are exactly $N/2$ elements b in A such that $\nu(b|a) = 2/N$: given that the current state of the Markov chain is a , the law of the next state is uniform on a subset of states of size $N/2$,
- and two more conditions, called *UI1* and *UI2*, to be defined later.

A sequence (a_1, \dots, a_l) of length $l \geq 1$ is said to be *nice* if it is in the support of the Markov chain: $\nu(a_1, \dots, a_l) > 0$. For instance any sequence of length 1 is nice, and $N^2/2$ sequences of length 2 are nice. The proof is now split in 3 parts: we first define the information structures $(u^l)_{l \geq 1}$ and some payoff structures $(g^p)_{p \geq 1}$. Then we define the conditions *UI1* and *UI2* and show that they imply the conclusions of theorem 3. Finally, we show, via the probabilistic method, the existence of a Markov chain ν satisfying all our conditions.

6.1. Information and payoff structures $(u^l)_{l \geq 1}$ and $(g^p)_{p \geq 1}$.

Definition 9. For $l \geq 1$, define the information structure $u^l \in \Delta(K \times C^l \times D^l)$ by: for each state k in K , signal $c = (c_1, \dots, c_l)$ in C^l of player 1 and signal $d = (d_1, \dots, d_l)$ in D^l for player 2,

$$u^l(k, c, d) = \nu(c_1, d_1, c_2, d_2, \dots, c_l, d_l) \left(\frac{c_1}{N+1} \mathbf{1}_{k=1} + \frac{1-c_1}{N+1} \mathbf{1}_{k=0} \right).$$

The following interpretation of u^l holds: first select $(a_1, a_2, \dots, a_{2l}) = (c_1, d_1, \dots, c_l, d_l)$ in A^{2l} according to the Markov chain ν (i.e. uniformly among the nice sequences of length $2l$), then tell (c_1, c_2, \dots, c_l) (the elements of the sequence with odd indices) to player 1, and (d_1, d_2, \dots, d_l)

(the elements of the sequence with even indices) to player 2. Finally choose the state $k = 1$ with probability $c_1/(N+1)$, and state $k = 0$ with the complement probability $1 - c_1/(N+1)$.

Notice that the definition is not symmetric among players, the first signal c_1 of player 1 is uniformly distributed and plays a particular role. The marginal of u^l on K is uniform, and the marginal of u^{l+1} over $(K \times C^l \times V^l)$ holds : condition 2) of theorem 3 is satisfied.

We now show that condition 3) of the theorem holds. Recall that n -order beliefs are defined inductively as conditional laws. Precisely, the first order beliefs θ_1^i of player i is the conditional law of k given her signal. The n -order belief θ_n^i of player i is the conditional law of $(\omega, \theta_{n-1}^{-i})$ given her signal. In this construction, conditional laws are seen as random variables taking values in space of probability measures.

Lemma 2. *For all $l > p$, the joint distribution of $(\omega, \theta_{2p}^1, \theta_{2p}^2)$ induced by the information structure u^l is independent of l .*

Proof. We use the notation $\mathcal{L}(X|Y)$ for the conditional law of X given Y , and the identification $(a_1, \dots, a_{2l}) = (c_1, d_1, \dots, c_l, d_l)$. At first, note that by construction k and (a_2, \dots, a_{2l}) are conditionally independent given a_1 , so that the sequence $(k, a_1, a_2, \dots, a_{2l})$ is a Markov process. It follows that $\theta_1^1 = \mathcal{L}(k|c_1, \dots, c_l) = \mathcal{L}(k|c_1)$. The Markov property implies that

$$\theta_1^2 = \mathcal{L}(k|d_1, \dots, d_l) = \mathcal{L}(k|d_1), \quad \theta_2^2 = \mathcal{L}(d, \theta_1^1(c_1)|d_1, \dots, d_l) = \mathcal{L}(k, \theta_1^1(c_1)|d_1),$$

and therefore we have

$$\theta_2^1 = \mathcal{L}(k, \theta_1^2(d_1)|c_1, \dots, c_l) = \mathcal{L}(k, \theta_1^2(d_1)|c_1, c_2).$$

By induction, and applying the same argument (future and past of a Markov process are conditionally independent given the current position), we deduce that for all $n \geq 1$,

$$\theta_{2n}^1 = \mathcal{L}(k, \theta_{2n-1}^2|c_1, \dots, c_{\min(l, n+1)}), \quad \theta_{2n+1}^1 = \mathcal{L}(k, \theta_{2n}^2|c_1, \dots, c_{\min(l, n+1)}),$$

$$\theta_{2n-1}^2 = \mathcal{L}(k, \theta_{2n-2}^1|d_1, \dots, d_{\min(l, n)}), \quad \theta_{2n}^2 = \mathcal{L}(k, \theta_{2n-1}^1|d_1, \dots, d_{\min(l, n)}).$$

As a consequence, for all $n \leq p$, these conditional laws do not depend on which u^l we are using as soon as $l > p$. \square

Let us give already a very rough intuition of the conditions *UI1* and *UI2* and the Bayesian games that we will consider. The players will be asked to report their signals, and payoffs will highly depend on whether the reported sequence is nice or not. And, thanks to the conditions *UI1* and *UI2*, the chain will be such that if $(c_1, d_1, \dots, c_l, d_l)$ is selected according to ν and player 2 only knows (d_1, \dots, d_l) , any deviation of player 2 to some $(d_1, \dots, d_{r-1}, d_r, \dots, d'_l)$, with $d'_r \neq d_r$, will satisfy:

$$\begin{aligned}
\nu((c_1, d_1, \dots, c_r, d'_r) \text{ is nice}) &\simeq 1/2, \\
\nu((c_1, d_1, \dots, c_r, d'_r, c_{r+1}) \text{ is nice}) &\simeq 1/4, \\
\nu((c_1, d_1, \dots, c_r, d'_r, c_{r+1}, d'_{r+1}) \text{ is nice}) &\simeq 1/8,
\end{aligned}$$

etc..., and similar conditions for deviations of player 1.

Definition 10. Consider a sequence (a_1, \dots, a_l) of elements of A which is not nice, i.e. such that $\nu(a_1, \dots, a_l) = 0$. We say that the sequence is not nice because of player 1 if $\min\{t \in \{1, \dots, l\}, \nu(a_1, \dots, a_t) = 0\}$ is odd, and not nice because of player 2 if $\min\{t \in \{1, \dots, l\}, \nu(a_1, \dots, a_t) = 0\}$ is even.

A sequence (a_1, \dots, a_l) is now either nice, or not nice because of player 1, or not nice because of player 2. A sequence of length 2 is either nice, or not nice because of player 2.

Definition 11. For $p \geq 1$, define the payoff structure $g^p : K \times C^p \times D^{p-1} \rightarrow [-1, 1]$ such that for all k in K , $c' = (c'_1, \dots, c'_p)$ in C^p , $d' = (d'_1, \dots, d'_{p-1})$ in D^{p-1} :

$$\begin{aligned}
g^p(k, c', d') &= g_0(k, c'_1) + h^p(c', d'), \text{ with} \\
g_0(k, c'_1) &= -\left(k - \frac{u'_1}{N+1}\right)^2 + \frac{N+2}{6(N+1)}, \\
h^p(c', d') &= \begin{cases} \varepsilon & \text{if } (c'_1, d'_1, \dots, c'_p) \text{ is nice,} \\ 5\varepsilon & \text{if } (c'_1, d'_1, \dots, c'_p) \text{ is not nice because of player 2,} \\ -5\varepsilon & \text{if } (c'_1, d'_1, \dots, c'_p) \text{ is not nice because of player 1.} \end{cases}
\end{aligned}$$

One can check that $|g^p| \leq 5/6 + 5\varepsilon \leq 8/9$. Regarding the g_0 part of the payoff, consider a decision problem for player 1 where: c_1 is selected uniformly in A and the state is selected to be $k = 1$ with probability $c_1/(N+1)$ and $k = 0$ with probability $1 - c_1/(N+1)$. Player 1 observes c_1 but not k , and he choose c'_1 in A and receive payoff $g_0(k, c'_1)$. We have $\frac{c_1}{N+1}g_0(1, c'_1) + (1 - \frac{c_1}{N+1})g_0(0, c'_1) = \frac{1}{(N+1)^2}(c'_1(2c_1 - c'_1) + (N+1)((N+2)/6 - c_1))$. To maximize this expected payoff, it is well known that player 1 should play his belief on k , i.e. $c'_1 = c_1$. Moreover, if player 1 chooses $c'_1 \neq c_1$, its expected loss from not having chosen c_1 is at least $\frac{1}{(N+1)^2} \geq 10\varepsilon$. And the constant $\frac{N+2}{6(N+1)}$ has been chosen such that the value of this decision problem is 0.

Consider now $l \geq 1$ and $p \geq 1$. By definition, the Bayesian game $\Gamma(u^k, g^p)$ is played as follows: first, $(c_1, d_1, \dots, c_l, d_l)$ is selected according to the law ν of the Markov chain, player 1 learns (c_1, \dots, c_l) , player 2 learns (d_1, \dots, d_l) and the state is $k = 1$ with probability $c_1/(N+1)$ and $k = 0$ otherwise. Then *simultaneously* player 1 chooses c' in C^p and player 2 chooses d' in D^{p-1} , and finally the payoff to player 1 is $g^p(k, c', d')$. Notice that by the previous paragraph about g_0 , it is always strictly dominant for player 1 to report correctly his first signal, i.e. to choose $c'_1 = c_1$. We will show in the next section that if $l \geq p$ and player 1 simply plays the

sequence of signals he received, player 2 can not do better than also reporting truthfully his own signals, leading to a value not lower than the payoff for nice sequences, that is ε . On the contrary in the game $\Gamma(u^l, g^{l+1})$, player 1 has to report not only the l signals he has received, but also an extra-signal c'_{l+1} that he has to guess. In this game we will prove that if player 2 truthfully reports his own signals, player 1 will incur the payoff -5ε with probability at least (approximately) $1/2$, and this will result in a low value. These intuitions will prove correct in the next section, under some conditions *UI1* and *UI2*.

6.2. Conditions UI and values. To prove that the intuitions of the previous paragraph are correct, we need to ensure that players have incentives to report their true signals, so we need additional assumptions on the Markov chain.

Notations and definition: Let $l \geq 1$, $m \geq 0$, $c = (c_1, \dots, c_l)$ in C^l and $d = (d_1, \dots, d_m)$ in D^m . We write :

$$\begin{aligned} a^{2q}(c, d) &= (c_1, d_1, \dots, c_q, d_q) \in A^{2q} && \text{for each } q \leq \min\{l, m\}, \\ a^{2q+1}(c, d) &= (c_1, d_1, \dots, c_q, d_q, c_{q+1}) \in A^{2q+1} && \text{for each } q \leq \min\{l-1, m\}. \end{aligned}$$

For $r \leq \min\{2l, 2m+1\}$,

we say that c and d are *nice at level r* , and we write $c \smile_r d$, if $a^r(c, d)$ is nice.

In the next definition we consider an information structure $u^l \in \Delta(K \times C^l \times D^l)$ and denote by \tilde{c} and \tilde{d} the respective random variables of the signals of player 1 and 2.

Definition 12.

We say that the conditions *UI1* are satisfied if for all $l \geq 1$, all $c = (c_1, \dots, c_l)$ in C^l and $c' = (c'_1, \dots, c'_{l+1})$ in C^{l+1} such that $c_1 = c'_1$, we have

$$u^l \left(c' \smile_{2l+1} \tilde{d} \mid \tilde{c} = c, c' \smile_{2l} \tilde{d} \right) \in [1/2 - \alpha, 1/2 + \alpha] \quad (6.1)$$

and for all $m \in \{1, \dots, l\}$ such that $c_m \neq c'_m$, for $r = 2m - 2, 2m - 1$,

$$u^l \left(c' \smile_{r+1} \tilde{d} \mid \tilde{c} = c, c' \smile_r \tilde{d} \right) \in [1/2 - \alpha, 1/2 + \alpha]. \quad (6.2)$$

We say that the conditions *UI2* are satisfied if for all $1 \leq p \leq l$, for all $d \in D^l$, for all $d' \in D^{p-1}$, for all $m \in \{1, \dots, p-1\}$ such that $d_m \neq d'_m$, for $r = 2m - 1, 2m$

$$u^l \left(\tilde{c} \smile_{r+1} d' \mid \tilde{d} = d, \tilde{c} \smile_r d' \right) \in [1/2 - \alpha, 1/2 + \alpha]. \quad (6.3)$$

To understand the conditions *UI1*, consider the Bayesian game $\Gamma(u^l, g^{l+1})$, and assume that player 2 truthfully reports his sequence of signals and that player 1 has received the signals (c_1, \dots, c_l) in C^l . (6.1) states that if the sequence of reported signals $(c'_1, \tilde{d}_1, \dots, c'_l, \tilde{d}_l)$

is nice at level $2l$, then whatever the last reported signal c'_{l+1} , the conditional probability that $(c'_1, \tilde{d}_1, \dots, c'_l, \tilde{d}_l, c'_{l+1})$ is still nice is in $[1/2 - \alpha, 1/2 + \alpha]$, i.e. close to $1/2$. Regarding (6.2), first notice that if $c' = c$, then by construction $(c'_1, \tilde{d}_1, \dots, c'_l, \tilde{d}_l)$ is nice and $u^l(c' \smile_{r+1} \tilde{d} \mid \tilde{c} = c, c' \smile_r \tilde{d}) = u^l(c \smile_{r+1} \tilde{d} \mid \tilde{c} = c) = 1$ for each $r = 1, \dots, 2l - 1$. Assume now that for some $m = 1, \dots, l$, player 1 misreports his m^{th} -signal, i.e. reports $c'_m \neq c_m$. (6.2) requires that given that the reported signals were nice so far (at level $2m - 2$), the conditional probability that the reported signals are not nice at level $2m - 1$ (integrating c'_m) is close to $1/2$, and moreover if the reported signals are nice at this level $2m - 1$, adding the next signal \tilde{d}_m of player 2 has probability close to $1/2$ to keep the reported sequence nice. Conditions *UI2* have a similar interpretation, considering the Bayesian games $\Gamma(u^l, g^p)$ for $p \leq l$, assuming that player 1 reports truthfully his signals and that player 2 plays d' after having received the signals d .

Proposition 1. *Conditions UI1 and UI2 imply :*

$$\forall l \geq 1, \forall p \in \{1, \dots, l\}, \quad \text{val}(u^l, g^p) \geq \varepsilon. \quad (6.4)$$

$$\forall l \geq 1, \quad \text{val}(u^l, g^{l+1}) \leq -\varepsilon. \quad (6.5)$$

As a consequence of this proposition, under conditions *UI1* and *UI2* we easily obtain condition 1) of theorem 3 :

Corollary 2. *If $l \neq p$ then $d(u^l, u^p) \geq 2\varepsilon$.*

Proof. Assume $l > p$, then $d(u^l, u^p) \geq \text{val}(u^l, g^{p+1}) - \text{val}(u^p, g^{p+1}) \geq \varepsilon - (-\varepsilon)$. ■

Proof of proposition 1. We assume that *UI1* and *UI2* hold. We fix $l \geq 1$, work on the probability space $K \times C^l \times D^l$ equipped with the probability u^l , and denote by \tilde{c} and \tilde{d} the random variables of the signals received by the players.

1) We first prove (6.4), and consider the game $\Gamma(u^l, g^p)$ with $p \in \{1, \dots, l\}$. We assume that player 1 chooses the truthful strategy. Fix $d = (d_1, \dots, d_l)$ in D^l and $d' = (d'_1, \dots, d'_{p-1})$ in D^{p-1} , and assume that player 2 has received the signal d and chooses to report d' .

Define the non-increasing sequence of events:

$$A_n = \{\tilde{c} \smile_n d'\}.$$

We will prove by backward induction that:

$$\forall n = 1, \dots, p, \quad \mathbb{E}[h^p(\tilde{c}, d') \mid \tilde{d} = d, A_{2n-1}] \geq \varepsilon. \quad (6.6)$$

If $n = p$, $h^p(\tilde{c}, d') = \varepsilon$ on the event A_{2p-1} , implying the result. Assume now that for some n such that $1 \leq n < p$, we have : $\mathbb{E}[h^p(\tilde{c}, d') \mid \tilde{d} = d, A_{2n+1}] \geq \varepsilon$. Since we have a non-increasing

sequence of events, $\mathbb{1}_{A_{2n-1}} = \mathbb{1}_{A_{2n+1}} + \mathbb{1}_{A_{2n-1}} \mathbb{1}_{A_{2n}^c} + \mathbb{1}_{A_{2n}} \mathbb{1}_{A_{2n+1}^c}$, so by definition of the payoffs, $h^p(\tilde{c}, d') \mathbb{1}_{A_{2n-1}} = h^p(\tilde{c}, d') \mathbb{1}_{A_{2n+1}} + 5\varepsilon \mathbb{1}_{A_{2n-1}} \mathbb{1}_{A_{2n}^c} - 5\varepsilon \mathbb{1}_{A_{2n}} \mathbb{1}_{A_{2n+1}^c}$.

First assume that $d'_n = d_n$. By construction of the Markov chain, $u^l(A_{2n+1}|A_{2n-1}, \tilde{d} = d) = 1$, implying that $u^l(A_{2n+1}^c|A_{2n-1}, \tilde{d} = d) = u^l(A_{2n}^c|A_{2n-1}, \tilde{d} = d) = 0$. As a consequence,

$$\begin{aligned} \mathbb{E}[h^p(\tilde{c}, d')|\tilde{d} = d, A_{2n-1}] &= \mathbb{E}[h^p(\tilde{c}, d') \mathbb{1}_{A_{2n+1}}|\tilde{d} = d, A_{2n-1}] \\ &= \mathbb{E}[\mathbb{E}[h^p(\tilde{c}, d')|\tilde{d} = d, A_{2n+1}] \mathbb{1}_{A_{2n+1}}|\tilde{d} = d, A_{2n-1}] \\ &\geq \varepsilon. \end{aligned}$$

Assume now that $d'_n \neq d_n$. Assumption UI2 implies that :

$$\begin{aligned} u^l(A_{2n}^c|A_{2n-1}, \tilde{d} = d) &\geq 1/2 - \alpha, \\ u^l(A_{2n} \cap A_{2n+1}^c|A_{2n-1}, \tilde{d} = d) &\leq (1/2 + \alpha)^2, \\ u^l(A_{2n+1}|A_{2n-1}, \tilde{d} = d) &\geq (1/2 - \alpha)^2. \end{aligned}$$

It follows that :

$$\begin{aligned} \mathbb{E}[h^p(\tilde{c}, d')|\tilde{d} = d, A_{2n-1}] &= \mathbb{E}[\mathbb{E}[h^p(\tilde{c}, d')|\tilde{d} = d, A_{2n+1}] \mathbb{1}_{A_{2n+1}}|\tilde{d} = d, A_{2n-1}] \\ &\quad + 5\varepsilon u^l(A_{2n}^c|A_{2n-1}, \tilde{d} = d) - 5\varepsilon u^l(A_{2n} \cap A_{2n+1}^c|A_{2n-1}, \tilde{d} = d) \\ &\geq \varepsilon \left(\frac{1}{4} - \alpha + \alpha^2 \right) + 5\varepsilon \left(\frac{1}{2} - \alpha \right) - 5\varepsilon \left(\frac{1}{4} + \alpha + \alpha^2 \right) \\ &= \varepsilon \left(\frac{3}{2} - 11\alpha - 4\alpha^2 \right) \geq \varepsilon, \end{aligned}$$

And (6.6) follows by backward induction.

Since A_1 is an event which holds almost surely, we deduce that $\mathbb{E}[h^p(\tilde{c}, d')|\tilde{d} = d] \geq \varepsilon$. Hence the truthful strategy of player 1 guarantees the payoff ε in $\Gamma(u^l, g^p)$.

2) We now prove (6.5) and consider the Bayesian game $\Gamma(u^l, g^{l+1})$, assuming that player 2 chooses the truthful strategy. Fix $c = (c_1, \dots, c_l)$ in C^l and $c' = (c'_1, \dots, c'_{l-1})$ in C^{l-1} , and assume that player 1 has received the signal c and chooses to report c' . We will show that the expected payoff of player 1 is not larger than $-\varepsilon$, and assume w.l.o.g. that $c'_1 = c_1$. Consider the non-increasing sequence of events :

$$B_n = \{c' \smile_n \tilde{d}\}.$$

We will prove by backward induction that:

$$\forall n = 1, \dots, l, \quad \mathbb{E}[h^{l+1}(c', \tilde{d})|\tilde{c} = c, B_{2n}] \leq -\varepsilon.$$

If $n = l$, we have $\mathbb{1}_{B_{2l}} = \mathbb{1}_{B_{2l+1}} + \mathbb{1}_{B_{2l}} \mathbb{1}_{B_{2l+1}^c}$, and $h^{l+1}(c', \tilde{d}) \mathbb{1}_{B_{2l}} = \varepsilon \mathbb{1}_{B_{2l+1}} - 5\varepsilon \mathbb{1}_{B_{2l}} \mathbb{1}_{B_{2l+1}^c}$. UI1 implies that $|u^l(B_{2l+1} | \tilde{c} = c, B_{2l}) - \frac{1}{2}| \leq \alpha$, and it follows that :

$$\begin{aligned} \mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c, B_{2l}] &= \varepsilon u^l(B_{2l+1} | \tilde{c} = c, B_{2l}) - 5\varepsilon u^l(B_{2l+1}^c | u = \hat{u}, B_{2l}) \\ &\leq \varepsilon \left(\frac{1}{2} + \alpha\right) - 5\varepsilon \left(\frac{1}{2} - \alpha\right) \leq -\varepsilon. \end{aligned}$$

Assume now that for some $n = 1, \dots, l-1$, we have $\mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c, B_{2n+2}] \leq -\varepsilon$. We have $\mathbb{1}_{B_{2n}} = \mathbb{1}_{B_{2n+2}} + \mathbb{1}_{B_{2n}} \mathbb{1}_{B_{2n+1}^c} + \mathbb{1}_{B_{2n+1}} \mathbb{1}_{B_{2n+2}^c}$, and by definition of h^{l+1} ,

$$h^{l+1}(c', \tilde{d}) \mathbb{1}_{B_{2n}} = h^{l+1}(c', \tilde{d}) \mathbb{1}_{B_{2n+2}} - 5\varepsilon \mathbb{1}_{B_{2n}} \mathbb{1}_{B_{2n+1}^c} + 5\varepsilon \mathbb{1}_{B_{2n+1}} \mathbb{1}_{B_{2n+2}^c}.$$

First assume that $c'_{n+1} = c_{n+1}$, then $u^l(B_{2n+2} | B_{2n}, \tilde{c} = c) = 1$. Then :

$$\begin{aligned} \mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c, B_{2n}] &= \mathbb{E}[h^{l+1}(c', \tilde{d}) \mathbb{1}_{B_{2n+2}} | \tilde{c} = c, B_{2n}], \\ &= \mathbb{E}[\mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c, B_{2n+2}] \mathbb{1}_{B_{2n+2}} | \tilde{c} = c, B_{2n}] \leq -\varepsilon. \end{aligned}$$

Assume on the contrary that $c'_{n+1} \neq c_{n+1}$, assumption UI1 implies that :

$$\begin{aligned} u^l(B_{2n+1}^c | B_{2n}, \tilde{c} = c) &\geq 1/2 - \alpha, \\ u^l(B_{2n+1} \cap B_{2n+2}^c | B_{2n}, \tilde{c} = c) &\leq (1/2 + \alpha)^2, \\ u^l(B_{2n+2} | B_{2n}, \tilde{c} = c) &\geq (1/2 - \alpha)^2. \end{aligned}$$

It follows that :

$$\begin{aligned} \mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c, B_{2n}] &= \mathbb{E}[\mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c, B_{2n+2}] \mathbb{1}_{B_{2n+2}} | \tilde{c} = c, B_{2n}] \\ &\quad - 5\varepsilon u^l(B_{2n+1}^c | B_{2n}, \tilde{c} = c) + 5\varepsilon u^l(B_{2n+1} \cap B_{2n+2}^c | B_{2n}, \tilde{c} = c) \\ &\leq -\varepsilon \left(\frac{1}{4} - \alpha + \alpha^2\right) - 5\varepsilon \left(\frac{1}{2} - \alpha\right) + 5\varepsilon \left(\frac{1}{4} + \alpha + \alpha^2\right) \leq -\varepsilon. \end{aligned}$$

By induction, we obtain $\mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c, B_2] \leq -\varepsilon$. Since B_2 holds almost surely here, we get $\mathbb{E}[h^{l+1}(c', \tilde{d}) | \tilde{c} = c] \leq -\varepsilon$, showing that the truthful strategy of player 2 guarantees that the payoff of the maximizer is less or equal to $-\varepsilon$, and concluding the proof. \blacksquare

6.3. Existence of an appropriate Markov chain. Here we conclude the proof of Theorem 3 by showing the existence of an even integer N and a Markov chain with law ν on $A = \{1, \dots, N\}$ satisfying our conditions :

- 1) the law of the first state of the Markov chain is uniform on A ,
- 2) for each a in A , there are exactly $N/2$ elements b in A such that $\nu(b|a) = 2/N$,
- 3) UI1 and UI2.

Denoting by $P = (P_{a,b})_{(a,b) \in A^2}$ the transition matrix of the Markov chain, we have to prove the existence of P satisfying 2) and 3). The proof is non constructive and uses the following probabilistic method, where we select independently for each a in A , the set $\{b \in A, P_{a,b} > 0\}$

uniformly among the subsets of A with cardinal $N/2$. We will show that when N goes to infinity, the probability of selecting an appropriate transition matrix does not only become positive, but converges to 1.

Formally, denote by \mathcal{S}_A the collection of all subsets $S \subseteq A$ with cardinality $|S| = \frac{1}{2}N$. We consider a collection $(S_a)_{a \in A}$ of i.i.d. random variables uniform distributed over \mathcal{S}_A defined on a probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$. For all a, b in A , let

$$X_{a,b} = \mathbf{1}_{\{b \in S_a\}} \text{ and } P_{a,b} = \frac{2}{N} X_{a,b}.$$

By construction, P is a transition matrix satisfying 2). Theorem 3 will now follow directly from the following proposition.

Proposition 2.

$$\mathbb{P}_N (P \text{ induces a Markov chain satisfying UI1 and UI2 }) \xrightarrow[n \rightarrow \infty]{} 1.$$

In particular, the above probability is strictly positive for all sufficiently large N .

The rest of this section is devoted to the proof of proposition 2.

We start with probability bounds based on Hoeffding's inequality.

Lemma 3. *For any $a \neq b$, each $\gamma > 0$*

$$\mathbb{P}_N \left(\left| |S_a \cap S_b| - \frac{1}{4}N \right| \geq \gamma N \right) \leq \frac{1}{2} e^4 N e^{-2\gamma^2 N}.$$

Proof. Consider a family of i.i.d. Bernoulli variables $(\tilde{X}_{i,j})_{i=a,b, j \in A}$ of parameter $\frac{1}{2}$ defined on a space $(\Omega, \mathcal{F}, \mathbb{P})$. For $i = a, b$, define the events $\tilde{L}_i = \{ \sum_{j \in A} \tilde{X}_{i,j} = \frac{N}{2} \}$ and the set-valued variables $\tilde{S}_i = \{ j \in A \mid \tilde{X}_{i,j} = 1 \}$. It is straightforward to check that the conditional law of $(\tilde{S}_a, \tilde{S}_b)$ given $\tilde{L}_a \cap \tilde{L}_b$ under \mathbb{P} is the same as the law of (S_a, S_b) under \mathbb{P}_N . It follows that

$$\begin{aligned} \mathbb{P}_N \left(\left| |S_a \cap S_b| - \frac{1}{4}N \right| \geq \gamma N \right) &= \mathbb{P} \left(\left| |\tilde{S}_a \cap \tilde{S}_b| - \frac{1}{4}N \right| \geq \gamma N \mid \tilde{L}_a \cap \tilde{L}_b \right) \\ &\leq \frac{\mathbb{P} \left(\left| |\tilde{S}_a \cap \tilde{S}_b| - \frac{1}{4}N \right| \geq \gamma N \right)}{\mathbb{P} \left(\tilde{L}_a \cap \tilde{L}_b \right)}. \end{aligned}$$

Using Hoeffding inequality, we have

$$\mathbb{P} \left(\left| |\tilde{S}_a \cap \tilde{S}_b| - \frac{1}{4}N \right| \geq \gamma N \right) = \mathbb{P} \left(\left| \sum_{j \in A} \tilde{X}_{a,j} \tilde{X}_{b,j} - \frac{1}{4}N \right| \geq \gamma N \right) \leq 2e^{-2\gamma^2 N}.$$

On the other hand, using Stirling approximation³, we have

$$\mathbb{P}\left(\tilde{L}_a \cap \tilde{L}_b\right) = \left(\frac{1}{2^N} \frac{N!}{\left(\frac{N!}{2}\right)^2}\right)^2 \geq \left(\frac{2^{N+1} N^{-\frac{1}{2}}}{2^N e^2}\right)^2 = \frac{4}{Ne^4}.$$

We deduce that $\mathbb{P}_N\left(\left||S_a \cap S_b| - \frac{1}{4}N\right| \geq \gamma N\right) \leq \frac{1}{2}e^4 Ne^{-2\gamma^2 N}$. \square

Lemma 4. *For each $a \neq b$, for any subset $S \subseteq A$ and any $\gamma \geq \frac{1}{2N-2}$,*

$$\mathbb{P}_N\left(\left|\sum_{i \in S} X_{i,a} - \frac{1}{2}|S|\right| \geq \gamma N\right) \leq 2e^{-2N\gamma^2}, \text{ and } \mathbb{P}_N\left(\left|\sum_{i \in S} X_{i,a}X_{i,b} - \frac{1}{4}|S|\right| \geq \gamma N\right) \leq 2e^{-\frac{1}{2}N\gamma^2}.$$

Proof. For the first inequality, notice that $X_{i,a}$ are i.i.d. Bernoulli random variables with parameter $\frac{1}{2}$. The Hoeffding inequality implies that :

$$\mathbb{P}_N\left(\left|\sum_{i \in S} X_{i,a} - \frac{1}{2}|S|\right| \geq \gamma N\right) \leq 2e^{-2\gamma^2 \frac{N^2}{|S|}} \leq 2e^{-2N\gamma^2}.$$

For the second inequality, let $Z_i = X_{i,a}X_{i,b}$. Notice that all variables Z_i are i.i.d. Bernoulli random variables with parameter $p = \frac{1}{2} \left(\frac{\frac{N}{2}-1}{N-1}\right) = \frac{1}{4} - \frac{1}{4N-4}$. The Hoeffding inequality implies that

$$\begin{aligned} \mathbb{P}_N\left(\left|\sum_{i \in S} Z_i - \frac{1}{4}|S|\right| \geq \gamma N\right) &\leq \mathbb{P}_N\left(\left|\sum_{i \in S} Z_i - p|S|\right| \geq \frac{1}{2}\gamma N\right) \\ &\leq 2e^{-2\gamma^2 \frac{N^2}{|S|}} \leq 2e^{-\frac{1}{2}N\gamma^2}, \end{aligned}$$

where we used that $|S||p - \frac{1}{4}| \leq \frac{N}{4N-4} \leq \frac{\gamma N}{2}$ for the first inequality. \square

Definition 13. *For each $a \neq b$ and $c \neq d$, each $\gamma > 0$, define :*

$$\begin{aligned} Y_a &= 2 \sum_{i \in A} X_{i,a}, & Y^c &= 2 \sum_{i \in A} X_{c,i} = N, \\ Y_{a,b} &= 4 \sum_{i \in A} X_{i,a}X_{i,b}, & Y_a^c &= 4 \sum_{i \in A} X_{i,a}X_{c,i}, & Y^{c,d} &= 4 \sum_{i \in A} X_{c,i}X_{d,i}, \\ Y_{a,b}^c &= 8 \sum_{i \in A} X_{i,a}X_{i,b}X_{c,i}, & Y_a^{c,d} &= 8 \sum_{i \in A} X_{i,a}X_{c,i}X_{d,i}, & Y_{a,b}^{c,d} &= 16 \sum_{i \in A} X_{i,a}X_{i,b}X_{c,i}X_{d,i}. \end{aligned}$$

Lemma 5. *For each $a \neq b$ and $c \neq d$, each $\gamma \geq 64/N$, each of the variables $Z \in \{Y_a, Y^c, Y_{a,b}, Y^{c,d}, Y_a^c, Y_{a,b}^c, Y_a^{c,d}, Y_{a,b}^{c,d}\}$,*

$$\mathbb{P}_N(|Z - N| \geq \gamma N) \leq e^4 Ne^{-\frac{N}{32} \left(\frac{\gamma}{10}\right)^2}.$$

³We have $n^{n+\frac{1}{2}}e^{-n} \leq n! \leq en^{n+\frac{1}{2}}e^{-n}$ for each n .

Proof. In case $Z = Y_a$ or $Y_{a,b}$, the bound follows from Lemma 4 (for $S = A$). If case $Z = Y^c$, the bound is trivially satisfied. If $Z = Y^{c,d}$, the bound follows from Lemma 3.

In case $Z = Y_{a,b}^{c,d}$, notice that

$$Y_{a,b}^{c,d} = 16 \sum_{i \in S_c \cap S_d} Z_i, \quad \text{where } Z_i = X_{i,a} X_{i,b}.$$

All variables Z_i are i.i.d. Bernoulli random variables with parameter $p = \frac{1}{4} - \frac{1}{4N-4}$. Moreover, $\{Z_i\}_{i \neq c,d}$ are independent of $S_c \cap S_d$. Up to enlarge the probability space, we can construct a new collection of i.i.d. Bernoulli random variables Z'_i such that $Z'_i = Z_i$ for all $i \neq c, d$ and such that $\{(Z'_i)_{i \in A}, S_c \cap S_d\}$ are all independent. Then,

$$\left| Y_{a,b}^{c,d} - 16 \sum_{i \in S_c \cap S_d} Z'_i \right| \leq 32,$$

and, because $\frac{1}{2}\gamma N \geq 32$, we have

$$\mathbb{P}_N \left(\left| Y_{a,b}^{c,d} - N \right| \geq \gamma N \right) \leq \mathbb{P}_N \left(\left| \sum_{i \in S_c \cap S_d} Z'_i - \frac{1}{16} N \right| \geq \frac{1}{32} \gamma N \right).$$

Define the events

$$A = \left\{ \left| \frac{1}{4} |S_c \cap S_d| - \frac{N}{16} \right| \geq \frac{1}{160} \gamma N \right\}, \quad B = \left\{ \left| \sum_{i \in S_c \cap S_d} Z'_i - \frac{1}{4} |S_c \cap S_d| \right| \geq \frac{1}{40} \gamma N \right\}.$$

Then, the probability can be further bounded by

$$\leq \mathbb{P}_N(A) + \mathbb{P}_N(B) \leq \frac{1}{2} e^4 N e^{-2N(\frac{1}{40}\gamma)^2} + 2e^{-\frac{1}{2}N(\frac{1}{40}\gamma)^2} \leq e^4 N e^{-\frac{N\gamma^2}{3200}}$$

where the first bound comes from Lemma 3, and the second from the second bound in Lemma 4.

The remaining bounds have proofs similar (and simpler) to the case $Z = Y_{a,b}^{c,d}$. \square

Finally, we describe an event E that collects these bounds. Recall that $\alpha = 1/25$, and define for each $a \neq b$ and $c \neq d$,

$$\begin{aligned} E_{a,b,c,d} &= \left\{ \left| \frac{Y_{a,b}}{Y_a} - 1 \right| \leq 2\alpha \right\} \cap \left\{ \left| \frac{Y_{a,b}^c}{Y_a^c} - 1 \right| \leq 2\alpha \right\} \cap \left\{ \left| \frac{Y_{a,b}^{c,d}}{Y_a^{c,d}} - 1 \right| \leq 2\alpha \right\} \cap \left\{ \left| \frac{Y_{a,b}^{c,d}}{Y_a^{c,d}} - 1 \right| \leq 2\alpha \right\} \\ &\quad \cap \left\{ \left| \frac{Y_{a,b}^{c,d}}{Y^c} - 1 \right| \leq 2\alpha \right\} \cap \left\{ \left| \frac{Y_{a,b}^c}{Y^c} - 1 \right| \leq 2\alpha \right\} \cap \left\{ \left| \frac{Y_{a,b}^{c,d}}{Y^{c,d}} - 1 \right| \leq 2\alpha \right\}. \end{aligned}$$

Finally, let

$$E = \bigcap_{a,b,c,d: a \neq b \text{ and } c \neq d} E_{a,b,c,d}.$$

Lemma 6. *We have*

$$\mathbb{P}_N(E) > 1 - 7e^4 N^5 e^{-\frac{N}{2163200}} \xrightarrow{n \rightarrow \infty} 1.$$

Proof. Take $\gamma = \frac{\alpha}{1+\alpha} = \frac{1}{26}$ and let

$$F_{a,b,c,d} = \bigcap_{Z \in \{Y_a, Y_{a,b}, Y^{c,d}, Y^{c,d}, Y_a^c, Y_{a,b}^c, Y_a^{c,d}, Y_{a,b}^{c,d}\}} \{|Z - N| \leq \gamma N\}.$$

It is easy to see that $F_{a,b,c,d} \subseteq E_{a,b,c,d}$. The probability that $F_{a,b,c,d}$ holds can be bounded from Lemma 5 (as soon as $N \geq \frac{64}{\gamma} = 1664$), as

$$\mathbb{P}_N(F_{a,b,c,d}) \geq 1 - 7e^4 N e^{-\frac{N}{32 \cdot (260)^2}}.$$

The result follows since there are less than N^4 ways of choosing (a, b, c, d) . \square

Computations using the bound of lemma 6 show that $N = 52 \cdot 10^6$ is enough to have the existence of an appropriate Markov chain. So one can take $\varepsilon = 3 \cdot 10^{-17}$ in the statement of theorem 3. We conclude the proof of proposition 2 by showing that event E implies conditions $UI1$ and $UI2$.

Lemma 7. *If event E holds, then the conditions $UI1, UI2$ are satisfied.*

Proof. We fix the law ν of the Markov chain on A and assume that it has been induced, as explained at the beginning of section 6.3, by a transition matrix P satisfying E . For $l \geq 1$, we forget about the state in K and still denote by u^l the marginal of u^l over $C^l \times D^l$. If $c = (c_1, \dots, c_l) \in C^l$ and $d = (d_1, \dots, d_l) \in D^l$, we have $u^l(c, d) = \nu(c_1, d_1, \dots, c_l, d_l)$.

Let us begin with condition $UI2$ which we recall here: for all $1 \leq p \leq l$, for all $d \in D^l$, for all $d' \in D^{p-1}$, for all $m \in \{1, \dots, p-1\}$ such that $d_m \neq d'_m$, for $r = 2m-1, 2m$,

$$u^l(\tilde{c} \smile_{r+1} d' | \tilde{d} = d, \tilde{c} \smile_r d') \in [1/2 - \alpha, 1/2 + \alpha], \quad (6.3)$$

where (\tilde{c}, \tilde{d}) is a random variable selected according to u^l . The quantity $u^l(\tilde{c} \smile_{r+1} d' | \tilde{d} = d, \tilde{c} \smile_r d')$ is thus the conditional probability of the event $(\tilde{c}$ and d' are nice at level $r+1$) given that they are nice at level r and that the signal received by player 2 is d . We divide the problem into different cases.

Case $m > 1$ and $r = 2m-1$.

Note that the events $\{\tilde{c} \smile_{2m} d'\}$ and $\{\tilde{c} \smile_{2m-1} d'\}$ can be decomposed as follows :

$$\begin{aligned} \{\tilde{c} \smile_{2m-1} d'\} &= \{\tilde{c} \smile_{2m-2} d'\} \cap \{X_{d'_{m-1}, \tilde{c}_m} = 1\}, \\ \{\tilde{c} \smile_{2m} d'\} &= \{\tilde{c} \smile_{2m-2} d'\} \cap \{X_{d'_{m-1}, \tilde{c}_m} = 1\} \cap \{X_{\tilde{c}_m, d'_m} = 1\}. \end{aligned}$$

So $u^l \left(\tilde{c} \smile_{2m} d' | \tilde{d} = d, \tilde{c} \smile_{2m-1} d' \right) = u^l \left(X_{\tilde{c}_m, d'_m} = 1 | \tilde{d} = d, \tilde{c} \smile_{2m-1} d' \right)$, and the Markov property gives:

$$\begin{aligned} u^l \left(\tilde{c} \smile_{2m} d' | \tilde{d} = d, \tilde{c} \smile_{2m-1} d' \right) &= u^l \left(X_{\tilde{c}_m, d'_m} = 1 | X_{d'_{m-1}, \tilde{c}_m} = 1, X_{d_{m-1}, \tilde{c}_m} = 1, X_{\tilde{c}_m, d_m} = 1 \right), \\ &= \frac{\sum_{i \in U} X_{i, d'_m} X_{d'_{m-1}, i} X_{d_{m-1}, i} X_{i, d_m}}{\sum_{i \in U} X_{d'_{m-1}, i} X_{d_{m-1}, i} X_{i, d_m}}. \end{aligned}$$

This is equal to $\frac{1}{2} \frac{Y_{d_m, d'_m}^{d_{m-1}, d'_{m-1}}}{Y_{d_m}^{d_{m-1}, d'_{m-1}}}$ if $d'_{m-1} \neq d_{m-1}$, and to $\frac{1}{2} \frac{Y_{d_m, d'_m}^{d_{m-1}, d'_{m-1}}}{Y_{d_m}^{d_{m-1}, d'_{m-1}}}$ if $d'_{m-1} = d_{m-1}$. In both cases,

E implies (6.3).

Case $r = 2m$.

We have $u^l \left(\tilde{c} \smile_{2m+1} d' | \tilde{d} = d, \tilde{c} \smile_{2m} d' \right) = u^l \left(X_{d'_m, \tilde{c}_{m+1}} = 1 | \tilde{d} = d, \tilde{c} \smile_{2m} d' \right)$, and by the Markov property :

$$\begin{aligned} u^l \left(\tilde{c} \smile_{2m+1} d' | \tilde{d} = d, \tilde{c} \smile_{2m} d' \right) &= u^l \left(X_{d'_m, \tilde{c}_{m+1}} = 1 | X_{d_m, \tilde{c}_{m+1}} = 1, X_{\tilde{c}_{m+1}, d_{m+1}} = 1 \right), \\ &= \frac{\sum_{i \in U} X_{d'_m, i} X_{d_m, i} X_{i, d_{m+1}}}{\sum_{i \in U} X_{d_m, i} X_{i, d_{m+1}}} \\ &= \frac{1}{2} \frac{Y_{d_{m+1}}^{d'_m, d_m}}{Y_{d_{m+1}}^{d_m}} \in [1/2 - \alpha, 1/2 + \alpha]. \end{aligned}$$

Case $m = 1, r = 1$.

$$\begin{aligned} u^l \left(\tilde{c} \smile_2 d' | \tilde{d} = d, \tilde{c} \smile_1 d' \right) &= u^l \left(\tilde{c} \smile_2 d' | \tilde{d} = d \right), \\ &= u^l \left(X_{\tilde{c}_1, d'_1} = 1 | X_{\tilde{c}_1, d_1} = 1 \right), \\ &= \frac{\sum_{i \in U} X_{i, d'_1} X_{i, d_1}}{\sum_{i \in U} X_{i, d_1}} \\ &= \frac{1}{2} \frac{Y_{d_1, d'_1}}{Y_{d_1}} \in [1/2 - \alpha, 1/2 + \alpha]. \end{aligned}$$

Let us now consider condition UI1: we require that for all $l \geq 1$, all $c = (c_1, \dots, c_l)$ in C^l and $c' = (c'_1, \dots, c'_{l+1})$ in C^{l+1} such that $c_1 = c'_1$, we have

$$u^l \left(c' \smile_{2l+1} \tilde{d} | \tilde{c} = c, c' \smile_{2l} \tilde{d} \right) \in [1/2 - \alpha, 1/2 + \alpha] \quad (6.1)$$

and for all $m \in \{1, \dots, l\}$ such that $c_m \neq c'_m$, for $r = 2m - 2, 2m - 1$,

$$u^l \left(c' \smile_{r+1} \tilde{d} | \tilde{c} = c, c' \smile_r \tilde{d} \right) \in [1/2 - \alpha, 1/2 + \alpha]. \quad (6.2)$$

We start with (6.1).

$$\begin{aligned}
u^l \left(c' \smile_{2l+1} \tilde{d} \mid \tilde{c} = c, c' \smile_{2l} \tilde{d} \right) &= u^l \left(X_{\tilde{d}_l, c'_{l+1}} = 1 \mid \tilde{c} = c, c' \smile_{2l} \tilde{d} \right), \\
&= u^l \left(X_{\tilde{d}_l, c'_{l+1}} = 1 \mid X_{c_l, \tilde{d}_l} = 1, X_{c'_l, \tilde{d}_l} = 1 \right), \\
&= \frac{\sum_{i \in V} X_{i, c'_{l+1}} X_{c_l, i} X_{c'_l, i}}{\sum_{i \in V} X_{c_l, i} X_{c'_l, i}}.
\end{aligned}$$

This is $\frac{1}{2} \frac{Y_{c_l, c'_l}}{Y_{c_l, c'_l}}$ if $c'_l \neq c_l$, and $\frac{1}{2} \frac{Y_{c_l, c'_l}}{Y_{c_l, c'_l}}$ if $c'_l = c_l$. In both cases, (6.1) holds.

We finally consider (6.2) and distinguish several case.

Case $r = 2m - 1$ and $m = l$.

$$\begin{aligned}
u^l \left(c' \smile_{2l} \tilde{d} \mid \tilde{c} = c, c' \smile_{2l-1} \tilde{d} \right) &= u^l \left(X_{c'_l, \tilde{d}_l} = 1 \mid \tilde{c} = c, c' \smile_{2l-1} \tilde{d} \right), \\
&= u^l \left(X_{c'_l, \tilde{d}_l} = 1 \mid X_{c_l, \tilde{d}_l} = 1 \right), \\
&= \frac{\sum_{i \in V} X_{c'_l, i} X_{c_l, i}}{\sum_{i \in V} X_{c_l, i}}, \\
&= \frac{1}{2} \frac{Y_{c'_l, c_l}}{Y_{c_l, c'_l}} \in [1/2 - \alpha, 1/2 + \alpha].
\end{aligned}$$

Case $r = 2m - 1$ and $m < l$.

$$\begin{aligned}
u^l \left(c' \smile_{2m} \tilde{d} \mid \tilde{c} = c, c' \smile_{2m-1} \tilde{d} \right) &= u^l \left(X_{c'_m, \tilde{d}_m} = 1 \mid \tilde{c} = c, c' \smile_{2m-1} \tilde{d} \right), \\
&= u^l \left(X_{c'_m, \tilde{d}_m} = 1 \mid X_{c_m, \tilde{d}_m} = 1, X_{\tilde{d}_m, c_{m+1}} = 1 \right), \\
&= \frac{\sum_{i \in V} X_{c'_m, i} X_{c_m, i} X_{i, c_{m+1}}}{\sum_{i \in V} X_{c_m, i} X_{i, c_{m+1}}}, \\
&= \frac{1}{2} \frac{Y_{c'_m, c_m}^{c'_m, c_m}}{Y_{c_m, c'_m}^{c_m, c'_m}} \in [1/2 - \alpha, 1/2 + \alpha].
\end{aligned}$$

Case $r = 2m - 2$.

$$\begin{aligned}
u^l \left(c' \smile_{2m-1} \tilde{d} \mid \tilde{c} = c, c' \smile_{2m-2} \tilde{d} \right) &= u^l \left(X_{\tilde{d}_{m-1}, c'_m} = 1 \mid \tilde{c} = c, c' \smile_{2m-1} \tilde{d} \right), \\
&= u^l \left(X_{\tilde{d}_{m-1}, c'_m} = 1 \mid X_{c'_{m-1}, \tilde{d}_{m-1}} = X_{c_{m-1}, \tilde{d}_{m-1}} = X_{\tilde{d}_{m-1}, c_m} = 1 \right), \\
&= \frac{\sum_{i \in V} X_{i, c'_m} X_{i, c_m} X_{c'_{m-1}, i} X_{c_{m-1}, i}}{\sum_{i \in V} X_{i, c_m} X_{c'_{m-1}, i} X_{c_{m-1}, i}}.
\end{aligned}$$

This is $\frac{1}{2} \frac{Y_{c'_m, c_m}^{c'_{m-1}, c_{m-1}}}{Y_{c'_m}^{c'_{m-1}, c_{m-1}}}$ if $c_{m-1} \neq c'_{m-1}$, and $\frac{1}{2} \frac{Y_{c'_m}^{c_{m-1}, c_m}}{Y_{c'_m}^{c_{m-1}, c_m}}$ if $c_{m-1} = c'_{m-1}$. In both cases, it belongs to $[1/2 - \alpha, 1/2 + \alpha]$, concluding the proofs of lemma 7, proposition 2 and theorem 3. \square

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