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"A general theory of risk apportionment"

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A general theory of risk apportionment

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Abstract

Suppose that the conditional distributions of $\tilde{x}$ (resp. $\tilde{y}$) can be ranked according to the m-th (resp. n-th) risk order. Increasing their statistical concordance increases the (m, n) degree riskiness of $(\tilde{x}, \tilde{y})$, i.e., it reduces expected utility for all bivariate utility functions whose sign of the (m, n) cross-derivative is $(-1)^{m+n+1}$. This means in particular that this increase in concordance of risks induces a $m+n$ degree risk increase in $\tilde{x}+\tilde{y}$. On the basis of these general results, I provide different recursive methods to generate high degrees of univariate and bivariate risk increases. In the reverse-or-translate (resp. reverse-or-spread) univariate procedure, a m degree risk increase is either reversed or translated downward (resp. spread) with equal probabilities to generate a $m+1$ (resp. $m+2$) degree risk increase. These results are useful for example in asset pricing theory when the trend and the volatility of consumption growth are stochastic or statistically linked.

Keywords: Stochastic dominance, risk orders, prudence, temperance, concordance.

JEL codes: D81

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1 Introduction

The theory of stochastic dominance has been developed almost five decades ago initially to determine the conditions under which all (risk-averse) individuals dislike a specific change in risk.[1] Rothschild and Stiglitz (1970) have characterized the second-degree risk increase through either an integral condition and a more intuitive approach using the concept of mean-preserving spreads. For any interger m, Ekern (1980) has more generally defined the m-th degree risk increase as any change in risk that is disliked by any von Neumann-Morgenstern individual whose m-th derivative has the same sign as $(-1)^{m+1}$. For $m = 3$ and $m = 4$, this corresponds to the notions of prudence and temperance, respectively. Ekern characterizes the m-th degree riskiness through an integral condition, but he failed to provide an intuitive approach associated to it. For a long time, economists knew little about the m-th degree risk increase beyond its integral condition. A breakthrough came when citeeeckhoudtschlesinger2006 and Eeckhoudt et al. (2009) found a way to decompose a higher-degree risk increase into lower-degree risk increases through a process known as risk apportionment. The risk apportionment approach facilitates intuitive understanding of m-th degree risk increases in terms of the well-understood first-degree risk increases (leftward shifts in the probability mass) and/or second-degree risk increases (mean-preserving spreads). To illustrate, suppose that $\tilde{x}(2)$ is a m-th degree risk increase of $\tilde{x}(1)$, and $\tilde{y}(2)$ is a n-th degree risk increase of $\tilde{y}(1)$, then the 50-50 percent lottery $[\tilde{x}(1) + \tilde{y}(1), \tilde{x}(2) + \tilde{y}(2)]$ is a $(m+n)$-th degree risk increase of the 50-50 percent lottery $[\tilde{x}(1) + \tilde{y}(2), \tilde{x}(2) + \tilde{y}(1)]$. In short, when two versions of $\tilde{x}$ and $\tilde{y}$ must be combined in two equally likely states, it is always safer to apportion the riskier version of these two lotteries in different states, thereby compounding more risk with less risk. This simple concept of risk apportionment provides an intuitive approach to high orders of risk increases, as did the concept of mean-preserving spreads for second-degree risk increases.

My Theorem 2 in this paper generalizes this result in several directions. It uses the concept of increasing concordance introduced by Tchen (1980). An increase in concordance between two random variables is a change in their joint probability distribution that increases their correlation without changing their marginal distributions. Suppose that the m-th degree riskiness of $\tilde{x}$ and the n-th degree riskiness of $\tilde{y}$ are uncertain. This is modeled by assuming that $\tilde{x}$ is parametrized by θ, and that an increase in θ implies a m-th degree risk increase of $\tilde{x}(\theta)$, which denotes the random variable that is distributed as $\tilde{x}$ conditional on $\tilde{\theta} = \theta$. Similarly, assume that $\tilde{y}$ is parametrized by η, and that an increase in η implies a n-th degree risk increase of $\tilde{y}(\eta)$. The uncertainty affecting the riskiness of these two random variables is represented by a joint distribution function for $(\tilde{\theta}, \tilde{\eta})$. Our Theorem 2 states that increasing the concordance of this pair of random variables raises the $(m+n)$-th degree riskiness of $\tilde{x} + \tilde{y}$.

The basic idea conveyed in the recent literature on risk apportionment is that making different harms more statistically concordant deteriorates welfare ex ante. But the result by Eeckhoudt et al. (2009) is a special case of this idea because it is limited to two states for $(\tilde{\theta}, \tilde{\eta})$, equal state probabilities and perfect (negative or positive) correlations. In this

[1]See for example Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970), Menezes et al. (1980) and Whitmore (1989).

paper, I remove these appendages of the existing theory of risk apportionment to empower the initial idea that increasing the concordance in riskiness (rather than harms) deteriorate welfare. This generates more powerful and general results, as shown in my theorems 1 and 2, and in the various applications developed in this paper. I also go one step further in the generalization of the theory by extending it to non-monetary outcomes, in the spirit of Richard (1975), Tchen (1980), Epstein and Tanny (1980), Eeckhoudt et al. (2007) and Meyer and Strulovici (2013).[2]

Generalizing a theory has costs and benefits. The main cost is to make the result more complex and therefore potentially less intuitive. The impressive success of Eeckhoudt and Schlesinger (2006) comes precisely from the simplicity and elegance of their analysis, that is yet strong enough to yield the expected utility equivalence result. I preserve this elegance by exhibiting different recursive methods to build high order risk increases. However, one must recognize at the same time that the restrictive assumptions mentioned in this introduction have limited their applicability to other fields. More than ten years after the first publications of this new theory, researchers in finance and macroeconomics still fail to take notice of it. This paper offers an attempt to break this standoff.

The asset pricing literature on long run risks pioneered by Bansal and Yaron (2004) provides an interesting field of applications for the decision-theoretic results presented in this paper. For example, as is well-known, the persistence to the shocks affecting the trend of consumption growth magnifies the long run risk, and is an illustration of the fact that increasing the concordance between two first degree risks raises the second degree risk of their sum. A less well understood phenomenon emerges when persistent shocks to the volatility of growth are introduced, which helps solving the equity premium puzzle. In fact, this is an illustration of the result that introducing positive concordance to two second degree risks increases the fourth degree riskiness of their sum. As already shown by Gollier (2018), a positive serial correlation in volatility raises the equity premium only if the representative agent is temperant, i.e., has a utility function whose fourth derivative is negative. Following Tinang (2017), we can also examine the case of introducing a negative concordance between the trend and the volatility of consumption growth, which can be shown to yield third degree risk increase of future consumption.[3] In another direction, this new literature on high degree risk orders has been useful to determine the impact of a change in risk on optimal decisions such as saving, portfolio, insurance and self-protection choices.[4]

These results provide a simple iterative procedure to test for the sign of any derivative of the utility function. For example, a $(m+1)$-th degree risk increase can be obtained from a m-th degree risk increase though a simple "reverse-or-translate" randomized actions: with probability 0.5, one reverses it, and with probability 0.5, one translates it to smaller wealth levels. This is an application of Theorem 2 with $n = 1$, where the first degree risk increase on $\tilde{y}$ takes the form of a sure reduction of wealth. For example, as shown by Chiu (2005)

[2]See also Atkinson and Bourguignon (1982) and Moyes (2012) for a definition of concordance in social choice theory. As is well-known, there is an equivalence between the concepts of stochastic dominance and of inequality measures when the social welfare function is utilitarian, so that our results can easily be translated into that field of applications.

[3]There are a few papers which build a bridge between high degrees of risk increase and asset prices. For example, Jokung (2013) gives sufficient conditions for a m-th degree risk increase in aggregate consumption to reduce the price of equity at equilibrium in a Lucas economy.

[4]See for example Denuit et al. (2011), Jouini et al. (2013), Denuit et al. (2013), Liu (2014) and Nocetti (2016).

in case $m = 1$, a simple mean-preserving spread can be obtained by a "reverse-or-translate" procedure that randomizes a simple first-degree risk reduction at some wealth level (i.e. an upward transfer of probability mass) with the symmetric first-degree risk increase at some lower wealth level. Using this second-degree risk increase as the new basic ingredient, a third-degree risk increase can be obtained from a new "reverse-or-translate" randomization of its reversal at some wealth level and of its implementation at some lower level. This recursive method is not new, but it has been obscured in the literature by a misleading terminology. In Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009) for example, the qualifier "good" or "bad" was used to describe the m-th change in risk, a terminology that requires knowing the sign of the m-th derivative of the utility function. This means that the recursive method required knowing the sign of all derivatives up to $m+1$ to generate a meaningful result. In reality, Theorem 2 shows that one should replace the qualifier "good or "bad" by "safer" or "riskier" in the sense of the i-th order, a solution that abstains us from restricting the sign of the successive derivatives of the utility function to perform the recursive method described here. In short, high orders of risk aversion are about a preference for the disaggregation of risks, not about the disaggregation of bads.[5]

Theorem 2 can also be applied recursively with $n = 2$, thereby allowing to generate all odd (starting with $m = 1$) and even (starting with $m = 2$) orders of risk. It uses a randomization procedure that I call "reverse-or-spread" that randomizes between a reversal of a risk increase and its spread. For example, a fourth-degree risk increase can be obtained from a second-degree risk increase through a "reverse-or-spread" procedure that randomizes between its reversal at some wealth level and its spread around that wealth. We illustrate the randomized reverse-or-spread procedure in Table 1, starting from the bottom with a mean-preserving spread that transfers a probability mass p from outcome 0 to outcome -1 and +1 in a symmetric way. I build a fourth degree risk increase by combining a reversal and a spread of this second degree risk increase. With probability 1/2, this spread is reversed, which leads to a transfer of probability mass described in row "reverse" of $m = 4$. With probability 1/2, the initial mean-preserving spread is spread, with a probability 1/2 to be translated by 1 to the left, and with probability 1/2 to be translated by 1 to the right. This leads to a transfer of probability mass described in row "spread" of $m = 4$. The net transfer of probability mass is described in row "total", which describes a fourth degree risk increase. The iteration to $m = 6$ is also described in this table. As an illustration, this implies that lottery $(-3, 1/32; -1, 15/32; +1, 15/32; +3, 1/32)$ is sixth-degree riskier than lottery $(-2, 3/16; 0, 10/16; +2, 3/16)$.

[5] For a discussion, see Crainich et al. (2013), Ebert (2013) and Ebert et al. (2019).

outcome		-3	-2	-1	0	+1	+2	+3
m=6	total	p/32	-3p/16	15p/32	-10p/16	15p/32	-3p/16	p/32
	spread	p/32	-p/8	3p/16+p/32	-p/8-p/8	p/32+3p/16	-p/8	p/32
	reversal		-p/16	p/4	-3p/8	p/4	-p/16	
m=4	total		p/8	-p/2	3p/4	-p/2	p/8	
	spread		p/8	-p/4	p/8+p/8	-p/4	p/8	
	reversal			-p/4	p/2	-p/4		
m=2	total			p/2	-p	p/2		

Table 1: Example of m-th degree risk increases for $m \in \{2, 4, 6\}$ using the randomized reverse-or-spread procedure.

In Section 2, I summarize the basic concepts and properties of m-th degree risk orders and of comparative concordance. Section 3 is devoted to the derivation of the basic results of the paper. In Section 4, I describe some new applications of Theorem 2 in the Gaussian world. Based on these results, sections 5 and 6 are devoted to the development of recursive procedures to construct high orders of risk increases, respectively in the univariate and bivariate cases.

2 Preliminaries

In this paper, we combine two classical tools from stochastic dominance theory: m-th degree increases in risk, and statistical concordance. Let us first define a m-th degree risk increase. Consider a pair $(\tilde{x}_1, \tilde{x}_2)$ of random variables characterized by cumulative distribution functions (F_1, F_2) whose support is in $[a, b]$. To any cdf F_i, we can associate a family of functions $(F_i^1, F_i^2, F_i^3, ...)$ that are defined recursively as follows: $F_i^1 \equiv F_i$ and $\forall x \in [a, b]$, $\forall k \in \mathbb{N}_0$:

$$F_i^{k+1}(x) = \int_a^x F_i^k(t)dt. \tag{1}$$

The following proposition, in which $v^{(m)}$ denotes the m-th order derivative of v, is due to Ekern (1980).

Proposition 1. *(Ekern (1980)) Consider any integer $m \in \mathbb{N}_0$. The following statements are two equivalent definitions of $\tilde{x}_2$ being m-th degree riskier than $\tilde{x}_1$, i.e., $\tilde{x}_2 \succeq_m \tilde{x}_1$:*

- *The cdfs of $\tilde{x}_1$ and $\tilde{x}_2$ satisfy the following integral conditions:*

$$F_2^k(b) \quad = F_1^k(b) \quad \text{for } k = 1, 2, ..., m, \tag{2}$$

$$F_2^m(x) \quad \geq F_1^m(x) \quad \forall x \in [a, b]. \tag{3}$$

- *For any real-valued function v such that the sign of $v^{(m)}$ is $(-1)^{m+1}$, we have that $Ev(\tilde{x}_2)$ is smaller than $Ev(\tilde{x}_1)$.*

Equation (2) means that the first $m-1$ moments of $\tilde{x}_1$ and $\tilde{x}_2$ are equal, whereas equation (3) implies that the m-th moment of $\tilde{x}_2$ is larger (smaller) than the m-th moment of $\tilde{x}_1$ if

m is even (odd). In fact, the change in the expectation of v is obtained by m successive integrations by parts which yields the following property, assuming (2):

$$Ev(\tilde{x}_2) - Ev(\tilde{x}_1) = \int_a^b (-1)^m v^{(m)}(t) \left[F_2^m(t) - F_1^m(t)\right] dt. \tag{4}$$

The equivalence proposition above is the direct consequence of property (4). We say that the individual is m-th degree risk-averse if the m-th derivative of her utility function has the same sign as $(-1)^{m+1}$. Under this terminology, second, third and fourth degrees risk aversion correspond respectively to risk aversion, prudence and temperance. Therefore, the above proposition states that any m-th degree increase in risk is disliked by all m-th degree risk-averters.

We now turn to the analysis of stochastic dominance in the bivariate case. Suppose that the decision-maker extracts utility $u : \mathbb{R}^2 \to \mathbb{R}$ from the consumption of two goods. Ex ante, her expected utility is measured by $Eu(\tilde{x}, \tilde{y})$, where $(\tilde{x}, \tilde{y})$ is the pair of random variables measuring the consumption level of the two goods ex post. By extension of the definition of the univariate notion of m-th degree risk increase, we define the bivariate notion of (m, n) degree risk increase as follows.[6]

Definition 1. *Consider a pair $(m, n) \in \mathbb{N}^2 \setminus \{(0, 0)\}$. A change in the joint distribution of $(\tilde{x}, \tilde{y})$ is said to be a (m, n) degree risk increase if and only if it reduces the expectation of $u(\tilde{x}, \tilde{y})$ for any function $u : \mathbb{R}^2 \to \mathbb{R}$ whose (m, n) cross-derivative has the same sign as $(-1)^{m+n+1}$.*

Obviously, by the law of iterated expectations, the bivariate $(m, 0)$ degree risk increase is obtained when, for all y, $\tilde{x} \mid y$ undergoes a univariate m-th degree risk increase characterized by Proposition 1.

A well-known family of bivariate risk increases introduced in economics by de Finetti (1952), Epstein and Tanny (1980), Tchen (1980) and Atkinson and Bourguignon (1982) corresponds to $(1, 1)$ degree risk increases. For a reason that will be apparent in the next section, let us examine the impact of the change in the joint distribution of $(\tilde{\theta}, \tilde{\eta})$ on $Eg(\tilde{\theta}, \tilde{\eta})$, where g is a bivariate real-valued function. Let H_1 and H_2 represent respectively the initial and final joint probability distribution of $(\tilde{\theta}, \tilde{\eta})$. The following proposition has been independently demonstrated by Tchen (1980), Epstein and Tanny (1980) and Atkinson and Bourguignon (1982). It characterizes the $(1, 1)$ degree risk increase through an integral condition that is similar in spirit to what has been presented in Proposition 1 for $(m, 0)$ degree risk increases.

Proposition 2. *(Tchen (1980), Epstein and Tanny (1980)) Let H_i denote a joint cdf for the pair of random variables $(\tilde{\theta}_i, \tilde{\eta}_i)$, $i = 1, 2$. We have that $(\tilde{\theta}_2, \tilde{\eta}_2)$ is $(1, 1)$ degree riskier, i.e., is "more concordant", than $(\tilde{\theta}_1, \tilde{\eta}_1)$ if and only if H_1 and H_2 have the same marginal distributions and $H_2(\theta, \eta)$ is larger than $H_1(\theta, \eta)$ for all $(\theta, \eta) \in \mathbb{R}^2$.*

From Definition 1, if the two consumption goods θ and η are substitutes in the sense that the cross-derivative of the utility function $g(\theta, \eta)$ is negative, then consumers dislike any (1,1)

[6] We use the standard terminology in which the $(m, 0)$ cross-derivative of a function is just the m-th partial derivative of this function with respect to its first argument, with a symmetric terminology for its second argument.

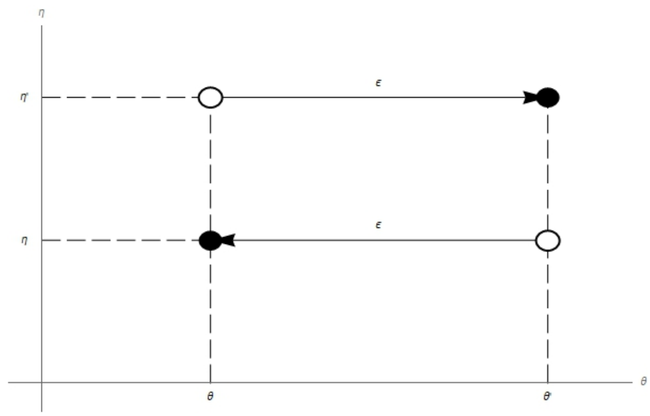

Figure 1: A simple increase in concordance between $\tilde{\theta}$ and $\tilde{\eta}$ expressed as two symmetric transfers of probability mass in the support of this pair of random variables.

risk increase of $(\tilde{\theta}, \tilde{\eta})$. Proposition 2 is a direct consequence of the following property that holds when H_1 and H_2 exhibit the same marginal distributions:

$$Eg(\tilde{\theta}_2, \tilde{\eta}_2) - Eg(\tilde{\theta}_1, \tilde{\eta}_1) = \iint \frac{\partial^2 g}{\partial\theta\partial\eta}(\theta, \eta)\,[H_2(\theta, \eta) - H_1(\theta, \eta)]\,d\theta d\eta. \tag{5}$$

Because it will play a critical role in this paper, we will follow the terminology of Tchen (1980) to refer to a $(1, 1)$ degree risk increase as "more concordance". This terminology is based on the following observation made by Tchen (1980) and Epstein and Tanny (1980): Any increase in concordance between $\tilde{\theta}$ and $\tilde{\eta}$ can be constructed by a sequence of elementary marginal-preserving transfers of probability masses described in Figure 1. Such a change in risk combines a first degree risk reduction of $\tilde{\theta} \mid \eta$ for some η, and the reversed first degree risk increase of $\tilde{\theta} \mid \eta'$ at a larger η'. It undoubtedly raises the statistical correlation of the pair of random variables. Moreover, we see that such an elementary change in the joint risk distribution reduces the expectation of $g(\tilde{\theta}, \tilde{\eta})$ only if

$$\theta' > \theta \text{ and } \eta' > \eta \quad \Rightarrow \quad g(\theta', \eta') - g(\theta', \eta) \leq g(\theta, \eta') - g(\theta, \eta). \tag{6}$$

This condition holds if and only if g is submodular, i.e., iff its cross-derivative is non-positive. This confirms that such a change in the bivariate distribution is an example of $(1, 1)$ degree risk increase. By definition if two random variables are independent, their degree of concordance is normalized to zero. Two random variables exhibit positive (negative) concordance when they are more (less) concordant than between the equivalent pair of independent random variables with the same marginals.

3 The main result

Consider an individual whose welfare ex ante is represented by a von Neumann-Morgenstern utility function $u : [a, b]^2 \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ with two goods whose quantity consumed ex post is measured by x and y, respectively. Ex ante, the uncertainty is quantified by a pair $(\tilde{x}, \tilde{y})$ of random variables. They are statistically related in the following sense. Suppose that $\tilde{x}$ is related to random variable $\tilde{\theta}$ whose support is in $S_\theta \subset \mathbb{R}$. Suppose also that $\tilde{x}(\theta) = \tilde{x} \mid \theta$ can be ranked according to the m-th risk order. More precisely, suppose that for all $(\theta, \theta') \in S_\theta^2$,

$$\theta' > \theta \quad \Rightarrow \quad \tilde{x}(\theta') \succeq_m \tilde{x}(\theta). \tag{7}$$

A larger θ implies a m-th degree risk increase in $\tilde{x} \mid \theta$. Symmetrically, we assume that $\tilde{y}$ is related to random variable $\tilde{\eta}$ whose support is in $S_\eta \subset \mathbb{R}$. Suppose also that $\tilde{y}(\eta) = \tilde{y} \mid \eta$ can be ranked according to the n-th risk order: For all $(\eta, \eta') \in S_\eta^2$,

$$\eta' > \eta \quad \Rightarrow \quad \tilde{y}(\eta') \succeq_n \tilde{y}(\eta). \tag{8}$$

A larger η implies a n-th degree risk increase in $\tilde{y} \mid \eta$. Conditional to (θ, η), the expected utility is equal to

$$G(\theta, \eta) = Eu(\tilde{x}(\theta), \tilde{y}(\eta)). \tag{9}$$

In parallel to the existing literature on risk apportionment, I assume that the statistical dependence in $(\tilde{x}, \tilde{y})$ exists only through the dependence in $(\tilde{\theta}, \tilde{\eta})$. Technically, this means that $\tilde{x}(\theta)$ and $\tilde{y}(\eta)$ are independent for all (θ, η) in $S_\theta \times S_\eta$.[7] We are interested in determining the impact of the concordance in $(\tilde{\theta}, \tilde{\eta})$ on the unconditional expected utility $Eu(\tilde{x}, \tilde{y})$. This is a generalization of the risk apportionment problem initially raised by Eeckhoudt and Schlesinger (2006) in its simplest representation. They intuited that individuals should prefer to apportion relatively good risks with relatively bad risks rather than to apportion them in the opposite way. Eeckhoudt et al. (2009) examined a special case of our framework in which there are only two states that are equally likely, where $(\tilde{\theta}, \tilde{\eta})$ are either perfectly positively or negatively correlated, and where u is additive.

Suppose that $(\tilde{\theta}_2, \tilde{\eta}_2)$ has more concordance than $(\tilde{\theta}_1, \tilde{\eta}_1)$. Observe that this increase in concordance in $(\tilde{\theta}, \tilde{\eta})$ has no effect on their marginal distribution, and therefore no effect on the marginal distributions of $\tilde{x}$ and $\tilde{y}$. Let us measure the increase in expected utility generated by this increase in concordance by ΔU:

$$\begin{aligned} \Delta U &= Eu(\tilde{x}(\tilde{\theta}_2), \tilde{y}(\tilde{\eta}_2)) - Eu(\tilde{x}(\tilde{\theta}_1), \tilde{y}(\tilde{\eta}_1)) \\ &= EG(\tilde{\theta}_2, \tilde{\eta}_2) - EG(\tilde{\theta}_1, \tilde{\eta}_1). \end{aligned} \tag{10}$$

By definition 1 or Tchen (1980), we know that any increase in concordance yields a negative ΔU if and only if G is submodular, i.e., if and only if condition (6) holds for all (θ, η) and (θ', η') belonging to $S_\theta \times S_\eta$. Now, remember that $\tilde{y}(\eta')$ is n-th degree riskier than $\tilde{y}(\eta)$. This implies that we can apply property (4) to both sides of inequality (6) to rewrite it as follows:

$$\int_a^b (-1)^n \left\{ E\left[\frac{\partial^n}{\partial y^n} u(\tilde{x}(\theta'), y)\right] - E\left[\frac{\partial^n}{\partial y^n} u(\tilde{x}(\theta), y)\right] \right\} \left[F^n_{y,\eta'}(y) - F^n_{y,\eta}(y)\right] dy \leq 0, \tag{11}$$

[7]In other words, for any (θ, η), $\tilde{x}$ and $\tilde{y}$ conditional to (θ, η) are statistically independent. The simplest example (with $m = n = 1$) is a pair $\tilde{x}(\theta) = \theta + \tilde{x}$ and $\tilde{y}(\eta) = \eta + \tilde{y}$ in which $\tilde{x}$ and $\tilde{y}$ are independent. Our modeling provides the most general formulation that allows to define the notion of increasing concordance in the degrees of riskiness of two random variables.

where $F_{y,\eta}$ is the cdf of random variable $\tilde{y}(\eta)$. But remember also that $\tilde{x}(\theta')$ is m-th degree riskier than $\tilde{x}(\theta)$. Using property (4) again, the above inequality can be rewritten as

$$\iint_{[a,b]^2} (-1)^{m+n} \frac{\partial^{m+n}}{\partial x^m \partial y^n} u(x,y) \left[F^m_{x,\theta'}(x) - F^m_{x,\theta}(x)\right] \left[F^n_{y,\eta'}(y) - F^n_{y,\eta}(y)\right] dxdy \leq 0. \tag{12}$$

Observe that both bracketed terms in the integrand of the above inequality are uniformly non-negative. This implies that the above condition is satisfied if and only if the sign of the (m,n) cross-derivative of u is $(-1)^{m+n+1}$. To sum up, ΔU is negative if and only if G is submodular, which is in turn true if and only if the sign of the (m,n) cross-derivative of u is $(-1)^{m+n+1}$. This result is summarized in the following theorem.

Theorem 1. *Suppose that θ is an index of m-th degree riskiness of $\tilde{x}(\theta)$, and that η is an index of n-th degree riskiness of $\tilde{y}(\eta)$. The following two statements are equivalent:*

1. *$(\tilde{x}(\tilde{\theta}_2), \tilde{y}(\tilde{\eta}_2))$ is (m,n) degree riskier than $(\tilde{x}(\tilde{\theta}_1), \tilde{y}(\tilde{\eta}_1))$;*

2. *$(\tilde{\theta}_2, \tilde{\eta}_2)$ is more concordant than $(\tilde{\theta}_1, \tilde{\eta}_1)$.*

In other words, increasing the concordance between the m-th degree riskiness of $\tilde{x}$ and the n-th degree riskiness of $\tilde{y}$ reduces the expectation of $u(\tilde{x}, \tilde{y})$ if and only if the (m,n) cross-derivative of u has the same sign as $(-1)^{m+n+1}$. This new result lies at the intersection between the recent developments on risk apportionment and the older literature on statistical concordance. In fact, if we take $\tilde{x}(\theta)$ and $\tilde{y}(\eta)$ to be degenerated at respectively θ and η for all (θ, η) in $[a,b]^2$, our Theorem 1 is equivalent to the result by Richard (1975), Tchen (1980), Epstein and Tanny (1980) and Atkinson and Bourguignon (1982) that is summarized in Proposition 2 above.

In the special case Cobb-Douglas utility functions, this theorem demonstrates that these changes in risk have an unambiguous impact on the co-moments of $(\tilde{x}, \tilde{y})$. Another important special case corresponds to the context in which $u(x,y)$ is a function U of $x+y$. In that case, the (m,n) cross-derivative of u is just the $(m+n)$-th derivative of U. The following theorem is then a direct consequence of Proposition 1 and Theorem 1.

Theorem 2. *Increasing the concordance between the m-th degree riskiness of $\tilde{x}$ and the n-th degree riskiness of $\tilde{y}$ increases the $(m+n)$-th degree riskiness of $\tilde{x} + \tilde{y}$.*

Therefore, increasing the concordance between the m-th degree riskiness of $\tilde{x}$ and the n-th degree riskiness of $\tilde{y}$ does not affect the moments of $\tilde{x} + \tilde{y}$ up to the $m+n-1$ moment. It raises (reduces) the $m+n$ moment if $m+n$ is even (odd). In the remainder of this paper, we first examine the application of this univariate case.

4 Illustrations in the Gaussian world

Let us illustrate Theorem 2 by three examples in which $\tilde{x}$ and $\tilde{y}$ are conditionally normal. Suppose first that $\tilde{x}(\theta)$ is $N(\theta, \sigma_x^2)$ and $\tilde{y}(\eta)$ is $N(\eta, \sigma_y^2)$. This implies that the set $\{\tilde{x}(\theta) | \theta \in S_\theta\}$ can be ranked under FSD, and the same property also holds for $\tilde{y}$. Suppose also that $(\tilde{\theta}, \tilde{\eta})$ is jointly normally distributed with mean 0, with $Var(\tilde{\theta}) = Var(\tilde{\eta}) = \sigma_\theta^2$, and with a correlation coefficient ρ. As is well-known, an increase in ρ yields an increase in concordance between $\tilde{\theta}$

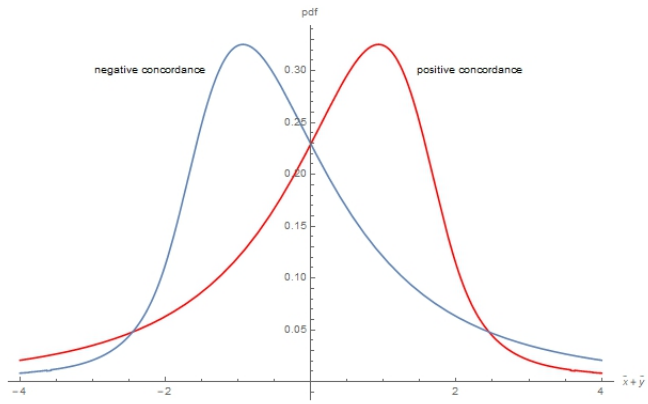

Figure 2: An example of the impact on the distribution of $\tilde{x}+\tilde{y}$ of an increase in concordance between the first degree riskiness of $\tilde{x}$ and the second degree riskiness of $\tilde{y}$. I assume that $\tilde{x}(\theta) = -\theta$ with $\tilde{\theta} \sim N(0,1)$, whereas $\tilde{y}(\eta) \sim N(0,\eta)$ with $\log(\tilde{\eta}) \sim N(0,1)$. The right-skewed function is the density of $\tilde{x}+\tilde{y}$ by assuming $\eta = \exp(-\theta)$ (negative concordance), whereas the left-skewed density function corresponds to the case with $\eta = \exp(\theta)$ (positive concordance).

and $\tilde{\eta}$. This implies that we are in a context in which Theorem 2 can be applied in the case $m = n = 1$: An increase in ρ implies a second degree risk increase in $\tilde{x} + \tilde{y}$. This is indeed the case since $\tilde{x} + \tilde{y}$ is normally distributed with mean 0 and variance $\sigma_x^2 + \sigma_y^2 + 2(1+\rho)\sigma_\theta^2$.

Let us now illustrate the case $m = 1$ and $n = 2$. Suppose that $\tilde{x}(\theta) = -\theta$ and that $\tilde{y}(\eta) \sim N(0,\eta)$. It is clear that an increase in θ or η yields respectively a first degree risk increase in $\tilde{x}(\theta)$ or a second degree risk increase in $\tilde{y}(\eta)$. Observe that conditional to (θ,η), the distribution of $\tilde{x}+\tilde{y}$ is $N(-\theta,\eta)$. Suppose now that θ and $\log(\eta)$ are $N(0,1)$. We consider two structures of correlation between the conditional mean $-\tilde{\theta}$ and the conditional variance $\tilde{\eta}$ of $\tilde{x} + \tilde{y}$. In one case, we have that $\eta = \exp(\theta)$, an extreme form of positive concordance. In the other case, we have that $\eta = \exp(-\theta)$, an extreme form of negative concordance. I draw in Figure 2 the density functions of $\tilde{x}+\tilde{y}$ in these two cases. It is obtained numerically since there is no analytical solution to conjugate the conditional distribution $N(-\theta, \exp(\pm\theta))$ with a normal prior for $\tilde{\theta}$. This example illustrates the fact that increasing the concordance between the first degree risk of $\tilde{x}$ and the second degree risk of $\tilde{y}$ raises the third degree risk of $\tilde{x} + \tilde{y}$, i.e., it raises its downside riskiness. In particular, it reduces its skewness. This is considered as undesirable by prudent (downside-risk-averse) individuals (Menezes et al. (1980)). In relation to these results, Tinang (2017) examines an asset pricing model with long run risks and a negative correlation between shocks to the expected growth rate of consumption and shocks to its volatility. If the representative agent is prudent, this helps solving the equity premium puzzle since the increased downside risk of a claim on aggregate consumption makes it less attractive.

Finally, I hereafter describe an example for $m = n = 2$. Suppose that $\tilde{x}(\theta)$ and $\tilde{y}(\eta)$ are both normally distributed with mean 0 and variance θ and η, respectively. These variances are uncertain. Suppose that $\log(\tilde{\theta})$ and $\log(\tilde{\eta})$ are both $N(0,1)$. I examine an extreme version of increasing concordance between the two second degree risk measures $\tilde{\theta}$ and $\tilde{\eta}$ of $\tilde{x}$ and $\tilde{y}$ respectively. Namely, I assume that the initial relation between the two parameters is

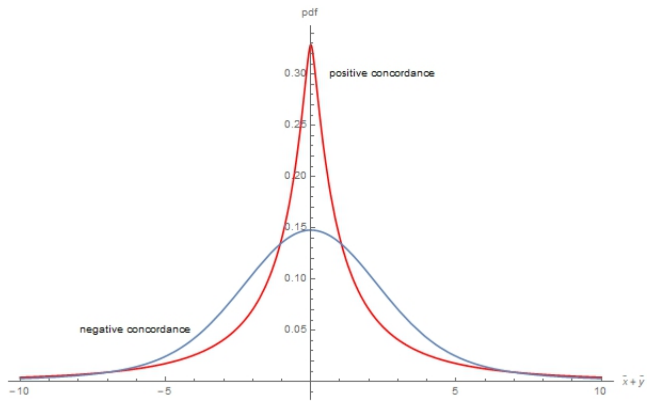

Figure 3: An example of the impact on the distribution of $\tilde{x}+\tilde{y}$ of an increase in concordance in the second degree riskiness of $\tilde{x}$ and $\tilde{y}$. I assume that $\tilde{x}(\theta) \sim N(0,\theta)$ with $\log(\tilde{\theta}) \sim N(0,1)$, whereas $\tilde{y}(\eta) \sim N(0,\eta)$ with $\log(\tilde{\eta}) \sim N(0,1)$. The fat-tailed curve is the density of $\tilde{x}+\tilde{y}$ by assuming $\eta = \theta$ (positive concordance), whereas the other density function corresponds to the case with $\eta = 1/\theta$ (negative concordance).

such that $\log(\theta) = -\log(\eta)$ (negative concordance), which is then shifted to $\log(\theta) = \log(\eta)$ (positive concordance). The initial and final unconditional densities for $\tilde{x}+\tilde{y}$ are drawn in Figure 3, which have been computed numerically in the absence of an analytical solution. They have the same first three moments, but the positive concordance yields a larger kurtosis. In fact, it is a consequence of Theorem 2 that it generates an increase in the fourth degree riskiness of $\tilde{x}+\tilde{y}$. This is disliked by all temperant individuals. An example of positive concordance in second degree riskiness can be found in Bansal and Yaron (2004) in which it is assumed that shocks to the variance of the per-period change in log consumption are persistent.[8] This positive concordance in the variance implies a fourth degree risk increase in the two-period change in log consumption. Because temperance is embedded in the model through the assumption of a power utility function, this can explain the equity premium puzzle. This means that a main driving force of the long run risk literature – namely, stochastic volatility – is based on an application of Theorem 2, which requires signing the fourth derivative of the utility function.

[8] In this literature on long run risks, it is assumed that the conditional variance is normally distributed, which is incompatible with the definition of the variance. I escape this difficulty in this numerical example by assuming a lognormal distribution for the conditional variance.

5 Recursive procedures to build simple m-th degree risk increases

Consider any pair $(m, n) \in \mathbb{N}_0^2$ and any quadruple $(\tilde{x}_m(1), \tilde{x}_m(2), \tilde{y}_n(1), \tilde{y}_n(2))$ of independent random variables. We then define a new pair $(\tilde{x}_{m+n}(1), \tilde{x}_{m+n}(2))$ as follows:[9]

$$\tilde{x}_{m+n}(1) = [\tilde{x}_m(1) + \tilde{y}_n(2), \tilde{x}_m(2) + \tilde{y}_n(1)] \tag{13}$$

$$\tilde{x}_{m+n}(2) = [\tilde{x}_m(2) + \tilde{y}_n(2), \tilde{x}_m(1) + \tilde{y}_n(1)]. \tag{14}$$

Theorem 2 implies the following result:

$$\left.\begin{array}{l} \tilde{x}_m(2) \succeq_m \tilde{x}_m(1) \\ \tilde{y}_n(2) \succeq_n \tilde{y}_n(1) \end{array}\right\} \Rightarrow \tilde{x}_{m+n}(2) \succeq_{m+n} \tilde{x}_{m+n}(1) \tag{15}$$

This result by Eeckhoudt et al. (2009) is a special case of this theorem because it is specific to the comparison of perfect positive/negative concordance in the degrees of riskiness, in a context with only two states of nature of equal probabilities, as in the existing literature.[10] Property (15) provides different recursive procedures to construct pairs of random variables that are comparable under the m-th degree of riskiness, for any $m \in \mathbb{N}_0$, as we show now. Deck and Schlesinger (2014, 2018) bring it to the lab in order to test for the sign of the first few derivatives of the utility function.

5.1 Case $n = 1$: Reverse-or-translate procedure

Consider an initial pair $(\tilde{x}_1(1), \tilde{x}_1(2))$ and a sequence $\{(\tilde{y}_1^m(1), \tilde{y}_1^m(2)) \mid m = 1, 2, ...\}$ such that in all these pairs, the second random variable is riskier than the first in the sense of the first-degree risk order. These ingredients can be used recursively in the above procedure to build a sequence of pairs of random variables $(\tilde{x}_{m+1}(1), \tilde{x}_{m+1}(2))$ such that:

$$\tilde{x}_{m+1}(1) = [\tilde{x}_m(1) + \tilde{y}_1^m(2), \tilde{x}_m(2) + \tilde{y}_1^m(1)] \tag{16}$$

$$\tilde{x}_{m+1}(2) = [\tilde{x}_m(2) + \tilde{y}_1^m(2), \tilde{x}_m(1) + \tilde{y}_1^m(1)]. \tag{17}$$

Using property (15) recursively, we obtain that, for all integers m, $\tilde{x}_m(2)$ is riskier than $\tilde{x}_m(1)$ in the sense of the m-th degree risk order. To illustrate, suppose that $\tilde{x}_1(1)$ and $\tilde{x}_1(2)$ are degenerated random variables at respectively w and $w - k_0 < w$. Suppose also that $\tilde{y}_1^m(1)$ and $\tilde{y}_1^m(2)$ are equal respectively to 0 and $-k_m < 0$ with certainty. The first stage of this recursive procedure applied in this case yields[11]

$$\tilde{x}_2(2) \sim [w - k_0 - k_1, w] \quad \succeq_2 \quad [w - k_1, w - k_0] \sim \tilde{x}_2(1). \tag{18}$$

Obviously, lottery $\tilde{x}_2(2)$ is a mean-preserving spread of $\tilde{x}_2(1)$.

[9] In this section, we use notation $[a, b]$ to refer to a lottery with two equally likely outcomes a and b.
[10] Chiu et al. (2012) explored the dual case of multiplicative risks with $n = 1$.
[11] In this section, we use notation $[a, b]$ to refer to a lottery with two equally likely outcomes a and b.

We can now move to applying procedure (13)-(14) to $m = 2$. Using properties (15) and (18) yields

$$\tilde{x}_3(2) \sim [\tilde{x}_2(2) - k_3, \tilde{x}_2(1)] \quad \succeq_3 \quad [\tilde{x}_2(1) - k_3, \tilde{x}_2(2)] \sim \tilde{x}_3(1). \tag{19}$$

We can similarly proceed to any higher order by following this recursive procedure. In Table 2, I use this procedure to produce pairs of lotteries that can be ranked by the m-th degree risk order, for $m = 1, 2, 3, 4$. A graphical representation of the same procedure is proposed in Figure 4, in which the distribution $\tilde{x}_1(2)$ is obtained from $\tilde{x}_1(1)$ by a transfer of probability mass ε from outcome w to outcome $w - k_0$.

outcome	$w-4k$	$w-3k$	$w-2k$	$w-k$	w
$\tilde{x}_4(2)$	1/8		3/4		1/8
$\tilde{x}_4(1)$		1/2		1/2	
$\tilde{x}_3(2)$		1/4		3/4	
$\tilde{x}_3(1)$			3/4		1/4
$\tilde{x}_2(2)$			1/2		1/2
$\tilde{x}_2(1)$				1	
$\tilde{x}_1(2)$				1	
$\tilde{x}_1(1)$					1

Table 2: Example of m-th degree risk increases for $m \in \{1, 2, 3, 4\}$ using procedure (24)-(25) with $\tilde{x}_1(1) = w$ with certainty, $k_0 = k_1 = k_2 = k_3$. Each row represents the probability distribution of $\tilde{x}_m(\theta)$, with the property that $\tilde{x}_m(2)$ is riskier than $\tilde{x}_m(1)$ in the sense of the m-th degree risk order.

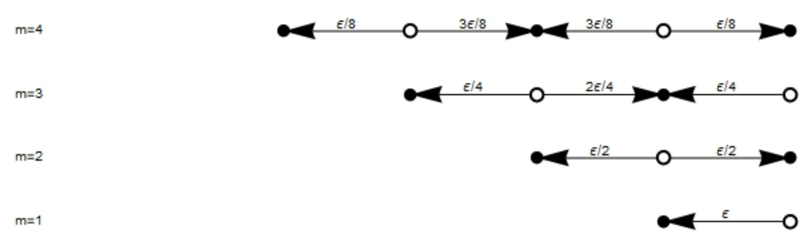

Figure 4: Reverse-or-translate procedure (16)-(17) with $k_0 = k_1 = k_2 = k_3$. The probability mass transferred is indicated above each arrowed segment. The change in distribution represented on row m is a m-th degree risk increase.

Various authors explored special cases of this recursive procedure by assuming that the successive derivatives of the utility function u alternate in sign. From Theorem 2, this assumption implies that all m-th degree risk increases are "bad", independent of the value of m. Procedure (16)-(17) raises the concordance of the "worse" conditional risk in $\tilde{x}_m$ with the worse conditional risk in $\tilde{y}_m$. The $(m+1)$-th degree risk aversion (risk aversion, prudence, temperance, edginess,...) describes a preference for the disaggregation of m-th degree harmful changes in risk on $\tilde{x}$ and the first-degree risk of $\tilde{y}$. Notice however that this story of aggregation of good and bad risks requires conditions stronger than in Theorem 2. Indeed, it requires that all successive derivatives of u up to m alternate in sign, whereas Theorem 2 just requires knowing the sign of its m-th derivative. In fact, the right terminology is that the $(m+1)$-th degree risk aversion describes a preference for the disaggregation of m-th degree risk increases in $\tilde{x}$ with first degree risk increases in $\tilde{y}$. Whether m-degree risk increases are perceived as good or bad is irrelevant for defining the $(m+1)$-degree risk aversion. For example, a prudent person ($u''' > 0$) will always dislike increasing the concordance between the second degree riskiness of $\tilde{x}$ and the first degree riskiness of $\tilde{y}$, independent of whether this person likes or dislikes first and second degree increases in risk. See Crainich et al. (2013), Ebert (2013) and Ebert et al. (2019) for an in-depth analysis of this point.

There exists an alternative way to analyze the change in risk generated by switching from $\tilde{x}_{m+1}(1)$ to $\tilde{x}_{m+1}(2)$ under procedure (16)-(17). By construction, it compounds a m-th degree risk reduction from $\tilde{x}_m(2)$ to $\tilde{x}_m(1)$ with a perfectly symmetric m-th degree risk increase at a lower wealth level by k_m, thus from $\tilde{x}_m(1) - k_m$ to $\tilde{x}_m(2) - k_m$. In short, it compounds a reversal of m-th degree risk increase and its translation at a lower wealth level. This "reverse-or-translate" procedure yields a $(m+1)$-th degree risk increase. By construction, the impact of such a reverse-or-translate operation on expected utility depends upon the relative intensity of the EU-impact of a given m-th degree risk increase applied at different wealth levels. The aversion to these second degree risk increases is the consequence of the decreasing sensitiveness to first-degree risk increases. Chiu (2005) made the same point. This mechanism can be applied recursively. An elementary third-degree risk increase combines an elementary mean-preserving spread at a low wealth level with the symmetric mean-preserving contraction at a larger wealth level. Eeckhoudt et al. (1995), Chiu (2005) and Eeckhoudt and Schlesinger (2006) also made that point.

5.2 Case $n = 2$: Reverse-or-spread procedure

Let us alternatively consider a sequence $\{(\tilde{y}_2^m(1), \tilde{y}_2^m(2)) \mid m = 1, 2, ...\}$ such that $\tilde{y}_2^m(2)$ is riskier than $\tilde{y}_2^m(1)$ in the sense of the second-degree risk order. These pairs are used as inputs in the following iterative procedure:

$$\tilde{x}_{m+2}(1) = [\tilde{x}_m(1) + \tilde{y}_2^m(2), \tilde{x}_m(2) + \tilde{y}_2^m(1)] \tag{20}$$

$$\tilde{x}_{m+2}(2) = [\tilde{x}_m(2) + \tilde{y}_2^m(2), \tilde{x}_m(1) + \tilde{y}_2^m(1)]. \tag{21}$$

If this procedure starts from $m = 1$ with $\tilde{x}_1(2) - \tilde{x}_1(1)$ being an elementary first degree risk increase as earlier in this section, one obtains elementary risk increases of odd degrees. For example, suppose that $\tilde{x}_1(1)$ and $\tilde{x}_1(2)$ are respectively equal to w and $w - k_0$ with certainty, and that $\tilde{y}_2^m(1)$ equals 0 with certainty whereas $\tilde{y}_2^m(2)$ is a zero-mean risk $\tilde{\varepsilon}_m$. Then property (15) applied for $m = 1$ and $n = 2$ implies the following property:

$$\tilde{x}_3(2) \sim [w - k_0 + \tilde{\varepsilon}_1, w] \quad \succeq_3 \quad [w + \tilde{\varepsilon}_1, w - k_0] \sim \tilde{x}_3(1). \tag{22}$$

This is already in Eeckhoudt and Schlesinger (2006), who showed that prudence can be defined as a preference for reducing the concordance between a sure loss and a zero-mean risk.

If one starts the procedure from $m = 2$ with $\tilde{x}_2(1) = w$ with certainty and $\tilde{x}_2(2) = w + \tilde{\varepsilon}_0$, one obtains elementary risk increases of even degrees. For example, the first step of the procedure yields the following result:

$$\tilde{x}_4(2) \sim [w + \tilde{\varepsilon}_0 + \tilde{\varepsilon}_2, w] \quad \succeq_4 \quad [w + \tilde{\varepsilon}_0, w + \tilde{\varepsilon}_2] \sim \tilde{x}_4(1). \tag{23}$$

This is also in Eeckhoudt and Schlesinger (2006) who showed that temperance ($u'''' < 0$) is a preference for reducing the concordance between two zero-mean risks. This recursive procedure generalizes their method to any risk order. Under this alternative procedure, the $(m+2)$-th degree risk increase is obtained from the m-th degree risk increase by compounding two changes in distribution: With probability 1/2, this m-th degree risk increase is reversed. And with probability 1/2, this m-th degree risk increase is either shifted to the right or to the left by k_m with equal probabilities. This is why I refer to this recursive method as the "reverse-or-spread" procedure. Table 1 presented in the introduction illustrates this procedure in the even case.

These two iterative procedures are related to Gollier (2019) in which I define variance stochastic orders. In that paper, I suppose that one is unsure about the number $\tilde{v}$ of i.i.d. risks $\tilde{\varepsilon}_m$ in final wealth $w + \sum_{m=1}^{\tilde{v}} \tilde{\varepsilon}_m$, with $E\tilde{\varepsilon}_m = 0$ and $Var(\tilde{\varepsilon}_m) = 1$. This means that the variance of final wealth is uncertain. I showed in that paper that if this variance $\tilde{v}$ undergoes a n-th degree risk increase, then final wealth becomes riskier in the sense of the $2n$-th risk order. A special case of this result is presented in property (23) for $n = 2$. In Gollier (2019), I also examine the case in which w is random. I show that an increase in concordance between $\tilde{w}$ and the n-th degree riskiness of $\tilde{v}$ reduces (resp. increases) the riskiness of final wealth in the sense of the $(2n + 1)$-th degree risk order if n is a even (resp. odd) number. For example (case $n = 1$), reducing the concordance between the conditional mean and the conditional variance raises the third degree riskiness.[12] In particular, it reduces the skewness of final wealth. An illustration of this result is presented in equation (22).

6 Recursive procedures to build simple (m, n) degree risk increases

It is now easy to generalize this recursive procedure to bivariate stochastic dominance (and by extension, to multivariate stochastic dominance). In order to construct an elementary (m, n) degree risk increase, we compound with increasing concordance an elementary m-th degree risk increase in $\tilde{x}$ and an elementary n-th degree risk increase in $\tilde{y}$. More precisely, suppose that $\tilde{x}_m(2)$ is m-th degree riskier than $\tilde{x}_m(1)$, and that $\tilde{y}_n(2)$ is n-th degree riskier than $\tilde{y}_n(1)$. Then, consider the following two pairs of random variables:

$$(\tilde{X}_m(1), \tilde{Y}_n(1)) = \begin{cases} (\tilde{x}_m(2), \tilde{y}_n(1)) & \text{with prob. } 1/2; \\ (\tilde{x}_m(1), \tilde{y}_n(2)) & \text{with prob. } 1/2, \end{cases} \tag{24}$$

[12]Notice that a first-degree risk increase in $\tilde{v}$ corresponds to a lower v in the degenerate case. Thus, in the degenerate case examined here, an reduction in the concordance between $\tilde{w}$ and $\tilde{v}$ implies an increase in the concordance between $\tilde{w}$ and the first degree riskiness of $\tilde{v}$.

and

$$(\tilde{X}_m(2), \tilde{Y}_n(2)) = \begin{cases} (\tilde{x}_m(1), \tilde{y}_n(1)) & \text{with prob. } 1/2; \\ (\tilde{x}_m(2), \tilde{y}_n(2)) & \text{with prob. } 1/2. \end{cases} \tag{25}$$

By construction, shifting from $(\tilde{X}_m(1), \tilde{Y}_n(1))$ to $(\tilde{X}_m(2), \tilde{Y}_n(2))$ raises the concordance between the m-th degree riskiness of $\tilde{X}$ and the n-th degree riskiness of $\tilde{Y}$. We can thus conclude from Theorem 1 that $(\tilde{X}_m(2), \tilde{Y}_n(2))$ is (m, n) degree riskier than $(\tilde{X}_m(1), \tilde{Y}_n(1))$.

This procedure is illustrated in Figure 5 for $(m, n) \in \{0, 1, 2, 3\}^2$. Consider for example the case $n = 1$, where $\tilde{y}_1(2)$ is obtained from $\tilde{y}_1(1)$ by shifting some probability mass from outcome y to outcome $y - k < y$. In that case, the elementary $(m, 1)$ degree risk increase described by rules (24)-(25) combines two equally likely transformations: A reversal of the m-th degree risk increase of $\tilde{x}$ conditional to $\tilde{y} = y$, and a downward translation of this m-th degree risk increase by k. Consider alternatively the case $n = 2$ in which $\tilde{y}_2(2)$ is obtained from $\tilde{y}_2(1)$ by transferring some probability mass from outcome y to be split equally and transferred at $y - k$ and $y + k$. In that case, the elementary $(m, 2)$ degree risk increase combine two equally likely transformations: A reversal of the m-th degree risk increase of $\tilde{x}$ conditional to $\tilde{y} = y$, and a mean-preserving spread of the original m-th increase in risk of $\tilde{x}$ at $y - k$ and $y + k$, respectively.

The special case of $(2, 2)$ degree risk increase is particularly interesting. As shown in Figure 5, it is obtained by shifting the same probability masses from the middle of each side of a square, half of them going to the center of the square, and the remaining half going to its corners. As for the "first degree" concordance of Tchen (1980) corresponding to cell $(1, 1)$, the marginal distributions of $\tilde{x}$ and $\tilde{y}$ are unaffected by this change in the joint distribution. It generates a mean-preserving spread in the probability-weighted index of second degree riskiness (from 0-2-0 to 1-0-1) in both dimensions. The aversion towards such "second degree" concordance corresponds to the condition that the $(2, 2)$ cross derivative of u is negative. An example of third degree concordance is described in the upper right cell of Figure 5.

Rule (24)-(25) is not fully recursive since it jumps from the m-th and n-th degree risk increases of $\tilde{x}$ and $\tilde{y}$ directly to the (m, n) degree risk increase of $(\tilde{x}, \tilde{y})$. It would be useful to obtain a $(m, n+1)$ or $(m+1, n)$ degree risk increase directly from a (m, n) degree risk increase. Suppose that $Z(2) = (\tilde{x}_m(2), \tilde{y}_n(2))$ is (m, n) degree riskier than $Z(1) = (\tilde{x}_m(1), \tilde{y}_n(1))$. From this pair of bivariate risk contexts and a positive scalar h, let us define a new pair $(Z^h(1), Z^h(2))$ by using the following "horizontal reverse-or-translate" procedure:

$$Z^h(1) = \begin{cases} (\tilde{x}_m(2), \tilde{y}_n(2)) & \text{with prob. } 1/2; \\ (\tilde{x}_m(1) - h, \tilde{y}_n(1)) & \text{with prob. } 1/2. \end{cases} \tag{26}$$

and

$$Z^h(2) = \begin{cases} (\tilde{x}_m(1), \tilde{y}_n(1)) & \text{with prob. } 1/2; \\ (\tilde{x}_m(2) - h, \tilde{y}_n(2)) & \text{with prob. } 1/2. \end{cases} \tag{27}$$

$Z^h(2)$ is obtained from $Z^h(1)$ by compounding two equally likely changes in risk based on the initial (m, n) degree risk increase from $Z(1)$ to $Z(2)$: Its reversal, and its leftward translation by h. To determine the attitude towards this horizontal reverse-or-translate change in risk, suppose that one is initially confronted to two equally likely states of nature, one yielding

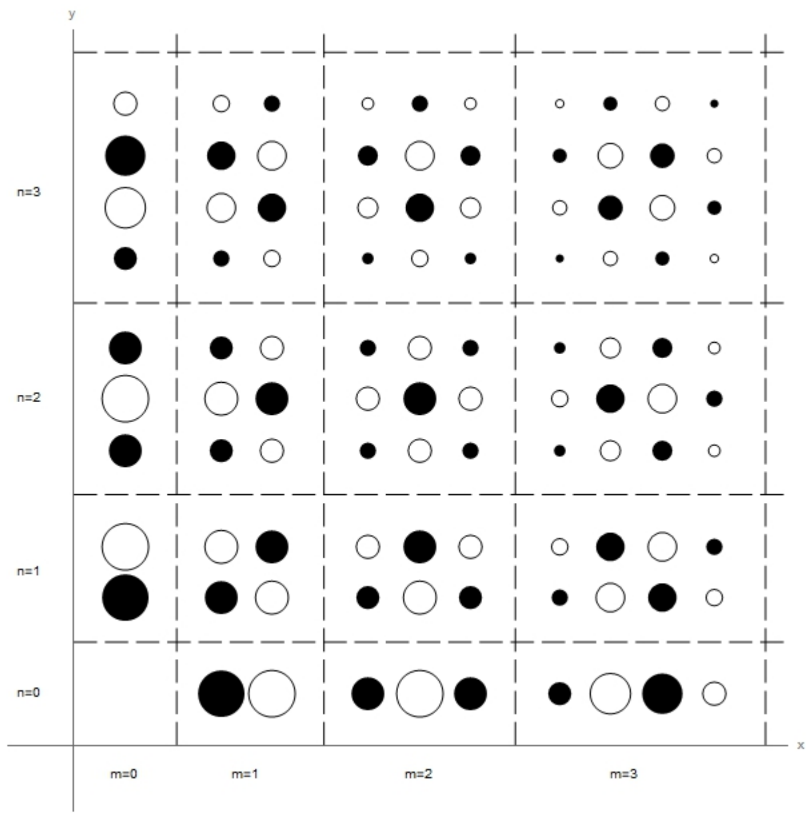

Figure 5: A method to construct bivariate (m, n) degree risk increases, (m, n) in $\{0, 1, 2, 3\}^2$. Each rectangle represents a specific elementary (m, n) degree risk increase. In each of these rectangle, circles and disks are located where probability masses are respectively taken away and transferred, whereas their surfaces are proportional to these probability masses. These elementary bivariate risk increases are obtained by combining univariate m and n degree risk increases described in the lower row and left column.

$Z(1)$, and the other yielding the (m, n) degree riskier $Z(2)$. Suppose now that one must bear a sure reduction of x by h in one of these two states. Does one prefer to bear this loss in the low risk state 1, or in the riskier state 2? It is easy to show that the new change in risk belongs to the class of $(m+1, n)$ degree risk increases and should therefore be disliked by any individual whose utility function has a sign of its $(m+1, n)$ derivative equaling $(-1)^{m+n}$. In Figure 5, any cell describing a $(m+1, n)$ degree risk increase can be obtained from the cell to its left by compounding its reversal and its leftward translation by h. The symmetric vertical reverse-or-translate procedure yields the symmetric property in the y dimension. It is also easy to extend this analysis by substituting the sure loss by a zero-mean risk, in x or in y. These horizontal or vertical reverse-or-spread procedures generate jumps from one cell in Figure 5 to two cells to the right, or to two cells to the top.

Theorem 1 is linked to a growing literature on the apportionment of bi-dimensional risks with bivariate utility. Eeckhoudt et al. (2007) have examined the bi-dimensional version of compounding sure losses and zero-mean risks discussed in Eeckhoudt and Schlesinger (2006). It is linked to what is presented here in the context of two equally likely states with perfect positive/negative concordance. Tsetlin and Winkler (2009) improved this approach by offering a full-fledged recursive methodology corresponding to the reverse-and-translate procedure (26)-(27), and by expanding it to more than two dimensions. They also pushed the argument to infinity to show that the systematic preference for compounding "good" with "bad" requires the multivariate utility function to be a mixture of exponential functions of weighted sums of the attributes. In this paper, I remove the unnecessary jargon of "good" and "bad" to replace it by "less risky" or "more risky" in the sense of some specific risk order, a move that allows me to focus on a specific cross-derivative of the utility function rather than on all cross-derivatives up to a specific integer. My Theorem 1 also expands the recursive methodology in other directions, such as more than two states for $(\tilde{\theta}, \tilde{\eta})$, marginal increases in concordance, marginal first-degree shifts in the distribution of $\tilde{y}$, or higher risk orders for the change in $\tilde{y}$.

Moyes (2012) examined some low degree bivariate risk increases such as the $(1, 2)$ order in the context of the distribution of wealth and health. Denuit and Eeckhoudt (2010) had the idea to use the concept of increasing concordance to express the concept of apportioning losses. They also used a bi-dimensional version of the precedence relationship introduced by Chiu (2005) to build what I call the reverse-or-translate procedure (26)-(27). Because they combine the reversal and the translation whereas I compound them with equal probabilities, these two procedures do not generate the same elementary bi-dimensional risk increases. Compared to this work, as to all other existing papers in the literature, my results explore the crucial idea of *increasing the concordance in the degrees of riskiness* of the two risks. My work also clarifies the recursive nature of the Denuit-Eeckhoudt methodology, and it shows how it can be used with higher-order transformation such as the reverse-or spread procedure.

7 Conclusion

The objective of this paper is to combine the two independent ideas of statistical concordance (Epstein and Tanny (1980)) and m-th degree riskiness (Ekern (1980)) to produce a general theory of the aversion to increasing the concordance in different degree risks. When an individual bears two additive risks, she dislikes any increase in the statistical concordance of the two outcomes. This is due to risk aversion. More generally, I showed that risk aversion implies that any increase in concordance in the first degree riskiness of the two risks is always disliked. But suppose alternatively that the expected value of one risk is statistically related to the variance of the other risk. Then, increasing the concordance between the expectation of the first risk and the variance of the other risk tends to reduce the skewness of their sum. More generally, I showed that increasing the concordance of the first degree riskiness of the first risk and the second degree riskiness of the second risk raises the third degree riskiness of their sum, which is disliked by prudent individuals. And increasing the concordance between the m-th degree riskiness of the first risk and the n-th degree riskiness of the second risk raises the (m+n)-th degree riskiness of their sum.

Over the last two decades, some complex stochastic relationships among various macroeconomic variables emerged as a key ingredient to solve the classical puzzles that exist in asset pricing theory. For example, in the literature about long run risks, serial correlations to the trend of growth and to its volatility tend to distort high-order moments of the distribution of future consumption. This paper provides a new decision-theoretic framework to discuss the relations that prevail between sophisticated probabilistic concordances among different random variables, the riskiness of their functional combination, and their impact on preferences.

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