Multiproduct Mergers and Quality Competition

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Abstract. We investigate mergers in markets where quality differences between products are central. In our model, firms may sell multiple products, and merging and non-merging firms may reposition their product lines by adding or removing products following a merger. We find that such mergers are materially different from those studied in the existing literature. Mergers without synergies may raise consumer surplus, but only when the pre-merger industry structure satisfies certain observable features. Synergies may lower consumer surplus. Mergers are more readily profitable when an industry exhibits multiple qualities, and mergers between small numbers of firms with small market shares may be profitable. Some non-merging firms may benefit while others lose following a merger. We also provide a new measure of industry concentration: the Quality-adjusted Herfindahl-Hirschman Index extends the standard Herfindahl-Hirschman Index to markets in which quality differences are central.

In many industries products differ substantially in terms of quality and these differences play a central role in consumer decision making. Multiproduct firms are common in such industries, although firms may instead specialize in low or high quality. Examples include consumer markets for automobiles, personal electronics, and air travel, as well as business markets for processed materials, computer servers, and various types of industrial machinery.

Although competition authorities around the world recognize the importance of accounting for quality in merger policy, the theoretical literature has lagged behind.[1] We provide a quantity-setting framework for assessing mergers in these markets, allowing for a general demand system for quality, asymmetric firms which may reposition their product lines by adding or removing products following a merger, and the possibility that a merged entity has a better cost structure than any of its constituent firms.

These mergers raise a number of questions. How do they affect the overall level of output and the equilibrium product mix? What does observed product-line repositioning, by either

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[1]The 2013 OECD document *The Role and Measurement of Quality in Competition Analyses* notes, "While the importance of quality is undisputed and issues about quality are mentioned pervasively in competition agency guidelines and court decisions, there is no widely-agreed framework for analysing it which often renders its treatment superficial."

merging firms or their rivals, tell us about the likely welfare effects of a merger? Do synergistic cost reductions always benefit consumers? Under what conditions are mergers profitable? What is the correct way to define markets? In answering these questions and others, we find that such mergers are materially different from those that have been previously studied.

Our first results involve the effects of mergers on industry supply, prices, and consumer surplus. We show that a merger that exhibits no cost synergies always leads to a reduction in the total industry supply (that is, the sum of low-quality and high-quality output declines). Nevertheless, such a merger may increase consumer surplus by inducing a favorable change in the equilibrium product mix, in sharp contrast to what happens in single-product markets (Farrell and Shapiro, 1990). Indeed, consumer-surplus increasing mergers can also be profitable both for merger participants (insiders) and non-participants (outsiders), which cannot happen in a single-product world.

Although the presence of multiple qualities necessarily provides additional flexibility for how mergers may affect consumer surplus, it not true that anything can happen following a merger in a multiproduct industry. As already noted, total output always declines. Moreover, if consumer surplus increases, it is always "led by quality", meaning that it increases due to a decline in the price of high quality and despite an increase in the price of low quality, never the other way around. Finally, consumer surplus can only increase if the pre-merger market structure satisfies certain observable necessary conditions.

The first observable necessary condition that the pre-merger market structure must satisfy is that firms are asymmetric. Thus to have a complete picture of mergers it is essential to allow for asymmetric firms. The second condition is that some firms sell only high-quality products. As we explain later, the presence of such firms creates a specific type of strategic linkage between total supply and high-quality supply. Third, despite the required presence of some high-quality only firms in the market, the merging firms themselves must not sell only high quality—if they do then consumer surplus falls.

Our results hold even though we allow both merging and non-merging firms to reposition their product lines following a merger, by either adding or removing products. Such repositioning is widely recognized as an important consideration.[2] For example, if two firms selling only high quality merge, our results show that consumer surplus decreases even if rivals respond by introducing their own high-quality products. This provides support for the Federal Trade Commission's (FTC) challenge to the approval of the merger between (high-quality) organic

[2] For example, in a November 9, 1995 speech, Deputy Assistant Attorney General of the Antitrust Division of the US Department of Justice, Carl Shapiro, argued that merger assessment should "try to account for any likely and timely changes in prices or product offerings by non-merging parties, including product repositioning and entry." Similarly, the 2014 Office of Fair Trading document *Competing on Quality* identifies "anticipating merger effects with post-merger brand repositioning" as an important issue in markets where quality is central.

grocers Whole Foods and Wild Oats; one justification for the original approval was the district court's view that rival grocers could introduce their own organic food lines.[3]

We also consider how merger-induced synergies influence market outcomes. In some cases, synergies work as expected and raise consumer surplus but in other cases they may induce unfavorable changes in the equilibrium quality mix and lower consumer surplus. Whether synergies are present or not, demand curvature and the differences in such curvature across products of different qualities play an important role. This is reminiscent of studies of the welfare effects of third-degree price discrimination (Schmalensee (1981), Varian (1985), Cowan (2007), Aguirre, Cowan, and Vickers (2010)).

Our second set of results explores the classic question of whether firms can use horizontal mergers to raise their profits. Salant, Switzer, and Reynolds (1983), Levin (1990), and Cheung (1992) have argued that the competitive responses of outsiders to mergers frequently render mergers unprofitable to insiders (in the absence of synergies or fixed-cost savings). Their results imply that, for general demand systems, insiders must control at least 50% of pre-merger market output in order to gain from merging. This in turn implies that (absent strong synergies) no two-firm merger is profitable unless it leads to a monopoly.

Allowing for multiple quality-differentiated products suggests that mergers are profitable in a wider variety of circumstances. Insiders need not control significant pre-merger market share for a merger to be profitable. For example, suppose some firms, each of which produces only low quality, are considering a merger. These insiders' pre-merger market share—either in terms of total production or restricted to low-quality production—need not exceed any particular threshold for the merger to be profitable. In some cases mergers involving as few as two firms are profitable even as the number of rivals becomes unboundedly large.

Our third main contribution is the construction of a new measure of industry concentration. We propose a version of the famous Herfindahl-Hirschman Index (HHI) that is appropriate for quality differentiated markets and which sheds light on questions of market definition by connecting market share data to a measure of industry participants' profits (namely, the industry gross profit margin).[4] This measure has a low information requirement and can be applied without cost data or demand-elasticity estimates.

The Quality-adjusted Herfindahl-Hirschman Index (QHHI) is a weighted average of two separate HHI measures, one involving firms' overall market shares (across low and high quality combined) and the other involving market shares of high-quality output. The weights can be

[3]Boberg and Woodbury (2009) explain that the district court's view was that rival grocers "had already proven themselves adept at repositioning through the addition and expansion of organic produce sections, perishable meat sections and other products, and through reformatting and redesigning stores."

[4]Market definition is more subtle when quality matters. The 2018 OECD document *Considering Non-price Effects in Merger Control* cautions, "a narrow reliance on rigid market definitions, and the resulting market shares generated from these definitions, is not likely to be meaningful when multiple non-price dimensions of competition exist."

computed using revenue data. Like the HHI, the QHHI provides a first-pass assessment of the market power in an industry and how much a merger might increase that power. By design it accounts for the fact that mergers may have asymmetric effects on market power across the multiproduct industry and also appropriately weights these asymmetric changes, which the HHI necessarily cannot do. We furthermore provide a theoretical basis for why mergers between multiproduct firms may be worse for consumers than other types of mergers.

Our final results involve the profits of outsider firms. We consider the "external effects" welfare measure in which a merger is deemed beneficial if it raises the sum of consumer surplus and outsiders' profits. We provide conditions under which the external effect is positive based on simple measures of (correctly defined) market share.

Certain outsiders may benefit from a merger while others lose. Although intuitive, this is not typically predicted by other models. The reason is that such models exhibit either a single homogenous product and all outsiders agree that a merger is beneficial if it reduces aggregate supply, or else firms sell differentiated products but mergers raise or lower all prices.

Despite the ubiquity of multiproduct firms and the importance of product quality differences, much of the theoretical literature on mergers focuses on single-product firms in markets with a single quality.[5] These studies typically assume quantity competition (Perry and Porter, 1985; Levin, 1990; McAfee and Williams, 1992) or price competition with horizontally differentiated products competition (Deneckere and Davidson, 1985). An exception is Nocke and Schutz (2018), who consider mergers between multiproduct firms offering portfolios of horizontally differentiated products, whereas we focus on multiproduct firms offering product lines composed of vertically differentiated products. They reduce multiproduct competition to a single dimension and argue that many of the classic results from single-product markets hold, such as mergers reducing consumer surplus in the absence of synergies.

In our model, mergers influence both the product-quality mix in the market and product lines. But mergers may have other important effects on product quality, for example by changing the investment incentives of firms to enhance the quality of a given product line, as in Motta and Tarantino (2018) and Federico, Langus, and Valletti (2018), who find that a merger must exhibit significant innovation synergies for it to increase consumer surplus.

The remainder of our manuscript is laid out as follows. Section 1 presents the model. Section 2 assesses the impact of mergers on prices, quantities, and consumer surplus, both with and without synergies. Section 3 investigates the profitability of mergers. Section 4 presents the Quality-adjusted Herfindahl-Hirschman Index. Section 5 provides an assessment of external effects and the profitability of mergers for outsiders.

[5] Work on multiproduct competition without mergers includes Brander and Eaton (1984), Gal-Or (1983), Champsaur and Rochet (1989), Anderson and De Palma (1992), Johnson and Myatt (2006, 2016, 2018), Anderson and Çelik (2015), Rhodes (2015), Johnson (2017), and Armstrong and Vickers (2018). Rhodes and Zhou (2017) study multiproduct competition that arises after mergers between firms that sell unrelated products.

1. Model

Here we describe a market in which there are two vertically differentiated products of quality $q_L > 0$ and $q_H > q_L$. Buyers are indexed by θ. A consumer of type θ is willing to pay at most $v(\theta, q)$ for a single unit of a product with quality q, where $v(\theta, q)$ is increasing in both of its arguments and satisfies the usual sorting condition: $v(\theta, q_H) - v(\theta, q_L)$ is increasing in θ. Consumers have quasilinear preferences and purchase a single unit of the product that offers the greatest non-negative surplus, and otherwise buy no product. Amongst a unit mass of potential buyers, for $z \in [0, 1]$ we let $\theta(z)$ be the buyer type for which there are z buyers with higher values of θ, where $\theta(z)$ is strictly decreasing and twice differentiable in z. If θ is distributed according to $F(\theta)$ then $\theta(z) = F^{-1}(1 - z)$.

Firm i has a constant marginal cost $c_1^i \geq 0$ for low-quality products and a constant marginal cost $c_2^i \geq c_1^i$ for high-quality products. Firms simultaneously set outputs. An equilibrium is a set of quantities for each firm i such that i is maximizing its profits taking as given the output of all other firms. We do not assume that each firm supplies a positive quantity of both products but instead allow for completely arbitrary equilibrium outcomes.

1.1. **Market Clearing Prices.**

Let Z_1^i denote the total number of units *of low and high quality combined* that firm i produces, and let Z_2^i denote the number of high-quality units that i produces. Necessarily, $Z_1^i \geq Z_2^i$ and the number of low-quality units that i produces is $Z_1^i - Z_2^i \geq 0$. Working with Z_1^i and Z_2^i as the strategic variables is particularly convenient.

Let $Z_1 = \sum_i Z_1^i$ be the industry supply of units of either quality (so Z_1 is the total number of units available on the market). Let $Z_2 = \sum_i Z_2^i$ be the industry supply of high-quality units. Given these quantities, market-clearing prices are as follows. The marginal buyer of the low-quality product is indifferent between purchasing that product and nothing. Because the total number of units for sale is Z_1, the marginal buyer is of type $\theta(Z_1)$ and hence the price of the low-quality good is

$$P_1(Z_1) = v(\theta(Z_1), q_L).$$

Rather than directly deriving the price of the high-quality good we will instead find the price that the marginal consumer would pay to "upgrade" from low to high quality. This upgrade price depends on the type of the consumer who is indifferent between buying low or instead high quality. Given that there are Z_2 high-quality products available, this marginal consumer has type $\theta(Z_2)$, and is willing to pay P_2 to upgrade to high quality, where

$$P_2(Z_2) = v(\theta(Z_2), q_H) - v(\theta(Z_2), q_L).$$

Conceptually, we imagine that there exist Z_1 "baseline" units that consumers purchase at price P_1 and that there are Z_2 "upgrades" available for purchase at price P_2. The total price for a high-quality good is thus $P_1 + P_2$.

1.2. **Product Lines.** Firm i produces Z_1^i baseline units and $Z_2^i \leq Z_1^i$ upgrades, meaning it produces $Z_1^i - Z_2^i \geq 0$ low-quality products. Hence the profit π^i of firm i is given by

$$\begin{aligned}\pi^i =&(Z_1^i - Z_2^i)[P_1(Z_1) - c_1^i] + Z_2^i[P_1(Z_1) + P_2(Z_2) - c_2^i] \\ =&Z_1^i[P_1(Z_1) - c_1^i] + Z_2^i[P_2(Z_2) - (c_2^i - c_1^i)]. \end{aligned} \tag{1}$$

This says that firm i sells Z_1^i baseline units at margin $P_1(Z_1) - c_1^i$ and also sells Z_2^i upgrades at margin $P_2(Z_2) - (c_2^i - c_1^i)$, where $c_2^i - c_1^i$ is firm i's "upgrade cost" from low quality to high quality. This approach has been used by Johnson and Myatt (2003, 2006).

By construction $P_1(Z_1)$ depends only on the number of baseline units Z_1, not the number of upgrades Z_2, and the upgrade price $P_2(Z_2)$ depends only on the number of upgrades Z_2, not the total number of products Z_1. This implies that firm i can separately maximize its profits from baseline units and from upgrades: Z_1^i and Z_2^i can be chosen independently, subject only to the "upgrade constraint" $Z_1^i \geq Z_2^i$. In other words, baseline units and upgrade are neither substitutes nor complements, although firms do need to obey their upgrade constraints.

If firm i is selling both products then its upgrade constraint is not binding and at the equilibrium quantities firm i must be satisfying the two independent first-order conditions

$$\begin{aligned} P_1(Z_1) + Z_1^i P_1'(Z_1) &= c_1^i, \text{ and} \\ P_2(Z_2) + Z_2^i P_2'(Z_2) &= c_2^i - c_1^i. \end{aligned}$$

A firm that in equilibrium sells only low-quality products (so that $Z_2^i = 0$) must satisfy only the first condition above (for it to be optimal not to sell any high-quality goods it must be that $P_2(Z_2) \leq c_2^i - c_1^i$). For a firm that sells only high-quality products it has a binding upgrade constraint ($Z_1^i = Z_2^i = Z^i$) and must satisfy the single first-order condition

$$[P_1(Z_1) + P_2(Z_2)] + Z^i \left[P_1'(Z_1) + P_2'(Z_2)\right] = c_2^i.$$

If there were no firms selling only high-quality products, then the equilibrium values of Z_1 and Z_2 could be found independently simply by using the first-order conditions given above.

Our focus is on measures of industry change such as consumer surplus, allowing for endogenous product lines that may change in response to mergers. Previous work has focused on the determinants of product lines.[6] Although there are several forces at play, some simple intuition that can be relayed here comes from the case of multiplicatively separable preferences ($v(q, \theta) = \theta q$). With symmetric firms ($c_1^i = c_1^j$ and $c_2^i = c_2^j$ for each i and j), in

[6] Johnson and Myatt (2003, 2006, 2018) consider both symmetric and asymmetric settings.

equilibrium both products are sold by each firm if and only if the quality-adjusted upgrade costs are increasing in quality: $c_1^i/q_L < (c_2^i - c_1^i)/(q_H - q_L)$. With asymmetric firms, this condition is necessary but not sufficient for firm i to offer both products.[7]

In later analysis the curvature of $P_k(Z)$ will be useful, defined as

$$\sigma_k(Z) = -\frac{ZP_k''(Z)}{P_k'(Z)}, \text{ for } k \in \{1, 2\}.$$

Assumption 1 (Decreasing Marginal Revenue)**.** *In each market $k \in \{1, 2\}$ marginal revenue is strictly decreasing. That is, for each k and any $Z^i \in [0, Z]$,*

$$P_k(Z) + Z^i P_k'(Z)$$

is strictly decreasing in Z. This is equivalent to $P_k(Z)$ being log-concave, that is $\sigma_k(Z) < 1$.

We close with an important lemma from earlier work (Johnson and Myatt, 2006).

Lemma 1 (Existence and Uniqueness)**.** *There exists an equilibrium and it is unique.*

2. The Impact of Mergers on Consumer Welfare

In this section we address the classic question of what effect horizontal mergers have on consumer surplus. Economists and regulators often posit that horizontal mergers have two primary effects: a *market-power effect* from consolidation of decisions regarding key choice variables such as output, which harms consumers, and a *synergy effect* from improvements in the merging firms' cost structure, which benefits consumers.

Thus, it is often presumed that a merger which exhibits no synergies must harm consumers. The classic analysis of Farrell and Shapiro (1990) forcefully makes this point by showing that, in a homogenous product market in which all firms sell a single product of the same quality and compete in quantities, mergers which are not accompanied by strict reductions in the marginal cost structures of the merging parties must reduce aggregate output and therefore reduce consumer surplus. The market we consider is the same except that we have two quality levels and allow for multiproduct firms.

We begin by considering the impact of mergers on consumer surplus when there are no synergies. After that, we investigate the effects of synergies.

[7]Furthermore, if $c_1^i = c_2^i$, so that it is no more costly to sell high quality than low quality, then firm i only sells high-quality products in equilibrium.

2.1. **Mergers with No Synergies.** Following Farrell and Shapiro (1990), we say that when there are *no synergies* the merged firm's cost of producing any output vector equals the minimum-cost method of producing it using the merging firms' pre-merger technologies.

Definition 1 (No Synergies)**.** *Suppose that all firms $i \in \mathcal{I}$ merge to create a firm with marginal costs for low-quality products given by c_1^m and marginal costs for high-quality products given by c_2^m. This merger exhibits* no synergies *if $c_1^m = \min_{i \in \mathcal{I}} c_1^i$ and $c_2^m = \min_{i \in \mathcal{I}} c_2^i$.*

Observe that a merger with no synergies nonetheless may allow the merged entity to have a more attractive variable cost structure than any of its constituent firms alone. The merged entity may wish to reallocate (or rationalize) its output, for example by shifting all of its low-quality output to the plant with the lowest marginal cost for low quality products. Also observe that such an improved variable cost structure may alter the merged entity's optimal product mix or product line in addition to its optimal level of overall production.[8]

We now provide several results addressing the impact of a merger on consumer surplus. We first consider how aggregate output Z_1 (across all firms and qualities combined) responds.

Proposition 1. *A merger with no synergies leads to a strict reduction in aggregate output, measured across all firms and qualities: Z_1 strictly decreases (and so the price of the low-quality good strictly increases).*

Proposition 1 is the analogue of a leading result of the single-product, identical quality analysis of Farrell and Shapiro (1990). Although an intuitive result, proving it in a multiproduct setting is much more subtle. For example, we must allow firms to adjust their product lines.

Importantly, in a multiproduct industry total output Z_1 is not a sufficient statistic for consumer surplus. Rather, both total output and the industry product mix between low and high quality matter; Z_2 must also be known. If a merger were to increase Z_2, then the price of the high-quality good might decrease and some consumers would be better off. In principle, this might raise overall consumer surplus. Because the number of low quality units is $Z_1 - Z_2$, any increase in consumer surplus must be "led by quality", meaning it increases due to an expansion of high-quality supply and despite a contraction of low-quality supply.

Can a merger with no synergies indeed increase the industry output of high quality? The answer is yes, although certain observable necessary conditions must be satisfied.

Lemma 2. *For a merger with no synergies, a necessary condition for the aggregate output of high-quality products (and also consumer surplus) to strictly increase is that, pre-merger, at least one firm produces only high-quality products.*

[8]For example, a multiproduct firm acquiring a firm that only produces low-quality products but which has a lower marginal cost will experience an increase in its cost of upgrading to high quality and will, all else equal, increase its total production but reduce its supply of high-quality goods.

To illustrate this result, consider a simple example in which no firm sells only high quality and a merger occurs involving firms selling only low quality. We know from Proposition 1 that aggregate output must go down. Given that no firm's upgrade constraint was binding pre-merger, each firm not involved in the merger will raise its own total supply. However, doing so further relaxes each multiproduct firm's upgrade constraint—no firm changes its supply of high-quality products and so the merger does not affect Z_2.

The situation is different for a firm selling only high-quality products. As mentioned in Section 1, such a firm i with output $Z_1^i = Z_2^i = Z^i$ has first-order condition

$$[P_1(Z_1) + P_2(Z_2)] + Z^i\,[P_1'(Z_1) + P_2'(Z_2)] = c_2^i.$$

Firm i's output is influenced by both Z_1 and Z_2. All else fixed, an equilibrium reduction in Z_1 would raise firm i's marginal revenue and induce it to expand its supply of high-quality products. The implication is that—even if all merging firms produce only low-quality products—the post-merger equilibrium may exhibit a greater supply of high quality.

The following proposition gives conditions on the pre-merger industry structure under which the output of high-quality products definitely does or does not increase following a merger.

Proposition 2. *For a merger with no synergies:*

(1) Both consumer surplus and the aggregate output of high-quality products strictly decrease if each merging firm produces only high quality before the merger.

(2) The aggregate output of high-quality products strictly increases if, pre-merger, at least one firm is producing only high-quality products and

- *(a) each merging firm produces only low-quality products, or*
- *(b) at most one merging firm sells high-quality products and its cost for low quality is equal to the merged firm's cost of low quality.*[9]

Item 1 of Proposition 2 indicates that if all firms involved in the merger produce only high quality, then Z_2 cannot increase and so consumer surplus must fall. The intuition is that the direct cause of the reduction in Z_1 predicted by Proposition 1 is the enhanced market power of the merging firms: the merging firm's combined high-quality output is reduced which—precisely because each merging firm only produces high-quality products and so has a binding upgrade constraint—leads to an equal decrease in these firms' total output. Although other firms will respond and potentially raise their own supply of high-quality goods in the post-merger equilibrium, any such increases are not enough to counteract the reduction enacted by the merging firms.

[9]For items 2a and 2b we require that at least one firm selling only high-quality products "strictly" does so, meaning that the firm would have a strict incentive to raise its output of high-quality units if it were allowed to violate its upgrade constraint. Precisely, for such a firm i facing an equilibrium value Z_2 and its own optimal quantity of high-quality products Z_2^i, we mean that $P_2(Z_2) + Z_2^i P_2'(Z_2) > c_2^i - c_1^i$.

Thus we know that for a merger to raise Z_2 there must be some firms producing only high-quality products but the merger should not only involve such firms. Item 2a from Proposition 2 is now quite intuitive. If each merging firm produces only low quality, then the direct effect of the merger is for the merged entity to curtail its output of low-quality products, reducing Z_1. As explained above, such a reduction encourages those firms producing only high-quality products to expand their output. Although firms producing both products may in turn produce fewer high-quality goods, the equilibrium effect is an increase in Z_2. The same reasoning guarantees that Z_2 increases if one merging firm produces high quality (item 2b), subject to the additional requirement that it does not have a higher cost of producing low-quality products than other merging firms.[10]

We emphasize that Lemma 2 and Proposition 2 hold even though firms may reposition their product lines following a merger. Most pointedly, if each of the merging parties sells only high quality pre-merger, then the aggregate output of high-quality products declines and so must consumer surplus—even if rivals respond by introducing their own high-quality products.

As noted in the introduction, this provides theoretical support for the FTC's challenge of the district court's approval of the merger between organic grocers Whole Foods and Wild Oats. The ability of traditional or lower quality grocers to, for example, introduce their own organic lines was part of the court's justification for the approval of the merger (Boberg and Woodbury, 2009). Our results suggest that even if rival grocers would indeed so respond, the merger (assuming no synergies) would nonetheless increase all prices and harm consumers.

Such a repositioning defense might also be weak if the merging parties were selling only low-quality products. Such a merger might induce high-quality rivals to expand their product lines to include low quality.[11] But such repositioning would by no means imply that the merger increased consumer surplus. Indeed, to some extent it might imply the opposite.[12]

We conclude our discussion of the conditions under which a merger may increase high-quality output by further observing that asymmetries across firms are absolutely crucial for such an increase, highlighting the importance of allowing for such asymmetries in our analysis.

Corollary 1. *Suppose that pre-merger all firms have the same product line. Then a merger with no synergies strictly increases both prices and hence strictly decreases consumer surplus.*

[10] If its cost of producing such products were higher, then the upgrade cost of the merged firm, $c_2^m - c_1^m$, would be higher than that of the firm producing high-quality, which would cause a direct effect of the merger to be a lowering of the merged firm's high-quality output, potentially overturning the result. We return to this point in Section 2.2.

[11] This would happen if the merger induced an equilibrium reduction in Z_1 sufficiently large that it led firms producing only high quality pre-merger to expand their total output such that their upgrade constraints did not bind post-merger, meaning that they would be selling both products post-merger.

[12] By this we mean the following. Consider a high-quality firm responding to a merger by raising its total output. Initially, such increases occur due to its expansion of its high-quality output. But at the point where it introduces a low-quality product, further increases in its total output occur solely due to expansion of its low-quality supply.

We now turn to the more refined question of whether any increase in Z_2 can be substantial enough to raise consumer surplus. To show that this can indeed happen, we consider the following example based on multiplicatively separable preferences.

Example 1. *Consumers have multiplicatively separable preferences, so that a consumer of type θ who purchases a product of quality q has utility $v(\theta, q) = \theta q$. Also, $\theta \sim F(\theta)$ with $F(\theta) = \theta^{1/\alpha}$ for $\alpha > 0$.*

For Example 1, the demand functions for low quality and upgrades to high quality are

$$P_1(Z_1) = q_L(1 - Z_1)^\alpha, \text{ and}$$
$$P_2(Z_2) = (q_H - q_L)(1 - Z_2)^\alpha.$$

These are log-concave and so satisfy our assumption of decreasing marginal revenue.

In our examples all firms have the same cost c_1 for low quality. One firm i has upgrade costs $c_2^i - c_1^i = c_2 - c_1$ such that it produces high quality, but the other two firms have upgrade costs so high that they only produce low quality.

There are two possible mergers, one involving both low quality firms and one involving a low quality firm and the high quality firm. The effect on consumer surplus is the same for each type of merger.[13] For various parameterizations the effects of mergers are reported in Table 1. All of the mergers raise consumer surplus, with such increases ranging from just over 1% to almost 8.5%.[14] All mergers are profitable when the acquirer is high quality. The mergers involving both low quality firms are less profitable and may incur a loss. But for any of these mergers, there may be fixed-cost savings such as from the shuttering of a manufacturing facility, possible compensating for the loss in variable profits.

Consumers overall and outsiders both gain from the mergers in Table 1, which cannot occur in a single-product market. Indeed, here most consumers purchase high quality and so benefit; consumer surplus gains are not limited to a small "luxury segment." For the mergers that benefit insiders, total surplus increases despite the absence of synergies. For the mergers involving the high-quality firm, the merged firm sells only high quality—the merged firm removes the low-quality product from its portfolio and yet consumer surplus increases.

As in our discussion of the repositioning defense following Proposition 2, this suggests caution in making inferences regarding consumer surplus based on either the observed addition or removal of products from industry participants' portfolios.[15]

[13]The reason is that, due to our cost assumptions in these examples, each merger's effect on consumers is equivalent to removing a low quality firm from the market.

[14]We note that there are many parameterizations of Example 1 that generate much larger consumer-surplus gains and which are profitable. However, the required parameters are somewhat extreme and so we do not emphasize these.

[15]The 2018 OECD document *Considering non-price effects in merger control* states that, "The theory of harm associated with the impact of an anticompetitive merger on product positioning is relatively simple: unilateral effects

Case	%CSΔ	%ProfitΔ(L)	%ProfitΔ(H)
(i)	1.10	0.17	0.62
(ii)	2.97	-4.85	0.69
(iii)	8.46	-14.31	0.70

TABLE 1. Percentage changes in consumer surplus %CSΔ and variable profit of the merging parties for a low-quality acquirer %ProfitΔ(L) or high-quality acquirer %ProfitΔ(H), based on Example 1. Parameters for the three cases are given by $(\alpha, c_1, c_2, q_L, q_H)$: case (i) $(0.3, 0.25, 0.25, 1, 11)$, case (ii) $(0.2, 0.3, 0.3, 1, 11)$, and case (iii) $(0.1, 0.15, 0.25, 1, 21)$.

We now generalize the insights from Example 1, showing how curvature properties of demand influence the size of any increase in Z_2. Our parameterizations above exhibit $\alpha \in (0,1)$ such that the demand for low quality is concave and more concave than the demand for upgrades. That is, recalling our earlier definition of curvature, $\sigma_1(Z_1) < 0$ and $\sigma_1(Z_1) < \sigma_2(Z_2)$.

We begin with the concept of an "infinitesimal merger" involving two firms (introduced by Farrell and Shapiro (1990)). One firm is the acquirer and the other is the "target," with the target having weakly higher costs than the acquirer for both products. Because of this assumption on costs a merger is equivalent to removing the target from the market. We define an infinitesimal merger as a small exogenous reduction in the target firm's output, beginning from the pre-merger industry configuration, with all other firms including the acquirer adjusting their own outputs in response, resulting in a new industry equilibrium.[16] If we were to continue reducing the target firm's output to zero, then by integrating these output changes along the entire path we would arrive at the correct equilibrium configuration corresponding to a complete merger.

As we will see, this approach allows us to isolate the effect of demand curvature on the change in consumer surplus following a merger. Consumer surplus can be written as

$$\int_{\theta(Z_1)} \left[v\left(\theta, q_L\right) - P_1\left(Z_1\right)\right] dF\left(\theta\right) + \int_{\theta(Z_2)} \left[\left(v\left(\theta, q_H\right) - v\left(\theta, q_L\right)\right) - P_2\left(Z_2\right)\right] dF\left(\theta\right).$$

Denote the changes to total output Z_1 and high-quality output Z_2 from an infinitesimal merger by dZ_1 and dZ_2. Then consumer surplus increases if and only if

$$-Z_1 P_1'(Z_1) dZ_1 - Z_2 P_2'(Z_2) dZ_2 > 0.$$

Given that $dZ_1 < 0$ this can only hold if $dZ_2 > 0$. Supposing this is the case, consumer surplus increases if and only if

$$\left|\frac{dZ_2}{dZ_1}\right| > \frac{Z_1 P_1'(Z_1)}{Z_2 P_2'(Z_2)}. \tag{2}$$

may arise, allowing firms to reduce variety in a post-merger product offering, and avoid introducing future variety." We are not saying this is incorrect but merely that caution is required.

[16] If the target is a multiproduct firm, we imagine lowering its total output and its upgrade output at a fixed rate which is proportional to the pre-merger ratio of the target's total output to its upgrade output.

In other words, conditional on the infinitesimal merger indeed raising high-quality output, consumer surplus increases whenever the magnitude of the relative increase in high quality supply (compared to the decrease in total supply) is sufficiently large.

Proposition 3. *Suppose at least one firm is producing only high-quality products. Consider an infinitesimal merger in which the target's pre-merger ratio of high-quality output to total output is sufficiently small.*

(1) The total supply of high quality increases, that is $dZ_2 > 0$.

(2) The relative increase in high quality supply, given by $|dZ_2/dZ_1|$, is larger whenever:

(a) the demand for low-quality products is more concave ($\sigma_1(Z_1)$ is smaller),

(b) the demand for upgrades to high quality is more convex ($\sigma_2(Z_2)$ is larger).

We now explain the economic intuition behind Proposition 3. For the sake of clean expository math and intuition, we assume there are no multiproduct firms and hence that the target sells only low quality. Recall that a firm i producing only high quality with output $Z_1^i = Z_2^i = Z^i$ has first-order condition

$$[P_1(Z_1) + P_2(Z_2)] + Z^i [P_1'(Z_1) + P_2'(Z_2)] = c_2^i.$$

If we sum up the first-order conditions over all firms producing high quality, and use the fact that their total output equals Z_2, then the implicit function theorem reveals that a small equilibrium industry decrease in Z_1 leads to the following equilibrium increase in Z_2:

$$\left|\frac{dZ_2}{dZ_1}\right| = \frac{P_1'(Z_1)\left[n^H - \frac{Z_2}{Z_1}\sigma_1(Z_1)\right]}{P_2'(Z_2)\left[n^H - \sigma_2(Z_2) + 1\right] + P_1'(Z_1)}, \tag{3}$$

where n^H denotes the number of firms producing high-quality products.

Consistent with Proposition 3 and our numerical examples, the response given in (3) is larger when $\sigma_1(Z_1)$ is smaller and $\sigma_2(Z_2)$ is larger. The reason is that when $\sigma_1(Z_1)$ is smaller, an equilibrium decrease in Z_1 leads to a larger increase in the combined marginal revenues of firms making high-quality products. To equilibrate, Z_2 must increase. But when $\sigma_2(Z_2)$ is larger, marginal revenue is less sensitive to such increases, meaning that the expansion of high-quality output must be larger to re-equilibrate the industry after the merger.

We now provide conditions under which consumer surplus increases from a complete merger if it increases from an infinitesimal merger. To do so while continuing to emphasize the role of demand curvature, we specialize to the case in which $\sigma_1(Z_1)$ and $\sigma_2(Z_2)$ are constant as functions of output but possibly different from one another.

Proposition 4. *Suppose that there are no multiproduct firms and that the demand system exhibits constant curvatures $\sigma_1 \leq 0$ and $\sigma_2 \leq 0$. If an infinitesimal merger involving a low-quality target firm would raise consumer surplus (at the pre-merger output levels), then a complete merger would also raise consumer surplus.*

Proposition 4 ensures that if demand curvatures are such that $|dZ_2/dZ_1|$ as given in (3) exceeds $Z_1 P_1'(Z_1)/Z_2 P_2'(Z_2)$, so that an infinitesimal merger raises consumer surplus as required by (2), then a complete merger raises consumer surplus. We emphasize that this demand specification is readily generated using non-separable preferences. Moreover, it is easy to provide examples based on this specification that lead to profitable mergers, gains for outsiders, and consumer-surplus increases, just as for Example 1.

Returning to the more general case, we observe that there is a lower bound that the difference in demand curvature $\sigma_2(Z_2) - \sigma_1(Z_1)$ must exceed, if consumer surplus is to increase.

Remark 1. *A necessary condition for an infinitesimal merger to raise consumer surplus is that* $\sigma_2(Z_2) - \sigma_1(Z_1) > 1$ *(at the equilibrium quantities). An implication is that consumer surplus decreases from such a merger if consumer preferences are multiplicative with* $\theta \sim F(\theta)$*, where* $F(\theta)$ *induces a demand curve that has increasing or positive curvature.*

We now connect our results to third-degree price discrimination. Much work has been done on bans on such monopolistic price discrimination, including Robinson (1933), Schmalensee (1981), Varian (1985), Cowan (2007), and Aguirre, Cowan, and Vickers (2010). The focus of this literature is the effect on total surplus. A beneficial aspect of a ban is that it results in better allocation of any given number of units, whereas a potential harm is that total output may fall. In such cases, the effect of a merger on consumer surplus is related to the effect on total surplus of a ban on price discrimination. This literature is also connected to our work in that the curvature of demand, and in particular differences in the curvature of demand across different markets, plays a key role in the analysis and conclusions.

2.2. **The Effects of Synergies.** For a merger involving firms $\mathcal{I}$, a synergy for product k means that the merged firm m's marginal cost c_k^m is lower than $\min_{i \in \mathcal{I}} c_k^i$. We will show that in some situations synergies work entirely as expected using the intuition from single-product markets, but in other situations are more nuanced and can even lower consumer surplus.[17]

We begin by identifying when synergies work as in a single-product setting, unambiguously raising consumer surplus. For analytical convenience we state our results in terms of small synergies, that is small reductions in post-merger marginal costs as just discussed.

Proposition 5. *A small synergy increases consumer surplus in the following three cases.*

(1) The merged firm only produces high quality and its cost of high quality decreases.

(2) No firm produces only high quality, and

  (a) the merged firm's cost of high quality decreases, or

  (b) the merged firm only produces low quality and its cost of low quality decreases.

[17] Williamson (1968) explores cost efficiencies as a merger defense. Chen and Li (2018) show that synergies may raise industry prices when the pattern of substitution between differentiated products satisfies certain conditions.

For the case in which the merged firm produces only high quality there is a strategic linkage between the two markets. However, because the merged firm is upgrade-constrained, the direct effect of the synergy is an equal increase in m's output in both the baseline and upgrade markets. Although other firms respond by lowering their own outputs, the response is not enough to overturn the direct effect; both prices fall as a result of such a synergy.

For the two cases in which no firms produce only high quality, the result holds because (i) there is no strategic linkage between the baseline and upgrade markets because no firm is upgrade-constrained, and (ii) the synergy only affects one marginal cost of relevance. To understand condition (ii) it is easiest to understand how it might fail. If the merged firm m sold both low- and high-quality products, then a synergy that lowered its cost of producing low quality c_1^m would also *raise* its cost of upgrading to high quality $c_2^m - c_1^m$ and hence would influence the merging firm's product mix. Such a synergy might lower consumer surplus.

In cases beyond those in Proposition 5, synergies have more complicated effects on consumer surplus. Our discussion has already identified why this may be. Either the merging firm sells both products and the synergy lowers the marginal cost of low quality but therefore raises upgrade costs, or there is a strategic linkage because the merged firm is not only producing high quality but some other firms are. These other cases may involve consumer-surplus tradeoffs, with one price falling but the other rising.

Recalling our earlier discussion, we know that the change in consumer surplus for some small changes in outputs dZ_1 and dZ_2 is positive if and only if

$$-Z_1 P_1'(Z_1) dZ_1 - Z_2 P_2'(Z_2) dZ_2 > 0.$$

In situations where a synergy increases the supply of high quality but reduces total supply, the resulting change in consumer surplus is more readily positive when $|dZ_1/dZ_2|$ is smaller, as was also true for a merger with no synergies (Proposition 3). In situations where a synergy increases total output but lowers upgrade supply, the resulting change in consumer surplus is more readily positive when $|dZ_1/dZ_2|$ is larger.

Proposition 6 connects these relative output changes to curvature properties of demand.

Proposition 6. *Assume the merged firm produces both products.*

(1) A small synergy that reduces the merged firm's cost of the low-quality product leads to an increase in total output but a reduction in the supply of high quality.

(2) A small synergy that reduces the merged firm's cost of the high-quality product leads to a (weak) decrease in total output but an increase in upgrade supply.

(3) In both cases, $|dZ_1/dZ_2|$ is increasing in $\sigma_1(Z_1)$ and decreasing in $\sigma_2(Z_2)$.

The intuition for Proposition 6 is straightforward. A general fact is that the sensitivity of marginal revenue to further industry output increases (in either the baseline market or the upgrade market) is larger when demand is more convex and smaller when it is more concave. Hence, when a multiproduct firm experiences a decrease in the cost of the low-quality product, industry output of Z_1 will respond more when marginal revenue is less sensitive, that is when demand for low-quality products is more concave. This same synergy increases the firm's upgrade costs, and the corresponding decrease in Z_2 will be smaller when marginal revenue is more sensitive, that is when demand for upgrades is more convex. An opposing logic applies when the synergy lowers the cost of the firm's high-quality product.

We can derive precise thresholds that determine whether a synergy will raise consumer surplus. Details are in the proof of Proposition 6. For example, a synergy that reduces a multiproduct firm's cost of low quality raises consumer surplus if and only if

$$\frac{1+n-\sigma_1(Z_1)}{1+n-n^L-\sigma_2(Z_2)} < \frac{Z_1}{Z_2},$$

where n is the number of post-merger firms and n^L is the number of such firms only producing low quality. This confirms that synergies can indeed lower consumer surplus.

3. Merger Profitability

In this section we address the classic question of whether firms can use horizontal mergers to raise their profits. One benefit to insiders of such mergers is enhanced market power that may lead to lower industry output and higher prices. But competitive responses may limit any such gains if outsiders take advantage of the opportunity presented to be more aggressive, for example by expanding output as in the standard quantity-setting framework.

Indeed, Salant, Switzer, and Reynolds (1983) have argued that the competitive response of outsiders is frequently strong enough to dominate, causing many such mergers to be unprofitable for insiders. In a single-product quantity-setting industry with linear demand and symmetric firms with constant marginal costs, a necessary condition for a merger to be profitable is that the merging parties have at least an 80% pre-merger market share. Both Levin (1990) and Cheung (1992) have extended this analysis, showing that, even allowing for general demand functions, a 50% threshold is necessary.

Others have attacked this conclusion and its implication that a merger involving only two firms is unprofitable unless the industry is a duopoly pre-merger. Perry and Porter (1985) show that if mergers allow firms to lower their average costs by combining their capital stocks, then pairs of small firms with limited market shares may find it profitable to merge.[18]

We will show that the profitability of horizontal mergers is meaningfully altered when there are products of different qualities in the market, even maintaining most aspects of Salant,

[18]Daughety (1990) shows that if merging firms become Stackelberg leaders then smaller mergers may be profitable. Deneckere and Davidson (1985) show that many mergers are profitable when firms set prices.

Switzer, and Reynolds (1983) such as no synergies, linear demand, and constant marginal costs. At the same time, however, we will demonstrate some close analogues between merger profitability in markets with single products and markets with multiple products.

3.1. **Linear Demand.** We begin by assuming that consumers have multiplicative preferences with θ distributed uniformly on $[0,1]$. This is a special case of Example 1 with $\alpha = 1$ and implies that the demand functions are linear:

$$P_1(Z_1) = q_L(1 - Z_1), \text{ and}$$
$$P_2(Z_2) = (q_H - q_L)(1 - Z_2).$$

We place no restrictions on the cost structures of outsider firms. We do assume that the $k+1$ insiders have symmetric costs, which rules out profitability gains from output rationalization.[19] We also assume there are no synergies.

Let n^L be the number of firms that are producing only low quality, n^H the number producing only high quality, and n^M the number of multiproduct firms, where all variables reference the pre-merger industry structure of firms with positive output. We assume each of these firms is active post-merger and offers the same product line as before the merger. Where appropriate we provide conditions on parameters that ensure this.

There are several different types of mergers that we will consider, beginning with the case in which each of the $k+1$ merging firms produces only low quality. After that, we show that other types of mergers exhibit similar properties.

Lemma 3. *Consider a merger between $k{+}1$ firms that each produce only low-quality products. Consider the ratio of the merged firm's profit to the sum of insiders' pre-merger profits. An increase in the number of low-quality n^L or multiproduct n^M firms reduces this ratio by strictly more than an increase in the number of high-quality firms n^H does.*

To see how this lemma implies that the presence of multiple qualities makes mergers more readily profitable, consider an initial market structure with only the low quality product and fix the number of insiders and the total number of firms. Now suppose that the high quality product becomes available and that (i) each insider i has upgrade cost $c_2^i - c_1^i$ sufficiently high that it does not produce high quality, but (ii) at least one outsider i that formerly made only low-quality products has upgrade costs $c_2^i - c_1^i$ such that it now chooses to make only high-quality products.[20] Then if the merger was profitable before this change it is also profitable after (and may be profitable after even if it was not profitable before).

[19] Levin (1990) and Fauli-Oller (2002) have noted that reallocation opportunities may make mergers more profitable.

[20] As discussed in Section 1, in the multiplicatively separable framework we are considering here a sufficient condition for firm i to make only high-quality products is that it has an upgrade cost $c_2^i - c_1^i$ of zero.

Below we provide intuition for why a high quality firm poses less of a constraint on a merger's profitability than either a multiproduct or low-quality firm. But first we show that this effect can be powerful.

Proposition 7. *Suppose that no firms produce both products ($n^M = 0$). A sufficient condition for a merger that involves all firms producing low quality to be profitable is that*

$$n^L \geq \left(\frac{q_H}{q_H - q_L} \right)^2 ,$$

regardless of the number of firms n^H producing only high quality.

Proposition 7 stands in stark contrast to the existing literature and along with Lemma 3 suggests that quality differences may make mergers amongst quantity-setting firms more profitable than previously recognized. As an illustration of the implications of this result, suppose that $n^L = 2$ and recall that in the single-product setting of Salant, Switzer, and Reynolds (1983) a merger between two firms is never profitable unless the pre-merger industry is a duopoly. In our setting, such a two-firm merger is profitable so long as $q_H \gtrsim 3.41 q_L$, even if the total number of firms producing high quality (and hence the total number of firms in the industry) becomes infinitely large.[21]

The intuition for Proposition 7 is as follows. When the low-quality firms merge the direct effect is a reduction in total output Z_1. Firms producing only high quality respond by raising their own output but because they are upgrade-constrained this means they must raise both their baseline and upgrade outputs. In other words, the profits of the high-quality firms depend on what happens in two markets and hence output increases have effects in both of these markets; this blunts their competitive response to the merger. Considering the limit as the number of responding firms n^H is very large, it is not possible that the post-merger level of Z_1 is the same as the pre-merger level. If it were, then, because Z_2 has increased, the price of high-quality products would be lower than before the merger which would imply that these firms are losing money post-merger.

So far we have emphasized how quality differences alter the usual conclusions about merger profitability in quantity-setting markets. The following proposition continues this theme and at the same time indicates some similarities with classic results.

Proposition 8. *Consider a merger of $k+1$ firms that each produce only low-quality products. A necessary condition for this merger to be profitable is that it involves at least 80% of the firms that, pre-merger, produce low-quality products. That is, a necessary condition is that*

$$\frac{k+1}{n^L + n^M} \geq \frac{8}{10}.$$

[21] To ensure that each firm has positive output certain conditions must hold. For example, suppose all outsiders are symmetric with marginal cost for high quality c_2^H, and let c_1^L be the marginal cost of low-quality firms. Both insiders and outsiders have positive output if $c_1^L / q_L < c_2^H / q_H$ and $c_2^H / (q_H - q_L) < 1$.

Remarkably, this is the same lower bound for merger profitability discovered by Salant, Switzer, and Reynolds (1983). However, in their analysis all firms in the industry have the same costs and so a necessary (and sufficient) condition for profitability is that the merging firms have at least 80% market share. In contrast, we emphasize that what matters is not insiders' share of output (either of the entire market or the low quality segment). Rather, what matters is their share of all firms that produce low-quality products. In this sense, the emphasis of the existing literature on the market shares of insiders is somewhat misplaced.

The spirit of our results above holds for other types of mergers. When the insiders produce only high-quality products rather than low-quality products, our main results hold exactly.

Remark 2. *Consider a merger involving $k+1$ firms producing only high-quality products. Swapping the n^L and the n^H terms, Lemma 3 and Propositions 7 and 8 hold exactly.*

For example, a necessary condition for profitability is that at least 80% of firms producing high-quality products take part in the merger, $(k+1)/(n^H + n^M) \geq 8/10$. Additionally, mergers involving as few as two firms producing only high-quality can be profitable even when the number of firms n^L producing low-quality products grows infinitely large

Mergers involving only multiproduct firms or mixed mergers are more complicated analytically but lead to similar qualitative results. We do not present formal results but instead note that a merger involving a fixed number of firms, each of which is a multiproduct firm pre-merger, may be profitable even when there are an infinite number of firms.

3.2. **Nonlinear Demand and the 50% Benchmark.** Here we briefly address merger profitability with nonlinear demand. Finding exact sufficient conditions for profitability is not straightforward with asymmetric firms and so we focus on necessary conditions for mergers to be profitable. We assume the insiders share the same pre-merger cost structure.

Proposition 9. *A necessary condition for a merger between $k+1$ low-quality firms or $k+1$ high-quality firms to be profitable is that*

$$\frac{k}{n^M + n^j - 2} \geq \frac{1}{2},$$

where $n^j = n^L$ or n^H if the insiders are low quality or high quality, respectively.

At least roughly, a 50% benchmark does apply—but only in terms of the ratio of insiders to the number of firms producing the same quality as the insiders. Additionally, it is possible to demonstrate that if demand is concave instead of merely log-concave, then a different threshold can be derived that is slightly harder to satisfy. Thus we find some support for the argument in Fauli-Oller (2002) (made in a single-product setting) that mergers are more profitable when demand is convex.

4. The Quality-adjusted Herfindahl-Hirschman Index

A common measure of industry concentration is the Herfindahl-Hirschman Index (HHI), which—in a single-product market—is defined solely in terms of the market shares of firms:

$$HHI = \sum_i (s^i)^2,$$

where s^i is the market share of firm i. Here we propose a version of this famous index that is appropriate for quality differentiated markets and which sheds light on questions of market definition by connecting market share data to a measure of industry participants' profits.

As a preliminary step we review a theoretical justification for using the HHI in single-product markets so that we can appropriately extend it to a multiproduct setting. In such a single-product market firm i's first-order condition can be used to write i's profits (at the equilibrium quantities) as

$$(P(Z) - c^i)Z^i = -P'(Z)(Z^i)^2.$$

Summing this across all firms to derive industry (variable) profit and then dividing by industry revenue $ZP(Z)$ gives

$$\frac{\text{Industry Profit}}{\text{Industry Revenue}} = \frac{-P'(Z)\sum_i (Z^i)^2}{ZP(Z)} = \left(\frac{-ZP'(Z)}{P(Z)}\right)\sum_i (s^i)^2 = \frac{HHI}{\epsilon}, \tag{4}$$

where the market share of firm i is $s^i = Z^i/Z$ and $\epsilon = -P(Z)/(ZP'(Z))$ is the elasticity of (inverse) demand. Thus, the HHI divided by ϵ is the gross profit margin of the industry.[22] As such the HHI connects market share data to a measure of industry profits, as pointed out by Cowling and Waterson (1976).

We now extend the standard HHI to a Quality-adjusted Herfindahl-Hirschman Index (QHHI).

4.1. Constructing the QHHI.

Following steps similar to those above, we compute the gross profit margin of a multiproduct industry. At the equilibrium outputs, firm i's profit is

$$(P_1(Z_1) - c_1^i)Z_1^i + (P_2(Z_2) - (c_2^i - c_1^i))Z_2^i = -P_1'(Z_1)(Z_1^i)^2 - P_2'(Z_2)(Z_2^i)^2,$$

using the first-order conditions for baseline units and upgrades for firm i. Note that this expression applies for any firm with any product line, although for a firm selling only high-quality products it is derived using the single first-order condition that firm i satisfies, specifying $Z_1^i = Z_2^i$. And, for firms producing only low-quality products, the first-order condition for upgrades is not satisfied but this does not cause any problems because $Z_2^i = 0$.

[22] The HHI divided by ϵ is also equal to the share-weighted sum of Lerner ratios in the single-product industry.

Hence, we can always correctly write industry profits as $\sum_i[-P_1'(Z_1)(Z_1^i)^2 - P_2'(Z_2)(Z_2^i)^2]$. Industry revenue is $Z_1P_1(Z_1) + Z_2P_2(Z_2)$. The gross profit margin of the industry is

$$\begin{aligned}\frac{\text{Industry Profit}}{\text{Industry Revenue}} &= \frac{\sum_i [-P_1'(Z_1)(Z_1^i)^2 - P_2'(Z_2)(Z_2^i)^2]}{Z_1P_1(Z_1) + Z_2P_2(Z_2)} \\ &= \frac{Z_1P_1(Z_1)\left[\frac{1}{\epsilon_1}\sum_i(s_1^i)^2\right] + Z_2P_2(Z_2)\left[\frac{1}{\epsilon_2}\sum_i(s_2^i)^2\right]}{Z_1P_1(Z_1) + Z_2P_2(Z_2)} \\ &= \frac{R_1}{R_1+R_2}\left[\frac{1}{\epsilon_1}\sum_i(s_1^i)^2\right] + \frac{R_2}{R_1+R_2}\left[\frac{1}{\epsilon_2}\sum_i(s_2^i)^2\right], \qquad (5)\end{aligned}$$

where $\epsilon_k = -P_k(Z_k)/(Z_kP'(Z_k))$ is the elasticity of either total demand ($k = 1$) or upgrade demand ($k = 2$), $R_k = Z_kP(Z_k)$ is the revenue associated with either baseline units ($k = 1$) or upgrades ($k = 2$), and $s_k^i = Z_k^i/Z_k$ is firm i's share of total market supply ($k = 1$) or upgrade supply ($k = 2$).

The traditional HHI in effect assumes that the elasticity of demand is constant and focuses solely on market shares. With multiple products, ignoring the demand elasticities is harder to justify because they give important comparative margin information across the baseline and upgrade markets. However, if we suppose that $\epsilon_1 = \epsilon_2 = \epsilon$ for some constant $\epsilon > 0$, as is the case when consumers have multiplicative preferences and demand exhibits constant elasticity, then using (5) as a stepping stone we define the QHHI as follows.

$$QHHI = \frac{R_1}{R_1+R_2}\sum_i(s_1^i)^2 + \frac{R_2}{R_1+R_2}\sum_i(s_2^i)^2 = \frac{R_1}{R_1+R_2}HHI_1 + \frac{R_2}{R_1+R_2}HHI_2.$$

The QHHI is a weighted average of the HHI measured in terms of total output (HHI_1) and in terms of upgrades (HHI_2). One weight ($R_1/(R_1+R_2)$) is the proportion of revenue generated by baseline units and the other weight ($R_2/(R_1+R_2)$) is the proportion of revenue generated by upgrades. This indicates that a higher HHI_1 is more concerning when more of the total revenue is generated from baseline units, whereas a higher HHI_2 is more concerning when more of the total revenue is generated from upgrade units.

4.2. **Applying the QHHI.** The QHHI, like the traditional HHI, is meant to provide a quick first-pass assessment of an industry. It does require slightly more information than the HHI, as it utilizes revenue information in addition to market-share data, but is still simpler than more elaborate options such as SSNIP tests.[23] Knowing the market prices for low- and high-quality units allow the computation of the implicit upgrade price $P_2(Z_2)$ and hence both R_1 and R_2. We emphasize that R_1 is not the revenue associated with the low-quality

[23]The SSNIP test (Katz and Shapiro, 2003; O'Brien and Wickelgren, 2003) and its multiproduct firm extension (Moresi, Salop, and Woodbury, 2008) require detailed cost data as well as estimates of certain diversion ratios.

products and R_2 is not the revenue associated with complete high-quality products, although they are readily computed from price and quantity data for these products.

What would happen if an enforcement agency ignored the multiproduct nature of the market and computed a single HHI measure based on total market shares? This corresponds to using (only) HHI_1 as a concentration index rather than QHHI. This means overweighting shares of total output and ignoring the concentration of the upgrade market, which is problematic except in the special case in which the industry is fully symmetric.[24] Similarly, only looking at HHI_2 means overweighting the upgrade market, which is also likely to be problematic outside of the fully symmetric case.

The QHHI probably provides the most guidance in cases where an agency finds that HHI_1 is high but HHI_2 is low (or the other way around), by suggesting a weighting procedure. Another possibility is that an agency would compute an HHI for low-quality goods, either exclusively or in addition to either or both HHI_1 and HHI_2. However, concentration in the low-quality market is not a direct input into the QHHI. The reason that low-quality units do not comprise the correct market is that such units also compete with the baseline units that are ultimately upgraded into high-quality goods. Thus, concentration in the low-quality market could be very high and yet there might be little cause for concern if such units represent a small share of total units.

In terms of assessing actual changes to the QHHI consequent to a merger, we suggest—as is often done in single-product markets for a first-pass assessment—computing the post-merger market shares in the naive fashion of assuming that the merging firms are replaced with one firm that has market shares and output equal to the combined pre-merger market shares of the insiders, and that outsiders' market shares do not change.

To illustrate, consider a merger between firm 1 and firm 2 and let s_k^i denote their pre-merger market shares (for $i = 1, 2$ and $k = 1, 2$). Then, mechanically, the increases in HHI_1 and HHI_2 are given by $\Delta HHI_1 = 2s_1^1 s_1^2$ and $\Delta HHI_2 = 2s_2^1 s_2^2$, leading to a change in QHHI denoted by $\Delta QHHI$, where

$$\Delta QHHI = \left(\frac{R_1}{R_1 + R_2}\right) 2s_1^1 s_1^2 + \left(\frac{R_2}{R_1 + R_2}\right) 2s_2^1 s_2^2.$$

The expression for $\Delta QHHI$ makes it clear that changes in concentration must be tracked at both the level of total output and also upgrade output. The only exception is when $s_1^1 s_1^2 = s_2^1 s_2^2$ which does not seem particularly likely outside of certain special cases.[25]

[24] If all firms have identical cost structures and consequently the same outputs then $s_1^i = s_2^i$ for each firm i. This means that $HHI_1 = HHI_2$ and because QHHI is a weighted average of these two measures there is no loss of information from using a traditional HHI measure instead of the QHHI.

[25] Note for example that $s_1^1 s_1^2 = s_2^1 s_2^2$ is not implied merely by firms 1 and 2 being symmetric; such symmetry only implies that $s_1^1 = s_1^2$ and $s_2^1 = s_2^2$.

$\Delta QHHI$ suggests that mergers which involve changes in concentration in the upgrade market may be more concerning or deserving of additional scrutiny than mergers which only affect concentration in the overall market. That is, fixing the market shares s_1^1 and s_1^2, a merger between two firms that each produce high quality ($s_2^1 s_2^2 > 0$) has a higher value of $\Delta QHHI$ than a merger in which at most one firm produces high quality ($s_2^1 s_2^2 = 0$).

Under certain conditions, a full equilibrium analysis confirms that such mergers are indeed worse for consumers. In the spirit of using $\Delta QHHI$ as a first-pass assessment, we will intentionally "shut down" the prospect of a beneficial shift in the equilibrium product mix by assuming that each firm either sells only low quality or else is a multiproduct firm.

Proposition 10. *Consider a market in which no firm produces only high quality. Consider any two mergers, each involving two firms and no synergies, where each potential insider has the same total market share. Then amongst all such possible mergers:*

(1) the worst for consumer surplus is a merger involving two multiproduct firms, and
(2) all other mergers are equivalent in terms of their effect on consumer surplus.

Under the conditions given, a merger involving two multiproduct firms reduces both total supply and the supply of upgrades, whereas the other mergers only lower total supply.

5. External Effects and Outsider Profits

In this section we first examine *external effects* of mergers, where the external effect is defined as the merger-induced change in the sum of consumer surplus and outsider profits. After that, we explore in more detail the profits of outsiders alone, emphasizing that mergers can have disparate effects on different outsiders.

5.1. External Effects.

When the external effect of a merger is positive then—assuming that only mergers which benefit insiders are proposed—the merger must raise overall surplus. As such the analysis of external effects provides another view on the welfare effects of a merger, complementing our earlier results on consumer surplus and the QHHI.

To assess the external effect of a merger, we will use the idea of an infinitesimal merger from earlier. However, here we will be more general and open to the prospect that the merger may generate synergies and hence lead to arbitrary equilibrium changes in total industry output dZ_1 and industry high-quality output dZ_2.

For such small but arbitrary changes, what is the net effect on outsider firms and consumers combined? As in Section 4, we let the market share of firm i in the baseline market be denoted by s_1^i and in the upgrade market by s_2^i, and write $s_1^{\mathcal{I}}$ and $s_2^{\mathcal{I}}$ as the corresponding shares of the insiders combined. In the appendix, we show that the external effect $d\mathcal{E}$ is

$$\begin{aligned} d\mathcal{E} =& Z_1 P_1'(Z_1) \left\{ \sum_{i \in \mathcal{O}} s_1^i \left[1 - s_1^i \sigma_1 (Z_1) \right] - s_1^{\mathcal{I}} \right\} dZ_1 \\ &+ Z_2 P_2'(Z_2) \left\{ \sum_{i \in \mathcal{O}} s_2^i \left[1 - s_2^i \sigma_2 (Z_2) \right] - s_2^{\mathcal{I}} \right\} dZ_2. \end{aligned} \tag{6}$$

This says that the external effect is the sum of an external effect in the baseline market plus an external effect in the upgrade market, where these two effects have a similar form. If the upgrade market does not exist (that is, if there are only low-quality goods) and the merger lowers total output so that $dZ_1 < 0$, then the external effect (6) is positive if and only if

$$s_1^{\mathcal{I}} < \sum_{i \in \mathcal{O}} s_1^i \left[1 - s_1^i \sigma_1 (Z_1) \right].$$

This is the same condition that Farrell and Shapiro (1990) find: an output-decreasing merger has a positive external effect whenever the market share of insiders is not too big.

We now discuss the key economic implications of external effects in quality-differentiated markets. First, as is evident from (6), it is important to correctly define markets. Specifically, although market shares for both total supply and high-quality supply matter, market shares for low-quality supply do not. This is the same conclusion reached in our earlier assessment of the QHHI in Section 4. As we noted there, low-quality market shares are not crucial because low-quality products compete not only against other low-quality products but also against the baseline units that are ultimately upgraded to high quality.

For example, consider a merger that is anticipated to lower both Z_1 and Z_2 and hence to unambiguously raise both prices. Such an outcome might occur if two multiproduct firms merged and no firms were selling only high-quality products, as suggested by Lemma 2. With $dZ_1 < 0$ and $dZ_2 < 0$, a sufficient condition for the external effect to be positive is that

$$s_1^{\mathcal{I}} < \sum_{i \in \mathcal{O}} s_1^i \left[1 - s_1^i \sigma_1 (Z_1) \right] \text{ and } s_2^{\mathcal{I}} < \sum_{i \in \mathcal{O}} s_2^i \left[1 - s_2^i \sigma_2 (Z_2) \right].$$

In the special case of linear demand, these conditions reduce to

$$s_1^{\mathcal{I}} < \sum_{i \in \mathcal{O}} s_1^i \text{ and } s_2^{\mathcal{I}} < \sum_{i \in \mathcal{O}} s_2^i.$$

Although this requires that the market share of insiders is not too big in terms of total supply or in terms of upgrade supply, it puts no requirements on insiders' share of the low-quality market itself. The reverse situation might arise if insiders have a small market share in the low-quality market but have large overall market shares and large shares in the upgrade market—their merger might exhibit a negative external effect.[26]

Second, even when correctly defining markets, a merger may exhibit a positive external effect even if insiders have substantial market share for either total supply or upgrade supply (but not both), even when both dZ_1 and dZ_2 are negative. For example, consider insiders with a large share of the total market but a small share of the upgrade market. From inspection of (6), if $Z_2 P_2'(Z_2) dZ_2$ is large compared to $Z_1 P_1'(Z_1) dZ_1$, then the external effect could be positive. Or, if insiders have a large share of the high-quality market but have a small overall market share, and if $Z_1 P_1'(Z_1) dZ_1$ is large compared to $Z_2 P_2'(Z_2) dZ_2$, then the external effect could also be positive.

Third, in some cases of interest a positive external effect can be associated with a large market share for insiders. For example, suppose that $dZ_1 < 0$ but $dZ_2 > 0$. This situation arises if insiders only produce low quality but some outsiders produce only high quality (Proposition 2). Or, if synergies were considered, this might occur if two multiproduct firms merge and experience a reduction in their costs of making high-quality products. In such cases, a sufficient condition for the external effect to be positive is that

$$s_1^{\mathcal{I}} < \sum_{i \in \mathcal{O}} s_1^i \left[1 - s_1^i \sigma_1 \left(Z_1\right)\right] \text{ and } \sum_{i \in \mathcal{O}} s_2^i \left[1 - s_2^i \sigma_2 \left(Z_2\right)\right] < s_2^{\mathcal{I}}.$$

Ideally insiders have a high share of high-quality output but a small share of total output.

The following proposition summarizes some of the results discussed above.

Proposition 11. *For a given set of insiders to a merger, consider arbitrary small changes in industry output* (dZ_1, dZ_2)*. The total effect of these changes on the welfare of consumers and the profits of outsiders combined is positive if and only if* $d\mathcal{E}$ *as given in* (6) *is positive. For given output levels* Z_1 *and* Z_2*,* $d\mathcal{E}$ *is larger when*

(1) the market share of each outsider is smaller (so that insiders have larger combined market share $s_k^{\mathcal{I}}$*) when* dZ_k *is positive,*

(2) the market share of each outsider is larger (so that insiders have smaller combined market share $s_k^{\mathcal{I}}$*) when* dZ_k *is negative.*

5.2. **Outsider Profits.** We now turn from external effects to just the outsiders. In single-product quantity-setting markets each outsider benefits from a merger if and only if it leads to lower industry output; although different outsiders may gain or lose more than others, if one gains (or loses) then they all do.[27] We begin by providing cases in which the same is true in our multiproduct environment.

[26]For instance, insiders might either sell only or mostly high-quality products. If the industry supply of low-quality products is small compared to total output, then the shares $s_1^{\mathcal{I}}$ and $s_2^{\mathcal{I}}$ could be large.

[27]A similar effect emerges in studies of price competition, where typically all prices move together.

Proposition 12. *If a merger with no synergies occurs, then the profits of any given outsider i increase if any of the following hold.*

(1) The outsider i sells only low-quality products pre-merger,
(2) The merger lowers the supply of high-quality goods Z_2,
(3) Consumers have multiplicative preferences with θ distributed uniformly on $[0, 1]$ (demand is linear) and there is an insider with weakly lower costs than other insiders.

Item 1 is a corollary of Proposition 1, which ensures that a merger with no synergies lowers total output. Because the profits of firms that sell only low-quality products are decreasing in the equilibrium level of Z_1 and independent of Z_2, such firms are better off from any merger without synergies. Item 2 follows because a reduction in the equilibrium supply of high-quality products Z_2 benefits all firms that sell high-quality products, as does the reduction in Z_1. (Lemma 2 and Proposition 2 provide sufficient conditions for Z_2 to decrease.)

Item 3 is more subtle because a merger might increase Z_2 even as it lowers Z_1. Firms selling high-quality products would then lose in the upgrade market but gain in the baseline market. However, as suggested in Section 2, any increase in Z_2 is larger when there is a larger difference between the curvature in the upgrade market and that in the baseline market, and with linear demand in each market this difference is zero. In this light it is not too surprising that all outsiders gain with linear demand.

Nonetheless, with nonlinear demand there are cases in which mergers with no synergies lower the profits of some outsiders.[28] And for arbitrary mergers exhibiting synergies there are few constraints on how Z_1 and Z_2 change, meaning it is not hard to find cases in which some but not all outsiders lose from a merger. To see this, consider arbitrary small equilibrium changes dZ_1 and dZ_2. As shown in the appendix, for any outsider i the effect on i's profit is

$$d\pi^i = Z_1 P_1'(Z_1) s_1^i \left[2 - s_1^i \sigma_1 (Z_1)\right] dZ_1 + Z_2 P_2'(Z_2) s_2^i \left[2 - s_2^i \sigma_2 (Z_2)\right] dZ_2. \tag{7}$$

One possibility, evident from (7), is that a merger whose primary effect is to increase output in the upgrade market will harm firms that sell high-quality products but benefit other firms. To be precise, suppose that the merger leads to a small decrease in Z_1 but a significant increase in Z_2. This could easily happen if two multiproduct firms merged and experienced synergies that were more significant in the upgrade market, for example. For a firm with non-trivial market shares in the upgrade market, profit would fall.

[28] By allowing for two asymmetric firms selling only high quality, each of the numerical examples provided for Example 1 can be extended so that the higher cost firm loses profits as an outsider.

6. Conclusion

We have provided a framework for assessing merger activity in markets where quality differences are central. Our framework allows for asymmetric firms which may sell multiple products, but also for firms that specialize in either low or high quality. Both merging and non-merging firms may reposition their product lines by adding or removing products following a merger. Using this framework, we address classic topics including the welfare effects of mergers, the profitability of mergers, and market definition.

The mergers we study exhibit materially different effects from those studied in the existing literature. For example, a merger without synergies may raise consumer surplus (and benefit both insider and outsider firms). Synergies, when present, may lower consumer surplus. We also find that mergers are more profitable in the presence of multiple qualities. And our analysis of external effects shows that the external effect may be larger when insiders' market share is larger, while our analysis of the effects of mergers on outsider profits reveals that some outsiders may gain while others lose—neither of these outcomes typically occurs in studies of single-product markets.

At the same time, we found that in certain situations our predictions match well those based on models in which quality differences are not important and each firm sells a single product. For instance, a merger with no synergies involving firms that each sell only high quality unambiguously raises all prices and harms consumers (Proposition 2). And when a merged firm sells only high quality, synergies that reduce its costs certainly benefit consumers (Proposition 5). An implication of these two results is that mergers involving firms that produce only high quality must exhibit strong synergies if consumers are to benefit.

We introduced a new measure of industry concentration. The QHHI requires little information and so permits a low-cost initial merger assessment. By design, the QHHI considers how a merger may change concentration across the different quality segments of the market. One prediction from our analysis of this new measure of concentration is that mergers involving two multiproduct firms are deserving of greater scrutiny than other types of mergers.

Appendix For Online Publication: Omitted Proofs

We require several lemmas. For lemmas 4 and 5, we let Z^* and Z^{**} denote two distinct equilibria, one representing the industry before a merger and the other representing the industry after the merger; it doesn't matter which one is which. Similarly, Z_k^* and Z_k^{**} represent industry upgrades in market $k \in \{1,2\}$, and Z_k^{i*} and Z_k^{i**} represent firm i's outputs.

We alert the reader that in this appendix we sometimes use the term "upgrades" to refer both to market $k=1$ and $k=2$, whereas in the body of the manuscript we reserved the term upgrades for market $k=2$. We do this to avoid unnecessarily lengthening the proofs.

Lemma 4. *Suppose that $Z_k^{**} \geq Z_k^*$ and $Z_k^{i**} \geq Z_k^{i*}$ where at least one inequality is strict.*

(1) If, ignoring its monotonicity constraint, firm i has a weak incentive to raise its output Z_k^{i**}, then it has a strict incentive to raise its output Z_k^{i*} when either (i) it is not involved in the merger, or (ii) it is involved in the merger but has weakly lower cost for upgrade k when it chooses Z_k^{i*}.

(2) If, ignoring its monotonicity constraint, firm i has a weak incentive to lower Z_k^{i*}, then it has a strict incentive to lower Z_k^{**i} when either (i) it is not involved in the merger, or (ii) it is involved but has weakly higher cost for upgrade k when it chooses Z_k^{i**}.

Proof. We prove the first claim; the second can be proven similarly. Because firm i has a weak incentive to raise Z_k^{i**} ignoring its upgrade constraint,

$$P_k(Z_k^{**}) + Z_k^{i**} P_k'(Z_k^{**}) \geq C_k^i,$$

where C_k^i is firm i's upgrade cost for k (c_1^i if $k=1$ and $c_2^i - c_1^i$ if $k=2$). (Recall what we mentioned just before the statement of this lemma: in this appendix we will let the term "upgrade" refer both to $k=1$ and $k=2$, in order to save space.) Since $Z_k^{**} \geq Z_k^*$ and $Z_k^{i**} \geq Z_k^{i*}$ with at least one being strict, decreasing marginal revenue and $P_k' < 0$ imply

$$P_k(Z_k^*) + Z_k^{i*} P_k'(Z_k^*) > C_k^i.$$

Finally, it is clear that this result continues to hold for the merged firm if the merged firm's upgrade cost C_k^m satisfies $C_k^m < C_k^i$ when choosing Z_k^{i*}. ■

Lemma 5. *Suppose that $Z_k^{**} \geq Z_k^*$ and $Z_k^{i**} \geq Z_k^{i*}$ where at least one inequality is strict.*

(1) If $k=1$ then $Z_1^{i**} = Z_2^{i**}$ provided firm i is not involved in the merger.

(2) If $k=2$ and $Z_2^{i**} > 0$, then $Z_1^{i*} = Z_2^{i*}$ provided either (i) firm i is not involved in the merger, or (ii) firm i is involved but has weakly lower cost for upgrade 2 when it chooses Z_2^{i*}.

Proof. First consider $k=1$. Because i could raise Z_1^{i*} but does not it must have a weak incentive to lower Z_1^{i*} (ignoring its upgrade constraint). Using the second claim in Lemma 4, i has a strict incentive to lower Z_1^{i**}. Because i does not lower Z_1^{i**} it must be upgrade constrained and hence $Z_1^{i**} = Z_2^{i**}$. Second consider $k=2$. Because i could lower Z_2^{i**} but does not it must have a weak incentive to raise Z_2^{i**} (again ignoring its upgrade constraint). Using the first claim in Lemma 4, i has a strict incentive to raise Z_2^{i*}. (This is also true if i faces a lower cost for upgrade 2 when it chooses Z_2^{i*}.) Because i does not raise Z_2^{i*} it must be upgrade-constrained and hence $Z_1^{i*} = Z_2^{i*}$. ■

In the proofs of Propositions 1 and 2 and Lemma 2, let Z_1 and Z_2 represent post-merger equilibrium outputs, and let $\tilde{Z}_1$ and $\tilde{Z}_2$ represent pre-merger equilibrium outputs.

Proof of Proposition 1: We begin by showing that $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$ is impossible.

First, we prove that if to the contrary $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$, then any firm i not involved in the merger weakly decreases its supply of upgrade $k = 1, 2$. Suppose $k = 1$ and that to the contrary $Z_1^i > \tilde{Z}_1^i$. The first part of Lemma 5 says that $Z_1^i = Z_2^i$. There are then two subcases to consider. In the case where $\tilde{Z}_1^i = \tilde{Z}_2^i$, such that i is upgrade-constrained both pre- and post-merger, we immediately obtain a contradiction because Z_1 and Z_2 have weakly increased and hence firm i cannot have optimally strictly increased its output. In the case where $\tilde{Z}_1^i > \tilde{Z}_2^i$, it must be true that $Z_2^i > \tilde{Z}_2^i$ (because $Z_1^i > \tilde{Z}_1^i$ and $Z_1^i = Z_2^i$); but then the second part of Lemma 5 says that $\tilde{Z}_1^i = \tilde{Z}_2^i$, which is also a contradiction. Since in both cases we obtain a contradiction, we conclude that $Z_1^i \leq \tilde{Z}_1^i$. Now suppose $k = 2$ and that contrary to what we claimed above, $Z_2^i > \tilde{Z}_2^i$. The second part of Lemma 5 says that $\tilde{Z}_1^i = \tilde{Z}_2^i$, meaning that $Z_2^i > \tilde{Z}_2^i = \tilde{Z}_1^i$. But by necessity $Z_1^i \geq Z_2^i$, and so $Z_1^i \geq Z_2^i > \tilde{Z}_2^i = \tilde{Z}_1^i$, so that $Z_1^i > \tilde{Z}_1^i$, which we showed just above ($k = 1$) cannot be. We conclude that $Z_2^i \leq \tilde{Z}_2^i$.

Second, we consider the merged firm. Without loss of generality, suppose the merger involves firms 1 and 2 and that firm 1 has weakly lower cost for the low-quality good whilst firm 2 has weakly lower cost for the high-quality good. We will refer to the merged firm as m. There are two possible cases, according to whether $Z_2^m = 0$ or $Z_2^m > 0$. In the case where $Z_2^m = 0$, it must be true that $\tilde{Z}_2^1 = \tilde{Z}_2^2 = 0$ otherwise $Z_2 \geq \tilde{Z}_2$ definitely cannot hold (since we have just shown that non-merging firms weakly lower their upgrade supply). However $Z_1 \geq \tilde{Z}_1$ then implies that $Z_1^m \leq \tilde{Z}_1^1$. Since $\tilde{Z}_1^2 > 0$ as well, we must have $\tilde{Z}_1^1 + \tilde{Z}_1^2 > Z_1^m$, but this contradicts the fact that $Z_1 \geq \tilde{Z}_1$ (since again, non-merging firms weakly lower their baseline supply). In the case where $Z_2^m > 0$, since firm 2 has weakly lower upgrade costs than m, and because we are supposing $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$, firm 2 must have positive pre-merger output, $\tilde{Z}_2^2 > 0$. Summing m's two first order conditions (or else using its single first order condition if it is upgrade constrained), m's output choices satisfy

$$P_1(Z_1) + Z_1^m P_1'(Z_1) + P_2(Z_2) + Z_2^m P_2'(Z_2) = c_2^m = c_2^2.$$

By the same logic, pre-merger firm 2's choices must satisfy

$$P_1(\tilde{Z}_1) + \tilde{Z}_1^2 P_1'(\tilde{Z}_1) + P_2(\tilde{Z}_2) + \tilde{Z}_2^2 P_2'(\tilde{Z}_2) = c_2^2.$$

Because both firms 1 and 2 have positive supply of upgrade $k = 1$, for aggregate outputs not to have fallen it must at least be that $Z_1^m > \tilde{Z}_1^2$, and similarly that $Z_2^m \geq \tilde{Z}_2^2$. But this ranking of outputs means, because of decreasing marginal revenue, the fact that $P_1' < 0, P_2' < 0$, and $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$, that both of the equations immediately above cannot be satisfied.

We conclude, from all of the work above, that if $Z_1 \geq \tilde{Z}_1$, then it must also be that $Z_2 < \tilde{Z}_2$. The final step of the proof is to show that $Z_1 \geq \tilde{Z}_1$ and $Z_2 < \tilde{Z}_2$ is also impossible.

First, let U denote the set of all firms which are not involved in the merger and which strictly increased their supply of baseline output following the merger. Hence for a firm $i \in U$ it is the case that $Z_1^i > \tilde{Z}_1^i$. According to the first part of Lemma 5, firm i has $Z_1^i = Z_2^i$. Since by definition $\tilde{Z}_1^i \geq \tilde{Z}_2^i$, it must be that $Z_2^i - \tilde{Z}_2^i \geq Z_1^i - \tilde{Z}_1^i > 0$. Let U_1 and U_2 be the total increases in baseline and upgrade supplies, respectively, over all firms in U. Thus,

$$U_1 = \sum_{i \in U} \left(Z_1^i - \tilde{Z}_1^i \right) > 0, \text{ and } U_2 = \sum_{i \in U} \left(Z_2^i - \tilde{Z}_2^i \right) > 0.$$

Note that $U_2 \geq U_1 > 0$, based on the logic given just above.

Second, let D denote the set of all firms which are not involved in the merger and which strictly decreased their supply of upgrade output following the merger. Hence for a firm $i \in D$ it is the case that $Z_2^i < \tilde{Z}_2^i$. According to the second part of Lemma 5, firm i has $Z_1^i = Z_2^i$. Since by definition $\tilde{Z}_1^i \geq \tilde{Z}_2^i$, it must be that $0 > Z_2^i - \tilde{Z}_2^i \geq Z_1^i - \tilde{Z}_1^i$. Define D_1 and D_2 as the decreases in baseline and upgrade outputs for these firms:

$$D_1 = \sum_{i \in D} \left(Z_1^i - \tilde{Z}_1^i \right) < 0, \text{ and } D_2 = \sum_{i \in D} \left(Z_2^i - \tilde{Z}_2^i \right) < 0.$$

Note that it must be that $0 > D_2 \geq D_1$. Additionally, it must be that $D_2 + U_2 \geq D_1 + U_1$.

Third, consider the merging firms. As above, we will suppose without loss of generality that the merger involves firms 1 and 2 and that firm 1 has weakly superior technology for producing the low-quality good and that firm 2 has weakly superior technology for producing the high-quality good. We will view the merger as removing firm 2's output from the market but also endowing firm 1 with firm 2's better technology for the high-quality product. From this perspective, the total output change involving the merging firms in market k is given by the change in firm 1's output minus the lost output of firm 2, $(Z_k^1 - \tilde{Z}_k^1) - \tilde{Z}_k^2$. Let $\Delta Z_k^1 = Z_k^1 - \tilde{Z}_k^1$ denote the change in firm 1's output in market k.

We can now complete the proof and show that it is not possible that $Z_1 \geq \tilde{Z}_1$ and $Z_2 < \tilde{Z}_2$. These inequalities require, as necessary conditions, that

$$\Delta Z_1^1 + \left(D_1 + U_1 - \tilde{Z}_1^2 \right) \geq 0, \text{ and} \tag{8}$$

$$\Delta Z_2^1 + \left(D_2 + U_2 - \tilde{Z}_2^2 \right) < 0. \tag{9}$$

As indicated above, $D_2 + U_2 \geq D_1 + U_1$. Also it must be that $\tilde{Z}_1^2 \geq \tilde{Z}_2^2$. Therefore, $D_2 + U_2 - \tilde{Z}_2^2 \geq D_1 + U_1 - \tilde{Z}_1^2$.

One possibility is that $D_1 + U_1 - \tilde{Z}_1^2 \geq 0$, which implies $D_2 + U_2 - \tilde{Z}_2^2 \geq 0$. To satisfy (9), ΔZ_2^1 must be negative. But then using the same logic as for firms in set D, it must also be that $0 > \Delta Z_2^1 \geq \Delta Z_1^1$. But this implies that $\Delta Z_1^1 + D_1 + U_1 - \tilde{Z}_1^2 \leq \Delta Z_2^1 + D_2 + U_2 - \tilde{Z}_2^2 < 0$ which means (8) cannot hold. (Note that part of the logic given for firms in D appeals to Lemma 5 for the case of $k = 2$, which will also hold for the merged firm because the merged firm is firm 1 with firm 2's superior technology for the high-quality good which means firm 1 operates post-merger with a reduced upgrade cost compared to pre-merger.)

The other possibility is that $0 > D_1 + U_1 - \tilde{Z}_1^2$. For (8) to hold, it must be that $\Delta Z_1^1 > 0$. Following logic given for firms in set U, it must be that $\Delta Z_2^1 \geq \Delta Z_1^1 > 0$. But then it must be that $\Delta Z_2^1 + D_2 + U_2 - \tilde{Z}_2^2 \geq \Delta Z_1^1 + D_1 + U_1 - \tilde{Z}_1^2 \geq 0$, which means (9) cannot hold. ■

Proof of Lemma 2: Suppose to the contrary that no firm's monotonicity constraint binds before the merger, and yet $Z_2 > \tilde{Z}_2$. We will prove that no firm wishes to strictly increase its output of upgrade $k = 2$ following the merger, yielding a contradiction.

We only do this for the merging firms because simpler logic applies to other firms. Without loss of generality, suppose that the merger involves firms 1 and 2, and that firm 2 has a weakly lower cost for the high-quality good. We may view the merger as removing firm 1's output from the market, and endowing firm 2 with a cost $\min(c_1^1, c_1^2)$ for low quality and a cost c_2^2 for high quality. To complete the proof, it is then sufficient to show that, following the merger, firm 2 does not strictly raise its supply of upgrades. On the way to a contradiction, suppose that in fact $Z_2^2 > \tilde{Z}_2^2$. Because firm 2 could lower Z_2^2 but does not it must have a weak incentive to raise Z_2^2 (ignoring its upgrade constraint). Moreover firm 2's upgrade cost when it chooses Z_2^2 is $c_2^2 - \min(c_1^1, c_1^2)$, which is weakly higher than its upgrade cost $c_2^2 - c_1^2$ when it chooses $\tilde{Z}_2^2$. Hence using the first claim in Lemma 4, firm 2 must have a strict incentive to raise $\tilde{Z}_2^2$. But this contradicts the assumption that firm 2 was not upgrade-constrained prior to the merger. ■

Proof of Proposition 2: We first prove item 1. We know from Proposition 1 that Z_1 strictly decreases following the merger. Suppose for the sake of contradiction that aggregate high-quality output weakly increases, that is $Z_2 \geq \tilde{Z}_2$.

We start by showing that any firm i that, pre-merger, produces both qualities (and hence is not involved in the merger) must weakly lower its upgrade output after the merger. Suppose to the contrary that $Z_2^i > \tilde{Z}_2^i > 0$. The second claim in Lemma 5 implies that $\tilde{Z}_1^i = \tilde{Z}_2^i$, but this contradicts the assumption that i's upgrade constraint does not bind pre-merger.

Next, we show that any firm i that, pre-merger, produces only low-quality or produces both qualities must strictly raise its baseline units after the merger. Suppose to the contrary that $Z_1^i \leq \tilde{Z}_1^i$. The first claim in Lemma 5 implies that $\tilde{Z}_1^i = \tilde{Z}_2^i$, which again contradicts the assumption that i's upgrade constraint does not bind pre-merger.

We now consider firms that were producing only high-quality pre-merger, which includes the firms involved in the merger. Without loss of generality, suppose that the merger involves firms 1 and 2, and that firm 1 has a weakly lower cost for the low-quality good. We may view the merger as eliminating firm 2, and endowing firm 1 with a cost c_1^1 for low quality and a cost $\min(c_2^1, c_2^2)$ for high quality. Note that one effect of the merger is to eliminate the pre-merger output of firm 2, given by $\tilde{Z}_1^2 = \tilde{Z}_2^2 > 0$.

Let D denote the set of firms that were upgrade-constrained pre-merger and which strictly reduce their supply of baseline units following the merger: $i \in D$ if and only if $\tilde{Z}_1^i = \tilde{Z}_2^i$ and $Z_1^i < \tilde{Z}_1^i$. Since by definition $Z_1^i \geq Z_2^i$, it must be that $Z_2^i - \tilde{Z}_2^i \leq Z_1^i - \tilde{Z}_1^i < 0$ for each $i \in D$.

Let U denote the set of firms that were upgrade-constrained pre-merger and which strictly increase their supply of upgrades following the merger: $i \in U$ if and only if $\tilde{Z}_1^i = \tilde{Z}_2^i$ and $Z_2^i > \tilde{Z}_2^i$. (We know that no firm which is not upgrade-constrained pre-merger increases its upgrade supply, and so U contains all firms that increase their upgrade supply). Since by definition $Z_1^i \geq Z_2^i$, it must be that $0 < Z_2^i - \tilde{Z}_2^i \leq Z_1^i - \tilde{Z}_1^i$.

Finally, since we assumed $Z_2 \geq \tilde{Z}_2$, the increased upgrade supply of firms in U must at least weakly exceed the combined lost upgrade supply of firm 2 and also firms in D. That is

$$0 < \tilde{Z}_1^2 = \tilde{Z}_2^2 \leq \sum_{i \in U} \left(Z_2^i - \tilde{Z}_2^i \right) + \sum_{i \in D} \left(Z_2^i - \tilde{Z}_2^i \right) \leq \sum_{i \in U} \left(Z_1^i - \tilde{Z}_1^i \right) + \sum_{i \in D} \left(Z_1^i - \tilde{Z}_1^i \right).$$

However this says that firms in D and U have increased their baseline output by more than firm 2's pre-merger output. Using Proposition 1, Z_1 must have strictly decreased. Hence there must exist some firm $i \notin U \cup D$ that has strictly lowered its output Z_1^i. But any firm $i \notin D$ which strictly lowers its baseline output must have been producing only low-quality or both qualities prior to the merger, and we showed above that there are no such firms.

We arrived at this contradiction by assuming $Z_2 \geq \tilde{Z}_2$. Hence we conclude that this merger strictly lowers the market supply of high-quality products.

We now prove item 2a. We know from Proposition 1 that $Z_1 < \tilde{Z}_1$. Suppose for the sake of contradiction that in fact $Z_2 \leq \tilde{Z}_2$. Using the techniques already developed, it is easy to show that any firm i that produces both products pre-merger has weakly higher upgrade output post-merger.

Now consider any firm i that strictly produces only high-quality goods pre-merger. We claim that i sells strictly more upgrades post-merger. First, suppose firm i only sells high quality post-merger. Because Z_1 has strictly decreased and Z_2 has weakly decreased, firm i must be selling strictly more units. Secondly, suppose instead that firm i sells both low- and high-quality goods post-merger. Suppose also, on the way to a contradiction, that $Z_2^i \leq \tilde{Z}_2^i$. Because by assumption firm i has a strict incentive to raise $\tilde{Z}_2^i$ ignoring its upgrade constraint, a slight adaptation of the proof of Lemma 4 shows that it has a strict incentive to raise Z_2^i. But since this is feasible and yet firm i doesn't do it, we obtain a contradiction.

In summary, the total upgrade supply of non-merging firms strictly increases. Because each non-merging firm was producing zero upgrades, this contradicts $Z_2 \leq \tilde{Z}_2$.

Item 2b follows from observing that if the high-quality insider has weakly lower costs for low quality than the low-quality insiders, then the merger is equivalent to eliminating the other insiders from the market. Z_1 strictly falls. If Z_2 weakly fell, then all high-quality firms would strictly increase their output in the post-merger equilibrium and each multiproduct firm would weakly increase its upgrade output, contradicting Z_2 weakly falling. (This is true for a firm that strictly sells only high quality; the claim weakly holds otherwise.) ■

Proof of Proposition 3: Let i^m be the firm whose output is reduced by the infinitesimal merger. Recall that for every unit decrease in firm i^m's total output we reduce its upgrade output by $r \equiv Z_2^{i^m}/Z_1^{i^m}$ units. All other firms apart from i^m choose their outputs optimally. Denote by $\mathcal{L}$ those firms which produce only low quality, $\mathcal{H}$ those which produce only high quality, and $\mathcal{M}$ those which produce strictly positive amounts of both low and high quality. Let n^L, n^H and n^M denote the number of each of those groups of firms, with $n = n^L + n^H + n^M$.

Summing the first order conditions that pertain to total output over all the n firms,

$$nP_1(Z_1) + \left(Z_1 - Z_1^{i^m}\right) P_1'(Z_1) + n^H P_2(Z_2) + \left(\sum_{i\in\mathcal{H}} Z^i\right) P_2'(Z_2) = \sum_{i\in\mathcal{L},\mathcal{M}} c_1^i + \sum_{j\in\mathcal{H}} c_2^j. \quad (10)$$

Summing the first order conditions that pertain to upgrade supply over the $n^H + n^M$ firms,

$$n^H P_1(Z_1) + \left(\sum_{i\in\mathcal{H}} Z^i\right) P_1'(Z_1) + \left(n^H + n^M\right) P_2(Z_2) + \left(Z_2 - rZ_1^{i^m}\right) P_2'(Z_2) = \sum_{i\in\mathcal{M}} \left(c_2^i - c_1^i\right) + \sum_{j\in\mathcal{H}} c_2^j. \quad (11)$$

Summing the first order condition that pertains to upgrade supply over the n^H firms,

$$n^H P_1(Z_1) + \left(\sum_{i\in\mathcal{H}} Z^i\right) P_1'(Z_1) + n^H P_2(Z_2) + \left(\sum_{i\in\mathcal{H}} Z^i\right) P_2'(Z_2) = \sum_{j\in\mathcal{H}} c_2^j. \quad (12)$$

For a unit decrease in $Z_1^{i^m}$ we totally differentiate equations (10)–(12) and solve to obtain

$$dZ_1 = Y^{-1}\left\{ r(1-\phi)\left[n^H - \sigma_2(Z_2)\sum_{i\in\mathcal{H}} \frac{Z^i}{Z_2}\right] - \left[n^H(1-\phi) + n^M + 1 - \alpha_2\sigma_2(Z_2)\right] \right\} \quad (13)$$

and

$$dZ_2 = Y^{-1}\left\{ \phi\left[n^H - \sigma_1(Z_1)\sum_{i\in\mathcal{H}} \frac{Z^i}{Z_1}\right] - r\left[n - (1-\phi)n^H + 1 - \alpha_1\sigma_1(Z_1)\right] \right\}, \quad (14)$$

where $\phi \equiv P_1'(Z_1)/\left[P_1'(Z_1) + P_2'(Z_2)\right]$, and where $Y > 0$ is given by

$$Y = \left[n - (1-\phi)n^H + 1 - \alpha_1\sigma_1(Z_1)\right]\left[n^H(1-\phi) + n^M + 1 - \alpha_2\sigma_2(Z_2)\right] - (1-\phi)\left[n^H - \left(\sum_{i\in\mathcal{H}} \frac{Z^i}{Z_2}\right)\sigma_2(Z_2)\right]\phi\left[n^H - \left(\sum_{i\in\mathcal{H}} \frac{Z^i}{Z_1}\right)\sigma_1(Z_1)\right], \quad (15)$$

with

$$\alpha_1 = \frac{\sum_{i\in\mathcal{L},\mathcal{M}} Z_1^i + \phi\sum_{i\in\mathcal{H}} Z^i}{Z_1}, \quad \alpha_2 = \frac{\sum_{i\in\mathcal{M}} Z_2^i + (1-\phi)\sum_{i\in\mathcal{H}} Z^i}{Z_2}.$$

For part (1), note that using Assumption 1 and the fact that $n^H \geq 1$, all else fixed dZ_2 is strictly positive at $r = 0$, strictly negative at $r = 1$, and is strictly decreasing in r.

For part (2a), note that $\sigma_1(Z_1)$ only affects $|dZ_2/dZ_1|$ through its effect on the numerator of the dZ_2 expression, and its derivative with respect to $\sigma_1(Z_1)$ is $-\phi\left(\sum_{i\in\mathcal{H}} Z^i/Z_1\right) + r\alpha_1$, which is strictly negative provided r is sufficiently small, given our assumption that at least one firm produces a strictly positive amount of only high quality.

For part (2b), note that $\sigma_2(Z_2)$ only affects $|dZ_2/dZ_1|$ through its effect on the numerator of the dZ_1 expression, and its derivative with respect to $\sigma_2(Z_2)$ is $-r(1-\phi)\left(\sum_{i\in\mathcal{H}} Z^i/Z_2\right)+\alpha_2$. This is strictly positive given $r < 1$, the definition of ϕ, and our assumption that at least one firm produces a strictly positive amount of only high quality. Since $dZ_1 < 0$ and $dZ_2 > 0$ for r sufficiently small, $|dZ_2/dZ_1|$ is increasing in $\sigma_2(Z_2)$ ■

Proof of Proposition 4: An infinitesimal merger raises consumer surplus if and only if

$$-Z_1 P_1'(Z_1)\, dZ_1 - Z_2 P_2'(Z_2)\, dZ_2 > 0.$$

Using (3), when the target firm is low quality this is equivalent to

$$\frac{Z_1}{Z_2}\left[n^H - \sigma_2 + 1 + \frac{P_1'(Z_1)}{P_2'(Z_2)}\right] - n^H + \frac{Z_2}{Z_1}\sigma_1 < 0 \ . \tag{16}$$

The lefthand side of (16) is increasing in Z_1, given that dZ_2/dZ_1 satisfies (3). To see this, note that clearly Z_1/Z_2 is increasing in Z_1, and also $(Z_2\sigma_1)/Z_1$ is increasing in Z_1 provided that $\sigma_1 \leq 0$. Also note that the derivative of $P_1'(Z_1)/P_2'(Z_2)$ with respect to Z_1 is

$$\frac{P_1'(Z_1)}{P_2'(Z_2)}\left[-\frac{\sigma_1}{Z_1} + \frac{\sigma_2}{Z_2}\frac{dZ_2}{dZ_1}\right] \geq 0 \ ,$$

given our assumptions that $\sigma_1 \leq 0$ and $\sigma_2 \leq 0$.

A complete merger is a series of infinitesimal mergers which progressively reduce Z_1. We have proved that if (16) holds at pre-merger outputs, it also holds at all (lower) outputs along the merger path. Hence a complete merger strictly increases consumer surplus. ■

Proof of Remark 1: Using the proof of Proposition 3 a necessary condition for consumer surplus to increase is $n^H \geq 1$. Assume throughout that this holds (and hence $Z_2 > 0$). We first prove nevertheless that an infinitesimal merger strictly reduces consumer surplus if $\sigma_1(Z_1) \geq \sigma_2(Z_2) - 1$. Recall that $dCS < 0$ if and only if

$$-Z_1 P_1'(Z_1)\, dZ_1 - Z_2 P_2'(Z_2)\, dZ_2 < 0 \ . \tag{17}$$

Using equations (13) and (14), all else fixed, the lefthand side of (17) is linear in r. Hence it is sufficient to show that (17) holds at both $r = 0$ and $r = 1$. It is clear that (17) holds at $r = 1$ because in that case $dZ_1, dZ_2 < 0$. Now consider $r = 0$. Because $Z_1 \geq Z_2$ and $-P_1'(Z_1)\, dZ_1 < 0$, it is sufficient to prove that $-P_1'(Z_1)\, dZ_1 - P_2'(Z_2)\, dZ_2 < 0$, which for

$r = 0$ is equivalent to

$$n^M - \left(\sum_{i\in\mathcal{M}} \frac{Z_2^i}{Z_2}\right)\sigma_2(Z_2) + 1 + (1-\phi)\left(\sum_{i\in\mathcal{H}} Z^i\right)\left[\frac{\sigma_1(Z_1)}{Z_1} - \frac{\sigma_2(Z_2)}{Z_2}\right] > 0 \ . \qquad (18)$$

We now show that (18) holds. The first two terms are weakly positive. Moreover

$$\begin{aligned}(1-\phi)\left(\sum_{i\in\mathcal{H}} Z^i\right)\left[\frac{\sigma_1(Z_1)}{Z_1} - \frac{\sigma_2(Z_2)}{Z_2}\right] &\geq (1-\phi)\left(\sum_{i\in\mathcal{H}} Z^i\right)\left[\sigma_2(Z_2)\frac{Z_2 - Z_1}{Z_1 Z_2} - \frac{1}{Z_1}\right] \\ &\geq -(1-\phi)\left(\sum_{i\in\mathcal{H}} \frac{Z^i}{Z_2}\right) > -1 \ ,\end{aligned}$$

where the first inequality uses $\sigma_1(Z_1) \geq \sigma_2(Z_2) - 1$, the second inequality uses $\sigma_2(Z_2) \leq 1$, and the third inequality uses $\phi < 1$ and $\sum_{i\in\mathcal{H}} Z^i \leq Z_2$.

We have thus proved that $\sigma_1(Z_1) < \sigma_2(Z_2) - 1$ is necessary for consumer surplus to increase.

We now consider the special case of multiplicative preferences such that $\sigma_1(Z) = \sigma_2(Z) = \sigma(Z)$. The necessary condition for consumer surplus to increase is then $\sigma(Z_2) - \sigma(Z_1) > 1$. Firstly if curvature is increasing, $Z_1 \geq Z_2$ implies that $\sigma(Z_1) \geq \sigma(Z_2)$ and the necessary condition fails. Secondly if demand has positive curvature then $\sigma(Z_1) \geq 0$ and $\sigma(Z_2) \leq 1$ such that the necessary condition again fails. ■

Proof of Proposition 5: Note that equations (10)–(12) from the proof of Proposition 3 still determine equilibrium outputs, although we can model the synergies as occurring after the merger is consummated and hence can set $Z_1^{im} = 0$.

First consider a synergy that reduces c_2^i for a firm $i \in \mathcal{H}$. Given a unit decrease in c_2^i, we can totally differentiate (10)-(12) and then solve to obtain that

$$\begin{aligned} dZ_1 &= -\phi\left[P_1'(Z_1)Y\right]^{-1}\left\{n^M + 1 - \frac{Z_2 - \sum_{i\in\mathcal{H}} Z^i}{Z_2}\sigma_2(Z_2)\right\} \\ dZ_2 &= -(1-\phi)\left[P_2'(Z_2)Y\right]^{-1}\left\{n^L + n^M + 1 - \frac{Z_1 - \sum_{i\in\mathcal{H}} Z^i}{Z_1}\sigma_1(Z_1)\right\},\end{aligned}$$

where ϕ and Y are defined in the proof of Proposition 3. Since $dZ_1 > 0$ and $dZ_2 > 0$, the prices of both low and high quality strictly decrease, and consumer surplus strictly increases.

Second consider $n^H = 0$ and a synergy that reduces c_2^i for a firm $i \in \mathcal{M}$. For a unit decrease in c_2^i, the total derivative of (10) gives $dZ_1 = 0$, and the total derivative of (11) gives

$$dZ_2 = -\left\{P_2'(Z_2)\left[n^M + 1 - \sigma_2(Z_2)\right]\right\}^{-1} > 0.$$

Hence the price of low quality is unchanged, the price of high quality strictly decreases, and thus consumer surplus strictly increases.

Third consider $n^H = 0$ and a synergy that reduces c_1^i for a firm $i \in \mathcal{L}$. Given a unit decrease in c_1^i, the total derivative of (11) gives $dZ_2 = 0$, and the total derivative of (10) gives

$$dZ_1 = -\left\{ P_1'(Z_1)\left[n^L + n^M + 1 - \sigma_1(Z_1)\right] \right\}^{-1} > 0.$$

The prices of both goods strictly decrease, and consumer surplus strictly increases. ■

Proof of Proposition 6: As in the proof of Proposition 5, equations (10)–(11), and when $n^H \geq 1$ equation (12), specialized to the case $Z_1^{im} = 0$ determine equilibrium outputs.

First, for a unit decrease in c_1^i, we can totally differentiate and then solve to obtain

$$\begin{aligned} dZ_1 &= -\left[P_1'(Z_1)\,Y\right]^{-1}\left[n^H + n^M + 1 - \sigma_2(Z_2)\right] \\ dZ_2 &= \left[P_2'(Z_2)\,Y\right]^{-1}\left[n + 1 - \sigma_1(Z_1)\right], \end{aligned}$$

where Y is the same as in equation (15) from earlier. Note that $dZ_1 > 0$ and $dZ_2 < 0$, and that $|dZ_1/dZ_2|$ is increasing in $\sigma_1(Z_1)$ and decreasing in $\sigma_2(Z_2)$.

Second, given a unit decrease in c_2^i, we can totally differentiate and solve to get

$$dZ_1 = \phi\left[P_1'(Z_1)\,Y\right]^{-1}\left[n^H - \left(\sum_{i\in\mathcal{H}} \frac{Z^i}{Z_2}\right)\sigma_2(Z_2)\right] \tag{19}$$

$$dZ_2 = -\left[P_2'(Z_2)\,Y\right]\left[n - (1-\phi)\,n^H + 1 - \frac{Z_1 - (1-\phi)\sum_{i\in\mathcal{H}} Z^i}{Z_1}\sigma_1(Z_1)\right]. \tag{20}$$

Note that $dZ_1 \leq 0$ and $dZ_2 > 0$, and that $|dZ_1/dZ_2|$ is increasing in $\sigma_1(Z_1)$ and decreasing in $\sigma_2(Z_2)$. ■

Lemma 3 is implicit in the proof of the following result.

Lemma 6. *A merger of $k+1$ firms producing only low-quality products is profitable if and only if*

$$k + 1 \leq \left[\frac{1 + n^L + n^M + \gamma}{1 + n^L + n^M + \gamma - k}\right]^2, \text{ where } \gamma = \frac{[n^H(n^M+1)q_L]}{[(n^M+1)q_H + n^H(q_H - q_L)]}.$$

Proof. The merger of these firms means that, in effect, k such firms cease production. Conceptually we imagine that firm i is the merging firm that "survives" the merger. Letting Z_1^{**} denote the total post-merger industry output and Z_1^{i**} firm i's post-merger output, this i's post-merger first-order condition is

$$P_1(Z_1^{**}) + Z_1^{i**}P_1'(Z_1^{**}) = q_L(1 - Z_1^{**}) - q_L Z_1^{i**} = c_1^i.$$

This allows firm i's post-merger profits—and hence the post-merger profits of the $k+1$ merging firms—to be written as

$$(P_1(Z_1^{**}) - c_1^i)Z_1^{i**} = q_L(Z_1^{i**})^2.$$

Using similar computations, the pre-merger profit of each of the $k+1$ insiders is $q_L(Z_1^{i*})^2$, where Z_1^{i*} is such a firm's pre-merger output. Thus, a merger is profitable if and only if

$$(Z_1^{i**})^2 \geq (k+1)(Z_1^{i*})^2.$$

We now solve for Z_1^{i**}/Z_1^{i*}. Let $\mathcal{K}$ be the set of the k firms who cease production following the merger. Note that by symmetry these firms' pre-merger output is kZ_1^{i*}. We model the merger as a series of infinitesimal mergers, which reduce these firms' output from kZ_1^{i*} to 0, and let $Z_1^{\mathcal{K}}$ denote their output at an arbitrary point along this merger 'path'.

Using firm i's first order condition and the fact that $P_1'(Z_1) = -q_L$, an infinitesimal merger induces firm i to change its output by $dZ_1^i = -dZ_1$. Using equation (13) from the proof of Proposition 3, substituting $r = \sigma_1(Z_1) = \sigma_2(Z_2) = 0$ and replacing n^L with $n^L - k$ (since there are k fewer low-quality firms who optimize over their output) we obtain

$$dZ_1 = -\frac{\left[n^H(1-\phi) + n^M + 1\right]}{\left[n^L - k + n^M + 1\right]\left[n^H(1-\phi) + n^M + 1\right] + n^H\phi\left[n^M + 1\right]},$$

where $\phi = q_L/q_H$. Since dZ_1^i is the same at all points along the merger path,

$$\frac{Z_1^{i**}}{Z_1^{i*}} = \frac{Z_1^{i*} + \int_0^{kZ_1^{i*}} dZ_1^i dZ_1^{\mathcal{K}}}{Z_1^{i*}} = 1 + \frac{k}{n^L - k + n^M + 1 + \gamma}.$$

Substituting this into the above profitability condition then gives the stated result. ■

Proof of Proposition 7: The righthand side of the inequality in Lemma 6 is decreasing in γ. At the same time γ is increasing in n^H and satisfies $\lim_{n^H\to\infty}\gamma = \left(n^M+1\right)q_L/\left(q_H - q_L\right)$. Hence by substituting $n^M = 0$, $k = n^L - 1$, and $\gamma = q_L/\left(q_H - q_L\right)$ into the inequality in Lemma 6 we obtain a sufficient condition for the merger to be profitable. Rearranging this condition leads to the one stated in the proposition. ■

Proof of Proposition 8: The righthand side of the inequality in Lemma 6 is decreasing in γ. Hence by substituting $\gamma = 0$ we obtain the following necessary condition for profitability:

$$k + 1 \leq \left[\frac{1 + n^L + n^M}{1 + n^L + n^M - k}\right]^2. \tag{21}$$

Salant, Switzer, and Reynolds (1983) show that this cannot hold if $(k+1)/(n^L + n^M) < 0.8$. The necessary condition stated in our proposition then follows immediately. ■

Proof of Remark 2: This proof is omitted because it follows using the same steps in the proofs of Lemma 3 and Propositions 7 and 8. ■

Proof of Proposition 9: For brevity we only prove this result for a merger between $k+1$ low-quality firms. We prove that if the stated condition fails to hold, an infinitesimal merger reduces the joint profit of the merger insiders.

Closely following the proof of Lemma 6, let firm i be the surviving insider, and $Z_1^{\mathcal{K}}$ be the output of the k other insiders that are being removed from the market (at an arbitrary point along the merger path). The joint profit of the merger insiders $\mathcal{I}$ is $\left(Z_1^i + Z_1^{\mathcal{K}}\right)\left[P_1\left(Z_1\right) - c_1^i\right]$, and its change following a unit decrease in $Z_1^{\mathcal{K}}$ is

$$d\pi^{\mathcal{I}} = \left(dZ_1^i - 1\right)\left[P_1\left(Z_1\right) - c_1^i\right] + \left(Z_1^i + Z_1^{\mathcal{K}}\right) P_1'\left(Z_1\right) dZ_1.$$

Since firm i is optimizing, we can use its first order condition to replace $P_1\left(Z_1\right) - c_1^i$ and to derive that $dZ_1^i = -\left[1 - \left(Z_1^i/Z_1\right)\sigma_1\left(Z_1\right)\right] dZ_1$. Hence we can write

$$d\pi^{\mathcal{I}} = Z_1^i P_1'\left(Z_1\right)\left\{1 + \left(2 + \frac{Z_1^{\mathcal{K}}}{Z_1^i} - \frac{Z_1^i}{Z_1}\sigma_1\left(Z_1\right)\right) dZ_1\right\}.$$

A sufficient condition for the merger to be unprofitable is that, at all points along the merger path, the curly-bracketed term is strictly positive. Since $dZ_1 < 0$ this is harder to achieve when $Z_1^{\mathcal{K}}/Z_1^i$ is larger. $Z_1^{\mathcal{K}}/Z_1^i$ reaches its maximum pre-merger (when it equals k), so we obtain the following sufficient condition for the merger to be unprofitable:

$$1 + \left(2 + k - \frac{Z_1^i}{Z_1}\sigma_1\left(Z_1\right)\right) dZ_1 > 0.$$

We have already solved for dZ_1 in equation (13). Substituting $r = 0$ and $Z_1^{i^m} = Z_1^{\mathcal{K}}$ (and replacing n^L with $n^L - k$ since k fewer low-quality firms are optimizing) observe that

$$dZ_1 \geq -\left\{n^L - k + n^M + 1 - \frac{Z_1 - \left(\sum_{i\in\mathcal{H}} Z^i\right) - Z_1^{\mathcal{K}}}{Z_1}\sigma_1\left(Z_1\right)\right\}^{-1}.$$

Hence our sufficient condition for the merger to be unprofitable reduces to

$$n^L - 2k + n^M - 1 > \frac{Z_1 - \left(\sum_{i\in\mathcal{H}} Z^i\right) - Z_1^{\mathcal{K}} - Z_1^i}{Z_1}\sigma_1\left(Z_1\right).$$

Since the righthand side is bounded above by 1, a sufficient condition for the merger to be unprofitable is that $k < \left(n^L + n^M - 2\right)/2$. The reverse of this inequality is then a necessary condition for the merger to be profitable. ■

Proof of Proposition 10: Because no firm sells only high quality products either before or after the merger, equilibrium supplies can be determined by separately considering the baseline market and the upgrade market. Because all mergers under consideration exhibit firms with the same market shares in the overall market, all firms involved in the mergers have the same baseline costs c_1^i. A merger is equivalent to removing one of these firms from the market but (because the firms have the same baseline costs) any such removal leads to the same reduction in Z_1. For the merger involving two multiproduct firms, because there are no synergies and the upgrade market can be considered separately, Z_2 must fall. However, no merger not involving two multiproduct firms has an effect on the upgrade market because the number of firms active there does not change. ■

Proof of Proposition 11: We prove that that for any firm $i \in \mathcal{O}$,

$$d\pi^i = Z_1^i P_1'(Z_1)\left[2 - \frac{Z_1^i}{Z_1}\sigma_1(Z_1)\right] dZ_1 + Z_2^i P_2'(Z_2)\left[2 - \frac{Z_2^i}{Z_2}\sigma_2(Z_2)\right] dZ_2. \tag{22}$$

Summing (22) over all firms $i \in \mathcal{O}$ and adding the expression for dCS in the text, we obtain the total external effect $d\mathcal{E}$ in Equation (6). The rest of the proposition then follows.

We now prove (22). First consider a firm $i \in \mathcal{L}$. Its profit is $\pi^i = Z_1^i [P_1(Z_1) - c_1^i]$, and so

$$d\pi^i = dZ_1^i \left[P_1(Z_1) - c_1^i\right] + Z_1^i P_1'(Z_1)\, dZ_1. \tag{23}$$

The firm's first order condition is $P_1(Z_1) - c_1^i + Z_1^i P_1'(Z_1) = 0$, and the total derivative of this is $dZ_1^i = -[1 - (Z_1^i/Z_1)\,\sigma_1(Z_1)]\, dZ_1$. Substituting these into (23) and noting that $Z_2^i = 0$, we obtain the expression in (22).

Second, a firm $i \in \mathcal{M}$ has profit $\pi^i = Z_1^i [P_1(Z_1) - c_1^i] + Z_2^i [P_2(Z_2) - (c_2^i - c_1^i)]$. Since the firm chooses Z_1^i and Z_2^i independently, we can follow the same steps as we did for a low-quality firm and derive the expression for $d\pi^i$ in (22).

Third, consider a firm $i \in \mathcal{H}$. Its profit is $\pi^i = Z^i [P_1(Z_1) + P_2(Z_2) - c_2^i]$, and so

$$d\pi^i = dZ^i \left[P_1(Z_1) + P_2(Z_2) - c_2^i\right] + Z^i \left[P_1'(Z_1)\, dZ_1 + P_2'(Z_2)\, dZ_2\right]. \tag{24}$$

The firm's first order condition is

$$P_1(Z_1) + P_2(Z_2) - c_2^i + Z^i \left[P_1'(Z_1) + P_2'(Z_2)\right] = 0,$$

and its total derivative is

$$dZ^i = -\phi\left[1 - \frac{Z_1^i}{Z_1}\sigma_1(Z_1)\right] dZ_1 - (1-\phi)\left[1 - \frac{Z_2^i}{Z_2}\sigma_2(Z_2)\right] dZ_2,$$

where $\phi = P_1'(Z_1) / [P_1'(Z_1) + P_2'(Z_2)]$. Plug these into (24) and rearrange to get (22). ■

Proof of Proposition 12: Parts (1) and (2) have already been explained in the text. For part (3), using equation (22) we wish to show that

$$Z_1^i q_L dZ_1 + Z_2^i (q_H - q_L)\, dZ_2 < 0.$$

Since from Proposition 1 $dZ_1 < 0$ for any merger without synergies, and since $Z_1^i \geq Z_2^i$ for each $i \in \mathcal{O}$, it is sufficient to prove that $q_L dZ_1 + (q_H - q_L)\, dZ_2 < 0$. Using the expressions for dZ_1 and dZ_2 derived earlier in (13) and (14), this simplifies to

$$-\phi\left[n^M + 1\right] - r(1-\phi)\left[n^L + n^M + 1\right] < 0,$$

which is clearly satisfied. ■

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