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Revenue guarantees in auctions with a (correlated) common prior and additional information*

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Abstract

This paper considers auction environments with a (possibly correlated) common prior over bidders' values, where each bidder may have additional information (e.g., through information acquisition). Under certain conditions, we characterize the optimal mechanisms in terms of the expected revenue that is guaranteed given whatever additional information is available to the bidders. Even if the values are correlated,

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we do not necessarily have full-surplus extraction, and moreover, the optimal mechanism resembles those in the independently distributed cases. Specifically, we show that (i) a second-price auction is optimal among all the efficient mechanisms, and (ii) it is rate-optimal among all the mechanisms.

Keywords: Mechanism design, Auction, Correlated private information, Information acquisition, Revenue guarantee

1 Introduction

In many real auctions, bidders' valuations for auctioned objects exhibit correlation.[1] Despite the practical importance of optimal auction mechanism design with correlated private information, however, most of the papers in the Bayesian mechanism design literature focus on independently distributed valuations, with little focus on the correlated cases. This may be partly because of the extremely positive result obtained by Crémer and McLean (1985, 1988) in "generic" correlated environments.[2] Crémer and McLean (1985, 1988) show that, with a generic correlated distribution, *any* allocation rule is implementable without *any* information rent, and in this sense, the first-best outcome for the seller is always possible, even though the valuations are the bidders' private information. This observation is very different from the independent case (e.g., Myerson (1981)), where implementable allocation rules must be monotonic and bidders earn some information rent.

[1] For example, imagine auctions of oil tracts, spectra, sovereign bonds, and so on.

[2] See also McAfee and Reny (1992), Heifetz and Neeman (2006), Chen and Xiong (2013), and Gizatulina and Hellwig (2014).

Within the literature, this stark difference or discontinuity between the independent and correlated cases has been considered to be rather perplexing.[3] On the one hand, it seems that the bidders' valuations are correlated in many auctions. On the other hand, however, the observations in independent cases may seem to be more sensible, in that the bidders' private information restricts implementable objectives and the seller's ability to extract information rent. Also, the optimal mechanism obtained by Crémer and McLean (1985, 1988), which may be interpreted as a combination of a second-price auction and side bets, is often criticized as highly unrealistic.

A crucial (implicit) assumption in Crémer and McLean (1985, 1988) is that the bidders cannot have additional information about each other's valuation (in addition to the correlated common prior). For example, this means that each bidder cannot engage in information acquisition about other bidders' valuations. This assumption is crucial for their side-bet mechanism to extract information rents, where each bidder "bets" on other bidders' valuations. Naturally, if a bidder can acquire additional information about other bidders, he would have a strong incentive to do so, because then he could earn positive (possibly large) information rent. This means, in turn, that the seller can no longer extract all of the surplus of the bidders.[4] Also, this assumption of no information acquisition seems unrealistic in many auction environments. For example, in the auctions of oil tracts, bidders (e.g., oil companies) often have the technologies to acquire more precise information about the tracts before the bidding stage.

[3] See, for example, Crémer and McLean (1985, 1988), McAfee and Reny (1992), and Milgrom (2004).

[4] This impossibility of full-surplus extraction is formally observed by Bikhchandani (2010).

This suggests that the possibility of additional information (e.g., through information acquisition) plays a crucial role in mechanism design with correlated information. Therefore, in this paper, we assume that each bidder may have arbitrary additional information (about others' private information), and that the seller does not know what kind of additional information is available to each bidder. Given such "uncertainty" or "ambiguity" about additional information, our goal is to characterize the highest expected revenue that can be *guaranteed* given whatever additional information the bidders may have. Although there may be many other ways to model the bidders' additional information, such a "pessimistic" approach may be reasonable when bidders have more expertise than the seller (e.g., an auction of oil tracts), so that it is difficult for the seller to know what kind of information acquisition technologies are ever available to the bidders. More generally, avoiding any *ad hoc* restriction on the bidders' possible additional information, we can avoid the optimal mechanism highly dependent on the structure of additional information. This feature may be preferable in view of the "detail-freeness" in Wilson (1987).

The paper is structured as follows. In Section 2, we introduce a single-good, private-value auction model with (correlated) common prior over bidders' valuations.[5] We formally define the bidders' (arbitrary) additional information and the concept of revenue guarantee. In private-value auctions, each agent knows his willingness-to-pay for the object, and in this sense, such additional information is payoff-irrelevant. However, it could be impor-

[5] Although some results can be extended to non-auction environments or interdependent-value (or common-value) environments, the main part of the paper focuses on this simple setting in order to convey clearer intuition. See Supplementary Materials.

tant in determining his (possibly high-order) belief about the other bidders' private information, and hence important for determining his behavior. In particular, the side-bet mechanism of Crémer and McLean (1985, 1988) can no longer guarantee the first-best level of expected revenue with additional information, and moreover, it often fails to be optimal in terms of revenue guarantee.

Section 3 provides characterization of the (exact or approximate) highest revenue guarantee in various settings. In Section 3.1, we show that, from amongst all efficient[6] auction mechanisms, a second-price auction is optimal in the sense of revenue guarantee. Therefore, the result provides a rationale for a benevolent principal (e.g., a government selling its asset) to use a second-price auction regardless of the valuation distribution; a simple and common auction format. Note that this result is not driven by a standard revenue-equivalence argument. Because of a correlated prior and additional information, we cannot apply the standard revenue-equivalence theorem in this environment. Theorem 1 of Bergemann and Morris (2005) implies that, in a quasilinear environment where the designer's goal is to implement a social choice function (such as an efficient allocation rule) regardless of the agents' additional information, then only a dominant-strategy incentive compatible social choice function is implementable. Their result does not answer, however, if a dominant-strategy mechanism is revenue-maximizing amongst all mechanisms that implement the social choice function. Our result shows that, for an efficient allocation rule, it is indeed the case that a second-price

[6] An auction mechanism is efficient if a highest-value bidder always wins the object, regardless of additional information of the bidders.

auction is revenue-maximizing.[7]

In Section 3.2, we show that, even if an arbitrary (possibly inefficient) auction mechanism is allowed, under certain regularity conditions, a second-price auction is still approximately optimal. More precisely, taking the highest value among all the bidders as the benchmark revenue,[8] we show that the difference between the benchmark expected revenue and the expected revenue in a second-price auction is $O\left(\frac{1}{N}\right)$, and furthermore, for *any* mechanism, the difference cannot be smaller than $O\left(\frac{1}{N}\right)$. Following the language in statistics, a second-price auction is *rate-optimal*.[9] Chung and Ely (2007) is an important precursor regarding this result. They show that (i) if the bidders have arbitrary (high-order) beliefs about each other's private information, which is not necessarily consistent with any common prior, then a dominant-strategy mechanism is optimal in revenue guarantee among all auction mechanisms; and (ii) if the bidders' (high-order) beliefs are consistent with a common prior, there is a (counter)example of a value distribution such that a dominant-strategy mechanism is strictly suboptimal. Regarding (i), our result is partly stronger in the sense that the set of possible beliefs the bidders may possess in our environment is smaller. In particular, in our

[7]As a natural extension, we investigate whether the same result holds for some other dominant-strategy incentive compatible social choice functions. See Supplementary Materials.

[8]An alternative natural benchmark revenue may be the second-best revenue *under no additional information*. However, notice that they are the same in the correlated environment where Crémer and McLean (1988) apply.

[9]Investigation of convergence rates is ubiquitous in mechanism design / market design to study asymptotic performances of mechanisms. However, establishing the rate-optimality of those mechanisms (in environments where exactly optimal mechanisms are unknown) seems less popular, maybe because it is much harder in certain environments. An important exception is Andreyanov and Sadzik (2016) in mechanism design in an exchange environment, who establish the rate-optimality of their *σ-Walrasian Equilibrium* mechanism.

model, the bidders' beliefs are always consistent with the original correlated prior regardless of additional information, while in the model of Chung and Ely (2007), the bidders may believe very different priors from each other.[10] Such heterogeneous priors may be difficult to justify in some contexts, such as those where some data about past auctions of similar objects is publicly available.[11] On the other hand, their result is stronger than ours in the sense that they obtain exact optimality, rather than approximate optimality (and in this sense, the two papers are complementary). For this point, recall that their result (ii) shows that exact optimality of a dominant-strategy mechanism is impossible in our environment. Our result shows that, nevertheless, a dominant-strategy mechanism is not "too far" from the optimum.

The common qualitative feature of these results is that, even though we consider Bayesian incentive compatibility with a correlated common prior, the optimal mechanism in revenue guarantee is a dominant-strategy incentive compatible mechanism. With independent common priors, Myerson (1981) shows optimality of a second-price auction (with a reserve price). Our results may be interpreted as a generalization in correlated environments when revenue guarantee is concerned.[12]

It should be noted that the literature already raises a number of critiques on the surplus extraction result by Crémer and McLean (1988), based on, for example, a possibility of collusion (Laffont and Martimort (2000)), risk-averse

[10] Börgers (2013) raises a concern about revenue-maximizing Bayesian mechanism design with risk-neutral agents and without a common prior.

[11] Furthermore, our common-prior model admits a natural interpretation that the bidders' additional information comes from their information acquisition. See Supplementary Materials.

[12] As in Segal (2003), this type of result has been conjectured in the literature.

bidders (Robert (1991)), and bidders' information acquisition (Bikhchandani (2010)). The current paper builds on the idea raised by Bikhchandani (2010) suggesting the importance of information acquisition in correlated auction environments. Leaving aside the fact that we consider different kinds of "challenges" to the designer in order to circumvent the surplus-extraction result, our marginal contribution is characterization of (exact / approximate) optimal mechanisms in a general auction setting given the possibility of additional information.[13]

At the methodological level, the idea that agents may know more than the available information to the "outside observer" is studied by Forges (1993) and Bergemann and Morris (2013) in game theory (i.e., for arbitrarily fixed games). In the sense that essentially no restriction is made in terms of which additional information may be available, that studied by Bergemann and Morris (2013) is the closest to our approach. Although our revenue-guarantee problem can be seen as a mechanism-design application of their concept of *Bayes correlated equilibrium*, we develop a different methodology to analyze our mechanism design problem. We discuss why such a different methodology is necessary in mechanism design in Section 3.

Finally, Section 4 concludes the paper.

In Supplementary Materials sections, we discuss four related topics. First, we discuss the relationship between one of our assumptions and exchangeability. Second, we present the third application (sequential pricing). Third, we endogenize the information acquisition decisions of the bidders, and show that we obtain similar results as in Section 3 under certain conditions. This

[13]A notable exception is Laffont and Martimort (2000) who characterize the optimal mechanism given the possibility of collusion, but they focus on a two-agent, two-type case.

suggests that the worst-case evaluation adopted in this paper may indeed be relevant if the bidders themselves choose their information structure. Finally, we discuss some possible generalizations such as non-auction environments and common / interdependent valuations.

2 Auction environment

There are N bidders, $i = 1, \ldots, N$. Each bidder i knows his value $v_i \in V_i = [0, 1]$ for the object ("private values"). Let $V = \prod_i V_i$. His utility is given by $v_i q_i - p_i$, where $q_i \in [0, 1]$ is the probability that he is assigned the object, and $p_i \in \mathbb{R}$ is his payment. The set of feasible allocations is denoted by $X = \{(q, p) = (q_i, p_i)_{i=1}^N | \sum q_i \leq 1\}$. Given $v = (v_i)_{i=1}^N \in V$, let $v^{(k)}$ denote the k-th highest value among $v_1, \ldots, v_N$.

As in the standard Bayesian mechanism design approach, we assume that the bidders commonly know a probability distribution over the values, denoted by $F \in \Delta(V)$, and F is known to the designer as well. The assumption that the distribution over the values is common knowledge may be considered to be a reasonable assumption in some cases, for example, when "similar" goods have been auctioned many times and the data for the value distributions is publicly available. We also assume that F admits a density that is everywhere positive.

If F exhibits certain correlation, and moreover, if the bidders do not have any additional information, then as in Crémer and McLean (1988), the designer can attain full-surplus extraction based on their "side-bet" mechanism. However, if the bidders may have additional information, then such a mechanism may raise revenue much lower than other mechanisms (and recall that Bikhchandani (2010) shows that, indeed, each bidder has a significant incentive for acquiring additional information). We consider the designer who is extremely pessimistic with respect to the possibility of the bidders' additional information, and aims to construct an auction mechanism that guarantees a good amount of expected revenue given whatever additional information the bidders have.

First, let $S = \prod_i S_i$, each S_i is a measurable space, and $G \in \Delta(V \times S)$. Each S_i is interpreted as the space of additional information available to bidder i, and G is a joint probability distribution over $V \times S$. We assume that each i observes both $v_i \in V_i$ and $s_i \in S_i$ before playing a mechanism, and G is commonly known among the bidders.

Definition 1. (S, G) is F-*feasible* if the consistency between F and G is maintained in the sense that, for each measurable $\tilde{V} \in V$, we have $G(\tilde{V} \times S) = F(\tilde{V})$.

Let $G_i(\cdot|v_i, s_i) \in \Delta(V_{-i} \times S_{-i})$ denote i's conditional probability distribution over $V_{-i} \times S_{-i}$ given his own signal v_i, s_i.

An auction mechanism is denoted by $\Gamma = \langle M, q, p\rangle$, where each M_i is a message set for each i, $M = \prod_i M_i$, and $(q, p) : M \to X$ is an outcome function. We assume that each M_i has a message that corresponds to "opting-out", and whenever i chooses that message, he is assigned $(q_i, p_i) = (0, 0)$.

Given $(V, F; S, G)$ such that (S, G) is F-feasible, the bidders play a Bayesian equilibrium in mechanism Γ. Let $\sigma_i : V_i \times S_i \to M_i$ be i's (pure) strategy in

Γ. We say that $\sigma^* = (\sigma_i^*)_{i=1}^N$ is a Bayesian equilibrium if, for each i, v_i, s_i, m_i:

$$\begin{aligned} & \int_{V_{-i}\times S_{-i}} v_i q_i(\sigma_i^*(v_i, s_i), \sigma_{-i}^*(v_{-i}, s_{-i})) - p_i(\sigma_i^*(v_i, s_i), \sigma_{-i}^*(v_{-i}, s_{-i}))\; dG_i(v_{-i}, s_{-i}|v_i, s_i) \\ \geq & \int_{V_{-i}\times S_{-i}} v_i q_i(m_i, \sigma_{-i}^*(v_{-i}, s_{-i})) - p_i(m_i, \sigma_{-i}^*(v_{-i}, s_{-i}))\; dG_i(v_{-i}, s_{-i}|v_i, s_i). \end{aligned}$$

The designer evaluates an auction mechanism according to its worst-case expected revenue across (S, G).

Definition 2. The *revenue guarantee* of mechanism Γ is

$$R(\Gamma) = \inf_{(S,G):F\text{-feasible}} \int_{V\times S} \left[\sum_i p_i(\sigma^*(v, s))\right] dG.$$

That is, for any (S, G) that is F-feasible, there exists a Bayesian equilibrium σ^* such that expected revenue at least as high as $R(\Gamma)$ is attained.

Our objective is maximization of $R(\Gamma)$. Of course, based on the revelation principle, even though the designer does not know (S, G) originally, he can always extract that information for free from the bidders (because (S, G) is common knowledge among them). Therefore, in principle, we can first obtain the optimal mechanism for *each* (S, G), and then minimize that value function across all F-feasible (S, G).

However, this direct approach is in general very difficult, because (i) each bidder has multi-dimensional private information, and (ii) the distribution G may not be "nicely regular" even if F is. Therefore, we take a different approach, by first characterizing its lower bound. For this purpose, we consider a second-price auction. Because truth-telling is a dominant-strategy equilibrium in this auction mechanism, *regardless of* (S, G), it attains expected revenue $E[v^{(2)}]$, where $v^{(2)}$ is the second-highest valuation among N bidders.[14] That is, its revenue guarantee is $E[v^{(2)}]$.

In general, it is not necessarily the case that this lower bound is tight. However, in the two applications studied in this paper (and in some other cases in Supplementary Materials), we show that this lower bound is exactly / approximately tight.

Remark 1. In this definition, we evaluate a mechanism based on its worst-case expected revenue in terms of all additional information structures, (S, G), but on its "best-case" expected revenue in terms of equilibrium selection. Such a best-case approach in equilibrium selection is standard in mechanism design. An alternative approach is to consider the worst case also in terms of equilibrium selection.[15] In the current context of private-value auction environments, however, the difference is small. As we find later, dominant-strategy mechanisms are often optimal or close to optimal, and such a dominant-strategy mechanism can be slightly trembled so that truth-telling is strictly dominant (and hence the equilibrium is unique regardless of (S, G)).

[14] More specifically, given $v = (v_1, \ldots, v_N)$, $v^{(1)} = \max\{v_1, \ldots, v_N\}$ and $v^{(2)} = \max[\{v_1, \ldots, v_N\} \setminus \{v^{(1)}\}]$. Then, $E[v^{(2)}] = \int_v v^{(2)} dF(v)$, which depends on F but not on G.

[15] As we discuss later, in that case, the solution concept would be equivalent to the worst-case Bayes correlated equilibria (Bergemann and Morris (2013)). Du (2016), Bergemann, Brooks, and Morris (2016a), Bergemann, Brooks, and Morris (2016b) apply this solution concept to (pure) common-value auction environments.

3 Revenue guarantee

To explain the critical (S, G) that attains the revenue guarantee, first, observe that such (S, G) is not a trivial one (e.g., S is singleton and G is essentially equivalent to F), because for such (S, G), surplus extraction as in Crémer and McLean (1988) would be possible. More generally, any (S, G) that exhibits much correlation would not be such a worst-case information structure. This includes, as a special case, the "full-information" (S, G), that is, $S_i = V$ for all i and G assigns probability one on the even that $s_1 = \ldots = s_N = v$ (recall that v is the profile of the bidders' values) for all v. In such a full-information structure, the principal can again achieve the full-surplus extraction.

This means that the critical (S, G) is "somewhere between" no-information and full-information. As we see in each subsection, G rather exhibits certain *(conditional) independence* conditions. More specifically, the agents' value profile $v = (v_i)_{i \in I}$ is independently distributed in G conditional on any realization of payoff-irrelevant signal profile $s = (s_i)_{i \in I}$. This conditional independence implies optimality of dominant-strategy mechanisms in the applications below. More generally, under such critical G, the conditional distribution of $v|s$ does not satisfy the *belief-determines-preference property* (Neeman (2004)), even if the standard type space (i.e., that without any additional information) has this property.

In game theory, Forges (1993) and Bergemann and Morris (2013) introduce several versions of incomplete-information correlated equilibria, and at the conceptual level, the revenue-guarantee problem in auction can be seen as a mechanism-design application of the concept of robust prediction in Bergemann and Morris (2013). However, we use a different methodology to identify the optimal mechanism. To explain this, recall that, in their robust prediction, Bergemann and Morris (2013) develop the concept of *Bayes correlated equilibrium* to identify all Bayesian equilibrium outcomes given any (F-feasible) (S, G). Their approach has a great advantage in predicting possible outcomes in the sense that they do not need to consider all possible F-feasible (S, G). Rather, as in the complete-information correlated equilibrium (Aumann (2013)), the Bayes correlated equilibria are simply characterized by a number of inequalities that correspond to the obedience conditions for the agents to follow the "mediator's recommendation" (in this sense, their approach treats additional information in an "implicit" manner).[16] In mechanism design, there is a difficulty in treating correlated equilibrium as a solution concept: as far as I am aware, it is not possible to apply the revelation principle (at least in a straightforward manner) to focus on direct mechanisms in seeking optimal mechanisms. Therefore, despite the usefulness of their concept in the prediction in any fixed games, we take another, more "explicit" way to predict the agents' behavior given each specific (S, G), and to identify (S, G) that corresponds to the worst-case scenario for the designer. This explicit approach allows us to apply the standard revelation principle for Bayesian mechanisms.[17]

[16] Based on this solution concept, Bergemann, Brooks, and Morris (2015) study monopoly pricing, and Bergemann, Brooks, and Morris (2017) study expected revenue in first-price auction. Du (2016), Bergemann, Brooks, and Morris (2016a), Bergemann, Brooks, and Morris (2016b) study the optimal mechanism in (pure) common-value auction with the worst Bayes correlated equilibrium.

[17] Of course, there may be an appropriate version of revelation principle for correlated equilibrium, which allows us to directly apply the Bayes correlated equilibria of Bergemann and Morris (2013). Whether or not such a version of revelation principle exists is an open question.

3.1 Revenue guarantee by efficient mechanism

First, we study the maximum expected revenue that can be guaranteed from amongst all efficient auction mechanisms. Such a question may be relevant when the designer is a public entity selling its asset: its primary concern may be to allocate the asset in the most efficient way, but if there are multiple ways to sell the asset efficiently the entity may desire to achieve higher revenue.[18] If F satisfies independence, then it is the expected value of the second highest value, which is, for example, achieved by a second price auction. If F has certain correlation, then as in Crémer and McLean (1985, 1988), full surplus extraction is possible (and hence the expected revenue is the expected value of the highest value) if there is no additional information. However, if additional information is available to the bidders, full surplus extraction is no longer possible. Furthermore, we show that, if any additional information is allowed, then given any F (independent or correlated), the highest expected revenue we can guarantee is the expected value of the second highest value.

Definition 3. $\Gamma = \langle M, q, p \rangle$ is *efficient* given F if, for any (S, G) that is F-feasible, there is a Bayesian equilibrium σ^* such that, for each v, s:

$$\sum_{i | v_i = v^{(1)}} q_i(\sigma^*(v, s)) = 1.$$

Given any F, the set of efficient mechanisms is nonempty, because a second price auction is efficient. Other examples of efficient mechanisms include (not necessarily *pivotal*) VCG mechanisms which satisfy ex post individual

[18]Note that, with a correlated prior and arbitrary additional information, the standard revenue equivalence result does not generally hold, and hence, there may be multiple efficient mechanisms with different revenue levels.

rationality. There are also non-dominant-strategy mechanisms that are efficient. A simple example is a second-price auction with side bets, where the side bets are separated from the winner determination and their expected payments do not violate ex post (with respect to v) individual rationality.

We show that $E(v^{(2)})$ is the highest revenue we can guarantee from amongst all efficient mechanisms, which implies that a second-price auction is optimal for this problem.

Theorem 1. $E(v^{(2)})$ is the highest expected revenue we can guarantee from amongst all efficient mechanisms.

Proof. We consider the following information structure. Given each v distributed according to F, if there is i such that $v_i > v_{-i}^{(1)} (= \max_{j \neq i} v_j)$, then every agent observes (i, v_{-i}) as a public (among the agents) information.[19]

More precisely, we let $S_j = \{1, \ldots, N\} \times [0,1]^{N-1} (\ni (i, v_{-i}))$ for each j, and G be the following. Let $\delta(s) \in \Delta(S)$ be a Dirac measure for $s \in S$ (i.e., for any measurable $A \subseteq S$, $\delta(A) = 1$ if $s \in A$, and $\delta(A) = 0$ if $s \notin A$). Then, for each measurable $B \subseteq V \times S$:

$$G(B) = \sum_{i=1}^{N} \int_{(v,s) \in B} 1\{v_i > v_{-i}^{(1)}\} d\delta((i, v_{-i}), \ldots, (i, v_{-i})) dF.$$

In this information structure, $(i, v_{-i}) \in S_j$ indicates that i is the highest-value bidder, v_{-i} are the losers' values, and this signal is common knowledge among the bidders. Thus, the only asymmetric information among the bidders is the highest bidder i's value, v_i, which is known solely to i.[20]

[19] We ignore ties without loss of generality.

[20] Similar information structures are considered in **?** and **?** although in quite differ-

To characterize the maximum revenue from amongst all efficient mechanisms, we first consider the following "relaxed" problem. Imagine that the designer can also observe (i, v_{-i}), but not v_i (except that $v_i > v_{-i}^{(1)}$). Then, in an efficient mechanism, i must always win conditional on s, which implies that the maximum payment the mechanism can charge is $v_{-i}^{(1)} (= v^{(2)})$. The maximum expected revenue in this relaxed problem is thus $E[v^{(2)}]$.

□

3.2 Approximate revenue guarantee

Next, we study the maximum expected revenue that can be guaranteed from amongst all (not necessarily efficient) mechanisms. As opposed to the first result, we do not characterize the maximum expected revenue that can be guaranteed. Instead, under certain condition on F, we show that the second-price auction is *rate-optimal* with respect to N in the sense that: (i) $E[v^{(1)}] - E[v^{(2)}] = O\left(\frac{1}{N}\right)$; and (ii) for any mechanism Γ, its revenue guarantee satisfies $E[v^{(1)}] - R(\Gamma) \geq O\left(\frac{1}{N}\right)$, i.e., no mechanism can achieve a strictly faster convergence rate than $\frac{1}{N}$.[21]

ent contexts (I thank an anonymous referee in a previous submission). **?** considers the worst-case information structure amongst those satisfy the affiliation property in a specific English auction (which is equivalent to a second-price auction with private values), while we show that, given this information structure, a second-price auction is optimal amongst all efficient mechanisms. **?** considers the worst-case information structure amongst all private-value type spaces, which includes minimization with respect to the value distribution (i.e., F in our notation), and shows that a second-price auction (with "surveying" of losers) is optimal. Modulo many differences in the details, our result is partly stronger than his: we show optimality of a second-price auction for *any* F, while **?** shows it for the *worst-case* F.

[21] Given that we obtain this rate-optimality result based on some specific (S, G), it may be tempting to conjecture that, for a "better" choice of (S, G), the lower bound becomes a tight bound. However, this is in general impossible: recall a coutnerexample by Chung and Ely (2007) where the revenue guarantee of any dominant-strategy mechanism is strictly

To facilitate understanding of the result of this section, first, suppose that each v_i follows an independent uniform distribution over $[0,1]$. Then, we have $E[v^{(1)}] = \frac{N}{N+1}$ and $E[v^{(2)}] = \frac{N-1}{N+1}$, and hence $E[v^{(1)}] - E[v^{(2)}] = O\left(\frac{1}{N}\right)$ (i.e., (i) above). Regarding (ii), recall that Myerson (1981) shows that a second-price auction with a reserve price $\frac{1}{2}$ is optimal, which attains expected revenue $\frac{N-1+\left(\frac{1}{2}\right)^N}{N+1} = E[v^{(1)}] - O\left(\frac{1}{N}\right)$. Therefore, no mechanism can achieve strictly better than $E[v^{(1)}] - O\left(\frac{1}{N}\right)$.

The goal of this section is to show that, even if v is correlated according to F, we obtain a similar result under certain conditions. We now introduce the key assumption of this section.

Assumption 1. There exists (i) a measurable set Θ, (ii) $\mu \in \Delta(\Theta)$, and (iii) $H_i^\theta \in \Delta([0,1])$ for each $\theta \in \Theta$, that satisfies the following.

(I) For any measurable $A_i \subseteq [0,1]$ for i:

$$F\left(\prod_{i=1}^{N} A_i\right) = \int_\Theta \prod_{i=1}^{N} H_i^\theta(A_i)\ d\mu.$$

(II) There exists $0 < a < b < \infty$ such that, for each $\theta \in \Theta$, H_i^θ admits differentiable density h_i^θ with $h_i^\theta(x) \in [a,b]$ for $x \in [0,1]$.[22]

The assumption says that, even though we do not *a priori* assume an independent distribution, F can be written as a mixture of conditionally independent distributions with bounded density.

Although it is not an innocuous assumption, we believe that it is not too restrictive either. Notice that any F with a full-support density can

suboptimal.

[22]Note that we necessarily have $a < 1 < b$.

always be represented as a mixture of independent distributions (as $(H_i^\theta)_{i,\theta}$ above with property (I)). In this sense, the assumption may be interpreted as regularizing each such independent distribution, in a standard way in the literature (that is, with a smooth and bounded density: (II)).[23] Moreover, a special case of the property (I) is *exchangeability*, a standard assumption in applied probability theory with large samples such as Bayesian statistics. See Supplementary Materials for a more formal relationship between our Assumption 1 and exchangeability.

Theorem 2. Under Assumption 1, we have $E[v^{(1)}] - E[v^{(2)}] = O(\frac{1}{N})$, and $E[v^{(1)}] - R(\Gamma) \geq O(\frac{1}{N})$ for any mechanism Γ.

Proof. First, we show $E[v^{(1)}] - E[v^{(2)}] = O(\frac{1}{N})$. Because $E[v^{(k)}] = E[E[v^{(k)}|\theta]]$ for each $k = 1, 2$, where the inner expectation is with respect to $\prod_i H_i^\theta$ and the outer expectation is with respect to μ, it suffices to show that $E[v^{(1)}|\theta] - E[v^{(2)}|\theta] = O(\frac{1}{N})$ for each θ.

Fix an arbitrary θ, and let $H^{(k),\theta}$ denote the cdf of $v^{(k)}|\theta$ for $k = 1, 2$.[24] Then,

$$\begin{aligned} H^{(1),\theta}(x) &= \prod_{i=1}^{N} H_i^\theta(x), \\ H^{(2),\theta}(x) &= \Pr(v^{(1)} \leq x) + \sum_{i=1}^{N} (1 - H_i^\theta(x)) \prod_{j \neq i} H_j^\theta(x), \end{aligned}$$

[23] Note that we do not even need other regularity properties in the literature, such as the monotone hazard rate condition, although additional regularity property may enable us to obtain a better convergence result.

[24] H_i^θ is originally introduced as a probability measure in Assumption 1, but in what follows it also denotes a corresponding cdf. I believe this abuse does not cause any confusion.

and thus,

$$\begin{aligned} E[v^{(1)} - v^{(2)}|\theta] &= \int_0^1 xdH^{(1),\theta}(x) - \int_0^1 xdH^{(2),\theta}(x) \\ &= \int_0^1 H^{(2),\theta}(x) - H^{(1),\theta}(x)dx \\ &= \int_0^1 \sum_{i=1}^N (1 - H_i^\theta(x)) \prod_{j \neq i} H_j^\theta(x)dx. \end{aligned}$$

Because $1 - H_i^\theta(x) \leq b(1-x)$ and $H_j^\theta(x) \leq 1 - a(1-x)$, we have

$$\begin{aligned} E[v^{(1)} - v^{(2)}|\theta] &\leq \int_0^1 Nb(1-x)(1-a(1-x))^{N-1}dx \\ &= bN \int_{1-a}^1 \frac{1-y}{a} y^{N-1}dy \\ &\leq \frac{b}{a} \cdot \frac{1}{N+1}, \end{aligned}$$

and similarly, because $1-H_i^\theta(x) \geq a(1-x)$ and $H_j^\theta(x) \geq \max\{0, 1-b(1-x)\}$, we have

$$\begin{aligned} E[v^{(1)} - v^{(2)}|\theta] &\geq \int_{1-\frac{1}{b}}^1 Na(1-x)(1-b(1-x))^{N-1}dx \\ &= aN \int_0^1 \frac{1-y}{b} y^{N-1}dy \\ &= \frac{a}{b} \cdot \frac{1}{N+1}. \end{aligned}$$

Next, we show $E[v^{(1)}] - R(\Gamma) \geq O(\frac{1}{N})$. Consider the following information structure: each bidder "observes" $\theta \in \Theta$ as public (among them) information, while each v_i is i's private information.

More precisely, we let $S_i = \Theta$ for each i, and G be the following. Let

$\delta(s) \in \Delta(S)$ be a Dirac measure for $s \in S$. Then, for each measurable $A \subseteq V \times S$:

$$G(A) = \int_\theta \int_{(v,s)\in A} dH^\theta d\delta(\theta, \dots, \theta) d\mu,$$

where $H^\theta = \prod_i H_i^\theta$.

In this information structure, θ indicates the "realized" value of the fundamental, which is common knowledge among the bidders. Obviously, the designer can also know this θ for free.

Conditional on θ, the bidders' values are then conditionally independently distributed according to H^θ.[25] Thus, the problem is then a standard optimal auction problem with independently distributed values. The maximum expected revenue given θ is achieved by virtual-value maximization, which is given by

$$R^\theta \equiv \int_v \left(\max_{i=0,\dots,N} \gamma_i(v) \right) dH^\theta(v),$$

where $\gamma_0(v) \equiv 0$ and $\gamma_i(v) = v_i - \frac{1-H_i^\theta(v_i)}{h_i^\theta(y_i)}$ for each $i = 1, \dots, N$. We use $\int_\theta R^\theta d\mu(\theta)$ as an upper bound of $R(\Gamma)$ for any mechanism Γ.

Fix $\varepsilon > 0$, θ, and v such that $v^{(2)} \geq 1 - \varepsilon$. To achieve R^θ, the bidder with the highest virtual value must win, while a second-price auction chooses the bidder with the highest value, and given the potential asymmetry of the value distributions, these two bidders may be different. In terms of the virtual value, therefore, the optimal mechanism can achieve a higher virtual value

[25] Farinha Luz (2013) studies the optimal mechanisms under conditionally independent value distributions.

than the second-price auction by at most

$$\begin{aligned} \sup_{v^{(1)},x,i,j} \quad & \gamma_j(x) - \gamma_i(v^{(1)}) \\ \text{sub. to} \quad & i,j \in \{1,\ldots,N\},\ v^{(1)} \geq x \geq 1-\varepsilon. \end{aligned}$$

By Taylor's theorem, $\gamma_i(x) = 1-(1-x)(2+o(1-x))$ for all $x \geq 1-\varepsilon$,[26] implying that the above expression is bounded from above by $o(\varepsilon^2)$.

Because $\int_\theta R^\theta d\mu(\theta)$ is an upper bound of $R(\Gamma)$ for any mechanism Γ, for a sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} R(\Gamma) - E[v^{(2)}] &\leq \int_\theta \left[\Pr(v^{(2)} \leq 1-\varepsilon|\theta) + \Pr(v^{(2)} \geq 1-\varepsilon|\theta)o(\varepsilon^2)\right] d\mu \\ &\leq (1-a\varepsilon)^N + Nb\varepsilon(1-a\varepsilon)^{N-1} + o(\varepsilon^2). \end{aligned}$$

Thus, by taking $\varepsilon = \frac{1}{\sqrt{N}}$, we obtain

$$\begin{aligned} (1-a\varepsilon)^N + Nb\varepsilon(1-a\varepsilon)^{N-1} &= (1-\frac{a}{\sqrt{N}} + b\sqrt{N})(1-\frac{a}{\sqrt{N}})^{N-1} \\ &= \exp(\log(1-\frac{a}{\sqrt{N}} + b\sqrt{N}) + (N-1)\log(1-\frac{a}{\sqrt{N}})) \\ &\leq \exp(\log(1+b\sqrt{N}) - (N-1)\frac{a}{\sqrt{N}}) \\ &\leq \exp(\frac{a}{2}\sqrt{N} + \gamma - a\sqrt{N} + a) \\ &\leq o(\frac{1}{N}), \end{aligned}$$

where the first inequality is because $\log(1+x) \leq x$ for all $x > -1$, the second inequality is because $\log(1+x) \leq cx + c - 1 - \log c$ for all $x > -1$ and all $c > 0$, where we set $c = \frac{a}{2b}$ and $\gamma = \frac{a}{2b} - 1 - \log\left(\frac{a}{2b}\right)$, and the last inequality

[26]Note that the bound is uniform across N.

is because $\exp(-dx) \leq \frac{1}{1+dx+dx^2} \leq \frac{1}{dx^2}$ for all $d, x > 0$.

Therefore, we obtain

$$R(\Gamma) - E[v^{(2)}] \leq o(\frac{1}{N}),$$

which implies

$$\begin{aligned} E[v^{(1)}] - R(\Gamma) &= E[v^{(1)}] - E[v^{(2)}] - (R(\Gamma) - E[v^{(2)}]) \\ &\geq O(\frac{1}{N}) + o(\frac{1}{N}) \\ &= O(\frac{1}{N}). \end{aligned}$$

□

Remark 2. Obviously, a second-price auction is just one of the rate-optimal mechanisms. This may raise the question as to whether or not "many" mechanisms are rate-optimal. Although there are indeed multiple such mechanisms, these must be all "close to" a second-price auction in the following sense: first, except for a probability at most $o(\frac{1}{N})$, the highest-value bidder must win the auction; second, the expected information rent for the winner i must not deviate from that in a second-price auction by more than $o(\frac{1}{N})$. These are the consequence of characterization of R^{θ}, i.e., the upper-bound expected revenue if the principal could know θ. In this sense, although there are multiple rate-optimal mechanisms, we argue that this "second-price property" is close to necessary and sufficient for optimal revenue guarantee.

For example, a first-price auction is known to achieve asymptotically the same expected revenue as a second-price auction for a class of symmetric (but not necessarily independent) value distributions, so-called *asymptotic revenue equivalence* (Bali and Jackson (2002)).[27] This suggests that a first-price auction may also be rate-optimal, at least for a restricted class of F.[28]

As another example, a posted-price mechanism is dominant-strategy incentive compatible and its revenue guarantee approaches $E[v^{(1)}]$ as $N \to \infty$, but with a much slower convergence rate (hence, it is *not* rate-optimal).[29] Imagine a procedure where the seller specifies a price in advance, and if there exist buyers whose values are above the price, then one of those buyers is randomly chosen as the winner (otherwise, no winner). To provide an idea of the convergence rate, suppose that $v_i \sim U(0,1)$ independently (perhaps for some given θ). Its expected revenue given price p is $p(1-p^N)$, and hence, with the optimal price of $p^* = \frac{1}{N+1}^{\frac{1}{N}}$, we obtain

$$
\begin{aligned}
p^*(1-(p^*)^N) &\leq (\frac{1}{N})^{\frac{1}{N}} \\
&= \exp(-\frac{\log N}{N}) \\
&\leq \frac{1}{1+\frac{\log N}{N}} \\
&= \frac{N}{N+\log N}.
\end{aligned}
$$

[27]Gavish, Fibich, and Gavious (2017) conjecture that a similar asymptotic result holds with a specific convergence rate even with asymmetric (and not necessarily independent) distributions.

[28]Bergemann, Brooks, and Morris (2017) identify a maximization problem that provides the revenue guarantee of a first-price auction, and obtain a solution when each bidder's value is binary (a closed-form solution is obtained with two bidders). They also discuss why it would be difficult to generalize their approach to the case with more than binary values.

[29]I thank Drew Fudenberg for this remark.

Thus,

$$
\begin{aligned}
E[v^{(1)}] - p^*(1-(p^*)^N) &\geq \frac{N}{N+1} - \frac{N}{N+\log N} \\
&= \frac{N(\log N - 1)}{(N+1)(N+\log N)} \\
&\geq \frac{\log N - 1}{4N} \\
&= O(\frac{\log N}{N}),
\end{aligned}
$$

and therefore, the convergence rate is much slower than $O(\frac{1}{N})$.

To evaluate the difference in convergence rates especially for relatively small values of N, imagine again an iid uniform distribution over $[0,1]$. In this case, we have: $E[v^{(1)}] = \frac{N}{N+1}$, $E[v^{(2)}] = \frac{N-1}{N+1}$, $R^\theta = \frac{N-1}{N+1} + \frac{1}{(N+1)2^N}$, and $p^*(1-(p^*)^N) = (\frac{1}{N+1})^{\frac{1}{N}} \frac{N}{N+1}$. The differences are plotted in the left panel of Figure 1 (up to $N = 10$).

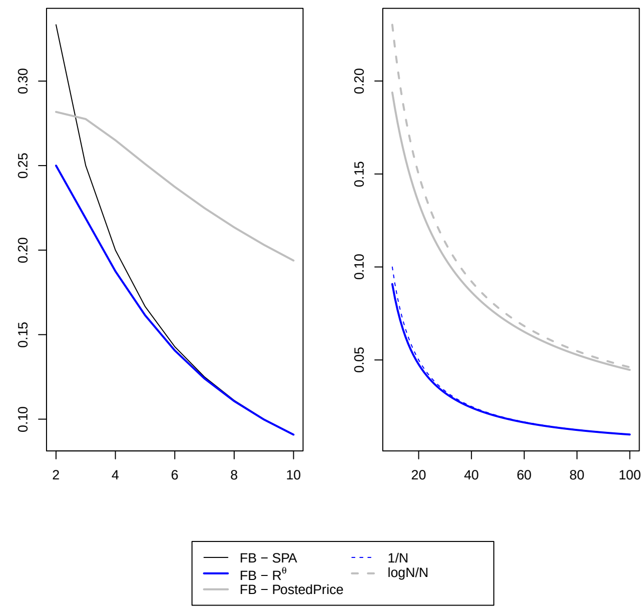

Figure 1

As is clearly seen, the convergence between $E[v^{(1)}] - E[v^{(2)}]$ and $E[v^{(1)}] - R^\theta$ occurs very quickly (even around $N = 5$), while $E[v^{(1)}] - p^*(1 - (p^*)^N)$ is still much above.

The right panel extends these graphs until around $N = 100$, and the difference between $E[v^{(1)}]$ and the posted-price revenue seems still significant. To help understanding the source of the difference, the figure also has two additional plots: $\frac{1}{N}$ and $\frac{\log N}{N}$. The difference between $E[v^{(1)}]$ and $E[v^{(2)}]$ converges to $\frac{1}{N}$, while the difference between $E[v^{(1)}]$ and the posted-price revenue converges to $\frac{\log N}{N}$ (with a significant difference between $\frac{1}{N}$ and $\frac{\log N}{N}$ even around $N = 100$). This suggests that the approximation argument above is valid not only at the limit ($N \to \infty$) but for much smaller values of N.

4 Conclusion

In this paper, we consider auction environments with a (possibly correlated) common prior over bidders' values, where each bidder may have additional information (e.g., through information acquisition). Under certain conditions, we characterize the optimal mechanisms in terms of the expected revenue that is guaranteed given whatever additional information is available to the bidders. Specifically, we show that a second-price auction is optimal from amongst all the efficient mechanisms, and it is rate-optimal from amongst all the mechanisms.

Although the paper focuses on simple auction environments, the idea of decomposing a (possibly correlated) distribution into multiple independent distributions (with the interpretation of the agents' additional information) could be useful in other mechanism design contexts, as we discuss in Supplementary Material.

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Supplementary Materials

(not for publication)

A Assumption 1 and Exchangeability

In Section 3.2, we show the rate-optimality of a second-price auction under Assumption 1; a possibility of representing F as a mixture of conditionally independent distributions. As we discuss there, Assumption 1 is related to exchangeability, a common assumption in applied probability theory with large samples. Here, we discuss a more formal relationship between Assumption 1 and exchangeability.

Recall that exchangeability is about the distribution of a countably infinite sequence of random variables. Let $F_{\mathbb{N}} \in \Delta([0,1]^{\mathbb{N}})$ denote a probability distribution over a countably infinite sequence of random variables, each taking a value in $[0,1]$. A possible (though not necessary) interpretation is that there are potentially an infinite number of bidders $i = 1, 2, \ldots$, and $F_{\mathbb{N}}$ provides a (possibly correlated) joint distribution over the values of all those bidders. Consider any finite permutation of the identities of the bidders. Let $\tilde{F}_{\mathbb{N}}$ denote the joint distribution over the values of all the bidders, but with the permuted identities. We say that $F_{\mathbb{N}}$ is *exchangeable* if $F_{\mathbb{N}} = \tilde{F}_{\mathbb{N}}$ for every finite permutation. We say that F (a distribution over N bidders' values) is *exchangeable* if it can be expressed as a marginal of an exchangeable $F_{\mathbb{N}}$ on N variables.[30]

[30]Note that, by exchangeability, "which" N variables one fixes does not make any difference.

The following theorem, called de Finetti theorem, says that exchangeable $F_{\mathbb{N}}$ can be represented as a mixture of conditionally independent and identical distributions.

Lemma 1. (de Finetti theorem) If $F_{\mathbb{N}}$ is exchangeable, then there exists a profile of (i) a measurable set Θ, (ii) $\mu \in \Delta(\Theta)$, and (iii) $H^{\theta} \in \Delta([0,1])$ for each $\theta \in \Theta$, such that, for any measurable $A_i \subseteq [0,1]$ for $i \in \mathbb{N}$, we have:

$$F_{\mathbb{N}}(\prod_{i=1}^{\infty} A_i) = \int_{\Theta} \prod_{i=1}^{\infty} H^{\theta}(A_i)\ d\mu.$$

Therefore, Property (I) in Assumption 1 is obtained by assuming exchangeability of F. We obtain the same result as in the main text.

Theorem. Let $F_{\mathbb{N}} \in \Delta([0,1]^{\mathbb{N}})$ be exchangeable, and assume there exists $0 < a < b < \infty$ such that, for each $\theta \in \Theta$, H_i^{θ} admits differentiable density h_i^{θ} with $h_i^{\theta}(x) \in [a,b]$ for $x \in [0,1]$. Then, we have $E[v^{(1)}] - E[v^{(2)}] = O(\frac{1}{N})$, and $E[v^{(1)}] - R(\Gamma) \geq O(\frac{1}{N})$ for any mechanism Γ.

B Revenue guarantee in sequential sales

The third application considers a sequential-sales setting. Consider a discrete-time ($t = 1, 2, \ldots$) dynamic environment where each agent i is available only at time $t = i$. The designer (a seller) has an indivisible good, and attempts to time the sale of it in order to maximize expected revenue. Although our model introduced in Section 2 is static, we treat this dynamic environment as an application of our model by requiring that a feasible mechanism should be such that the allocation for agent i given a message profile $m = (m_1, \ldots, m_N)$,

$(q_i(m), p_i(m))$, can only be a function of $m^i \equiv (m_1, \ldots, m_i)$ but not of $m_{i+1}, \ldots, m_N$.

Because the allocation for agent 1 can only be a function of his own message, it is in general impossible to extract the "full" surplus. However, if F exhibits certain correlation as in Crémer and McLean (1985), Crémer and McLean (1988), or McAfee and Reny (1992), and if we do not allow for any additional information for the agents, then no information rent would be left for all of the other agents $i \neq 1$. In this sense, much of the surplus could still be extracted by a similar side-bets mechanism as in those papers.

If the agents have additional information, however, then this surplus extraction result no longer holds. Furthermore, we show that the highest revenue guarantee is attained by a dominant-strategy mechanism. More specifically, it is a sequential posted-price mechanism where the price for each agent i is adjusted by the previous agents' reports, $v_1, \ldots, v_{i-1}$. Therefore, this is an instance where the revenue-maximizing seller finds it optimal (exactly, rather than approximately) to use a dominant-strategy mechanism.

First, we characterize the optimal dominant-strategy mechanism, which is obtained by a backward-induction argument. Let $v^i = (v_1, \ldots, v_i)$. First, consider the last buyer, $i = N$. If each agent $j < N$ has been assigned the good with probability q_j, then the optimal dominant-strategy mechanism solves:

$$
\begin{aligned}
R_N(q_1,\ldots,q_{N-1}) = \max_{q_N(v^N),p_N(v^N)} \quad & E(q_N(v^N)p_N(v^N)) \\
\text{sub. to} \quad & q_N(v^N)(v_N - p_N(v^N)) \geq q_N(v^{N-1}, v_N')(v_N - p_N(v^{N-1}, v_N')), \\
& q_N(v^N)(v_N - p_N(v^N)) \geq 0, \\
& \sum_{j=1}^{N-1} q_j + q_N(v^N) \leq 1.
\end{aligned}
$$

By an induction argument, let $R_{i+1}(q_1,\ldots,q_i)$ be the expected revenue from the agents $k = i+1,\ldots,N$ in the optimal dominant-strategy mechanism given that each agent $j \leq i$ has been assigned the good with probability q_j. Now, regarding the allocation for agent i, the optimal dominant-strategy mechanism solves:

$$
\begin{aligned}
R_i(q_1,\ldots,q_{i-1}) = \max_{q_i(v^i),p_i(v^i)} \quad & E(q_i(v^i)p_i(v^i) + R_{i+1}(q_1,\ldots,q_i(v^i))) \\
\text{sub. to} \quad & q_i(v^i)(v_i - p_i(v^i)) \geq q_i(v^{i-1}, v_i')(v_i - p_i(v^{i-1}, v_i')), \\
& q_i(v^i)(v_i - p_i(v^i)) \geq 0, \\
& \sum_{j=1}^{i-1} q_j + q_i(v^i) \leq 1.
\end{aligned}
$$

Therefore, we obtain the following.

Lemma 2. The expected revenue of the optimal dominant-strategy mechanism is given by R_1.

We now show that R_1 is indeed the highest revenue guarantee.

Theorem 3. The highest revenue guarantee among all feasible mechanisms in the sequential-sales environment is R_1.

Proof. We consider the information structure such that bidder i observes $v^i = (v_1, \ldots, v_i)$. More precisely, we let $S_i = V^i \equiv V_1 \times \cdots \times V_i$ for each i, and G be the following. Let $\delta_j(s_j) \in \Delta(S_j)$ be a Dirac measure for $s_j \in S_j$. Then, for each measurable $A \subseteq V \times S$:

$$G(A) = \int_{(v,s)\in A} \prod_{i=1}^{N} d\delta_i(v^i) dF.$$

Suppose that there exists a feasible mechanism $\Gamma = (M, \tilde{q}, \tilde{p})$ and its Bayesian equilibrium σ given this information structure that yields a strictly higher expected revenue than R_1.

In the following, we only consider the case where σ is a pure-strategy equilibrium, but a similar logic also applies to mixed-strategy equilibria. For each i and v, let $q_i(v^i) = \tilde{q}_i(\sigma^i(v^i))$ and $p_i(v^i) = \tilde{p}_i(\sigma^i(v^i))$.

First, consider the last buyer, $i = N$. Because this buyer knows the realized v^N, the Bayesian equilibrium condition is that, for each $m_N \in M_N$:

$$\begin{aligned} & \tilde{q}_N(\sigma_N(v^N), \sigma^{N-1}(v^{N-1}))(v_N - \tilde{p}_N(\sigma_N(v^N), \sigma^{N-1}(v^{N-1}))) \\ \geq \; & \tilde{q}_N(m_N, \sigma^{N-1}(v^{N-1}))(v_N - \tilde{p}_N(m_N, \sigma^{N-1}(v^{N-1}))), \end{aligned}$$

which implies that, for each $v'_N \in V$:

$$q_N(v^N)(v_N - p_N(v^N)) \geq q_N(v^{N-1}, v'_N)(v_N - p_N(v^{N-1}, v'_N)).$$

Therefore, the expected revenue from this buyer is at most:

$$\begin{aligned}
\max_{q_N(v^N),p_N(v^N)} \quad & E(q_N(v^N)p_N(v^N)) \\
\text{sub. to} \quad & q_N(v^N)(v_N - p_N(v^N)) \geq q_N(v^{N-1}, v'_N)(v_N - p_N(v^{N-1}, v'_N)), \\
& q_N(v^N)(v_N - p_N(v^N)) \geq 0, \\
& \sum_{j=1}^{N-1} q_j(v^j) + q_N(v^N) \leq 1,
\end{aligned}$$

which is $R_N(q_1(v^1), \ldots, q_{N-1}(v^{N-1}))$.

By an induction argument, suppose that $R_{i+1}(q_1(v^1), \ldots, q_i(v^i))$ is an upper-bound expected revenue from the agents $k = i+1, \ldots, N$. Now, regarding the allocation for agent i, because this agent knows the realized v^i, the Bayesian equilibrium condition is that, for each $m_i \in M_i$:

$$\begin{aligned}
& \tilde{q}_i(\sigma_i(v^i), \sigma^{i-1}(v^{i-1}))(v_i - \tilde{p}_i(\sigma_i(v^i), \sigma^{i-1}(v^{i-1}))) \\
\geq \quad & \tilde{q}_i(m_i, \sigma^{i-1}(v^{i-1}))(v_i - \tilde{p}_i(m_i, \sigma^{i-1}(v^{i-1}))),
\end{aligned}$$

which implies that, for each $v'_i \in V$:

$$q_i(v^i)(v_i - p_i(v^i)) \geq q_i(v^{i-1}, v'_i)(v_i - p_i(v^{i-1}, v'_i)).$$

Thus, the expected revenue is at most:

$$\begin{aligned}
\max_{q_i(v^i),p_i(v^i)} \quad & E(q_i(v^i)p_i(v^i) + R_{i+1}(q_1, \ldots, q_i(v^i))) \\
\text{sub. to} \quad & q_i(v^i)(v_i - p_i(v^i)) \geq q_i(v^{i-1}, v_i')(v_i - p_i(v^{i-1}, v_i')), \\
& q_i(v^i)(v_i - p_i(v^i)) \geq 0, \\
& \sum_{j=1}^{i-1} q_j + q_i(v^i) \leq 1,
\end{aligned}$$

which is $R_i(q_1(v^1), \ldots, q_{i-1}(v^{i-1}))$.

Therefore, the expected revenue is at most R_1, which contradicts our initial supposition. □

C Endogenous information acquisition

In the previous sections, we model the bidders' additional information structure in a general way, so that the results would not depend on *ad hoc* restrictions. However, it also seems reasonable to ask if such a critical information structure can be "endogenously chosen" by the bidders' equilibrium information acquisition decisions.[31] Here, we show a simple model of information acquisition in the context of the efficient auction of Section 3.1, with which the seller's best expected revenue can be made arbitrarily close to $E(v^{(2)})$,

[31] An alternative question is whether or not such a critical information structure can be an endogenous outcome of the bidders' communication or information sharing. I thank Juuso Valimaki for suggesting that our critical (S, G) may be obtained if the bidders play an English auction before they play the current mechanism. Interestingly, Vartiainen (2013) shows that such an information structure eliminates the (non-committed) seller's incentive of "re-auctioning", which is a different kind of robustness concern.

and in this sense, a second-price auction is "virtually" optimal.[32]

Assume that each bidder i simultaneously chooses an information acquisition action $a_i \in A$ after he learns v_i and observes the mechanism, but before playing the mechanism. Let $A = \{(S_0, G_0), (S_1, G_1)\}$, where (i) S_0 is singleton (and G_0 collapses to F) so that (S_0, G_0) corresponds to "no information acquisition", and (ii) (S_1, G_1) is the critical information presented in the proof of Theorem 1. The set A can contain more alternatives, but it is not essential in the following argument. Bidder i's information acquisition cost enters into his payoff function in an additively separable manner, is zero for (S_0, G_0), and is $c_i \geq 0$ for (S_1, G_1), where c_i is his private information. For simplicity, assume that each c_i follows a distribution γ independently from (c_{-i}, v), and that the probability of ties in v is zero.

Intuitively, if the seller offers a second-price auction or any other dominant-strategy mechanism, then no one engages in information acquisition (except for the zero-cost type, which is indifferent). On the other hand, if the seller offers a Cremer-McLean mechanism, then a bidder may have an incentive to pay the cost to acquire the critical information to "protect" himself from surplus extraction. Characterizing the optimal mechanism would be, in general, difficult given a multi-dimensional type space and endogenous information acquisition. However, as long as each c_i is small in expectation, we can show that a second-price auction, which does not induce costly information acquisition, is close to being optimal.

Proposition 1. For any mechanism and its Bayesian equilibrium where the

[32]Admittedly, the model presented here may look somewhat "extreme". It is beyond the scope of this paper to examine whether or not there exists a "more natural" information acquisition model that achieves the same result.

highest-value bidder always wins, the expected revenue cannot be greater than $E(v^{(2)}) + E(c_i)N$.

Proof. Fix an arbitrary mechanism and its Bayesian equilibrium where the highest-value bidder always wins. In the following, we only consider the case with a pure-strategy equilibrium for simplicity. The same argument holds for the case with a mixed-strategy equilibrium, but with more complicated notation.

Let $U_i(v_i, c_i)$ denote bidder i's equilibrium expected payoff given his value type v_i and cost type c_i, i.e.:

$$U_i(v_i, c_i) = E_{v_{-i},c_{-i}}[q_i^*(v, c, s(\alpha^*(v, c)))v_i - p_i^*(v, c, s(\alpha^*(v, c))) - \alpha_i^*(v_i, c_i)c_i|v_i, c_i],$$

where $\alpha_i^*(v_i, c_i) \in \{0, 1\}$ represents bidder i's equilibrium probability of acquiring the critical information given (v_i, c_i), $s(\alpha^*(v, c)) = (s_j(\alpha_j^*(v_i, c_i)))_{j=1}^N$, and q_i^*, p_i^* represent the implied probability of winning and expected payment of bidder i.

First, consider the case with $c_i = 0$. His possible deviation is to acquire the critical information, and then in the mechanism, (i) behave the same way as supposed in the equilibrium as long as his auction payoff $E_{c_{-i}}[q_i^*(v, c, s(\alpha^*(v, c)))v_i - p_i^*(v, c, s(\alpha^*(v, c)))|v, c_i = 0]$ is non-negative, and (ii) refuse participation otherwise. This is always a weakly profitable deviation, and is strictly profitable if case (ii) happens. Therefore, $E_{c_{-i}}[q_i^*(v, c, s(\alpha^*(v, c)))v_i - p_i^*(v, c, s(\alpha^*(v, c)))|v, c_i = 0]$ should be non-negative for any v, which implies $E_{c_{-i}}[p_i^*(v, c, s(\alpha^*(v, c)))|v, c_i = 0] \leq v_i$. Furthermore, we obtain the following restriction on his payment.

Lemma 3. For each v such that $v_i > v_{-i}^{(1)}$, $E_{c_{-i}}[p_i^*(v, c, s(\alpha^*(v, c)))|v, c_i =$

$0] \leq v_{-i}^{(1)}$.

Proof. Suppose contrarily that, for some v with $v_i > v_{-i}^{(1)}$, $E_{c_{-i}}[p_i^*(v, c, s(\alpha^*(v, c)))|v, c_i = 0] = x_i(v) > v_{-i}^{(1)}$.

Bidder i's alternative possible deviation is to acquire the critical information, and then in the mechanism, behave as if his value type is $v_i' \in (v_{-i}^{(1)}, x_i(v))$. Because efficiency implies that he wins for sure both in the equilibrium and also in this deviation:

$$\begin{aligned} v_i - x_i(v) &\geq v_i - E_{c_{-i}}[p_i^*(v_i', v_{-i}, c, s_i(1), s_{-i}(\alpha_{-i}^*(v_{-i}, c_{-i})))|v, c_i = 0] \\ &= v_i - x_i(v_i', v_{-i}) \\ &\geq v_i - v_i', \end{aligned}$$

which implies $x_i(v) \leq v_i'$, which contradicts $v_i' \in (v_{-i}^{(1)}, x_i(v))$. □

The lemma implies that:

$$U_i(v_i, 0) \geq E_{v_{-i}}[1\{v_i > v_{-i}^{(1)}\}(v_i - v_{-i}^{(1)})|v_i].$$

Now consider bidder i with type (v_i, c_i). His possible deviation is to acquire the critical information, and then in the mechanism, behave as if his type is $(v_i, 0)$. Thus:

$$\begin{aligned} U_i(v_i, c_i) &\geq U_i(v_i, 0) - c_i \\ &\geq E_{v_{-i}}[1\{v_i > v_{-i}^{(1)}\}(v_i - v_{-i}^{(1)})|v_i] - c_i. \end{aligned}$$

Therefore, the seller's expected revenue is:

$$
\begin{aligned}
& E(v^{(1)}) - \sum_i E_{v_i, c_i}[U_i(v_i, c_i)] \\
\leq \;& \int_c \int_v 1\{v_i > v_{-i}^{(1)}\}(v_i - (v_i - v_{-i}^{(1)})) + \sum_i c_i dF \; d\Gamma \\
\leq \;& E(v^{(2)}) + E(c_i)N.
\end{aligned}
$$

□

Therefore, as $E(c_i)$ vanishes, a second-price auction becomes close to being optimal. Of course, if $E(c_i)$ is very large, then a second-price auction is far from optimal, but this is not surprising. If information acquisition is prohibitively costly, we return to the Cremer-McLean case where there is no information acquisition and hence full extraction is possible.

D Discussion: Some possible generalizations

Although we consider private-value auction environments, the methodology developed in this paper can be generalized into a number of other environments. In this section, we propose some possible generalizations based on the efficient-auction problem in Section 3.1.

First, consider the following interdependent/common-value environment. Instead of valuation v_i, imagine that bidder i has a signal $t_i \in T_i = [0, 1]$, where the common prior for $t = (t_i)_{i=1}^N$ is denoted by F, and i's valuation is $v_i(t)$. Assume that (i) the bidders are symmetric in the sense that $v_i(t) = v_j(t')$ if t' satisfies $t'_i = t_j$, $t'_j = t_i$ and $t'_k = t_k$ for $k \neq i, j$, (ii) v_i is strictly increasing in t_i, and (iii) the following *single-crossing* condition holds (Maskin (1992) and Dasgupta and Maskin (2000)). For each $i \neq j$, $t_i < t'_i$ and t_{-i}, $v_i(t_i, t_{-i}) \geq v_j(t_i, t_{-i})$ implies $v_i(t'_i, t_{-i}) > v_j(t'_i, t_{-i})$.

Under these conditions, efficiency means that the bidder with the highest t_i wins. Then, with an analogous information structure (S, G) as in the proof of the theorem (i.e., where every bidder knows who should be the winner and all the losers' t's), the best expected revenue is given by the optimal ex post incentive compatible mechanism. For example, an English ascending auction is one such mechanism.

More generally, consider a general mechanism design environment where each agent has a payoff type $t_i \in T_i$, and the principal desires to implement some social choice function that only depends on the bidders' payoff types (e.g., efficient allocation), $\phi : T \to Q$, where Q denotes a general non-monetary allocation space. Note that we do not have any restriction on each T_i and Q (e.g., not necessarily intervals). Each agent's payoff is $V_i(t, q) - p_i$ where t is a payoff type profile, $q \in Q$ is an allocation and p_i is his payment to the mechanism. This is a *separable* environment of Bergemann and Morris (2005), where their result implies the following.

Lemma 4. Suppose that, for any (S, G), there exists a transfer function for each i, $p_i : S \times T \to \mathbb{R}$, such that truth-telling (of realized $(s, t) \in S \times T$) is a Bayesian equilibrium in mechanism $(\phi, p_1, \ldots, p_N)$.[33] Then, there exists a monetary transfer function for each i, $p_i^{EP} : T \to \mathbb{R}$, such that truth-telling (of realized $t \in T$) is an ex post equilibrium in mechanism $(\phi, p_1^{EP}, \ldots, p_N^{EP})$.

[33] One may imagine, though not necessarily, that this corresponds to the revenue-maximizing mechanism among those which implement ϕ, when the principal knows (S, G).

Proof. By the proof of their Proposition 1, ϕ is ex post implementable if ϕ is Bayesian (or *interim*) implementable with every information structure such that all but one agent's payoff types are common knowledge. Note that such an information structure can be represented as a feasible (S, G), and hence, their result immediately applies. □

The result does not exclude a possibility that there is a mechanism that Bayesian implements ϕ with higher expected revenue than that of the ex post incentive compatible mechanism (with p_i^{EP}). Recall that, with arbitrary (S, G), our environment does not generally admit such a "revenue equivalence" sort of property. Nevertheless, under appropriate choice of (S, G), we can indeed have revenue equivalence, and hence obtain optimality of the ex post mechanism (with p_i^{EP}) in the sense of revenue guarantee.[34]

To show this, we first observe that any F can be represented as a mixture of independent (though not necessarily identical) distributions, i.e., there exists a measurable space Θ, a probability measure μ over Θ and a family of independent distributions $\{H_\theta\}_{\theta\in\Theta}$ over T such that $F(\tilde{T}) = \int_\Theta H_\theta(\tilde{T})d\mu$.[35] Then, we can apply appropriate versions of revenue-equivalence results. For example:

Theorem 4. 1. Suppose that $Q \subseteq [0,1]^N (\ni (q_i)_{i=1}^N)$, T_i is a compact interval of $\mathbb{R}$, $V_i(t, q) = v_i(t)q_i$ where v_i is increasing in t_i, and satisfies

[34] Note the two different roles of additional information in this argument. First, considering (S, G) used in Bergemann and Morris (2005), the property of implementable ϕ is derived. Then, considering different (S, G), revenue guarantee is characterized.

[35] There can be multiple ways to decompose F into independent distributions. A trivial way is to set $\Theta = T$ and each H_t as a Dirac distribution on $t \in T$. Whether or not there exists a more "nontrivial" or "looking natural" decomposition would depend on F and the specificity of environments. I thank Gabriel Carroll for this point.

single-crossing: for each $i \neq j$, $t_i < t'_i$ and t_{-i}, $v_i(t_i, t_{-i}) \geq v_j(t_i, t_{-i})$ implies $v_i(t'_i, t_{-i}) > v_j(t'_i, t_{-i})$. Then, for any F that is full-support and any ϕ that is implementable given any (S, G), the optimal revenue guarantee is given by an ex post mechanism.[36]

2. Suppose that Q is the set of all probability distributions over a finite set of social alternatives $\{1, \ldots, K\}$, $T_i = \mathbb{R}^K$, $V_i(t, q) = \sum_{k=1}^{K} t_i(k)q(k)$, and ϕ is efficient (i.e., $\phi(t)$ maximizes $\sum_i V_i(t, q)$ for all t). Then, for any F, the optimal revenue guarantee is given by a dominant-strategy mechanism.[37]

3. Suppose that T_i is a convex subset of $\mathbb{R}^d$, and $V_i(t, q)$ is convex in t_i. Then, for any F that is full-support and any ϕ that is implementable given any (S, G), the optimal revenue guarantee is given by an ex post mechanism.[38]

Even if revenue equivalence in the above sense does not hold, our result may still be generalized. Specifically, consider an "auction-like" separable environment where a (non-monetary) allocation $q \in Q$ is not simultaneously relevant to multiple agents, in the following sense: there exist $(b_i)_{i=1}^N \in \mathbb{R}^N$ such that $v_i(t, q) \neq b_i$ for some i implies $v_j(t, q) = b_j$ for $j \neq i$. In this case, agent i is not necessarily the "winner" in an appropriate sense (e.g., $v_i(t, q)$ can be less than b_i), but a crucial property is that only i's value matters as long as q is to be chosen.

[36]See Mookherjee and Reichelstein (1992) for the private-value case, and Jehiel and Moldovanu (2001) for the interdependent-value case.

[37]See Krishna and Perry (2000).

[38]See Krishna and Maenner (2001).

For example, consider an allocation problem of an indivisible *bad*,[39] where b_i denotes i's (publicly known) benefit of a noxious facility such as nuclear power plants or garbage disposal. An allocation is $q \in Q = \{0, 1, \ldots, N\}$, where $q = 0$ means no facility (hence every agent earns 0), and $q = i$ means that the facility is located at i's location, which makes j's payoff $b_j - 1\{i = j\}c_j$. This c_i may be privately known by agent i (i.e., $t_i = c_i$), or alternatively, agent i only knows a noisy information t_i about the cost, and the actual cost may depend also on the others' information t_{-i} (hence denoted by $c_i(t)$).

Another, related, example may be a variant of the partnership dissolution problem of Cramton, Gibbons, and Klemperer (1987). Consider a company owned by N shareholders, and assume that, with the same ownership structure ($q = 0$), each shareholder i's payoff is publicly known as $-b_i(\geq 0)$. The principal designs a mechanism to reallocate the shares among the agents or keep the status quo (hence, $q \in Q = \{0, 1, \ldots, N\}$). If i is assigned the full ownership ($q = i$), then he earns $\pi_i(t)$, potentially as a function of all the shareholders' payoff-relevant information. If i loses his ownership ($q \neq 0, i$), then he earns 0. After renormalization of each i's payoffs by adding b_i, this example also fits the "auction-like" separable environment.

For those examples, q simply indicates who is relevant (as in an auction), but q can be more complex in other cases. For example, the authority may choose a firm which operates in a regulated industry, through a reverse auction. Then the regulation contract between the authority and the winning firm may depend on the other relevant information.[40] In this case, we may

[39] This is called a NIMBY ("not-in-my-backyard") problem. Often for this problem, not only efficiency but also fairness is a central issue. See, for example, Sakai (2012).

[40] For example, a higher-cost firm may produce less goods. See Laffont and Tirole (1986) and Laffont and Tirole (1987).

let $Q = \{1, \ldots, N\} \times \mathbb{R}_+ (\ni (q^1, q^2))$, where $q^1 \in \{1, \ldots, N\}$ indicates which firm wins and $q^2 \in \mathbb{R}_+$ represents how much this firm produces.

Instead of this strong assumption that only one agent is relevant, we can allow for multidimensional payoff types, interdependence, non-monotone payoff functions, and so on, with which the standard revenue equivalence results may not hold.

Theorem 5. In the "auction-like" separable environment as above, the highest revenue guarantee among all mechanisms that implements ϕ (given every feasible (S, G)) is achieved by an ex post incentive compatible mechanism.

Proof. Define $T^1 \subseteq T$ as the set of all type profiles $t \in T$ with which any agent $j \neq 1$ is irrelevant, i.e., $v_j(t, \phi(t)) = b_j$ for $j \neq 1$. Then, inductively from $i = 2$ to N, define $T^i \subseteq T \setminus (\bigcup_{j<i} T^j)$ as the set of all type profiles $t \in T \setminus (\bigcup_{j<i} T^j)$ with which any agent $j \neq i$ is irrelevant, i.e., $v_j(t, \phi(t)) = b_j$ for $j \neq i$. Note that $\{T^i\}_{i=1}^N$ is a partition of T.

We consider the following information structure. Given each t distributed according to F, if $t \in T^i$, then every agent j observes (i, t_{-i}) as a public (among the agents) information.

More precisely, we let $S_j = \{(i, t_{-i}) | i \in \{1, \ldots, N\}, t_{-i} \in T_{-i}\}$ for every agent j, and G be the following. Let $\delta(s) \in \Delta(S)$ be a Dirac measure for $s \in S$. Then, for each measurable $B \subseteq T \times S$:

$$G(B) = \sum_{i=1}^{N} \int_{(t,s)\in B} 1\{t \in T^i\} d\delta((i, t_{-i}), \ldots, (i, t_{-i})) dF.$$

In this information structure, $(i, t_{-i}) \in S_j$ indicates that i is the relevant agent (i.e., $t \in T^i$), and t_{-i} are all the irrelevant agents' payoff types, and this signal is common knowledge among the bidders. Thus, the only asymmetric information among the bidders is the relevant agent i's payoff type, t_i, which is known solely to i.

Consider the relaxed problem where the designer can always observe this additional information (i, t_{-i}) whenever $t \in T^i$. In the optimal mechanism, we collect b_j from agent $j \neq i$, and for i's payment, his incentive compatibility and participation constraint given t_{-i} must be satisfied. More precisely, the optimal mechanism is given by:

$$\begin{aligned}
\max_{p} \quad & \int_t \sum_i p_i(t) dF(t) \\
\text{sub. to} \quad & \forall i,\ \forall t \notin T^i,\ b_i - p_i(t_i, t_{-i}) = 0, \\
& \forall i,\ \forall t \in T^i,\ v_i(t_i, t_{-i}, \phi(t_i, t_{-i})) - p_i(t_i, t_{-i}) \\
& \qquad\qquad \geq \max\{0, v_i(t_i, t_{-i}, \phi(t_i', t_{-i})) - p_i(t_i', t_{-i})\}.
\end{aligned}$$

Let $p^* = (p_i^*(t))_{i,t}$ solve this problem. Let $p^{EP} = (p_i^{EP}(t))_{i,t}$ denote the payment scheme in the optimal ex post incentive compatible mechanism (that implements ϕ). Our goal is to show that mechanism (ϕ, p^*) is ex post incentive compatible, and hence achieves the same expected revenue as p^{EP}. The only non-trivial part is to show that, given t_{-i} such that $(t_i, t_{-i}) \notin T^i$, agent i with type t_i has no incentive to pretend to be another type t_i'.

Note that, in p^{EP}, we have $b_i - p_i^{EP}(t) = 0$ for $t \notin T^i$: if $b_i - p_i^{EP}(t) = \varepsilon > 0$ for some i and $t = (t_i, t_{-i}) \notin T^i$, then for any $t_i' \in T_i$, $v_i(t_i', t_{-i}, \phi(t_i', t_{-i})) - p_i^{EP}(t_i', t_{-i}) \geq \varepsilon$ by ex post incentive compatibility. But this means that the principal can increase $p_i^{EP}(t_i', t_{-i})$ by ε for all t_i' without violating any constraint.

Given each i and t_{-i}, let $T_i^*(t_{-i}) \subseteq T_i$ denote the set of t_i such that $p_i^*(t_i, t_{-i}) > p_i^{EP}(t_i, t_{-i})$. Then, consider a new payment scheme $p^{**} = (p_i^{**}(t))_{i,t}$ such that:

$$p_i^{**}(t) = \max\{p_i^*(t), p_i^{EP}(t)\}.$$

We show that a mechanism (ϕ, p^{**}) is ex post incentive compatible, implying that the expected revenues in p^{**}, p^*, p^{EP} are all the same, as desired.

For each i, t such that $p_i^{**}(t) = p_i^{EP}(t)$, ex post incentive compatibility continues to be satisfied, because $p_i^{**}(t_i', t_{-i}) \geq p_i^{EP}(t_i', t_{-i})$ for all $t_i' \neq t_i$, and p_i^{EP} is ex post incentive compatible. For each i, t such that $p_i^{**}(t) = p_i^*(t) > p_i^{EP}(t)$, we have $t \in T^i$ (otherwise, $p_i^*(t) = p_i^{EP}(t) = b_i$), and thus, for each $t_i' \neq t_i$:

$$\begin{aligned} v_i(t_i, t_{-i}, \phi(t_i, t_{-i})) - p_i^{**}(t_i, t_{-i}) &= v_i(t_i, t_{-i}, \phi(t_i, t_{-i})) - p_i^*(t_i, t_{-i}) \\ &\geq v_i(t_i, t_{-i}, \phi(t_i', t_{-i})) - p_i^*(t_i', t_{-i}) \\ &\geq v_i(t_i, t_{-i}, \phi(t_i', t_{-i})) - p_i^{**}(t_i', t_{-i}), \end{aligned}$$

implying ex post incentive compatibility. Therefore, p^{**} is indeed ex post incentive compatible. $\square$