Collusion between two-sided platforms*

Yassine Lefouili
Toulouse School of Economics,
University of Toulouse Capitole.
yassine.lefouili@tse-fr.eu

Joana Pinho
Católica Porto Business School and CEGE,
Universidade Católica Portuguesa
jpinho@porto.ucp.pt

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Abstract: We study the price and welfare effects of collusion between two-sided platforms and show that they depend on whether collusion occurs on both sides or a single side of the market, and whether users single-home or multi-home. Our most striking result is that one-sided collusion leads to lower (resp. higher) prices on the collusive (resp. competitive) side if the cross-group externalities exerted on the collusive side are positive and sufficiently strong. One-sided collusion may, therefore, benefit the users on the *collusive* side and harm the users on the *competitive* side. Our findings have implications regarding cartel detection and damages actions.

Keywords: Collusion; Two-sided markets; Cross-group externalities.

JEL Codes: L41, D43.

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1 Introduction

Several cartels involving newspaper publishers have been uncovered all around the world. In 1969, a U.S. District Court convicted of monopolization the two daily newspapers of general circulation in Tucson, Arizona, for jointly setting subscription and advertising rates.[1] In 1996, several Venezuelan newspapers were convicted of forming a cartel to fix advertising rates for movie theaters.[2] In 2005, the Brazilian antitrust authority fined the four largest newspapers in Rio de Janeiro for forming a cartel, after a simultaneous increase in cover prices by 20%.[3] In 2010, the Croatian antitrust authority established that nine publishers of daily newspapers engaged in concerted practices that translated into a uniform increase in newspapers' cover prices.[4] In 2014, the Hungarian antitrust authority convicted the four major newspaper publishers in the country of price-fixing conspiracy.[5] Also in 2014, the Montenegrin antitrust authority convicted the three major daily newspaper publishers in the country for price-fixing conspiracy.[6]

Newspapers are two-sided platforms that enable the interaction between two distinct types of agents: advertisers and readers. As pointed out by Evans and Schmalensee (2013, p. 2), *"a number of results for single-sided firms, which are the focus of much of the applied antitrust economics literature, do not apply directly to multi-sided platforms."* However, the theoretical literature on collusion in two-sided markets is remarkably scarce, which is striking given the empirical evidence on collusion in these markets.[7] In particular, our understanding of *imperfect* collusion among two-sided platforms, i.e., collusion that does not yield the monopoly outcome, is very limited.

In this paper, we study the price and welfare effects of collusion between two horizontally

[1]Citizen Publishing Co. v. United States 394 U.S. 131 (1969).

[2]See page 9 of the 2005 report by the Venezuelan antitrust authority available at: `http://www.oecd.org/daf/competition/prosecutionandlawenforcement/38835563.pdf`.

[3]CADE - Processo Administrativo no. 08012.002097/99-81.

[4]CCA vs. daily newspaper publishers: UP/I 030-02/2008-01/072.

[5]Gazdasági Versenyhivatal (GVH) - Case Number: Vj/23/2011.

[6]Agency for Protection of Competition - Case Number: 02-UPI-68/1-14. In this case, there was even a written agreement signed by three of the convicted publishers, where they combined to simultaneously increase the retail price of newspapers.

[7]There are several examples of collusion between two-sided platforms outside the newspaper market. The most famous example dates back to 2002, when the two largest worldwide fine art auction houses, Christie's and Sotheby's, were fined for coordinating vendor commission rates (COMP/E-2/37.784). More recently, in 2019, the four issuers of restaurant vouchers in France were fined for sharing information on the number of issued tickets (individual market shares) from 2010 to 2015, which allowed them to detect deviations in the issuing fees charged to the companies (AdC Décision 19-D-25). This was not the first time collusive behavior was detected in this market: in 2002, three of these firms had already been fined for market sharing agreements and for setting a uniform commission rate to restaurateurs (AdC Décision 01-D-41).

differentiated platforms, allowing for *any* degree of collusion. Our baseline model is an infinitely repeated version of the canonical Armstrong (2006)'s model, with single-homing on both sides and either positive or negative cross-group externalities. We first consider the scenario in which platforms engage in *two-sided collusion*, that is, collusion on the prices set on both sides of the market. We show that the most profitable collusive agreement involves a price structure that minimizes the platforms' incentives to deviate from the agreement. Using this result, we show that (optimal) collusion distorts the price structure (relative to the static Nash equilibrium) by leading to more rent extraction from the side in which the degree of differentiation is higher. We also establish that two-sided collusion may either lead to higher prices on both sides of the market or to *lower* prices on one side of the market (the one with the lower degree of differentiation) and higher prices on the other side of the market. The latter scenario occurs when the degree of differentiation on one of the sides is sufficiently low (relative to cross-group externalities) and the discount factor is not too large.

We then consider the scenario in which platforms engage in *one-sided collusion*, i.e., they set their prices cooperatively on one side of the market and non-cooperatively on the other side. Such a collusive behavior can be explained by the existence of coordination or antitrust costs that make it optimal for platforms to collude on a single side of the market, and has been documented empirically in the case of newspapers. For instance, using data from the Italian daily newspaper market from 1976 to 2003, Argentesi and Filistrucchi (2007) found empirical evidence that the four biggest newspapers colluded on cover prices, but found no evidence for collusion on advertising rates.[8]

One-sided collusive agreements affect the prices on the non-cooperative side of the market because of the existence of cross-group externalities. If increasing the price on the collusive side softens competition on the non-cooperative side, the most profitable one-sided collusive agreement leads to supra-competitive prices on both sides of the market. This happens when the cross-group externalities exerted on the collusive side are negative. By contrast, if increasing the price on the collusive side strengthens competition on the non-cooperative side, the price on one of the two sides will be above its static Nash level, while the price on the other side will be below its static Nash level. This scenario occurs when the cross-group externalities exerted on the collusive side are positive. Interestingly, if these externalities are sufficiently high (relative to the degree of differentiation on the

[8]There are also real-world examples of two-sided platforms active in other markets and convicted for collusion on a single side of the market. For instance, the German TV groups Prosieben and RTL were convicted in 2007 for collusion on the advertising side and, as discussed before, the four issuers of restaurant vouchers in France were fined in 2019 for cooperation on the issuing side.

collusive side), the price on the collusive side is *below* its static Nash level, while the price on the non-cooperative side is *above* its static Nash level. As a result, one-sided collusion may benefit the users on the collusive side and harm the users on the non-cooperative side.

Next, we extend our analysis to a setting in which there is single-homing on one side of the market and multi-homing on the other side. A key difference between this extension and the baseline model (assuming full market coverage) is that *total* demand on the multi-homing side increases (resp. decreases) when prices on the multi-homing side decrease (resp. increase). Therefore, collusion in that setting can affect total welfare while it does not in our baseline model. We first show that two-sided collusion has no impact on the price on the multi-homing side but leads to a price increase on the single-homing side. Consequently, users on the multi-homing side and total welfare are not affected while users on the single-homing side are harmed. Turning to one-sided collusion we show that, as in the baseline model, collusion on a single side leads to a decrease in the price on that side and an increase in the price on the other side if the network externalities received by the collusive side are positive and large enough. We further establish that when collusion occurs on the single-homing (resp. multi-homing) side only, it raises total welfare if and only if the cross-group externalities received by the collusive side are strong enough (resp. positive and large enough).

Related literature. Ruhmer (2011) is the closest paper to ours. She also considers a repeated version of Armstrong's model but her setting is substantially less general than ours. First, in the context of two-sided collusion, she focuses on perfect collusion (i.e., collusion at the monopoly prices) while we allow for imperfect collusion as well. This is natural when platforms are differentiated: in this case, perfect collusion may not be sustainable while (profitable) collusion at other prices could be. The distinction between perfect and imperfect two-sided collusion turns out to be crucial: a focus on perfect collusion leads to the prediction that prices always increase if platforms collude on both sides of the market, while this is not always true under imperfect collusion. Second, in the context of one-sided collusion, Ruhmer (2011) focuses on the profitability and sustainability of a very specific collusive agreement in which platforms set the price on the collusive side at the maximum level that allows them to fully cover that side of the market (which is above the static Nash level). In contrast, we do not restrict the type of one-sided collusive agreements that platforms can achieve and show that they may find it optimal to decrease the price on the collusive side below its static Nash level. This explains, in particular, why one-sided collusion may be unprofitable in Ruhmer's setting, while this is never the case in our setting. Finally, we examine both the scenario in which there is single-homing on both sides and a competitive bottleneck scenario with multi-homing on a single side, while Ruhmer only deals with the former. Our analysis of the case where users on one side are allowed to multi-home brings additional insights as it allows us to have a demand expansion effect on the multi-homing side.

Our paper is also related to the work of Dewenter *et al.* (2011) who build a model to investigate the welfare effects of collusion between newspaper publishers. They consider a static setting where newspapers compete in prices in the reader market and in quantities in the advertising market, and compare the platforms' profits when there is two-sided perfect collusion, one-sided perfect collusion (on the advertising side) and two-sided competition. In contrast, we investigate, in a *dynamic* setting, the most profitable sustainable agreement, allowing for intermediate degrees of collusion and analyzing the incentives for platforms to comply with the collusive agreement. Dewenter *et al.* (2011) find that, when newspapers only collude on the advertisers' side, the price is lower on the non-cooperative side while it is higher on the collusive side (as compared to the static Nash prices). By contrast, we show that one-sided collusion may also lead to a price lower than the competitive price on the *collusive* side.[9]

Another paper our work is related to is Boffa and Filistrucchi (2014). These authors build a model of collusion between two TV channels and use it to show that prices above the two-sided monopoly price may prevail on one side of the market as a means to enhance cartel sustainability. However, they assume that the price on the viewer side is zero and study collusion in quantities, which makes their paper complementary to ours. Moreover, they focus on the case of two-sided collusion while we also deal with one-sided collusion.

There is also a small literature on collusion with network externalities in one-sided markets. Pal and Scrimitore (2016) show that the relationship between market concentration and collusion sustainability depends on the strength of network externalities. In the same vein, Song and Wang (2017) show that the presence of strong network externalities can reverse the traditional result that collusion between firms is easier with differentiated products (Deneckere, 1983). Finally, Rasch (2017) studies the relationship between firms' incentives to introduce compatibility and collusion and finds that it is non-monotonic.

Finally, our paper is also linked to the work by Choi and Gerlach (2013) on firms' incentives to collude when they interact in multiple markets and demands in these markets

[9] In both papers, the users on the collusive side may benefit from a one-sided collusive agreement, but the mechanisms driving this result in the two papers are different. In Dewenter *et al.* (2011), this result may hold despite the price increase on the collusive side because of an indirect feedback effect: the price decrease on the non-cooperative side leads to more participation on that side, which benefits the users on the collusive side. In contrast, in our baseline model, the result that one-sided collusion may benefit the users on the collusive side is driven by the direct impact on the price paid by these users.

are interrelated. The main goal of Choi and Gerlach (2013) is, however, fundamentally different from ours. They focus on antitrust enforcement issues and, in particular, on whether the discovery of a cartel in one market favors the emergence or collapse of a cartel in another market. Moreover, they restrict their attention to homogeneous goods, which implies in particular that collusion at the monopoly price is sustainable whenever some collusion is sustainable. In contrast, we consider a setting with differentiated platforms and possibly imperfect collusion, and abstract away from antitrust enforcement issues.

The remainder of the paper is organized as follows. In Section 2, we investigate the price and welfare effects of two-sided and one-sided collusion in a setting with single-homing on both sides of the market. In Section 3, we extend our analysis to the scenario in which there is single-homing on one side of the market and (partial) multi-homing on the other side of the market. We discuss some of our assumptions and derive the policy implications of our findings in Section 4. Finally, we conclude in Section 5. Most of the proofs are relegated to the Appendix.

2 Baseline model: Single-homing on both sides

We consider an infinitely repeated version of Armstrong's (2006) model with single-homing on both sides of the market. There are two platforms in the market, A and B, that enable the interaction between two groups of users, 1 and 2. Users on each side are uniformly distributed along the interval $[0, 1]$ and platforms are located at the extremes: $x^A = 0$ and $x^B = 1$. Platform $i \in \{A, B\}$ sets a subscription fee p_j^i to the users on each side of the market $j \in \{1, 2\}$. There is single-homing on both sides of the market and the utility of an agent on side j located at $x \in [0, 1]$ that joins platform i is:

$$u_j^i(x, p_1^i, p_2^i, p_1^{-i}, p_2^{-i}) = k_j + \alpha_j n_{-j}^i(p_1^i, p_2^i, p_1^{-i}, p_2^{-i}) - t_j \left|x^i - x\right| - p_j^i, \tag{1}$$

where: k_j is the intrinsic benefit that an agent on side j gets from joining a platform; α_j captures the benefit (which can be positive or negative) that an agent on side j enjoys from the existence of an agent on the other side of the market that joined the same platform; and $t_j > 0$ measures the degree of differentiation between platforms on side j.

The demand addressed to platform i on side j is:[10]

$$n_j^i(p_1^i, p_2^i, p_1^{-i}, p_2^{-i}) = \frac{1}{2} + \frac{\alpha_j(p_{-j}^{-i} - p_{-j}^i) + t_{-j}(p_j^{-i} - p_j^i)}{2(t_1 t_2 - \alpha_1 \alpha_2)}. \tag{2}$$

Platforms interact for an infinite number of periods and have a common discount factor $\delta \in (0,1)$. In each period, $\tau \in \{0, 1, 2...\}$, they simultaneously set membership fees, p_j^i. Platforms have constant marginal production costs, which, for simplicity, are normalized to zero. Thus, the per-period profit function of platform $i \in \{A, B\}$ is:

$$\pi^i(p_1^i, p_2^i, p_1^{-i}, p_2^{-i}) = p_1^i n_1^i(p_1^i, p_2^i, p_1^{-i}, p_2^{-i}) + p_2^i n_2^i(p_1^i, p_2^i, p_1^{-i}, p_2^{-i}). \tag{3}$$

Assumption 1

i. $4t_1t_2 > (\alpha_1 + \alpha_2)^2$.

ii. $k_1 > \frac{3t_1 - \alpha_1 - 2\alpha_2}{2}$ *and* $k_2 > \frac{3t_2 - 2\alpha_1 - \alpha_2}{2}$.

Assumption 1 ensures that the static game has a unique (symmetric) Nash equilibrium with full coverage of both sides of the market (Armstrong, 2006).[11]

Let us first recall the Nash equilibrium of the stage game.

Lemma 1 (Armstrong, 2006) *If platforms set prices non-cooperatively, they choose equal prices, $p_j^N = t_j - \alpha_{-j}$ for $j \in \{1, 2\}$, fully cover both market sides, and get equal market shares on each side of the market. Their individual profit is given by $\pi^N = \frac{t_1 + t_2 - \alpha_1 - \alpha_2}{2}$.*

Proof. See Armstrong (2006) for the determination of the Nash prices and profits. Market $j \in \{1, 2\}$ is fully covered if and only if $u_j^i\left(\frac{1}{2}, p_1^N, p_2^N, p_1^N, p_2^N\right) \geq 0 \;\Leftrightarrow\; k_j > \frac{3t_j - \alpha_j - 2\alpha_{-j}}{2}$, which holds by Assumption 1. Note also that Assumption 1 guarantees that the expression of the Nash profit is positive. ■

[10] For details, see Armstrong (2006). Under full market coverage, the demands addressed to the two platforms on side $j \in \{1, 2\}$ are related in the following way: $n_j^i = 1 - n_j^{-i}$, for $i \in \{A, B\}$.

[11] As pointed out by Armstrong (2006), Assumption 1.i ensures that the second-order conditions for the individual profit-maximization are satisfied. In addition, it implies that the second-order conditions for the maximization problem under two-sided collusion are also satisfied (see footnote 40).

2.1 Two-sided collusion

Suppose that, at the beginning of period $\tau = 0$, platforms may agree to collude using grim trigger strategies that imply a permanent reversion to the static Nash prices in case of a deviation from the collusive agreement.

In this section, we consider the scenario in which platforms seek to collude on both sides of the market. Let us first examine the prices under the most profitable collusive agreement among those that are sustainable. We restrict our attention to symmetric agreements, i.e., such that the two platforms set equal prices on each market side ($p_j^A = p_j^B$, for $j \in \{1,2\}$). Denote by:

$$\pi\left(p_1, p_2\right) = \pi^i\left(p_1, p_2, p_1, p_2\right)$$

the profit of platform $i \in \{A, B\}$ if the two platforms set equal prices $p_j^A = p_j^B = p_j$ on each side $j \in \{1,2\}$. The most profitable sustainable symmetric agreement involves prices that solve the following maximization program:

$$\max_{(p_1,p_2)\in\mathbb{R}^2} \pi\left(p_1, p_2\right)$$

subject to the incentive compatibility constraint (hereafter, ICC):

$$\frac{\pi\left(p_1, p_2\right)}{1-\delta} \geq \pi^d\left(p_1, p_2\right) + \frac{\delta}{1-\delta}\pi^N, \tag{4}$$

where $\pi^d(p_1, p_2) = max_{(p_1^i, p_2^i)}\pi^i(p_1^i, p_2^i, p_1, p_2)$ is the optimal deviation profit if the collusive prices are (p_1, p_2).

2.1.1 Preliminaries

For any given $\delta \in (0,1)$, denote by

$$I\left(\delta\right) = \left\{\left(p_1, p_2\right) \in \mathbb{R}^2 \mid \frac{\pi\left(p_1, p_2\right)}{1-\delta} \geq \pi^d\left(p_1, p_2\right) + \frac{\delta}{1-\delta}\pi^N\right\}$$

the set of price pairs such that the ICC is satisfied, and by

$$\bar{I}\left(\delta\right) = \left\{\left(p_1, p_2\right) \in \mathbb{R}^2 \mid \frac{\pi\left(p_1, p_2\right)}{1-\delta} = \pi^d\left(p_1, p_2\right) + \frac{\delta}{1-\delta}\pi^N\right\}$$

the set of price pairs such that the ICC is binding. Moreover, define

$$\pi^c(\delta) = \max_{(p_1,p_2)\in I(\delta)} \pi(p_1, p_2)$$

and

$$\delta^m = \frac{\pi^d(p_1^m, p_2^m) - \pi^m}{\pi^d(p_1^m, p_2^m) - \pi^N} \tag{5}$$

where (p_1^m, p_2^m) is the unique solution to the unconstrained maximization program

$$\max_{(p_1,p_2)\in\mathbb{R}^2} \pi(p_1, p_2),$$

and $\pi^m = \pi(p_1^m, p_2^m)$ is the profit each firm derives from perfect collusion.

The following preliminary results are useful for the subsequent analysis. Lemma 2 shows that the ICC is binding for sufficiently small values of the discount factor and that the collusive profit is (weakly) increasing in the platform's discount factor.

Lemma 2 *The prices and profits under the most profitable sustainable agreement satisfy the following properties:*

(i) If $\delta \in (0, \delta^m)$ and $(p_1^c(\delta), p_2^c(\delta))$ is a pair of prices in $I(\delta)$ such that $\pi^c(\delta) = \pi(p_1^c(\delta), p_2^c(\delta))$, then $(p_1^c(\delta), p_2^c(\delta)) \in \bar{I}(\delta)$.

(ii) $\pi^c(\delta) < \pi^c(\delta') < \pi^m$ for any $\delta, \delta' \in (0, \delta^m)$ such that $\delta < \delta'$; and $\pi^c(\delta) = \pi^m$, for any $\delta \in [\delta^m, 1)$.

Proof. See Appendix. ■

The next lemma shows that the price structure under the most profitable sustainable agreement minimizes the platforms' incentives to deviate (among all possible price structures for a given collusive profit).

Lemma 3 *Consider $\delta \in (0, \delta^m)$ and let $(p_1^c(\delta), p_2^c(\delta))$ be a pair of prices in $I(\delta)$ such that $\pi^c(\delta) = \pi(p_1^c(\delta), p_2^c(\delta))$. Then, $(p_1^c(\delta), p_2^c(\delta))$ is necessarily a solution to the following constrained minimization program:*

$$\min_{(p_1,p_2)\in\mathbb{R}^2} \pi^d(p_1, p_2)$$

subject to

$$\pi\left(p_1, p_2\right) = \pi^c\left(\delta\right).$$

Proof. See Appendix. ■

2.1.2 The most profitable sustainable agreement

We now make use of the previous results to derive the price and welfare effects of (optimal) two-sided collusion.

Assumption 2

i. $k_1 \geq \frac{2t_1-\alpha_1}{2}$ *and* $k_2 \geq \frac{2t_2-\alpha_2}{2}$;

ii. $\min\left\{2t_2k_1 + (\alpha_1+\alpha_2)k_2, (\alpha_1+\alpha_2)k_1 + 2t_1k_2\right\} \geq \frac{4t_1t_2-(\alpha_1+\alpha_2)^2}{2}$.

We need this assumption on the stand-alone values to ensure that the following result holds.[12]

Lemma 4 *Both sides of the market are fully covered under the most profitable sustainable two-sided collusive agreement.*

Proof. See Appendix. ■

If platforms set equal prices, there is full coverage of side $j \in \{1, 2\}$ if and only if the utility of the indifferent consumer, located at $x = \frac{1}{2}$, is non-negative. Thus, the maximum price that platforms can charge on side j for this side to be fully covered is:[13]

$$p_j^m \;= p_j^N + u_j^i\left(\tfrac{1}{2}, p_1^N, p_2^N, p_1^N, p_2^N\right) \;= k_j + \frac{\alpha_j}{2} - \frac{t_j}{2}. \tag{6}$$

The maximum individual profit under full coverage of both sides of the market is:[14]

$$\pi^m = \frac{p_1^m + p_2^m}{2} = \frac{k_1+k_2}{2} - \frac{\pi^N}{2}. \tag{7}$$

[12] This simplifies the analysis by reducing the number of possible demand configurations under collusion.
[13] For the expression of the Nash prices, p_j^N, see Lemma 1.
[14] Assumptions 1 and 2 imply that $(\pi^N, \pi^m] \neq \emptyset$.

In order to compare collusive and competitive prices, we proceed in two steps. First, we use our characterization of the optimal price structure under collusion (Lemma 3) to write collusive prices as functions of collusive profits. Second, we rely on the monotonicity of the collusive profits with respect to platforms' discount factor (Lemma 2) to derive the monotonicity of collusive prices with respect to the discount factor, which allows us to compare these prices to their competitive counterparts. The following lemma shows that the way prices under the most profitable sustainable agreement relate to the profit generated by this agreement depends on how the degrees of differentiation t_1 and t_2 compare to $(\alpha_1 + \alpha_2)/2$. Before stating the result, notice that a scenario in which both t_1 and t_2 would be below $(\alpha_1 + \alpha_2)/2$ is not possible due to Assumption 1,[15] which leaves us with three possible scenarios.

Lemma 5 *For any $\delta \in (0,1)$, there exists a unique pair of prices $(p_1^c(\delta), p_2^c(\delta)) \in I(\delta)$ satisfying $\pi^c(\delta) = \pi(p_1^c(\delta), p_2^c(\delta))$. Furthermore:*

(i) If $t_1 < \frac{\alpha_1+\alpha_2}{2} < t_2$, the collusive prices are:

$$(p_1^c(\delta), p_2^c(\delta)) = \begin{cases} \left(\frac{\alpha_1-\alpha_2}{2} + \frac{2t_1-\alpha_1-\alpha_2}{t_1+t_2-\alpha_1-\alpha_2}\pi^c(\delta), \frac{\alpha_2-\alpha_1}{2} + \frac{2t_2-\alpha_1-\alpha_2}{t_1+t_2-\alpha_1-\alpha_2}\pi^c(\delta)\right) & \text{if} \quad 0 < \delta \leq \tilde{\delta}_2 \\ (2\pi^c(\delta) - p_2^m, p_2^m) & \text{if} \quad \tilde{\delta}_2 < \delta < \delta^m \\ (p_1^m, p_2^m) & \text{if} \quad \delta^m \leq \delta < 1, \end{cases} \tag{8}$$

where $p_j^m = \frac{2k_j-t_j+\alpha_j}{2}$, $\tilde{\delta}_2$ is the solution of $\pi^c(\tilde{\delta}_2) = \frac{2k_2-t_2+\alpha_1}{2t_2-\alpha_1-\alpha_2}\pi^N$, and δ^m is the solution of $\pi^c(\delta^m) = \pi^m$, with π^m given by (7).

(ii) If $t_2 < \frac{\alpha_1+\alpha_2}{2} < t_1$, the collusive prices are:

$$(p_1^c(\delta), p_2^c(\delta)) = \begin{cases} \left(\frac{\alpha_1-\alpha_2}{2} + \frac{2t_1-\alpha_1-\alpha_2}{t_1+t_2-\alpha_1-\alpha_2}\pi^c(\delta), \frac{\alpha_2-\alpha_1}{2} + \frac{2t_2-\alpha_1-\alpha_2}{t_1+t_2-\alpha_1-\alpha_2}\pi^c(\delta)\right) & \text{if} \quad 0 < \delta \leq \tilde{\delta}_1 \\ (p_1^m, 2\pi^c(\delta) - p_1^m) & \text{if} \quad \tilde{\delta}_1 < \delta < \delta^m \\ (p_1^m, p_2^m) & \text{if} \quad \delta^m \leq \delta < 1, \end{cases} \tag{9}$$

where $\tilde{\delta}_1$ is the solution of $\pi^c(\tilde{\delta}_1) = \frac{2k_1-t_1+\alpha_2}{2t_1-\alpha_1-\alpha_2}\pi^N$.

[15] To see why, note first that $(t_1 + t_2)^2 \geq (\alpha_1 + \alpha_2)^2$. This, combined with the assumption $4t_1t_2 > (\alpha_1 + \alpha_2)^2$, implies that $(t_1 + t_2)^2 > (\alpha_1 + \alpha_2)^2$ and, therefore, that $t_1 + t_2 > \alpha_1 + \alpha_2$. The latter excludes the scenario in which both t_1 and t_2 are less than (or equal to) $(\alpha_1 + \alpha_2)/2$.

(iii) If $t_1 \geq \frac{\alpha_1+\alpha_2}{2}$ *and* $t_2 \geq \frac{\alpha_1+\alpha_2}{2}$, *the collusive prices are given by (8) if* $k_2(2t_1-\alpha_1-\alpha_2)-k_1(2t_2-\alpha_1-\alpha_2) < \pi^N(\alpha_2-\alpha_1)$, *and by (9) if* $k_2(2t_1-\alpha_1-\alpha_2)-k_1(2t_2-\alpha_1-\alpha_2) > \pi^N(\alpha_2-\alpha_1)$.

Proof. See Appendix. ■

This lemma allows us to understand how collusive prices depend on the discount factor and, therefore, how they compare to competitive prices. To this end, denote $P^c(\delta) = p_1^c(\delta) + p_2^c(\delta)$ and $S^c(\delta) = p_2^c(\delta) - p_1^c(\delta)$ the *total price* and the *price structure* under the most profitable sustainable collusive agreement; and $P^N = P^c(0)$ and $S^N = S^c(\delta)$ the total price and the price structure under the competitive (Nash) equilibrium.

From Lemma 5 it follows that

$$S^c(\delta) = \alpha_2 - \alpha_1 + \frac{2\,(t_2-t_1)}{t_1+t_2-\alpha_1-\alpha_2}\pi^c(\delta).$$

for δ sufficiently small, i.e., before any of the collusive prices $p_1^c(\delta)$ and $p_2^c(\delta)$ reaches it maximum level. Using the expressions for p_1^N, p_2^N and π^N provided in Lemma 1, we can rewrite the above expression as

$$S^c(\delta) = S^N + (t_2-t_1)\,\frac{\pi^c(\delta)-\pi^N}{\pi^N} \tag{10}$$

Thus, the sign of $S^c(\delta)-S^N$, which captures the impact of collusion on the price structure, is the same as the sign of t_2-t_1 (over the considered range of δ). This implies that (optimal) collusion distorts the price structure by leading to more rent extraction from the side with the larger degree of differentiation, relative to the side with the smaller degree of differentiation.[16] This finding is in line with the traditional result in the (one-sided) Hotelling setting that, *ceteris paribus*, collusion is easier to sustain in markets with larger product differentiation (see e.g. Chang, 1991).

Moreover, equation (10) shows that $S^c(\delta)$ is increasing (resp. decreasing) in δ if t_2 is greater (resp. lower) than t_1. This is useful for understanding the impact of δ on collusive prices $p_1^c(\delta)$ and $p_2^c(\delta)$. To see why, assume that $t_1 > (\alpha_1+\alpha_2)/2$,[17] and consider the

[16] In the special case $t_1 = t_2$, the additional rents extracted by two platforms that collude are evenly distributed between the two sides, i.e. the price structure under collusion is the same as under competition.

[17] This does not entail any loss of generality because Assumption 1 implies that $t_1+t_2 > \alpha_1+\alpha_2$, which in turn implies that we cannot have both $t_1 \leq (\alpha_1+\alpha_2)/2$ and $t_2 \leq (\alpha_1+\alpha_2)/2$.

following decomposition:

$$\frac{dp_2^c}{d\delta} = \underbrace{\frac{1}{2}\frac{dP^c}{d\delta}}_{\text{total price effect}} + \underbrace{\frac{1}{2}\frac{dS^c}{d\delta}}_{\text{price structure effect}}.$$

For any $\delta \in [0, \tilde{\delta}_1]$, the total price effect is always positive because:

$$\frac{1}{2}\frac{dP^c}{d\delta} = \frac{d\pi^c}{d\delta};$$

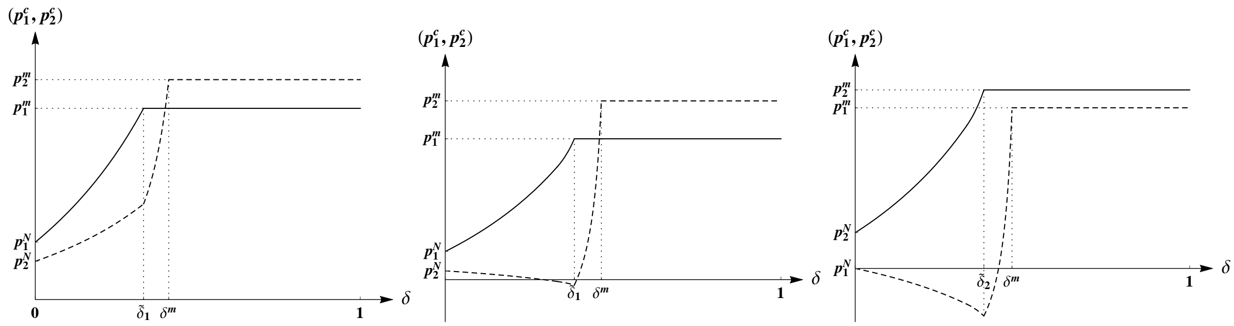
while the price structure effect has the same sign as $t_2 - t_1$:

$$\frac{1}{2}\frac{dS^c}{d\delta} = \frac{t_2 - t_1}{2\pi^N}\frac{d\pi^c}{d\delta}$$

Thus, if $t_2 \geq t_1$ then $dS^c/d\delta$ is weakly positive and, therefore, $dp_2^c/d\delta$ is positive. However, if $t_2 < t_1$ then $dS^c/d\delta$ is negative and, therefore, the sign of $dp_2^c/d\delta$ is *a priori* ambiguous. To sign the total effect of the discount factor in this case we need to distinguish between two possible cases. Consider first the scenario in which $1 + \frac{t_2 - t_1}{2\pi^N} \geq 0$ or, equivalently, $t_2 \geq (\alpha_1 + \alpha_2)/2$. In this case, the positive total price effect (weakly) outweighs the negative price structure effect and, therefore, $p_2^c(\delta)$ is (weakly) *increasing* in δ over $[0, \tilde{\delta}_1]$. Consider now the scenario in which $1 + \frac{t_2 - t_1}{2\pi^N} < 0$ or, equivalently, $t_2 < (\alpha_1 + \alpha_2)/2$. In this case, the negative price structure effect outweighs the positive total price effect and, therefore, $p_2^c(\delta)$ is *decreasing* in δ over $[0, \tilde{\delta}_1]$, which implies that the collusive price $p_2^c(\delta)$ is *lower* than the competitive (Nash) price on that side over the considered range. Thus, the distortion of the price structure induced by collusion in this scenario is so strong that it results in a decrease in the price of side 2 (despite the increase in the total price). Note, however, that $p_2^c(\delta)$ is always increasing over $[\tilde{\delta}_1, \delta^m]$ as $p_2^c(\delta) = 2\pi^c(\delta) - p_1^m$ over that interval. Therefore, we reach the following conclusion: (*i*) if $t_2 > (\alpha_1 + \alpha_2)/2$, then $p_2^c(\delta)$ is increasing over $[0, \delta^m]$; (*ii*) if $t_2 < (\alpha_1 + \alpha_2)/2$ then $p_2^c(\delta)$ is decreasing over $[0, \tilde{\delta}_1]$ and increasing over $[\tilde{\delta}_1, \delta^m]$.[18] The following figure plots the collusive prices as functions of the discount factor.

Using the above analysis, we can show the following result about the comparison of the collusive prices generating the most profitable two-sided sustainable agreement and the competitive prices.

[18]In the knife-edge case where $t_2 = (\alpha_1 + \alpha_2)/2$, $p_2^c(\delta)$ is constant over $[0, \tilde{\delta}_1]$ and increasing over $[\tilde{\delta}_1, \delta^m]$.



(a) $t_1 > \frac{\alpha_1+\alpha_2}{2} \wedge t_2 > \frac{\alpha_1+\alpha_2}{2}$ (b) $t_2 < \frac{\alpha_1+\alpha_2}{2} < t_1$ (c) $t_1 < \frac{\alpha_1+\alpha_2}{2} < t_2$

Figure 1: Collusive prices under the most profitable sustainable agreement.

Proposition 1 *Two-sided collusion can lead to either higher prices on both sides of the market or higher prices on one side and lower prices on the other side. More specifically:*

(i) If $t_1 > \frac{\alpha_1+\alpha_2}{2}$ *and* $t_2 > \frac{\alpha_1+\alpha_2}{2}$ *then* $p_1^c(\delta) > p_1^N$ *and* $p_2^c(\delta) > p_2^N$ *for any* $\delta \in (0, 1]$.

(ii) If $t_2 < \frac{\alpha_1+\alpha_2}{2} < t_1$ *then* $p_1^c(\delta) > p_1^N$ *for any* $\delta \in (0, 1]$ *and there exists a threshold* $\hat{\delta}_2 \in (\tilde{\delta}_1, \delta^m)$ *such that* $p_2^c(\delta) < p_2^N$ *for any* $\delta \in (0, \hat{\delta}_2)$*, and* $p_2^c(\delta) > p_2^N$ *for any* $\delta \in (\hat{\delta}_2, 1]$.

(iii) If $t_1 < \frac{\alpha_1+\alpha_2}{2} < t_2$ *then* $p_2^c(\delta) > p_2^N$ *for any* $\delta \in (0, 1]$ *and there exists a threshold* $\hat{\delta}_1 \in (\tilde{\delta}_2, \delta^m)$ *such that* $p_1^c(\delta) < p_1^N$ *for any* $\delta \in (0, \hat{\delta}_1)$*, and* $p_1^c(\delta) > p_1^N$ *for any* $\delta \in (\hat{\delta}_1, 1]$.

Proof. See Appendix. ■

The fact that the market is fully (and symmetrically) covered under both competition and collusion in the considered parameter constellation, combined with the assumption that users single-home on both sides of the market, implies that, in our baseline model, (*i*) collusion does not affect total welfare (which means that the effect on aggregate users surplus is necessarily negative), and (*ii*) the impact of collusion on the users on each side is fully determined by the comparison of the collusive and competitive prices. Thus, we get the following result.

Corollary 1 *In the model with single-homing on both sides, two-sided collusion does not affect total welfare. Furthermore, it can be either detrimental to users on both sides of the market, or detrimental to users on one side and beneficial to users on the other side. More specifically:*

(i) If $t_1 > \frac{\alpha_1+\alpha_2}{2}$ and $t_2 > \frac{\alpha_1+\alpha_2}{2}$, users on both sides of the market are harmed by collusion for any $\delta \in (0,1]$.

(ii) If $t_2 < \frac{\alpha_1+\alpha_2}{2} < t_1$, side-1 users are harmed by collusion for any $\delta \in (0,1]$, while side-2 users benefit from collusion if $\delta \in (0,\hat{\delta}_2)$ and are harmed by collusion if $\delta \in (\hat{\delta}_2, 1]$.

(iii) If $t_1 < \frac{\alpha_1+\alpha_2}{2} < t_2$, side-2 users are harmed by collusion for any $\delta \in (0,1]$, while side-1 users benefit from collusion if $\delta \in (0,\hat{\delta}_1)$ and are harmed by collusion if $\delta \in (\hat{\delta}_1, 1]$.

2.2 One-sided collusion

Let us now investigate the most profitable sustainable agreement when platforms collude on a single side of the market. Without loss of generality, suppose that platforms collude over the price on side 1 and set non-cooperatively the price on side 2. We restrict the analysis to symmetric collusive agreements, i.e., such that the platforms set the same price on the collusive side (i.e., $p_1^A = p_1^B = p_1$). Thus, given $\delta \in (0,1)$, the most profitable sustainable one-sided symmetric agreement features a price on side 1 that solves:

$$\max_{p_1} \left\{ \pi^A(p_1, p_2^A, p_1, p_2^B) + \pi^B(p_1, p_2^A, p_1, p_2^B) \right\} \tag{11}$$

subject to the following constraints:

$$\begin{cases} p_2^A = \operatorname{argmax}_{\tilde{p}_2}\ \pi^A\left(p_1, \tilde{p}_2, p_1, p_2^B\right) \\ p_2^B = \operatorname{argmax}_{\tilde{p}_2}\ \pi^B\left(p_1, p_2^A, p_1, \tilde{p}_2\right) \\ p_1 \in \left\{ p_1 \in \mathbb{R} \mid \frac{\pi^A\left(p_1, p_2^A, p_1, p_2^B\right)}{1-\delta} \geq \max_{(\tilde{p}_1, \tilde{p}_2)} \pi^A(\tilde{p}_1, \tilde{p}_2, p_1, p_2^B) + \frac{\delta}{1-\delta}\pi^N \right\} \equiv I^{oc}\left(\delta, p_2^A, p_2^B\right). \end{cases} \tag{12}$$

Combining the first two constraints, we obtain:[19]

$$p_2^A = p_2^B = p_2 = \frac{t_1 t_2 - \alpha_1\alpha_2}{t_1} - \frac{\alpha_1}{t_1} p_1 \equiv g\left(p_1\right), \tag{13}$$

and can rewrite the above maximization program as:

$$\max_{p_1} \left\{ \pi^A(p_1, p_2, p_1, p_2) + \pi^B(p_1, p_2, p_1, p_2) \right\}$$

[19] The second-order conditions corresponding to the choice of p_2 are satisfied, as $\frac{d^2\pi^i}{d(p_2^i)^2} = -\frac{t_1}{t_1 t_2 - \alpha_1\alpha_2} < 0$.

subject to:

$$\begin{cases} p_2 = g(p_1) \\ p_1 \in I^{oc}(\delta, p_2, p_2). \end{cases}$$

For a given $\delta \in (0,1)$, let $p_1^{oc}(\delta)$ denote the solution to this constrained maximization program, and $p_2^{oc}(\delta)$ the corresponding price on the non-cooperative side 2.[20] Moreover, define:

$$\Delta p_j \equiv p_j^{oc}(\delta) - p_j^N$$

as the effect of one-sided collusion on the price on side $j \in \{1,2\}$. Even though platforms collude only on side 1, the price on side 2 is also affected by collusion due to the existence of cross-group externalities. Notice that:

$$\Delta p_2 = p_2^{oc}(\delta) - p_2^N = g(p_1^{oc}(\delta)) - g(p_1^N) = \int_{p_1^N}^{p_1^{oc}(\delta)} g'(p_1) dp_1. \tag{14}$$

Thus, if $g'(p_1) > 0$, then Δp_1 and Δp_2 have the same sign; if $g'(p_1) < 0$, then Δp_1 and Δp_2 have opposite signs. In other words, if increasing the price on the collusive side strengthens competition on the non-cooperative side (i.e. $g'(p_1) < 0$), the price on one side of the market will be supra-competitive, while the other will be infra-competitive.[21] In contrast, if increasing the price on the collusive side softens competition on the non-cooperative side, (i.e. $g'(p_1) > 0$), prices are supra-competitive on both sides of the market. More precisely, from condition (14), we conclude that, if $g'(p_1) > 0$, prices on both sides are either supra-competitive or infra-competitive. However, as there is no scope for demand expansion on either side (because both sides are already fully covered under competition, by Assumption 1), platforms would never find it optimal to adopt a one-sided collusive scheme that induces below-Nash prices on both sides of the market.[22]

Lemma 6 *The most profitable one-sided sustainable agreement leads to price variations across sides that are related as follows:*

$$\Delta p_2 = -\frac{\alpha_1}{t_1} \Delta p_1. \tag{15}$$

[20] The existence and uniqueness of the solution are established later.

[21] Please notice that we are not stating that platforms will set a supra-competitive price on the (cooperative) side 1. Indeed, as we will see below, this many not be the case.

[22] In the case where there is multi-homing on one side of the market, studied in Section 3, this is no longer the case.

Proof. Deriving function g, given in (13), with respect to p_1 and replacing it in (14), we obtain:

$$\Delta p_2 = \int_{p_1^N}^{p_1^{oc}(\delta)} g'(p_1)dp_1 = -\frac{\alpha_1}{t_1}\left[p_1^{oc}(\delta) - p_1^N\right] = -\frac{\alpha_1}{t_1}\Delta p_1$$

■

Let us explore the intuition behind Lemma 6. Suppose, for instance, that platforms set a supra-competitive price on the collusive side, i.e., $p_1^{oc} > p_1^N$. Then, users on side 1 are more valuable to platforms under collusion than under competition. As a result, platforms would like to increase their market share on side 1, as compared to the competitive scenario. As p_1 is fixed by the collusive agreement, the only way for a platform to conquer more side-1 users without triggering a punishment from the rival platform is to increase the attractiveness of its platform to these users, by changing the number of users on side 2. If $\alpha_1 > 0$, side-1 users like the presence of side-2 users, and platforms have, therefore, incentives to decrease p_2. In contrast, if $\alpha_1 < 0$, each platform has incentives to increase p_2 to attract less side-2 users, and increase its attractiveness to side-1 users. Naturally, if $p_1^{oc} < p_1^N$, the reasoning is exactly the opposite: as collusion makes side-1 users less valuable to platforms, they use p_2 to decrease the value that side-1 users get from joining a platform.

To gain further insights we need to distinguish between the scenario in which the ICC:

$$\frac{\pi^A\left(p_1, g\left(p_1\right), p_1, g\left(p_1\right)\right)}{1-\delta} \geq \max_{(\tilde{p}_1,\tilde{p}_2)} \pi^A(\tilde{p}_1, \tilde{p}_2, p_1, g\left(p_1\right)) + \frac{\delta}{1-\delta}\pi^N$$

is binding (*imperfect one-sided collusion*) and the scenario in which it is not (*perfect one-sided collusion*). Given $\delta \in (0,1)$, let $\pi^{oc}(\delta)$ be the highest sustainable profit under one-sided collusion, and π^{om} be the firm's profit when the ICC is not binding. As in the case of two-sided collusion, one can show that there exists a unique threshold $\delta^{om} \in (0,1)$ such that $\pi^{oc}(\delta) < \pi^{om}$ if and only if $\delta < \delta^{om}$, and that $\pi^{oc}(\delta)$ is increasing in δ over $[0, \delta^{om}]$.[23]

Let us first consider that *perfect* one-sided collusion is sustainable, i.e., $\delta \geq \delta^{om}$. In this scenario, firms can pick the price they want without caring about sustainability issues. Let p_1^{om} denote the firms' optimal price on side 1 in this case, i.e., $p_1^{om} = p_1^{oc}(\delta)$ for any $\delta \geq \delta^{om}$. Using (13), we know that, if $\alpha_1 < 0$, a decrease in p_1 would lead to a decrease

[23]This follows from the fact that an increase in δ does not affect the firms' objective function but relaxes the constraints (or, equivalently, widen the subspace of prices over which firms maximize their joint profits), combined with the fact the ICC is binding for δ lower than δ^{om}.

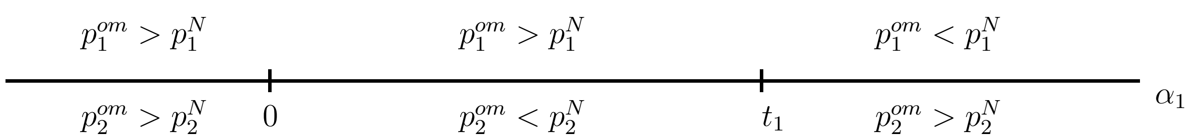
in p_2 and, therefore, would be unprofitable. Therefore, if $\alpha_1 < 0$ then $p_1^{om} > p_1^N$ and $p_2^{om} > p_2^N$. In contrast, if $\alpha_1 > 0$, an increase in p_1 is followed by a decrease in p_2. Thus, charging an above-Nash price on the collusive side is only profitable if the gain on this side, $p_1^{om} - p_1^N$, outweighs the loss on side 2, $\frac{\alpha_1}{t_1}\left(p_1^{om} - p_1^N\right)$, which is the case if and only if $\alpha_1 < t_1$. If, instead, $\alpha_1 > t_1$, side-1 users are so valuable that platforms decrease p_2 so much (to increase their attractiveness on side 1) that the profit loss on side 2 outweighs the profit gain on side 1. These results are summarized in the following proposition.

Proposition 2 *Assume that* $\delta \geq \delta^{om}$.

(i) If $\alpha_1 < 0$, *the prices under the most profitable one-sided agreement are above their static Nash levels on both sides of the market.*

(ii) If $\alpha_1 > 0$, *the prices under the most profitable one-sided agreement are such that the price on one side is above its static Nash level while the price on the other side is below its static Nash level.*

More precisely, the following holds:



When the cross-group externalities exerted on the collusive side are positive ($\alpha_1 > 0$), the relative price variation on the two sides due to collusion depends on the ratio between the strength of these externalities and the degree of differentiation on the collusive side, $\frac{\alpha_1}{t_1}$. If $\alpha_1 > t_1$, the price variation due to collusion is higher in the non-cooperative side: $|\Delta p_2| > |\Delta p_1|$. In contrast, if $0 < \alpha_1 < t_1$, the price variation is higher in the collusive side: $|\Delta p_1| > |\Delta p_2|$. This is due to the fact that an additional side-2 agent attracts $\frac{\alpha_1}{t_1}$ additional side-1 users to a platform (Armstrong, 2006). If $\alpha_1 = t_1$, any price change in the collusive side is accompanied by a change of the same magnitude but on the opposite direction on side 2. Therefore, if $\alpha_1 = t_1$, the (one-period) collusive profit coincides with the static Nash profit, corresponding to the conjecture of Evans and Schmalensee (2008) that if platforms *"agree to fix prices on one side only, the cartel members will* *tend to compete the supracompetitive profits away on the other side."* (p. 689) We prove, however, that this only happens in that very particular case.[24] More importantly, our analysis points out a fundamental problem with the logic behind Evans and Schmalensee's conjecture: their claim hinges on the implicit assumption that platforms colluding on a single of side of a market will seek to increase their rents (above their competitive level) on that side. However, our analysis shows that platforms colluding on a single side of the market may prefer to make *infra-competitive* profits on that side and increase their rents on the *competitive* side of the market.

To show that the price comparison under one-sided collusion and competition provided in Proposition 2 extends to the case of imperfect one sided-collusion (i.e., for $\delta < \delta^{om}$), we make Assumption 3,

Assumption 3 *The stand-alone values on both sides, k_1 and k_2, are sufficiently high for both market sides to be fully covered under the most profitable sustainable one-sided collusive agreement.*

Lemma 7 *Let $\tilde{u}_j^N \equiv u_j^i(\frac{1}{2}, p_1^N, p_2^N, p_1^N, p_2^N)$ denote the utility of the side-j agent located at $x = \frac{1}{2}$ if platforms set the static Nash prices, and $\pi^{oc}(\delta)$ denote the highest collusive profit that platforms can sustain for a given δ. The corresponding collusive prices $(p_1^{oc}(\delta), p_2^{oc}(\delta))$ are as follows:*

1. If $-\frac{\tilde{u}_2^N}{\tilde{u}_1^N} t_1 \leq \alpha_1 \leq t_1$:

$$(p_1^{oc}(\delta), p_2^{oc}(\delta)) = \begin{cases} \left(\frac{2t_1}{t_1-\alpha_1}\pi^{oc}(\delta) - \frac{t_1t_2-\alpha_1\alpha_2}{t_1-\alpha_1}, \frac{t_1t_2-\alpha_1\alpha_2}{t_1-\alpha_1} - \frac{\alpha_1}{t_1-\alpha_1}\pi^{oc}(\delta)\right) & \text{if} \quad 0 < \delta \leq \tilde{\delta}^{om} \\ \left(p_1^m, \frac{t_1t_2-\alpha_1\alpha_2}{t_1} - \frac{\alpha_1}{t_1}p_1^m\right) & \text{if} \quad \tilde{\delta}^{om} \leq \delta < 1, \end{cases} \tag{16}$$

where p_1^m is given by (6), and $\tilde{\delta}^{om}$ is implicitly defined by $\pi^{oc}(\tilde{\delta}^{om}) = \frac{2k_1(t_1-\alpha_1)-(t_1-\alpha_1)^2+2(t_1t_2-\alpha_1\alpha_2)}{4t_1}$.

2. If $\alpha_1 < -\frac{\tilde{u}_2^N}{\tilde{u}_1^N} t_1$ or $\alpha_1 > t_1$:

$$(p_1^{oc}(\delta), p_2^{oc}(\delta)) = \begin{cases} \left(\frac{2t_1}{t_1-\alpha_1}\pi^{oc}(\delta) - \frac{t_1t_2-\alpha_1\alpha_2}{t_1-\alpha_1}, \frac{t_1t_2-\alpha_1\alpha_2}{t_1-\alpha_1} - \frac{\alpha_1}{t_1-\alpha_1}\pi^{oc}(\delta)\right) & \text{if} \quad 0 < \delta \leq \hat{\delta}^{om} \\ \left(p_1^N - \frac{t_1}{\alpha_1}\tilde{u}_2^N, p_2^m\right) & \text{if} \quad \hat{\delta}^{om} \leq \delta < 1, \end{cases} \tag{17}$$

[24] Dewenter *et al.* (2011) also find that the claim by Evans and Schmalensee (2008) is true only in a very special case in their model.

where p_2^m is given by (6), and $\hat{\delta}^{om}$ is implicitly defined by $\pi^{oc}(\hat{\delta}^{om}) = \frac{2k_2(\alpha_1 - t_1) + 3t_1t_2 - \alpha_1\alpha_2 - \alpha_1 t_2 - \alpha_2 t_1}{4\alpha_1}$.

Proof. See Appendix. ■

From the previous proposition and the definition of δ^{om} it follows that:

$$\delta^{om} = \begin{cases} \tilde{\delta}^{om} & if \quad -\frac{\tilde{u}_2^N}{\tilde{u}_1^N} t_1 \le \alpha_1 \le t_1 \\ \hat{\delta}^{om} & if \quad \alpha_1 < -\frac{\tilde{u}_2^N}{\tilde{u}_1^N} t_1 \ \vee \ \alpha_1 > t_1 \end{cases}$$

As $\pi^{oc}(\delta)$ is increasing in δ for $\delta < \delta^{om}$ and $p_1^{oc}(0) = p_1^N$ and $p_2^{oc}(0) = p_2^N$, it is straightforward to derive the monotonicity of the prices $(p_1^{oc}(\delta), p_2^{oc}(\delta))$ under one-sided collusion with respect to δ over the interval $[0, \delta^{om}]$. More precisely, there are three possible scenarios: (i) if $\alpha_1 < 0$, the prices on both sides increase in δ, (ii) if $0 \le \alpha_1 < t_1$, the price on side 1 increases in δ while the price on side 2 decreases in δ, and (iii) if $\alpha_1 \ge t_1$, the price on side 1 decreases in δ while the price on side 2 increases in δ. This, combined with the fact that $p_1^{oc}(0) = p_1^N$ and $p_2^{oc}(0) = p_2^N$, leads to the following result.

Proposition 3 *The comparison of prices under one-sided collusion (on side* 1*) and static Nash prices depends on α_1 and t_1 as follows:*

(i) If $\alpha_1 < 0$, prices under one-sided collusion are above their static Nash levels on both sides: $p_1^{oc}(\delta) > p_1^N$ and $p_2^{oc}(\delta) > p_2^N$.

(ii) If $0 \le \alpha_1 < t_1$, the price on the collusive side under one-sided collusion is above its static Nash level while the price on the competitive side under one-sided collusion is below its static Nash level: $p_1^{oc}(\delta) > p_1^N$ and $p_2^{oc}(\delta) \le p_2^N$.

(iii) If $\alpha_1 \ge t_1$, the price on the collusive side under one-sided collusion is below its static Nash level while the price on the competitive side under one-sided collusion is above the static Nash level: $p_1^{oc}(\delta) \le p_1^N$ and $p_2^{oc}(\delta) > p_2^N$.

Argentesi and Filistrucchi (2007) provide empirical support for the existence of one-sided collusion on the reader side of the newspaper market. Under the assumption that readers are not affected (neither positively nor negatively) by advertising and that there is single-homing on both sides of the market, they find that the markups on the reader side are greater than those in the counterfactual competitive scenario while the markups on the advertising side are the same. This empirical finding is in line with the prediction of Proposition 3 in the special case $\alpha_1 = 0$.[25]

We can now state the welfare effects of one-sided collusion. Again, the fact that the market is fully and symmetrically covered under both collusion and competition implies that collusion is neutral for total welfare and that its impact on the users of a given side is determined solely by its effect on prices.

Corollary 2 *In the model with single-homing on both sides, one-sided collusion on side* 1 *does not affect total welfare, and can either harm users on both sides of the market or benefits users on one of the two sides and harm users on the other side. More specifically:*

(i) If $\alpha_1 < 0$, users on both market sides are harmed. Users on the collusive side are more harmed than users on the competitive side if and only if $|\alpha_1| > t_1$.

(ii) If $0 \leq \alpha_1 < t_1$, users on the collusive side are harmed by collusion, while users on the competitive side benefit from collusion.

(iii) If $\alpha_1 \geq t_1$, users on the collusive side benefit from collusion, while users on the competitive side are harmed by collusion.

3 Extension: Competitive bottleneck setting

Let us now study the price and welfare effects of collusion when users on one side of the market can join both platforms (i.e., multi-home), while users on the other side of the market continue to join just one platform (i.e., single-home). We rely for this on a repeated version of the competitive bottleneck model considered by Belleflamme and Peitz (2019). Without loss of generality, let side 1 be the side where users can multi-home. Figure 2 presents the demand on each side of the market, where $\tilde{x}_1^j$ denotes the consumer on side 1 that is indifferent between joining platform $j \in \{A, B\}$ and not joining this platform; while $\tilde{x}_2$ is the agent on side 2 that is indifferent between joining platforms A and B.

We will focus on the scenario where, on side 1, users that join a single platform co-exist with users that join both platforms, i.e., $0 < \tilde{x}_1^B < \tilde{x}_1^A < 1$. We will refer to this situation as *partial multi-homing* on side 1.

[25] Argentesi and Filistrucchi (2007) justify the use of an empirical model with single-homing on both sides by the observation that, in each day of the week, 84% of advertisers put an ad in only one of the four newspapers they consider. Note, however, that their empirical finding is also consistent with the prediction of the competitive bottleneck model in the next section regarding the impact of two-sided collusion on prices (if we consider, following most of the literature, that there is single-homing on the reader side and multi-homing on the advertising side).

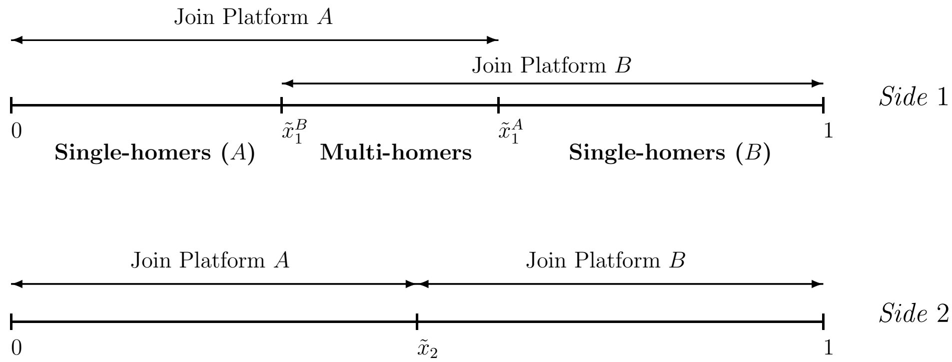


Figure 2: Demand configuration with multi-homing on side 1 and single-homing on side 2.

The utility function for single-homers (a subset of the users on side 1 and all users on side 2) is the same as in the baseline model.[26] For the specification of the utility function of the multi-homers, we follow Belleflamme and Peitz (2019). We assume, as they do in their baseline model, that multi-homers get the sum of the stand-alone values from each platform (which is assumed to be the same across platforms, k_1).[27] Moreover, multi-homers can interact with all users on side 2 and, therefore, benefit from a total network externality of α_1. Finally, multi-homers pay the membership fee to both platforms and their transportation cost is the sum of the transportation costs of joining the two platforms separately, i.e., $t_1 x + t_1(1-x) = t_1$. Thus, the utility function of a multi-homer on side 1 is:

$$u_1^{ib}(p_1^A, p_1^B) = 2k_1 + \alpha_1 - t_1 - p_1^A - p_1^B,$$

which, in contrast to the utility function of single-homers, does not depend on the location of the agent.

To present shorter mathematical expressions, let us introduce the following additional notation:

$$\Omega \equiv 8t_1t_2 - \alpha_1^2 - 6\alpha_1\alpha_2 - \alpha_2^2. \tag{18}$$

Assumption 4 (Belleflamme and Peitz, 2019)[28]

[26] See Expression (1)

[27] In contrast, Armstrong and Wright (2007) and Rasch (2007) assume that joining a second platform does not generate any extra stand-alone benefit. A more general assumption that would encompass their assumption and the one we make as particular cases, would be to consider that the stand-alone value of a multi-homer is given by $(1+\gamma)k_1$, where γ is between 0 and 1 (see Appendix A.4 of Belleflamme and Peitz, 2019). However, this would imply additional notations, without providing richer results.

[28] This assumption guarantees that the second-order conditions for the individual profit maximization problem are satisfied and that there is no tipping in equilibrium (part i). It also ensures that, under Nash

i. $\Omega > 0$.

ii. $2t_1 - \alpha_1 - \alpha_2 < 2k_1 < 4t_1 - \alpha_1 - \alpha_2$.

iii. $2(\alpha_1 + \alpha_2)k_1 + 4t_1k_2 > 6(t_1t_2 - \alpha_1\alpha_2) - (\alpha_1 - \alpha_2)^2$.

Under this assumption, the demand functions for platform $i \in \{A, B\}$ on sides 1 and 2 are given, respectively, by:[29]

$$n_1^{ib}(p_1^i, p_2^i, p_1^{-i}, p_2^{-i}) = \frac{\alpha_1}{t_1}\left[\frac{1}{2} + \frac{\alpha_2(p_1^{-i} - p_1^i) + t_1(p_2^{-i} - p_2^i)}{2(t_1t_2 - \alpha_1\alpha_2)}\right] + \frac{k_1 - p_1^i}{t_1} \tag{19}$$

and

$$n_2^{ib}(p_1^i, p_2^i, p_1^{-i}, p_2^{-i}) = \frac{1}{2} + \frac{\alpha_2(p_1^{-i} - p_1^i) + t_1(p_2^{-i} - p_2^i)}{2(t_1t_2 - \alpha_1\alpha_2)}. \tag{20}$$

When platforms set symmetric prices, i.e., $p_1^A = p_1^B = p_1$ and $p_2^A = p_2^B = p_2$, the individual demands become: $n_1^b(p_1) = \frac{2k_1+\alpha_1-2p_1}{2t_1}$ and $n_2^b = \frac{1}{2}$. A key difference with our baseline model is the existence of a *demand expansion* effect on the multi-homing side, captured by the fact that n_1^b is decreasing in p_1.

As in the baseline model, we assume that the marginal cost to serve each side of the market is constant and normalized to zero. Thus, the individual profit function of platform $i \in \{A, B\}$ is:

$$\pi^{ib} = p_1^i n_1^{ib} + p_2^i n_2^{ib}. \tag{21}$$

We first provide the prices, demands and profits in the static Nash equilibrium of the game.

Proposition 4 (Belleflamme and Peitz, 2019). *The static Nash equilibrium in the competitive bottleneck model considered above is such that:*

- *platforms set symmetric prices,* $p_1^{Nb} = \frac{2k_1+\alpha_1-\alpha_2}{4}$ *and* $p_2^{Nb} = t_2 - \frac{\alpha_1(2k_1+\alpha_1+3\alpha_2)}{4t_1}$.
- *there is partial multi-homing on side* 1 *and full coverage of side* 2.
- *the number of users joining each platform is* $n_1^{Nb} = \frac{2k_1+\alpha_1+\alpha_2}{4t_1}$ *and* $n_2^{Nb} = \frac{1}{2}$.
- *the profit of each platform is* $\pi^{Nb} = \frac{4{k_1}^2+\Omega}{16t_1}$.

competition, there is partial multi-homing on side 1 (part ii) and full market coverage of side 2 (part iii).

[29] For details, see Belleflamme and Peitz (2019).

Proof. See Belleflamme and Peitz (2019). ■

Let us now provide the expressions of the aggregate surplus of users on each side of the market and total welfare when platforms set symmetric prices, p_1 and p_2. The aggregate surplus of users on the multi-homing side (side 1) is:

$$\begin{aligned} CS_1^b &= \int_0^{1-n_1^b} (k_1 + \frac{\alpha_1}{2} - p_1 - t_1 x)dx + \int_{1-n_1^b}^{n_1^b} (2k_1 + \alpha_1 - 2p_1 - t_1)dx + \int_{n_1^b}^{1} (k_1 + \frac{\alpha_1}{2} - p_1 - t_1(1-x))dx \\ &= n_1^b \left(2k_1 + \alpha_1 - 2p_1 - n_1^b t_1\right), \end{aligned} \tag{22}$$

while the aggregate surplus of users on the single-homing side (side 2) is:

$$\begin{aligned} CS_2^b &= \int_0^{\frac{1}{2}} (k_2 + \alpha_2 n_1^b - p_2 - t_2 x)dx + \int_{\frac{1}{2}}^{1} [k_2 + \alpha_2 n_1^b - p_2 - t_2(1-x)]dx \\ &= k_2 + \alpha_2 n_1^b - p_2 - \frac{t_2}{4}. \end{aligned} \tag{23}$$

Thus, total welfare with symmetric prices is:

$$W^b = CS_1^b + CS_2^b + \pi^{Ab} + \pi^{Bb} = n_1^b(2k_1 + \alpha_1 + \alpha_2 - n_1^b t_1) + k_2 - \frac{t_2}{4}, \tag{24}$$

The following lemma is useful for our subsequent welfare analysis as it shows that the impact of a (symmetric) change in prices on welfare is solely driven by its impact on the number of multi-homers.

Lemma 8 *If platforms set symmetric prices (i.e., $p_1^A = p_1^B$ and $p_2^A = p_2^B$) that induce partial multi-homing on side 1 and full market coverage on side 2, total welfare is greater than under Nash competition if and only if the number of multi-homers is greater than under Nash competition ($n_1^b > n_1^{Nb}$).*

Proof. See Appendix. ■

As in the baseline model, we assume that platforms interact for an infinite number of periods and, in each period, they choose the price to charge on each side of the market and have a common discount factor, $\delta \in (0, 1)$. Again, for any $\delta \in (0, 1)$, we will study the most profitable sustainable collusive agreement both when platforms collude on the two sides of the market and when they only collude on one side of the market. In contrast to the baseline setting, in the competitive bottleneck model, the two sides of the market are not symmetric and, therefore, a one-sided collusive agreement is expected to be qualitatively different depending on whether it targets the price on side 1 or side 2. Thus, we will analyze three possible collusive scenarios: (i) platforms set both prices cooperatively (*two-sided collusion*); (ii) platforms set the price on side 2 cooperatively and the price on side 1 non-cooperatively (*collusion on the single-homing side only*); and, finally, (iii) platforms set the price on side 1 cooperatively and the price on side 2 non-cooperatively (*collusion on the multi-homing side only*). As in the baseline model, we assume that platforms adopt grim trigger strategies to punish deviations from the collusive path, i.e., they permanently revert to the Nash equilibrium of the stage game (Proposition 4) if one platform defects.

We will focus on the constellations of model parameters for which there is partial multi-homing on side 1 and full market coverage of side 2 under all the considered competitive scenarios (i.e. competition on both sides, two-sided collusion, and one-sided collusion on side 1 or side 2).

3.1 Two-sided collusion

Assume that platforms cooperatively set symmetric prices on both sides of the market. Hence, for a given δ, they choose prices p_1^{cb} and p_2^{cb} that solve:

$$\max_{(p_1,p_2)\in\mathbb{R}^2} \left\{\pi^{Ab}(p_1,p_2,p_1,p_2)+\pi^{Bb}(p_1,p_2,p_1,p_2)\right\}$$

subject to the incentive compatibility constraint:

$$\frac{\pi^{ib}(p_1,p_2,p_1,p_2)}{1-\delta}\geq \pi^{db}\left(p_1,p_2\right)+\frac{\delta}{1-\delta}\pi^{Nb},$$

where $\pi^{db}(p_1,p_2)=max_{(p_1^i,p_2^i)}\pi^{ib}(p_1^i,p_2^i,p_1,p_2)$ is the optimal deviation profit if the collusive prices are p_1 and p_2.

The following proposition characterizes prices and profits under the most profitable sustainable collusive agreement.

Proposition 5 *In the competitive bottleneck model, for a given* $0<\delta<1$*, the most profitable agreement among the sustainable and symmetric two-sided collusive agreements is such that:*

- *platforms charge the Nash price on the multi-homing side, $p_1^{cb}(\delta) = p_1^{Nb}$, and a supra-competitive price on the single-homing side:*

$$p_2^{cb}(\delta) = \begin{cases} p_2^{Nb} + \frac{\Omega}{2t_1}\frac{\delta}{1-\delta} & \textit{if} \quad 0 < \delta < \delta^{mb} \\ p_2^{mb} & \textit{if} \quad \delta^{mb} \leq \delta < 1, \end{cases} \tag{25}$$

where $p_2^{mb} = k_2 - \frac{t_2}{2} + \frac{\alpha_2(2k_1+\alpha_1+\alpha_2)}{4t_1}$ and $\delta^{mb} \equiv 1 - \frac{2\Omega}{\Omega+2k_1(\alpha_1+\alpha_2)+4k_2t_1+2(t_1t_2-\alpha_1\alpha_2)}$,

- *the number of users that join each platform on each side of the market is the same as in the static Nash equilibrium.*

- *the individual profit is:*

$$\pi^{cb}(\delta) = \begin{cases} \pi^{Nb} + \frac{\Omega}{2t_1}\frac{\delta}{1-\delta} & \textit{if} \quad 0 < \delta < \delta^{mb} \\ \pi^{mb} & \textit{if} \quad \delta^{mb} \leq \delta < 1, \end{cases}$$

where $\pi^{mb} = \frac{2k_2-t_2}{4} + \frac{(2k_1+\alpha_1+\alpha_2)^2}{16t_1}$.

Proof. See Appendix. ■

When platforms collude on both sides of the market, they set the static Nash price on the multi-homing side.[30] The intuition behind this finding is that, for *fixed* demands on the single-homing side, firms do not compete on the multi-homing side. Interestingly, our subsequent analysis will show that a crucial condition for prices to remain unchanged on the multi-homing side is that firms collude on *both* sides of the market.

The result that the price on the multi-homing side is not affected by two-sided collusion, combined with our assumption that the single-shoming side of the market is fully (and symmetrically) covered under both competition and two-sided collusion, implies that the latter has no effect on the demands on any side of the market. As a result, two-sided collusion causes no harm to users on side 1, as they pay the same price and benefit from the same network externalities as under competition. In contrast, users on side 2 are harmed by collusion, as they pay a higher price and receive the same network externalities. Finally, from Lemma 8, it follows that two-sided collusion does not affect total welfare since it does not affect the number of multi-homers.

[30]This is somewhat reminiscent of the result in Gössl and Rasch (2016) that in a (one-sided) Hotelling model with elastic demand, collusion between two firms charging two-part tariffs does not affect their linear prices. However, the mechanism at work in Gössl and Rasch (2016) is different from the one behind our result.

Corollary 3 *In the competitive bottleneck model, two-sided collusion:*

- *has no impact on users on the multi-homing side.*
- *harms users on the single-homing side.*
- *has no impact on total welfare.*

3.2 Collusion on the single-homing side only

Let us now characterize the most profitable collusive agreement when platforms collude on the single-homing side and choose the price on the multi-homing side non-cooperatively. The formulation of the optimization problem is similar to the one presented in the baseline model in the scenario of one-sided collusion. The only difference concerns the expression for the demand function on side 1. Thus, given $\delta \in (0,1)$, the most profitable sustainable symmetric agreement when platforms collude on the single-homing side features p_2^{c2b} that solves:

$$\max_{p_2 \in \mathbb{R}} \left\{ \pi^{Ab}(p_1^A, p_2, p_1^B, p_2) + \pi^{Bb}(p_1^A, p_2, p_1^B, p_2) \right\} \tag{26}$$

subject to the constraints:

$$\begin{cases} p_1^A = \text{argmax}_{\tilde{p}_1} \ \pi^{Ab}\left(\tilde{p}_1, p_2, p_1^B, p_2\right) \\ p_1^B = \text{argmax}_{\tilde{p}_1} \ \pi^{Bb}\left(p_1^A, p_2, \tilde{p}_1, p_2\right) \\ p_2 \in \left\{ p_2 \in \mathbb{R} \mid \frac{\pi^{Ab}\left(p_1^A, p_2, p_1^B, p_2\right)}{1-\delta} \geq \max_{(\tilde{p}_1, \tilde{p}_2)} \pi^{Ab}(\tilde{p}_1, \tilde{p}_2, p_1^B, p_2) + \frac{\delta}{1-\delta}\pi^{Nb} \right\}. \end{cases} \tag{27}$$

Solving the FOCs corresponding to the maximization problems underlying the first two constraints, we get that the price charged by platform $i \in \{A, B\}$ on side 1 relates to the collusive price charged on side 2 as follows:

$$p_1^i = \frac{(2k_1 + \alpha_1)(t_1 t_2 - \alpha_1 \alpha_2)}{4t_1 t_2 - 3\alpha_1 \alpha_2} - \frac{\alpha_2 t_1}{4t_1 t_2 - 3\alpha_1 \alpha_2} p_2 \equiv f\left(p_2\right), \quad i \in \{A, B\}. \tag{28}$$

Thus, given the collusive price on side 2, p_2^{c2b}, the price on the non-cooperative side 1 is $p_1^{c2b} = f(p_2^{c2b})$.

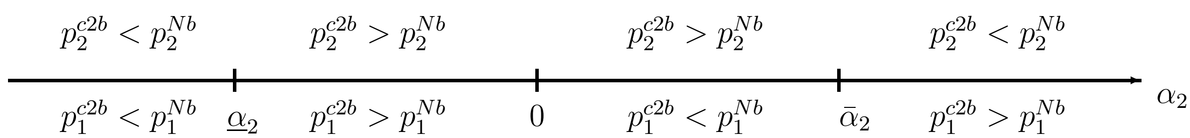
Lemma 9 *In the competitive bottleneck model, the price variations induced by (one-sided) collusion on the single-homing side relate as follows:*

$$\Delta p_1 = -\frac{\alpha_2 t_1}{4t_1 t_2 - 3\alpha_1\alpha_2}\Delta p_2 \quad \textit{where} \quad \Delta p_j = p_j^{c2b} - p_j^{Nb}. \tag{29}$$

Proof. This can be easily shown following the same steps as in the proof of Lemma 6. ■

Notice the similarity between Lemmata 6 and 9. In particular, note that in both settings, the way signs of the price variation on the two sides of the market are interrelated depends only on the sign of the cross-group externalities received by the users on the *cooperative* side.[31] Also, the intuition presented for Lemma 6 still applies to Lemma 9.

Proposition 6 *In the competitive bottleneck model, if platforms collude only on the single-homing side, prices relate to their static Nash counterparts as follows:*[32]



Proof. See Appendix. ■

A first remark to be made is that the qualitative effects of one-sided collusion on prices in this setting are close to those obtained under one-sided collusion in the baseline model (Proposition 2). The only exception concerns the case where users on the competitive side exert a sufficiently strong negative externality on users on the collusive side. More precisely, in the competitive bottleneck setting, when $\alpha_2 < \underline{\alpha}_2$, platforms charge infra-competitive prices on *both* sides of the market under one-side collusion over the single-homing side. Such a scenario is clearly unprofitable in the baseline setting as decreasing both prices would lead to a decrease in margins on both sides without leading to an increase in demand (on any of the two sides). This is no longer true in the competitive bottleneck setting. From Lemma 9, if $\alpha_2 < 0$, setting a supra-competitive (resp. infra-competitive) price on the collusive side leads to a supra-competitive (resp. infra-competitive) price on the competitive side.

[31] Notice that Assumption 4.i. implies that $4t_1t_2 - 3\alpha_1\alpha_2 > 0$

[32] The analytical expressions for $\underline{\alpha}_2$ and $\bar{\alpha}_2$ are provided in (57), in the Appendix.

As a result, setting $p_2 < p_2^{Nb}$ in this case has two *negative* effects on profits: it decreases revenues on the single-homing side (as there is no demand expansion on this side) and it decreases the revenue *per* user on the multi-homing side (as $p_2 < p_2^{Nb}$ implies $p_1 < p_1^{Nb}$ in this case); but it also has a *positive* effect: demand expands on the multi-homing side (as more users join both platforms).[33] Platforms will set an infra-competitive price on side 2 if and only if this positive effect outweighs the negative effects on profits, which is the case if and only if α_2 is sufficiently negative. More precisely, when $\alpha_2 < \underline{\alpha}_2$, users on side 2 strongly dislike the presence of users on side 1, which limits the extent to which it is profitable to expand demand on side 1 (as this will strongly decrease the willingness to pay of users on side 2).

Deriving the welfare effects of collusion in this setting is less straightfoward than in the baseline model. Let us start with the way collusion affects users on the collusive side (side 2). As the agreement has no impact on the demand on this side, we only need to examine the effects on the price p_2 and on the externalities exerted by users on side 1 on side 2-users. It is immediate to see that, from the perspective of users on side 2, these are opposite effects. Simple computations allow us to conclude that, regardless of the value for α_2, collusion always harms the indifferent consumer (located at $x = \frac{1}{2}$).[34] Since all users on side 2 are affected in the same way by collusion (they experience the same variation in prices and in network externalities), we can conclude that collusion harms all users on side 2. Thus, even when users on side 2 pay an infra-competitive price, they are worse off under collusion because the negative impact on the cross-group externalities they receive dominates the positive price effect.

Let us now address the impact of collusion on users on the competitive side (side 1). To do that, consider a (hypothetical) situation where platforms set symmetric prices p_1 and p_2, and platform A serves $1 - \tilde{x}_1$ users on side 1, while platform B serves $\tilde{x}_1$ users (see Figure 3). Assume that side 2 is fully covered. Suppose now that both platforms decrease p_1 by the same amount, and that side 2 remains covered. An immediate consequence is that the number of multi-homers increases, say to $x \in (\tilde{x}_1', 1 - \tilde{x}_1')$ with $\tilde{x}_1' < \tilde{x}_1$. Users that did not change their decision about joining one platform or both, i.e., all users except those located at $x \in [\tilde{x}_1', \tilde{x}_1] \cup [1 - \tilde{x}_1, 1 - \tilde{x}_1']$, are better off with the decrease in p_1, as they

[33] As demand on side 2 is not affected by collusion, the externality that users on side 2 exert on users on side 1 is also not affected by collusion. As a result, a decrease in p_1 surely increases demand on side 1.

[34] More precisely, replacing the expressions for the collusive price, given in (63), and Nash prices, given in Proposition 4, in the utility function (1), we obtain: $u_2^j\left(\frac{1}{2}, f(p_2^{c2b}(\delta)), p_2^{c2b}(\delta), f(p_2^{c2b}(\delta)), p_2^{c2b}(\delta)\right) - u_2^j\left(\frac{1}{2}, p_1^{Nb}, p_2^{Nb}, p_1^{Nb}, p_2^{Nb}\right) = -\frac{\Gamma^2\delta\Omega}{2t_1\left[\alpha_2^2\Omega + \Gamma^2(1-\delta)\right]}$, with $\Gamma \equiv 4t_1t_2 - 3\alpha_1\alpha_2 - \alpha_2^2$ and $j \in \{A, B\}$. As, by Assumption 4i, $\Omega > 0$, we conclude that $u_2^j\left(\frac{1}{2}, f(p_2^{c2b}(\delta)), p_2^{c2b}(\delta), f(p_2^{c2b}(\delta)), p_2^{c2b}(\delta)\right) < u_2^j\left(\frac{1}{2}, p_1^{Nb}, p_2^{Nb}, p_1^{Nb}, p_2^{Nb}\right)$.

pay a lower price, incur the same transportation costs, and benefit from the same network externalities. The only doubt could arise with respect to users that initially are single-homers but decide to multi-home when p_1 decreases, since everything changes for them (price, transportation cost, stand-alone value and externalities). Notice, however, that if these users start preferring to multi-home, this is because their utility is greater than if they continued to single-home. As explained just before, if they continued to single-home, they will be better off with the decrease in p_1. Therefore, by transitivity, they also benefit from the reduction in p_1. Thus, the surplus of users on side 1 is the higher the lower is p_1 as long as side 2 remains fully covered.

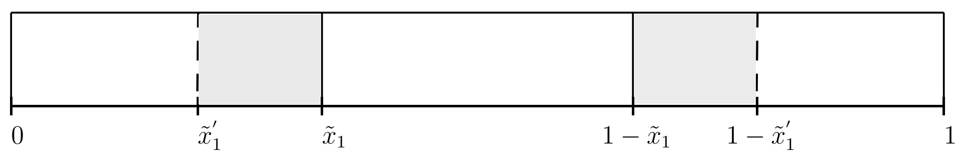


Figure 3: Impact of an increase in p_1 on the demand on side 1 (for n_2 fixed).

Finally, from Lemma 8 it follows that one-sided collusion on the single-homing side raises (resp. reduces) total welfare whenever it leads to an increase (resp. decrease) in the number of multi-homers on side 1. Given that side 2 is fully covered, this happens if platforms set an infra-competitive (resp. supra-competitive) price on side 1 (i.e. $p_1^{c2b} < p_1^{Nb}$). Proposition 6 provides the conditions under which this occurs.

Corollary 4 *In the competitive bottleneck model, collusion on the single-homing side (only):*

- *always harms users on the single-homing side;*
- *benefits users on the multi-homing side if and only if these users exert sufficiently strong (positive or negative) externalities on users on the single-homing side (i.e., $\alpha_2 < \underline{\alpha}_2$ or $\alpha_2 > \bar{\alpha}_2$);*
- *raises total welfare if and only if users on the multi-homing side exert sufficiently strong (positive or negative) externalities on users on the single-homing side (i.e., $\alpha_2 < \underline{\alpha}_2$ or $\alpha_2 > \bar{\alpha}_2$).*

It follows immediately that, if $\alpha_2 \in (\underline{\alpha}_2, \bar{\alpha}_2)$, collusion on the single-homing side reduces aggregate consumer surplus (i.e. the sum of the aggregate surplus of users on each side of the market), $CS^b = CS_1^b + CS_2^b$, as users on both sides of the market are worse off. However, when $\alpha_2 \notin (\underline{\alpha}_2, \bar{\alpha}_2)$, the impact on aggregate consumer surplus is no longer straightforward, as users on the multi-homing side are better off but users on the single-homing side are worse off. The following proposition provides a necessary and sufficient condition under which collusion has a positive effect on aggregate consumer surplus.

Proposition 7 *In the competitive bottleneck model, if platforms collude on the single-homing side only, aggregate consumer surplus increases if users on the (collusive) single-homing side strongly dislike the presence of users on the multi-homing side, $\alpha_2 < \underline{\alpha}_2 < 0$.*

Proof. See Appendix. ■

3.3 Collusion on the multi-homing side only

Last, let us study the most profitable sustainable symmetric agreement when platforms collude on the multi-homing side (side 1) and set the price on the single-homing side (side 2) non-cooperatively. In this case, for a given $\delta \in (0,1)$, platforms choose the price p_1^{c1b} that solves:

$$\max_{p_1} \left\{ \pi^{Ab}(p_1, p_2^A, p_1, p_2^B) + \pi^{Bb}(p_1, p_2^A, p_1, p_2^B) \right\} \tag{30}$$

subject to:

$$\begin{cases} p_2^A = \mathrm{argmax}_{\tilde{p}_2}\ \pi^{Ab}\left(p_1, \tilde{p}_2, p_1, p_2^B\right) \\ p_2^B = \mathrm{argmax}_{\tilde{p}_2}\ \pi^{Bb}\left(p_1, p_2^A, p_1, \tilde{p}_2\right) \\ p_1 \in \left\{ p_1 \in \mathbb{R} \mid \frac{\pi^{Ab}\left(p_1, p_2^A, p_1, p_2^B\right)}{1-\delta} \geq \max_{(\tilde{p}_1, \tilde{p}_2)} \pi^{Ab}(\tilde{p}_1, \tilde{p}_2, p_1, p_2^B) + \frac{\delta}{1-\delta}\pi^{Nb} \right\}. \end{cases}$$

Solving the FOCs underlying the first two constraints, we get:

$$p_2^i = \frac{t_1 t_2 - \alpha_1 \alpha_2}{t_1} - \frac{\alpha_1}{t_1} p_1 \equiv h\left(p_1\right), \quad i \in \{A, B\}. \tag{31}$$

Thus, given the collusive price on side 1, the price on the competitive side is $p_1^{c2b} = h(p_1^{c1b})$.

Lemma 10 *In the competitive bottleneck model, if platforms only collude on the multi-homing side, a change in the price on collusive side leads to a change in the price on the competitive side as follows:*

$$\Delta p_2 = -\frac{\alpha_1}{t_1} \Delta p_1 \quad \textit{where} \quad \Delta p_j = p_j^{c1b} - p_j^{Nb}. \tag{32}$$

Again, the key determinant for the impact of one-sided collusion on prices is the externality that users on the competitive side exert on users on the collusive side, α_1. Once more, platforms set a supra-competitive price on one side of the market and an infra-competitive price on the other if and only if users on the collusive side enjoy the presence of users on the competitive side (i.e., $\alpha_1 > 0$). Otherwise, both prices are above or below the static Nash level.

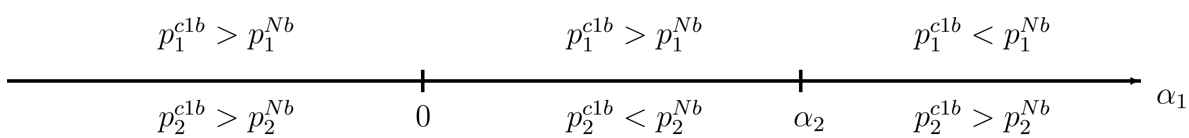
To limit the number of possible scenarios, we exclude in the rest of this section the uninteresting scenario in which both network externalities are non-positive.

Assumption 5 *$\alpha_1 > 0$ or $\alpha_2 > 0$.*

The following proposition compares prices in the current regime with those under competition.

Proposition 8 *In the competitive bottleneck model, if platforms collude on the multi-homing side only, the following holds:*

- *If $\alpha_2 > 0$, prices compare to their static Nash counterparts as follows:*



- *If $\alpha_2 < 0$, platforms set an infra-competitive price on the collusive side and a supra-competitive price on the non-cooperative side ($p_1^{c1b} < p_1^{Nb}$ and $p_2^{c1b} > p_2^{Nb}$).*[35]

Proof. See Appendix . ■

Propositions 2 and 8 are very similar and the main intuition behind the former applies to the latter.

As in the case of collusion on the single-homing side (only), users on side 1 are harmed by collusion on the multi-homing side (only) whenever they pay a supra-competitive price. From Proposition 8, this is the case whenever $\alpha_1 < \alpha_2$.

[35]If $\alpha_2 < 0$, we must have $\alpha_1 > 0$ and, therefore, $\alpha_1 > \alpha_2$. Thus, only the rightmost region of parameters in the line presented for the case of $\alpha_2 > 0$ exists.

Let us now analyze the impact of collusion on the multi-homing side (only) on users on the single-homing side. When $\alpha_2 < 0$, these users are surely harmed by collusion because they pay a higher price than under competition $\left(p_2^{c1b} > p_2^{Nb}\right)$ and also get stronger negative externalities (as they dislike the presence of users on side 1 whose number is higher under collusion). When $\alpha_2 > 0$ and $\alpha_1 < 0$, these users are also harmed by collusion, since they pay a higher price $\left(p_2^{c1b} > p_2^{Nb}\right)$ and get weaker positive externalities (as they enjoy the presence of users on side 1, whose number is lower under collusion). Finally, when $\alpha_2 > 0$ and $\alpha_1 > 0$, the impact of collusion on users on side 2 is not straightforward because either they pay a lower price but benefit from weaker positive network externalities (if $\alpha_1 < \alpha_2$), or they pay a higher price but benefit from stronger positive externalities (if $\alpha_1 > \alpha_2$). However, comparing the utility of the indifferent user (located at $\tilde{x}_2 = \frac{1}{2}$) under collusion and competition, we find that it is always lower under collusion.[36]

To complete the welfare analysis of the impact of collusion on the multi-homing side, it remains to examine the impact of this type of collusion on total welfare. Recall that collusion is welfare-improving if and only if it increases the number of multi-homers on side 1, which is the case if platforms charge an infra-competitive price on this side. Thus, from Proposition 8, we know that collusion on the multi-homing side increases total welfare if and only if $\alpha_1 > \alpha_2 > 0$.

Corollary 5 *In the competitive bottleneck model, collusion on the multi-homing side (only):*

- *harms users on the single-homing side if $\alpha_1 \neq \alpha_2$ and does not affect them if $\alpha_1 = \alpha_2$;*
- *benefits users on the multi-homing side if and only if $\alpha_1 > \alpha_2$;*
- *raises total welfare if and only if $\alpha_1 > \alpha_2 > 0$.*

4 Discussion

4.1 Demand expansion

In our competitive bottleneck model, there is a demand expansion effect on the multi-homing side. However, the assumptions we made do not allow for such an effect on a

[36]More precisely, replacing the expressions (70) and (70) for prices under one-sided collusion on the multi-homing side and Nash prices (Proposition 4), in the utility function of users on side 2, given in (1), we obtain: $u_2^j\left(\frac{1}{2}, p_1^{c1b}(\delta), p_2^{c1b}(\delta), p_1^{c1b}(\delta), p_2^{c1b}(\delta)\right) - u_2^j\left(\frac{1}{2}, p_1^{Nb}, p_2^{Nb}, p_1^{Nb}, p_2^{Nb}\right) = -\frac{\delta\Omega(\alpha_1-\alpha_2)^2}{2t_1[(1-\delta)(\alpha_1-\alpha_2)^2+\Omega]} \leq 0$, as $\Omega > 0$ (Assumption 4i). Thus, all users on side 2 are harmed when platforms collude on the multi-homing side whenever $\alpha_1 \neq \alpha_2$, and are not affected by collusion if $\alpha_1 = \alpha_2$.

side in which there is single-homing, neither in our baseline setting nor in the competitive bottleneck model. While this feature allows us to have a tractable model and derive neat results, it also imposes limitations. Note, however, that the striking result that, under one-sided collusion, the price on the collusive side may be lower than the static Nash price on that side is likely to be strengthened if we allowed for demand expansion on a single-homing side. To see why, note that, in our setting, firms' incentives to set a price below the Nash level are solely driven by the incentive to soften competition on the other side. In an environment where, following a decrease in the price on side j, there is a demand expansion on that side and potentially also on side $-j$ (if $\alpha_{-j} > 0$), our result is even more likely to hold.

4.2 Endogenous choice of the collusive side(s)

In our setting, platforms can always sustain some degree of collusion in equilibrium both when they collude on the two market sides, and when they just collude on one side of the market. Our model also suggests that platforms should always prefer to collude on both sides (since this is the most profitable scenario). However, as mentioned before, there is evidence of platforms being convicted of just coordinating the price on one market side. Coordination costs and the possibility of they being (prohibitively) higher when platforms coordinate two prices instead of one may underlie actual platforms' choices.[37] Relatedly, platforms may engage in one-sided collusion to attempt to reduce the risk of being caught and punished by antitrust authorities.[38]

In the context of one-sided collusion, a natural question that arises concerns the choice of the collusive side. While a general treatment of this issue is outside the scope of this paper, we provide two special cases where we are able to determine the platforms' choice in the baseline (single-homing) model. Collusion on side 1 yields the same outcome as competition if $\alpha_1 = t_1$, and the same outcome as two-sided collusion if $\alpha_1 = -t_1$ and $\delta < \delta^m$.[39] Therefore, platforms (weakly) prefer to collude on side 2 in the former case while they (weakly) prefer to collude on side 1 in the latter case. There are other reasons

[37] One (perhaps simplistic) way of incorporating these ingredients in our model would be to introduce a fixed coordination cost. It follows straightforwardly that: if this cost is not much higher when platforms coordinate prices on both sides of the market than when they just coordinate one price, platforms will settle a two-sided collusive agreement; while, if the coordination cost is larger enough under two-sided collusion, platforms will settle a one-sided collusive agreement. For intermediate values of this coordination cost, platforms' choice may depend on the discount factor.

[38] See Charistos (2018) for an analysis of collusion between advertising-selling platforms in the presence of an antitrust authority.

[39] Both results follow from Lemma 6.

outside our model that may also affect the choice of the side to collude on. For instance, it may be harder for platforms to coordinate prices on one side of the market than on the other one. For example, in the case of newspapers, coordinating cover prices may probably be easier than coordinating ad prices (as the latter are likely to be more heterogeneous). Moreover, monitoring might be easier on one side of the market than on the other one. Considering again the newspapers example, cover prices are typically more transparent and, therefore, easier to monitor, than ad prices.

4.3 Optimal punishment

The punishment mechanism considered in this paper (i.e., permanent reversion to Nash competition after a deviation) is not the optimal one (Abreu, 1986). While the determination of the optimal mechanism is outside the scope of the paper, we believe that most, if not all, of the main insights about the price effects of collusion would carry over to the case where firms would use such a mechanism. First, note that the scope for perfect collusion is larger with the optimal punishment mechanism than with grim trigger strategies. Therefore, all the results in our perfect two-sided collusion and perfect one-sided collusion scenarios not only hold, but also extend to some of the parameters under which there is imperfect collusion in our setting. Second, consider the case of imperfect two-sided collusion (under the optimal punishment mechanism) in a single-homing environment. In our setting with grim trigger strategies, collusive prices may either be higher than the competitive prices on both sides of the market or higher on one side and lower on the other side. This result hinges on the fact that collusion increases the total price charged by the platforms but, at the same time, distorts the price structure in a way that minimizes the incentive to deviate (for a give total price). This mechanism would still hold with any punishment mechanism: first, with full market coverage (under collusion), the increase in the total price is only driven by the increase in profits and not by the specific punishment mechansim that is considered and, second, the proof of Lemma 3 shows that there is a distortion of the price structure in a way that minimizes the deviation profit regardless of the punishment mechanism. The only question which remains open is whether the distortion of the price structure under the optimal mechanism can be large enough for the collusive price to be lower than the competitive price on one of the sides, as is the case in our setting. Third, consider the case of one-sided collusion in a single-homing environment and both two-sided and one-sided collusion in a competitive bottleneck scenario. Note that in all these cases, our results regarding the qualitative impact of collusion on prices would hold under any punishment mechanism leading to the property that prices are monotonic (i.e. there is no change in monotonicity) when the discount factor moves from 0 to the critical level under which perfect collusion becomes sustainable.

4.4 Policy implications

Our findings have several implications regarding the detection of collusion in two-sided markets and private damages actions by users of colluding platforms. First, a key lesson from our analysis of one-sided collusion is that higher prices on a given side of the market are neither a necessary nor a sufficient condition for the existence of collusion on that side. In particular, a decrease in prices on a given side should not be seen as a signal or evidence that firms do not collude on that side. This is a novel illustration of the importance of accounting for the two-sided nature of a market when running an antitrust analysis. Second, and relatedly, the computation of private damages should also account for the peculiarities of two-sided markets. Specifically, under one-sided collusion, users on the competitive side shoud be allowed to seek damages from the colluding platforms as they may be harmed by their collusive behavior on the other side of the market. Third, our findings show that in a competitive bottleneck environment, the fact that a price does *not* change on the multi-homing side of the market should not be interpreted by competition authorities as indicating that firms collude on the single-homing side only. On the contrary, such an observation should be considered as strong evidence that firms collude on *both* sides of the market as it is not consistent with plaforms' (predicted) behavior under one-sided collusion.

5 Conclusion

We investigate collusion between two-sided platforms in a single-homing environment and a competitive bottleneck setting. Our findings show that collusion on a given side of the market can lead to either an increase, a decrease or no change in prices on that side depending on (i) whether collusion occurs on the other side as well, (ii) whether there is multi-homing on one side of the market, and (iii) whether the network externalities received by the collusive side are positive or negative, and how large they are. One of the main takeaways of our paper is that collusion on a single side of the market can lead to lower prices and higher user surplus on the *collusive* side and higher prices and lower user surplus on the *competitive* side when the network externalities received by the collusive side are positive and large enough. Another key takeaway is that it is important to understand the effect of collusion on the price structure as this can explain counterintuitive behaviors of colluding platforms.

We believe that our results can help antitrust authorities understand better the changes in pricing behavior that are consistent with one-sided and two-sided collusion between platforms. They can also be useful to judges who need to decide who was harmed by a cartel involving platforms in private damages cases.

Finally, our results also provide interesting insights into the effects of collusion on prices in a multi-product setting with demand linkages. When the parameters capturing cross-group externalities in our model are positive, the latter can be reinterpreted as a model in which two firms selling two complementary products compete against each other. Our results show in particular that single-product collusion in such an environment can lead to a decrease in the price of the product for which there is collusion and an increase in the price of the product for which there is competition.

Appendix

Proof of Lemma 2.

(i) Note first that, for any $\delta \in (0,1]$, ICC (4) can be rewritten as:

$$\frac{1}{\delta}\left[\pi^d(p_1,p_2)-\pi(p_1,p_2)\right] \leq \pi^d(p_1,p_2)-\pi^N.$$

Let $(0,\delta^m)$ and consider $(p_1^c(\delta),p_2^c(\delta)) \in I(\delta)$ such that $\pi(p_1^c(\delta),p_2^c(\delta)) = \pi^c(\delta)$. Assume by way of contradiction that $(p_1^c(\delta),p_2^c(\delta)) \notin \bar{I}(\delta)$. Then

$$\frac{1}{\delta}\left[\pi^d(p_1^c(\delta),p_2^c(\delta))-\pi(p_1^c(\delta),p_2^c(\delta))\right] < \pi^d(p_1^c(\delta),p_2^c(\delta))-\pi^N,$$

i.e., the constraint is not binding at the optimum. Then, by a (standard) continuity argument, there exists $\epsilon > 0$ such that

$$\frac{1}{\delta}\left[\pi^d(p_1,p_2)-\pi(p_1,p_2)\right] < \pi^d(p_1,p_2)-\pi^N$$

for any $(p_1,p_2) \in [p_1^c(\delta)-\epsilon, p_1^c(\delta)+\epsilon] \times [p_2^c(\delta)-\epsilon, p_2^c(\delta)+\epsilon]$. This implies that the pair of prices $(p_1^c(\delta),p_2^c(\delta))$ is a local maximum of $\pi(p_1,p_2)$. However, straightforward computations show that $\pi(p_1,p_2)$ does not have a local maximum but its global maximum, which is uniquely reached at (p_1^m,p_2^m). This implies that $(p_1^c(\delta),p_2^c(\delta)) = (p_1^m,p_2^m)$, which in turn implies that

$$\frac{1}{\delta}\left[\pi^d(p_1^m,p_2^m)-\pi(p_1^m,p_2^m)\right] < \pi^d(p_1^m,p_2^m)-\pi^N$$

or, equivalently,

$$\delta > \frac{\pi^d(p_1^m,p_2^m)-\pi(p_1^m,p_2^m)}{\pi^d(p_1^m,p_2^m)-\pi^N} = \delta^m,$$

which leads to a contradiction. Thus, it must hold that $(p_1^c(\delta),p_2^c(\delta)) \in \bar{I}(\delta)$.

(ii) From the ICC

$$\frac{1}{\delta}\left[\pi^d(p_1,p_2)-\pi(p_1,p_2)\right] \leq \pi^d(p_1,p_2)-\pi^N$$

and the fact that $\pi^d(p_1,p_2)-\pi(p_1,p_2) \geq 0$ it follows that $\delta < \delta' \Rightarrow I(\delta) \subseteq I(\delta') \Rightarrow \pi^c(\delta) \leq \pi^c(\delta')$. Moreover, for δ,δ' such that $0 < \delta < \delta' \leq \delta^m$, it must hold that $\pi^c(\delta) \neq \pi^c(\delta')$. To see why, assume by way of contradiction that $\pi^c(\delta) = \pi^c(\delta')$ and consider $(p_1^c(\delta),p_2^c(\delta)) \in I(\delta)$ such that $\pi(p_1^c(\delta),p_2^c(\delta)) = \pi^c(\delta)$. Since $I(\delta) \subseteq I(\delta')$, we have

$(p_1^c(\delta), p_2^c(\delta)) \in I(\delta')$. This, combined with $\pi(p_1^c(\delta), p_2^c(\delta)) = \pi^c(\delta')$ and (i), implies that $(p_1^c(\delta), p_2^c(\delta)) \in \bar{I}'(\delta)$, i.e.

$$\frac{1}{\delta'}\left[\pi^d(p_1^c(\delta), p_2^c(\delta)) - \pi(p_1^c(\delta), p_2^c(\delta))\right] = \pi^d(p_1^c(\delta), p_2^c(\delta)) - \pi^N$$

which can be rewritten as

$$\delta' = \frac{\pi^d(p_1^c(\delta), p_2^c(\delta)) - \pi(p_1^c(\delta), p_2^c(\delta))}{\pi^d(p_1^c(\delta), p_2^c(\delta)) - \pi^N}$$

because $(p_1^c(\delta), p_2^c(\delta)) \neq \left(p_1^N, p_2^N\right)$ (this follows from the fact that $\left(p_1^N, p_2^N\right) \notin \bar{I}(\delta)$ for $\delta > 0$ and (i)). Since $(p_1^c(\delta), p_2^c(\delta)) \in \bar{I}(\delta)$ (from (i)), we also have

$$\delta = \frac{\pi^d(p_1^c(\delta), p_2^c(\delta)) - \pi(p_1^c(\delta), p_2^c(\delta))}{\pi^d(p_1^c(\delta), p_2^c(\delta)) - \pi^N}.$$

Therefore, $\delta = \delta'$, which leads to a contradiction, which completes the proof.

Proof of Lemma 3.

Assume, by way of contradiction, that $(p_1^c(\delta), p_2^c(\delta))$ is not a solution to the constrained minimization program. Denoting $(\hat{p}_1(\delta), \hat{p}_2(\delta))$ a solution to that program, we then have

$$\pi^d(\hat{p}_1(\delta), \hat{p}_2(\delta)) < \pi^d(p_1^c(\delta), p_2^c(\delta)).$$

Therefore

$$\begin{aligned}\frac{\pi(\hat{p}_1(\delta), \hat{p}_2(\delta))}{1-\delta} &= \frac{\pi(p_1^c(\delta), p_2^c(\delta))}{1-\delta} = \\ &= \pi^d(p_1^c(\delta), p_2^c(\delta)) + \frac{\delta}{1-\delta}\pi^N > \pi^d(\hat{p}_1(\delta), \hat{p}_2(\delta)) + \frac{\delta}{1-\delta}\pi^N,\end{aligned}$$

which implies that

$$\frac{1}{\delta}\left[\pi^d(\hat{p}_1(\delta), \hat{p}_2(\delta)) - \pi(\hat{p}_1(\delta), \hat{p}_2(\delta))\right] < \pi^d(\hat{p}_1(\delta), \hat{p}_2(\delta)) - \pi^N.$$

Again, by a continuity argument, there exists $\mu > 0$ such that

$$\frac{1}{\delta}\left[\pi^d(p_1, p_2) - \pi(p_1, p_2)\right] < \pi^d(p_1, p_2) - \pi^N$$

for any $(p_1, p_2) \in [\hat{p}_1(\delta) - \mu, \hat{p}_1(\delta) + \mu] \times [\hat{p}_2(\delta) - \mu, \hat{p}_2(\delta) + \mu]$. There are only two possible scenarios, which both lead to a contradiction:

- If $\pi(p_1, p_2)$ reaches a local maximum at $(\hat{p}_1(\delta), \hat{p}_2(\delta))$ then it is necessarily the case that $(\hat{p}_1(\delta), \hat{p}_2(\delta)) = (p_1^m, p_2^m)$, and, therefore, $\pi(\hat{p}_1(\delta), \hat{p}_2(\delta)) = \pi^m > \pi^c(\delta)$ because $\delta \in (0, \delta^m)$, a contradiction.

- If $\pi(p_1, p_2)$ does not reach a local maximum at $(\hat{p}_1(\delta), \hat{p}_2(\delta))$ then there exists $(\breve{p}_1, \breve{p}_2) \in [\hat{p}_1(\delta) - \mu, \hat{p}_1(\delta) + \mu] \times [\hat{p}_2(\delta) - \mu, \hat{p}_2(\delta) + \mu]$ such that

$$\pi(\breve{p}_1, \breve{p}_2) > \pi(\hat{p}_1(\delta), \hat{p}_2(\delta)) = \pi^c(\delta)$$

Since $(\breve{p}_1, \breve{p}_2) \in I(\delta)$, this contradicts the fact that $\pi^c(\delta) = \max_{(p_1,p_2)\in I(\delta)} \pi(p_1, p_2)$. □

Proof of Lemma 4.

We are focusing on symmetric collusive agreements, i.e., such that platforms set the same price on each side of the market, i.e., $p_j^A = p_j^B = p_j$, $j \in \{1, 2\}$. Let $\tilde{x}_j$ denote the consumer on side j that is indifferent between joining platform A and not joining any platform.

1. We start by deriving the conditions that ensure that, if platforms fully serve side 2 (i.e., $\tilde{x}_2 = \frac{1}{2}$), it is also profitable to fully serve side 1.

Given that $\tilde{x}_2 = \frac{1}{2}$ and platforms set symmetric prices, the user on side 1 that is indifferent between joining platform A and not joining any platform is such that:

$$u_1^A(\tilde{x}_1, p_1, p_2, p_1, p_2) = 0 \;\Leftrightarrow\; \tilde{x}_1 = \frac{2k_1 - 2p_1 + \alpha_1}{2t_1}. \tag{33}$$

If side 1 is not fully covered, i.e. $\tilde{x}_1 \leq \frac{1}{2}$, the individual (collusive) profit is:

$$\pi^c(p_1, p_2) = p_1\tilde{x}_1 + \frac{p_2}{2} = p_1\frac{2k_1 - 2p_1 + \alpha_1}{2t_1} + \frac{p_2}{2}. \tag{34}$$

As π^c is strictly increasing in p_2, platforms will choose the highest price that leaves the consumer that is indifferent between joining platforms A and B, $\tilde{x}_2 = \frac{1}{2}$, with zero utility:

$$u_2^A(\tfrac{1}{2}, p_1, p_2, p_1, p_2) = 0 \;\Leftrightarrow\; p_2 = k_2 + \alpha_2\frac{2k_1 - 2p_1 + \alpha_1}{2t_1} - \frac{t_2}{2}. \tag{35}$$

Solving the FOC corresponding to the maximization of π^c with respect to p_1, $\frac{\partial \pi^c}{\partial p_1} = 0$, we obtain:

$$p_1 = \frac{2k_1 + \alpha_1}{4}$$

Replacing this expression in (35) and (33), we obtain respectively:

$$p_2 = \frac{4k_2t_1 - 2t_1t_2 + 2k_1\alpha_2 + \alpha_1\alpha_2}{4t_1}$$

and:

$$\tilde{x}_1 = \frac{2k_1 + \alpha_1}{4t_1}.$$

As a result, there is a local maximum of π^c with partial coverage of side 1 if:

$$\tilde{x}_1 < \frac{1}{2} \;\Leftrightarrow\; k_1 < t_1 - \frac{\alpha_1}{2}.$$

Notice that we could replicate this analysis assuming that platforms fully serve side 1 (i.e., $\tilde{x}_1 = \frac{1}{2}$), and derive the condition that ensures that it is also profitable to fully serve side 2. Similarly, we would conclude that there is a local maximum of π^c with partial coverage of side 2 if:

$$k_2 < t_2 - \frac{\alpha_2}{2}.$$

2. Let us now see under which conditions platforms prefer to partially serve *both* sides, i.e. $\tilde{x}_1 < \frac{1}{2}$ and $\tilde{x}_2 < \frac{1}{2}$, instead of fully serving them.

If platforms set symmetric prices, the user on side $j \in \{1, 2\}$ that is indifferent between joining platform A and not joining any platform is such that:

$$u_j^A(\tilde{x}_j, p_1, p_2, p_1, p_2) = 0 \;\Leftrightarrow\; k_j + \alpha_j\tilde{x}_{-j} - t_j\tilde{x}_j - p_j = 0.$$

Solving the corresponding system of two equations, we obtain:

$$\begin{cases} u_1^A(\tilde{x}_1, p_1, p_2, p_1, p_2) = 0 \\ u_2^A(\tilde{x}_2, p_1, p_2, p_1, p_2) = 0 \end{cases} \Leftrightarrow \begin{cases} \tilde{x}_1 = \frac{\alpha_1(k_2 - p_2) + t_2(k_1 - p_1)}{t_1t_2 - \alpha_1\alpha_2} \\ \tilde{x}_2 = \frac{\alpha_2(k_1 - p_1) + t_1(k_2 - p_2)}{t_1t_2 - \alpha_1\alpha_2} \end{cases}.$$

If $\tilde{x}_j \leq \frac{1}{2}$, the individual collusive profit is given by:

$$\pi^c = p_1\tilde{x}_1 + p_2\tilde{x}_2.$$

Solving the FOCs corresponding to the maximization of π^c, we obtain:

$$p_1 = \frac{k_2 t_1(\alpha_1 - \alpha_2) + k_1\left(2t_1t_2 - \alpha_1\alpha_2 - \alpha_2^2\right)}{4t_1t_2 - (\alpha_1+\alpha_2)^2} \;\wedge\; p_2 = \frac{k_1 t_2(\alpha_2 - \alpha_1) + k_2\left(2t_1t_2 - \alpha_1\alpha_2 - \alpha_1^2\right)}{4t_1t_2 - (\alpha_1+\alpha_2)^2}.$$

Given these prices:

$$\tilde{x}_j = \frac{2t_{-j}k_j + (\alpha_1+\alpha_2)k_{-j}}{4t_1t_2 - (\alpha_1+\alpha_2)^2}.$$

Thus, there is an interior local maximum with partial coverage of both market sides iff:

$$\tilde{x}_j < \frac{1}{2} \;\Leftrightarrow\; 2t_{-j}k_j + (\alpha_1+\alpha_2)k_{-j} < \frac{4t_1t_2 - (\alpha_1+\alpha_2)^2}{2}$$

Thus, for the two platforms to prefer to fully cover the two market sides, we must have:

$$\min\left\{2t_2k_1 + (\alpha_1+\alpha_2)k_2, (\alpha_1+\alpha_2)k_1 + 2t_1k_2\right\} \geq \frac{4t_1t_2 - (\alpha_1+\alpha_2)^2}{2}.$$

□

Proof of Lemma 5.

Assume, w.l.o.g., that platform A deviates from the collusive agreement, i.e., sets prices that maximize its individual profit, given that platform B charges the collusive prices (p_1^c, p_2^c). Its profit function is then:

$$\pi^A = \frac{-t_2(p_1)^2 - t_1(p_2)^2 - p_1p_2(\alpha_1+\alpha_2) + p_1(t_2p_1^c + \alpha_1p_2^c + t_1t_2 - \alpha_1\alpha_2) + p_2(\alpha_2p_1^c + t_1p_2^c + t_1t_2 - \alpha_1\alpha_2)}{2(t_1t_2 - \alpha_1\alpha_2)}. \quad (36)$$

The FOCs corresponding to the maximization of π^A are:

$$\frac{\partial \pi^A}{\partial p_j} = 0 \Leftrightarrow \frac{1}{2} - \frac{2t_{-j}p_j + (\alpha_1+\alpha_2)p_{-j} - t_{-j}p_j^c - \alpha_j p_{-j}^c}{2(t_1t_2 - \alpha_1\alpha_2)} = 0, \quad j \in \{1, 2\}.$$

Combining the two FOCs, we obtain:[40]

$$p_j^d(p_1^c, p_2^c) = \frac{[2t_1t_2 - \alpha_{-j}(\alpha_1+\alpha_2)]p_j^c + p_{-j}^c t_j(\alpha_j - \alpha_{-j}) + (2t_j - \alpha_1 - \alpha_2)(t_1t_2 - \alpha_1\alpha_2)}{4t_1t_2 - (\alpha_1+\alpha_2)^2}. \tag{37}$$

Replacing these prices in (36), we obtain the deviation profit (for given p_1^c and p_2^c):

$$\pi^d(p_1^c, p_2^c) = \frac{1}{2\left[4t_1t_2 - (\alpha_1+\alpha_2)^2\right]}\Big\{t_2(p_1^c)^2 + t_1(p_2^c)^2 + (\alpha_1+\alpha_2)p_1^cp_2^c + \left[t_2(2t_1 - \alpha_1 + \alpha_2) - \alpha_2(\alpha_1+\alpha_2)\right]p_1^c + \\ + \left[t_1(2t_2 + \alpha_1 - \alpha_2) - \alpha_1(\alpha_1+\alpha_2)\right]p_2^c + (t_1 + t_2 - \alpha_1 - \alpha_2)(t_1t_2 - \alpha_1\alpha_2)\Big\}. \tag{38}$$

From Lemma 4, both sides of the market are fully covered under the most profitable collusive agreement. Thus, if platforms charge prices (p_1, p_2), their individual per-period collusive profit is:

$$\pi^c(p_1, p_2) = \frac{p_1 + p_2}{2}. \tag{39}$$

From Lemma 3, collusive prices (p_1^c, p_2^c) solve the following constrained minimization program:

$$\min_{(p_1,p_2)\in R^2} \pi^d(p_1, p_2) \qquad \text{s.t.} \qquad \pi^c = \frac{p_1 + p_2}{2}.$$

For a given collusive profit π^c, replacing $p_2^c = 2\pi^c - p_1^c$ in (38) and solving the FOC corresponding to the minimization of $\pi^d(p_1, 2\pi^c - p_1)$ with respect to p_1, we obtain:[41]

$$p_1^c = \frac{\alpha_1 - \alpha_2}{2} + \frac{2t_1 - \alpha_1 - \alpha_2}{t_1 + t_2 - \alpha_1 - \alpha_2}\pi^c.$$

Thus, if δ is sufficiently low, collusive prices are:

$$\left(p_1^c(\delta), p_2^c(\delta)\right) = \left(\frac{\alpha_1 - \alpha_2}{2} + \frac{2t_1 - \alpha_1 - \alpha_2}{t_1 + t_2 - \alpha_1 - \alpha_2}\pi^c(\delta), \frac{\alpha_2 - \alpha_1}{2} + \frac{2t_2 - \alpha_1 - \alpha_2}{t_1 + t_2 - \alpha_1 - \alpha_2}\pi^c(\delta)\right). \tag{40}$$

[40] Assumption 1 implies that the second-order conditions are satisfied: $\frac{\partial^2\pi^d}{\partial(p_j^d)^2} = -\frac{t_2}{t_1t_2 - \alpha_1\alpha_2} < 0$ and $\frac{\partial^2\pi^d}{\partial(p_1^d)^2}\frac{\partial^2\pi^d}{\partial(p_2^d)^2} - \left(\frac{\partial^2\pi^d}{\partial p_1^d \partial p_2^d}\right)^2 = \frac{4t_1t_2 - (\alpha_1+\alpha_2)^2}{4(t_1t_2 - \alpha_1\alpha_2)^2} > 0$.

[41] The second-order is satisfied, $\frac{d^2\pi^d}{dp_1^{c2}}(p_1^c, 2\pi^c - p_1^c) = \frac{t_1 + t_2 - \alpha_1 - \alpha_2}{4t_1t_2 - (\alpha_1+\alpha_2)^2} > 0$, meaning that our candidate is, indeed, a minimum.

These prices are valid as long as they induce full coverage of side $j \in \{1,2\}$, i.e., :

$$u_j^i\left(\tfrac{1}{2}, p_1^c, p_2^c, p_1^c, p_2^c\right) \geq 0 \;\Leftrightarrow\; (2t_j - \alpha_1 - \alpha_2)\pi^c(\delta) \leq (2k_j - t_j + \alpha_{-j})\pi^N. \tag{41}$$

1. If $t_1 > \frac{\alpha_1+\alpha_2}{2}$ and $t_2 > \frac{\alpha_1+\alpha_2}{2}$, collusive prices are increasing in δ on both sides of the market. In this case, we can rewrite condition (41) for full coverage of side j as follows:

$$\pi^c(\delta) < \frac{(2k_j - t_j + \alpha_{-j})\pi^N}{2t_j - \alpha_1 - \alpha_2} \equiv \tilde{\pi}_j.$$

Combining Lemma 2 with $\pi^c(0) = \pi^N$ and the continuity of $\pi^c(\delta)$, we conclude that, as long as $\tilde{\pi}_j \in \left(\pi^N, \pi^m\right)$, $\exists \tilde{\delta}_j \in (0,1)$ such that $\pi^c(\delta) \leq \tilde{\pi}_j, \forall \delta \leq \tilde{\delta}_j$. For $\pi^c(\delta) > \tilde{\pi}_j$, the price on side j is no longer given by (40). We need, therefore, to know in which side of the market the collusive price reaches its maximum for a lower value of δ. Note that:

$$\tilde{\pi}_1 < \tilde{\pi}_2 \;\Leftrightarrow\; k_2(2t_1 - \alpha_1 - \alpha_2) - k_1(2t_2 - \alpha_1 - \alpha_2) > \pi^N(\alpha_2 - \alpha_1).$$

1.1. If $k_2(2t_1 - \alpha_1 - \alpha_2) - k_1(2t_2 - \alpha_1 - \alpha_2) < \pi^N(\alpha_2 - \alpha_1)$, the collusive price on side 1 reaches its maximum level (i.e., that ensures full coverage of this side) for lower values of δ. Thus, expressions (40) are valid for $\delta < \tilde{\delta}_1$. For $\delta > \tilde{\delta}_1$, we have that $p_1^c = p_1^m$ and, therefore, $p_2^c = \pi^c - p_1^m$. Again, the price on side 2 can not exceed the level that ensures full coverage of this side, p_2^m. Thus, $\exists \delta^m \in (\tilde{\delta}_1, 1)$ such that $p_2^c = p_2^m$, for $\delta \geq \delta^m$.

1.2. If $k_2(2t_1 - \alpha_1 - \alpha_2) - k_1(2t_2 - \alpha_1 - \alpha_2) > \pi^N(\alpha_2 - \alpha_1)$, the price on side 2 reaches its maximum level, p_2^m, for lower values of δ. Thus, expressions (40) are only valid for $\delta < \tilde{\delta}_2$. For $\delta > \tilde{\delta}_2$, we have that $p_2^c = p_2^m$ and $p_1^c = \pi^c - p_2^m$. The price on side 1 must be lower than p_1^m, to ensure full coverage of this side. Thus, $\exists \delta^m \in (\tilde{\delta}_2, 1)$ such that $p_1^c = p_1^m$, for $\delta \geq \delta^m$.

2. If $t_1 < \frac{\alpha_1+\alpha_2}{2} < t_2$, using (40), we conclude that, for sufficiently low values of δ, p_1^c is decreasing in δ and p_2^c is increasing in δ. Thus, the maximum level for the collusive price will be achieved on side 2 for lower values of the discount factor. The analysis is then similar to case **1.2**.

3. If $t_2 < \frac{\alpha_1+\alpha_2}{2} < t_1$, for low enough values of δ, p_1^c is increasing in δ while p_2^c is decreasing in δ. Thus, the maximum level for the collusive price will be achieved on side 1 for lower values of the discount factor, and the analysis is similar to case **1.1**. □

Proof of Proposition 1.

If $t_2 > (\alpha_1 + \alpha_2)/2$ then $p_2^c(\delta)$ is increasing in δ as long as $p_2^c(\delta) < p_2^m$, and then is constant. This implies that $p_2^c(\delta) > p_2^c(0) = p_2^N$ for any $\delta \in (0, 1]$. Likewise, if $t_1 > (\alpha_1 + \alpha_2)/2$ then $p_1^c(\delta) > p_1^N$ for any $\delta \in (0, 1]$. Consider now the scenario in which $t_2 < (\alpha_1 + \alpha_2)/2$. In this case, $p_2^c(\delta)$ is decreasing over $(0, \tilde{\delta}_1)$, which implies that $p_2^c(\delta) < p_2^N$ for any $\delta \in (0, \tilde{\delta}_1)$. Moreover, $p_2^c(\delta)$ is increasing over $\left[\tilde{\delta}_1, \delta^m\right]$ and $p_2^c(\tilde{\delta}_1) < p_2^N < p_2^m = p_2^c(\delta^m)$, which implies the existence of $\hat{\delta}_2 \in (\tilde{\delta}_1, \delta^m)$ such that $p_2^c(\delta) < p_2^N$ for any $\delta \in (0, \hat{\delta}_2)$ and $p_2^c(\delta) > p_2^N$ for any $\delta \in (\hat{\delta}_2, \delta^m]$. Finally, note that $p_2^c(\delta) = p_2^m > p_2^N$ for any $\delta \in (\delta^m, 1]$. Therefore, $p_2^c(\delta) < p_2^N$ for any $\delta \in (0, \hat{\delta}_2)$ and $p_2^c(\delta) > p_2^N$ for any $\delta \in (\tilde{\delta}_2, 1]$. Likewise, if $t_1 < (\alpha_1 + \alpha_2)/2$, then there exists $\hat{\delta}_1 \in (\tilde{\delta}_2, \delta^m)$ such that $p_1^c(\delta) < p_1^N$ for any $\delta \in (0, \hat{\delta}_1)$ and $p_1^c(\delta) > p_1^N$ for any $\delta \in (\tilde{\delta}_1, 1]$. $\square$

Proof of Lemma 7.

Let (p_1^{oc}, p_2^{oc}) denote the (unique) solution of the maximisation program (11) suubject to (12). From (13), we know that $p_2^{oc} = g(p_1^{oc}, \alpha_1, \alpha_2) = \frac{t_1 t_2 - \alpha_1 \alpha_2}{t_1} - \frac{\alpha_1}{t_1} p_1^{oc}$. In what follows, we will analyse separately three scenarios, divided according to the value of α_1.

If platforms set p_1^{oc} and p_2^{oc} inducing full market coverage, their individual profit is:

$$\pi^{oc} = \frac{p_1^{oc} + p_2^{oc}}{2} = \frac{p_1^{oc} + \left(\frac{t_1 t_2 - \alpha_1 \alpha_2}{t_1} - \frac{\alpha_1}{t_1} p_1^{oc}\right)}{2} = \frac{t_1 t_2 - \alpha_1 \alpha_2}{2t_1} + \frac{t_1 - \alpha_1}{2t_1} p_1^{oc}. \tag{42}$$

1. $0 \leq \alpha_1 < t_1$.

In this case, π^{oc} given in (42) is increasing on p_1^{oc}. Thus, platforms will set a supra-competitive price on side 1 and an infra-competitive price on side 2 (as $\alpha_1 < 0$). It follows, therefore, that if side 1 is fully covered, the condition for side 2 to be fully covered under Nash competition (Assumption 1) implies that side 2 is also fully covered under one-sided collusion. As platforms charge symmetric prices, they equally share both sides of the market. Thus, side 1 is fully covered if and only if the indifferent consumer, located at $x = \frac{1}{2}$, gets a non-negative utility:

$$u_1^i\left(\frac{1}{2}, p_1^{oc}, p_2^{oc}, p_1^{oc}, p_2^{oc}\right) \geq 0 \;\Leftrightarrow\; p_1^{oc} \leq k_1 + \frac{\alpha_1 - t_1}{2} = p_1^m, \tag{43}$$

where p_1^m is the maximum price that platforms can charge on side 1 for this side to be fully covered, as seen in (6).

Solving (42) with respect to p_1^{oc}, we obtain that, for a given collusive profit $\pi^{oc}(\delta)$, the price on side 1 is:

$$p_1^{oc}(\delta) = \frac{2t_1}{t_1 - \alpha_1}\pi^{oc}(\delta) - \frac{t_1 t_2 - \alpha_1 \alpha_2}{t_1 - \alpha_1}. \tag{44}$$

Replacing this expression in (13), we obtain the price on side 2:

$$p_2^{oc}(\delta) = \frac{t_1 t_2 - \alpha_1 \alpha_2}{t_1 - \alpha_1} - \frac{\alpha_1}{t_1 - \alpha_1}\pi^{oc}(\delta), \tag{45}$$

Replacing (44) in the condition for full coverage of side 1, (43), we obtain:

$$p_1^{oc}(\delta) \leq p_1^m \;\Leftrightarrow\; \pi^{oc}(\delta) \leq \frac{2k_1(t_1 - \alpha_1) - (t_1 - \alpha_1)^2 + 2(t_1 t_2 - \alpha_1 \alpha_2)}{4t_1} \equiv \tilde{\pi}^{om}. \tag{46}$$

Let $\tilde{\delta}^{om}$ be the value for the discount factor for which the most sustainable collusive profit coincides with $\tilde{\pi}^{om}$, i.e., $\pi^{oc}(\tilde{\delta}^{om}) = \tilde{\pi}^{om}$. If $\delta \leq \tilde{\delta}^{om}$, the prices are given by (44) and (45). If $\delta > \tilde{\delta}^{om}$:

$$p_1^{oc}(\delta) = p_1^m \quad \text{and} \quad p_2^{oc}(\delta) = \frac{t_1 t_2 - \alpha_1 \alpha_2}{t_1} - \frac{\alpha_1}{t_1}p_1^m = \frac{t_1(\alpha_1 + 2t_2) - \alpha_1(\alpha_1 + 2\alpha_2 + 2k_1)}{2t_1}.$$

2. $\alpha_1 > t_1$

In this case, π^{oc} given in (42) is decreasing in p_1^{oc}. Thus, $p_1^{oc} \leq p_1^N$. As $p_2^{oc} = g(p_1^{oc}, \alpha_1, \alpha_2) = \frac{t_1 t_2 - \alpha_1 \alpha_2}{t_1} - \frac{\alpha_1}{t_1}p_1^{oc}$ and $p_2^N = g(p_1^N, \alpha_1, \alpha_2)$, we conclude that $p_2^{oc} \geq p_2^N$. Thus, if side 2 is fully covered, the condition for full coverage under Nash competition (Assumption 1) ensures that side 1 is also fully covered under one-sided collusion (as $p_1^{oc} \leq p_1^N$). As platforms set symmetric prices, side 2 is fully covered if and only if:

$$u_2^i\left(\frac{1}{2}, p_1^{oc}, p_2^{oc}, p_1^{oc}, p_2^{oc}\right) \geq 0 \;\Leftrightarrow\; \tilde{u}_2^N + \frac{\alpha_1}{t_1}(p_1^{oc} - p_1^N) \geq 0 \;\Leftrightarrow\; p_1^{oc} \geq p_1^N - \frac{t_1}{\alpha_1}\tilde{u}_2^N \equiv \hat{p}_1, \tag{47}$$

where:

$$\tilde{u}_2^N = k_2 - \frac{3t_2 - 2\alpha_1 - \alpha_2}{2} \tag{48}$$

is the utility of the indifferent consumer on side 2 under Nash competition. Using (44), we can rewrite the previous inequality as follows:

$$\pi^{oc}(\delta) \leq \frac{(\alpha_1 - t_1)(2k_2 + \alpha_2 - t_2) + 2(t_1 t_2 - \alpha_1 \alpha_2)}{4\alpha_1} \equiv \hat{\pi}^{oc} \tag{49}$$

Let $\hat{\delta}^{om}$ be the value for the discount factor for which the most sustainable collusive profit coincides with $\hat{\pi}^{om}$, i.e., $\pi^{oc}(\hat{\delta}^{om}) = \hat{\pi}^{om}$. If $\delta \leq \hat{\delta}^{om}$, the prices are given by (44) and (45). If $\delta > \hat{\delta}^{om}$:

$$p_1^{oc}(\delta) = p_1^N - \frac{t_1}{\alpha_1}\tilde{u}_2^N \quad \text{and} \quad p_2^{oc}(\delta) = p_2^m.$$

3. $\alpha_1 < 0$

In this case, π^{oc} given in (42) is increasing in p_1^{oc}. Thus, $p_1^{oc} > p_1^N$ for $\delta > 0$. From Lemma 6, we also conclude that $p_2^{oc} > p_2^N$.

The expressions for prices, (44) and (45), are valid as long as consumers located at $x = \frac{1}{2}$ on each market side get positive utility. Thus, side 2 is fully covered if and only if:

$$u_2^i\left(\frac{1}{2}, p_1^{oc}, p_2^{oc}, p_1^{oc}, p_2^{oc}\right) \geq 0 \;\Leftrightarrow\; p_1^{oc}(\delta) \leq \hat{p}_1,$$

where $\hat{p}_1$ is given in (47). Thus, side 1 is fully covered if $p_1^{oc} \leq p_1^m$, given in (6), and side 2 is fully covered if $p_1^{oc} \leq \hat{p}_1$. Furthermore:

$$p_1^m < \hat{p}_1 \;\Leftrightarrow\; \alpha_1 > -\frac{\tilde{u}_2^N}{\tilde{u}_1^N} t_1.$$

Hence:

- If $-\frac{\tilde{u}_2^N}{\tilde{u}_1^N} t_1 \leq \alpha_1 < 0$, given δ, the most collusive prices are given by (16).
- If $\alpha_1 < -\frac{\tilde{u}_2^N}{\tilde{u}_1^N} t_1$, given δ, the most collusive prices are given by (17).

□

Proof of Lemma 8.

Looking at the expression for total welfare when platforms set symmetric prices, (24), it follows that it is quadratic in n_1^b and globally concave. In addition, total welfare would be maximized at:

$$\frac{dW^b}{dn_1^b} = 0 \;\Leftrightarrow\; n_1^{*b} = \frac{2k_1 + \alpha_1 + \alpha_2}{2t_1}.$$

As, under Assumption 4ii, $n_1^{*b} > 1$, it follows that W^b is strictly increasing $\forall n_1^b \in (0,1)$.
□

Proof of Proposition 5.

If platforms set symmetric prices, $p_1^A = p_1^B = p_1^{cb}$ and $p_2^A = p_2^B = p_2^{cb}$, their individual profit is:

$$\pi^{cb}(p_1^{cb}, p_2^{cb}) = \frac{\alpha_1 + 2k_1}{2t_1}p_1^{cb} - \frac{(p_1^{cb})^2}{t_1} + \frac{p_2^{cb}}{2}.$$

Suppose, without loss of generality, that platform A unilaterally deviates from the collusive agreement by choosing prices p_1 and p_2 that maximize its individual profit (while platform B is setting prices p_1^{cb} and p_2^{cb}). Then, its profit is:

$$\begin{aligned}\pi^A(p_1, p_2; p_1^{cb}, p_2^{cb}) =& \frac{1}{8t_1(t_1t_2 - \alpha_1\alpha_2)}\Big\{ - 4(2t_1t_2 - \alpha_1\alpha_2){p_1}^2 \\ &+ \left[-\alpha_1\alpha_2^2 + (\alpha_1 + 2k_1)(4t_1t_2 - 3\alpha_1\alpha_2) - 4t_1(\alpha_1 + \alpha_2)p_2 + 4\alpha_1 t_1 p_2^{cb}\right] p_1 \\ &- t_1\left[4t_1p_2 - 4t_1(p_2^{cb} + t_2) - \alpha_2(2k_1 - 3\alpha_1 - \alpha_2)\right] p_2\Big\}.\end{aligned} \tag{50}$$

Solving the corresponding FOCs, we obtain the following deviation prices (for given collusive prices):

$$\begin{aligned}p_1^{db}(p_1^{cb}, p_2^{cb}) =& \frac{\alpha_2(\alpha_1 - \alpha_2)}{\Omega}p_1^{cb} + \frac{t_1(\alpha_1 - \alpha_2)}{\Omega}p_2^{cb} + \frac{(4k_1 + \alpha_1 - \alpha_2)(t_1t_2 - \alpha_1\alpha_2)}{\Omega} \\ p_2^{db}(p_1^{cb}, p_2^{cb}) =& \frac{(4t_1t_2 - \alpha_1^2 - 3\alpha_1\alpha_2)}{t_1\Omega}(\alpha_2 p_1^c + t_1 p_2^c) \\ &+ \frac{(t_1t_2 - \alpha_1\alpha_2)\left[4t_1t_2 - \alpha_1^2 - 3\alpha_1\alpha_2 - 2k_1(\alpha_1 + \alpha_2)\right]}{t_1\Omega}\end{aligned}$$

Replacing these expressions in (50), we obtain the maximum deviation profit (for given p_1^{cb} and p_2^{cb}):

$$\begin{aligned}\pi^{cb}(p_1^{cb}, p_2^{cb}) =& \frac{1}{t_1\Omega}\Big[t_1^2(p_2^c)^2 + t_1(2t_1t_2 + k_1\alpha_1 - k_1\alpha_2 - 2\alpha_1\alpha 2)p_2^c + \alpha_2^2(p_1^c)^2 \\ &+ \alpha_2(2t_1t_2 + k_1\alpha_1 - k_1\alpha_2 - 2\alpha 1\alpha_2)p_1^c + 2\alpha_2 t_1 p_1^c p_2^c \\ &+ (t_1t_2 - \alpha_1\alpha_2)\left(2{k_1}^2 + t_1t_2 + k_1\alpha_1 - k_1\alpha_2 - \alpha_1\alpha_2\right)\Big]\end{aligned} \tag{51}$$

It is straightforward to see that Lemma 3 still applies and allows us to determine the price structure under the most profitable sustainable (two-sided) collusive agreement. Thus, p_1^{cb} and p_2^{cb} are those that minimize the deviation profit, π^{db}. Thus, for a given collusive profit

$\pi^{cb}(p_1^{cb}, p_2^{cb}) = \pi$, the collusive prices solve the following constrained minimisation problem:

$$\min_{(p_1^{cb}, p_2^{cb})} \pi^{db}(p_1^{cb}, p_2^{cb}) \quad \text{s.t.} \quad \pi^{cb}(p_1^{cb}, p_2^{cb}) = \pi,$$

whose Lagrangean function is:

$$\mathcal{L}(p_1^{cb}, p_2^{cb}, \lambda) = \pi^{db}(p_1^{cb}, p_2^{cb}) + \lambda \left[\pi^{cb}(p_1^{cb}, p_2^{cb}) - \pi\right].$$

Solving the corresponding FOCs, we obtain 3 candidates to constrained minimum but just the following one satisfies the SOCs:

$$p_1^{cb} = p_1^N, \quad p_2^{cb} = 2\pi - \frac{(2k_1 + \alpha_1 - \alpha_2)(2k_1 + \alpha_1 + \alpha_2)}{8t_1}, \quad \lambda = \frac{1}{2} + \frac{8t_1\pi - 2{k_1}^2}{\Omega}. \tag{52}$$

Replacing these expressions in (51), we obtain the deviation profit, for a given collusive profit π:

$$\pi^{db}(\pi) = \frac{16k_1^4 - 8{k_1}^2(16t_1\pi - \Omega) + (16t_1\pi + \Omega)^2}{64t_1\Omega}.$$

The (collusive) profit $\pi \geq \pi^N$ is sustainable iff the following ICC is satisfied:

$$\frac{\pi}{1-\delta} \geq \pi^{db}(\pi) + \frac{\delta}{1-\delta}\pi^{Nb} \Leftrightarrow$$
$$\frac{\pi - \pi^{Nb}}{4(1-\delta)\Omega}\left[(3\delta+1)\Omega + 4(1-\delta){k_1}^2 - 16t_1(1-\delta)\pi\right] \geq 0 \;\Leftrightarrow\; \pi \leq \frac{1}{4t_1}\left[{k_1}^2 + \frac{1+3\delta}{4(1-\delta)}\Omega\right].$$

Thus, given δ, the most profitable collusive profit is:

$$\pi^{cb}(\delta) = \frac{1}{4t_1}\left[{k_1}^2 + \frac{1+3\delta}{4(1-\delta)}\Omega\right]. \tag{53}$$

Replacing this expression in (53), we obtain the collusive prices:

$$p_1^{cb} = p_1^N \quad \text{and} \quad p_2^{cb}(\delta) = p_2^N + \frac{\Omega}{2t_1}\frac{\delta}{1-\delta} \tag{54}$$

Notice, however, that the above expressions are only valid if the market is fully covered and there is partial multi-homing on side 1. As $p_1^{cb} = p_1^N$ and the number of users on side 2 is the same under collusion and competition, the conditions for market coverage and partial multi-homing on side 1 are the same as under Nash competition (Assumption 4). As a result, we only need to check that side 2 is fully covered, i.e., that the utility of the consumer located at $x = \frac{1}{2}$ is non-negative. Replacing the collusive prices in (1), we get:

$$u_2^i\left(\frac{1}{2}, p_1^{cb}, p_2^{cb}(\delta), p_1^{cb}, p_2^{cb}(\delta)\right) \geq 0$$
$$\Leftrightarrow \ \delta \leq 1 - \frac{2\Omega}{2k_1(\alpha_1+\alpha_2)+4k_2t_1+2(t_1t_2-\alpha_1\alpha_2)+\Omega} \equiv \delta^{mB}.$$

Thus, expressions (53) and (54) are only valid for $\delta \leq \delta^{mB}$. For $\delta^{mb} < \delta < 1$, we have $p_1^{cb} = p_1^N$, $p_2^{cb}(\delta) = p_2^{cb}(\delta^{mb})$ and $\pi^{cb}(\delta) = \pi^{cb}(\delta^{mb})$. □

Proof of Proposition 6.

1. (Perfect collusion) Start by assuming that platforms are sufficiently patient for the ICC in (27) to be not binding. In this case, platforms set the price on side 2 that maximises their joint profit, which we denote by p_2^{m2b}. Replacing $p_1^A = p_1^B = f(p_2)$, given in (31), in the platforms' joint profit, (30), we get:

$$\pi^J(p_2) = \frac{1}{t_1(4t_1t_2-3\alpha_1\alpha_2)^2}\Big\{ -2\alpha_2^2t_1^2p_2^2 - 2t_1\left[\alpha_1\alpha_2^2k_1 - 4(t_1t_2-\alpha_1\alpha_2)(2t_1t_2-\alpha_1\alpha_2)\right]p_2 + (2k_1+\alpha_1)^2(t_1t_2-\alpha_1\alpha_2)(2t_1t_2-\alpha_1\alpha_2)\Big\}. \tag{55}$$

As $\frac{d^2\pi^J}{dp_2^2} = -\frac{4\alpha_2^2t_1}{(4t_1t_2-3\alpha_1\alpha_2)^2} < 0, \forall p2$, it follows that π^J is globally concave. As a result, the maximiser of π^J, p_2^{m2b}, is above the Nash level, p_2^{Nb}, iff:[42]

$$\left.\frac{d\pi^J}{dp_2}\right|_{p_2=p_2^{Nb}} > 0 \Leftrightarrow \ \Gamma > 0. \tag{56}$$

In the LHS of the last inequality we have a second-order polynomial in α_2 whose roots are:

$$\bar{\alpha}_2 = \frac{1}{2}\left(-3\alpha_1 + \sqrt{9\alpha_1^2+16t_1t_2}\right) \quad \text{and} \quad \underline{\alpha}_2 = \frac{1}{2}\left(-3\alpha_1 - \sqrt{9\alpha_1^2+16t_1t_2}\right). \tag{57}$$

It is straightforward to show that $\underline{\alpha}_2 < 0 < \bar{\alpha}_2$. Thus:

$$p_2^{m2b} > p_2^{Nb} \Leftrightarrow \alpha_2 \in]\underline{\alpha}_2, \bar{\alpha}_2[. \tag{58}$$

Combining this with (29), we get the comparison between the price on side 1 under collusion and competition.

[42] Please note that the parameter $\Gamma = 4t_1t_2 - 3\alpha_1\alpha_2 - \alpha_2^2$ was defined in section 3.2.

Let us determine the critical discount factor, i.e., the value for δ above which the ICC is not binding. Solving the FOC corresponding to the maximization of (55), we get:

$$p_2^{m2b} = -\frac{\alpha_1(k_1 - 4\alpha_1)}{2t_1} + \frac{4t_1t_2^2}{\alpha_2^2} - \frac{6\alpha_1 t_2}{\alpha_2}. \tag{59}$$

Replacing in (55), we get the individual (perfect) collusive profit:

$$\pi^{m2b} = \pi^{Nb} + \frac{\Gamma^2}{16\alpha_2^2 t_1} \tag{60}$$

where π^{Nb} is given in Proposition 4 and Γ is defined in section 3.2. Solving the FOCs corresponding to the individual profit maximization if the rival firm abides by the collusive agreement, we can obtain the unilateral deviation profit:

$$\pi^{dm2b} = \pi_2^{m2b} + \frac{(t_1t_2 - \alpha_1\alpha_2)(2t_1t_2 - \alpha_1\alpha_2)(4t_1t_2 - \alpha_2(3\alpha_1 + \alpha_2))^2}{2\alpha_2^4 t_1 \Omega}.$$

Thus, the critical discount factor is:

$$\delta^{m2b} = \frac{\pi^{dm2b} - \pi^{m2b}}{\pi^{dm2b} - \pi^{Nb}} = 1 - \frac{\alpha_2^2\Omega}{\alpha_2^2\Omega + 8(t_1t_2 - \alpha_1\alpha_2)(2t_1t_2 - \alpha_1\alpha_2)}. \tag{61}$$

2. (Imperfect collusion) Consider now that platforms are little patient and, therefore the ICC in (27) binds. As a result, that the collusive price on side 2, p_2^{c2b}, is the solution of:

$$\frac{\pi^J(p_2)}{2} - (1-\delta)\pi^{d2b}(p_2) - \delta\Pi^N = 0, \tag{62}$$

where π^J is given by (55) and $\pi^{d2b}(p_2)$ is the maximum profit that a platform can obtain by unilaterally deviating from the agreement, while the rival sets prices $p_1 = f(p_2)$ and p_2. To compute $\pi^{d2b}(p_2)$, suppose, without loss of generality, that platform A deviates from the collusive agreement, while platform B sets the collusive price on side 2, p_2^{c2b}, and the corresponding competitive price on side 1, $p_1^{c2b} = f(p_2^{c2b})$. More precisely, platform A fix prices p_1^{d2b} and p_2^{d2b} that maximize its individual profit:

$$\max_{p_1, p_2} \ \pi^{Ab}\left(p_1, p_2, f(p_2^{c2b}), p_2^{c2b}\right).$$

Solving the corresponding FOCs, we obtain:

$$p_1^{d2b}(p_2^{c2b}) = \frac{1}{\Omega(4t_1t_2 - 3\alpha_1\alpha_2)}\Big\{2k_1(t_1t_2 - \alpha_1\alpha_2)\left(8t_1t_2 - 5\alpha_1\alpha_2 - \alpha_2^2\right)$$
$$+ 2(\alpha_1 - \alpha_2)(t_1t_2 - \alpha_1\alpha_2)(2t_1t_2 - \alpha_1\alpha_2) + t_1(\alpha_1 - \alpha_2)\left(4t_1t_2 - 3\alpha_1\alpha_2 - \alpha_2^2\right)p_2^{cb2}\Big\}$$

and

$$p_2^{d2b}(p_2^{c2b}) = \frac{1}{\Omega(4t_1t_2 - 3\alpha_1\alpha_2)t_1}\Big\{2(t_1t_2 - \alpha_1\alpha_2)(2t_1t_2 - \alpha_1\alpha_2)\left(4t_1t_2 - \alpha_1^2 - 3\alpha_1\alpha_2 - 2\alpha_1k_1\right)$$
$$+ t_1\left(4t_1t_2 - 3\alpha_1\alpha_2 - \alpha_1^2\right)\left(4t_1t_2 - 3\alpha_1\alpha_2 - \alpha_2^2\right)p_2^{c2b}\Big\}.$$

Replacing these expressions in $\pi^{Ab}\left(p_1, p_2, f(p_2^{c2b}), p_2^{c2b}\right)$, we get $\pi^{d2b}(p_2)$. After plugging this expression in (62), we obtain an equation in p_2, whose solution (besides $p_2 = p_2^{Nb}$) is:

$$p_2^{c2b}(\delta) = \frac{1}{4t_1\left(\Lambda - \Gamma^2\delta\right)}\Big\{\Lambda\left[4t_1t_2 - \alpha_1(\alpha_1 + 3\alpha_2 + 2k_1)\right]$$
$$+ \Gamma\Big[2\alpha_1k_1\Gamma + \alpha_1\alpha_2\left(3\alpha_1^2 + 26\alpha_1\alpha_2 + 3\alpha_2^2\right)$$
$$+ 4t_1t_2\left(12t_1t_2 - 18\alpha_1\alpha_2 - \alpha_1^2 - \alpha_2^2\right)\Big]\delta\Big\} \tag{63}$$

where $\Gamma \equiv 4t_1t_2 - 3\alpha_1\alpha_2 - \alpha_2^2$ and $\Lambda \equiv 8(t_1t_2 - \alpha_1\alpha_2)(2t_1t_2 - \alpha_1\alpha_2) > 0$. Differentiating this expression with respect to δ, we obtain:[43]

$$\frac{dp_2^{c2b}}{d\delta} > 0 \Leftrightarrow \frac{\Gamma\Lambda\Omega\left(\Gamma + \alpha_2^2\right)}{2t_1\left(\Lambda - \Gamma^2\delta\right)^2} > 0 \Leftrightarrow \Gamma > 0,$$

which is exactly the same condition as the one we obtained for perfect collusion, (56). As $p_2^{c2b}(0) = p_2^{Nb}$, platforms will set a supra-competitive price on the collusive side (side 2) iff $\Gamma > 0$, which, as seen above, is true iff $\alpha_2 \in]\underline{\alpha}_2, \bar{\alpha}_2[$. The comparison for p_1 follows combining this result with (29).

□

Proof of Proposition 7.

If $\alpha_2 \in (\underline{\alpha}_2, \bar{\alpha}_2)$, collusion on the single-homing side damages aggregate consumer surplus, $CS^b = CS_1^b + CS_2^b$, as consumer surplus on both sides of the market decreases.

Consider now that $\alpha_2 \notin (\underline{\alpha}_2, \bar{\alpha}_2)$. Let us focus on the scenarios where platforms set

[43] Notice that $\Gamma + \alpha_2^2 = 3(t_1t_2 - \alpha_1\alpha_2) + t_1t_2 > 0$.

symmetric prices ($p_j^A = p_j^B = p_j$). Replacing expression (19) in (22) and (23) and adding up the obtained expressions, we get:

$$CS^b(p_1, p_2) = \frac{(\alpha_1 + 2k_1 - 2p_1)(\alpha_1 + 2\alpha_2 + 2k_1 - 2p_1) + t_1(4k_2 - 4p_2 - t_2)}{4t_1}$$

For price combinations that satisfy the FOCs on the competitive side, (31), we get:

$$\tilde{CS}^b(p_2) = \frac{[(\alpha_1 + 2k_1)(2t_1t_2 - \alpha_1\alpha_2) + 2\alpha_2 t_1 p_2]\left[(\alpha_1 + 4\alpha_2 + 2k_1)(2t_1t_2 - \alpha_1\alpha_2) - 2\alpha_1\alpha_2^2 + 2\alpha_2 t_1 p_2\right]}{4t_1(4t_1t_2 - 3\alpha_1\alpha_2)^2} + k_2 - p_2 - \frac{t_2}{4}$$

which is a globally concave function in p_2, as it is a quadratic function with coefficient $\frac{\alpha_2^2 t_1}{(4t_1t_2 - 3\alpha_1\alpha_2)^2} > 0$ in p_2^2. Differentiating $\widetilde{CS}^b$ with respect to p_2, we obtain:

$$\frac{\partial \widetilde{CS}^b}{\partial p_2} = \frac{2\alpha_2 t_1 t_2(13\alpha_1 + 2\alpha_2 + 2k_1) - \alpha_1\alpha_2^2(10\alpha_1 + 3\alpha_2 + 2k_1) - 16t_1^2t_2^2 + 2\alpha_2^2 p_2 t_1}{(4t_1t_2 - 3\alpha_1\alpha_2)^2}$$

Evaluating this derivative at the Nash price, given in Lemma 1, we obtain:

$$\left.\frac{\partial \widetilde{CS}^b}{\partial p_2}\right|_{p_2 = p_2^{Nb}} < 0 \;\Leftrightarrow\; 2\alpha_2 k_1 < (\alpha_1 + \alpha_2)(\alpha_1 - 2\alpha_2) + \Omega. \tag{64}$$

- If $\alpha_2 < \underline{\alpha}_2 < 0$, this condition is trivially satisfied (recall that, by assumption 4, $\Omega > 0$). Thus, p_2^{Nb} is at the decreasing branch of $\widetilde{CS}^b$. As, from Proposition 6, $p_2^{c2b} < p_2^{Nb}$, we conclude that aggregate surplus is greater when platforms only collude on single-homing side than under Nash competition.

- If $\alpha_2 > \bar{\alpha}_2$, we can rewrite (64) as follows:

$$\left.\frac{\partial \widetilde{CS}^b}{\partial p_2}\right|_{p_2 = p_2^{Nb}} < 0 \;\Leftrightarrow\; 2k_1 < \frac{8t_1t_2 - 7\alpha_1\alpha_2 - 3\alpha_2^2}{\alpha_2}.$$

  Let us show that, if $\alpha_2 > \bar{\alpha}_2$, the expression in the RHS is negative and, therefore, the inequality is never satisfied. As $\alpha_2 > 0$, this turns out to prove that $f(\alpha_2) = 8t_1t_2 - 7\alpha_1\alpha_2 - 3\alpha_2^2 < 0$:

$$\alpha_2 < \frac{-\sqrt{49\alpha_1^2 + 96t_1t_2} - 7\alpha_1}{6} \equiv \underset{\sim}{\alpha}_2 \;\vee\; \alpha_2 > \frac{\sqrt{49\alpha_1^2 + 96t_1t_2} - 7\alpha_1}{6} \equiv \tilde{\alpha}_2$$

  As f is concave downward, $\underset{\sim}{\alpha}_2 < 0 < \tilde{\alpha}_2$, $\bar{\alpha}_2 > 0$, $f(\bar{\alpha}_2) < 0$, we conclude that

$f(\alpha_2) < 0,\ \forall \alpha_2 > \bar{\alpha}_2$ and, consequently, $\left.\frac{\partial \widetilde{CS}^b}{\partial p_2}\right|_{p_2 = p_2^{Nb}} > 0$.

Let $\underline{p_2}$ be the (global) minimum of $\widetilde{CS}^b$. We have shown that $p_2^{Nb} > \underline{p_2}$ and know that $p_2^{c2b}(\delta) < p_2^{Nb}$ (Proposition 6). Let us now compare $p_2^{c2b}(\delta)$ to $\underline{p_2}$. As, for $\alpha_2 > \bar{\alpha}_2$, $p_2^{c2b}(\delta)$ is decreasing in δ (see the proof of Proposition 6), it suffices to compare $p_2^{m2b} = p_2^{c2b}(\delta^{m2b})$, where δ^{m2b} is given in (61) and $\underline{p_2}$:

$$p_2^{m2b} > \underline{p_2} \ \Leftrightarrow\ k_1 > \frac{2t_1t_2 - 2\alpha_1\alpha_2 - \alpha_2^2}{\alpha_2}.$$

Simple algebra allows us to show that Assumption 4ii ensures that this condition is satisfied (for $\alpha_2 > \bar{\alpha}_2$).

As a result, $p_2^{c2b}(\delta)$ is at the increasing branch of $\widetilde{CS}^b$. Combining this with the fact that $p_2^{c2b}(\delta) < p_2^{Nb}$, we conclude that total consumer surplus is lower when platforms collude over the price to charge on the single-homing side than under Nash competition, if $\alpha_2 > \bar{\alpha}_2$.

$\square$

Proof of Proposition 8.

From (31), if platforms set price p_1 on side 1, the price on side 2 is $p_2 = h(p_1)$. Replacing $p_2 = h(p_1)$ in (21), we find that, for a given p_1, the platforms' individual profit is:

$$\pi^{ibc1}(p_1) = \pi^{ib}\left(p_1, h(p_1), p_1, h(p_1)\right) = \frac{1}{2t_1}\left(t_1t_2 - \alpha_1\alpha_2 + 2k_1p_1 - 2p_1^2\right). \tag{65}$$

Suppose, without loss of generality, that platform A unilaterally deviates from the collusive agreement, by choosing prices, $\tilde{p}_1$ and $\tilde{p}_2$, that maximize its individual profit while the rival is charging p_1 and $p_2 = h(p_1)$:

$$\max_{\tilde{p}_1, \tilde{p}_2} \ \pi^{Ab}\left(\tilde{p}_1, \tilde{p}_2, p_1, h(p_1)\right).$$

Solving the corresponding FOCs, we get the deviation prices (for a given p_1):[44]

$$p_1^{dc1b}(p_1) = \frac{2(t_1t_2 - \alpha_1\alpha_2)(2k_1 + \alpha_1 - \alpha_2)}{\Omega} - \frac{(\alpha_1 - \alpha_2)^2}{\Omega} p_1$$

$$p_2^{dc1b}(p_1) = \frac{(t_1t_2 - \alpha_1\alpha_2)\left[\Omega - (\alpha_1 + \alpha_2)(2k_1 + \alpha_1 - \alpha_2)\right]}{t_1\Omega} - \frac{(\alpha_1 - \alpha_2)\left[\Omega - (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)\right]}{2t_1\Omega} p_1,$$

Replacing these prices in $\pi^{Ab}(\tilde{p}_1, \tilde{p}_2, p_1, h(p_1))$, we get the unilateral deviation profit (for a given p_1):

$$\pi^{dc1b}(p_1) = \frac{1}{t_1\Omega}\Big\{2(t_1t_2 - \alpha_1\alpha_2)\left[{k_1}^2 + k_1(\alpha_1 - \alpha_2) + 2(t_1t_2 - \alpha_1\alpha_2)\right] \\ - (\alpha_1 - \alpha_2)\left[k_1(\alpha_1 - \alpha_2) + 4(t_1t_2 - \alpha_1\alpha_2)\right] p_1 + (\alpha_1 - \alpha_2)^2 p_1^2\Big\}. \quad (66)$$

1. (Perfect collusion) Consider first the case wherein platforms' discount is so high that the ICC is not binding. Then, platforms set p_1 that maximises their joint profit (antecipating that the price on side 2 will be $p_2 = h(p_1)$). More precisely, platforms will choose p_1 that maximises (65). Solving the corresponding FOCs, we obtain:

$$p_1^{m1b} = p_1^{Nb} - \frac{\alpha_1 - \alpha_2}{4}.$$

As a result:

$$p_1^{m1b} > p_1^{Nb} \;\Leftrightarrow\; \alpha_1 < \alpha_2. \quad (67)$$

and:

$$p_2^{m1b} = h(p_1^{m1b}) = p_2^{Nb} + \frac{\alpha_1(\alpha_1 - \alpha_2)}{4t_1}.$$

Hence:

$$p_2^{m1b} > p_2^{Nb} \;\Leftrightarrow\; (\alpha_1 > 0 \;\wedge\; \alpha_1 > \alpha_2) \vee (\alpha_1 < 0 \;\wedge\; \alpha_1 < \alpha_2) \quad (68)$$

Replacing $p_1 = p_1^{m1b}$ in (65), we obtain the perfect collusive profit:

$$\pi^{m1b} = \pi^{Nb} + \frac{(\alpha_1 - \alpha_2)^2}{16t_1},$$

which, as expected, always exceeds the two-sided competition profit.
To get the expression for the critical discount factor, δ^{m1b}, it is only missing to derive the

[44] The expression for Ω in given by (18).

expression for the deviation profit. Replacing $p_1 = p_1^{m1b}$ in (66), we get:

$$\pi^{dm1b} = \frac{k_1^2\Omega + 16(t_1t_2 - \alpha_1\alpha_2)^2}{4t_1\Omega}$$

Thus, the critical discount factor is:

$$\delta^{m1b} = \frac{\pi^{dm1b} - \pi^{m1b}}{\pi^{dm1b} - \pi^{Nb}} = 1 - \frac{\Omega}{8(t_1t_2 - \alpha_1\alpha_2) + \Omega}.$$

2. (Imperfect collusion) Consider now that platforms are not sufficiently patient so that the ICC is binding, i.e., $\delta < \delta^{m1b}$. For a given p_1, the expressions for profits under collusion, deviation and competition are respectively given in (65), (66) and Proposition 4. Replacing them in the ICC (in equality) and solving it with respect to p_1, we obtain (beyond the trivial solution $p_1 = p_1^{Nb}$):

$$p_1^{c1b}(\delta) = p_1^{Nb} - \frac{(\alpha_1 - \alpha_2)\Omega\delta}{2\left[(1-\delta)(\alpha_1 - \alpha_2)^2 + \Omega\right]} \tag{69}$$

The corresponding price on side 2 is:

$$p_2^{c1b}(\delta) = h(p_1^{c1b}(\delta)) = p_2^{Nb} + \frac{\alpha_1(\alpha_1 - \alpha_2)\Omega\delta}{2t_1\left[(1-\delta)(\alpha_1 - \alpha_2)^2 + \Omega\right]}. \tag{70}$$

It follows immediately that the conditions to have supra-competitive prices on each side of the market are exactly the same as those obtained for the case of perfect collusion, (67) and (68).

Finally, replacing $p_1 = p_1^{c1b}$ in (65), we obtain the most sustainable collusive profit for a given $\delta < \delta^{m1b}$ when platforms only collude on the multi-homing side:

$$\pi^{c1b}(\delta) = \pi^{Nb} + \frac{2\Omega(\alpha_1 - \alpha_2)^2(t_1t_2 - \alpha_1\alpha_2)\delta(1-\delta)}{t_1\left[(1-\delta)(\alpha_1 - \alpha_2)^2 + \Omega\right]^2}.$$

□

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