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A two-dimensional control problem arising from dynamic contracting theory.*

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Abstract

We study a corporate finance dynamic contracting model in which the firm's growth rate fluctuates and is impacted by the unobservable effort exercised by the manager. We show that the principal's problem takes the form of a two-dimensional Markovian control problem. We prove regularity properties of the value function that are instrumental in the construction of the optimal contract that implements full effort, which we derive explicitly. These regularity results appear in some recent economic studies but with heuristic proofs that do not clarify the importance of the regularity of the value function at the boundaries.

Keywords: Principal-agent problem, two-dimensional control problem, regularity properties.

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1 Introduction

Stochastic control models in economics aim at obtaining qualitative properties of value functions and at deriving optimal control policies in order to analyse various economic questions and to propose quite explicit recommendations. To meet this objective, economists always follow a standard route which consists in building value functions in two steps:

i) derive the associated HJB equation whose solution gives a candidate value function, and, as a by-product, a candidate optimal policy, if any,

ii) apply a verification theorem based on Itô's formula which asserts that a *smooth* solution to the HJB equation coincides with the value function.

The key of this approach is to show that the HJB equation admits a solution that is regular enough to apply the Itô's formula needed in the verification theorem. However, it is well-known that the value function of a stochastic control problem is generally a solution to the associated HJB equation in some weak sense such as in the viscosity sense. For one-dimensional stochastic control problems, a recent literature has given results to check at hand the regularity of value functions. For instance, Strulovici and Szydlowski (2015) avoid the concept of viscosity solutions and use a shooting method to prove regularity results. Pham (2007) shows that the value functions of a class of optimal switching problems are differentiable by means of viscosity solutions. Yet, the concept of viscosity solution does not give a clear set of conditions to derive regularity results for multi-dimensional problems, even if has proved to be very efficient to provide numerical approximations of value functions, which forces to argue case by case.

Motivated by economic relevance, two-dimensional stochastic control problems have emerged recently from dynamic contracting in corporate finance.[1] In a two-dimensional setting, existence of derivatives of value functions, regularity properties and existence of optimal controls can be very challenging. In this paper, we provide a complete solution of a two-dimensional control problem arising from the optimal exit problem under moral hazard. Our mathematical results go beyond our model and complement the heuristic derivation of regularity results made in recent economic studies on dynamic contracting in corporate finance.

Dynamic contracting models in corporate finance are based on the premise that two factors drive the relationship between firm's owner (principal) and firm's manager (agent). First, owners delegate tasks to managers. Second, incentives of managers and those of owners are not fully aligned. Firm's manager may take some actions providing him private benefits and having a negative externality on the firm's cash flows. This impacts firm's owner payoff. Those actions taken by firm's manager are typically unobservable. Hence, the firm's owner problem is to find the best contract that aligns the interest of the firm's manager with her own. Clearly, the mathematical formulation of the problem depends on the modeling of the cash flows. A common assumption is to model cash flows generated by the firm as the increment of an arithmetic Brownian motion

$$dY_t \,=\, \mu\, dt + \sigma\, dZ_t, \tag{1}$$

[1] See e.g. Williams (2009), Strulovici (2011), Faingold and Vasama (2014), De Marzo and Sannikov (2017), Vasama (2017).

where Z is a standard Brownian motion. The process Y represents the cumulative cash flows, its increment dY_t the cash flows over a period $[t, t+dt)$, and its drift μ the firm's growth rate.[2] In this environment, shocks on cash flows are identically, independently distributed. As pointed out in the literature[3], dynamics (1) is merely for the sake of tractability and does not account for elementary stylized facts. For instance, cash flows are usually serially correlated over time. Also, cumulated cash flows up to any given time corresponds to the difference between two increasing processes, cash inflows and cash outflows, and, as such should be modeled as a finite variation process, clearly a property that does not satisfy dynamics (1). The attraction of modeling cumulative cash flows as an arithmetic Brownian motion comes from the simple form taken by incentive compatibility conditions. The principal problem reduces then to a tractable one-dimensional Markov control problem whose value function has well established regularity properties.[4]

In this paper we consider a setting in which, the firm's growth rate fluctuates and follows a Brownian motion with volatility σ. Specifically, the cumulative cash flows process Y follows the dynamics

$$dY_t = X_t\, dt,$$

where $X_t = x+\sigma Z_t$. In this setting cash flows are serially correlated over time and cumulative cash flows have finite variations. In sharp contrast with the environment defined by (1), the principal is concerned with the random growth rate of the firm (that may induce him to liquidate it for pure profitability reasons) and by the agent's actions. These two concerns are very much interconnected. We show that the principal's problem takes the form of a Markovian two-dimensional control problem with state variables, the so-called continuation value of the agent, and the level of the firm's growth rate or profitability. Following the literature, we solve the firm's owner problem in the set of contracts that induce the manager to exert full effort all the time. We establish all the required regularity properties of the associated value function. We point out at each step of our analysis the novelty of our results and explain how they complement recent studies. Notably, we clarify the importance of the regularity of the value function at the boundaries in solving the two-dimensional control problem.

The outline of the paper is as follows. Section 2 develops the mathematical model and writes the principal's control problem. Section 3 derives the incentive compatibility conditions and the Markovian representation of the principal's problem. Section 4 contains our main results. We derive regularity properties of the value function of the principal's problem and characterize the optimal contract in the class of contracts inducing effort at any time. Section 5 discusses our results and presents open questions for future research.

2 The model

Principal and agent. We consider a firm that hires a manager to operate a project. The firm's owner, or the principal, has access to unlimited funds and the manager, or agent, is protected by limited liability. The agent and the principal both agree on the same discount

[2] See the seminal paper De Marzo and Sannikov (2006).

[3] See for instance, Zhang (2009).

[4] See e.g. Sannikov (2008), De Marzo and Sannikov (2006) and Strulovici and Szydlowski (2015).

rate r. We assume that, at any time t, the project produces observable cash flows if and only if the manager is in charge. In particular, the project is abandoned when the manager is fired and we assume without loss of generality that its scrap value is zero. The cumulative cash flows process $(Y_t)_{t\geq 0}$ and the growth rate process $(X_t)_{t\geq 0}$ evolve as

$$dY_t = X_t\,dt \text{ and } dX_t = -\delta a_t\,dt + \sigma dZ_t^a, \quad X_0 = x \tag{2}$$

where δ and σ are positive constants, Z_t^a is a Brownian motion, and $a_t \in [0,1]$ is the agent's unobservable action. The unobservable action $a_t = 0$ is called the effort action, the unobservable action $a_t > 0$ is called the shirking action. Thus, shirking has a negative effect $-\delta a_t$ on the growth rate. Whenever the agent shirks, he receives a private benefit $Ba_t\,dt$ where B is a positive constant.

Probabilistic model. Formally, we consider the probability space $\Omega = \mathcal{C}([0,\infty),\mathbb{R})$, the set of continuous real functions on $[0,+\infty)$ endowed with the Wiener measure denoted by $\mathbb{P}^0$. Let $Z = (Z_t)_{t\geq 0}$ be a Brownian motion under $(\mathbb{P}^0, \mathcal{F}_t)$ where $\mathcal{F}_t$ is the completion of the natural filtration generated by Z. Under $\mathbb{P}^0$, we assume that the project's growth rate evolves as

$$dX_t = \sigma dZ_t.$$

Thus, $\mathbb{P}^0$ corresponds to the probability distribution of the growth rate when the agent chooses at any time the effort action. For any action process $a = (a_t)_{t\geq 0}$ which is assumed to be a $\mathcal{F}_t$ adapted process with values in $[0,1]$, we define

$$\gamma_t^a = \exp\left[\int_0^t -\left(\frac{\delta a_s}{\sigma}\right) dZ_s - \frac{1}{2}\int_0^t \left(\frac{\delta a_s}{\sigma}\right)^2 ds\right].$$

Because the action process a is bounded, the process $(\gamma_t^a)_{t\geq 0}$ is an $\mathcal{F}_t$- martingale. We then define a probability $\mathbb{P}^a$ on Ω such that

$$\frac{d\mathbb{P}^a}{d\mathbb{P}^0}|\mathcal{F}_t = \gamma_t^a.$$

The process $(Z_t^a)_{t\geq 0}$ with

$$Z_t^a = Z_t + \int_0^t \left(\frac{\delta a_s}{\sigma}\right) ds$$

is a Brownian motion under $\mathbb{P}^a$. Therefore, any action process a induces a probability measure $\mathbb{P}^a$ on Ω for which the dynamics of cash flows is given by Equation (2).

Problem formulation. Following the literature (see for instance Zhu (2012)), a contract is a triplet (C, τ_L, a) that specifies nonnegative transfers $C = (C_t)_{t\geq 0}$ (remuneration) from the principal to the agent, a stopping time τ_L at which the project is liquidated and an action process a that the principal recommends to the agent. The process C is $\mathcal{F}^X$-adapted, nondecreasing (reflecting agent's limited liability), τ_L is an $\mathcal{F}^X$-stopping time, and, for any action process a, we assume

$$\mathbb{E}^a\left(\int_0^{\tau_L} e^{-rs} dC_s\right) < +\infty.$$

Throughout the paper $\mathcal{F}^X$ denotes the $\mathbb{P}^a$-augmentation of the filtration generated by $(X_t)_{t\geq 0}$ and $\mathcal{T}^X$ the set of $\mathcal{F}^X$-stopping times.

For a fixed contract $\Gamma = (C, \tau_L, a)$. The agent's expected profit and the principal's expected profit associated to Γ are respectively,

$$V_A(\Gamma) = \mathbb{E}^a \left(\int_0^{\tau_L} e^{-rt}(Ba_t\, dt + dC_t) \right),$$

and

$$V_P(\Gamma) = \mathbb{E}^a \left(\int_0^{\tau_L} e^{-rt}(X_t\, dt - dC_t) \right).$$

An *incentive-compatible* action process $a^*(C, \tau_L) = (a_t^*(C, \tau_L))_{t\geq 0}$ is an agent best reply in term of effort to a given remuneration and liquidation policy (C, τ_L). That is, for any action process a, the action process $a^*(C, \tau_L)$ satisfies

$$\mathbb{E}^a \left(\int_0^{\tau_L} e^{-rt}(Ba_t\, dt + dC_t) \right) \leq \mathbb{E}^{a^*(C,\tau_L)} \left(\int_0^{\tau_L} e^{-rt}(Ba_t^*(C, \tau_L)\, dt + dC_t) \right).$$

We say that a contract (C, τ_L, a) is incentive compatible or (C, τ_L) induces an effort strategy $a^*(C, \tau_L)$ if $a = a^*(C, \tau_L)$. An optimal contract is an incentive compatible contract that maximizes the expected principal's profit at date 0 subject to delivering to the agent a payoff larger than her reservation utility $w_0 > 0$. The principal problem is then to find, if any, an optimal contract. Formally, the principal studies the problem

$$\sup_{C,\tau_L} \mathbb{E}^{a^*(C,\tau_L)} \left(\int_0^{\tau_L} e^{-rt}(X_t\, dt - dC_t) \right) \tag{3}$$

$$\text{s.t } \mathbb{E}^{a^*(C,\tau_L)} \left(\int_0^{\tau_L} e^{-rt}(Ba_t^*(C, \tau_L)\, dt + dC_t) \right) \geq w_0. \tag{4}$$

We refer inequality (4) to the agent's participation constraint.

3 Incentive compatibility and Markov formulation

This section develops in our setting a result due to Sannikov (2008) and generalized by Cvitanic, Possamaï and Touzi (2016) and (2017): the continuation value to the agent (defined below) characterizes the incentive compatible actions and allows for a Markov formulation of the principal's problem (3)-(4).

Fix a contract Γ and assume for a while that a is incentive compatible in order to have the same set of information for both players, namely $\mathcal{F}_t^X = \mathcal{F}_t$. Let us define $W^\Gamma = (W_t^\Gamma)_{t\geq 0}$ with

$$W_t^\Gamma = \mathbb{E}^a \left(\int_t^{\tau_L} e^{-r(s-t)}(Ba_s ds + dC_s) \,|\, \mathcal{F}_t^X \right).$$

The process W^Γ corresponds to the agent's continuation value process associated to contract Γ. Because C is an increasing process, $W_t^\Gamma \geq 0$ for all $t \leq \tau_L$ with $W_{\tau_L}^\Gamma = 0$ by construction. Moreover, if one of the two processes $(a_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ is nonzero then, τ_L is the first hitting time of 0 for the agent's continuation value. The following holds.

Lemma 1 *The continuation value process W^Γ satisfies under $\mathbb{P}^a$ the dynamics*

$$dW_t^\Gamma = (rW_t^\Gamma - Ba_t)\,dt + \beta_t(\Gamma)\,dZ_t^a - dC_t \quad \textit{for} \quad t \le \tau_L, \tag{5}$$

where the process $\beta(\Gamma) = (\beta_t(\Gamma))_{t\ge 0}$ is $\mathcal{F}^X$ predictable and uniquely defined. It is called hereafter the sensitivity process.

Proof of Lemma 1. By assumption (2), the process

$$Y_t = e^{-rt}W_t^\Gamma + \int_0^t e^{-rs}(Ba_s ds + dC_s) = \mathbb{E}^a\left(\int_0^{\tau_L} e^{-rs}(Ba_s ds + dC_s)|\mathcal{F}_t^X\right)$$

is a uniformly integrable martingale under $\mathbb{P}^a$. By the martingale Representation theorem, there exists a unique $\mathcal{F}_t^X$ predictable process $\beta(\Gamma)$ such that

$$Y_t = Y_0 + \int_0^t e^{-rs}\beta_s(\Gamma)\,dZ_s^a,$$

with

$$\mathbb{E}^a\left(\int_0^{\tau_L} e^{-2rs}\beta_s(\Gamma)^2\,ds\right) < +\infty.$$

Then, Itô's formula, yields (5). □

Thus, a contract $\Gamma = (C, \tau_L, a)$ defines a unique sensitivity process $\beta(\Gamma) = (\beta_t(\Gamma))_{t\ge 0}$ by the representation theorem for Brownian martingale that yields (5), the dynamics of the continuation value process under $\mathbb{P}^a$. We could interpret Lemma 1 in the framework of BSDE as follows: for any given incentive compatible contract $\Gamma = (C, \tau_L, a)$, there exists an unique pair of $\mathcal{F}_t^X$ adapted process $(W_t(\Gamma), \beta_t(\Gamma))$ such that

$$\begin{cases} W_{\tau_L}^\Gamma &= 0, \\ dW_t^\Gamma &= (rW_t^\Gamma - Ba_t)\,dt + \beta_t(\Gamma)\,dZ_t^a - dC_t. \end{cases}$$

However, the question of characterizing incentive-compatible contracts that satisfy the agent's participation constraint (4) remains unanswered: we have to characterize the set $\Gamma(w_0)$ of contracts Γ for which W_0^Γ is greater than w_0.

To solve this problem, the idea of Sannikov (2008) is to see the sensitivity process $\beta(\Gamma)$ as a control. To this end, let us consider the class of $\mathcal{F}^X$ measurable processes $\beta = (\beta_t)_{t\ge 0}$ such that

$$\mathbb{E}^a\left(\int_0^\infty e^{-2rs}\beta_s^2\,ds\right) < +\infty, \tag{6}$$

and, for any fixed increasing process $(C_t)_{t\ge 0}$, let us consider the process $W^\beta = (W_t^\beta)_{t\ge 0}$ that satisfies the controlled stochastic differential equation under $\mathbb{P}^a$,

$$dW_t^\beta = (rW_t^\beta - Ba_t)\,dt + \beta_t\,dZ_t^a - dC_t \;\text{ with }\; W_0^\beta \ge w_0.$$

We would like the process $(W_t^\beta)_{t\le\tau_L}$ to play the role of the agent continuation value associated to some contract $\Gamma \in \Gamma(w_0)$. This requires $W_t^\beta \ge 0$ up to the termination date of the contract Γ. Therefore, we introduce

$$\tau_0^\beta(C) = \inf\{t \ge 0\,,\, W_t^\beta = 0\}.$$

Condition (6) implies that $\left(e^{-rt}W_t^\beta + \int_0^t e^{-rs}(Ba_s\,ds + dC_s) = W_0^\beta + \int_0^t e^{-rs}\beta_s\,dZ_s^a\right)_{t\le\tau_0^\beta(C)}$ is a uniformly integrable martingale under $\mathbb{P}^a$. Therefore, Optional sampling Theorem gives

$$\begin{aligned} w_0 &\le W_0^\beta \\ &= \mathbb{E}^a\left(\int_0^{\tau_0^\beta(C)} e^{-rs}(Ba_s ds + dC_s)\right) \\ &= W_0^{\tilde{\Gamma}} \end{aligned} \tag{7}$$

where $\tilde{\Gamma} = ((C_t)_t, \tau_0^\beta(C), a)$ belongs to $\Gamma(w_0)$. On the other hand, any contract $\Gamma \in \Gamma(w_0)$ can be uniquely written in the form $((C_t)_t, \tau_0^{\beta(\Gamma)}(C), a)$ by uniqueness of the BSDE representation.

Let us recall that we have assumed so far that a is incentive compatible. The next lemma characterizes incentive compatible contracts as a deterministic function of the control process β. To ease notations, we write in the sequel W in place of W^β.

Lemma 2 *A contract $\Gamma = (C, \tau_0^\beta(C), a)$ is incentive compatible if and only if $a_t = \mathbb{1}_{\beta_t<\sigma\lambda}$ with $\lambda = \frac{B}{\delta}$.*

Proof of Lemma 2. For any contract $\Gamma = (C, \tau_0^\beta(C), a)$, we will show that

$$\mathbb{E}^{a^*}\left(\int_0^{\tau_0^\beta(C)} e^{-rt}(Ba_t^* dt + dC_t)\right) \ge \mathbb{E}^a\left(\int_0^{\tau_0^\beta(C)} e^{-rt}(Ba_t dt + dC_t)\right). \tag{8}$$

where $a_t^* = \mathbb{1}_{\beta_t<\lambda\sigma}$.

Let W^*, the continuation value associated to action strategy a^*. We have, under $\mathbb{P}^{a^*}$

$$dW_t^* = (rW_t^* - Ba_t^*)dt \;+\; \beta_t dZ_t^{a^*} - dC_t.$$

with

$$W_0^* = \mathbb{E}^{a^*}\left(\int_0^{\tau_0^\beta(C)} e^{-rs}(Ba_t^* + dC_t)\,dt\right).$$

Under $\mathbb{P}^a$,

$$dW_t^* = (rW_t^* \;+\; (\frac{\delta}{\sigma}\beta_t - B)\mathbb{1}_{\beta_t<\lambda\sigma} - \frac{\delta}{\sigma}\beta_t a_t)\,dt + \beta_t dZ_t^a - dC_t$$

with

$$W_0^* = \mathbb{E}^a\left(\int_0^{\tau_0^\beta(C)} e^{-rs}(Ba_t dt + dC_t)\right) + \mathbb{E}^a\left(\int_0^{\tau_0^\beta(C)} e^{-rt}\psi(a_t,\beta_t)dt\right)$$

where $\psi(a,\beta) = (\frac{\delta}{\sigma}\beta - B)a - (\frac{\delta}{\sigma}\beta - B)\mathbb{1}_{\beta<\sigma\lambda}$. Now, if $\beta < \lambda\sigma$ then, for any $a \in [0,1]$, $\psi(a,\beta) = (\frac{\delta}{\sigma}\beta - B)(a-1) \ge 0$. If $\beta \ge \lambda\sigma$ then, for any $a \in [0,1]$, $\psi(a,\beta) = (\frac{\delta}{\sigma}\beta - B)a \ge 0$. Thus, (8) is satisfied. □

Therefore, the principal's problem is to find a contract $\Gamma = (C, \tau_0^\beta(C), \mathbb{1}_{\beta<\lambda\sigma})$ that maximizes his expected profit at date 0. This leads to the following Markov formulation of problem (3)-(4).

$$V_P(x, w_0) = \max_{w \geq w_0} V_P(x, w) \tag{9}$$

where

$$V_P(x, w) = \sup_{C,\beta} \mathbb{E}^{a^*}\left(\int_0^{\tau_0^\beta(C)} e^{-rs}(X_s\, ds - \, dC_s)\right)$$

$$\text{with } a^* = (a_t^*)_{t\geq 0} \text{ and } a_t^* = \mathbb{1}_{\beta<\sigma\lambda},$$

such that

$$dX_t = -\delta \mathbb{1}_{\beta_t<\sigma\lambda}\, dt + \sigma dZ_t \text{ with } X_0 = x, \tag{10}$$

$$dW_t = (rW_t - B\mathbb{1}_{\beta_t<\sigma\lambda})\, dt + \beta_t dZ_t - dC_t \text{ with } W_0 = w. \tag{11}$$

The last result of this section shows that the optimal remuneration scheme postpones payments. It is a key result because it allows the principal to focus only on a remuneration scheme that consists in a terminal lump-sum payment.

Lemma 3 *It is always optimal for the principal to postpone payments and to pay the agent only at liquidation time with a lump-sum payment.*

Proof of Lemma 3. First, observe that, from (7), the Principal's value function (9) can be re-written as $V_P(x, w_0) = \max_{w\geq w_0}(v(x, w) - w)$ where

$$v(x, w) = \sup_{C,\beta} \mathbb{E}^{a^*}\left(\int_0^{\tau_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\, ds\right) \quad \text{s.t (10) and (11).} \tag{12}$$

The amount $v(x, w)$ corresponds to the total surplus generated by the project in our moral hazard framework.

Second, note that $\tau_0^\beta(C) = \sigma_0^\beta \wedge \tilde{\tau}_0^\beta(C)$ where for any fixed increasing process $(C_t)_{t\geq 0}$, we have

$$\tilde{\tau}_0^\beta(C) = \inf\{t \geq 0,\, W_{t-}^\beta = 0\},$$

and

$$\sigma_0^\beta = \inf\{t \geq 0,\, (\Delta C)_t = W_{t-}^\beta \text{ and } (\Delta C)_t > 0\}.$$

Third, with no loss of generality, a remuneration process can be written under the form $(C_t)_{t<\tau_0^\beta(C)} + W_{(\tau_0^\beta(C))^-}\mathbb{1}_{t=\tau_0^\beta(C)}$. Therefore, A control policy can be viewed as a pair (C, β) and a stopping time τ at which the Principal pays $W_{\tau-}^\beta$ and liquidate. Thus, we have

$$v(x, w) = \sup_{C,\beta,\tau} \mathbb{E}^{a^*}\left(\int_0^{\tau\wedge\tilde{\tau}_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\, ds\right).$$

Now, noting that $\tilde{\tau}_0^\beta(C) \leq \tilde{\tau}_0^\beta(0)$ where the latter stopping time corresponds to the situation where the principal postpones payments up to liquidation, we have

$$v(x,w) \geq \sup_{\beta,\tau} \mathbb{E}^{a^*} \left(\int_0^{\tau \wedge \tilde{\tau}_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds \right) \tag{13}$$

$$\geq \mathbb{E}^{a^*} \left(\int_0^{(\sigma_0^\beta \wedge \tilde{\tau}_0^\beta(C)) \wedge \tilde{\tau}_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds \right)$$

$$= \mathbb{E}^{a^*} \left(\int_0^{\sigma_0^\beta \wedge \tilde{\tau}_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds \right). \tag{14}$$

Taking the supremum over the controls C, β in (14) yields $v(x,w)$. It then follows from (13) that

$$v(x,w) = \sup_{\beta,\tau} \mathbb{E}^{a^*} \left(\int_0^{\tau \wedge \tilde{\tau}_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds \right), \tag{15}$$

which proves that it is optimal to postpone payments. □

We summarize our findings as follows. The principal solves the maximization problem

$$\max_{w \geq w_0} (v(x,w) - w)$$

where

$$v(x,w) = \sup_{\beta,\tau} \mathbb{E}^{a^*} \left(\int_0^{\tau \wedge \tilde{\tau}_0^\beta} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds \right), \tag{16}$$

with

$$\tilde{\tau}_0^\beta = \inf\{t \geq 0,\, W_{t^-} = 0\};$$

The Markov process (X, W) is defined by

$$dX_t = -\delta\mathbb{1}_{\beta_t < \sigma\lambda}\, dt + \sigma dZ_t \text{ with } X_0 = x,$$

$$dW_t = (rW_t - B\mathbb{1}_{\beta_t < \sigma\lambda})\, dt + \beta_t dZ_t \text{ with } W_0 = w.$$

The supremum in (16) is taken over the class of $\mathcal{F}^X$-adapted processes β such that

$$\mathbb{E}^{a^*} \left(\int_0^\infty e^{-2rs}\beta_s^2\, ds \right) < +\infty$$

and over stopping time $\tau \in \mathcal{T}^X$.

Solving problem (16) remains very challenging and so far an open question, to the best of our knowledge. The next section restricts the analysis to contracts that are incentive compatible with the full effort action process $a_t = 0$ for every t.

4 Full effort contracts.

We focus on the *full-effort* contracts, that is the class of contracts that induces the agent to exert effort at any time. It follows from Lemma 2 that the full-effort action process $a = 0$ is incentive compatible if and only if $\beta_t \geq \lambda\sigma$. Restricting the analysis to contracts that incentivize the full-effort action leads to re-write problem (16) as follows

Find a contract $\Gamma = (W_{\tau-}\mathbb{1}_{t=\tau}, \tau \wedge \tilde{\tau}_0^{\beta}, 0)$ solution to

$$v(x,w) = \sup_{\beta,\tau} \mathbb{E}^0 \left(\int_0^{\tau\wedge\tilde{\tau}_0^{\beta}} e^{-rs} X_s \, ds \right) \tag{17}$$

such that

$$dX_t = \sigma dZ_t \quad \text{with} \quad X_0 = x, \tag{18}$$

$$dW_t = rW_t + \beta_t dZ_t \quad \text{with} \quad W_0 = w, \quad \text{and} \quad \beta_t \geq \lambda\sigma \tag{19}$$

where

$$\tilde{\tau}_0^{\beta} = \inf\{t \geq 0,\, W_{t^-} = 0\}.$$

Problem (17) boils to a two-dimensional optimal exit decision, hence optimal stopping theory is from now the key mathematical tool.

Let us consider the sub-solution to problem (17)-(19) where the constraint on the incentives contract is binding (that is when $\beta_t = \lambda\sigma$). This yields the two-dimensional constrained optimal stopping problem

$$u(x,w) = \sup_{\tau} \mathbb{E}^0 \left(\int_0^{\tau\wedge\tilde{\tau}_0^{\lambda\sigma}} e^{-rs} X_s \, ds \right) \tag{20}$$

such that

$$dX_t = \sigma dZ_t \text{ with } X_0 = x,$$

$$dW_t = rW_t \, dt + \lambda\sigma dZ_t \text{ with } W_0 = w,$$

and

$$\tilde{\tau}_0^{\lambda\sigma} = \inf\{t \geq 0,\, W_{t^-} = 0\}.$$

Observe also that the unconstrained stopping problem

$$v_0(x) = \sup_{\tau} \mathbb{E}^0 \left(\int_0^{\tau} e^{-rs} X_s \, ds \right) \tag{21}$$

corresponds to the firm value in a frictionless world in which there are no asymmetry of information and private benefits. Problem (21) is a standard real option problem that has an explicit solution (see for instance Dixit and Pindyck (1994)). We have

$$v_0(x) = \frac{x}{r} - \frac{x^*}{r} e^{\theta(x-x^*)}, \text{ with } \theta = \frac{-\sqrt{2r}}{\sigma} \text{ and } x^* = \frac{1}{\theta}.$$

The threshold x^* is the (expected) profitability threshold below which it is optimal to trigger the firm liquidation in the frictionless world. That is, the stopping time

$$\tau^* = \inf\{t \geq 0,\, X_t \leq x^*\}$$

is optimal for (21).

We are ready to state our main result.

Theorem 1 *The following holds*

(i) For all $(x,w) \in \mathbb{R}\times\mathbb{R}^+$, $v(x,w) = u(x,w)$. Furthermore, $u(x,w) = v_0(x)$ for all $(x,w) \in \mathbb{R}\times\mathbb{R}^+$ such that $w \geq \lambda(x-x^*)$.

(ii) The contract $((W_{\tau^*_-}\mathbb{1}_{t=\tau^*})_{t\geq 0}, \tau^* \wedge \tilde{\tau}_0^{\lambda\sigma}, 0)$ is a solution to problem (17), (18), (19).

Thus, the principal optimally postpones payments and pays to the agent the amount $W_{\tau^*_-}$ if $\tau^* \leq \tilde{\tau}_0^{\lambda\sigma}$ or nothing if $\tau^* > \tilde{\tau}_0^{\lambda\sigma}$. Either the principal stops at the frictionless threshold τ^*, or stops at $\tilde{\tau}_0^{\lambda\sigma}$ because, the cost of incentivizing the agent is too high.

The proof of Theorem 1 is challenging and requires a series of steps. We comment each step, pointing out the novelty of our results. To ease the reading, we develop into the appendix the most technical arguments.

Proof of Theorem 1. To alleviate notations, we write in the sequel τ_0 in place of $\tilde{\tau}_0^{\lambda\sigma}$. We use whenever needed the following notations: $(X_t^x)_{t\geq 0}$ (resp. $(W_t^w)_{t\geq 0}$) denotes the process $(X_t)_{t\geq 0}$ starting at $X_0 = x$ (resp. the process $(W_t)_{t\geq 0}$ starting at $W_0 = w$), and τ^* the stopping time $\inf\{t \geq 0 \,:\, X_t^x = x^*\}$ (resp. τ_0^w, the stopping time $\inf\{t\geq 0 \,:\, W_{t^-}^w = 0\}$).

We start with the study of constrained optimal stopping problem (20). We show the following.

Proposition 1 *The exit time $\tau_R = \tau^* \wedge \tau_0$ of the open rectangle $R = (x^*, +\infty)\times(0,+\infty)$ is optimal for (20). That is,*

$$u(x,w) = \mathbb{E}^0\left(\int_0^{\tau_R} e^{-rs}X_s\,ds\right).$$

Moreover, if $w \geq \lambda(x-x^*)$ then, $u(x,w) = v_0(x)$.

Proof of Proposition 1. We first show that $u(x,w) = v_0(x)$ for every $w \geq \lambda(x-x^*)$. Note that $(W_t - \lambda X_t)_{t\geq 0}$ is an increasing process up to time τ_0 because $d(W_t - \lambda X_t) = rW_t\,dt \geq 0$, and thus $W_t - w \geq \lambda(X_t - x)$. Therefore, if $w \geq \lambda(x-x^*)$, we have $W_t \geq \lambda(X_t - x^*)$. As a consequence, the first time the agent's continuation value W_t hits zero will occur after the first time the cash-flows hit the threshold x^* almost surely. Thus, for $w \geq \lambda(x-x^*)$,

$$\begin{aligned}
v_0(x) &\geq u(x,w)\\
&\geq \mathbb{E}^0\left(\int_0^{\tau^*\wedge\tau_0} e^{-rs}X_s\,ds\right)\\
&= \mathbb{E}^0\left(\int_0^{\tau^*} e^{-rs}X_s\,ds\right)\\
&= v_0(x).
\end{aligned}$$

Second, we show that for all $x > x^*$ and $w > 0$, $u(x,w)$ is strictly positive. Let $0 < w < \lambda(x - x^*)$ and $\varepsilon = \lambda(x - x^*) - w$. Let us introduce the finite stopping time

$$\tau_\varepsilon = \inf\{t \geq 0,\ X_t = x^* + \frac{\varepsilon}{\lambda}\}.$$

Because, $W_t \geq \lambda(X_t - x^*) - \varepsilon$, we have $\tau_0 \geq \tau_\varepsilon$ almost surely. Dynamic programming principle implies

$$\begin{aligned} u(x,w) &\geq \mathbb{E}^0\left(\int_0^{\tau_\varepsilon} e^{-rs} X_s\, ds\right) + \mathbb{E}^0(e^{-r\tau_\varepsilon} u(X_{\tau_\varepsilon}, W_{\tau_\varepsilon})) \\ &\geq \mathbb{E}^0\left(\int_0^{\tau_\varepsilon} e^{-rs} X_s\, ds\right) \\ &= \frac{x}{r} - \frac{x^* + \frac{\varepsilon}{\lambda}}{r} e^{\theta(x - (x^* + \frac{\varepsilon}{\lambda}))} \\ &> 0 \quad \text{because } \varepsilon > 0. \end{aligned}$$

This latter result allows us to conclude the proof. Because $u > 0$ on R, the process

$$M_t = e^{-r(t\wedge\tau_R)} u(X_{t\wedge\tau_R}, W_{t\wedge\tau_R}) + \int_0^{t\wedge\tau_R} e^{-rs} X_s\, ds$$

is a martingale according to the optimal stopping theory. Optional sampling theorem gives for all $t \geq 0$,

$$u(x,w) = \mathbb{E}^0\left(e^{-rt\wedge\tau_R} u(X_{t\wedge\tau_R}, W_{t\wedge\tau_R}) + \int_0^{t\wedge\tau_R} e^{-rs} X_s\, ds\right).$$

Note that $\tau_R \leq \tau^*$ which is the hitting time of x^* by a Brownian motion. Therefore, τ_R is almost surely finite. Moreover, $u = 0$ on the boundaries of R. Letting t to $+\infty$ gives

$$u(x,w) = \mathbb{E}^0\left(\int_0^{\tau_R} e^{-rs} X_s\, ds\right).$$

This concludes the proof. □

To prove Theorem 1, it remains to show that functions u and v coincide. The road map is as follows. We consider the HJB equation formally associated to the value function v, that is

$$\max(\max_{\beta \geq \lambda\sigma} \mathcal{L}(\beta)v, -v) = 0 \text{ on } \mathbb{R} \times \mathbb{R}_+, \tag{22}$$

with boundary conditions $v(x,0) = 0$ and where $\mathcal{L}(\beta)$ is the differential operator

$$\mathcal{L}(\beta)V \equiv -rV(x,w) + x + rw\frac{\partial V}{\partial w}(x,w) + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}(x,w) + \frac{1}{2}\beta^2\frac{\partial^2 V}{\partial w^2}(x,w) + \sigma\beta\frac{\partial^2 V}{\partial x \partial w}(x,w).$$

We prove that u is a smooth solution to (22). Then, a standard verification argument based on Itô's formula yields $u = v$. The novelty of our analysis is to establish required continuity and smoothness properties of the value function u. The more involved results are about regularity properties of u with respect to w. Notably, the fact that u is locally Lipschitz with respect to w ((assertion ii) in the Proof of Proposition 2 and Lemma 4) and the existence and uniqueness of $\frac{\partial u}{\partial w}(x,0)$ (Proposition 3).

Proposition 2 *Value function u is jointly continuous over $[x^*, +\infty) \times [0, \infty)$ and C^∞ over $R = (x^*, +\infty) \times (0, +\infty)$. Furthermore, it satisfies*

$$\max(\mathcal{L}(\lambda\sigma)u, -u) = 0 \tag{23}$$

almost everywhere on $\mathbb{R} \times \mathbb{R}_+$.

Proof of Proposition 2. Clearly, $u(x) = 0$ for $x \leq x^*$. To show that u is jointly continuous over $[x^*, +\infty) \times [0, \infty)$, we prove that, for any $(x, w) \in [x^*, +\infty) \times [0, +\infty)$,

i) u is Lipschitz with respect to x, uniformly in w,

ii) u is locally Lipschitz with respect to w.

According to Proposition 1, we have for every $x > x_0$,

$$u(x, w) - u(x_0, w) = \mathbb{E}^0\left(\int_0^{\tau^{*,x}\wedge\tau_0} e^{-rs} X_s^x \, ds\right) - \mathbb{E}^0\left(\int_0^{\tau^{*,x_0}\wedge\tau_0} e^{-rs} X_s^{x_0} \, ds\right)$$

where $\tau^{*,x} = \inf\{t \geq 0,\ x + \sigma Z_t \leq x^*\}$. Because the stopping time $\tau^{*,x} \wedge \tau_0$ is suboptimal starting from $X_0 = x_0$, we get

$$\begin{aligned} u(x, w) - u(x_0, w) &\leq \mathbb{E}^0\left(\int_0^{\tau^{*,x}\wedge\tau_0} e^{-rs}(x - x_0) \, ds\right) \\ &\leq \frac{x - x_0}{r}. \end{aligned}$$

Thus assertion (i) is proven. The proof assertion (ii) is more involved and relies on the following lemma proved in the appendix.

Lemma 4 *For every couple $(x, w) \in (x^*, +\infty) \times (0, +\infty)$, there is a constant C such that $u(x, w) \leq C(1 + x)w$.*

For every $w > w_0$, we have $\tau_0^w \geq \tau_0^{w_0}$ a.s. and thus by Strong Markov Property, we have using $u(x^*, w) = 0$ for all $w > 0$,

$$\begin{aligned} u(x, w) - u(x, w_0) &= \mathbb{E}^0\left(\int_0^{\tau^*\wedge\tau_0^w} e^{-rs} X_s^x \, ds\right) - \mathbb{E}^0\left(\int_0^{\tau^*\wedge\tau_0^{w_0}} e^{-rs} X_s^{x_0} \, ds\right) \\ &= \mathbb{E}^0\left(e^{-r\tau_0^{w_0}} \mathbb{1}_{\tau_0^{w_0} \leq \tau^*} u(X_{\tau_0^{w_0}}, (w - w_0)e^{r\tau_0^{w_0}})\right) \\ &\leq C\mathbb{E}^0(X_{\tau_0^{w_0}} + 1)(w - w_0), \end{aligned}$$

where the last inequality follows from Lemma 4. Now, observe that for every $t \geq 0$, we have $W_t - w_0 \leq \lambda(X_t^x - x)$ and thus $X_{\tau_0^{w_0}} \leq x$ which ends the proof of assertion (ii) and, in turn, the proof that u is jointly continuous over $[x^*, +\infty) \times [0, \infty)$.

Now, from optimal stopping theory, the continuous value function u is a viscosity solution to (23) (see for instance, Pham (2010), Theorem 4.3.1). We show that, for any $\epsilon > 0$, u

satisfies (23) over $R_\epsilon = (x^*, +\infty) \times (\epsilon, +\infty)$ in a classical sense. To this end, we introduce a deterministic transformation of the process (X, W) where

$$\begin{cases} dX_t &= \sigma dZ_t, \\ dW_t &= rW_t\,dt + \lambda\sigma dZ_t. \end{cases}$$

Such transformation unveils a parabolic nature of the problem and is similar to the method of characteristics in PDE analysis. Given $(x, w) \in R_\epsilon$, let us define

$$S_t = \lambda X_t - W_t - \lambda x^* \text{ with } S_0 = s = \lambda(x - x^*) - w.$$

We have

$$\begin{cases} dS_t &= -rW_t\,dt, \\ dW_t &= rW_t\,dt + \lambda\sigma dZ_t. \end{cases}$$

Consider the function $\hat{u}(s, w) = u(x^* + \frac{1}{\lambda}(w + s), w)$. The function $\hat{u}$ is jointly continuous because u is jointly continuous. By results on interior regularity for solution to parabolic PDE (see, Krylov (2008), Ch 2, Sect. 4, Corollary 3), for any $\epsilon > 0$, the solution on any rectangle $\hat{R}_\epsilon = (0, +\infty) \times (\epsilon, +\infty)$ to

$$rw\frac{\partial f}{\partial s} = rw\frac{\partial f}{\partial w} + \frac{\lambda^2}{2}\frac{\partial^2 f}{\partial w^2} + x^* + \frac{1}{\lambda}(w + s)$$

with boundary condition $f = \hat{u}$ on $\partial\hat{R}_\epsilon$, is $C^\infty(\hat{R}_\epsilon)$ and coincides with $\hat{u}$. Therefore, for any $\epsilon > 0$, $\hat{u}$ is $C^\infty(\hat{R}_\epsilon)$ which, in turn, implies that u is $C^\infty(R_\epsilon)$ and satisfies $\mathcal{L}(\lambda\sigma)u = 0$ on the set R where $u > 0$ or, equivalently, u satisfies

$$\max(\mathcal{L}(\lambda\sigma)u, -u) = 0$$

almost everywhere. This ends the proof of Proposition 2. □

We need additional properties to prove that value function u is also a smooth solution to (22) almost everywhere on $\mathbb{R} \times \mathbb{R}_+$.

Because, $u = 0$ and $\mathcal{L}(\beta)u \leq 0$ for every $\beta \geq \lambda\sigma$ on the set $\{x \leq x^*\}$, it is enough to prove that u satisfies $\max_{\beta \geq \lambda\sigma} \mathcal{L}(\beta)u = 0$ over $R = (x^*, +\infty) \times (0, +\infty)$. We use the following result that we prove in the Appendix.

Proposition 3 *For any $x > x^*$, $\frac{\partial u}{\partial w}(x, 0)$ exists and is finite. Moreover, for any $(x, w) \in R$, the value function u satisfies*

(i) $\frac{\partial u}{\partial w}(x, w) = \mathbb{E}^0\left(\mathbb{1}_{\tau_0 \leq \tau^*}\frac{\partial u}{\partial w}(X_{\tau_0}, 0)\right) \geq 0,$

(ii) $\frac{\partial^2 u}{\partial w^2}(x, w) < 0,$

(iii) $(\frac{\partial^2 u}{\partial x \partial w} + \lambda\frac{\partial^2 u}{\partial w^2})(x, w) < 0.$

Proposition 3 appears in the literature in different settings, (see for instance, Faingold and Vasama (2014), De Marzo and Sannikov (2017), and Vasama (2017)). Assertion (ii) corresponds to a concavity property of the value function with respect to the agent's continuation value w. This property is standard in one dimensional agency models.[5] Together with assertion (iii), it implies that choosing $\beta > \lambda\sigma$ is suboptimal in the (two-dimensional) HJB equation (22). In their respective settings, recent contributions provide heuristic justifications of properties (ii) and (iii) based on a stochastic representation for $\frac{\partial v}{\partial w}$, this latter representation is obtained by differentiating the HJB equation associated to the value function v of the principal's problem. In particular, the regularity properties of the value function v over R needed to establish the stochastic representation are not proven. The main forgetting is the proof of the existence and finiteness of $\frac{\partial v}{\partial w}(x,0)$ that is instrumental in the proof of assertions (i), (ii) and (iii). To the best of our knowledge our paper is the first that offers a complete proof of Proposition 3.

The next Proposition follows from Propositions 2 and 3 and concludes the proof of Theorem 1.

Proposition 4 *Value function u satisfies the HJB equation*

$$\max_{\beta \geq \lambda\sigma} \mathcal{L}(\beta)V = 0, \quad \text{on } R, \tag{24}$$

with boundary conditions $V(x^*, w) = 0$ and $V(x, 0) = 0$. Therefore, the two value functions u and v coincide.

Proof of Proposition 4. Let us consider any function V regular solution to (24) with $V(x^*, w) = 0$ and $V(x, 0) = 0$ and such that for any $(x, w) \in R$, the mapping

$$\beta \longrightarrow \frac{1}{2}\beta^2 \frac{\partial^2 V}{\partial w^2}(x,w) + \sigma\beta \frac{\partial^2 V}{\partial x \partial w}(x,w) \text{ takes its maximum over } [\lambda\sigma, \infty) \text{ at } \beta = \lambda\sigma. \tag{25}$$

Then, such a function V is clearly a smooth solution to (24). It is easy to check that, a sufficient condition for (25) is that $\frac{\partial^2 V}{\partial w^2} < 0$ and $\frac{\partial^2 V}{\partial x \partial w} + \lambda \frac{\partial^2 V}{\partial w^2} < 0$. From Proposition 3, value function u satisfies these two properties. Together with Proposition 2, we get that u satisfies (24) on R. Finally, the fact that functions u and v coincide follows from a standard verification result. This ends the proof of Proposition 4 and Theorem 1. □

5 Concluding remarks

In this paper, we have built a dynamic contracting model in corporate finance with moral hazard in which the principal is concerned by the agent's action on the random growth rate of the firm. In contrast to standard environment, cash flows are serially correlated over time and cumulative cash flows have finite variations. This led us to study a two-dimensional control problem. We derived the Markovian formulation of the principal's problem and

[5]See the seminal papers of Sannikov (2008) and De Marzo and Sannikov (2006).

proved regularity properties of the associated value function that allow to derive the optimal contract. These regularity properties appeared in previous studies in different environment. Their proofs remained up to now heuristic.

We derived the optimal contract of principal's problem in the class of full effort contracts. We know from previous studies that, when cash flows are defined as the increment of an arithmetic Brownian motion, we can find restrictions on the parameters of the model that ensure never inducing shirking is indeed optimal.[6] This remark has been taken as a rationale for restricting attention to full effort contracts in economic applications. We show below that this result does not extend to our setting in which the firm's growth rate fluctuates. To see this point, let us consider the HJB equation formally associated to the value function v in (15), that is

$$\max(\max_{\beta}(\mathcal{L}^0(\beta)V + (B(1 - \frac{\partial V}{\partial w}(x,w)) - \delta\frac{\partial V}{\partial x}(x,w))\mathbb{1}_{\beta<\lambda\sigma}), -V) = 0 \tag{26}$$

and consider (x,w) with $w \geq \lambda(x - x^*)$. It is easy to see that v_0 does not satisfy the HJB equation (26). Indeed, a direct computation yields

$$\max_{\beta}(\mathcal{L}^0(\beta)v_0(x) + (B - \delta\frac{\partial v_0}{\partial x}(x)) = B - \frac{\delta}{r}(1 - e^{\theta(x-x^*)}) > 0,$$

where the inequality holds for any (x,w) such that $w \geq \lambda(x - x^*)$ and x in a right neighborhood of x^*. It then follows from Theorem 1 that, incentivizing the agent to exert full effort at any time cannot be optimal. The economic intuition is simple: when the realized expected growth rate is close to the profitability threshold that triggers liquidation in a frictionless world, incentivizing the agent becomes very costly and taking action $a = 0$ is no longer optimal. Clearly, this situation does not occur in a setting in which the profitability of the firm is constant.

Problem (26) relates to optimal control problems with discontinuous coefficients. Typically, a class of problems about which we know very little. Characterizing the optimal contract in a moral hazard environment with random growth rate and identifying when to release pressure on a firm's manager are clearly important economic issues. This and related questions must await for future work.

6 Appendix

Proof of Lemma 4. We start with the following observation: for every couple $(x,w) \in (x^*, +\infty) \times (0, +\infty)$, there is some $C > 0$ such that $u(x,w) \leq C(1 + x)$. Indeed,

$$\begin{aligned}
u(x,w) &\leq \mathbb{E}^0\left(\int_0^\infty e^{-rs}|x + \sigma Z_s|\,ds\right) \\
&\leq \frac{x}{r} + \sigma\mathbb{E}^0\left(\int_0^\infty e^{-rs}|Z_s|\,ds\right) \\
&= \frac{x}{r} + \sigma\sqrt{\frac{2}{\pi}}\mathbb{E}^0\left(\int_0^\infty e^{-rs}\sqrt{s}\,ds\right) \\
&\leq C(1 + x).
\end{aligned}$$

[6] See DeMarzo and Sannikov (2006), Zhu (2012).

Therefore, Lemma 4 holds for $w \geq 1$. Let us now consider $w \in (0,1)$. We decompose $u(x,w)$ as follows $u(x,w) = u_1(x,w) + u_2(x,w)$, with

$$u_1(x,w) = \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_0^w < \tau_1^w\}} \int_0^{\tau^* \wedge \tau_0^w} e^{-rs}(x + \sigma Z_s)\, ds \right),$$
$$u_2(x,w) = \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} \int_0^{\tau^* \wedge \tau_0^w} e^{-rs}(x + \sigma Z_s)\, ds \right).$$

On the event $\{\tau_0^w < \tau_1^w\}$, we have for every $t \leq \tau^* \wedge \tau_0^w < \tau_1^w$ the inequality $X_t \leq \frac{1}{\lambda} + x$. Therefore,

$$u_1(x,w) \leq \left(\frac{1}{\lambda} + x \right) \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_0^w < \tau_1^w\}} \int_0^{\tau_0^w} e^{-rs}\, ds \right).$$

Conditioning the process $(W_t)_{t \in [0, \tau_1^w]}$ on the event $\{\tau_0^w < \tau_1^w\}$ and using Doob h-transform (see Rogers and Williams (2000) for a definition) makes $(W_t)_{t \in [0, \tau_1^w]}$ a diffusion absorbed at 0 with generator

$$\tilde{\mathcal{L}} = \frac{\lambda^2 \sigma^2}{2} \frac{\partial^2 .}{\partial w^2} + \left(rw + \lambda \sigma \frac{h'(w)}{h(w)} \right) \frac{\partial .}{\partial w}$$

where

$$\begin{aligned} h(w) &= \mathbb{P}^0(\tau_0^w < \tau_1^w) \\ &= \frac{\int_w^1 e^{-rs^2}\, ds}{\int_0^1 e^{-rs^2}\, ds}. \end{aligned}$$

Let us denote $\tilde{\tau}_0^w = \inf\{t \geq 0,\, \tilde{W}_t^w = 0\}$ where

$$d\tilde{W}_t = \left(r\tilde{W}_t + \lambda \sigma \frac{h'(\tilde{W}_t)}{h(\tilde{W}_t)} \right) dt + \lambda \sigma dZ_t.$$

We have

$$\begin{aligned} \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_0^w < \tau_1^w\}} \int_0^{\tau_0^w} e^{-rs}\, ds \right) &= \mathbb{P}^0(\tau_0^w < \tau_1^w) \mathbb{E}^0 \left(\int_0^{\tilde{\tau}_0^w} e^{-rs}\, ds \right) \\ &\leq \mathbb{E}^0 \left(\int_0^{\tilde{\tau}_0^w} e^{-rs}\, ds \right) \\ &= \tilde{\phi}(w). \end{aligned}$$

The function $\tilde{\phi}$ satisfies

$$\tilde{\mathcal{L}} \tilde{\phi} - r\tilde{\phi} = 0$$

with $\tilde{\phi}(0) = 0$. Because $rw + \lambda\sigma \frac{h'(w)}{h(w)} < 0$ and $\tilde{\phi}$ is nondecreasing, we deduce that $\tilde{\phi}$ is convex and thus satisfies $\tilde{\phi}(w) \leq Cw$ which implies $u_1(x,w) \leq C(1+x)w$.
Now, we decompose u_2 as follows:

$$\begin{aligned} u_2(x,w) &= \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} \int_0^{\tau^* \wedge \tau_1^w} e^{-rs}(x + \sigma Z_s)\, ds \right) + \mathbb{E}^0 \left(\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} \int_{\tau^* \wedge \tau_1^w}^{\tau^* \wedge \tau_0^w} e^{-rs}(x + \sigma Z_s)\, ds \right) \\ &= u_3(x,w) + u_4(x,w). \end{aligned}$$

Because for every $s \le \tau_1^w$, we have $X_s \le \frac{1}{\lambda} + x$, it follows that

$$\begin{aligned} u_3(x,w) &\le \left(\frac{1}{\lambda} + x\right) \int_0^\infty e^{-rs}\, ds\, \mathbb{P}^0(\tau_0^w > \tau_1^w) \\ &= \frac{\frac{1}{\lambda} + x}{r} \mathbb{P}^0(\tau_0^w > \tau_1^w) \\ &= \frac{\frac{1}{\lambda} + x}{r} (1 - h(w)) \\ &\le C(1+x)w. \end{aligned}$$

Finally, Strong Markov property implies

$$\begin{aligned} u_4(x,w) &= \mathbb{E}^0\left(\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} e^{-r(\tau^* \wedge \tau_1^w)} u(X_{\tau^* \wedge \tau_1^w}, W_{\tau^* \wedge \tau_1^w})\right) \\ &\le \mathbb{E}^0\left(\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} e^{-r\tau_1^w} u(X_{\tau_1^w}, 1)\right) \\ &\le C(1+x)\mathbb{P}^0(\tau_0^w > \tau_1^w) \\ &\le C(1+x)w, \end{aligned}$$

where the first inequality holds because $u(x^*, W_{\tau^*}) = 0$. This ends the proof of Lemma 4. $\square$

Proof of Proposition 3. We show that, for w sufficiently small, $u(x,w) = c(x)\, w + o(w)$ where $c(x)$ is a real constant. This will imply the existence of $\frac{\partial u}{\partial w}(x,0)$. We have

$$u(x,w) \;=\; \mathbb{E}^0\Big(\int_0^{\tau_0^w} e^{-rs}(x + Z_s)\, ds\Big) - \mathbb{E}^0\Big(\int_{\tau_R}^{\tau_0^w} e^{-rs}(x + Z_s)\, ds\Big). \tag{27}$$

Observe that

$$\mathbb{E}^0\left(\int_0^{\tau_0^w} e^{-rs}(x + Z_s)\, ds\right) = \frac{x}{r}(1 - h(w)) \;+\; \mathbb{E}^0\left(\int_0^{\tau_0^w} e^{-rs} Z_s\, ds\right), \tag{28}$$

where, the function $h(w) \equiv \mathbb{E}^0(e^{-r\tau_0^w})$ is twice continuously differentiable over $(0,\infty)$ and satisfies the ordinary differential equation

$$\begin{aligned} \frac{\sigma^2\lambda^2}{2} h'' + rwh' - rh = 0, \\ h(0) = 1, \quad \lim_{w \to +\infty} h(w) = 0. \end{aligned}$$

It follows that, for w sufficiently small, $1 - h(w) = -h'(0^+)\, w \,+\, o(w)$. We now study the second term on the rhs of (28). We have

$$\mathbb{E}^0\left(\int_0^{\tau_0^w} e^{-rs} Z_s\, ds\right) = -\mathbb{E}^0\left(\int_{\tau_0^w}^\infty e^{-rs} Z_s ds\right) = -\frac{1}{r}\mathbb{E}^0(e^{-r\tau_0^w} Z_{\tau_0^w}) \tag{29}$$

where the second equality follows from the strong Markov property. The process $(M_t)_{t \ge 0}$ with

$$M_t \equiv e^{-rt} W_t = we^{-rt} + re^{-rt} \int_0^t W_s\, ds + \lambda\sigma e^{-rt} Z_t$$

is a uniformly integrable martingale under $\mathbb{P}^0$, thus the optional sampling theorem yields

$$0 = w\mathbb{E}^0(e^{-r\tau_0^w}) + r\mathbb{E}^0\left(e^{-r\tau_0^w}\int_0^{\tau_0^w} W_s\,ds\right) + \lambda\sigma\mathbb{E}^0\left(e^{-r\tau_0^w}Z_{\tau_0^w}\right). \tag{30}$$

Using (29) and (30) one gets

$$\begin{aligned}
\mathbb{E}^0\left(\int_0^{\tau_0^w} e^{-rs}Z_s\,ds\right) &= \frac{1}{\lambda\sigma r}w\mathbb{E}^0(e^{-r\tau_0^w}) + \frac{1}{\lambda\sigma}\mathbb{E}^0\left(e^{-r\tau_0^w}\int_0^{\tau_0^w} W_s\,ds\right)\\
&= \frac{1}{\lambda\sigma}\left(\frac{1}{r}wh(w) + \mathbb{E}^0\left(\int_0^{\infty} e^{-r\tau_0^w}\mathbb{1}_{s\leq\tau_0^w}W_s\,ds\right)\right)\\
&= \frac{1}{\lambda\sigma}\left(\frac{1}{r}wh(w) + \mathbb{E}^0\left(\int_0^{\infty} \mathbb{E}^0(e^{-r\tau_0^w}|\mathcal{F}_s)\mathbb{1}_{s\leq\tau_0^w}W_s\,ds\right)\right)\\
&= \frac{1}{\lambda\sigma}\left(\frac{1}{r}wh(w) + \mathbb{E}^0\left(\int_0^{\tau_0^w} e^{-rs}h(W_s)W_s\,ds\right)\right),
\end{aligned}$$

where the last equality follows from the strong Markov property. Below, we prove that, for w sufficiently small,

$$g(w) \equiv \mathbb{E}^0\left(\int_0^{\tau_0^w} e^{-rs}h(W_s)W_s\,ds\right) \;=\; cw \;+\; o(w).$$

First, we show that

$$g_\infty(w) \equiv \mathbb{E}^0\left(\int_0^{\infty} e^{-rs}h(W_s)W_s\,ds\right)$$

is a well defined bounded function and consequently the random variable $\int_0^\infty e^{-rs}h(W_s)W_s\,ds$ is integrable. To this extent, let us consider the real function $\theta \equiv k - h$ where $k(w) = \dfrac{\int_w^\infty e^{-\frac{r}{\sigma^2\lambda^2}t^2}\,dt}{\int_0^\infty e^{-\frac{r}{\sigma^2\lambda^2}t^2}\,dt}$ is the smooth solution to

$$\frac{\sigma^2\lambda^2}{2}k'' + rwk' = 0, \tag{31}$$

$$k(0) = 1, \;\lim_{w\to+\infty} k(w) = 0. \tag{32}$$

Note that the function θ is twice continuously differentiable and bounded over $(0,\infty)$, satisfies $\theta(0) = \lim_{w\to+\infty}\theta(w) = 0$ together with

$$\frac{\sigma^2\lambda^2}{2}\theta'' + ry\theta' \;=\; -rh \leq 0.$$

Then, Itô's formula gives

$$\begin{aligned}
0 &= \mathbb{E}^0(\theta(W_{\tau_0^w}))\\
&= \theta(w) - \mathbb{E}^0\left(\int_0^{\tau_0^w} h(W_s)\,ds\right).
\end{aligned}$$

It follows that $\theta \geq 0$ over $[0,\infty)$ and, thus $h(w)w \leq k(w)w$ over $[0,\infty)$. We deduce that $\lim_{w\to\infty} wk(w) = 0$ and thus $\lim_{w\to\infty} wh(w) = 0$. Therefore, the function $w \to wh(w)$ is bounded on $[0,\infty)$. It follows that, g_∞ is a well defined bounded function on $[0,\infty)$. Now, let f be a bounded C^2 solution[7] to the differential equation, .

$$\frac{\sigma^2\lambda^2}{2} f^{''} + rwf^{'} - rf + wh(w) = 0, \quad f(0) = 0. \tag{33}$$

Itô's formula yields for every $T > 0$,

$$\mathbb{E}^0\left[e^{-r(T\wedge\tau_0^w)} f(W_{T\wedge\tau_0^w})\right] = f(w) - \mathbb{E}^0\left(\int_0^{T\wedge\tau_0^w} e^{-rs} h(W_s)W_s\,ds\right)$$

Observe that

$$\mathbb{E}^0\left[e^{-r(T\wedge\tau_0^w)} f(W_{T\wedge\tau_0^w})\right] \leq ||f||_\infty e^{-rT}.$$

Because $\int_0^\infty e^{-rs}h(W_s)W_s\,ds$ is integrable and $f(0) = 0$, the monotone convergence theorem gives

$$\lim_{T\to+\infty} \mathbb{E}^0\left(\int_0^{T\wedge\tau_0^w} e^{-rs} h(W_s)W_s\,ds\right) = g(w).$$

Letting T goes to $+\infty$, we thus have $f = g$. Therefore, g is a bounded, twice continuously differentiable function over $(0,\infty)$ and thus $g(w) = g'(0^+)w + o(w)$.

Summing up our results, we have obtained

$$\tilde{u}(x,w) \equiv \mathbb{E}^0\left(\int_0^{\tau_0^w} e^{-rs}(x + Z_s)\,ds\right) = c(x)w + o(w).$$

We now turn to the second term of (27). We show that, for w sufficiently small,

$$R(x,w) \equiv \mathbb{E}^0\left(\int_{\tau_R}^{\tau_0^w} e^{-rs}(x + Z_s)\,ds\right) = o(w).$$

We have

$$R(x,w) = \mathbb{E}^0\left(\mathbb{1}_{\{\tau^*<\tau_0^w\}} \int_{\tau^*}^{\tau_0} e^{-rs}(x + Z_s)\,ds\right) = \mathbb{E}^0\left(\mathbb{1}_{\{\tau^*<\tau_0^w\}} e^{-r\tau^*}\tilde{u}(x^*, W_{\tau^*})\right),$$

where the equality comes from the strong Markov property. Proceeding analogously as in Lemma 4, we prove the inequality $\tilde{u}(x,w) \leq c(x)w$ and thus

$$|R(x,w)| \leq c(x)\mathbb{E}^0(\mathbb{1}_{\{\tau^*<\tau_0^w\}} e^{-r\tau^*} W_{\tau^*}).$$

Let us consider again the uniformly martingale $(M_t)_{t\geq 0}$ under $\mathbb{P}^0$, and let us define the equivalent probability measure $\hat{\mathbb{P}}^0$, such that

$$\frac{d\hat{\mathbb{P}}^0}{d\mathbb{P}^0}|\mathcal{F}_t = \frac{M_t^w}{w},$$

[7]The proof of the existence of a bounded solution to (33) is left as an exercise to the reader.

with

$$\frac{d\hat{\mathbb{P}}^0}{d\mathbb{P}^0}|\mathcal{F}_\infty = \frac{M^w_\infty}{w}.$$

Then, we deduce from $\lim_{w\longrightarrow 0}\mathbb{P}^0(\tau^*\leq\tau_0^w)=0$ that $\mathbb{E}^0(1\!\!1_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}W_{\tau^*}) = w\hat{\mathbb{P}}^0(\tau^*\leq\tau_0^w) = o(w)$. This ends the proof that $\frac{\partial u}{\partial w}(x,0)$ exists and is finite.

Proof of assertion (i). We have for $\varepsilon>0$,

$$u(x,w+\varepsilon)-u(x,w) = \mathbb{E}^0\left(\int_0^{\tau_R^\varepsilon} e^{-rs}X_s\,ds\right) - \mathbb{E}^0\left(\int_0^{\tau_R} e^{-rs}X_s\,ds\right)$$

where $\tau_R^\varepsilon = \inf\{t\geq 0,\,(x+\sigma Z_t, w+\varepsilon+\int_0^t rW_s\,ds+\lambda\sigma Z_t)\notin R\}$. Strong Markov property gives for the first term

$$\mathbb{E}^0\left(\int_0^{\tau_R^\varepsilon} e^{-rs}X_s\,ds\right) = \mathbb{E}^0\left(\int_0^{\tau_R} e^{-rs}X_s\,ds\right) + \mathbb{E}^0\left(e^{-r(\tau^*\wedge\tau_0)}u(X_{\tau^*\wedge\tau_0}, W^{w+\varepsilon}_{\tau^*\wedge\tau_0})\right).$$

Using $u(x^*,w)=0$ for all $w>0$, we get

$$\frac{1}{\varepsilon}(u(x,w+\varepsilon)-u(x,w)) = \frac{1}{\varepsilon}\mathbb{E}^0\left(e^{-r\tau_0}u(X_{\tau_0}, W^{w+\varepsilon}_{\tau_0})1\!\!1_{\tau^*\geq\tau_0}\right).$$

Now, observe that $W^{w+\varepsilon}_{\tau_0} = \varepsilon e^{r\tau_0}$ and thus

$$\frac{1}{\varepsilon}(u(x,w+\varepsilon)-u(x,w)) = \mathbb{E}^0\left(\frac{u(X_{\tau_0},\varepsilon e^{r\tau_0})}{\varepsilon e^{r\tau_0}}1\!\!1_{\tau^*\geq\tau_0}\right)\geq 0. \tag{34}$$

We know that $\frac{\partial u}{\partial w}(x,0)$ exists and is finite. Then, the dominated convergence Theorem yields assertion (i) by letting ε tend to zero in (34).

Proof of Assertion (ii). We note that $X^x_{\tau_0^{w_0}}\geq X^x_{\tau_0^{w_1}}$ for any $w_0\leq w_1$. Indeed, using

$$\lambda dX_t = dW_t - rW_t\,dt,$$

we obtain

$$\lambda(X^x_{\tau_0^{w_0}} - X^x_{\tau_0^{w_1}}) = w_1 - w_0 + r\int_{\tau_0^{w_0}}^{\tau_0^{w_1}} W_s\,ds \geq 0.$$

According to assertion (i),

$$\begin{aligned}
\frac{\partial u}{\partial w}(x,w_0) &= \mathbb{E}^0\left(1\!\!1_{\tau_0^{w_0}\leq\tau^*}\frac{\partial u}{\partial w}(X^x_{\tau_0^{w_0}},0)\right)\\
&\geq \mathbb{E}^0\left(1\!\!1_{\tau_0^{w_0}\leq\tau^*}\frac{\partial u}{\partial w}(X^x_{\tau_0^{w_1}},0)\right)\\
&\geq \mathbb{E}^0\left(1\!\!1_{\tau_0^{w_1}\leq\tau^*}\frac{\partial u}{\partial w}(X^x_{\tau_0^{w_1}},0)\right)\\
&= \frac{\partial u}{\partial w}(x,w_1).
\end{aligned}$$

Thus, the function $\frac{\partial u}{\partial w}$ is a decreasing function of w on R. Because we know that u is twice continuously differentiable over R, we get assertion (ii).

Proof of assertion (iii). Let us consider f defined as

$$f(x) = \frac{\partial u}{\partial w}(x, \lambda(x-c)) \text{ for } x \geq x^*.$$

To prove assertion (iii), we show that f is decreasing for any c such that $(x, \lambda(x-c))$ is in R. Take $x_0 \leq x_1$ and $w_i = \lambda(x_i - c)$ for $i = 0, 1$. From assertion (ii), we have

$$f(x_0) = \mathbb{E}^0\left(\mathbb{1}_{\tau_0^{w_0} \leq \tau^{*,x_0}} \frac{\partial u}{\partial w}(X^{x_0}_{\tau_0^{w_0}}, 0)\right)$$

Proceeding as previously,

$$\lambda(X^{x_0}_{\tau_0^{w_0}} - X^{x_1}_{\tau_0^{w_1}}) = w_1 - w_0 + r\int_{\tau_0^{w_0}}^{\tau_0^{w_1}} W_s\, ds \geq 0.$$

Thus,

$$f(x_0) \geq \mathbb{E}^0\left(\mathbb{1}_{\tau_0^{w_0} \leq \tau^{*,x_0}} \frac{\partial u}{\partial w}(X^{x_1}_{\tau_0^{w_1}}, 0)\right)$$

We will end the proof by showing that $\mathbb{1}_{\tau_0^{w_0} \leq \tau^{*,x_0}} \geq \mathbb{1}_{\tau_0^{w_1} \leq \tau^{*,x_1}}$ or equivalently that $\{\tau_0^{w_0} > \tau^{*,x_0}\} \subset \{\tau_0^{w_1} > \tau^{*,x_1}\}$ which will imply $f(x_0) \geq f(x_1)$. On the set $\{\tau_0^{w_0} > \tau^{*,x_0}\}$, we have

$$\begin{aligned} X^{x_1}_{\tau^{*,x_0}} &= x^* + x_1 - x_0 \\ W^{w_1}_{\tau^{*,x_0}} &= W^{w_0}_{\tau^{*,x_0}} + (w_1 - w_0)e^{r\tau^{*,x_0}}. \end{aligned}$$

Therefore,

$$\begin{aligned} W^{w_1}_{\tau^{*,x_0}} - \lambda X^{x_1}_{\tau^{*,x_0}} &= W^{w_0}_{\tau^{*,x_0}} + (w_1 - w_0)e^{r\tau^{*,x_0}} - \lambda(x^* + x_1 - x_0) \\ &\geq \lambda(x_1 - x_0)(e^{r\tau^{*,x_0}} - 1) - \lambda x^* \\ &\geq -\lambda x^* \end{aligned}$$

and thus for all $t \geq \tau^{*,x_0}$, we have $W^{w_1}_t \geq \lambda(X^{x_1}_t - x^*)$ which implies $\tau_0^{w_1} > \tau^{*,x_1}$ and thus the result.

7 References

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