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Impact of advertising on brand's market-shares in the automobile market: a multi-channel attraction model with competition and carryover effects

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Abstract

This article presents a new approach to measure the impact of multi-channel advertising investments on brands' market shares in the main segment of the French automobile market. We propose a multi-channel attraction model with adstock, in order to take into account the advertising carryover effect and the competition. This model allows to distinguish between short term and long term effect of the advertising. As, from a mathematical point of view, a vector of market shares is a composition belonging to the simplex space, i.e. subject to positivity and summing up to one contraints, we take benefit from the compositional data analysis (CODA) literature to estimate properly this model. We show how to determine the carryover parameters for each channel (outdoor, press, radio and television) in a multivariate way. We consider several model specifications with more or less complexity (cross effects between brands), including Dirichlet models, and we compare them using goodness-of-fit and prediction accuracy measures. We explain how to built confidence and prediction ellipsoids in the space of market shares. The impact of each channel on market shares is measured in terms of direct and cross elasticities. We conclude that in this market, radio only has a contemporaneous impact whereas outdoor, press and television have a large decay effect. Moreover, the advertising elasticities vary across brands and channels, and can be negative. It also turns out that positive interactions do exist between certain brands for certain media.

Keywords: Market response model, fully extended multiplicative competitive interaction model, carryover effect, adstock, Koyck model, compositional data analysis, automobile market, multi channel advertising.

1 Introduction

A lot of consumer goods belong to competitive markets where the final performance of a brand does not depend only on the supplied product and marketing actions of the brand but also on competitors actions (supply side), and on the socioeconomic context (demand side). Danaher et al.

[5] show that the competitive interference effects on sales are strong and diminish the advertising elasticity. Then, cross effects of advertising between brands should be considered. In the case of the automobile market, which is the ultimate example of a durable good market subject to fierce competition, we can consider that the size of the market at time t is mainly determined by the demand side: the size and the wealth or welfare of the population, people's transportation behavior and mobility habits, governmental incentives to purchase a new environmentally-friendly car, etc. In this context, car manufacturers try to get the largest slice of the pie spending huge amounts of money in media investments through different channels in order to promote the quality of their vehicles, but also to enhance their values, their identity and their image. These elements advocate for modeling market shares instead of sales, taking into account cross effects between brands.

The effect of marketing mix variables (advertising, price, promotion, distribution) have been modeled since the 50's using the so-called market response models, where the response variable is usually the sales or the market shares of products or brands (see Hanssens et al. [9] for a review of existing models).

Three main categories of models are used in practice: linear, multiplicative and attraction models. Only the latter category complies with the constraints of positivity and summing up to one of market shares data, as emphasized by Cooper and Nakanishi [4] (p.28). Nevertheless, attraction models are not used systematically for market shares modeling, because of three main reasons.

1. The first one is that some authors have shown empirically that attraction models, as the multiplicative competitive interaction model (MCI) for example, do not give significantly better results than the others in terms of fitting and prediction accuracy (see for example Ghosh et al. [8] and Leeflang et al. [11]). Nevertheless, Naert et al. [18] have made the opposite claim a few years ago, suggesting that the conclusion can depend on the considered application.

2. The second reason is that the estimation of an attraction model is not straightforward: it is a non-linear model which can be linearized by a transformation, generally the log-centering transformation, also called centered log-ratio transformation (CLR) in the compositional data analysis literature (see Aitchison [1]). A simple estimation by ordinary least squares is generally run on the resulting coordinates, while it is obvious that the log-centered error terms cannot be orthonormal. Generalized least squares (GLS) and iterative generalized least squares (IGLS) have also been considered by several authors, but without concluding to a significant improvement of the estimation (see for example Ghosh et al. [8], Leeflang and Reuyl [11], and Cooper and Nakanishi [4], p.128).

3. The third reason is that they are often overparametrized. The classical MCI suggests that the impact of a marketing instrument is the same for all brands, which is often too restrictive. The differential MCI model (DMCI) includes brand specific parameters, leading to $D + DK$ parameters, where D is the number of brands and K the number of explanatory variables, but it ignores the potential cross effects between brands. The additional specification of cross effects, done in the so-called fully extended MCI model (FEMCI), leads to a huge number of parameters: $D(1 + DK)$. With the estimation on the CLR transformed model, only the centered version of these coefficients can be identified, although they are sufficient for interpreting the model, according to Cooper and Nakanishi [4] (p.145).

We argue in favor of the use of attraction models to model market shares. Our claims concerning the three previous points are the following:

1. From a mathematical point of view, a vector of market shares data is a "composition" (a vector of positive numbers where the relative information is of interest) and it belongs to the simplex space. The compositional nature of market shares must be considered to analyze them. Compositional data analysis (CODA) is a field in statistics which has developed a set of tools, including compositional regression models with the advantage of including brand specific (differential) parameters and flexible (non-symmetric and not necessarily negative) cross effects. Morais et al. [16] prove that these models are very close to the fully extended MCI model, developped in Cooper and Nakanishi [4].
2. Morais et al. [16] suggest to use another transformation than the CLR called isometric log-ratio (ILR) transformation, which is recommended in the CODA literature because it allows to obtain orthonormal transformed error terms, with non-constant variance between the obtained coordinates. The associated estimation method is easy to implement (e.g. with the R package **compositions**).
3. Concerning the DMCI model, it should not be used as proved in Morais et al. [16] because this model is not scale invariant[1]. Concerning the FEMCI model, we prove in the same article that if the normality assumption of error terms is not made on the CLR transformed error terms but on the ILR transformed error terms (which gives exactly the same results in the initial space), then the FEMCI model is identical to the CODA model presented in Morais et al. [17]. This assumption allows to recover the $D(1+DK)$ parameters to be estimated with only $(D-1)(1+(D-1)K)$ parameters. Moreover, in order to determine if cross effects for a given marketing instrument are really improving the model, we propose in Morais et al. [16] a model selection based on an adapted Fisher test.

In addition to the MCI model, the Dirichlet (DIR) regression model can also be used to model market shares respecting their compositional nature. Although rarely used in marketing, the DIR model is a flexible model allowing the specification of differential effects for example, and it is easy to implement (R package **DirichletReg**).

Once the type of market response model is chosen, we then need to determine how to take into account the dynamic aspect of the relationship between market shares and advertising. Some authors have emphasized the existence of short term and long term effect of advertising on sales (see for example Assmus et al. [2] and Lodish et al. [13]). In the case of durable and expensive goods like automobile, we can expect the advertising impact to be spread over several periods, with diminishing returns effect on sales.

This is called the carryover effect of advertising and it is usually integrated in market response models using a stock variable, built using a retention rate which can be estimated econometrically. In advertising research, this notion is also called "adstock" variable and was initiated by Broadbent (1979). The most commonly used adstock model is the Koyck model, defined as $Q_t = \mu + \beta Adstock_t + \epsilon_t$ where the adstock function is equal to $Adstock_t = (1-\lambda)(M_t + \lambda M_{t-1} + \lambda^2 M_{t-2} + \dots)$, Q_t is the demand at time t, M_t is the media investment at time t, and λ is the retention rate. Then,

[1] It means that if marketing instruments are used in thousands of euro or in euro, the DMCI model will not lead to the same results in terms of fitted market shares.

$\beta(1-\lambda)$ can be interpreted as the current (short term) effect of advertising and β the carryover (long term) effect of advertising, and we can say that $\theta\%$ of the advertising impact occurs in the $\log(1-\theta)/\log(\lambda)-1$ periods after advertising.

Vakratsas and Ambler [21] report that "*Clarke [3] (1976) and Assmus, Farley, and Lehmann [2] (1984), in meta-analytic studies, suggest that 90% of the advertising effects dissipate after three to fifteen months. Leone [12] (1995), in an empirical generalizations study, suggests that the range be narrowed to six to nine months*". However, Leone [12] also emphasizes the fact that the retention parameter λ "*should increase as the level of aggregation increases*". Note that advertising campaigns are often analyzed at the week level and for FMCGs (fast moving consumer goods), whereas in our application we are observing the monthly advertising budgets for a durable good. Then we can expect to find larger carryover effects of advertising.

Usually carryover effects are estimated in the case of market response models for sales and for only one marketing instrument (advertising in most of the cases), not for market share models with multi-channel advertising.

In this article, we first present a descriptive compositional analysis of the competition situation in the French automobile market, of the marketing mix habits and pricing strategy.

In a second phase, we develop a fully extended multi-channel attraction model with adstock (what we call the FEMCIAd model), which considers the cross effects of television, outdoor, radio and press advertising budgets between brands and their carryover effects. This model allows to distinguish between short-term and long-term effects of the advertising. As market shares are compositions belonging to the simplex, we take benefit from the compositional data analysis literature to estimate properly this model. We also explain how to determine the decay parameters of advertising in this case.

Then, we present an application to the main segment of the French automobile market where different specifications of market share models, including Dirichlet models, are compared in terms of complexity, goodness of fit and prediction accuracy. A residuals diagnostic is done and we explain how to build confidence and prediction ellipsoids in the space of market shares.

We then interpret the chosen model in terms of short term advertising elasticities of market shares by channel, and we conclude on practical findings for car manufacturers concerning marketing mix strategies.

2 A compositional data analysis of the French automobile market

We are working on a data base coming from the French registration data base, which contains the sales of all brands. It is important to note that what we call "sales" at time t actually correspond to registrations of new passenger vehicles at time t, which can correspond to purchases during the previous months, due to the delivery delay.

2.1 A market in 5 segments

The usual segmentation of the automobile market in Europe is done in five main segments for passenger cars, from A to E, according to the size of the chassis: small vehicles are in the A segment, and largest one in the E segment. Sport and luxury cars are grouped in another additional F segment.

Figure 1 represents the registrations (called sales below by abuse of language) in volume and the corresponding market shares of each segment from 2003 to 2015 in France. A strong seasonality exists in this market in terms of sales volumes but not in terms of market shares, meaning that the seasonality impacts all segments in a similar way. The economic crisis of 2008 led to a decrease of sales in most of the segments, especially in the E segment made of the most expensive cars, except for the A segment which gained market shares at the expense of the others. The scrapping incentive put in place by the French government from December 2008 to December 2010, and delimited by black dotted lines in Figure 1, has clearly boosted sales of the first two segments.

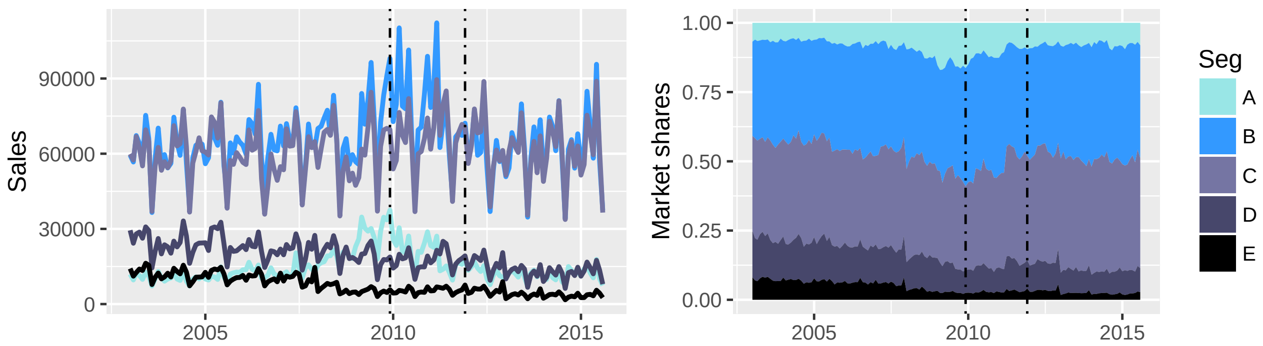

Figure 1: Segmentation of the French automobile market

For the rest of this article, we focus on the B segment which is the main segment in France (almost 40% of sales on average in the period) and includes for example the best-seller model Renault Clio.

2.2 Overview of competition in the B segment

From 2003 to 2015, 34 different brands sell vehicles in the B segment in France. However, three main brands take the lead of the market on the whole period: the three French manufacturers Renault, Peugeot and Citroën, as can be seen in Figure 12.

For the sake of simplicity and for confidentiality reasons, we focus on Renault, Peugeot and Citroën, and the remaining brands are grouped in an "Others" category (also denoted "ZZZ" below).

From Figure 2, we can easily see the order of magnitude of each brand in terms of sales across time. During the scrapping incentive period, the sales of the group of smaller brands Others increased quite a lot. Citroën also seems to take advantage from this incentive, contrary to Renault and Peugeot. We observe that the strong seasonality of sales is not visible on market shares, suggesting that this seasonality has the same impact on all brands.

In terms of market shares, it is easier to see the evolution of the competition with the 3D ternary diagram[2] presented in Figure 3, and we clearly observe a move in the direction of Others from 2003

[2]Example of interpretation of a 3D ternary diagram: the closer we are from the Others vertex, the higher the Others' market share is. If the point is on the face delimited by Citroën/Renault/Peugeot vertices, then the Others' market share is null. If the point is in the center of the tetrahedron, then all market shares are equal to 1/4.

to 2012. The market share of Others was around 35% in 2003 versus almost 50% in 2011-2012. We note that the supremacy of the three leaders was especially true at the beginning of the period (blue points) but things changed since the economic crisis (in green). The scrapping incentive (in anise green) benefits to Citroën and Others, as seen previously. At the end of the period we observe a slight backward step of the Others market share. Renault seems to be the brand the most affected by the competition of new or small brands (Others), according to 2D ternary diagrams in Figure 4.

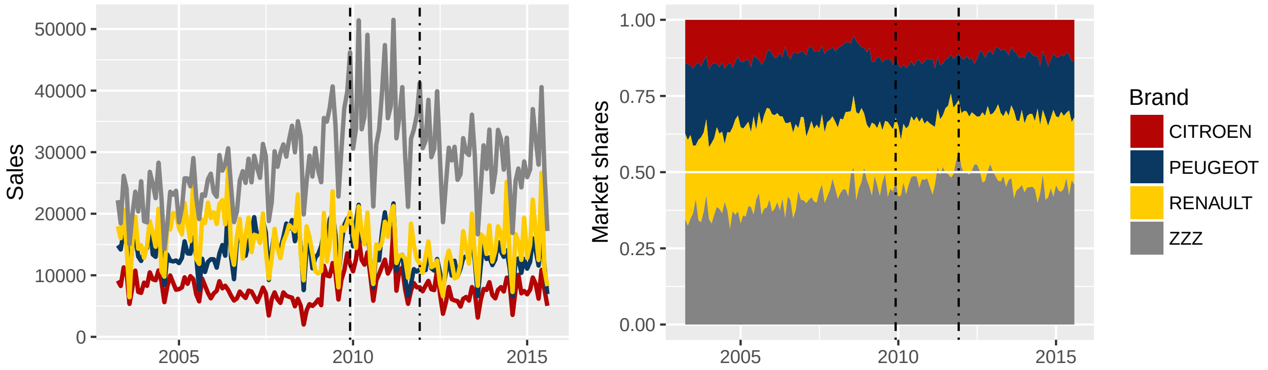

Figure 2: Sales and market shares - Citroën, Peugeot, Renault and Others

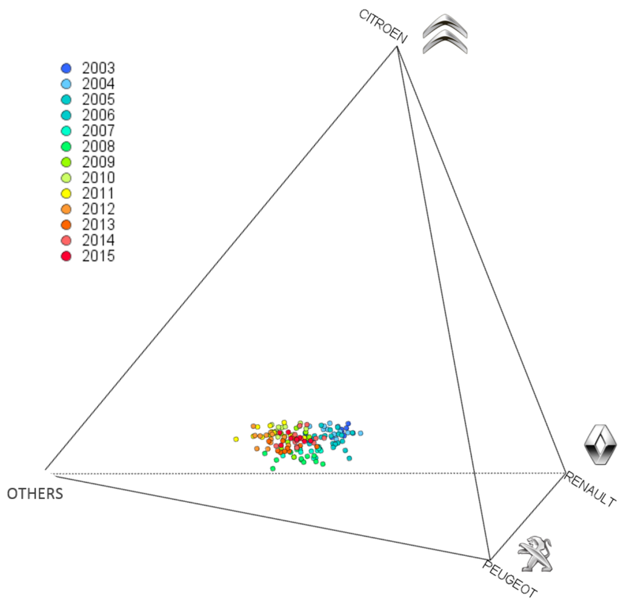

Figure 3: 3D ternary diagram of sales - Citroën, Peugeot, Renault and Others

2.3 Advertising budgets and channels

The question we want to answer is: “what is the impact of advertising on brands’ market shares?”. To do so, it is important to have an idea of who is spending the most relative to other brands, and

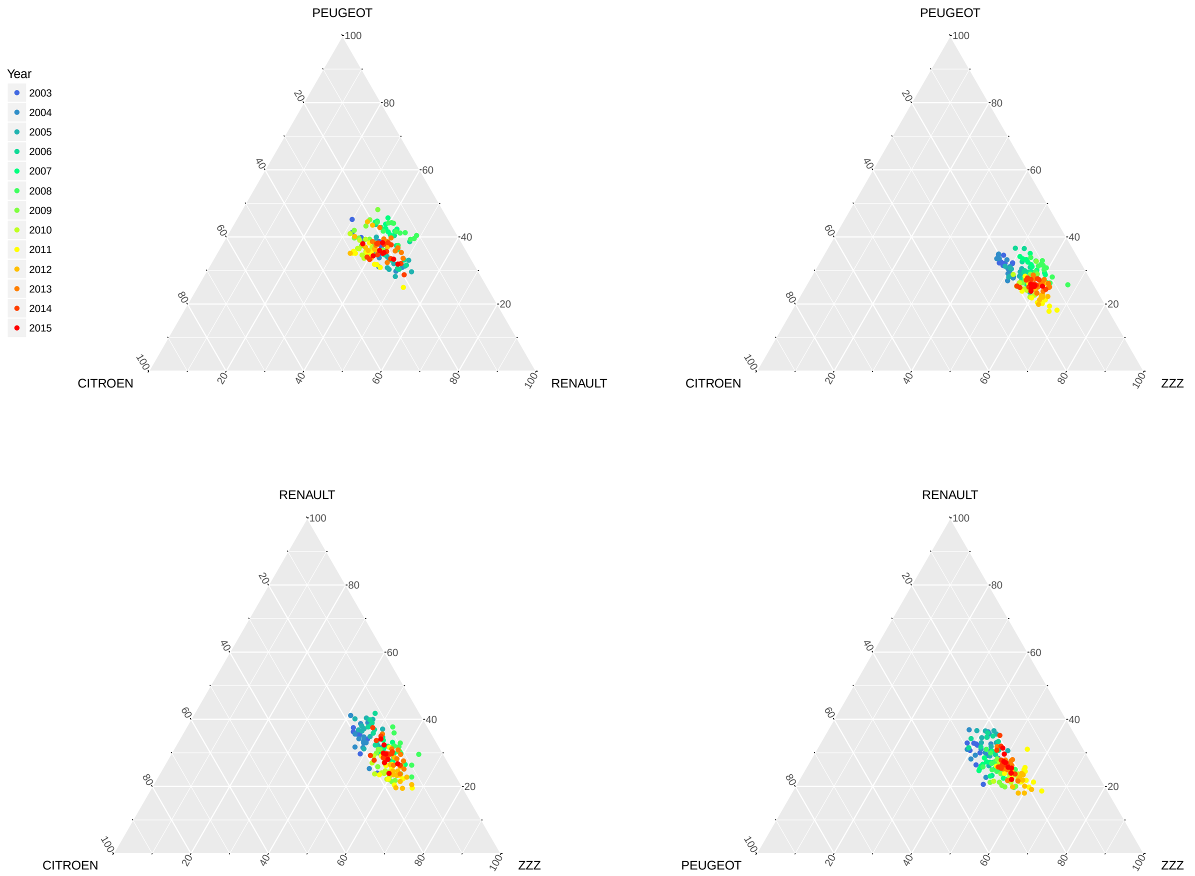

Figure 4: 2D ternary diagrams of sales - Citroën, Peugeot, Renault and Others

in which channel.

2.3.1 Aligning advertising with registrations

First of all, media investments, expressed in euro, are not coming from the registration data base but from a different data base of advertising tracking. Customers who register their vehicle at time t have not necessarily been exposed to media investments spent at time t before their purchase decision, because of the delay between the purchase act and the registration: the customer may have purchased his car at $t-2$. As we want to put in parallel the advertising potentially seen by customers and the purchase acts, we need to readjust media investment on registration time.

In order to do so, we use empirical knowledge about delivery times, which almost correspond to the difference of time between the purchase act and the registration. This delay evolves a little bit across years and across brands, but on average, the registrations of month t are made of 31% of sales of month t, 34% of sales of month $t-1$, 21% of sales of month $t-2$, and 14% of sales of month $t-3$ or before. Then, we can consider that the media investments corresponding to the vehicles registered at time t are made of 31% of month t media investments, 34% of month $t-1$ media investments, 21% of month $t-2$ media investments, and 14% of month $t-3$ media investments (see Table 5 in the appendix). We use these "weighted" media investments throughout this article.

2.3.2 Marketing mix between channels

Media investments are allocated among six channels: television, outdoor, press, radio, internet and cinema. Cinema's budget is marginal or null for most of the brands and the nature of advertising on internet has changed a lot between 2003 and 2015. Then, we choose not to consider these two channels.

The (weighted) media investments by channel are compared across brands in Figure 5, in euro and in share of voice. The media investments of Renault, Peugeot and Citroën are of the same order of magnitude for outdoor, radio and press channels, but in general Citroën spends less in television than the two leaders. Globally, advertising expenses vary a lot from one month to another for every channel and for all brands, much more than market shares. Moreover, we remark a net increase of outdoor, television and press shares-of-voice for the group Others between 2010 and 2013, which seems to be concomitant with the increase in Others market shares. Similarly, the TV share of voice of Renault is quite large before 2007 and after 2013, which are the periods where Renault has larger market shares.

Figure 13 in the appendix compares for each brand the advertising expenses by channel. Outdoor advertising (in orange) represents the main item of expenditure for all brands. Television is generally the second one (except for Citroën). We can remark an increase in the press budget for all brands at the end of the period. Note that here we are not interested in the impact of the composition across channels of the advertising budgets, but in the impact of the share of voice across brands for every channel.

Let us look at the relationship between sales and advertising by channel and by brand. In Figure 14 in the appendix, the graphs in the left column represent the volume of registrations of a brand as a function of the advertising budget in euro of this brand, while in the right column the graphs represent the market shares of brands as a function of shares of voice (relative media investment) for each channel. We observe a positive relationship between sales and advertising, in volume and in share. However, this positive relationship is even stronger, according to correlation coefficients, in shares than in volumes, which confirms our assumption that competition cannot be omitted, and that a compositional approach should be undertaken. Note that correlation coefficients indicated in Figure 14 are the correlation coefficients between the y and the x axes variables, all brands combined.

2.4 Pricing strategy

Registrations and catalogue prices are based on the same source: the registration data base. Then, prices at time t do correspond to vehicles registered at time t. We computed average prices in euro by brand for the B segment using the catalogue prices of each vehicle model of the brand and weighting by the corresponding sales volume of each vehicle model. The smallest brands grouped in Others tend to have a lower price on average, but it is less true at the end of the period. Renault had a price decline in 2012 due to the liquidation of the Clio's stock before the new version (Clio IV). Prices are increasing across time for all brands almost in the same way. Then, if we look at the relative prices, they are quite stable, as can be seen in Figure 6. They indicate the position of the brand inside the segment (low cost versus high quality). Figure 15 in the appendix shows

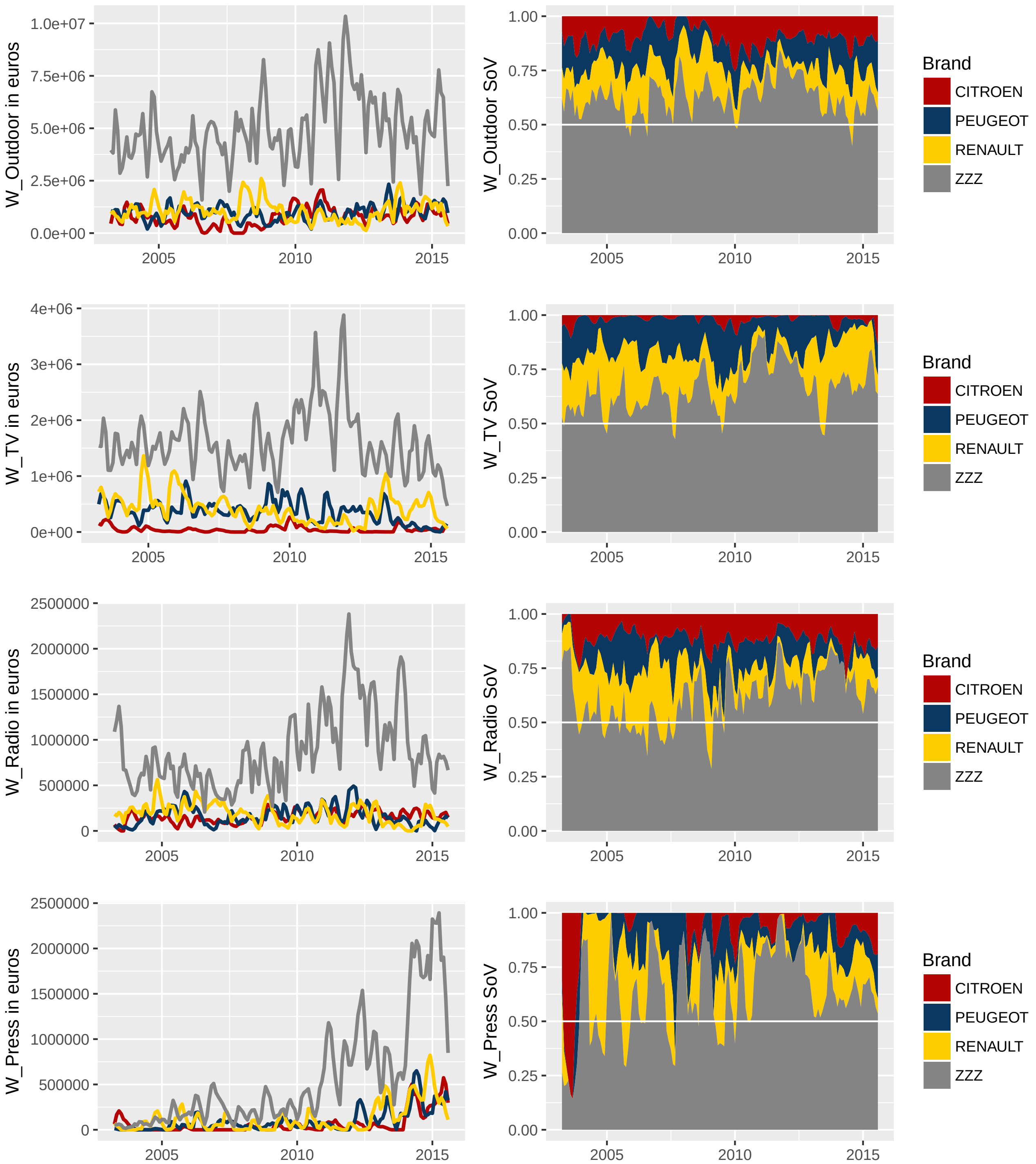

Figure 5: Advertising budgets and Share of Voice by channel - Citroën, Peugeot, Renault and Others

that, as for media investments, the relationship between sales and price is stronger (negatively) in shares than in volumes.

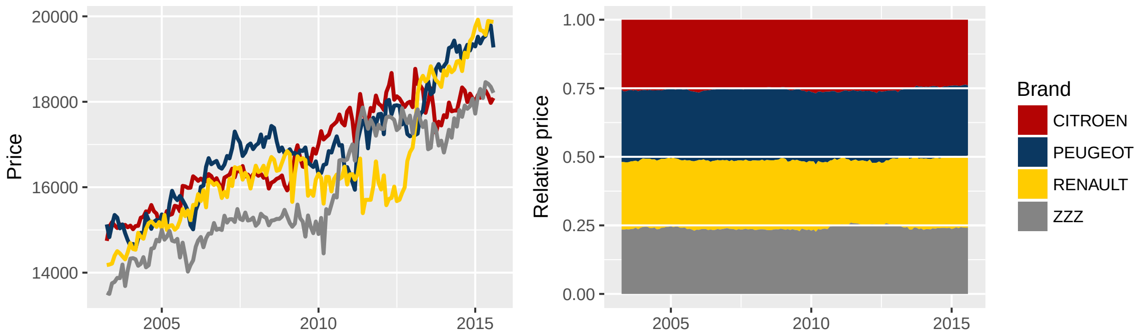

Figure 6: Catalogue prices and relative prices - Citroën, Peugeot, Renault and Others

3 Multi-channel attraction model with carryover effects

3.1 Extending the Koyck model to the multi-channel attraction case

In order to properly measure the link between advertising budgets by channel and market shares for the three leaders of the French B segment and the group of others brands, we develop a multi-channel attraction model specifying the carryover effect of advertising in a similar manner as in the Koyck model. The Koyck model (or geometric distributed lag (GL) model in Hanssens et al. [9]), the most famous adstock model, is defined by:

$$Q_t = \mu + \beta Adstock_t + \epsilon_t \tag{1}$$

where $Adstock_t = \beta(1-\lambda)\sum_{\tau=0}^{\infty}\lambda^{\tau}M_{t-\tau}$ is the adstock at time t, M_t represents the media investment at time t, and $0 \leq \lambda < 1$ is the decay parameter.

Let us now adapt this model with adstock for the multi-channel advertising case and an attraction model formulation of MCI type. We call this model MCIAd:

$$S_{jt} = \frac{a_j \prod_{c=1}^{C} Adstock_{cjt}^{b_c}\epsilon_{jt}}{\sum_{l=1}^{D} a_l \prod_{c=1}^{C} Adstock_{clt}^{b_c}\epsilon_{lt}} \tag{2}$$

where $Adstock_{cjt} = \prod_{\tau=0}^{\infty} M_{cj,t-\tau}^{\lambda_c^{\tau}(1-\lambda_c)}$ is the adstock at time t of brand j for the advertising channel c, and where each channel $c = 1, \ldots, C$ has its proper decay parameter λ_c (the same for all brands j). This model can be equivalently written with the operators of the simplex, as in the compositional data analysis literature (see Morais et al. [16] for brief definitions and Pawlowsky-Glahn and Buccianti [20] for more detail):

$$\mathbf{S}_t = \mathbf{a} \bigoplus_{c=1}^{C} b_c(1-\lambda_c) \odot \bigoplus_{\tau=0}^{\infty} \lambda_c^{\tau}\mathbf{M}_{c,t-\tau} \oplus \boldsymbol{\epsilon}_t,$$

where $\mathbf{S}_t, \mathbf{a}, \mathbf{M}_{c,t-\tau}, \boldsymbol{\epsilon}_t$ are respectively the compositions (the whole vectors) of market shares at time t, intercept terms, advertising of channel c at time $t-\tau$, and error terms at time t. The advantage of this presentation is that it looks like a linear model but with simplicial notations, and then it is easier to see the link with the transformed model. Indeed, the linearization of the MCI model in equation (2) is done by a transformation. The ILR (isometric log-ratio transformation) is the best choice because contrary to the log-centered (CLR) transformation usually used in marketing, the error terms of the coordinates are orthonormal and the OLS estimation can be applied without any issue, as explained in Morais et al. [16]. The ILR transformed MCIAd model can be written as follows:

$$S^*_{j't} = a^*_{j't} + \sum_{c=1}^{C} \sum_{\tau=0}^{\infty} \lambda_c^\tau (1-\lambda_c) b_c M^*_{cj',t-\tau} + \epsilon^*_{j't} \qquad \text{for } j' = 1, \ldots, D-1, \tag{3}$$

where $S^*_{j't}, M^*_{cj't}, \epsilon^*_{j't}$ are respectively the j'^{th} ILR coordinate of market shares, advertising of channel c and error terms.

Now, if we think that brands may have different advertising impacts and that cross effects may exist between brands, we can consider the CODA model (which is similar to the fully extended MCI model in the marketing literature) with a multi-channel carryover effects specification, which we call FEMCIAd:

$$S_{jt} = \frac{a_j \prod_{c=1}^{C} \prod_{m=1}^{D} Adstock_{cm,t}^{b_{cjm}} \epsilon_{jt}}{\sum_{l=1}^{D} a_l \prod_{c=1}^{C} \prod_{m=1}^{D} Adstock_{cm,t}^{b_{clm}} \epsilon_{lt}} \tag{4}$$

where $Adstock_{cjt} = \prod_{\tau=0}^{\infty} M_{cj,t-\tau}^{\lambda_c^\tau (1-\lambda_c)}$ is the adstock at time t of brand j for the advertising channel c, as in Equation (2), but here in equation (4) the adstocks of all brands are directly impacting the market share S_{jt}. The simplicial formulation of this model is

$$\mathbf{S}_t = \mathbf{a} \bigoplus_{c=1}^{C} \mathbf{B}_c (1-\lambda_c) \boxdot \bigoplus_{\tau=0}^{\infty} \lambda_c^\tau \mathbf{M}_{c,t-\tau} \oplus \boldsymbol{\epsilon}_t$$

The ILR transformed version of the FEMCIAd model is a system of $D-1$ equations, one by ILR coordinate $j' = 1, \ldots, D-1$, such as:

$$S^*_{j't} = a^*_{j't} + \sum_{c=1}^{C} \sum_{m'=1}^{D-1} \sum_{\tau=0}^{\infty} \lambda_c^\tau (1-\lambda_c) b^*_{cj'm'} M^*_{cm't} + \epsilon^*_{j't} \qquad \text{for } j' = 1, \ldots, D-1 \tag{5}$$

Each equation of this system can be estimated separately by OLS in order to get different variances for error terms. If we fit this model in a unique equation using dummy variables in order to estimate the $b^*_{cj'm'}$ parameters, it implies that one assumes that the variance of error terms are constant across coordinates.

The matrix of parameters $\mathbf{B}_{D\times D}$ can be recovered after inverse ILR transformation of the matrix of parameters $\mathbf{B}^*_{(D-1)\times(D-1)}$, using the following equation: $\mathbf{B} = \mathbf{V}\mathbf{B}^*\mathbf{V}'$, where $\mathbf{V}$ is the balance matrix used for the ILR transformation (see Morais et al. [17]).

3.2 Optimal advertising carryover parameters

In order to find the optimal adstock parameters, we have tested every combination of λ_c for each channel c, taking values from 0 to 0.9 with a step of 0.1 (10000 possible combinations). We also imposed a maximum τ lag of 24 months because after this delay, even with a strong decay parameter, the residual impact becomes negligible ($(1-0.9) \times 0.9^{24} \leq 0.008$). For each combination, we run the MCIAd and the FEMCIAd models with all the explanatory variables (the adstock functions of outdoor, press, radio and television, the price and the scrapping incentive), and we report the corresponding R^2, computed on the ILR transformed models.

The best MCI model is obtained for the decay parameters ($\lambda_{Outdoor} = 0.9; \lambda_{Press} = 0.9; \lambda_{Radio} = 0.8; \lambda_{TV} = 0.8$), and the best FEMCI model is obtained for ($\lambda_{Outdoor} = 0.9; \lambda_{Press} = 0.9; \lambda_{Radio} = 0; \lambda_{TV} = 0.9$), as shown in Figures 16 and 17 in the appendix. As the MCI model is a particular case of the FEMCI model, and because no particular λ_{Radio} gives better results, we choose to consider the decay parameters obtained in the FEMCI model (they also give very good results in the MCI model).

These decay parameters suggest that the half life of advertising in outdoor, television and press communications is about 5.6 months, while the advertising through the radio only has a contemporaneous effect within the month of diffusion. We demonstrate in the appendix A.6 how the half life can be computed for attraction models with carryover effects. The half life of advertising in outdoor, television and press may seem to be quite strong, but it is not surprising in a durable good market and it seems to be as expected by car manufacturers.

4 Final model specification and results

4.1 Comparison of model specifications

We want to explain brands' market shares by the advertising budgets[3] in television, outdoor, radio and press, the average prices and the scrapping incentive. Several models can be considered, with more or less complexity. In this section, we compare the MCI model, the Dirichlet model (DIR, see Morais et al. [17]) and the FEMCI model. The last two models specify brand-specific parameters for the marketing explanatory variables, while the first one does not. The FEMCI model is the only model specifying additionally cross effects between brands. All these models can be expressed in attraction formulation: the market share of brand j is defined as the relative attraction of brand j, that is the attraction of brand j divided by the sum of attractions of all brands of the market (see Morais et al. [17] for details).

MCI, FEMCI and DIR models are fitted with and without adstock variables in order to assess the relevance of the carryover effects. Note that in the case of the DIRAd model, adstock variables are computed additively whereas they are computed multiplicatively in the MCIAd and FEMCIAd models[4].

In order to enhance the importance of explanatory variables, these models are compared to naive models, called Constant MCI and Constant DIR models, where only brand-specific intercepts are used as explanatory variables. In the case of the MCI model, it is equivalent to estimate market shares to their closed geometric means (called "center" in compositional data analysis).

Models are adjusted on the period from 2005/01 to 2014/12 (in sample, $T = 120$) and are tested on the period from 2015/01 to 2015/08 (out of sample, $T' = 8$). The period 2003-2004 has been

[3]When the advertising budgets are null, they are replaced by one euro.

[4]Multiplicative adstock functions in equations (2) and (4) correspond to additive adstock functions in the ILR transformed models (3) and (5).

sacrificed for the computation of weighted media which uses 3 lags (from 2003/01 to 2003/03) and adstock variables which uses 21 lags (from 2003/04 to 2004/12).

We compare the considered models according to their goodness of fit (in sample) and their prediction accuracy (out of sample), reported in Table 1. The adjusted R^2 and the non adjusted R^2 are computed respectively on the in-sample and out-of-sample ILR coordinates of market shares, in the case of the MCI and FEMCI models. Then, they reflect the quality of the model on log ratios of shares, giving more importance to the relative error than to the absolute error. The compositional R^2_T (R-squared based on the total variance, see details in the appendix A.7) is computed for all models on the market shares directly, as the RMSE. Nevertheless, the R^2_T is a measure of quality on the log ratios of shares, whereas the RMSE gives the same importance to an error of 1 percentage point made on a share of 1% or on a share of 50%. We conclude from Table 1 that the FEMCIAd model is the best model in terms of goodness of fit according to all quality measures. This suggests that the carryover effect of the advertising does exist and that the advertising of each brand has potentially a different impact on the other brands.

Concerning the prediction accuracy, we can notice in Figure 7 that the market shares from January to August 2015 are not varying a lot and are very close to the center of the data from 2005 to 2014. Thus, the result is that, by chance, the Constant MCI and the Constant DIR models give very good results in terms of RMSE. The FEMCIAd model still is the best model of the MCI family[5] according to the R^2 on ILR coordinates. Note that the maximum R^2_T for prediction is higher than 1 for the FEMCI model, meaning that the variability of the predicted log ratios is larger than the variability of the real log ratios.

The complexity of each model can be appreciated through the number of parameters to be estimated. In the case of the models of the MCI family, the number of parameters to be estimated is lower than the number of parameters in the attraction form of the model, thanks to the use of the ILR transformation, and the number of observations to consider is $T(D-1)$. Then, in Table 1, we see that FEMCI and FEMCIAd require the estimation of 51 parameters for 360 in-sample ILR observations, which is feasible. In the case of the Dirichlet family models, the number of parameters to estimate is equal to the final number of parameters.

Table 1: Market share models accuracy (in sample: 2005-2014, out of sample: 2015)

		In sample					Out of sample		
Model	Param	Adj. R^2	R^2	Adj. R^2_T	R^2_T	$RMSE$	R^2	R^2_T	$RMSE$
Constant MCI	3	0.589	0.591	0	0	0.035	0.853	0	**0.018**
Constant DIR	4	-	-	-	0	0.035	-	0	0.019
MCI	11	0.727	0.734	0.315	0.350	0.029	0.835	0.385	0.022
MCIAd	11	0.786	0.792	0.464	0.491	0.025	0.722	0.139	0.026
DIR	28	-	-	-	0.422	0.024	-	0.478	0.029
DIRAd	28	-	-	-	0.561	0.021	-	0.640	0.025
FEMCI	51	0.789	0.819	0.488	0.557	0.024	0.623	**1.913**	0.029
FEMCIAd	51	**0.859**	**0.879**	**0.657**	**0.703**	**0.019**	**0.858**	0.585	0.026

Adj. R^2 and R^2 computed on the ILR coordinates (Dirichlet models not concerned).
Adj. R^2_T and R^2_T computed on the compositions of market shares.
$RMSE$ computed on the market shares.

The observed, fitted and predicted market shares of the different models are represented in Fig-

[5] We call "MCI family" the MCI and FEMCI type models, in opposition to Dirichlet models.

ure 7. We can notice that taking into account the carryover effect of advertising in the MCI model allows to give a better fit of the lowest point of Citroën in 2008 (MCIAd compared to MCI). The consideration of the cross effects between brands (FEMCI compared to MCI) results in a better adjustment of Renault's market shares in 2005-2006. Finally, the combination of cross effects and adstock specification of advertising (FEMCIAd compared to MCI, MCIAd and FEMCI) gives very satisfying fitted market shares during the whole period 2005-2015.

Moreover, using an adapted Fisher test (see Morais et al. [16]), we have tested whether the complexity of the "unconstrained" FEMCIAd gives significantly better results that the "constrained" model MCIAd where the explanatory variables impacts are the same for all brands, without cross effect. The Fisher statistic is equal to 5.51 while the 95% quantile is equal to 0.44, then we can largely reject H0 and conclude that FEMCIAd is better than MCIAd. We have also compared the FEMCIAd model and the constrained model where brand specific and cross effects parameters are defined for advertising channel but not for price, but the conclusion is the same (the Fisher statistics equals 1.76 and the 95% quantile equals 0.52).

The FEMCIAd model is then considered to be the best specification for the modeling of advertising impact on market shares in this application.

4.2 Residual diagnostic

Let us check that the FEMCIAd model residuals have good features. The residual diagnostic is done on the ILR residuals, for the FEMCIAd model and for the MCIAd model in order to have a benchmark. The first graph of Figure 8 suggests that there is no heteroscedasticity problem in the FEMCIAd model, while it is less clear for the MCIAd model. ILR residuals look normal according to the QQ-plot even if the tails are a little bit heavier. Finally, the graphs of the last column allow to conclude that the FEMCIAd model residuals are not autocorrelated, as confirmed by the Breusch-Godfrey test at order 1, contrary to those of the MCIAd model.

4.3 Confidence and prediction ellipsoids

Now, let us look more precisely at the prediction power of the FEMCIAd model. It is possible to construct confidence and prediction intervals for market shares using the ellipsoid of confidence and the ellipsoid of prediction of the ILR coordinates, as we demonstrate in Appendix A.9. Here it is important to estimate the FEMCIAd model as proposed in compositional data analysis with the ILR transformation, instead of the estimation in a unique equation with dummy variables proposed by Nakanishi and Cooper [19]. Indeed, while it does not change the predicted values, it does change the prediction intervals because the estimated values of standard deviations of parameters are different. We have run a test of equality of variance on residuals across ILR coordinates, and we conclude that the variance of ILR error terms should not be considered as equal, which reinforces the choice of this estimation method.

Figure 9 represents the confidence and prediction ellipsoids for the Citroën, Peugeot, Renault and Others market shares in January 2015, in the simplex $\mathcal{S}^4$.

A 95% marginal prediction interval for each share is obtained taking the minimum and the maximum values of the projection of the ellipsoid on the corresponding simplex edge in $\mathcal{S}^4$ (see for

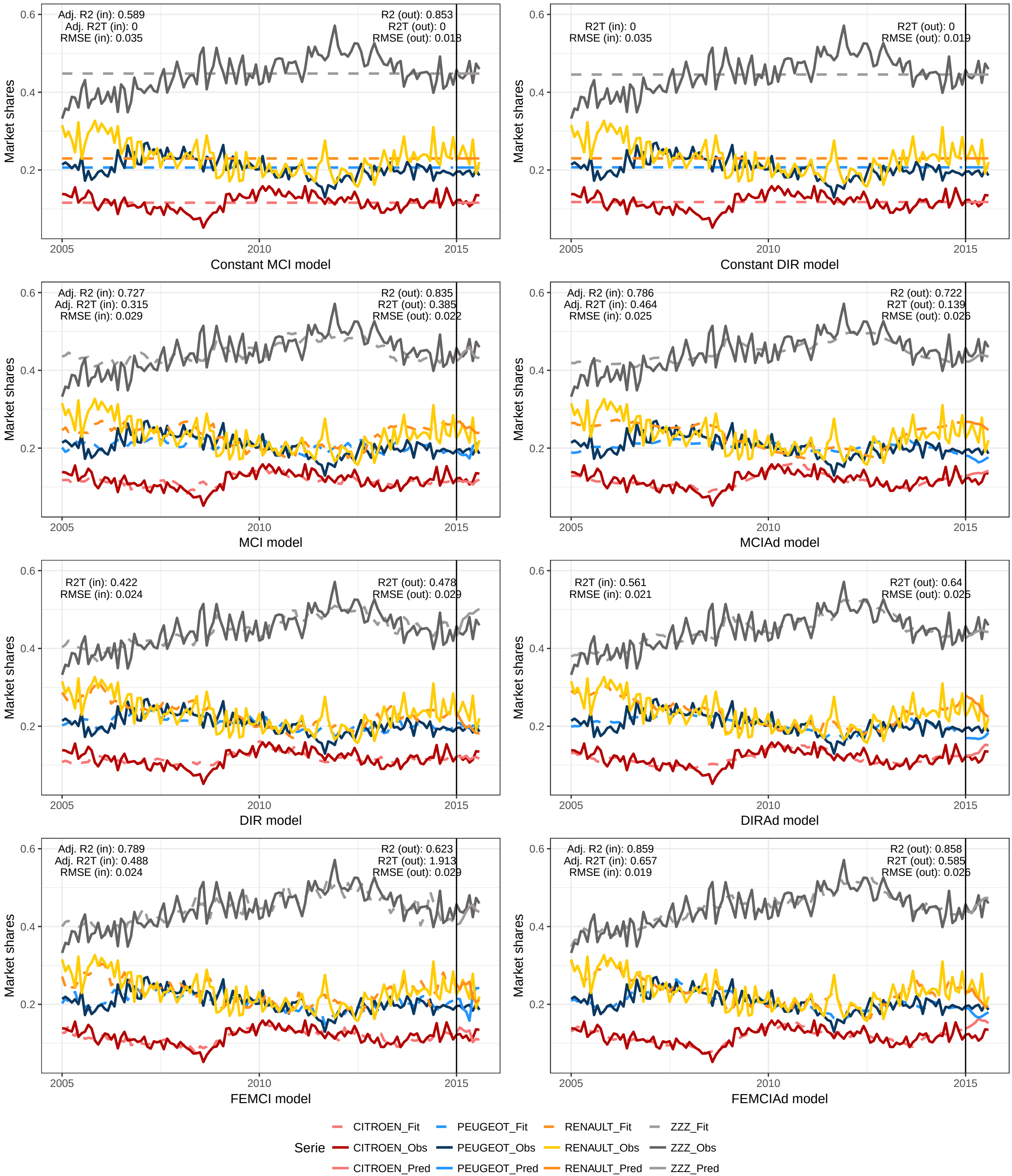

Figure 7: Observed, fitted and predicted market shares and accuracy measures

example the minimum and maximum for Renault delimited by dotted lines in Figure 9). These maximum and minimum points are computed for each prediction time and are represented in Figure

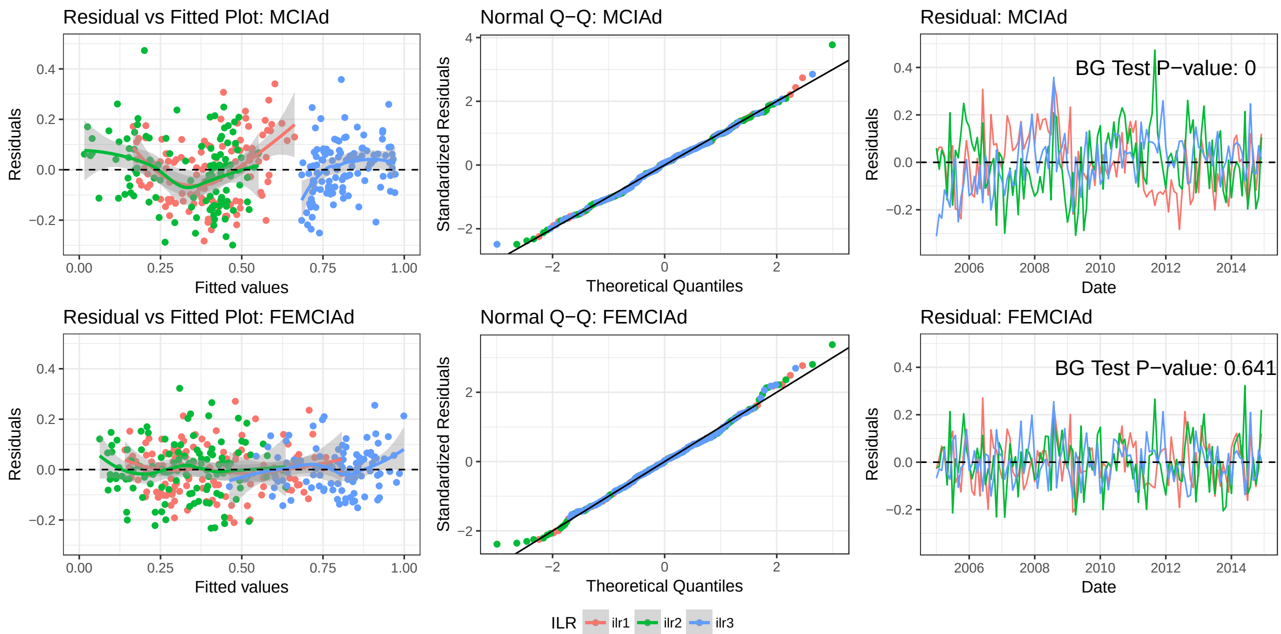

Figure 8: ILR residuals diagnostic of models for MCIAd and FEMCIAd models

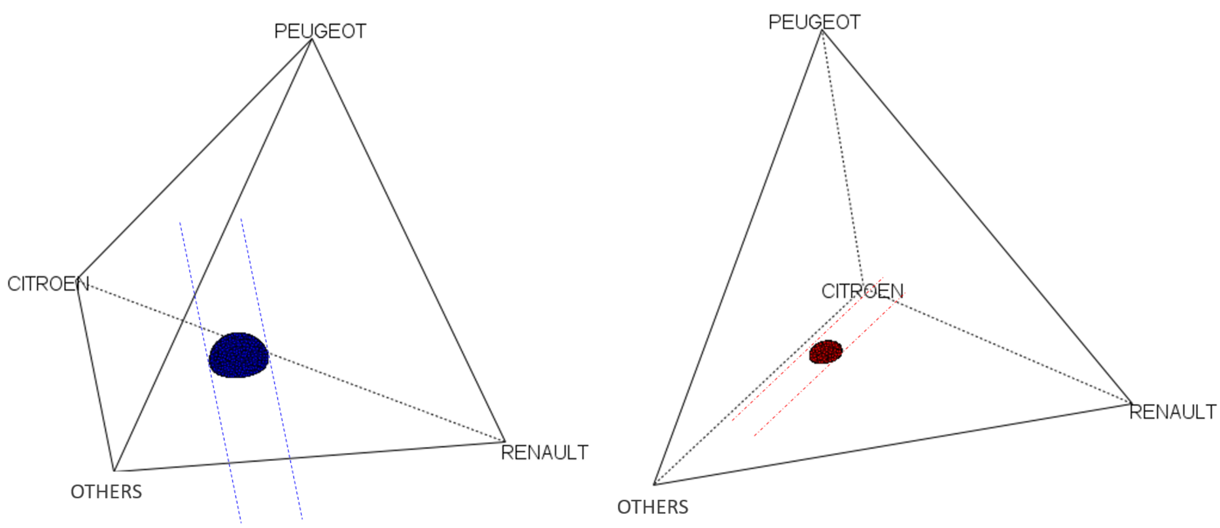

Figure 9: Prediction (left) and confidence (right) ellipsoids at 95% for market shares in January 2015

10. We observe that the true market shares (in grey) are always in the 95% prediction intervals and very close to the predictions (black). However, the FEMCIAd model tends to overestimate the Citroën market shares and underestimate the Peugeot market shares in 2015. The Renault's market share in June is surprisingly high at the expense of the Others market share, but the model does not succeed in reflecting this temporary change, while the predictions of the rest of the period are very accurate.

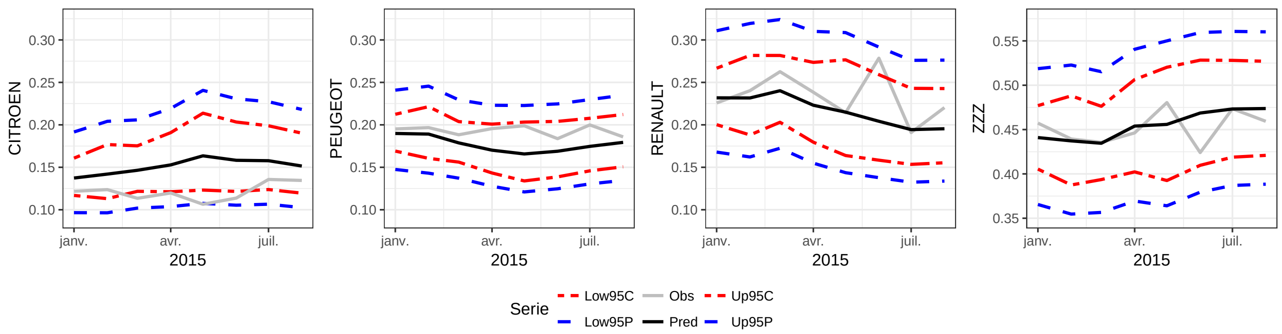

Figure 10: Observed (grey) and predicted (black) market shares with 95% confidence (red) and prediction (blue) intervals

4.4 Advertising elasticity of market shares

We now focus on the explanatory power of the FEMCIAd model. According to the analysis of variance, all explanatory variables are significant in this model, except the scrapping incentive (see ANOVA results in Table 6 in the appendix). Note that the scrapping incentive was generally significant in other models, suggesting that the specification of adstock and cross-effects are sufficient to take into account the perturbation of market shares during this special period of governmental incentive. Moreover, if SI is not included as explanatory variable in FEMCIAd, this model stays the best according to its accuracy measures which remain almost the same.

Then, for the purpose of interpreting the model, we fit the FEMCIAd model without the scrapping incentive, on the total period from 2005 to 2015, and we call the resulting model FEMCIAd_SI. All explanatory variables are strongly significant in this model according to the ANOVA (see Table 2) and the accuracy measures are fully satisfactory (see Table 3).

Table 2: Analysis of variance table for FEMCIAd_SI model

	Df	Pillai	approx F	num Df	den Df	Pr(>F)
(Intercept)	1	0.99	3902.2	3	102	$< 2.2e^{-16***}$
ilr(OutdoorAd)	3	1.34	28.1	9	312	$< 2.2e^{-16***}$
ilr(PressAd)	3	0.68	10.2	9	312	$9.384e^{-14***}$
ilr(RadioAd)	3	0.35	4.5	9	312	$1.374e^{-05***}$
ilr(TVAd)	3	0.58	8.3	9	312	$4.206e^{-11***}$
ilr(Price)	3	0.23	2.9	9	312	0.002926^{**}
Residuals	104					

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

ilr(X) denotes the vector of ILR coordinates of the variable X.

Table 3: Accuracy of the FEMCIAd_SI model (from 01-2005 to 08-2015)

	In sample				
Model	Adj. R^2	R^2	Adj. R_T^2	R_T^2	$RMSE$
FEMCIAd_SI	0.856	0.874	0.639	0.682	0.019

The short-term elasticities of market share S_j at time t relative to the brand l media investments in channel c at time t, $\check{X}_{clt}$, can be computed according to the following formula (see Morais et al. [16] for details):

$$elast(S_{jt}, \check{X}_{clt}) = \frac{\partial \log S_{jt}}{\partial \log \check{X}_{clt}} = \frac{\partial S_{jt}/S_{jt}}{\partial \check{X}_{clt}/\check{X}_{clt}} = (1-\lambda_c)(b_{cjl} - \sum_{m=1}^{D} S_{mt} b_{cml})$$

We talk about short-term elasticity because it only measures the relative impact of $\check{X}_{clt}$ on S_{jt} at time t, ignoring the future impacts on $S_{jt+1}, S_{jt+2}, \ldots$. As we are working on market shares, it is not possible to summarize the overall long term effect of an evolution of $\check{X}_{clt}$ on the $S_{jt'}$ for $t' \geq t$. However, it is of course possible to compute the elasticity of a market share S_j at time $t+\tau$ relative to the media $\check{X}_{cl}$ at time t:

$$elast(S_{jt+\tau}, \check{X}_{clt}) = \lambda_c^{\tau}(1-\lambda_c)(b_{cjl} - \sum_{m=1}^{D} S_{mt+\tau} b_{cml})$$

Table 4 presents the average short-term elasticities of market shares relative to outdoor, press, radio and television. Direct elasticities are on the diagonal and cross elasticities are extra diagonal. We remark that the largest elasticities (in bold) are often off-diagonal, which highlights the importance of cross effects between brands. For example, if Citroën increases by 1% its TV advertising budget, its market share increases on average by 0.0192%, but if Others increases by 1% its TV advertising budget, the Citroën market share decreases on average by 0.0372%.

For outdoor, press and television, we can compute "long-term elasticities", that is the elasticities of market shares relative to these channels' adstocks, but care must be taken for the interpretation: they correspond to the relative impact on market shares for a relative change in the adstock variable, which can come from changes in one or several lags of the media investments. Long-term elasticities for these three channels are equal to ten times the short-term elasticities (short-term elasticities divided by $1-\lambda$, with $\lambda = 0.9$). Note that in the case of radio, as the estimated retention rate is equal to zero, and in the case of price for which we assumed contemporaneous effect only, there is no long-term elasticities.

We can see in Figure 11 that the direct elasticities of the four channels are quite constant across time, which means that the average elasticities presented in Table 4 are good indicators of the actual elasticity at time t. Moreover, the Renault's elasticity of market share to its own outdoor and television advertising are very close (around 0.018) but an increase of 1% of TV represents less money than an increase of 1% in outdoor, as on average the budget of outdoor is much larger than the TV's budget. Note that Citroën has a particular profile regarding advertising elasticities: it has a very positive direct elasticity for the radio while other brands have a negative direct elasticity, and negative outdoor and press direct elasticities while other brands have positive direct elasticities. Peugeot has surprisingly quite low outdoor and television elasticities compared to its direct competitor Renault.

5 Conclusion

The aim of this article is to measure the impact of advertising investments through different channels (outdoor, press, radio and television), on brands' market shares in the French automobile

Table 4: Advertising short term elasticities of market shares (FEMCIAd_SI)

	Outdoor_Citroën	Outdoor_Peugeot	Outdoor_Renault	Outdoor_Others
S_Citroën	-0.0042	-0.0372	-0.0014	**0.0428**
S_Peugeot	-0.0133	0.0076	-0.0064	0.0121
S_Renault	0.0088	0.0354	0.0183	*-0.0624*
S_Others	0.0027	-0.0058	-0.0047	0.0078
	Press_Citroën	Press_Peugeot	Press_Renault	Press_Others
S_Citroën	-0.0092	-0.0008	-0.0039	**0.0139**
S_Peugeot	0.0053	0.0110	0.0027	-0.0190
S_Renault	-0.0007	*-0.0212*	0.0128	0.0091
S_Others	0.0012	0.0049	-0.0051	-0.0010
	Radio_Citroën	Radio_Peugeot	Radio_Renault	Radio_Others
S_Citroën	0.0639	0.0011	-0.0140	-0.0510
S_Peugeot	0.0085	-0.0031	0.0308	-0.0362
S_Renault	*-0.0768*	0.0099	-0.0124	**0.0793**
S_Others	0.0083	-0.0033	-0.0021	-0.0030
	TV_Citroën	TV_Peugeot	TV_Renault	TV_Others
S_Citroën	**0.0192**	0.0024	0.0157	*-0.0372*
S_Peugeot	-0.0014	0.0056	0.0014	-0.0056
S_Renault	-0.0091	0.0016	0.0184	-0.0109
S_Others	-0.0018	-0.0036	-0.0132	0.0185

The most positive (resp. negative) relative impact due to an increase in advertising is in bold (resp. italic).

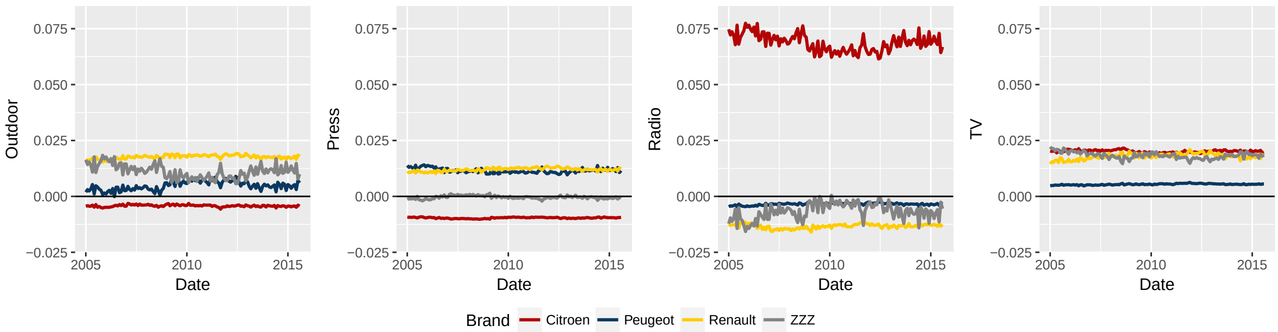

Figure 11: Short term direct elasticities of channels by brand

market, from 2005 to 2015. We focus on the main segment of this market, namely the B segment, and on the three leaders of this segment: Citroën, Peugeot and Renault, aggregating the other brands in a group.

After analyzing this market in terms of absolute and relative sales and media investments for the different brands, we emphasize the importance of taking into account the competition on the one hand, and the advertising carryover effect on the other hand. For this purpose, we build a multi-channel attraction model with carryover effects, called FEMCIAd model (for FEMCI model with adstock), combining the classical Koyck model and the CODA model presented in Morais et al. [17]. We stress the positive aspects of using an isometric log-ratio (ILR) transformation, coming from the compositional data analysis (CODA) literature, to estimate this type of models, instead of the usual centered log-ratio (CLR) transformation used in marketing. We explain how to determine the carryover parameters λ for several channels in a simultaneous way, and we conclude that outdoor, press and television advertisements have a large retention rate implying an advertising half life of 5.6 months, which seems realistic for a durable and expensive good such as automobile. On the contrary, the radio advertising appears to have only contemporaneous effect on market shares in the B segment of the automobile market.

Several model specifications are compared: with or without explanatory variables (constant models), with or without cross effects, with or without adstock variables, using a model belonging to the Dirichlet family or to the MCI family. According to goodness-of-fit (on the 2005-2014 period) and prediction accuracy (on 2015) measures, the FEMCIAd model, is considered to be the best model for our purpose. An adapted Fisher test confirms that the inclusion of cross effects improves significantly the model. The residual diagnostic suggests that the FEMCIAd model has good properties. Moreover, we draw the 95% confidence and prediction ellipsoids in the space of market shares (i.e. in the simplex), and we derive from that the 95% confidence and prediction intervals for the prediction period, in 2015.

In order to interpret the impact of each channel on brands' market shares, we compute direct and cross advertising elasticities. While television direct elasticities are positive for all brands, this is not the case for the other channels. Citroën has a different advertising impact profile from the other considered brands: the model suggests that Citroën can increase its market share diminishing its outdoor and press advertising budget in favor of the radio and television advertisements, whereas for Peugeot, Renault and the group of others, it suggests to increase the investments in outdoor, press and television and to reduce the radio communication.

Further research should be done in order to evaluate the elasticities significance, in particular for cross elasticities. A standardized interpretation could be proposed in order to be able to compare the marginal impact of channels whose total expenses are not of the same order of magnitude. Threshold effects in the advertising elasticities may also be considered. Moreover, the geometric distributed lagged adstock function of the Koyck model can be seen as too restrictive, and more flexible functions could be investigated. Finally, these elasticities can be useful for advertising budgeting optimization, like in the Dorfman-Steiner theorem [6], but the optimization problem has to be adapted for a multi-channel attraction model, as Morais [15] starts to do in chapter 4.

References

[1] Aitchison, J. *The statistical analysis of compositional data.* Monographs on statistics and applied probability. Chapman and Hall, 1986.

[2] Assmus, G., Farley, J. U., and Lehmann, D. R. How advertising affects sales: Meta-analysis of econometric results. *Journal of Marketing Research* (1984), 65–74.

[3] Clarke, D. G. Econometric measurement of the duration of advertising effect on sales. *Journal of Marketing Research* (1976), 345–357.

[4] Cooper, L., and Nakanishi, M. *Market-Share Analysis: Evaluating Competitive Marketing Effectiveness*. International Series in Quantitative Marketing. Springer, 1988.

[5] Danaher, P. J., Bonfrer, A., and Dhar, S. The effect of competitive advertising interference on sales for packaged goods. *Journal of Marketing Research 45*, 2 (2008), pp. 211–225.

[6] Dorfman, R., and Steiner, P. O. Optimal advertising and optimal quality. *The American Economic Review 44*, 5 (1954), 826–836.

[7] Friendly, M., Monette, G., Fox, J., et al. Elliptical insights: understanding statistical methods through elliptical geometry. *Statistical Science 28*, 1 (2013), 1–39.

[8] Ghosh, A., Neslin, S., and Shoemaker, R. A comparison of market share models and estimation procedures. *Journal of Marketing Research 21*, 2 (1984), pp. 202–210.

[9] Hanssens, D. M., Parsons, L. J., and Schultz, R. L. *Market response models: Econometric and time series analysis*, vol. 12. Springer Science & Business Media, 2003.

[10] Hijazi, R. H. Residuals and diagnostics in Dirichlet regression. *ASA Proceedings of the General Methodology Section* (2006), 1190–1196.

[11] Leeflang, P. S. H., and Reuyl, J. C. On the predictive power of market share attraction models. *Journal of Marketing Research 21*, 2 (1984), pp. 211–215.

[12] Leone, R. P. Generalizing what is known about temporal aggregation and advertising carryover. *Marketing Science 14*, 3_supplement (1995), G141–G150.

[13] Lodish, L. M., Abraham, M. M., Livelsberger, J., Lubetkin, B., Richardson, B., and Stevens, M. E. A summary of fifty-five in-market experimental estimates of the long-term effect of TV advertising. *Marketing Science 14*, 3 (1995), pp. G133–G140.

[14] Monti, G., Mateu-Figueras, G., Pawlowsky-Glahn, V., and Egozcue, J. Shifted-Dirichlet regression vs simplicial regression: a comparison. *Welcome to CoDawork 2015* (2015).

[15] Morais, J. *Impact of media investments on brands' market shares: a compositional data analysis approach*. Phd thesis, Toulouse School of Economics, Oct. 2017.

[16] Morais, J., Thomas-Agnan, C., and Simioni, M. Interpretation of explanatory variables impacts in compositional regression models. Working paper (in press), July 2017.

[17] Morais, J., Thomas-Agnan, C., and Simioni, M. Using compositional and Dirichlet models for market share regression. *Journal of Applied Statistics* (2017).

[18] Naert, P., and Weverbergh, M. On the prediction power of market share attraction models. *Journal of Marketing Research 18*, 2 (1981), pp. 146–153.

[19] Nakanishi, M., and Cooper, L. G. Simplified estimation procedures for MCI models. *Marketing Science 1*, 3 (1982), pp. 314–322.

[20] Pawlowsky-Glahn, V., and Buccianti, A. *Compositional data analysis: Theory and applications*. John Wiley & Sons, 2011.

[21] Vakratsas, D., and Ambler, T. How advertising works: what do we really know? *The Journal of Marketing* (1999), 26–43.

A Appendix

A.1 Brands ranking in segment B

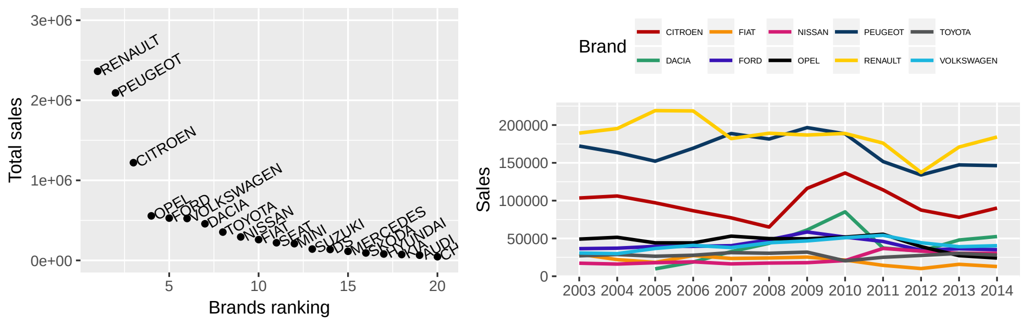

Figure 12: Brands ranking on total sales from 2003 to 2015 (left); Yearly sales of top 10 brands (right)

A.2 Aligning media investments on registrations

Table 5: Weighted media

Registration time	Purchase time	% Sales registered in t	Media	Weighted Media
	t	31%	M_t	$WM_t = 0.31M_t$
t	$t-1$	34%	M_{t-1}	$+0.34M_{t-1}$
	$t-2$	21%	M_{t-2}	$+0.21M_{t-2}$
	$t-3$	14%	M_{t-3}	$+0.14M_{t-3}$

A.3 Media investments by brand

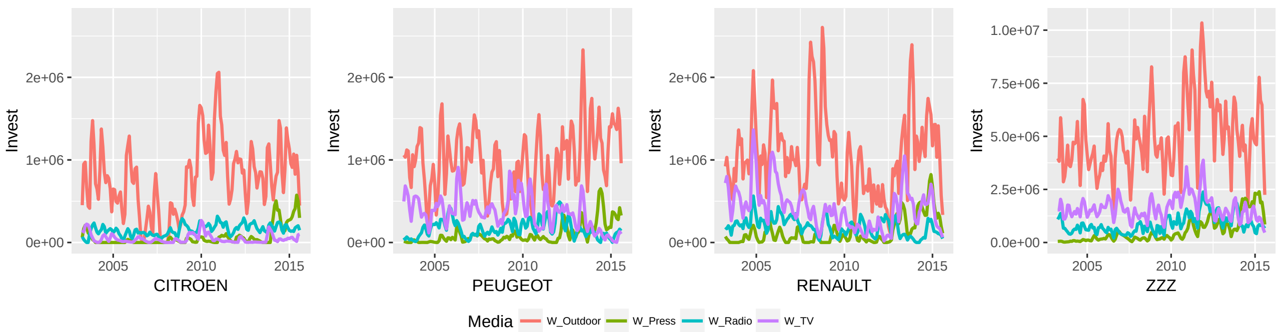

Figure 13: Media investments by brand

A.4 Correlations between explanatory variables and dependent variable, in volume and in share

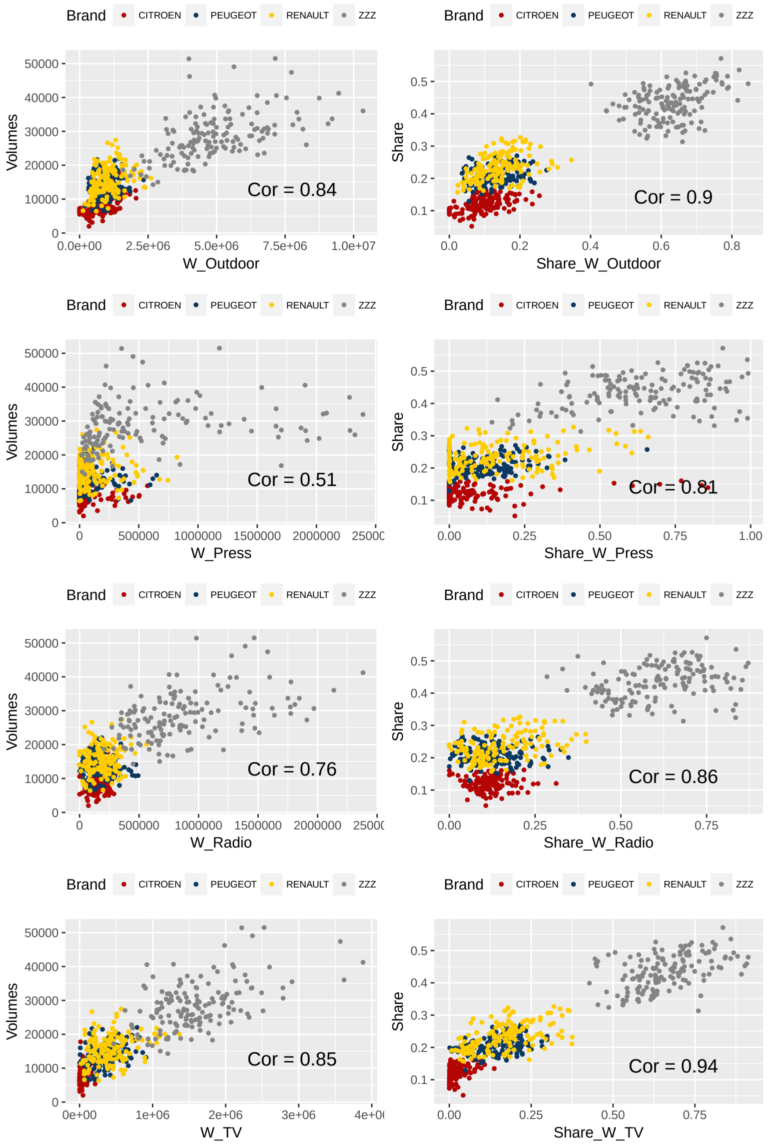

Figure 14: Relationship between sales and media (in volume and in share)

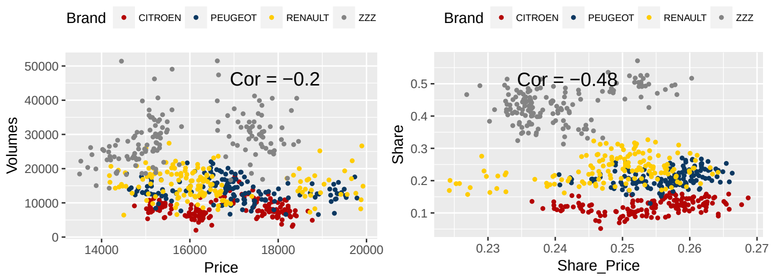

Figure 15: Relationship between sales and media (in volume and in share)

A.5 R^2 values for different adstock parameters

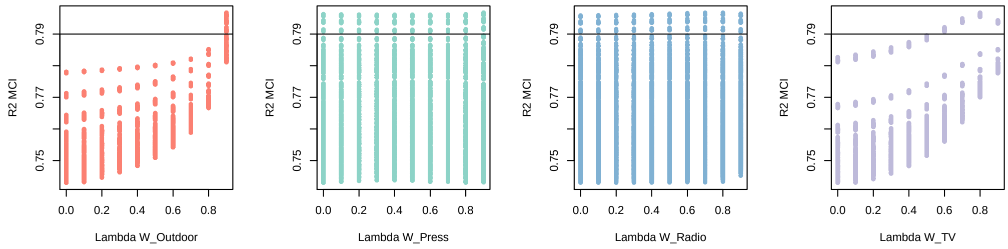

Figure 16: R^2 values of MCI model for different values of the adstock parameters of channels

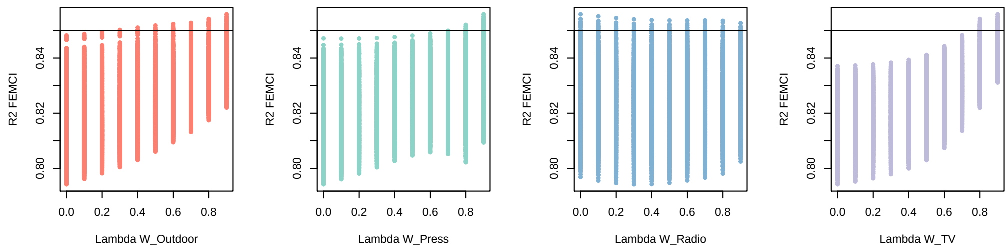

Figure 17: R^2 values of FEMCI model for different values of the adstock parameters of channels

A.6 Advertising half life in an attraction model with carryover effects

In a classical Koyck model as in equation (1), one can show easily that the half life of the advertising is equal to $\log(1-\theta)/\log(\lambda) - 1$ where $\theta = 1/2$. Indeed, if we consider the advertising investment A_t at time t, ignoring all other investments, we are looking for the period n such that a fraction θ of the long run impact of A_t occurs in n periods:

$$
\begin{aligned}
\theta A_t &= (1-\lambda)\sum_{\tau=0}^{n} \lambda^{\tau} A_t \\
\Leftrightarrow \theta &= (1-\lambda)\sum_{\tau=0}^{n} \lambda^{\tau} = (1-\lambda^{n+1}) \\
\Leftrightarrow \lambda^{n+1} &= 1-\theta \\
\Leftrightarrow (n+1)\log\lambda &= \log(1-\theta) \\
\Leftrightarrow n &= \frac{\log(1-\theta)}{\log\lambda} - 1
\end{aligned}
$$

In the case of the attraction model with carryover effects, as in equations (2) and (4), we can make the same demonstration based on the ILR transformed models of equations (3) and (5) which are similar to equation (1). Thus, it gives exactly the same result.

A.7 Quality measure for compositions

The compositional data analysis literature proposes a R^2 directly adapted to compositional data (see Hijazi [10], Monti et al. [14]). It uses the measure of the total variability of a set of compositions, based on the variance of log ratios. In terms of interpretation, it is similar to the classical R^2: it measures the proportion of the total variation explained by the model:

$$R_T^2 = \frac{totvar(\widehat{\mathbf{S}})}{totvar(\mathbf{S})} \quad \text{where } totvar(\mathbf{S}) = \frac{1}{2D}\sum_{j=1}^{D}\sum_{l=1}^{D} var(\log \frac{S_j}{S_l})$$

This measure is always positive but is not guaranteed to be lower than 1. Note that for the constant model (where the only explanatory variables are component-specific intercepts), R_T^2 equals zero for all models because there is no variability in $\widehat{\mathbf{S}}$.

A.8 ANOVA of FEMCIAd

Table 6: Analysis of variance table for FEMCIAd model

	Df	Pillai	approx F	num Df	den Df	Pr(>F)
(Intercept)	1	0.99	3883.01	3	101	0.0000***
ilr(OutdoorAd)	3	1.35	28.05	9	309	0.0000***
ilr(PressAd)	3	0.69	10.18	9	309	0.0000***
ilr(RadioAd)	3	0.35	4.55	9	309	0.0000***
ilr(TVAd)	3	0.58	8.30	9	309	0.0000***
ilr(Price)	3	0.23	2.91	9	309	0.0026***
SI	1	0.04	1.30	3	101	0.2799
Residuals	103					

A.9 Prediction intervals for market shares

In order to compute the confidence and prediction intervals of market shares for each prediction time, we use the fact that the confidence and prediction regions of the ILR coordinates are ellipsoids in dimension $D-1=3$. We denote $\mathbf{S}^* = ilr(\mathbf{S}) \sim \mathcal{N}_{D-1}(\mu^*, \mathbf{\Sigma}^*)$ where $\mathbf{\Sigma}^*$ is the covariance matrix of ILR error terms (estimated by the empirical covariance matrix of ILR residuals), and $\mathbf{S}^* = \mathbf{V}' \log \mathbf{S}$, where $\mathbf{V}$ is the balance matrix used for the ILR transformation.

The covariance matrix of the predicted ILR coordinates, $\hat{\mathbf{S}}^*$ is given by $Var_P\hat{\mathbf{S}}^* = Var\mathbf{X}^*\hat{\beta}^* + Var\epsilon^*$ where $Var\mathbf{X}^*\hat{\beta}^* = \mathbf{X}^*Var\hat{\beta}^*\mathbf{X}^{*\prime}$.

The points $\mathbf{S}^*$ in the confidence and prediction ellipsoids around the prediction $\hat{\mathbf{S}}^*$ are such that:

$$H = (\mathbf{S}^* - \hat{\mathbf{S}}^*)'\widehat{\Sigma_P}^{*-1}(\mathbf{S}^* - \hat{\mathbf{S}}^*)$$

follows a Hotelling distribution of parameters $p = D-1$ and $n-p$ with n the number of observations (see Friendly et al. [7]), which is equivalent to follow a $\frac{p(n-1)}{n-p}$ Fisher distribution. Thus, the ellipsoid of $\hat{\mathbf{S}}^*$ at a confidence level of $1-\alpha$ can be written:

$$\mathbb{P}((\mathbf{S}^* - \hat{\mathbf{S}}^*)'\widehat{\Sigma_P}^{*-1}(\mathbf{S}^* - \hat{\mathbf{S}}^*) \leq c_\alpha) = 1-\alpha$$

where $\widehat{\Sigma_P}^{*-1} = \widehat{Var_P}\hat{\mathbf{S}}^*$ for the prediction ellipsoid[6] and c_α verifies $\mathbb{P}(\mathcal{F}_{p,n-p} \leq \frac{n-p}{p(n-1)} c_\alpha) = 1 - \alpha$ (c_α is the $1-\alpha\%$ quantile of $\mathcal{F}_{p,n-p}$).

In order to visualize the prediction region in the simplex (which is the back transformed ellipsoid from the ILR coordinates space), we are going to simulate points on the ellipsoid, and back transform them to the simplex as follows. We first rewrite the ellipsoid equation:

$$\begin{aligned}
(\mathbf{S}^* - \hat{\mathbf{S}}^*)' \widehat{\Sigma_P}^{*-1} (\mathbf{S}^* - \hat{\mathbf{S}}^*) &= c_\alpha \\
\Leftrightarrow (\widehat{\Sigma_P}^{*-1/2} (\mathbf{S}^* - \hat{\mathbf{S}}^*))' (\widehat{\Sigma_P}^{*-1/2} (\mathbf{S}^* - \hat{\mathbf{S}}^*)) &= c_\alpha \\
\Leftrightarrow ||\widehat{\Sigma_P}^{*-1/2} (\mathbf{S}^* - \hat{\mathbf{S}}^*)||_E^2 &= c_\alpha \qquad (6)
\end{aligned}$$

It is equivalent to say that $\widehat{\Sigma_P}^{*-1/2}\mathbf{S}^*$ belongs to an hypersphere of center $\widehat{\Sigma_P}^{*-1/2}\hat{\mathbf{S}}^*$ and radius $\sqrt{c_\alpha}$ (where the "E" means Euclidean norm). If $D-1 = 3$, the points $\mathbf{U} = (sin(\Phi)cos(\Theta),\ sin(\Phi)sin(\Theta),\ cos(\Phi))'$, where Φ and Θ are independently uniformly distributed in $[0, 2\pi]$, are uniformly distributed on the sphere of center 0 and radius 1. Thus, the points $\mathbf{S}^*$ such that:

$$\widehat{\Sigma_P}^{*-1/2}\mathbf{S}^* = \widehat{\Sigma_P}^{*-1/2}\hat{\mathbf{S}}^* + \sqrt{c_\alpha}\mathbf{U} \ \Leftrightarrow\ \mathbf{S}^* = \hat{\mathbf{S}}^* + \sqrt{c_\alpha}\widehat{\Sigma_P}^{*1/2}\mathbf{U}$$

are distributed on the ellipsoid.

In order to come back to the simplex, we use the inverse ILR transformation. The points $ilr^{-1}(\mathbf{S}^*)$ are distributed on the prediction region in the simplex $\mathcal{S}^D$. Equation (6) is equivalent to the following equation:

$$||ilr^{-1}(\widehat{\Sigma_P}^{*-1/2}\mathbf{S}^*) \ominus ilr^{-1}(\widehat{\Sigma_P}^{*-1/2}\hat{\mathbf{S}}^*)||_A^2 = c_\alpha \qquad (7)$$

where the "A" stands for the Aitchison norm here. Then, we can say that the ellipsoid for ILR coordinates corresponds to an ellipsoid in the simplex, because equation (7) can also be written as:

$$||\widehat{\Sigma_P}^{-1/2} \boxdot (\mathbf{S} \ominus \hat{\mathbf{S}})||_A^2 = c_\alpha$$

Indeed, this is true because:

$$\begin{aligned}
ilr^{-1}(\widehat{\Sigma_P}^{*-1/2}\mathbf{S}^*) &= \mathcal{C}(\exp \mathbf{V}\widehat{\Sigma_P}^{*-1/2}\mathbf{S}^*) = \mathcal{C}(\exp \mathbf{V}\widehat{\Sigma_P}^{*-1/2}\mathbf{V}' \log \mathbf{S}) \\
&= \mathcal{C}(\exp \widehat{\Sigma_P}^{-1/2} \log \mathbf{S}) = \widehat{\Sigma_P}^{-1/2} \boxdot \mathbf{S}
\end{aligned}$$

where $\widehat{\Sigma_P}^{-1/2} = \mathbf{V}\widehat{\Sigma_P}^{*-1/2}\mathbf{V}'$ is a $D \times D$ matrix.

In our case, we generate 10000 couples of angles (Φ, Θ) and we compute the 10000 corresponding $\mathbf{S}^*$. Applying the ILR inverse transformation to these $\mathbf{S}^*$ allows us to represent the image of this ellipsoid in the simplex $\mathcal{S}^D$, for each prediction time t (see Figure 9). Then, a 95% marginal prediction interval for each share is obtained taking the minimum and the maximum values of the ellipsoid boundaries in $\mathcal{S}^4$ (see for example the minimum and maximum for Renault in Figure 9).

[6]Note that to built the confidence ellipsoid for the mean share, we take $\widehat{\Sigma_P}^{*-1} = \widehat{Var\epsilon^*}$.