# Allocating essential inputs*

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June 4, 2019

**Abstract**

Regulators must often allocate essential inputs, such as spectrum rights, transmission capacity or airport landing slots, which can transform the structure of the downstream market. These decisions involve a trade-off, as provisions aimed at fostering competition and lowering prices for consumers also tend to limit the proceeds from the sale of the inputs. We first characterize the optimal allocation, from the standpoints of consumer and total welfare. We then note that standard auctions yield substantially different outcomes. Finally, we show how various regulatory instruments can be used to implement the desired allocation.

*JEL Classification:* D47, D43, D44, D61, L13, L43, L42, L51, D43.

*Keywords:* Auctions, Market design, Essential inputs, Regulation, Antitrust.

## 1 Introduction

Over the past two plus decades, regulators have increasingly turned to competition as an alternative to direct price control. Regulators have done this by requiring, via auction, divestiture or other means, the allocation of essential inputs to multiple parties. Many

*We thank Maarten Janssen and Jon Levin for their comments; we also thank seminar participants at Oxford University (Nuffield College) and at the University of Vienna (Workshop on Auction Design). We gratefully acknowledge financial support from the European Research Council (ERC) under the European Community's Seventh Framework Programme (FP7/2007-2013) Grant Agreement N° 340903.

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recent auctions for essential inputs have been for reallocation or for new tranches of essential inputs, which affect downstream market concentration. When consumer welfare is adversely affected when an auction for an essential input increases downstream competition, an open auction cannot be relied on to allocate the input in a way that maximizes total welfare. For this reason, caps, set-asides and other measures are imposed to limit risk of an auction resulting in consolidation. These restrictions can be on spectrum bandwidth, transmission capacity or airport landing slots, which affect firms' costs or the quality of their offerings and can transform market structure.

In imposing such restrictions in an auction, regulators face a trade-off. On the one hand, they may seek to maximize consumer or social welfare. On the other hand, they may face political pressures to maximize revenue from the sale or lease of such resources. This tension can be particularly acute when the competing firms start with different levels of inputs and market shares, as the more established firms are then likely to be willing to outbid the weaker rivals in order to strengthen their market position. As a result, regulators adopt auctions that include specific provisions such as caps, set-asides and bidding credits, to balance these concerns.

In most spectrum auctions, the ex ante market structure includes three or four asymmetric incumbents, as well as potential challengers;[1] for airline slots between city pairs or electric transmission rights connecting two nodes, there can be as few as one or two incumbents. To capture these features, we consider a simplified and stylized model in which a single incumbent initially enjoys a cost advantage over a potential entrant. A regulator can allocate a divisible amount of newly released essential input, which can either widen or narrow the cost differential. This two-firm model reflects the regulators' trade-off mentioned above: awarding the newly released input to the incumbent tends to generate higher revenue, whereas awarding it to the challenger promotes competition in the market.

When the regulator's only objective is maximizing consumer surplus, or puts only a moderate weight on auction revenues or profits, the optimal policy is to allocate the resource so as to equalize the costs of the two firms, as consumer prices are lowest in that case. However, equalizing costs tends to minimize profits, and the willingness and

[1] In four-firm markets, there is usually a large gap (10% or more) between the market shares of the second and third largest operators; in three-firm markets, the dominant firm's market share can exceed 60%, as is the case in Switzerland.

ability of the firms to pay for the resource are thus also the lowest. Hence, when the regulator puts a large weight on profits or auction revenues, the optimal allocation limits the cost advantage of the incumbent, but no longer tries to equalize the firms' costs; it instead leaves an advantage to one firm – either one, if the incumbent's advantage can be overcome, and the incumbent otherwise. The winner then pays an amount equal to its operating profit.

Our baseline model is highly stylized; in particular, we assume that firms offer perfect substitutes and only differ in their costs. We show however that these insights carry over when the firms offer differentiated goods or services. Specifically, we consider an extended model of Hotelling competition with horizontal product differentiation and elastic individual demands, and show that the optimal allocation is close to that in the baseline model of Bertrand competition when products are not too differentiated.[2]

We then characterize the optimal allocation for the case in which the challenger's cost is private information. We find that the objective of the regulator and of the challenger are so conflicting that it is typically impossible to induce the challenger to reveal its costs in a useful way. Instead, the optimal allocation is similar to the previous one, except that the regulator must base its allocation decision on the expected value of its objective.

Next, we contrast this optimal allocation with the outcome that would arise in the types of auctions commonly used to allocate radio spectrum licenses. In particular, we compare the outcomes of sequential auctions, sealed-bid Vickrey-Clarke-Groves ("VCG") auctions, and simultaneous multi-round ascending ("SMRA") or clock auctions. In our framework, all these auctions exacerbate the incumbency advantage, as the incumbent always ends up winning all the newly released input. Moreover, revenues are the same in the VCG and clock auctions, and lower in a sequential auction.

Finally, we consider various measures that the regulator can adopt to promote competition, such as caps and set-asides, and discuss how they can be used to implement the optimal allocation.

The remainder of this section discusses the related literature. Section 2 presents the

[2]In a Cournot model with linear demand, the optimal allocation minimizes the sum of the two firms' costs; hence, it leads again to equalizing them when costs are a decreasing convex function of bandwidth.

model; Section 3 characterizes the optimal allocation in the case of perfect information, and Section 4 extends the analysis to the case where the challenger's cost is private information. Section 5 examines the outcome of standard auction formats used in these settings. Section 6 discusses the use of regulatory instruments to implement the optimal allocation. Section 7 concludes.

## Related Literature

Our insights are reminiscent of the literature on second-sourcing. Although the focus there was mostly on competition "for the market" rather than "in the market", it was recognized that the awarding of a contract or a procurement decision could affect the purchaser's ability to switch to alternative suppliers later on, or the suppliers' ability to compete effectively for subsequent contracts.[3] A few papers however consider the impact of the chosen market structure (e.g., monopoly or duopoly) on prices and welfare.[4] We build on this literature and study how the allocation of a (divisible) essential input can further affect the market structure and the outcome of competition.

Our paper relates to the large literature on optimal auction design, starting with Myerson (1981), the classic paper for single-object auctions.[5] Post-auction interaction generates externalities not only between the firms and their customers in the downstream market,[6] but also among the bidders in the auction: each bidder's payoff depends not only on what it wins, but also on what its rivals win. A series of papers, most notably by Jehiel and Moldovanu, have explored single-object auctions with such externalities.[7]

Among the most closely related works, Jehiel and Moldovanu (2003) consider several examples of auctions of fixed-size spectrum licenses, and discuss the likely market outcomes.[8] Mayo and Sappington (2016) explore a Hotelling model in which a single block

[3]For instance, Anton and Yao (1987) show that second-sourcing can be used to reduce suppliers' informational rents. Rob (1986), Laffont and Tirole (1988) and Riordan and Sappington (1989) consider the trade-off between such *ex post* savings and suppliers' *ex ante* R&D incentives.

[4]See, e.g., Dana and Spier (1994), McGuire and Riordan (1995) and Auriol and Laffont (1992).

[5]See also Maskin et al. (1989) and Armstrong (2000). Also, Milgrom (2004) provides sufficient conditions for the simultaneous ascending auction to result in a Pareto-optimal equilibrium.

[6]Borenstein (1988) shows that the resulting discrepancy between private and social benefits can lead to inefficient outcomes.

[7]See Jehiel and Moldovanu (2001), Jehiel and Moldovanu (2000), and Jehiel et al. (1996). Also, Varma (2003) and Goeree (2003) consider auctions in which bids convey signals that affect rivals' behavior after the auction. See Salant (2014) for a more extensive discussion

[8]See also Hoppe et al. (2006), who show that limiting the number of licenses to be auctioned may foster entry, by exacerbating free-riding among incumbents' preemption strategies.

of spectrum is available. They show that an auction is unlikely to result in an optimal allocation and consider various corrective handicapping policies. Klemperer (2004) warns regulators against the temptation of taking measures to increase auction revenues at the cost of discouraging entry, and suggests instead the Anglo-Dutch hybrid auction as a way to balance the trade-off between revenues and post-auction concentration. Cramton et al. (2011) note that provisions favoring entrants need not always sacrifice auction revenues; they provide one example, with one incumbent and several symmetric potential entrants, in which setting a license aside for the entrants does not affect auction revenues. Cramton et al. (2011) also argue that, absent provisions to handicap large bidders, entrants and small participants are unlikely to win new spectrum; hence, regulators should concentrate their efforts on achieving an efficient allocation rather than revenue maximization. Finally, Janssen and Karamychev (2009, 2010) study the impact of auctions on *ex post* prices.[9] Their analysis, however, focuses on firms' risk preferences rather than on the impact on market structure.[10]

A couple of papers consider multi-object auctions. Levin and Skrzypacz (2016) examine bidder incentives in a Combinatorial Clock Auction ("CCA"). They show that bidders may bid more aggressively on packages that they anticipate to lose, in order to increase the price paid by rivals. More closely related to our paper is Kasberger (2017), who examines auction designs that can achieve an optimal allocation, in a setting where Cournot competitors bid on the entire allocation across firms.

There is also some empirical work that sheds light on the benefits of competition. Landier and Thesmar (2012) evaluate the macroeconomic impact of the entry of the fourth telecom operator in France, Free. They find that entry benefited the population in several ways. First, it had an immediate effect on consumer prices, which increased the purchasing power of the population. Second, the price shock induced by the enhancement of competition created between 16,000 and 30,000 jobs in France. The authors argue that far from distressing the financial position of incumbents, the increased com-

[9]See also Janssen and Karamychev (2007), who show that auctions do not always select the most efficient firm.

[10]Other papers include Moldovanu and Sela (2001) in which a seller is conducting an all pay auction so as to maximize the sum of the bidder payments (or efforts). Eső et al. (2010) examines efficient capacity allocations when there is Cournot competition in the downstream market and Brocas (2013) examines optimal auction design of a single, indivisible object when there are externalities.

petition encouraged investments in the sector.[11] Hazlett and Muñoz (2009) conducted a large-scale cross-country analysis of spectrum awards and found a significant positive relation between market concentration and consumer prices. This suggests that the social benefits from encouraging entry can more than offset the loss of auction revenue from spectrum withholding or concentration.

# 2 Model

For the sake of exposition, we consider the case of mobile communication services, where spectrum constitutes an essential input, and study the optimal allocation of new spectrum. It should be clear that the analysis can be readily transposed to the access to key inputs in other industries.

Two firms, an incumbent $I$ and a new entrant $E$, compete à la Bertrand for a consumer demand $D(p)$. The operators have constant returns to scale, but their costs depend on how much bandwidth they have: the more spectrum a mobile operator has, the more data it can carry at a given cell-site; it can thus maintain a given network capacity with fewer cells, and thus at lower costs. The incumbent starts with more spectrum, and thus enjoys a lower cost:

$$\bar{c}_I = c(B_I) < \bar{c}_E = c(B_E),$$

where $B_i$ denotes the bandwidth initially available to firm $i$, $\bar{c}_i$ denotes its initial unit cost, and $c(\cdot)$ is a strictly decreasing function (i.e., $c'(\cdot) < 0$) that is common to both firms. Thus, prior to the allocation of new spectrum, the entrant obtains no profit, whereas the incumbent obtains a profit which, assuming that the entrant exerts effective competitive pressure (see condition (4) below), is equal to:

$$\Pi(B_I, B_E) \equiv [c(B_E) - c(B_I)] D(c(B_E)), \tag{1}$$

[11] Woroch (2018) using US regional data finds higher spectrum concentration is associated with higher penetration. However, he admittedly cannot control for endogeneity of spectrum concentration.

which increases with the bandwidth advantage of the incumbent:[12]

$$\partial_1 \Pi(B_I, B_E) = -D(c(B_E))\, c'(B_I) > 0, \tag{2}$$

$$\partial_2 \Pi(B_I, B_E) = \{D(c(B_E)) + [c(B_E) - c(B_I)]\, D'(c(B_E))\}\, c'(B_E) < 0. \tag{3}$$

Let $\Delta$ denote the amount of new spectrum available. Each firm $i$ can thus obtain an additional bandwidth $b_i \geq 0$, subject to $b_I + b_E \leq \Delta$. With this additional bandwidth, the cost of firm $i$ can lie anywhere in the range $[\underline{c}_i, \bar{c}_i]$, where

$$\underline{c}_i = c(B_i + \Delta)$$

denotes the lowest cost that firm $i$ can achieve with all the additional spectrum. We assume that cost differences are never so drastic that competition is ineffective; that is, the incumbent cannot charge its monopoly price, even if it obtains all the additional spectrum:

$$\bar{c}_E < p^m(\underline{c}_I), \tag{4}$$

where $p^m(c) \equiv \min_p \{p \mid p \in \arg\max_{\tilde{p}} (\tilde{p} - c)\, D(\tilde{p})\}$. This assumption ensures that the competitive price is always equal to the higher of the two costs:[13]

- As long as $B_I + b_I > B_E + b_E$, $I$ maintains a cost advantage (that is, $c_I = c(B_I + b_I) < c_E = c(B_E + b_E)$) and thus wins the downstream market. The profits are then $\pi_E = 0$ and $\pi_I = \Pi(B_I + b_I, B_E + b_E)$, and consumer surplus is equal to $S(c(B_E + b_E))$, where

$$S(p) \equiv \int_p^{+\infty} D(x)\, dx.$$

- If instead $B_I + b_I < B_E + b_E$, $E$ obtains a lower cost; the profits of the two firms are then $\pi_I = 0$ and $\pi_E = \Pi(B_E + b_E, B_I + b_I)$, and consumer surplus is equal to $S(c(B_I + b_I))$.

[12]In what follows, $\partial_i f(\cdot)$ denotes the partial derivative of $f(\cdot)$ with respect to its $i^{\text{th}}$ argument.
[13]It also ensures that demand is positive in the relevant range.

# 3 Complete Information

As a benchmark, this section characterizes the optimal spectrum allocation when costs are public information. We first consider the case where the regulator aims at maximizing consumer surplus, before considering the case where it aims at maximizing social welfare, accounting for a social cost of public funds.

## 3.1 Consumer Surplus

We first note that a regulator maximizing consumer surplus should seek to minimize the cost asymmetry among the two firms:

**Proposition 1** *To maximize consumer surplus, it is optimal to allocate all the additional spectrum among the two firms so as minimize their cost difference. The associated consumer price is*

$$p^S \equiv \max\{\underline{c}_E, \hat{c}\}$$

*where* $\underline{c}_E = c(B_E + \Delta)$ *and*

$$\hat{c} \equiv c\left(\frac{B_I + B_E + \Delta}{2}\right).$$

**Proof.** See Appendix A. ■

The intuition is straightforward. Maximizing consumer surplus amounts to minimizing the competitive price, which is equal to the lower of the two costs, $c_I = c(B_I + b_I)$ and $c_E = c(B_E + b_E)$. Hence, it is always optimal to distribute all the additional spectrum, and if there is enough spectrum to offset the initial cost difference, it is optimal to allocate this spectrum so as to equate the two costs (leading to $p = \hat{c}$). If instead it is impossible to do so, then it is optimal to minimize the cost asymmetry by allocating all the additional spectrum to the entrant (leading to $p = \underline{c}_E$). Interestingly, a former FCC Chief Technology Office indeed argued that equalizing spectrum holdings is essential for effective competition among carriers.[14]

[14] See Peha (2017).

## 3.2 Social Welfare

In practice, industry regulators may need to pay attention to firms' profitability and/or to the revenues generated by the scarce resources that they manage. First, firms would not operate at a loss absent socially costly subsidies. This concern does not affect the findings of Proposition 1, however: the described allocation remains optimal even when taking into account firms' budget constraints, as the entrant always obtains zero profit and the incumbent obtains a non-negative profit. Second, firms' financial contributions (e.g., in the form of – lump-sum – spectrum licensing fees) reduce the public budget deficit and/or lower distortionary taxes. To account for this concern, we now suppose that the regulator aims at maximizing social welfare, defined as the sum of consumer surplus and, with a weight $\lambda \geq 0$, the revenues generated from the allocation of the resource $\Delta$, subject to firms' viability constraints; that is:

- Any transfer $t$ obtained from the firms generates a social gain $\lambda t$, representing the social benefit from reducing budget deficit or distortionary taxes; social welfare is thus given by

  $$S + \lambda \left( t_I + t_E \right),$$

  where $t_i$ denotes the transfer obtained from firm $i = I, E$ and, as before, $S = S\left(p\right)$ denotes consumer surplus.

- The regulator must accommodate the firms' profitability constraints: for $i = I, E$,

  $$\pi_i - t_i \geq 0.$$

It follows that it is optimal to choose $t_i = \pi_i$ for $i = I, E$; social welfare can thus be expressed as

$$S + \lambda \Pi,$$

where $\Pi = \pi_I + \pi_E$ denotes total industry profit.

Obviously, it is again optimal to allocate all the additional bandwidth $\Delta$:

**Lemma 1** *It is socially optimal to allocate all the additional spectrum.*

**Proof.** When $\lambda > 0$, giving any residual bandwidth to the firm with the lower cost (or to either firm, if both have the same cost) would further reduce its cost and increase industry profit, and thus the obtained revenues, without any adverse effect on consumers. When $\lambda = 0$ (i.e., the regulatory objective focuses on consumer surplus), giving any residual bandwidth to the firm with the higher cost (or sharing it equally between the two firms, if both have the same cost) would reduce the price and enhance consumer surplus. ■

Therefore, without loss of generality, we can restrict attention to spectrum allocations of the form $b_I = \Delta - b_E$, for some $b_E \in [0, \Delta]$. Furthermore:

- If $b_E < (B_I - B_E + \Delta)/2$, this spectrum allocation yields a competitive equilibrium of the form

$$p = c_E > \hat{c} > c_I = \gamma(p), \tag{5}$$

  where

$$\gamma(p) \equiv c\left(B_I + B_E + \Delta - c^{-1}(p)\right)$$

  denotes the lower cost among the two firms, when the bandwidth is allocated so as to set the higher cost to $p$. The resulting social welfare that can be expressed as:

$$W(p;\lambda) \equiv S(p) + \lambda\left[p - \gamma(p)\right] D(p). \tag{6}$$

- If instead $b_E > (B_I - B_E + \Delta)/2$ (in which case the feasibility condition $\Delta \geq b_E$ requires $\Delta > B_I - B_E$), then

$$p = c_I > \hat{c} > c_E = \gamma(p), \tag{7}$$

  which, keeping $p$ constant, generates the same social welfare as the equilibrium described by (5) (the roles of the two firms are simply swapped).

Hence, looking for the optimal spectrum allocation amounts to maximizing $W(p;\lambda)$ in the range

$$p \in \left[p^S, \bar{c}_E\right],$$

where

$$p^S \equiv \max\left\{\underline{c}_E, \hat{c}\right\},$$

with the caveat that, if $p^S = \hat{c} > \underline{c}_E$, any price $p \in [\hat{c}, \bar{c}_I]$ can be achieved in two equivalent ways, by conferring the same cost advantage to either firm.

We have:

$$\begin{aligned}\frac{\partial W}{\partial p}(p;\lambda) &= -D(p) + \lambda\left[1-\gamma'(p)\right]D(p) + \lambda\left[p-\gamma(p)\right]D'(p) \\ &= \lambda D(p)\left[\rho(p) - \gamma'(p) - \frac{1}{\lambda}\right] \qquad (8)\end{aligned}$$

where

$$\rho(p) \equiv 1 - \frac{p-\gamma(p)}{\mu(p)}$$

and

$$\mu(p) \equiv -\frac{D(p)}{D'(p)}$$

denotes the market power function – see Weyl and Fabinger (2013); $\rho(p)$ can be interpreted as a competition index: it is equal to 1 when $p = \gamma(p)\,(=\hat{c})$, that is, when both firms face the same cost and thus exert perfect competition on each other, and would be instead equal to 0 if $p = p^m(\gamma(c))$, that is, if the firm with the higher cost were no longer exerting any competitive pressure on the other firm – our working assumption rules out this case, implying $\rho(p) < 1$).

To ensure that maximizing $W(p;\lambda)$ yields a unique solution, we will maintain the following regularity conditions:

**Assumption A:**

1. The unit cost function $c(\cdot)$ is (strictly decreasing and) weakly convex: $c''(B) \geq 0 > c'(B)$ for any $B \geq 0$.

2. The market power function is weakly decreasing in the relevant range: $\mu'(p) \leq 0$ for any $p \in \left[p^S, \bar{c}_E\right]$.

Assumption A.1 asserts that, while using more spectrum enables the firms to reduce their costs (i.e., $c'(\cdot) < 0$), this is less and less so as more bandwidth becomes available; as the next Lemma shows, it ensures that, while $\gamma(p)$ strictly decreases as $p$ increases, it does so at a decreasing rate. Assumption A.2 amounts to assuming that the demand function is log-concave; it also implies that the monopoly pass-through rate is lower than one; together with Assumption A.1, it ensures that the competition index $\rho(p)$

strictly decreases with $p$. We have:

**Lemma 2** *For any $p \in \left[p^S, \bar{c}_E\right]$:*

*(i)* $\gamma(p) \leq \hat{c} \leq p$ *(with strict inequalities for $p > \hat{c}$) and, under Assumption A.1:*

$$-1 \leq \gamma'(p) < 0 \leq \gamma''(p).$$

*(ii)* $0 < \rho(p) < 1$ *and, under Assumption A:*

$$\rho'(p) < 0.$$

**Proof.** See Appendix B. ■

The following Proposition characterizes the socially optimal allocation:

**Proposition 2** *Let*

$$\underline{\lambda} \equiv \frac{1}{\rho(p^S) - \gamma'(p^S)} \text{ and } \bar{\lambda} \equiv \frac{1}{\rho(\bar{c}_E) - \gamma'(\bar{c}_E)}.$$

*Under Assumption A, $\underline{\lambda} = 1/2$ if $p^S = \hat{c} \geq \underline{c}_E$ and $\underline{\lambda} > 1/2$ otherwise, $\bar{\lambda} > \underline{\lambda}$, and the spectrum allocation that maximizes social welfare yields a unique equilibrium price, $p^W$, which is as follows:*

- *if $\lambda \leq \underline{\lambda}$, then it is optimal to minimize the cost difference: $p^W = p^S$;*
- *if instead $\lambda \geq \bar{\lambda}$, then it is optimal to allocate all the additional bandwidth to the incumbent: $p^W = \bar{c}_E$;*
- *finally, if $\underline{\lambda} < \lambda < \bar{\lambda}$, then $p^W$ lies (strictly) between $p^S$ and $\bar{c}_E$, and is the unique solution to*

$$\rho(p) - \gamma'(p) = \frac{1}{\lambda}. \tag{9}$$

*Furthermore:*

  - *when $p^S = \hat{c} > \underline{c}_E$, there exists $\hat{\lambda} \in \left(1/2, \bar{\lambda}\right)$ such that, if $\lambda \leq \hat{\lambda}$, there are two optimal spectrum allocations, giving the same cost advantage to either firm;*

– *otherwise (i.e., either $p^S = \underline{c}_E \geq \hat{c}$, or $p^S = \hat{c} > \underline{c}_E$ and $\lambda > \hat{\lambda}$), the optimal spectrum allocation is unique and maintains a cost advantage to the incumbent.*

**Proof.** See Appendix C. ■

As long as the weight on profits remains small, maximizing social welfare still amounts to minimizing cost differences, as is the case when focusing on consumer surplus. This is no longer the case, however, when the weight placed on profit becomes significant. In particular, even if cost equalization is feasible (i.e., the additional bandwidth is large enough to offset the initial asymmetry: $\Delta \geq B_I - B_E$, and so $p^S = \hat{c}$), it is no longer optimal whenever $\lambda > 1/2$, that is, whenever the weight on profits is more than half of that on consumer surplus.[15] Hence, in the particular case where the regulatory objective is "total welfare", measured by the sum of consumer surplus and profits (i.e., $\lambda = 1$), it is optimal to maintain a cost advantage (for the incumbent when $p^W > \bar{c}_I$, for either firm otherwise). This insight is quite robust: in any setting in which firms compete in prices, cost equalization is a local minimum of "total welfare", measured by the sum of consumer surplus and profits. To see this, note that total welfare can be expressed as

$$\Pi + S = S(p) + (p - c) D(p),$$

where $c = \gamma(p)$. Starting from cost equalization, where $p = c = \hat{c}$, and using $S'(p) = -D(p)$, introducing a slight asymmetry ($dp > 0 > dc = \gamma'(\hat{c})\, dp$) increases total welfare:

$$d(\Pi + S) = (p - c) D'(c)\, dp - D(c)\, dc|_{p=c=\hat{c}} = -D(\hat{c})\, dc > 0.$$

As the weight on profit further increases, the socially optimal price increases, up to the point that it may become optimal to give all the additional spectrum to the incumbent, so as to maximize industry profit. This, however, can occur only when the entrant still maintains a significant competitive pressure on the incumbent and/or the weight on profits exceeds that on consumer surplus.[16] By contrast, an increase in

[15] Indeed, when $p^S = \hat{c}\,(> \underline{c}_E)$, $\rho\left(p^S\right) = -\gamma'\left(p^S\right) = 1$ and, thus, $\underline{\lambda} = 1/2$.

[16] To see this, note that, in the limit case where $\bar{c}_E = p^m(\underline{c}_I)$, we have $\rho(\bar{c}_E) = 0$ and, thus, $\bar{\lambda} = -1/\gamma'(\bar{c}_E) > 1$.

the bandwidth initially available to either firm, or in the additional bandwidth made available, leads to a reduction in the socially optimal price. More precisely:

• If $\lambda \leq \underline{\lambda}$, then $p^W = c(B_E + \Delta)$; $p^W$ thus does not depend on $B_I$, but strictly decreases as $B_E$ or $\Delta$ increases.

• If instead $\lambda \geq \bar{\lambda}$, then $p^W = c(B_E)$; $p^W$ thus only depends on $B_E$, and strictly decreases as $B_E$ increases.

• Finally, it is optimal to divide the additional bandwidth, and we have:

**Corollary 1** *As long as $\underline{\lambda} < \lambda < \bar{\lambda}$, the socially optimal price strictly increases with $\lambda$, but strictly decreases as the total bandwidth, $B_I + B_E + \Delta$, increases.*

*Furthermore, as long as $p^S = \underline{c}_E \geq \hat{c}$, or $p^S = \hat{c} > \underline{c}_E$ and $\lambda > \hat{\lambda}$, the unique optimal spectrum allocation maintains a cost advantage to the incumbent and is such that:*

- *any increase in $\lambda$ leads to a re-allocation of the additional bandwidth $\Delta$ in favor of the incumbent;*
- *any increase in the additional bandwidth $\Delta$ is shared between the two firms;*
- *any increase in the bandwidth initially available to one firm, $B_E$ or $B_I$, leads to a re-allocation of the additional bandwidth $\Delta$ in favor of the other firm, which is however limited so as to ensure that both firms end-up with a larger total bandwidth.*

**Proof.** See Appendix D. ■

## 3.3 Product Differentiation

Assuming the firms produce perfect substitutes allows for a simple analysis highlighting key determinants and features of the optimal allocation of an essential input. In this case, one firm wins all consumers, and can do so by offering only slightly more attractive prices. In practice, firms often differentiate themselves, which enables them to share the market. To check the robustness of the previous insights, we consider a standard model of product differentiation, and show the main features of the optimal allocation remain similar to that of our baseline model when firms offer close substitutes. Specifically, on the supply side we assume, as before, that firms face constant unit costs that are a decreasing function of bandwidth (i.e., $c_i = c(B_i) > 0$, where $c'(B) < 0$), but on the demand side we now consider a classic Hotelling setting of horizontal differentiation, with elastic individual demands:[17]

- a mass $M$ of consumers are uniformly distributed over the segment $[0,1]$, and transportation costs are linear in distance but *independent* of the quantity purchased;
- there is a unit mass of consumers, who each have an identical elastic demand $d(p) > 0$, where $p$ is the price charged per unit, and $d'(p) < 0$;
- the two firms $i = 1, 2$ are located respectively at $x = 0$ and $x = 1$.

We show that the optimal allocation converges to that of our baseline model when the differentiation parameter tends to vanish. We provide a full analysis in Appendix E, and only sketch the main steps here.

Firms share the market as long as their prices, $p_1$ and $p_2$, do not differ too much; consumers located at $x < \hat{x}$ then purchase from firm 1 whereas those located at $x > \hat{x}$ purchase from firm 2, where $\hat{x}$ is determined by:

$$s(p_1) - t\hat{x} = s(p_2) - t(1 - \hat{x}),$$

where

$$s(p) \equiv \int_p^{+\infty} d(v)\, dv$$

denotes individual consumer surplus, and $t$ denotes transportation costs per unit distance. Firm $i$'s profit is therefore $\Pi_i = M\hat{x}_i(p_1, p_2)\, \pi_i(p_i)$, where

$$\pi_i(p_i) \equiv (p_i - c_i)\, d(p_i)$$

denotes firm $i$'s per consumer profit and

$$\hat{x}_i(p_1, p_2) \equiv \frac{1}{2} + \frac{s(p_i) - s(p_j)}{2t}$$

[17] We assume that individual demand is such that a price equilibrium exists for any given unit costs. This is the case for inelastic demands and, by continuity, remains the case as long as demand is not too elastic.

denotes its market share. Intuitively, competition becomes tougher as $t$ tends to vanish, and costs must therefore be almost equal for the market to be shared. If instead a firm faces a higher cost $c$, then the other firm (with cost $\gamma(c) < c$) corners the market and charges a price $p^*(c) \in (\gamma(c), c)$ such that:

$$s(p^*(c)) = s(c) + t,$$

and satisfies

$$p^{*\prime}(c) = \frac{d(c)}{d(p^*(c))} > 0.$$

The more efficient firm thus obtains a profit equal to

$$\Pi^*(c) \equiv [p^*(c) - \gamma(c)] D(p^*(c)),$$

where $D(p) \equiv Md(p)$ denotes total demand at price $p$.

When the regulator focuses on consumer surplus (i.e., $\lambda > 0$), as $p^{*\prime}(c) > 0$ it is never optimal to have a cost handicap larger than what is needed for one firm to "barely" corner the market. It follows that, as $t$ vanishes, the optimal allocation converges to cost equalization. When instead $\lambda > 0$, total welfare can be expressed as

$$W^*(c;\lambda) \equiv S(p^*(c)) + \lambda [p^*(c) - \gamma(c)] D(p^*(c)),$$

where $S(p) \equiv Ms(p)$ denotes total consumer surplus. As $t$ tends to vanish, $p = p^*(c) \simeq c$ and thus $W^*(c;\lambda) \simeq W(p;\lambda)$, the welfare function studied in the baseline model of Bertrand competition (see (6)). If the regulator wants instead to maintain a shared-market equilibrium outcome, then costs should be almost equalized ($c_I \simeq c_E \simeq \hat{c}$, and thus $p_I^* \simeq p_E^* \simeq \hat{c}$), and total welfare thus converges to $S(\hat{c}) = W(\hat{c};\lambda)$. Hence, in both types of equilibrium (shared-market or cornered-market), total welfare converges to $W(p;\lambda)$; it follows that the optimal allocation converges towards that of the Bertrand baseline model (perfect substitutes).

Summarizing, we have:

**Proposition 3** *In the Hotelling model in which consumer demand is elastic and transportation costs, $t$, are linear in distance, for $t$ sufficiently small, the welfare maximizing* *spectrum allocation is arbitrarily close to that which maximizes welfare in the baseline model with perfect substitutes and Bertrand competition, and the resulting market equilibrium price is arbitrarily close to* $p^W$.

**Proof.** See Appendix E. ■

# 4 Incomplete information

This section studies how the above allocation must be adjusted when firms' costs are private information. As we will see, the usual techniques for eliciting this private information rely on monotonicity conditions that do not hold here. In particular, the impact of the initial cost handicap on firms' willingness to pay for additional spectrum depends critically on which firm eventually benefits from a lower cost.

To characterize the optimal allocation, we focus on the simple case where only one firm, say the entrant, has private information; that is, $B_I$ is common knowledge, whereas the initial handicap of the entrant,

$$\theta = B_I - B_E \left(\geq 0\right),$$

is: (i) drawn from a cumulative distribution function $F(\theta)$, with continuous density $f(\theta)$ on a support $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, where $\bar{\theta} > \underline{\theta} \geq 0$; and: (ii) only observed by $E$.

This parameter $\theta$ should not be interpreted literally as the difference in spectrum holdings (which is likely to be public information); rather, we use it as a proxy for the initial cost asymmetry between the two firms. In practice, an incumbent benefits from scale economies arising from its existing spectrum holdings and from its denser network of cell sites; it may also benefit from a better bargaining position when dealing with equipment suppliers, and possibly from superior know-how and expertise (due, e.g., from learning-by-doing). Firm $i$' cost can thus be expressed as $C(A_i)$, where $A_i$ denotes firm $i$'s total asset and is of the form $A_i = K_i + B_i$, where $K_i$ reflects firm $i$'s accumulated capital other than spectrum, and $K_I > K_E$. In this setting, the relevant cost handicap of the entrant is given by $\theta = K_I - K_E + B_I - B_E$, and is likely to be private information even if the spectrum holdings $B_I$ and $B_E$ are public knowledge. For the sake of exposition, and in line with our previous analysis, we simply denote by $B_I$

and $B_E$ the two firms' initial "total assets".

As before, it is optimal to allocate all the additional bandwidth; thus, if $E$ obtains $b_E = b$, $I$ obtains $\Delta - b$. The costs of the two firms are then respectively given by

$$c_I = c\left(B_I + \Delta - b\right) \text{ and } c_E = c\left(B_I - \theta + b\right),$$

and they coincide when

$$b = \hat{b}\left(\theta\right) \equiv \frac{\theta + \Delta}{2}.$$

The market price is given by[18]

$$p\left(b, \theta\right) \equiv \max\left\{c\left(B_I - \theta + b\right), c\left(B_I + \Delta - b\right)\right\}.$$

and the profit of the entrant is of the form $\pi_E\left(b, \theta\right) - t$, where $t$ denotes the fee charged by the regulator for the acquisition of bandwidth. Gross profit is given by:

$$\pi_E\left(b, \theta\right) \equiv \begin{cases} \pi\left(b, \theta\right) & \text{if } b > \hat{b}\left(\theta\right), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \pi\left(b, \theta\right) &\equiv \Pi\left(B_I - \theta + b, B_I + \Delta - b\right) \\ &= \left[c\left(B_I + \Delta - b\right) - c\left(B_I - \theta + b\right)\right] D\left(c\left(B_I + \Delta - b\right)\right). \end{aligned}$$

To maximize expected social welfare, without loss of generality the regulator can offer a direct mechanism, which determines the allocation of the additional spectrum, $b\left(\theta\right)$, and monetary transfers from the entrant, $t\left(\theta\right)$, and the incumbent, $t_I\left(\theta\right)$, as a function of the cost handicap reported by the entrant. This direct mechanism must be individually rational: for the incumbent, it is obviously optimal to set $t_I\left(\theta\right) = \pi_I\left(\theta\right) = \Pi\left(B_I + \Delta - b\left(\theta\right), B_I - \theta + b\left(\theta\right)\right)$ whenever $\lambda > 0$ (in the particular case where the regulator focuses on consumer surplus – i.e., $\lambda = 0$ – any arbitrary transfer is optimal);

[18]As before, we assume that cost differences are never so drastic that competition is ineffective.

for the entrant the individual rationality constraint is given by:

$$\forall\theta \in \Theta, \pi_E\left(b\left(\theta\right), \theta\right) - t\left(\theta\right) \geq 0. \tag{$IR$}$$

In addition, the direct mechanism must be incentive compatible:

$$\forall\theta, \tilde{\theta} \in \Theta, \pi_E\left(b\left(\theta\right), \theta\right) - t\left(\theta\right) \geq \pi_E\left(b(\tilde{\theta}), \theta\right) - t(\tilde{\theta}). \tag{$IC$}$$

The optimal allocation under complete information remains optimal whenever it is implementable, that is, whenever there exists transfers $\{t\left(\theta\right)\}_{\theta\in\Theta}$ satisfying $(IR)$ and $(IC)$. This would obviously be the case when the handicap of the entrant is too large to be overcome (i.e., $\underline{\theta} > \Delta$): the optimal allocation under complete information, $\left\{b^{FB}\left(\theta\right)\right\}_{\theta\in\Theta}$ (where the superscript "$FB$" refers to "first-best"), together with $t\left(\theta\right) = 0$, is then trivially implementable. This is also the case when the weight placed on revenues, $\lambda$, is so large that the optimal allocation simply consists in giving all the additional bandwidth to the incumbent.

From now on, we focus on the non-trivial case where, $\lambda$ being not too large, it is optimal to share the additional bandwidth $\Delta$ between the two firms. From Corollary 1, the regulator then wishes to allocate more spectrum to entrants with larger handicaps. Unfortunately, this may no longer be feasible under incomplete information, because the classic "single-crossing" property does not hold here; indeed, the following lemma shows that the entrant's willingness to pay for additional bandwidth varies non-monotonically with its handicap:

**Lemma 3** *Fix $b$ and $b' > b$, and let*

$$\delta\left(\theta\right) \equiv \pi_E\left(b', \theta\right) - \pi_E\left(b, \theta\right) \geq 0$$

*denote the additional profit obtained by the entrant when increasing its share of bandwidth* *from $b$ to $b'$, as a function of the entrant's handicap. We have:*

$$\delta'(\theta)\begin{cases} >0 & \textit{if } \hat{b}(\theta)<b, \\ <0 & \textit{if } \hat{b}(\theta)\in(b,b'), \\ =0 & \textit{if } \hat{b}(\theta)>b'. \end{cases}$$

**Proof.** See Appendix F. ■

The intuition is as follows. When $E$'s initial handicap, $\theta$, is small (namely, such that $\hat{b}(\theta) < b$), giving it a share $b$ of the additional bandwidth enables it to win the competition for the market, and increasing this share further to $b'$ thus increases its profit from $\pi(b,\theta)$ to $\pi(b',\theta)$. Hence, $\delta(\theta) = \pi(b',\theta) - \pi(b,\theta)$ and:

$$\delta'(\theta) \equiv \int_b^{b'} \frac{\partial^2 \pi}{\partial b \partial \theta}(x,\theta)\,dx,$$

where the integrand is positive. To see this, note first that an increase in the handicap $\theta$, which increases $E$'s cost, decreases its profit:

$$\frac{\partial \pi}{\partial \theta}(b,\theta) = c'(B_I - \theta + b)\,D(c(B_I + \Delta - b)) < 0.$$

However, an increase in $b$ tends to limit this impact of a larger handicap: (i) the convexity of the cost function $c(\cdot)$ implies that the cost increase is reduced (i.e., $c'(B_I - \theta + b)$ becomes less negative as $b$ increases); and (ii) transferring some of the additional bandwidth from $I$ to $E$ increases the market price (driven by $I$'s cost), which, in turn, reduces the quantity impacted by the cost increase (i.e., $D(c(B_I + \Delta - b))$ decreases as $b$ increases).

When $E$'s initial handicap $\theta$ is instead very large (namely, such that $\hat{b}(\theta) > b'$), $E$ obtains zero profit anyway, and thus remains unaffected by a further increase in its handicap. Finally, when the initial handicap is in the middle range (i.e., $b < \hat{b}(\theta) < b'$), increasing the bandwidth share from $b$ to $b'$ enables $E$ to win the competition. It follows that $\delta(\theta) = \pi(b',\theta)$, which, as noted above, is a decreasing function of the handicap, $\theta$.

This non-monotonicity of $E$'s willingness to pay for additional bandwidth substantially constrains the set of incentive-compatible allocations. In particular, the next

Lemma shows that it can generate bunching as long as the weight placed on revenues is not too large (namely, $\lambda \leq 1/2$, which ensures that cost equalization would be optimal under complete information):

**Lemma 4** *Suppose that the regulator puts a weight* $0 \leq \lambda \leq 1/2$ *on revenues. If a direct incentive-compatible mechanism* $\{b(\theta), t(\theta)\}_{\theta \in \Theta}$ *satisfies* $b(\hat{\theta}) = \hat{b}(\hat{\theta})$ *for some* $\hat{\theta} \in \Theta$*, then* $b(\theta) = \hat{b}(\hat{\theta})$ *for any* $\theta \in \Theta$ *("full bunching").*

**Proof.** See Appendix G. ∎

The intuition is as follows. As long as the weight on revenues does not exceed 1/2, under complete information it is optimal to seek to minimize the cost difference, that is, to choose $b(\theta)$ as close to $\hat{b}(\theta)$ as possible. However, if the handicap of the entrant is exactly offset for some particular type $\hat{\theta}$ (i.e., $b(\hat{\theta}) = b^{FB}(\hat{\theta})$), then incentive compatibility requires $b(\theta)$ to lie above $b(\hat{\theta})$ for $\theta < \hat{\theta}$, and below $b(\hat{\theta})$ for $\theta > \hat{\theta}$. The best such schedule is therefore $b(\theta) = b^{FB}(\hat{\theta})$.

The following proposition shows that when demand is inelastic,[19] the objectives of the regulator and of the entrant are so conflicting that the same bandwidth must always be allocated to the entrant, regardless of its handicap:

**Proposition 4** *If* $\underline{\theta} < \Delta$ *and demand is inelastic, then the optimal allocation under incomplete information consists of offering the same bandwidth* $b^{SB}$ *to the entrant, regardless of its type, where*

$$\hat{b}^{SB}(\lambda) \equiv \arg\max_{b \leq \Delta} E_\theta \left[ W\left( p(b, \theta); \lambda, \theta \right) \right] > b^{FB}(\underline{\theta}).$$

**Proof.** See Appendix H. ∎

The lessons from this analysis are two-fold. On the one hand, the qualitative insights of the previous section for the case of complete information appear robust: as long as the weight on revenues is not too large, it is socially desirable to allocate some of the bandwidth to the entrant (even if the incumbent ends up serving the market), and to do so in a way that limits the cost asymmetry between the two firms. On the other hand,

[19] That is, $D(p)$ is constant in the relevant price range.

accounting for firms' private information leads to the adoption of somewhat coarser mechanisms, because of the conflict of interests that arises between the regulator and the firms. In particular, under incomplete information the optimal mechanism exhibits full "bunching:" the allocation is the same, regardless of the magnitude of the handicap. Hence, the previous insights carry over, but rely on the overall distribution of the cost handicap, rather than on its actual realization.

We show in the Online Appendix that this insight extends to the case of an elastic demand when the handicap is sufficiently diverse (namely, $\underline{\theta} < \Delta < \bar{\theta}$) and attention is, moreover, restricted to bandwidth allocations that vary continuously, or monotonically, with the handicap of the entrant (see Online Appendix A). However, incentive compatibility does allow for discontinuous as well as non-monotonous allocations (see Online Appendix B); we provide an example with elastic (namely, linear) demand and a handicap taking two, not too diverse values (namely, $\underline{\theta} < \bar{\theta} < \Delta$), in which the optimal allocation varies with the handicap (see Online Appendix C).

# 5 Standard Auctions

Regulators use different auction formats for allocating multiple blocks of spectrum (which may be heterogenous): clock and combinatorial clock auctions (CCAs), simultaneous multi-round ascending auctions (SMRAs), and sequential auctions. To compare these auction formats, we adopt a discretized version of our framework and assume that the additional spectrum $\Delta$ is divided into $k$ equal blocks of size $\delta \equiv \Delta/k$; for the sake of exposition, we moreover focus on the case of complete information among the two bidders.

As regulators often allocate different tranches of spectrum at different times, we first consider sequential auctions (Section 5.1). To approximate CCAs, we then turn to simultaneous Vickrey-Clarke-Groves (VCG) auctions (Section 5.2).[20] Finally, to approximate SMRAs, we discuss the case of simultaneous clock auctions (Section 5.3).[21]

[20]A CCA starts with multiple clock rounds, each involving single package bids, and ends with a supplementary round with multipe package bids. VCG allocation rules then apply, except when the VCG outcome would not be in the core.

[21]In SMRAs and clock auctions, prices increase across multiple rounds of bidding. The main difference is that prices are set by bidders in SMRAs, and by the auctioneer in clock auctions. With homogenous blocks and symmetric information, the two auctions result in essentially the same out-

We show below that, in our setting, the outcomes of these auction formats depart drastically from the optimal allocations characterized above: while all the available spectrum is allocated, it is likely to go to the incumbent, thus reinforcing its initial advantage.

## 5.1 Sequential Auctions

We start with the case of sequential auctions. Specifically, we assume that $k$ successive auctions are organized, one for each of block, and that the outcome of each auction is publicly announced before the next auction takes place. As bidders have perfect information about each other, all classic auction formats (first-price or second-price sealed-bid auctions, as well as ascending or descending auctions) yield the same outcome. For the sake of exposition, we will refer to these auctions as "classic auctions". It is well-known that these auctions can generate multiple equilibria, as the losing bidder may bid more than its value without incurring a loss; to address this issue, we focus on coalition-proof Nash equilibria – see Bernheim et al. (1987); in our two-bidder setting, this amounts to focusing on Pareto-undominated Nash equilibria.

The following proposition shows that the incumbent wins all the auctions, and may even do so at zero price if the initial handicap of the entrant exceeds the size of the individual blocks:

**Proposition 5** *Suppose that $k$ blocks are sold sequentially using any classic auction format. At any coalition-proof Nash equilibrium, the incumbent wins all $k$ auctions; furthermore, if $B_I - B_E > \Delta/k$, then the incumbent acquires each block for free.*

**Proof.** See Appendix I. ■

The intuition is simple, and reminiscent of the insight of Vickers (1986) for patent races.[22] When a monopolistic incumbent bids against a potential entrant for a better technology, the incumbent's gain from preserving its monopoly position (and enjoying the better technology) exceeds the profit that the entrant would obtain in a duopoly situation, even when the new technology would enable the entrant to win the competition for the market. Likewise, here the incumbent, who by assumption is the initial leader,

come.

[22]Also see Gilbert and Newbery (1982) and Riordan and Salant (1994).

gains more from maintaining its incumbency advantage than an entrant gains from overtaking the incumbent.

Proposition 5 shows that, with sequential auctions, the incumbent can obtain all the additional spectrum for free when this spectrum is divided into sufficiently many small blocks. However, when the size of the blocks exceeds the initial handicap of the entrant, the incumbent pays for the first block a price equal to:

$$p^S(B_I, B_E) = \begin{cases} \sum_{h=0}^{m-1} \phi_h^k(B_E, B_I) - \sum_{h=1}^{m} \phi_h^k(B_I, B_E) & \text{if } k = 2m, \\ \sum_{h=0}^{m} \phi_h^k(B_E, B_I) - \sum_{h=1}^{m} \phi_h^k(B_I, B_E) & \text{if } k = 2m+1. \end{cases}$$

where

$$\phi_h^k(B_1, B_2) \equiv \Pi(B_1 + (k-h)\delta, B_2 + h\delta).$$

In the particular case where $B_I = B_E = B$, both firms bid the full benefit generated by the additional spectrum and the equilibrium price is thus equal to

$$p^S(B, B) = \Pi(B + \Delta, B).$$

## 5.2 VCG Auctions

This section considers a single, simultaneous VCG auction for all $k$ blocks, in which each bidder submits a sealed bid demand schedule specifying how much it would offer for every number of blocks it may wish to purchase. That is:

- Each firm $i = I, E$ submits a bid of the form[23]

  $$\beta_i = \{\beta_i(n_I, n_E)\}_{(n_I, n_E) \in \mathcal{A}},$$

  where $n_i \in \mathcal{K} \equiv \{0, 1, 2, ..., k\}$ denotes the number of blocks assigned to firm $i \in \{I, E\}$, and

  $$\mathcal{A} \equiv \{(n_I, n_E) \in \mathcal{K} \times \mathcal{K} \mid n_I + n_E \leq k\}.$$

[23]In theory, bids should be made for each entire allocation $(n_I, n_E)$. In practice, firm $i$ is often asked to submit bids for the various combinations of slots assigned to it (that is, $\beta_i = \{\beta_i(n_i)\}_{n_i \in \mathcal{K}}$). However, in our simple two-bidder setting, in which all $k$ blocks are always allocated, the distinction is moot.

- The resulting spectrum allocation, $n^V(\beta_I, \beta_E) = \left(n_I^V(\beta_I, \beta_E), n_E^V(\beta_I, \beta_E)\right)$ maximizes the sum of the offers over feasible allocations, i.e.,

$$n^V(\beta_I, \beta_E) = \arg\max_{n \in \mathcal{A}} \{\beta_I(n) + \beta_E(n)\}.$$

- Finally, the price paid by each bidder $i$ is the value that the other bidder would offer for bidder $i$'s blocks, and is thus equal to (where the subscript "$-i$" refers to firm $i$'s rival)

$$p_i^V(\beta_I, \beta_E) = \max_{n \in \mathcal{A}} \{\beta_{-i}(n)\} - \beta_{-i}\left(n^V(\beta_I, \beta_E)\right).$$

It is well-known that it is a dominant strategy for each firm to bid its full value for each package. The following proposition shows that, in this equilibrium, the incumbent again wins all the blocks. However, it always pays a positive price whenever the additional spectrum is large enough to offset the handicap of the entrant:

**Proposition 6** *In a simultaneous VCG auction for the k blocks, the incumbent wins all the blocks and pays a price equal to the entrant's profits from winning all the blocks:*

$$p^V(B_I, B_E) = \begin{cases} \Pi(B_E + \Delta, B_I) & \text{if } \Delta > B_I - B_E, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** See Appendix J. ■

The underlying logic is the same as for sequential auctions, and results again in allocating all the additional spectrum to the incumbent. Furthermore, if the additional bandwidth is not large enough to fully offset the initial handicap of the entrant, then revenues are zero in both auctions. Otherwise, Proposition 6 shows that the incumbent must now pay a positive price, independent of the block size. The following proposition shows that this price is typically higher than in sequential auctions:

**Proposition 7** *Revenues are always at least as high in a VCG auction than in a sequential auction, and strictly higher in the case where $\Delta > B_I - B_E$; furthermore, VCG revenues are independent of the block size, whereas a sequential auction brings no revenue if the size of the blocks is sufficiently small.*

**Proof.** See Appendix K. ■

### 5.3 Clock Auctions

Now consider a two-bidder clock auction. The auctioneer posts a price per block, and bidders announce how many blocks they want at that price. The posted price is initially set to zero and increases by increments as long as there is excess demand; as price increases, bidders can maintain or reduce the number of blocks they demand, but not augment it. When the market clears, each bidder obtains its desired number of blocks at the clearing price.[24]

In this auction, it is a dominant strategy for the entrant to bid for all $k$ blocks as long as the clock price $p$ satisfies

$$p < p^E \equiv \frac{\Pi(B_E + \Delta, B_I)}{k},$$

and to drop out once the posted price tops $p^E$. By contrast, the dominant strategy of the incumbent is to bid for all $k$ blocks as long as the clock price $p$ satisfies

$$p < p^I \equiv \frac{\Pi(B_I + \Delta, B_E)}{k}.$$

As $p^I > p^E$ whenever $B_I > B_E$, the incumbent wins the clock auction at price $p^E$, which is the same outcome as with a VCG auction. Thus, we have:

**Proposition 8** *Auction outcomes are the same with a simultaneous VCG auction and a clock auction.*

## 6 Policy Implications

The above analysis characterizes the optimal allocation of essential inputs and compares it with the outcomes of standard auctions, with and without measures (such as caps and set-asides) designed to maintain ex post competition. We show that the outcome of standard auctions typically differs drastically from the optimal allocation. With pure

[24]When demand abruptly drops below the clearing level, various tie-breaking rules apply, which often involve a random element.

Bertrand competition, the leading firm tends to win all the bandwidth, whereas the optimal allocation limit the cost differences of the firms whenever the weight on revenues is not too large. With product differentiation, the optimal outcome is again to limit the bandwidth asymmetries when products are not too differentiated.[25] More generally, any auction which maximizes revenue is unlikely to maximize social welfare unless maximizing industry profit happens to coincide with maximizing consumer welfare.

While we focus for simplicity on a setting in which a single incumbent faces a single challenger, the insights can shed some light on regulators' actual policy choices. In a number of recent spectrum auctions, regulators have faced quite explicit choices between running a competitive auction and avoiding consolidation in the market. This was the case in a 2019 auction in Switzerland, in which three competitors are vying for the 30 MHz available in the 700 MHz band (a relevant band for 5G networks).[26] The smallest challenger, SALT, stated that "with less than 10 MHz [in that band] no competitive and nationwide 5G network can be operated", thus suggesting that, in order to maintain competition, the regulator should allocate one third of the spectrum to each incumbent.[27]

In practice, as shown in Table (1), the larger incumbents tend to win more spectrum, despite the fact that most European auctions have had caps or set-asides to favor challengers.[28] This suggests that increased concentration in spectrum holdings, and in downstream market power is a risk that tends to increase over time. This section discusses regulatory instruments and policy to address this issue.

[25] In a Cournot model with linear inverse demand $P(Q) = a - bQ$, total consumer and producer surplus is $W = \frac{(2a-c_1-c_2)^2}{18b}$; it is therefore maximized when firms have equal bandwidth whenever unit costs are weakly convex in bandwidth. There, as well, standard auctions will not achieve this result – see Kasberger (2017).

[26] A similar situation arose in the US Incentive Auction, where the regulator (the FCC) had to determine how much spectrum to reserve for Sprint and T-Mobile, the two firms with limited holdings; the FCC reserved a spectrum adequate for one, but not both, of the two firms: the reserve was limited to three blocks, and each laggard needed to win two blocks in order to become economically efficient. In a 2013 auction, the Austrian regulator had to decide on caps in the allocation of 28 blocks among three incumbents. The adopted cap of 14 blocks resulted in very high revenues and nearly forced market consolidation.

[27] See https://www.handelszeitung.ch/unternehmen/telekommunikation-vertragswidrig-salt-pruft-klage-gegen-upc.

[28] See the Appendix for a list of the auctions included in this measure and further details about these auctions.

| | |
|---|---|
| Correlation of MS and total spectrum won | 0.674 |
| Correlation of MS and HF spectrum won | 0.537 |
| Correlation of MS and LF spectrum won | 0.696 |

Table 1: Correlation of market shares (MS) and total, high and low band frequency (HF,LF) winnings

## 6.1 Regulatory Instruments

Regulators can adopt different measures to promote competition – the most common being spectrum caps, set-asides and bidding preferences or discounts. A spectrum cap is a limit on the total amount of spectrum a firm can have. A set-aside reserves some spectrum for target groups such as "challengers" or entrants. A bidding preference provides a discount off the final auction price to such challengers. Each of these provisions can take different forms. For example, a cap can be on the amount available in the auction or on overall spectrum holdings, including what bidders have prior to the auction. A set-aside can also take the form of floors, i.e., minimum spectrum packages.[29]

Caps and set-asides impose similar restrictions on the set of feasible allocations. To see this, let $B_E \geq 0$ and $B_I > B_E$ denote, as before, the amount of spectrum initially owned by the entrant and the incumbent, and by $\Delta$ the additional amount made available. Introducing an overall cap $K$ limits the additional bandwidth $b_i$ that firm $i$ can obtain, that is:

$$B_i + b_i \leq K.$$

Obviously, a cap $K$ has no effect if it exceeds $B_I + \Delta\,(> B_E + \Delta)$. Introducing instead a "binding" cap $K < B_I + \Delta$ *de facto* reserves a bandwidth

$$S\left(K\right) \equiv \Delta - \left(K - B_I\right)$$

for the entrant. However, compared with the imposition of such a cap $K$, introducing a set-aside $S\left(K\right)$ further restricts the set of feasible allocations, as it also prevents the entrant from having less than $S\left(K\right)$ of additional bandwidth; that is, both instruments can be used to put the same *upper* bound on $I$'s share of the additional spectrum, $b_I$,

[29] For instance, floors have been used in the UK – see https://www.ofcom.org.uk/about-ofcom/latest/media/media-releases/2011/ofcom-prepares-for-4g-mobile-auction; for a discussion of this case, see Myers (2013).

but in addition a set-aside puts a *lower* bound on $E$'s share of this spectrum, $b_E$. As a result, as discussed below, caps and set-asides have different impacts on the outcome of an auction.

Some auctions, such as the early US auctions, included spectrum caps on overall spectrum holdings, of the form $B_i + b_i \leq K$, as discussed above.[30] Other auctions have used caps on the amount of spectrum each firm can acquire in the auction, of the form $b_i \leq k$. However, these two variants can have very different effects on product market competition. Indeed, whereas an overall cap can be effective in limiting the acquisition of additional spectrum by the incumbent, an auction cap puts more stringent limitations on the entrant, and may actually keep it from overtaking the incumbent.

Set-asides are often accompanied by reserve prices, which tend to discourage entry – in our Bertrand setting, any positive reserve price would deter entry. Entry was indeed discouraged in the 2013 4G auctions in Austria and limited in the UK, which relied on various forms of set-asides accompanied by reserve prices. In the same vein, while our analysis suggests that all spectrum should be allocated, this does not always occur in practice, as reserve prices sometimes result in unsold blocks. For instance, this has been the case in 4G auctions in Spain and Portugal, in which some of the most valuable (900 MHz) spectrum remained unsold.[31]

When the initial handicap of the entrant is very large, it is optimal to allocate all the additional spectrum to the entrant. Overall caps, set-asides and bidding credits can all be used to accomplish this.[32] In what follows, we concentrate on the more interesting case where the regulator finds it optimal to share the additional spectrum between the two firms.

Consider first the case where the regulator focuses on consumer surplus. From Proposition 1, it is optimal to equalize the costs of the two firms; that is, the optimal allocation

[30]See https://www.fcc.gov/node/189694 for a discussion of US spectrum caps.

[31]See http://www.minetad.gob.es/telecomunicaciones/es-ES/ResultadosSubasta/Informe_Web_29072011_fin_de_subasta.pdf and http://www.anacom.pt/streaming/Final_Report_Auction.pdf?contentId=1115304&field=ATTACHED_FILE for results of the 2011 Spanish and Portuguese multi-band auctions. Also, in France, the regulator took a number of years to reduce the reserve price before awarding a fourth 3G license. See https://www.arcep.fr/?id=8562.

[32]Specifically, an overall cap $K = B_I$, a set-aside $S = \Delta$, or a bidding credit reflecting $I$'s profit with the additional spectrum, would all result in $E$ winning all the additional spectrum. When the regulator cares about revenues as well, however, this would need to be complemented with a tax (e.g., an unconditional tax on $I$'s equilibrium profit).

$\left(b_I^S, b_I^S\right)$ is such that:

$$c_I\left(B_I + b_I^S\right) = c_E\left(B_E + b_E^S\right) = \hat{c} = c\left(\frac{B_I + B_E + \Delta}{2}\right).$$

This could be achieved with an overall cap set to $K^S = (B_I + B_E + \Delta)/2$. By contrast, a set-aside $S^S = S\left(K^S\right)$ would not work, as it would put the entrant ahead of the incumbent (as $B_E + S^S \left(= K^S\right) > B_I$), and thus result in the entrant winning all the additional spectrum. A bidding credit would not be effective either, as it would result in either firm winning all the additional spectrum ($I$ if the bidding credit is too low, and $E$ otherwise).

When instead the regulator also cares about revenues, Proposition 2 applies, and two cases arise.

• If the handicap of the entrant and/or the weight placed on revenues is not too large, then there are two optimal allocations, which: (i) confer a competitive advantage to one firm and lead to the same consumer price, $p^W$ – more specifically, one firm (the "loser") ends-up with an overall holding equal to $B_l^W \equiv c^{-1}\left(p^W\right)$, whereas the other firm (the "winner") accumulates a total amount of spectrum equal to $B_w^W \equiv c^{-1}\left(\gamma\left(p^W\right)\right) > B_l^W$; and: (ii) appropriate the winner's equilibrium profit, $\pi^W \equiv \left(p^W - c\left(K^W\right)\right) D\left(p^W\right)$. This optimal allocation could be achieved by setting aside an amount $S^W = B_I - B_E$ for the entrant, and introducing an overall cap set to $K^W = B_w^W$. The set-aside is designed to offset the initial handicap of the entrant; as a result, both firms are willing to bid up to $\pi^W$ to reach the overall cap.[33] Interestingly, neither instrument alone suffices to achieve a desired outcome. Relying only on a set-aside would again result in one firm (either one) winning all the additional spectrum (the same would be obtained with a bidding credit). Relying only on a cap could achieve the desired spectrum allocation (by setting the cap to $K^W$), but it would leave a positive rent to the incumbent;[34] a second instrument would be required to deal with this issue.[35]

[33]Consider, for instance, a simple clock auction with the set-aside $S^W$ and the overall cap $K^W$ (with the rule that any unbid spectrum is allocated to the losing bidder). As $E$ secures an amount $B_I - B_E$ for free, both firms are then willing to bid up to $\pi^W / \left(K^W - B_I\right)$ to obtain the additional amount $K^W - B_I$. As a result, either firm wins and accumulates an overall holding of $K^W = B_w^W$, and the other firm obtains an overall holding of $B_l^W$, leading to the market price $p^W$.

[34]For instance, in a simple clock auction $I$ would be willing to bid up to $\pi^W / \left(K^W - B_I\right)$, whereas $E$ is only willing to bid up to $\pi^W / \left(K^W - B_E\right)$. The conclusion follows from $B_E < B_I$.

[35]This, for instance, could be an unconditional tax on $I$'s profit, as described in footnote 32. Alter-

• If the handicap of the entrant and/or the weight placed on revenues is large enough, the unique optimal allocation is such that $c_E^W = c\left(B_l^W\right) > \max\{\bar{c}_I, \underline{c}_E\}$, and thus satisfies: $B_E^W = B_l^W < B_I$ – that is, it is no longer optimal to offset the initial handicap of the entrant. In this case, setting aside an amount $S^W = B_l^W - B_E$ for the entrant, or alternatively introducing a cap set to $K^W = B_w^W$, would both achieve the desired spectrum allocation. However, either instrument would again need to be complemented with another instrument designed to limit the rent left to the incumbent.[36]

## 6.2 Regulatory Experience and Policy

Most countries have adopted auctions and other spectrum assignment procedures, including caps and set-asides, designed to promote *ex post* competition in the market for mobile communications services. However, these provisions have tended to have limited long-run impact, once the first set of spectrum allocations was completed.

The three initial waves of spectrum allocations resulted in 4 - 5 mobile operators in most European countries and at least 4 operators in almost all of the US and Canada,[37] and often as many as 5 or 6.[38] Since the 3G auctions, however, consolidation has been the rule in much of Europe, including in Austria, Germany, Italy, the Netherlands, Switzerland and the UK, despite various measures employed to promote competition.[39] The Netherlands set aside two prime, low frequency 4G blocks for entrants,[40] which did attract two new bidders; however, only one entrant won any blocks, and after the

natively, it could take the form of a reserve price – a non-linear or discriminatory reserve price may however be needed.

[36]Combining a set-aside with a cap no longer suffices here to achieve the desired allocation *and* to appropriate the profit $\pi^W$; that is, the additional instrument should again be a tax on the incumbent's equilibrium profit, and/or some form of a reserve price.

[37]For the first generation of analog cellular in the Americas and in Europe, regulators tended to award one license to the incumbent local exchange carrier and one to an entrant. Additional operators entered during the second and third waves of spectrum allocations, starting in the late 1980's and running through the early 2000's (first for 2G, and then for 3G spectrum).

[38]The US and Canada, unlike European countries, awarded regional and not national licenses.

[39]The UK had 5 mobile network operators ("MNOs"), but even after set-asides in the recent 4G auction, only 4 remain, and there is talk of further consolidation. Germany had 6 winners after their 3G auction, but two winners abandoned their licenses, and subsequent to a recent merger, there are now only 3 MNOs left. The Netherlands had 5 MNOs, but mergers resulted in 3 players. The Austrian 3G auction had 6 winners. By the time the Austrians auctioned off 4G spectrum, a little over 10 years after the 3G auctions, there were only three MNOs remaining in the market. Finally, the 2013 re-auction of legacy spectrum in Austria nearly left Austria with two viable MNOs. See Salant (2014)) for a discussion.

[40]By contrast, Germany turned down requests of entrants for set-asides.

auction it signed a network-sharing agreement with the one incumbent (T-Mobile) that failed to win any low frequency blocks, and eventually merged with it. Austria failed to attract any bidders for the two blocks provisionally set aside for entrants. The UK's provisions for a floor mentioned in footnote 29 also failed to induce meaningful changes in the competitive structure.

The US, too, has seen continuing consolidation.[41] Since the FCC abandoned overall spectrum caps in 2003, the two largest MNOs have acquired most of the spectrum that has been auctioned. The HHI has increased from 2151 in 2003 to 3027 at the end of 2013. In a very recent auction for AWS-3 spectrum, AT&T and Verizon spent 6 and 10 times as much as the third largest MNO in the auction (T-Mobile), and no other MNO spent even 1% of what AT&T spent. In the most recent 600 MHz auction, the reserve price of $1.25 per MHzPOP[42] has deterred the weakest incumbent, Sprint, from even participating.[43]

Finally, Canada at one time had 4 national operators, which was reduced to 3 via merger. Despite having conducted a series of auctions with provisions including caps and regulations on wholesale prices, in the hope of attracting more competitors, no fourth national operator has emerged.

Regulatory agencies face a great deal of uncertainty. Experience in the past several years suggests that set-asides are not very likely to attract new entrants when incumbents start with a large amount of spectrum, and significant sunk investments that an entrant would need to duplicate. This suggests that caps that limit further consolidation may be preferable to set-asides or other measures to favor entrants. That is, the goal is limiting risk of excessive consolidation, as attracting entrants has proven difficult. A few auctions, the UK 4G auction, the US Incentive Auction, and the Canadian 600 MHz auction, have included what are effectively set-asides for smaller players.[44]

[41] Among the large regional and national carriers that at one time existed: (i) Cingular, BellSouth, Ameritech, and Leapwireless have all been absorbed by AT&T; (ii) BellAtlantic, NYNEX, USWest, Airtouch , GTE, Cincinnati Bell, and Alltel have been absorbed by Verizon; (iii) Western Wireless, Voicestream, Omnipoint and Powertel formed T-Mobile, which subsequently acquired MetroPCS; (iv) Sprint merged with Nextel; and (v) US Cellular is still independent, but has sold off most of its larger markets.

[42] The term "per MHzPOP" is used to compare prices of different sized blocks in different countries or regions. Literally, one MHzPOP is a license of one MHz covering an area with a population of one.

[43] In addition, the spectrum reserved for challengers sold eventually for essentially the same price than the non-reserved spectrum - less than 1% overall difference, and, in nearly 20% of the PEAs the reserved spectrum was more expensive than the non-reserved spectrum.

[44] The UK had spectrum "floors" for both entrants and smaller players. The US Incentive Auction

# 7 Conclusion

This paper characterizes the optimal allocation of a scarce resource (e.g., spectrum rights) between an incumbent and a challenger, for a regulator seeking to maximize the social surplus. The main insight is that the regulator wants to limit the dominance of the incumbent, and ensure that the challenger exerts an effective competitive pressure. More specifically, when the regulator focuses on consumer surplus, and does not care about auction revenues, it tries to equalize firms' competitiveness. When instead the regulator seeks to maximize social welfare, taking into account auction revenues, it finds it optimal to maintain some asymmetry among the competitors.

Further, we find that there is a tension between the regulator's objective and the challenger's incentives to report its handicap. More specifically, a regulator may want to provide a weaker challenger more spectrum than a stronger challenger. However, doing so gives the stronger challenger an incentive to try to act as if it is weak. As a result of this tension, the optimal auction is likely to exhibit "bunching", in that the challenger ends up with the same allocation, regardless of its initial handicap.

The finding that the regulator wishes to limit dominance contrasts sharply with the outcome of standard types of auctions, such as sequential, clock and VCG auctions, which all result in increasing dominance: in the Bertrand competition setting that we consider, the incumbent always obtains all the additional spectrum.[45] Furthermore, while the spectrum allocations are the same, revenues are lower in a sequential auction.

Finally, we examine some policy implications. When the regulator's objective only includes consumer surplus, a cap on firms' overall spectrum holdings can suffice to achieve the desired allocation. By contrast, neither a cap on the amount of spectrum that any firm can win in the auction, nor a set-aside reserved for the challenger are helpful – auction-specific caps could actually be counter-productive, as they may limit the challenger's ability to reduce its handicap. When the regulator also cares about auction revenues, an overall cap needs to be complemented with a set-aside or with

included "reserved" and "non-reserved" spectrum. Interestingly, in that auction, Verizon never bid, AT&T stopped bidding in the after the first of four stages, and there was on average a 1% difference in the average price of set-aside and non-set-aside licenses. The 2019 Canadian auction includes set-asides for "those bidders that are registered . . . as facilities-based providers ."

[45] This may not always hold when firms compete à la Cournot competition, in which case the packaging of blocks can also affect the outcome.

another instrument designed to limit the incumbent's rent.

While this paper has focused on spectrum auctions, similar issues arise in many other sectors. We discuss a few below.

*Sports broadcasting rights.*

Sports broadcasting rights, and, in particular, soccer rights are often regulated, especially in Europe. The reason for competition authorities to intervene is the concern that a concentration of broadcasting rights could create or reinforce the dominance of the rights holder.[46] Indeed, the fraction of rights owned by a provider affects the perceived value of its offering relative to that of its rivals. Suppose, for example, that the value a consumer derives from provider $i$'s offering is of the form:

$$u + v(r_i),$$

where $r_i$ denotes the fraction of rights own by firm $i$. Formally, the increase in firm $i$'s perceived quality stemming from an increase in $r_i$ then has the same effect as the cost reduction resulting from an increase in the bandwidth $b_i$ in our model.

*Train scheduling.*

Another application is the allocation of train slots. The frequency of service offered on a given route will affect the average wait time, and thus the cost imposed on customers. Many riders, e.g., because they purchased their tickets in advance or because they benefit from loyalty programs, will therefore favor the carrier offering the most frequent service. Suppose, for example, that the net value offered by firm $i$ is of the form:

$$u - c(s_i),$$

where $s_i$ denotes the number of train slots allocated to firm $i$. The reduction in customers' costs stemming from an increase in $s_i$ then has the same effect as the reduction in firm $i$'s own operating cost in our model.

[46] See, e.g., `http://ec.europa.eu/competition/elojade/isef/case_details.cfm?proc_code=1_38173`.

*Electric transmission.*

Electricity transmission rights raises similar issues in regions in which energy supply is limited and relatively costly.[47] The particular details of electricity markets differ quite a bit from spectrum,[48] but the basic message that the allocation of transmission rights can affect post-auction competition applies.

Similar issues arise with the allocation of many other scarce resources, such as landing slots (and other airport facilities, such as gates, kerosene tanks, and so forth), gas pipeline capacity, concessions to operate in given areas (e.g., highway service stations), or when zoning regulations put constraints on commercial activities or on the number (and/or the size) of supermarkets.[49]

[47] See Joskow and Tirole (2000) and Loxley and Salant (2004).

[48] See for example `http://pjm.com/markets-and-operations/ftr.aspx`, and also Salant (2005).

[49] In France, for instance, zoning regulations have prevented the entry of new retail chains.

# Appendix

## A Proof of Proposition 1

It is obviously optimal to allocate all the additional spectrum: allocating any residual bandwidth equally among the two firms reduces for sure the resulting competitive price and thus benefits consumers.

Without loss of generality, we can thus restrict attention to spectrum allocations of the form $b_E \in [0, \Delta]$, $b_I = \Delta - b_I$, yielding a competitive price equal to

$$\begin{aligned} p &= \max\left\{c\left(B_E + b_E\right), c\left(B_I + b_I\right)\right\} \\ &= \max\left\{c\left(B_E + b_E\right), c\left(B_I + \Delta - b_E\right)\right\}. \end{aligned}$$

Maximizing consumer surplus amounts to minimizing this competitive price; therefore:

- If $\Delta \geq B_I - B_E$, there is enough spectrum to offset the initial cost asymmetry; the optimal spectrum allocation thus equates the costs of the two firms:

  $$b_E = \hat{b} \equiv \frac{B_I - B_E + \Delta}{2} \text{ and } b_I = \Delta - \hat{b},$$

  leading to an equilibrium price equal to (where the superscript $S$ refers to consumer surplus):

  $$p^S = c_I = c_E = \hat{c}.$$

- If instead $\Delta < B_I - B_E$, there is not enough spectrum to offset the initial cost disadvantage of the entrant; to minimize this disadvantage, it it then optimal to give all the additional spectrum to the entrant:

  $$b_E = \Delta \text{ and } b_I = 0,$$

  leading to

  $$p^S = \underline{c}_E = c\left(B_E + \Delta\right) > \bar{c}_I = c\left(B_I\right).$$

# B Proof of Lemma 2

Part $(i)$. Any $p \in \left[p^S, \bar{c}_E\right]$ can be supported by an equilibrium by choosing $b_E \in \left[0, \hat{b}\right]$ and $b_I = \Delta - b_E$ such that $c_E = c\left(B_E + b_E\right) = p$ and $c_I = c\left(B_I + b_I\right) = \gamma\left(p\right)$. We have:

$$\gamma'\left(p\right) = \gamma'\left(c_E\right) = -\frac{c'\left(B_I + B_E + \Delta - c^{-1}\left(c_E\right)\right)}{c'\left(c^{-1}\left(c_E\right)\right)} = -\frac{c'\left(B_I + b_I\right)}{c'\left(B_E + b_E\right)},$$

and

$$\gamma''\left(p\right) = \gamma''\left(c_E\right) = \frac{c''\left(B_I + b_I\right) + \frac{c'\left(B_I + b_I\right)}{c'\left(B_E + b_E\right)} c''\left(B_E + b_E\right)}{\left[c'\left(B_E + b_E\right)\right]^2}.$$

Assumption A.1 then readily yields $\gamma'\left(p\right) < 0 < \gamma''\left(p\right)$; together with $b_E \leq \hat{b}$ (which implies $B_I + b_I \geq B_E + b_E$), it yields $\gamma'\left(p\right) \geq -1$, with a strict inequality if $b_E < \hat{b}$ and $c''\left(\cdot\right) > 0$.

Part $(ii)$. For any $p \in \left[p^S, \bar{c}_E\right]$, $p \geq \gamma\left(p\right)$ and thus $\rho\left(p\right) \leq 1$ (with strict inequalities for $p > p^S$). Furthermore, for any $c \geq 0$, the monopoly profit function

$$\pi^m\left(p; c\right) \equiv \left(p - c\right) D\left(p\right)$$

satisfies:

$$\frac{\partial \pi^m}{\partial c}\left(p; c\right) = -D'\left(p\right)\left[c + \mu\left(p\right) - p\right],$$

where, from Assumption A.2, the expression in brackets strictly decreases as $p$ increases. It follows that $\pi^m\left(p; c\right)$ is strictly quasi-concave and is maximal for $p = p^m\left(c\right)$, characterized by the first-order condition:

$$p^m = c + \mu\left(p^m\right).$$

As is well-known, the monopoly price $p^m\left(c\right)$ is moreover (weakly) increasing in $c$. As $\gamma\left(p\right) \geq \underline{c}_I$, the assumption $\bar{c}_E < p^m\left(\underline{c}_I\right)$ thus ensures that, for any $p \in \left[p^S, \bar{c}_E\right]$, we have:

$$p \leq \bar{c}_E < p^m\left(\underline{c}_I\right) \leq p^m\left(\gamma\left(p\right)\right).$$

The strict quasi-concavity of the profit function $\pi^m(p;c)$ then yields:

$$\gamma(p) + \mu(p) - p > 0,$$

that is, $\rho(p) > 0$. Finally, we have:

$$\rho'(p) = \frac{\mu'(p)\left[p - \gamma(p)\right] - \mu(p)\left[1 - \gamma'(p)\right]}{\mu^2(p)} < 0,$$

where the inequality stems from the part $(i)$ of the Lemma and Assumption A.2.

# C Proof of Proposition 2

The derivative of the welfare function $W(p;\lambda)$ can be expressed as $\partial W(p;\lambda)/\partial p = \lambda D(p)\phi(p;\lambda)$, where

$$\phi(p;\lambda) \equiv \rho(p) - \gamma'(p) - \frac{1}{\lambda}.$$

From Lemma 2, $\phi(p;\lambda)$ is strictly decreasing in $p$. Let

$$\underline{\lambda} \equiv \frac{1}{\rho(p^S) - \gamma'(p^S)} \text{ and } \bar{\lambda} \equiv \frac{1}{\rho(\bar{p}) - \gamma'(\bar{p})}.$$

When the additional bandwidth is large enough to overcome the handicap, $p^S = \hat{c}\,(\geq \underline{c}_E)$ and thus $\rho\left(p^S\right) = -\gamma'\left(p^S\right) = 1$; hence, $\underline{\lambda} = 1/2$. Otherwise (i.e., when $\Delta < B_I - B_E$), $p^S = \underline{c}_E > \hat{c} > \gamma(\underline{c}_E) = \bar{c}_I$ and thus:

$$\rho\left(p^S\right) = 1 - \frac{\underline{c}_E - \bar{c}_I}{\mu(\underline{c}_E)} < 1 \text{ and } -\gamma'\left(p^S\right) = \frac{c'(B_I)}{c'(B_E + \Delta)} < 1;$$

hence, $\underline{\lambda} > 1/2$.

Three cases can be distinguished:

• Case a: $\lambda \leq \underline{\lambda}$. We then have $\phi(p;\lambda) \leq 0$ in the relevant range $p \in \left[p^S, \bar{c}_E\right]$, implying that the optimal price is $p^W = p^S$; that is, it is still optimal to allocate all the additional bandwidth to the entrant, as when maximizing consumer surplus.

• Case b: $\lambda \geq \bar{\lambda}$. We then have $\phi(p;\lambda) \geq 0$ in the relevant range $p \in \left[p^S, \bar{c}_E\right]$, implying that the optimal price is $p^W = \bar{c}_E$; that is, it is instead optimal to allocate all the additional bandwidth to the incumbent, as if the objective were to maximize industry profit.

• Case a: $\underline{\lambda} < \lambda < \bar{\lambda}$. The optimal price, $p^W$, is then the unique solution in $p$ to (9); this optimal price lies strictly between $p^S$ and $\bar{c}_E$ and it is therefore optimal to share the additional bandwidth between the two firms. Furthermore:

- When $p^S = \underline{c}_E \geq \hat{c}$, there is not enough (or just enough, in case of equality) additional bandwidth to offset the initial cost asymmetry; $E$'s cost thus remains (weakly) higher than $I$'s cost, and so there is a unique optimal spectrum allocation, which consists in giving $b_E^W = c^{-1}\left(p^W\right) - B_E$ to the entrant and $b_I^W = \Delta - b_E^W$ to the incumbent.

- When instead $p^S = \hat{c} > \underline{c}_E$, there is again a unique optimal spectrum allocation whenever $p^W > \bar{c}_I$; when instead $p^W \leq \bar{c}_I$, there are two optimal bandwidth allocations, which consist in sharing the additional bandwidth so as to give a cost equal to $p^W$ to one operator and a cost equal to $\gamma\left(p^W\right)$ to the other operator. As $\phi(p;\lambda)$ increases with $\lambda$ and decreases in $p$, the solution to (9), $p^W$, increases with $\lambda$, from $p^S = \hat{c} < \bar{c}_I$ for $\lambda = \underline{\lambda}$ to $\bar{c}_E > \bar{c}_I$ for $\lambda = \bar{\lambda}$. Hence, there exists $\hat{\lambda} \in \left(\underline{\lambda}, \bar{\lambda}\right)$ such that $p^W > \bar{c}_I$ for $\lambda > \hat{\lambda}$.

# D Proof of Corollary 1

Building on the previous analysis, the optimal price, $p^W = p^W(\lambda)$, is the unique solution to $\phi(p;\lambda) = 0$, where $\phi(p;\lambda)$ strictly increases with $\lambda$ and, from Lemma 2, strictly decreases in $p$. It follows that $p^W(\lambda)$ strictly increases with $\lambda$. When the optimal spectrum allocation maintains a cost advantage to the incumbent, this implies a re-allocation of $\Delta$ which further favors the incumbent.

Turning to the impact of bandwidth, and using

$$\gamma(p;B) \equiv c\left(B - c^{-1}(p)\right),$$

$$\rho(p;B) \equiv 1 - \frac{p - \gamma(p;B)}{\mu(p)},$$

and

$$\phi(p;B) \equiv \rho(p;B) - \frac{\partial\gamma}{\partial p}(p;B) - \frac{1}{\lambda},$$

the optimal price can be expressed as $p^W = p^W(B)$, where $p_\phi(B)$ is the unique solution to

$$\phi(p; B) = 0. \tag{10}$$

The optimal price thus only depends on total available bandwidth, $B = B_I + B_E + \Delta$. In addition,

$$\frac{\partial p^W}{\partial B} = -\frac{\frac{\partial \phi}{\partial B}}{\frac{\partial \phi}{\partial p}}\left(p^W, B\right),$$

where

$$\begin{aligned}
\frac{\partial \phi}{\partial B}\left(p^W, B\right) &= \frac{1}{\mu\left(p^W\right)}\frac{\partial \gamma}{\partial B}\left(p^W; B\right) - \frac{\partial^2 \gamma}{\partial B \partial p}\left(p^W; B\right) \\
&= \frac{c'\left(B - c^{-1}\left(p^W\right)\right)}{\mu\left(p^W\right)} + \frac{c''\left(B - c^{-1}\left(p^W\right)\right)}{c'\left(c^{-1}\left(p^W\right)\right)}\mu\left(p^W\right) \\
&= \frac{c'\left(S_I\right)}{\mu\left(c\left(S_E\right)\right)} + \frac{c''\left(S_I\right)}{c'\left(S_E\right)},
\end{aligned}$$

where $S_I = B_I + b_I$ and $S_E = B_E + b_E$, respectively, denote the overall amount of spectrum eventually assigned to the incumbent and to the entrant, assuming that the incumbent is favored when the optimal price $p^W$ can be achieved in two symmetric ways, and

$$\begin{aligned}
\frac{\partial \phi}{\partial p}\left(p^W, B\right) =& \frac{\partial \rho}{\partial p}(p; B) - \frac{\partial^2 \gamma}{\partial p^2}(p; B) \\
=& \frac{\mu'(p)\left[p - c\left(B - c^{-1}(p)\right)\right]}{\mu^2(p)} - \frac{1 + \frac{c'\left(B - c^{-1}(p)\right)}{c'\left(c^{-1}(p)\right)}}{\mu(p)} \\
& - \frac{c''\left(B - c^{-1}(p)\right)}{\left[c'\left(c^{-1}(p)\right)\right]^2} - \frac{c'\left(B - c^{-1}(p)\right)c''\left(c^{-1}(p)\right)}{\left[c'\left(c^{-1}(p)\right)\right]^3} \\
=& \frac{\mu'\left(c\left(S_E\right)\right)\left[c\left(S_E\right) - c\left(S_I\right)\right]}{\mu^2\left(c\left(S_E\right)\right)} - \frac{1 + \frac{c'\left(S_I\right)}{c'\left(S_E\right)}}{\mu\left(c\left(S_E\right)\right)} - \frac{c''\left(S_I\right)}{\left[c'\left(S_E\right)\right]^2} - \frac{c'\left(S_I\right)c''\left(S_E\right)}{\left[c'\left(S_E\right)\right]^3},
\end{aligned}$$

where the last equality uses the fact that, by construction, $p^W = c_E = c(S_E)$ and $\gamma(c_E) = c_I = c(S_I)$. It follows that the derivative of $p^W$ with respect to total bandwidth can be expressed as:

$$\begin{aligned}
\frac{\partial p^W}{\partial B} &= -\frac{\frac{\partial \phi}{\partial B}}{\frac{\partial \phi}{\partial p}}\left(p^W, B\right) \\
&= \frac{\frac{c'(S_I)}{\mu(c(S_E))} + \frac{c''(S_I)}{c'(S_E)}}{\frac{c'(S_I)}{\mu(c(S_E))c'(S_E)} + \frac{c''(S_I)}{[c'(S_E)]^2} + \frac{1}{\mu(c(S_E))} + \frac{c'(S_I)c''(S_E)}{[c'(S_E)]^3} - \frac{\mu'(c(S_E))[c(S_E)-c(S_I)]}{\mu^2(c(S_E))}} \\
&= c'\left(S_E\right)\frac{A}{A+B},
\end{aligned}$$

with

$$\begin{aligned}
A &= \frac{c'\left(S_I\right)}{\mu\left(c\left(S_E\right)\right)c'\left(S_E\right)} + \frac{c''\left(S_I\right)}{\left[c'\left(S_E\right)\right]^2} > 0, \\
B &= \frac{1}{\mu\left(c\left(S_E\right)\right)} + \frac{c'\left(S_I\right)c''\left(S_E\right)}{\left[c'\left(S_E\right)\right]^3} - \frac{\mu'\left(c\left(S_E\right)\right)\left[c\left(S_E\right) - c\left(S_I\right)\right]}{\mu^2\left(c\left(S_E\right)\right)} > 0,
\end{aligned}$$

where the inequalities follows from $c'(\cdot) > 0 > c''(\cdot)$, $\mu(\cdot) > 0 > \mu'(\cdot)$ and $c(S_E) > c(S_I)$. Using

$$p^W = c\left(S_E\right),$$

we thus have:

$$0 < \frac{\partial S_E}{\partial B} = \frac{A}{A+B} < 1.$$

Therefore:

- An increase in $\Delta$ leads to an increase in both $b_E$ (as $\partial S_E/\partial B > 0$) and $b_I$ (as $\partial S_E/\partial B < 1$).
- An increase in $B_I$ leads to an increase in $b_E$ (as $\partial S_E/\partial B > 0$) and a reduction in $b_I$ (as $\partial S_E/\partial B < 1$).
- An increase in $B_E$ leads to an increase in $b_I$ (as $\partial S_E/\partial B < 1$) and thus to a reduction in $b_E$ (as $b_E + b_I = \Delta$).

# E Proof of Proposition 3

## E.1 Shared-market equilibria

We first study shared market equilibria, in which both firms attract some consumers. The location $\hat{x}$ of the customer who is indifferent between patronizing the two firms is determined by:

$$s(p_1) - t\hat{x} = s(p_2) - t(1-\hat{x}),$$

where $p_1$ and $p_2$ denote firms' prices,

$$s(p) \equiv \int_p^{+\infty} d(v)\, dv$$

denotes individual consumer surplus, and $t$ denotes transportation costs per unit distance. Consumers located at $x < \hat{x}$ then purchase from firm 1 and those consumers located at $x > \hat{x}$ purchase from firm 2. Firm $i$'s market share is therefore

$$\hat{x}_i(p_i, p_j) \equiv \frac{1}{2} + \frac{s(p_i) - s(p_j)}{2t},$$

where $i \neq j \in \{1, 2\}$, and its profit is

$$\Pi_i(p_i, p_j) \equiv M\hat{x}_i(p_i, p_j)\, \pi_i(p_i),$$

where

$$\pi_i(p_i) \equiv (p_i - c_i)\, d(p_i)$$

denotes firm $i$'s per consumer profit. The equilibrium prices, $p_1^*$ and $p_2^*$, and the associated equilibrium variables satisfy the first-order conditions, which, using $s(p) = -d(p)$, can be written as:

$$d(p_i^*)\, \pi_i(p_i^*) = 2t\hat{x}_i\left(p_i^*, p_j^*\right) \pi_i'(p_i^*) \leq 2t\hat{x}_i\left(p_i^*, p_j^*\right) d(p_i^*),$$

where the inequality follows from $\pi_i'(p_i^*) = d(p_i^*) + (p_i^* - c_i)\, d'(p_i^*) \leq d(p_i^*)$, as $d'(\cdot) < 0$ and active firms never sell below costs (i.e., $p_i^* \geq c_i$). Dividing by $d(p_i^*)$ and adding the resulting inequalities for the two firms yields $\pi_1(p_1^*) + \pi_2(p_2^*) < 2t$. As active firms never make a loss (i.e., $\pi^*(p_i^*) \geq 0$), this condition in turn implies that each firm $i$ obtains a total profit lower than $2t$:

$$\Pi_i^* \equiv \Pi_i\left(p_i^*, p_j^*\right) = M\hat{x}_i(p_i^*, p_j^*)\pi_i\left(p_i^*\right) \leq 2tM.$$

It follows that, when firms face different costs, the market cannot remain shared as $t$ tends to vanish. To see this, suppose that $c_i < c_j$. From the above analysis, in any shared-market equilibrium, $\Pi_i^* \leq 2tM$ and $p_j^* \geq c_j$. Hence, firm $i$'s profit tends to 0 as $t$ tends to vanish. But then, firm $i$ could corner the market by charging $p_i(t)$ such that $s(p_i) = s(c_j) + t$; as $t$ goes to 0, $p_i(t)$ tends to $c_j$ and firm $i$ could thus secure in this way (close to) $M(c_j - c_i)\pi_i(c_j)$, which is bounded away from 0, a contradiction.

## E.2 Cornered-market equilibria

We now characterize cornered market equilibria, in which one firm, say firm $i$, attracts all consumers; the other firm, $j$, thus makes zero profit. We first note that this requires asymmetric costs. To see this, suppose instead that both firms face the same cost $c$. Firm $i$ cannot be pricing below $c$, otherwise it would make a loss and profitably deviate by raising its price. But then, firm $j$ could profitably deviate by pricing slightly above cost, which, thanks to product differentiation, would enable it to gain a positive market share and earn a small but positive margin.

The two firms must therefore be facing different costs. A standard Bertrand argument ensures that the more efficient firm wins the market, and that the other firm does not charge more than its cost; that is, firm $j$ faces some cost $c$ and firm $i$ faces a lower cost, of the form $\gamma(c) < c$, and $p_j^* \leq c$. As usual, we will focus on trembling-hand perfect equilibria, and thus discard those equilibria in which the losing firm would price below its own cost. It follows that the candidate equilibrium is:

$$p_j^* = c \text{ and } p_i^* = p^*(c), \tag{11}$$

where $p^*(c)$ is such that $\hat{x}_i(p^*(c), c) = 1$, that is:

$$s(p^*(c)) = s(c) + t.$$

It thus satisfies $p^*(c) < c$ (as $s(p^*(c)) > s(c)$) and

$$0 < p^{*\prime}(c) = \frac{d(c)}{d(p^*(c))} < 1. \tag{12}$$

In this candidate equilibrium, firm $i$ obtains a profit equal to:

$$\Pi^*(c) \equiv [p^*(c) - \gamma(c)] D(p^*(c)),$$

where $D(p) \equiv Md(p)$ denotes total demand at price $p$.

For this to be an equilibrium, firm $i$ should not benefit from increasing its price (in which case it would share the market with firm $j$); we have:

$$\begin{aligned} \frac{1}{M}\frac{\partial \Pi_i}{\partial p_i}(p_i, p_j)\bigg|_{p_j^*=c, p_i^*=p^*(c)} &= d(p^*(c)) + [p^*(c) - \gamma(c)] d'(p^*(c)) - \frac{d(p^*(c))}{2t}\pi^*(c) \\ &\leq \frac{d(p^*(c))}{2t}[2t - \pi^*(c)]. \end{aligned}$$

Hence, whenever:

$$\pi^*(c) \geq 2t,$$

there exists indeed a cornered market equilibrium, in which the firm with cost $c$ prices at cost, and the other firm, with cost $\gamma(c)$, charges $p^*(c)$ and obtains a profit equal to $\pi^*(c)$.

## E.3 Welfare analysis

In a cornered market equilibrium, social welfare is equal to:

$$W^*(c;\lambda) \equiv S(p^*(c)) + \lambda \Pi^*(c),$$

where $S(p) \equiv Ms(p)$ denotes total consumer surplus.

When the regulator focuses on consumer surplus, we thus have:

$$\frac{\partial W^*}{\partial c}(c;\lambda) = -D(p^*(c))\, p^{*\prime}(c) < 0,$$

and so it is never optimal to have a cost handicap larger than what is needed for one firm to "barely" corner the market. It follows that, as $t$ vanishes, the optimal allocation converges to cost equalization.

When $\lambda > 0$, we have:

$$W^* (c; \lambda) \equiv S (p^* (c)) + \lambda [p^* (c) - \gamma (c)] D (p^* (c)) .$$

As $t$ tends to vanish, $p^* (c) \simeq c$ converges to $c$, and thus $W^* (c; \lambda) \simeq W (c; \lambda)$, where $W (p; \lambda)$ is the welfare function studied in the baseline model of Bertrand competition, given by (6). Furthermore, maintaining a shared-market equilibrium outcome as $t$ tends to vanish requires cost equalization (that is, $c_I = c_E = \hat{c}$); as the equilibrium price converges to cost (i.e., $p_I^* \simeq p_E^* \simeq \hat{c}$), it follows that total welfare converges to $S (\hat{c}) = W (\hat{c}; \lambda)$. Hence, in both types of equilibrium (shared-market or cornered-market), the equilibrium prices tend to cost and total welfare converges to $W (c; \lambda)$; it follows that the optimal allocation converges towards that of the Bertrand baseline model (perfect substitutes).

# F Proof of Lemma 3

We first derive some properties of the profit function

$$\pi (b, \theta) = [c (B_I + \Delta - b) - c (B_I - \theta + b)] D (c (B_I + \Delta - b)) .$$

We have:

$$\begin{aligned} \frac{\partial \pi}{\partial \theta} (b, \theta) &= c' (B_I - \theta + b) D (c (B_I + \Delta - b)) < 0, \\ \frac{\partial^2 \pi}{\partial b \partial \theta} (b, \theta) &= c'' (B_I - \theta + b) D (c (B_I + \Delta - b)) - c' (B_I - \theta + b) D' (c (B_I + \Delta - b)) c' (B_I + \Delta - b) \\ &> 0, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial\pi}{\partial b}(b,\theta)=&-\left[c'\left(B_I+\Delta-b\right)+c'\left(B_I-\theta+b\right)\right]D\left(c\left(B_I+\Delta-b\right)\right)\\&-\left[c\left(B_I+\Delta-b\right)-c\left(B_I-\theta+b\right)\right]D'\left(c\left(B_I+\Delta-b\right)\right)c'\left(B_I+\Delta-b\right)\\>&-c'\left(B_I+\Delta-b\right)D\left(c\left(B_I+\Delta-b\right)\right)\\&-c'\left(B_I+\Delta-b\right)\left[c\left(B_I+\Delta-b\right)-c\left(B_I-\theta+b\right)\right]D'\left(c\left(B_I+\Delta-b\right)\right)\\>&0,\end{aligned}$$

where the last inequality stems from the fact that competition remains effective (and thus $E$'s profit increases with the consumer price).

Fix $b$ and $b'>b$, and let $\delta(\theta)\equiv\pi_E(b',\theta)-\pi_E(b,\theta)\geq 0$ denote the additional profit obtained by the entrant when increasing its share of bandwidth from $b$ to $b'$. Using the above properties, we have:

- if $\hat{b}(\theta)<b$, then $\delta(\theta)=\pi(b',\theta)-\pi(b,\theta)$ and thus:

$$\delta'(\theta)\equiv\int_b^{b'}\frac{\partial^2\pi}{\partial b\partial\theta}(x,\theta)\,dx>0;$$

- if instead $b<\hat{b}(\theta)<b'$, then $\delta(\theta)=\pi(b',\theta)$ and thus:

$$\delta'(\theta)\equiv\frac{\partial\pi}{\partial\theta}(b',\theta)\,dx<0;$$

- finally, if $\hat{b}(\theta)>b'$, then $\delta(\theta)=0$ and thus no longer varies with $\theta$.

# G Proof of Lemma 4

Suppose that the regulator seeks to maximize $W=S+\lambda(t_I+t_E)$, with a weight $\lambda\in[0,1/2]$ on revenues (the case where the regulator focuses on consumer surplus corresponds to $\lambda=0$). As only the entrant has private information, from the revelation principle we can restrict attention to direct incentive-compatible mechanisms (DICMs for short) of the form $\{(b(\theta),t(\theta))\}_{\theta\in\Theta}$ for the entrant, together with a transfer $t_I(\theta)=\pi_I(\theta)=\Pi\left(B_I+\Delta-\hat{b}(\underline{\theta}),B_I-\theta+\hat{b}(\underline{\theta})\right)$ for the incumbent.

We first show that if a DICM satisfies $b(\hat{\theta}) \leq \hat{b}(\hat{\theta})$ for some $\hat{\theta} \in \Theta$, then the profile $\{b(\theta)\}_{\theta \in \Theta}$ remains (weakly) below $\hat{b}(\hat{\theta})$ for any $\theta$ higher than $\hat{\theta}$:

**Lemma 5** *If a DICM satisfies $b(\hat{\theta}) \leq \hat{b}(\hat{\theta})$ for some $\hat{\theta} \in \Theta$, then it satisfies $b(\theta) \leq \hat{b}(\hat{\theta})$ for any $\theta > \hat{\theta}$.*

**Proof.** Consider a DICM satisfying $b(\hat{\theta}) \leq \hat{b}(\hat{\theta})$ for some $\hat{\theta} \in \Theta$ (implying that an entrant of type $\hat{\theta}$ is expected to make zero profit in the market), and suppose that there exists $\theta > \hat{\theta}$ such that $b(\theta) > \hat{b}(\hat{\theta})$. As the entrant of type $\hat{\theta}$ would make a positive gross profit if it were to pick the option designed for type $\theta$, incentive compatibility requires:

$$-t(\hat{\theta}) \geq \pi(b(\theta), \hat{\theta}) - t(\theta).$$

Conversely, an entrant of type $\theta$ should not benefit from picking the option designed for the type $\hat{\theta}$, which requires:

$$\max\{\pi(b(\theta), \theta), 0\} - t(\theta) \geq -t(\hat{\theta}).$$

Adding-up these two conditions yields:

$$\max\{\pi(b(\theta), \theta), 0\} \geq \pi(b(\theta), \hat{\theta}), \tag{13}$$

a contradiction, as: (i) the assumption $b(\theta) > \hat{b}(\hat{\theta})$ ensures that $\pi(b(\theta), \hat{\theta}) > 0$; and: (ii) the monotonicity of $\pi(b, \theta)$ with respect to $\theta$, together with $\hat{\theta} < \theta$, implies $\pi(b(\theta), \hat{\theta}) > \pi(b(\theta), \theta)$. ■

Next, we show that, if $\lambda \leq 1/2$, then social welfare strictly decreases with the market price $p$ in the relevant range $p \in [\hat{c}(\theta), \bar{c}(\theta)]$, where $\bar{c}(\theta) \equiv c(B_I - \theta)$ and

$$\hat{c}(\theta) \equiv c(B_I + \Delta - \hat{b}(\theta)) = c(B_I - \theta + \hat{b}(\theta)).$$

Indeed, in the range in which $c_I < c_E$, after the auction the incumbent wins the competition and, as its profit can be appropriated through appropriate transfers, social welfare is given by:

$$W(p; \lambda, \theta) \equiv S(p) + \lambda [p - \gamma(p; \theta)] D(p),$$

where

$$\gamma\left(p;\theta\right) \equiv c\left(2B_I + \Delta - \theta - c^{-1}\left(p\right)\right).$$

We have:

**Lemma 6** *If* $\lambda \in [0, 1/2]$*, then, for any* $\theta \in \Theta$*,* $W\left(p;\lambda,\theta\right)$ *strictly decreases as* $p$ *increases in the relevant range* $\left[\hat{c}\left(\theta\right), \bar{c}\left(\theta\right)\right]$*.*

**Proof.** For $p > \hat{c}\left(\theta\right)$, we have:

$$\begin{aligned}
\frac{\partial W}{\partial p}\left(p;\lambda,\theta\right) &= -D\left(p\right) + \lambda\left[1 - \frac{\partial\gamma}{\partial p}\left(p;\theta\right)\right]D\left(p\right) + \lambda\left[p - \gamma\left(p;\theta\right)\right]D'\left(p\right) \\
&\leq -D\left(p\right) + \lambda\left[1 - \frac{\partial\gamma}{\partial p}\left(p;\theta\right)\right]D\left(p\right) \\
&\leq -D\left(p\right) + \frac{1}{2}\left[1 - \frac{\partial\gamma}{\partial p}\left(p;\theta\right)\right]D\left(p\right) \\
&= -\left[1 + \frac{\partial\gamma}{\partial p}\left(p;\theta\right)\right]\frac{D\left(p\right)}{2} \\
&< 0,
\end{aligned}$$

where the first inequality relies on $\lambda \geq 0$, $p > \gamma\left(p;\theta\right)$ and $D'\left(\cdot\right) \leq 0$, the second one follows from $\lambda \leq 1/2$ and $\partial\gamma/\partial p < 0$, and the strict inequality follows from $\partial\gamma/\partial p > -1$ for $p > \hat{c}\left(\theta\right)$. Furthermore, in the limit case $p = \hat{c}\left(\theta\right)$, where $\gamma = p$ and $\partial\gamma/\partial p = -1$, we have:

$$\frac{\partial W}{\partial p}\left(\hat{c}\left(\theta\right);\lambda,\theta\right) = -\left(1 - 2\lambda\right)D\left(\hat{c}\left(\theta\right)\right) \leq 0,$$

where the inequality is moreover strict as long as $\lambda < 1/2$. ■

As the market price decreases as $b$ becomes closer to $\hat{b}\left(\theta\right)$, we have:

**Corollary 2** *If* $\lambda \in [0, 1/2]$*, then, for any* $\theta \in \Theta$*, social welfare strictly increases with* $b$ *in the range* $b \leq \hat{b}\left(\theta\right)$*, and strictly decreases with* $b$ *in the range* $b \geq \hat{b}\left(\theta\right)$*.*

Building on this, we now proceed to prove Lemma 4 and thus consider a DICM $\left\{\left(b\left(\theta\right), t\left(\theta\right)\right)\right\}_{\theta\in\Theta}$ satisfying

$$b(\hat{\theta}) = \hat{b}(\hat{\theta}) \tag{14}$$

for some $\hat{\theta} \in \Theta$.

Consider first the case $\hat{\theta} = \underline{\theta}$. Lemma 5 implies that the profile $\{b(\theta)\}_{\theta \in \Theta}$ then lies everywhere below $\hat{b}(\underline{\theta})$ and thus, *a fortiori*, below $\hat{b}(\theta)$ for any $\theta \in \Theta$; that is, the entrant never wins the competition. Replacing the DICM $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ with the "full bunching" DICM $\left\{\tilde{b}(\theta) = \hat{b}(\underline{\theta}), \tilde{t}(\theta) = 0\right\}_{\theta \in \Theta}$ (together with $t_I(\theta) = \pi_I(\theta) = \Pi\left(B_I + \Delta - \hat{b}(\underline{\theta}), B_I - \theta + \hat{b}(\underline{\theta})\right)$) then increases expected welfare:

- the alternative allocation is obviously incentive compatible and individually rational, as it allocates the same option to every type of entrant (full bunching), and this option gives a non-negative net profit to any type of entrant;
- the alternative allocation (weakly) increases social welfare: it coincides with the original allocation for $\underline{\theta}$ and, for $\theta > \underline{\theta}$, it either remains equal to $b(\theta)$ $\left(< \hat{b}(\theta)\right)$ or increases towards $\hat{b}(\theta)$, in which case, from Corollary 2, it increases social welfare.

Consider now the case where $\hat{\theta} > \underline{\theta}$, and suppose that there exists $\theta < \hat{\theta}$ such that $b(\theta) < \hat{b}(\hat{\theta})(= b(\hat{\theta}))$. It follows from Lemma 5 and the monotonicity of $\hat{b}(\cdot)$ that $b(\theta) > \hat{b}(\theta)$: as $\theta < \hat{\theta}$, $b(\theta) \leq \hat{b}(\theta)$ would imply $b(\hat{\theta}) \leq \hat{b}(\theta) < \hat{b}(\hat{\theta})$, contradicting the working assumption (14). The monotonicity of $\hat{b}(\cdot)$ moreover implies $\hat{b}(\theta) < \hat{b}(\hat{\theta}) = b(\hat{\theta})$. Incentive compatibility thus requires:

$$\pi(b(\theta), \theta) - t(\theta) \geq \pi(\hat{b}(\hat{\theta}), \theta) - t(\hat{\theta}),$$
$$-t(\hat{\theta}) \geq -t(\theta).$$

Re-arranging these two conditions yields:

$$\pi(b(\theta), \theta) - \pi(\hat{b}(\hat{\theta}), \theta) \geq t(\theta) - t(\hat{\theta}) \geq 0,$$

contradicting the assumption $b(\theta) < \hat{b}(\hat{\theta})$ (as $\partial\pi/\partial b > 0$).

Therefore, in the range $\theta < \hat{\theta}$, $b(\theta)$ must lie (weakly) above $\hat{b}(\hat{\theta})(= b(\hat{\theta}))$. That is, we must have: $b(\theta) \geq b(\hat{\theta}) = \hat{b}(\hat{\theta}) > \hat{b}(\theta)$; incentive compatibility then requires:

$$\pi(b(\theta), \theta) - t(\theta) \geq \pi(\hat{b}(\hat{\theta}), \theta) - t(\hat{\theta}),$$
$$\pi(\hat{b}(\hat{\theta}), \hat{\theta}) - t(\hat{\theta}) \geq \pi(b(\theta), \hat{\theta}) - t(\theta),$$

and thus:

$$
\begin{aligned}
0 &\leq \left[\pi\left(b\left(\theta\right),\theta\right)-\pi\left(\hat{b}(\hat{\theta}),\theta\right)\right]-\left[\pi(b\left(\theta\right),\hat{\theta})-\pi(\hat{b}(\hat{\theta}),\hat{\theta})\right] \\
&= \int_{\hat{b}(\hat{\theta})}^{b(\theta)}\left[\frac{\partial\pi}{\partial b}\left(b,\theta\right)-\frac{\partial\pi}{\partial b}(b,\hat{\theta})\right]db \\
&= \int_{\hat{b}(\hat{\theta})}^{b(\theta)}\int_{\hat{\theta}}^{\theta}\frac{\partial^2\pi}{\partial b\partial\theta}\left(b,s\right)dsdb.
\end{aligned}
$$

As $\partial^2\pi/\partial b\partial\theta > 0$ and $\theta < \hat{\theta}$, this in turn implies $b\left(\theta\right) \leq \hat{b}(\hat{\theta})$; combined with the initial condition $b\left(\theta\right) \geq b(\hat{\theta}) = \hat{b}(\hat{\theta})$, this implies $b\left(\theta\right) = \hat{b}(\hat{\theta})$.

It follows that we must have $b\left(\theta\right) = \hat{b}(\hat{\theta})$ in the range $\theta \leq \hat{\theta}$ and, from Lemma 5, $b\left(\theta\right) \leq \hat{b}(\hat{\theta})$ in the range $\theta \geq \hat{\theta}$. The same reasoning as above then establishes that the best such profile corresponds to the bunching mechanism where $b\left(\theta\right) = \hat{b}(\hat{\theta})$ for every $\theta \in \Theta$.

# H Proof of Proposition 4

We thus assume here that demand is inelastic in the relevant price range, namely for $p \in \left[c\left(B_I+\Delta\right), c\left(B_I-\bar{\theta}\right)\right]$, and, without loss of generality, we normalize its size to 1. Social welfare is thus given by:

$$
W\left(p;\lambda,\theta\right) = v - p + \lambda\left[p-\gamma\left(p;\theta\right)\right] = v-\left(1-\lambda\right)p-\lambda\gamma\left(p;\theta\right), \tag{15}
$$

where $v$ denotes consumers' reservation price, and, when the entrant acquires enough spectrum to win the market, its gross profit is equal to:

$$
\pi\left(b,\theta\right) = c\left(B_I+\Delta-b\right)-c\left(B_I-\theta+b\right). \tag{16}
$$

Building on Lemma 5, the following lemma shows that "full bunching" is optimal:

**Lemma 7** *If $\underline{\theta} < \Delta$, $\lambda \in [0,1/2]$, and demand is inelastic in the relevant price range $\left[c\left(B_I+\Delta\right), c\left(B_I-\bar{\theta}\right)\right]$, then, without loss of generality, we can then restrict attention to direct mechanisms $\left\{\left(b\left(\theta\right), t\left(\theta\right)\right)\right\}_{\theta\in\Theta}$ such that, for every $\theta \in \Theta$, $t\left(\theta\right) = 0$ and $b\left(\theta\right) = \hat{b}$, for some $\hat{b} \in [0,\Delta]$.*

**Proof.** Consider a DICM $\{b(\theta), t(\theta)\}_{\theta\in\Theta}$. From Lemma 5, if the profile $b(\cdot)$ lies below the profile $\hat{b}(\cdot)$ for some handicap, then $b(\cdot)$ remains below $\hat{b}(\cdot)$ for any larger handicap. Let $\hat{\theta}$ denote the handicap threshold beyond which the profile $b(\cdot)$ lies below the profile $\hat{b}(\cdot)$; that is:

$$\hat{\theta} \equiv \begin{cases} \bar{\theta} & \text{if } b(\theta) > \hat{b}(\theta) \text{ for all } \theta \in \Theta, \\ \inf\left\{\theta \mid b(\theta) \leq \hat{b}(\theta)\right\} & \text{otherwise.} \end{cases}$$

By construction, $b(\theta) \leq \hat{b}(\hat{\theta})$ for any $\theta > \hat{\theta}$.[50] Therefore, any type $\theta > \hat{\theta}$ would obtain a net profit of $-t(\tilde{\theta})$ by picking the option designed for any other type $\tilde{\theta} > \hat{\theta}$ (as $\hat{b}(\theta) > \hat{b}(\hat{\theta}) \geq b(\tilde{\theta})$). Incentive compatibility then implies that the profile $t(\theta)$ is constant in the range $\theta > \hat{\theta}$; that is, there exists $\hat{t}$ such that $t(\theta) = \hat{t}$ for $\theta > \hat{\theta}$, and any type $\theta > \hat{\theta}$ obtains a net payoff equal to $-\hat{t}$ by picking any option $\left(b(\tilde{\theta}), t(\tilde{\theta})\right)$ designed for any type $\tilde{\theta} > \hat{\theta}$.

Consider first the case $\hat{\theta} = \underline{\theta}$; that is, the profile $\{b(\theta)\}_{\theta\in\Theta}$ lies everywhere below $\hat{b}(\underline{\theta})$, except possibly for $\theta = \underline{\theta}$). Replacing the DICM $\{(b(\theta), t(\theta))\}_{\theta\in\Theta}$ with $\left\{\tilde{b}(\theta) = \hat{b}(\underline{\theta}), \tilde{t}(\theta) = 0\right\}_{\theta\in\Theta}$ can only increase expected consumer surplus:

- the alternative mechanism is trivially incentive compatible, as it does not depend on the handicap, and it is individually rational, as it gives zero profit to the entrant, regardless of its handicap;

- the alternative allocation coincides with the optimal allocation under complete information for $\theta = \underline{\theta}$ and, for $\theta > \underline{\theta}$, it is (weakly) closer to that optimal allocation, as it satisfies

$$b(\theta) \leq \hat{b}(\underline{\theta}) = \tilde{b}(\theta) < \hat{b}(\theta).$$

Thus, if $\hat{\theta} = \underline{\theta}$ then we can restrict attention to a DICM of the form described in the statement of the Lemma.

We now focus on the case where $\hat{\theta} > \underline{\theta}$, which implies:

$$b(\underline{\theta}) > \hat{b}(\underline{\theta}), \tag{17}$$

[50] Fix $\theta > \hat{\theta}$. If $b\left(\hat{\theta}\right) \leq \hat{b}\left(\hat{\theta}\right)$, Lemma 5 directly yields $b(\theta) \leq \hat{b}\left(\hat{\theta}\right)$. If instead $b\left(\hat{\theta}\right) > \hat{b}\left(\hat{\theta}\right)$, Lemma 5 yields $b(\theta) \leq \hat{b}\left(\tilde{\theta}\right)$ for any $\tilde{\theta} \in \left(\hat{\theta}, \theta\right)$; hence, $b(\theta) \leq \lim_{\tilde{\theta}\to\hat{\theta}^-} \hat{b}\left(\tilde{\theta}\right) = \hat{b}\left(\hat{\theta}\right)$.

and, for any $\theta > \hat{\theta}$:

$$b(\theta) \leq \hat{b}(\hat{\theta}) < \hat{b}(\theta). \tag{18}$$

We first note that

$$b(\theta) \leq b(\underline{\theta}) \tag{19}$$

for any $\theta > \hat{\theta}$. To see this, suppose instead that there exists $\tilde{\theta} > \hat{\theta}$ for which $b(\theta) > b(\underline{\theta})$. Combined with (17) and (18), this yields:

$$\hat{b}(\underline{\theta}) < b(\underline{\theta}) < b(\theta) < \hat{b}(\theta).$$

Incentive compatibility thus requires:

$$\pi(b(\underline{\theta}), \underline{\theta}) - t(\underline{\theta}) \geq \pi(b(\theta), \underline{\theta}) - t(\theta),$$
$$-t(\theta) \geq -t(\underline{\theta}).$$

Combining these conditions yields $\pi(b(\underline{\theta}), \underline{\theta}) \geq \pi(b(\theta), \underline{\theta})$, contradicting the working assumption $b(\theta) > b(\underline{\theta})$.

Consider now the range $\theta < \hat{\theta}$, where $b(\cdot) > \hat{b}(\cdot)$; in this range, the entrant obtains a net profit equal to:

$$r(\theta) \equiv \pi(b(\theta), \theta) - t(\theta),$$

where the gross profit $\pi(b, \theta)$ is here given by (16). A revealed preference argument implies that the net profit $r(\theta)$ decreases with $\theta$ (as $\partial\pi/\partial\theta < 0$), and that $\lim_{\theta \to \hat{\theta}^-} r(\theta) = -\hat{t}$. Therefore, individual rationality boils down to $\hat{t} \leq 0$, and without loss of generality we can set $\hat{t} = 0$.

Furthermore, by choosing the option designed for a type $\tilde{\theta}$ "close enough" to its own type $\theta$ (so that $b(\tilde{\theta})$ not only exceeds $\hat{b}(\tilde{\theta})$, but also exceeds $\hat{b}(\theta)$), an entrant of type $\theta$ would obtain:

$$\varphi\left(\theta, \tilde{\theta}\right) \equiv \pi\left(b(\tilde{\theta}), \theta\right) - t(\tilde{\theta}).$$

The usual reasoning can then be used to show that incentive compatibility requires the profiles $\{b(\theta)\}_{\theta < \hat{\theta}}$ and $\{t(\theta)\}_{\theta < \hat{\theta}}$ to be (weakly) increasing (as the profit function satisfies Mirrlees' single-crossing property: $\partial^2 \pi / \partial\theta\partial b > 0$) and to satisfy:

$$t(\theta) = \pi(b(\theta), \theta) + \int_{\theta}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds.$$

Next, we show that without loss of generality we can restrict attention to DICMs such that $b(\underline{\theta}) \leq \hat{b}(\hat{\theta})$. To see this, it suffices to note that if $b(\underline{\theta}) > \hat{b}(\hat{\theta})$ (which implies that the profile $b(\cdot)$ lies strictly above $\hat{b}(\hat{\theta})$ for $\theta < \hat{\theta}$, and below $\hat{b}(\hat{\theta})$ for $\theta > \hat{\theta}$), then replacing the DICM $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ with $\{\tilde{b}(\theta) = \hat{b}(\hat{\theta}), \tilde{t}(\theta) = 0\}_{\theta \in \Theta}$ increases expected social welfare:

- the alternative mechanism is trivially incentive compatible (full bunching); it is moreover individually rational, as the single option gives every type of entrant a non-negative net profit (positive for $\theta < \hat{\theta}$, and zero for $\theta \geq \hat{\theta}$);
- the alternative allocation, $\tilde{b}(\theta)$, is closer (and strictly so for $\theta < \hat{\theta}$) to the optimal allocation under complete information, $\hat{b}(\cdot)$:
  - for $\theta < \hat{\theta}$, the alternative allocation is such that $\hat{b}(\theta) < \hat{b}(\hat{\theta}) = \tilde{b}(\theta) < b(\underline{\theta}) \leq b(\theta)$;
  - for $\theta > \hat{\theta}$, the alternative allocation is such that $b(\theta) \leq \hat{b}(\hat{\theta}) = \tilde{b}(\theta) < \hat{b}(\theta)$.

Finally, consider a DICM $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ such that $b(\underline{\theta}) \leq \hat{b}(\hat{\theta})$, and let

$$\left(b^-, t^-\right) \equiv \lim_{\theta \to \hat{\theta}^-} (b(\theta), t(\theta))$$

denote the left-sided limit of the profile $\{b(\theta), t(\theta)\}_{\theta \in \Theta}$ at $\hat{\theta}$ (this limit exists, as incentive-compatibility implies that $b(\theta)$ and $t(\theta)$ are both non-decreasing in this range). Note that, by construction, $b^- \geq \hat{b}(\hat{\theta})$ (as $b(\theta) > \hat{b}(\theta)$ in the range $\theta < \hat{\theta}$). We can distinguish two cases, according to whether or not $b^-$ lies strictly above $\hat{b}(\hat{\theta})$.

*Case a:* $b^- = \hat{b}(\hat{\theta})$. The same revealed preference argument as in the proof of Lemma 4 implies that the profile $\{b(\theta)\}_{\theta \in \Theta}$ must then coincide with $\hat{b}(\hat{\theta})$ for $\theta < \hat{\theta}$; furthermore, from Lemma 5 $b(\theta)$ lies below $\hat{b}(\hat{\theta})$ for $\theta > \hat{\theta}$. It follows that replacing the DICM $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ with the alternative DICM $\{\tilde{b}(\theta) = \hat{b}(\hat{\theta}), \tilde{t}(\theta) = 0\}_{\theta \in \Theta}$

can only increase expected consumer surplus, as the alternative mechanism is trivially incentive compatible and individually rational, and is closer to the optimal allocation under complete information.[51]

*Case b:* $b^- > \hat{b}(\hat{\theta})$. As by construction $\hat{b}(\underline{\theta}) < b(\underline{\theta}) \leq \hat{b}(\hat{\theta})$, there exists a (unique) type $\tilde{\theta} \in (\underline{\theta}, \hat{\theta}]$ such that

$$\hat{b}(\tilde{\theta}) = b(\underline{\theta}),$$

and the profile $\hat{b}(\cdot)$ lies strictly below $b(\underline{\theta})$ for $\theta < \tilde{\theta}$, whereas it lies strictly above $b(\underline{\theta})$ for $\theta > \tilde{\theta}$. Consider replacing the DICM $\{(b(\theta), t(\theta))\}_{\theta\in\Theta}$ with the alternative mechanism $\{\tilde{b}(\theta) = b(\underline{\theta}), \tilde{t}(\theta) = 0\}_{\theta\in\Theta}$. This alternative mechanism is again trivially incentive compatible and individually rational, and outside the range $\theta \in [\tilde{\theta}, \hat{\theta}]$, it is closer to the optimal allocation under complete information and thus increases social welfare:

- for $\underline{\theta} < \theta < \tilde{\theta}$, the alternative allocation is such that $\hat{b}(\theta) < b(\underline{\theta}) = \tilde{b}(\theta) \leq b(\theta)$;
- for $\theta > \hat{\theta}$, using (19), the alternative allocation is such that $b(\theta) \leq b(\underline{\theta}) = \tilde{b}(\theta) = \hat{b}(\tilde{\theta}) < \hat{b}(\theta)$.

Therefore, if $\tilde{\theta} = \hat{\theta}$, the alternative mechanism exhibits full bunching and outperforms the original DICM.[52]

We now show that, if instead $\tilde{\theta} < \hat{\theta}$, then the alternative mechanism also increases social welfare in the range $\theta \in [\tilde{\theta}, \hat{\theta}]$. From Lemma 6, it suffices to show that replacing $b(\theta)$ with $\tilde{b}(\theta)$ lowers the market price for any $\theta \in [\tilde{\theta}, \hat{\theta}]$.

As $\hat{b}(\underline{\theta}) < b(\underline{\theta}) = \tilde{b}(\theta) = \hat{b}(\tilde{\theta}) < \hat{b}(\theta) < b(\theta)$, incentive-compatibility requires:

$$\pi(b(\theta), \theta) - t(\theta) \geq -t(\underline{\theta}),$$

$$\pi(b(\underline{\theta}), \underline{\theta}) - t(\underline{\theta}) \geq \pi(b(\theta), \underline{\theta}) - t(\theta).$$

[51] If the distribution of the handicap $\theta$ is atomless at $\hat{\theta}$, a quicker argument consists in noting that the DICM is then equivalent to a DICM where $b(\theta) = b^- = \hat{b}\left(\hat{\theta}\right)$; Lemma 4 then applies.

[52] While the above inequalities only establish that $\tilde{b}(\theta) \leq b(\theta)$ for $\theta < \hat{\theta}$, in the particular case where $\tilde{\theta} = \hat{\theta}$ (which can arise only when $b(\underline{\theta}) = b(\hat{\theta})\ (< b^-)$), we have $\tilde{b}(\theta) < b(\theta)$ for $\theta$ close enough to $\hat{\theta}$ (as $\lim_{\theta\to\hat{\theta}^-} b(\theta) = b^- > \hat{b}(\hat{\theta})$). Hence, the alternative mechanism does strictly better than the original DICM.

Combining these two conditions yields

$$\phi\left(b\left(\theta\right),\theta\right) \leq \pi\left(b\left(\underline{\theta}\right),\underline{\theta}\right), \tag{20}$$

where

$$\phi\left(b,\theta\right) \equiv \pi\left(b,\underline{\theta}\right) - \pi\left(b,\theta\right)$$

satisfies, for $\theta > \tilde{\theta}$ (using the fact that $c'\left(\cdot\right)$ is negative but increasing):

$$\frac{\partial\phi}{\partial b}\left(b,\theta\right) = \frac{\partial\pi}{\partial b}\left(b,\underline{\theta}\right) - \frac{\partial\pi}{\partial b}\left(b,\theta\right) = c'\left(B_I - \theta + b\right) - c'\left(B_I - \underline{\theta} + b\right) < 0, \tag{21}$$

and:

$$\frac{\partial\phi}{\partial\theta}\left(b,\theta\right) = -\frac{\partial\pi}{\partial\theta}\left(b,\theta\right) = -c'\left(B_I - \theta + b\right) > 0.$$

From (21), the incentive compatibility condition (20) amounts to $b\left(\theta\right) \geq \beta\left(\theta\right)$, where, for any $\theta > \underline{\theta}$, the function $\beta\left(\theta\right)$ is the implicit solution to $\phi\left(\beta,\theta\right) = 0$. Note that, by construction:

- $\beta(\tilde{\theta}) = b\left(\underline{\theta}\right)$, as

  $$\phi(b\left(\underline{\theta}\right),\tilde{\theta}) = \pi(b\left(\underline{\theta}\right),\underline{\theta}) - \pi(b\left(\underline{\theta}\right),\tilde{\theta}) = \pi\left(b\left(\underline{\theta}\right),\underline{\theta}\right),$$

  where the second equality stems from $\pi(b\left(\underline{\theta}\right),\tilde{\theta}) = \pi(\hat{b}(\tilde{\theta}),\tilde{\theta}) = 0$.

- For $\theta \in [\tilde{\theta},\hat{\theta}]$, as $c'\left(\cdot\right) < 0$:

  $$\beta'\left(\theta\right) = -\frac{\frac{\partial\phi}{\partial\theta}\left(b,\theta\right)}{\frac{\partial\phi}{\partial b}\left(b,\theta\right)} = \frac{-c'\left(B_I - \theta + \beta\left(\theta\right)\right)}{-c'\left(B_I - \theta + \beta\left(\theta\right)\right) + c'\left(B_I - \underline{\theta} + \beta\left(\theta\right)\right)} > 1.$$

  Therefore, for $\theta \in (\tilde{\theta},\hat{\theta})$ (using respectively $b\left(\underline{\theta}\right) = \beta(\tilde{\theta})$, $b\left(\theta\right) \geq \beta\left(\theta\right)$, $\beta'\left(\cdot\right) > 1$, and $b\left(\underline{\theta}\right) = \hat{b}(\tilde{\theta}) = (\Delta + \tilde{\theta})/2$):

  $$b\left(\theta\right) - b\left(\underline{\theta}\right) = b\left(\theta\right) - \beta(\tilde{\theta}) \geq \beta\left(\theta\right) - \beta(\tilde{\theta}) > \theta - \tilde{\theta} = \theta + \Delta - 2\hat{b}(\tilde{\theta}),$$

  which in turn implies:

  $$\hat{b}(\tilde{\theta}) - \theta > \Delta - b\left(\theta\right).$$

It follows that in the range $\theta \in (\tilde{\theta}, \hat{\theta})$, replacing $b(\theta) > \hat{b}(\theta)$ with $\tilde{b}(\theta) = b(\underline{\theta}) < \hat{b}(\theta)$ increases social welfare, as it reduces the price from $c_I|_{b=b(\theta)} = c(B_I + \Delta - b(\theta))$ to $c_E|_{b=b(\underline{\theta})} = c(B_I - \theta + b(\underline{\theta}))$.

■

We can therefore restrict attention to "bunching" mechanisms which allocate the same bandwidth $\hat{b}$ to the entrant, regardless of its type. The resulting price is equal to

$$p(\hat{b}, \theta) = \begin{cases} c(B_I + \Delta - \hat{b}) & \text{if } \theta \leq \hat{\theta}, \\ c(B_I - \theta + \hat{b}) & \text{otherwise,} \end{cases}$$

where

$$\hat{\theta} = \hat{b}^{-1}(\hat{b}) = 2\hat{b} - \Delta.$$

Therefore, the optimal bandwidth lies between $\hat{b}(\underline{\theta})$ (which is lower than $\Delta$ by assumption) and $\min\{\hat{b}(\bar{\theta}), \Delta\}$ and aims at maximizing expected social welfare:

$$\hat{W}(\hat{b}, \lambda) \equiv E_\theta \left[ W(p(\hat{b}, \theta); \lambda, \theta) \right],$$

where, using (16), $\hat{W}(b, \lambda)$ can be expressed as

$$\begin{aligned} \hat{W}(b, \lambda) &\equiv \int_{\underline{\theta}}^{\bar{\theta}} [v - (1-\lambda) p(\theta, b) - \lambda \gamma(\theta, b)] f(\theta) d\theta \\ &= \int_{\underline{\theta}}^{2b-\Delta} [v - (1-\lambda) c(B_I + \Delta - b) - \lambda c(B_I - \theta + b)] f(\theta) d\theta \\ &+ \int_{2b-\Delta}^{\bar{\theta}} [v - (1-\lambda) c(B_I - \theta + b) - \lambda c(B_I + \Delta - b)] f(\theta) d\theta. \end{aligned}$$

We have:

$$\begin{aligned} \frac{\partial^2 \hat{W}}{\partial b^2}(b, \lambda) = &- \int_{\underline{\theta}}^{2b-\Delta} [(1-\lambda) c''(B_I + \Delta - b) + \lambda c''(B_I - \theta + b)] f(\theta) d\theta \\ &- \int_{2b-\Delta}^{\bar{\theta}} [(1-\lambda) c''(B_I - \theta + b) + \lambda c''(B_I + \Delta - b)] f(\theta) d\theta \\ &+ 4(1-2\lambda) c'(B_I + \Delta - b) f(2b - \Delta) \\ &< 0, \end{aligned}$$

and:

$$\begin{aligned}\frac{\partial \hat{W}}{\partial b}(\hat{b}(\underline{\theta}),\lambda) &= \int_{\underline{\theta}}^{\bar{\theta}} \Big[-(1-\lambda)\,c'(B_I-\theta+\hat{b}(\underline{\theta}))+\lambda c'(B_I+\Delta-\hat{b}(\underline{\theta}))\Big]\,f(\theta)\,d\theta \\ &\geq -(1-2\lambda)\,c'(B_I+\Delta-\hat{b}(\underline{\theta})) \\ &> 0.\end{aligned}$$

Therefore, the socially optimal threshold is uniquely defined and strictly larger than $\hat{b}(\underline{\theta}) = b^{FB}(\underline{\theta})$.

# I Proof of Proposition 5

As is well-known, in each (classic) auction, the higher-valuation bidder wins and pays a price equal to the lower-valuation bidder, where all valuations take into account the expected equilibrium outcome of subsequent auctions.

The proof proceeds by induction. We will label "auction $h$", for $h = 1, ..., k$, the auction taking place when $h$ blocks remain to be allocated (hence, auction "$k$" is the first auction, and auction "1" is the auction for the last block). Let $p_0(B_L, B_l) \equiv 0$ and $\Pi_0(B_L, B_l) \equiv \Pi(B_L, B_l)$, where $\Pi(\cdot, \cdot)$ is given by (1), and for every $h = 1, ..., k$, let $L_h$ and $l_h$ respectively denote the leader and the laggard (i.e., the firm with the larger and with the smaller bandwidth) at the beginning of auction $h$ – if both firms have the same bandwidth at the beginning of auction $h$, then select either firm as leader with probability 1/2.

We will use the following induction hypothesis $H_h$:

1. If $B_{L_h} > B_{l_h}$, then $L_h$ wins auction $h$ and obtains an expected net profit equal to $\Pi_h(B_{L_h}, B_{l_h}) = \Pi(B_{L_h} + h\delta, B_{l_h}) - p_h(B_{L_h}, B_{l_h})$, where

$$p_h(B_{L_h}, B_{l_h}) = \begin{cases} \Pi_{h-1}(B_{l_h} + \delta, B_{L_h}) & \text{if } B_{L_h} - B_{l_h} < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

whereas $l_h$ obtains zero expected net profit.

2. If $B_{L_h} = B_{l_h}$, then either firm wins auction $h$ and pays a price

$$p_h\left(B_{L_h}, B_{L_h}\right) = \Pi\left(B_{L_h} + h\delta, B_{L_h}\right).$$

Both firms obtain zero expected net profit.

We first check that $H_1$ holds:

- If $B_{L_1} \geq B_{l_1} + \delta$, then the laggard cannot obtain any profit in the product market, regardless of whether it wins the auction; hence, the leader obtains the last block for free.
- If instead $B_{L_1} < B_{l_1} + \delta$, then winning the auction gives the laggard a profit (gross of the price paid in the last auction) equal to $\Pi_0\left(B_{l_1} + \delta, B_{L_1}\right) = \Pi\left(B_{l_1} + \delta, B_{L_1}\right)$, and gives the leader a (gross) profit equal to $\Pi_0\left(B_{L_1} + \delta, B_{l_1}\right) = \Pi\left(B_{L_1} + \delta, B_{l_1}\right)$. Therefore:
  - If $B_{L_1} > B_{l_1}$, then the leader has a greater willingness to pay, as

$$\Pi\left(B_{L_1} + \delta, B_{l_1}\right) > \Pi\left(B_{l_1} + \delta, B_{l_1}\right) > \Pi\left(B_{l_1} + \delta, B_{L_1}\right),$$

    where the first and second inequalities respectively stem from (2) and (3). Hence, the leader obtains the last block for a price equal to $p_1\left(B_{L_1}, B_{l_1}\right) = \Pi\left(B_{l_1} + \delta, B_{L_1}\right)$.
  - If instead $B_{L_1} = B_{l_1}$, then both firms obtains the same (gross) profit from winning the auction, and thus bid the same amount, equal to this profit. Hence, $p_1\left(B_{L_1}, B_{L_1}\right) = \Pi\left(B_{L_1} + \delta, B_{L_1}\right)$, either firm wins at that price, and both firms obtain zero net profit.

Suppose now that $H_t$ holds for $t = 1, ..., h$, and consider auction $h+1$. If the leading firm $L_{h+1}$ wins, then it will be again the leader in the next round, and will enjoy a bandwidth advantage of at least $\delta$; therefore, according to the induction hypothesis, its profit from winning (gross of the price paid in auction $h+1$) is given by (taking into account that $p_h\left(B_{L_{h+1}}+\delta, B_{l_{h+1}}\right)=0$, as $\left(B_{L_{h+1}}+\delta\right)-B_{l_{h+1}} \geq \delta$):

$$\hat{\Pi}_L=\Pi\left(B_{L_{h+1}}+(h+1)\,\delta, B_{l_{h+1}}\right).$$

If instead the laggard firm $l_{h+1}$ wins auction $h+1$, it then becomes the leader in the next round if $B_{L_{h+1}}-B_{\ell_{h+1}}<\delta$, and otherwise remains the laggard (or becomes equally efficient as its rival, in which case it also obtains zero profit in the product market); therefore, according to the induction hypothesis, it obtains a profit (gross of the price paid in auction $h+1$) equal to:

$$\hat{\Pi}_l=\begin{cases}\Pi\left(B_{l_{h+1}}+(h+1)\,\delta, B_{L_{h+1}}\right)-p_h\left(B_{l_{h+1}}+\delta, B_{L_{h+1}}\right) & \text{if } B_{L_{h+1}}-B_{l_{h+1}}<\delta,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore:

- If $B_{L_{h+1}}=B_{l_{h+1}}$,

  $$\Pi\left(B_{l_{h+1}}+(h+1)\,\delta, B_{L_{h+1}}\right)=\Pi\left(B_{L_{h+1}}+(h+1)\,\delta, B_{l_{h+1}}\right)=\Pi\left(B_{L_{h+1}}+(h+1)\,\delta, B_{L_{h+1}}\right)$$

  and

  $$p_h\left(B_{l_{h+1}}+\delta, B_{L_{h+1}}\right)=p_h\left(B_{L_{h+1}}+\delta, B_{L_{h+1}}\right)=0,$$

  and thus $\hat{\Pi}_L=\hat{\Pi}_l$. Hence, both firms bid

  $$p_{h+1}=\Pi\left(B_{L_{h+1}}+(h+1)\,\delta, B_{L_{h+1}}\right),$$

  either firm wins, and both firms obtain zero net profit.

- If instead $B_{L_{h+1}}>B_{l_{h+1}}$, then $\hat{\Pi}_L>\hat{\Pi}_l$, as $p_h(\cdot)\geq 0$ and

  $$\Pi\left(B_{L_{h+1}}+(h+1)\,\delta, B_{l_{h+1}}\right)>\Pi\left(B_{l_{h+1}}+(h+1)\,\delta, B_{l_{h+1}}\right)>\Pi\left(B_{l_{h+1}}+(h+1)\,\delta, B_{L_{h+1}}\right),$$

  where the first and second inequalities respectively stem again from (2) and (3), and so the leading firm $L_{h+1}$ wins auction $h+1$. Furthermore:

  - When $B_{L_{h+1}}-B_{l_{h+1}}>\delta$, the lagging firm $l_{h+1}$ would remain behind and thus obtain zero profit even if it were to win; hence, it bids zero, that is, $p_{h+1} = 0$.

– When instead $B_{L_{h+1}} - B_{l_{h+1}} < \delta$, the lagging firm $l_{h+1}$ is willing to bid up to

$$\Pi\left(B_{l_{h+1}} + (h+1)\delta, B_{L_{h+1}}\right) - p_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right) = \Pi_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right).$$

The equilibrium price is thus equal to $p_{h+1} = \Pi_h\left(B_{l_{h+1}} + \delta, B_{L_{h+1}}\right)$, as in the induction hypothesis. It follows that the equilibrium payoffs are also as in the induction hypothesis.

Therefore, $H_{h+1}$ holds when $H_t$ holds for $t = 1, ..., h$. It follows that the incumbent firm $I$ wins all successive rounds. Furthermore, if $B_I - B_E \geq \delta$, then it obtains all the bandwidth at zero price. If instead $B_I - B_E < \delta$, then using the induction hypothesis we have:

$$p_k\left(B_I, B_E\right) = \begin{cases} \sum_{h=0}^{m-1} \phi_h^k\left(B_E, B_I\right) - \sum_{h=1}^{m} \phi_h^k\left(B_I, B_E\right) & \text{if } k = 2m, \\ \sum_{h=0}^{m} \phi_h^k\left(B_E, B_I\right) - \sum_{h=1}^{m} \phi_h^k\left(B_I, B_E\right) & \text{if } k = 2m+1. \end{cases}$$

where

$$\phi_h^k\left(B_1, B_2\right) \equiv \Pi\left(B_1 + (k-h)\delta, B_2 + h\delta\right).$$

Note that when $B_I = B_E = B$,

$$\phi_m^{2m}\left(B, B\right) = \Pi\left(B + m\delta, B + m\delta\right) = 0$$

and thus the equilibrium price is equal to

$$p_k\left(B, B\right) = \phi_0^k\left(B, B\right) = \Pi\left(B + \Delta, B\right).$$

## J Proof of Proposition 6

We now show that, for each firm $i = I, E$, it is a dominant strategy to bid $\beta_i^*(n) = \pi_i(n)$, where (using the subscript "$-i$" to refer to firm $i$'s rival):

$$\pi_i(n) \equiv \begin{cases} \Pi(B_i + n_i\delta, B_{-i} + n_{-i}\delta) & \text{if } B_i + n_i\delta > B_{-i} + n_{-i}\delta, \\ 0 & \text{otherwise.} \end{cases}$$

with $\Pi(\cdot,\cdot)$ given by (1).

To see this, consider an alternative strategy $\hat{\beta}_i$, and suppose that, for some bidding strategy of the other firm, $\beta_{-i}$, the bidding strategies $\beta_i^*$ and $\hat{\beta}_i$ lead to different outcomes. As the payments only depend on the bids through the spectrum allocation, this implies that $\beta_i^*$ and $\hat{\beta}_i$ lead to different spectrum allocations, which we will respectively denote by $n^*$ and $\hat{n}$. Likewise, let $\Pi_i^*$ and $\hat{\Pi}_i$ denote the net payoffs of firm $i$ associated with the bidding strategies $\beta_i^*$ and $\hat{\beta}_i$. We have:

$$\begin{aligned} \Pi_i^* - \hat{\Pi}_i &= \left\{\pi_i(n^*) - p_i^V(\beta_i^*, \beta_{-i})\right\} - \left\{\pi_i(\hat{n}) - p_i^V\left(\hat{\beta}_i, \beta_{-i}\right)\right\} \\ &= \left\{\pi_i(n^*) - \left[\max_{n\in\mathcal{A}}\{\beta_{-i}(n)\} - \beta_{-i}(n^*)\right]\right\} - \left\{\pi_i(\hat{n}) - \left[\max_{n\in\mathcal{A}}\{\beta_{-i}(n)\} - \beta_{-i}(\hat{n})\right]\right\} \\ &= \pi_i(n^*) - \pi_i(\hat{n}) - [\beta_{-i}(\hat{n}) - \beta_{-i}(n^*)] \\ &= \beta_i^*(n^*) - \beta_i^*(\hat{n}) - [\beta_{-i}(\hat{n}) - \beta_{-i}(n^*)]. \end{aligned}$$

But, by construction, as the bidding strategy $\beta_i^*$ leads to $n^*$, it must be the case that

$$\beta_i^*(n^*) + \beta_{-i}(n^*) \geq \beta_i^*(\hat{n}) + \beta_{-i}(\hat{n}).$$

It follows that $\Pi_i^* \geq \hat{\Pi}_i$, establishing that bidding $\beta_i^*(n) = \pi_i(n)$ is a dominant strategy for firm $i$.

Given these bidding strategies, the outcome maximizes

$$\max_{n\in\mathcal{A}} \Pi(B_I + n_I\delta, B_E + n_E\delta),$$

which is achieved for $n_I = k$ and $n_E = 0$. That is, the incumbent firm $I$ obtains all $k$

blocks, and pays a price equal to:

$$\max_{n \in \mathcal{A}} \{\beta_E(n)\} - \beta_E(k,0) = \begin{cases} \Pi(B_E + \Delta, B_I) & \text{if } \Delta > B_I - B_E, \\ 0 & \text{otherwise.} \end{cases}$$

# K Proof of Proposition 7

When the handicap of the entrant is too large to be offset by the additional spectrum (i.e., when $B_I - B_E \geq \Delta$), the equilibrium prices are zero in both types of auctions. We now focus on the more interesting case where $B_I - B_E < \Delta$. In the case of a multi-unit VCG auction, the price is then always positive and equal to

$$p^V = \Pi(B_E + \Delta, B_I) > 0.$$

By contrast, in the case of a sequential auction, the price remains zero when the lagging firm cannot catch-up with a single block of size $\delta = \Delta/k$. Hence, for any given $\Delta > 0$, the price remains zero when the spectrum is divided in sufficiently many blocks, namely, when

$$k \geq \bar{k} = \frac{\Delta}{B_I - B_E}.$$

Finally, when instead the lagging firm could catch up with a single block (i.e., $B_I - B_E < \delta = \Delta/k$), the price is of the form (using the induction hypothesis):

$$p_k(B_I, B_E) = \Pi_{k-1}(B_E + \delta, B_I) = \Pi(B_E + \Delta, B_I) - p_{k-1}(B_E + \delta, B_I),$$

where $p_{k-1}(B_E + \delta, B_I) > 0$. Hence, the revenue is again lower with sequential auctions.

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# Online Appendix
(Not for Publication)

This online Appendix first extends the bunching result of Proposition 4 (Section A). It then provides a partial characterization of socially optimal incentive-compatible mechanisms (Section B), before studying a counter-example in which bunching is not optimal (Section C).

## A Optimal bunching

To extend the result of Proposition 4, we now suppose that the handicap is sufficiently diverse, namely: $(\underline{\theta} <)\, \Delta < \bar{\theta}$.[53] Under this assumption, under complete information the optimal allocation gives some additional bandwidth to the incumbent when the handicap is low ($\theta$ close enough to $\underline{\theta}$), but gives instead the entire additional bandwidth to the entrant when its handicap is large ($\theta$ close enough to $\bar{\theta}$).

We now show that, under this assumption, the optimal mechanism exhibits again "full bunching" when attention is restricted to continuous or monotonic allocations:

**Proposition 9** *If $\underline{\theta} < \Delta < \bar{\theta}$, then within the set of direct incentive-compatible mechanisms (DICMs) $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ such that the profile $\{b(\theta)\}_{\theta \in \Theta}$ varies either continuously or monotonically with $\theta$, the optimal mechanism exhibits full bunching; that is, it is optimal to offer the same bandwidth to the entrant, regardless of its type.*

**Proof.** Consider a direct mechanism $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ satisfying feasibility (i.e., $b(\theta) \in [0, \Delta]$), individual rationality and incentive compatibility. We first note that $\bar{\theta} > \Delta$ implies $b(\theta)\,(\leq \Delta) < \hat{b}(\theta)$ whenever $\theta$ is sufficiently close to $\bar{\theta}$. Let

$$\hat{\theta} \equiv \inf \left\{ \theta \mid b(\theta) \leq \hat{b}(\theta) \right\}$$

denote the threshold beyond which the schedule $b(\cdot)$ remains below the schedule $\hat{b}(\cdot)$. If $\hat{\theta} = \underline{\theta}$ then, as in Lemma 7, replacing the DICM $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ with $\left\{ \tilde{b}(\theta) = \hat{b}(\underline{\theta}), \tilde{t}(\theta) = 0 \right\}_{\theta \in \Theta}$ can only increase expected consumer surplus, as the alternative mechanism is trivially

[53] Recall that when $\underline{\theta} > \Delta$, it is always optimal to allocate the entire additional bandwidth, $\Delta$, to the entrant, and this remains implementable (with zero transfers) under incomplete information.

incentive-compatible (full bunching) as well as individually rational (it gives zero profit to the entrant, regardless of its handicap), and it is (weakly) closer to the optimal allocation under complete information (it actually coincides with that allocation for $\theta = \underline{\theta}$). In what follows, we therefore focus on the case where $\hat{\theta} > \underline{\theta}$.

By construction, $b(\theta) > \hat{b}(\theta)$ for $\theta < \hat{\theta}$. Also, from Lemma 5, $b(\theta) \leq \hat{b}(\hat{\theta})$ for $\theta > \hat{\theta}$. Hence, when the schedule $b(\cdot)$ is continuous, we must have $b(\hat{\theta}) = \hat{b}(\hat{\theta})$, and Lemma 4 establishes that full bunching is optimal.

Consider now the case where the schedule $b(\cdot)$ is monotonic. If $b(\hat{\theta}) = \hat{b}(\hat{\theta})$, then Lemma 4 establishes again that full bunching is optimal. If instead $b(\hat{\theta}) \neq \hat{b}(\hat{\theta})$, then monotonicity implies that the profile $\{b(\theta)\}_{\theta\in\Theta}$ must be (weakly) decreasing:

- If $b(\hat{\theta}) < \hat{b}(\hat{\theta})$ then, for $\theta$ smaller than but close enough to $\hat{\theta}$ (and thus, $\hat{b}(\theta)$ close enough to $\hat{b}(\hat{\theta})$), we have:
$$b(\theta) > \hat{b}(\theta) > b(\hat{\theta}).$$
- If instead $b(\hat{\theta}) > \hat{b}(\hat{\theta})$, then, for $\theta$ larger than $\hat{\theta}$ we have:
$$b(\hat{\theta}) > \hat{b}(\hat{\theta}) > b(\theta).$$

It follows that the profile $\{b(\theta)\}_{\theta\in\Theta}$ (i) lies strictly above $\hat{b}(\hat{\theta})$ for $\theta < \hat{\theta}$, and below $\hat{b}(\hat{\theta})$ for $\theta > \hat{\theta}$, and (ii) it does strictly so in at least one of the ranges (for $\theta > \hat{\theta}$ when $b(\hat{\theta}) < \hat{b}(\hat{\theta})$, and for $\theta < \hat{\theta}$ when $b(\hat{\theta}) > \hat{b}(\hat{\theta})$). But then, replacing the DICM $\{(b(\theta), t(\theta))\}_{\theta\in\Theta}$ lies with the bunching mechanism $\{\tilde{b}(\theta) = \hat{b}(\hat{\theta}), \tilde{t}(\theta) = 0\}_{\theta\in\Theta}$ strictly increases expected consumer surplus, as $\tilde{b}(\theta)$ is weakly closer to $\hat{b}(\theta)$ for every $\theta \in \Theta$, and strictly so in one of the ranges. ■

# B Incentive-compatible mechanisms

The following proposition provides a complete characterization of direct incentive-compatible mechanisms; it shows in particular that:

- The entrant overtakes the incumbent when the handicap is sufficiently small, and the incumbent wins the market otherwise;

- Incentive compatibility allows for discontinuous and non-monotonic bandwidth allocations.

The proposition also provides a partial characterization of the optimal mechanism.

**Proposition 10** *(i) Any direct incentive-compatible mechanism (DICM)* $\{(b(\theta), t(\theta))\}_{\theta\in\Theta}$ *is such that, for some* $\hat{\theta} \in \Theta$ *and some* $\hat{t} \in \mathbb{R}$*:*

- *For* $\theta < \hat{\theta}$, $b(\theta) > \hat{b}(\theta)$, $b(\theta)$ *(weakly) increases with* $\theta$, *and*

$$t(\theta) = \hat{t} + \pi(b(\theta), \theta) + \int_{\theta}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds. \tag{22}$$

- *For* $\theta > \hat{\theta}$, $b(\theta) \leq \hat{b}(\hat{\theta})$ *and* $t(\theta) = \hat{t}$.

*(ii) Without loss of generality, we can further restrict attention to DICMs* $\{(b(\theta), t(\theta))\}_{\theta\in\Theta}$ *such that:*

- *For* $\theta > \hat{\theta}$, $b(\theta) = b(\underline{\theta})\,(\leq \hat{b}(\hat{\theta}))$ *and* $t(\theta) = 0$*; and*

- *For* $\theta = \hat{\theta}$, $b(\hat{\theta})$ *is the closest to* $\hat{b}(\hat{\theta})$ *between* $\lim_{\theta\to\hat{\theta}^-} b(\theta)$ *and* $b(\underline{\theta})$.

*Conversely, any direct mechanism satisfying the above conditions is individually rational, and it is incentive-compatible if and only:*

$$\pi(b(\underline{\theta}), \underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(\theta), \theta)\, d\theta. \tag{23}$$

**Proof.** To establish part (i) of the proposition, consider a DICM $\{b(\theta), t(\theta)\}_{\theta\in\Theta}$ and let

$$\hat{\theta} \equiv \begin{cases} \bar{\theta} & \text{if } b(\theta) > \hat{b}(\theta) \text{ for any } \theta \in \Theta, \\ \inf\left\{\theta \mid b(\theta) \leq \hat{b}(\theta)\right\} & \text{otherwise,} \end{cases}$$

denote the threshold beyond which the profile $\{b(\theta)\}_{\theta\in\Theta}$ remains below the profile $\left\{\hat{b}(\theta)\right\}_{\theta\in\Theta}$.

From Lemma 5, we know that $b(\theta)$ remains below $\hat{b}(\hat{\theta})$ in the range $\theta > \hat{\theta}$. Therefore, any type $\theta > \hat{\theta}$ would obtain a net profit of $-t(\tilde{\theta})$ by picking the option designed for another type $\tilde{\theta} > \hat{\theta}$ (as $\hat{b}(\theta) > \hat{b}(\hat{\theta}) \geq b(\tilde{\theta})$). Incentive compatibility then implies that the profile $t(\theta)$ is constant in the range $\theta > \hat{\theta}$; that is, there exists $\hat{t}$ such that $t(\theta) = \hat{t}$ for $\theta > \hat{\theta}$, and any type $\theta > \hat{\theta}$ obtains a net payoff equal to $-\hat{t}$ by picking any option $\left(b(\tilde{\theta}), t(\tilde{\theta})\right)$ designed for any type $\tilde{\theta} > \hat{\theta}$.

We now turn to the range $\theta < \hat{\theta}$, where $b(\theta) > \hat{b}(\theta)$. By choosing the option designed for a type $\tilde{\theta}$ "close enough" to its own type $\theta$ (so that $b(\tilde{\theta})$ not only exceeds $\hat{b}(\tilde{\theta})$, but also exceeds $\hat{b}(\theta)$), an entrant of type $\theta$ would obtain:

$$\varphi\left(\theta, \tilde{\theta}\right) \equiv \pi\left(b(\tilde{\theta}), \theta\right) - t(\tilde{\theta}),$$

where:

$$\pi(b, \theta) = \left[c(B_I + \Delta - b) - c(B_I - \theta + b)\right] D\left(c(B_I + \Delta - b)\right).$$

The usual reasoning can then be used to show that incentive compatibility requires the profiles $\{b(\theta)\}_{\theta<\hat{\theta}}$ and $\{t(\theta)\}_{\theta<\hat{\theta}}$ to be (weakly) increasing (as the profit function satisfies Mirrlees' single-crossing property: $\partial^2\pi/\partial\theta\partial b > 0$) and such that, by opting for the option $(b(\theta), t(\theta))$, a type $\theta < \hat{\theta}$ obtains a net profit equal to

$$r(\theta) \equiv \pi(b(\theta), \theta) - t(\theta) = r(\hat{\theta}) - \int_{\theta}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds, \tag{24}$$

which is decreasing in the range $\left[\underline{\theta}, \hat{\theta}\right]$, and such that $\lim_{\theta\to\hat{\theta}^-} r(\theta) = r(\hat{\theta})$.

To complete the proof of part (i), we now show that $r(\hat{\theta}) = -\hat{t}$. Indeed, if $r(\hat{\theta}) < -\hat{t}$, then an entrant of type $\hat{\theta}$ would strictly prefer any option designed for $\theta > \hat{\theta}$ to the option $(b(\hat{\theta}), t(\hat{\theta}))$. If instead $r(\hat{\theta}) > -\hat{t}$, two cases can be distinguished:

- If $b(\hat{\theta}) > \hat{b}(\hat{\theta})$, then a type $\theta$ slightly above $\hat{\theta}$ would strictly prefer the option $(b(\hat{\theta}), t(\hat{\theta}))$ to the option $(b(\theta) \leq \hat{b}(\hat{\theta}), t(\theta) = \hat{t})$.
- If $b(\hat{\theta}) \leq \hat{b}(\hat{\theta})$, then $r(\hat{\theta}) = -t(\hat{\theta})$, and incentive compatibility implies $t(\hat{\theta}) = \hat{t}$.

Replacing $r(\hat{\theta})$ by its value $-\hat{t}$ in (24) yields (22).

To establish part (ii) of the proposition, we first show that without loss of generality, we can restrict attention to profiles $\{b(\theta)\}_{\theta\in\Theta}$ that remain constant in the range $\theta > \hat{\theta}$. To see this, it suffices to note that replacing the profile $\{b(\theta)\}_{\theta>\hat{\theta}}$ with the constant profile $\{\tilde{b}(\theta)) = \underline{b}\}_{\theta > \hat{\theta}}$, where

$$\underline{b} \equiv \sup\{b(\theta) \mid \theta > \hat{\theta}\},$$

weakly improves expected consumer surplus:

- This does not affect incentive compatibility in the range $\theta > \hat{\theta}$, as any such type obtains $-\hat{t}$ anyway.
- This does not affect incentive compatibility in the range $\theta \leq \hat{\theta}$ either, as by construction we have, for any $\theta \leq \hat{\theta}$:

$$\pi(b(\theta), \theta) - t(\theta) \geq \sup_{\tilde{\theta} > \theta} \left\{ \pi\left(b(\tilde{\theta}), \theta\right) - t(\tilde{\theta}) \right\} = \pi(\underline{b}, \theta) - \hat{t}.$$

- Finally, this can only reduce the consumer price in the range $\theta > \hat{\theta}$; indeed, for any $\theta > \hat{\theta}$, the consumer price is initially given by

$$c_E = c(B_I - \theta + b(\theta)),$$

and thus can only decrease when $b(\theta)$ is replaced with $\underline{b} \geq b(\theta)$.

When $b(\hat{\theta}) \leq \hat{b}(\hat{\theta})$, the same reasoning applies to the entire range $\theta \geq \hat{\theta}$ (that is, including the type $\hat{\theta}$), and thus $b(\hat{\theta}) = \underline{b}$. When instead $b(\hat{\theta}) > \hat{b}(\hat{\theta})$, incentive compatibility implies that $b(\hat{\theta})$ cannot lie below $\lim_{\theta \to \hat{\theta}^-} b(\theta)$. Furthermore, as $b(\theta) > \hat{b}(\theta)$ for $\theta < \hat{\theta}$, it must be case that $\lim_{\theta \to \hat{\theta}^-} b(\theta) \geq \hat{b}(\hat{\theta})$; therefore, setting $b(\hat{\theta}) = \lim_{\theta \to \hat{\theta}^-} b(\theta)$ is better than any higher value for $b(\hat{\theta})$.

So far we have shown that any type $\theta \in \Theta$ obtains a net profit that is continuous in $\theta$ and weakly decreases as $\theta$ increases: it coincides with $r(\theta)$ given by (24) for $\theta \leq \hat{\theta}$, and with $r(\hat{\theta}) = -\hat{t}$ for $\theta \geq \hat{\theta}$. Therefore, individual rationality boils down to $\hat{t} \leq 0$, and without loss of generality we can set $\hat{t} = 0$.

To conclude the proof of part (ii), it remains to show that without loss of generality, we can further restrict attention to DICMs such that $\underline{b} = b(\underline{\theta}) \leq \hat{b}(\hat{\theta})$ and (23) holds. We do through a sequence of claims.

We first note that incentive compatibility requires $\underline{b} \leq b(\underline{\theta})$:

**Claim 1** $\underline{b} \le b(\underline{\theta})$.

***Proof.*** *By construction,* $\underline{b} \le \hat{b}(\hat{\theta})$ *(as* $b(\theta)$ *lies below* $\hat{b}(\hat{\theta})$ *in the range* $\theta > \hat{\theta}$*). Suppose now that* $\underline{b} > b(\underline{\theta}) \left(> \hat{b}(\underline{\theta})\right)$*. As type* $\underline{\theta}$ *should prefer* $(b(\underline{\theta}), t(\underline{\theta}))$ *to* $\left(\underline{b}, \hat{t} = 0\right)$*, we have:*

$$\pi(b(\underline{\theta}), \underline{\theta}) - t(\underline{\theta}) \ge \pi(\underline{b}, \underline{\theta}).$$

*Conversely, any type* $\theta > \hat{\theta}$ *should prefer* $(\underline{b}, 0)$ *to* $(b(\underline{\theta}), t(\underline{\theta}))$*, and thus (using* $\underline{b} \le \hat{b}(\hat{\theta}) < \hat{b}(\theta)$*):*

$$0 \ge -t(\underline{\theta}).$$

*Combining these two incentive compatibility conditions yields,* $\pi(b(\underline{\theta}), \underline{\theta}) \ge \pi(\underline{b}, \underline{\theta})$*, implying* $\underline{b} \le b(\underline{\theta})$*, a contradiction. Hence, we must have* $\underline{b} \le b(\underline{\theta})$*.* ■

Next, we show that we can restrict attention to DICMs such that $b(\underline{\theta}) \le \hat{b}(\hat{\theta})$:

**Claim 2** $b(\underline{\theta}) \le \hat{b}(\hat{\theta})$.

***Proof.*** *To see this, note that incentive compatibility requires* $b(\theta)$ *to be non-decreasing in the range* $\theta < \hat{\theta}$*. Hence, if* $b(\underline{\theta}) > \hat{b}(\hat{\theta})$*, then the alternative mechanism where* $(b(\theta), t(\theta))$ *is replaced with* $(\tilde{b}(\theta), \tilde{t}(\theta)) = (\hat{b}(\hat{\theta}), \hat{t} = 0)$ *in the range* $\theta \in [\underline{\theta}, \hat{\theta}]$*, is trivially incentive-compatible and individually rational, and would dominate the original DICM, as it gets closer to the optimal allocation under complete information in the range* $\theta \in [\underline{\theta}, \hat{\theta}]$*.* ■

The next step is to show that incentive compatibility imposes some bounds on $b(\underline{\theta})$ and $\underline{b}$:

**Claim 3** *Incentive compatibility requires the profile* $\{b(\theta)\}_{\theta \in \Theta}$ *to satisfy:*

$$\pi(\underline{b}, \underline{\theta}) \le -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(\theta), \theta)\, d\theta \le \pi(b(\underline{\theta}), \underline{\theta}). \tag{25}$$

***Proof.*** *To ensure that the type* $\underline{\theta}$ *does not prefer the option designed for a type* $\theta > \hat{\theta}$*, we must have:*

$$\pi(\underline{b}, \underline{\theta}) \le r(\underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(\theta), \theta)\, d\theta,$$

*which establishes the first condition in* (25).

*Conversely, to ensure that a type* $\theta > \hat{\theta}$ *does not prefer the option designed for* $\underline{\theta}$*, we must have (using* $b(\underline{\theta}) \leq \hat{b}(\hat{\theta})$*):*

$$0 \geq -t(\underline{\theta}) = -\pi(b(\underline{\theta}), \underline{\theta}) - \int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds,$$

*and thus:*

$$\pi(b(\underline{\theta}), \underline{\theta}) \geq -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds,$$

*which establishes the second condition in* (25). ■

Conversely, we now show that a mechanism $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ is incentive-compatible and individually rational if it satisfies the above properties:

**Claim 4** *A mechanism* $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ *is incentive-compatible and individually rational whenever, for some* $\hat{\theta} \in \Theta$ *and some* $\underline{b} \leq \hat{b}(\hat{\theta})$*:*

(*a*) *for* $\theta < \hat{\theta}$, $b(\theta) > \hat{b}(\theta)$, $b(\theta)$ *increases with* $\theta$*, and* $t(\theta)$ *satisfies* (22)*; and*

(*b*) *for* $\theta \geq \hat{\theta}$, $b(\theta) = \underline{b}$ *and* $t(\theta) = 0$*; and*

(*c*) *the profile* $\{b(\theta)\}_{\theta \in \Theta}$ *satisfies* $b(\underline{\theta}) \leq \hat{b}(\hat{\theta})$ *and* (25).

***Proof.*** *We first note that, from the above analysis, such a DICM is individually rational:*

- *For* $\theta \geq \hat{\theta}$*, opting for the option* $(\underline{b}, 0)$ *guarantees a net profit of zero.*
- *For* $\theta < \hat{\theta}$*, opting for the option* $(b(\theta), t(\theta))$ *gives a net profit equal to*

$$r(\theta) = \pi(b(\theta), \theta) - t(\theta) = -\int_{\theta}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds,$$

*which is decreasing in the range* $[\underline{\theta}, \hat{\theta}]$ *and satisfies* $r(\hat{\theta}) = 0$.

*We now turn to incentive compatibility. Consider first a type* $\theta < \hat{\theta}$*. Condition* (22) *ensures that such a type (weakly) prefers the option designed for it to any option designed* *for another $\tilde{\theta} < \hat{\theta}$. Therefore, incentive compatibility holds if, in addition, it does not prefer the option $(\underline{b}, 0)$ designed for $\theta > \hat{\theta}$, that is, if:*

$$r(\theta) \geq \pi(\underline{b}, \theta).$$

*We have:*

$$\begin{aligned}\frac{d}{d\theta}\left[r(\theta) - \pi(\underline{b}, \theta)\right] &= \frac{\partial \pi}{\partial \theta}(b(\theta), \theta) - \frac{\partial \pi}{\partial \theta}(\underline{b}, \theta) \\ &= \int_{\underline{b}}^{b(\theta)} \frac{\partial^2 \pi}{\partial b \partial \theta}(b(s), s)\, ds \\ &\geq 0,\end{aligned}$$

*where the inequality stems from $\underline{b} \leq b(\underline{\theta}) \leq b(\theta)$ and $\partial^2\pi/\partial b \partial \theta > 0$. Therefore, incentive compatibility holds for any type $\theta < \hat{\theta}$ if it holds for $\underline{\theta}$, that is, if*

$$\pi(\underline{b}, \theta) \leq r(\underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds,$$

*which amounts to the first inequality in* (25).

*Next, consider a type $\theta > \hat{\theta}$:*

- *By selecting an option designed for $\tilde{\theta} < \hat{\theta}$ such that $b(\tilde{\theta}) > \hat{b}(\theta)$ (which is feasible if $\lim_{\theta \to \hat{\theta}^-} b(\theta) > \hat{b}(\theta)$), it obtains*

  $$\varphi\left(\theta, \tilde{\theta}\right) = \pi\left(b(\tilde{\theta}), \theta\right) - t(\tilde{\theta}) = r(\tilde{\theta}) + \pi\left(b(\tilde{\theta}), \theta\right) - \pi\left(b(\tilde{\theta}), \tilde{\theta}\right),$$

  *where:*

  $$\begin{aligned}\frac{\partial \varphi}{\partial \tilde{\theta}}\left(\theta, \tilde{\theta}\right) &= \left[\frac{\partial \pi}{\partial b}\left(b(\tilde{\theta}), \theta\right) - \frac{\partial \pi}{\partial b}\left(b(\tilde{\theta}), \tilde{\theta}\right)\right]\frac{db}{d\theta}(\tilde{\theta}) = \int_{\tilde{\theta}}^{\theta} \frac{\partial^2 \pi}{\partial \theta \partial b}\left(b(\tilde{\theta}), s\right) ds \frac{db}{d\theta}(\tilde{\theta}) \geq 0, \\ \frac{\partial \varphi}{\partial \theta}\left(\theta, \tilde{\theta}\right) &= \frac{\partial \pi}{\partial \theta}\left(b(\tilde{\theta}), \theta\right) < 0.\end{aligned}$$

  *Therefore, incentive compatibility holds, as $\varphi\left(\theta, \tilde{\theta}\right) \leq \varphi(\theta, \hat{\theta}) < \varphi(\hat{\theta}, \hat{\theta}) = 0$.*

- *By selecting instead an option designed for $\tilde{\theta} < \hat{\theta}$ such that $b(\tilde{\theta}) \leq \hat{b}(\theta)$, an entrant of type $\theta$ obtains a net profit of $-t(\tilde{\theta})$; as $t(\tilde{\theta})$ increases with $\tilde{\theta}$, this net profit is* *maximal for $\tilde{\theta} = \underline{\theta}$. It follows that incentive compatibility holds if:*

$$0 \geq -t(\underline{\theta}) = -\pi(b(\underline{\theta}), \underline{\theta}) - \int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds,$$

*which amounts to the second inequality in* (25).

■

Note that (25) implies $\underline{b} \leq b(\underline{\theta})$. Thus, so far we have shown that attention could be restricted to DICMs such as described in the Proposition, except that we allow for $\underline{b} \leq b(\underline{\theta})$ and the bounds on transfers are given by (25).

We now show that we can further restrict the relevant class of DICMs:

**Claim 5** *Without loss of generality, we can restrict attention to DICMs such as described by Claim 4 that moreover satisfy*

$$\pi(b(\underline{\theta}), \underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds.$$

***Proof.*** *Suppose that the DICM $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ is instead such that*

$$\pi(b(\underline{\theta}), \underline{\theta}) > r(\underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds,$$

*and consider now the following alternative mechanism, where for every $\theta < \hat{\theta}$, $b(\theta)$ is replaced with*[54]

$$b(\theta, \alpha) \equiv \alpha \hat{b}(\theta) + (1 - \alpha) b(\theta).$$

*In this alternative mechanism, a type $\underline{\theta}$ obtains a net profit equal to*

$$r(\underline{\theta}, \alpha) \equiv -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(\theta, \alpha), \theta)\, d\theta,$$

*where:*

$$\frac{\partial r}{\partial \alpha}(\underline{\theta}, \alpha) = \left[b(\underline{\theta}) - \hat{b}(\underline{\theta})\right] \int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial^2 \pi}{\partial b \partial \theta}(b(\theta, \alpha), \theta)\, d\theta > 0.$$

[54]In the alternative mechanism, the bandwidth allocated to a type $\hat{\theta}$ can be taken equal to $b\left(\hat{\theta}, \alpha\right)$ when $b\left(\hat{\theta}\right) = \lim_{\theta \to \hat{\theta}^-} b(\theta)$, and to $\underline{b}$ otherwise.

*It follows that starting from $\alpha = 0$, an increase in $\alpha$:*

- *Reduces the consumer price when the entrant is of type $\theta < \hat{\theta}$, from $c_I|_{b=b(\theta)} = c(B_I + \Delta - b(\theta))$ to $c_I|_{b=b(\theta,\alpha)} = c(B_I + \Delta - b(\theta, \alpha))$.*
- *Relaxes the second constraint in (25), which becomes:*

$$\pi(\underline{b}, \underline{\theta}) \leq r(\underline{\theta}, \alpha),$$

*where $r(\underline{\theta}, \alpha) > r(\underline{\theta}, 0) = r(\underline{\theta})$.*

- *Keeps satisfying the first constraint in (25), for $\alpha$ small enough.*

*Therefore, without loss of generality, we can set restrict attention to DICMs satisfying*

$$\pi(b(\underline{\theta}), \underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds.$$

■

Note that the equality stated in Claim 5 implies $t(\underline{\theta}) = 0 \left(= \hat{t}\right)$. This, in turn, yields:

**Claim 6** *Without loss of generality, we can restrict attention to DICMs such as described by Claim 4 that moreover satisfy*

$$\pi(\underline{b}, \underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds.$$

***Proof.*** *Suppose that the DICM $\{(b(\theta), t(\theta))\}_{\theta \in \Theta}$ satisfies is such that*

$$\pi(b(\underline{\theta}), \underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds > \pi(\underline{b}, \underline{\theta}),$$

*implying $\underline{b} < b(\underline{\theta})\,(\leq \hat{b}(\hat{\theta}))$, and consider the alternative direct mechanism where, for $\theta > \hat{\theta}$, $b(\theta) = \underline{b}$ is replaced with $\tilde{b}(\theta) = b(\underline{\theta})$ – transfers being unchanged: $\tilde{t}(\theta) = t(\theta) = t(\underline{\theta}) = \hat{t} = 0$. The alternative mechanism is remains individually rational and incentive-compatible and it increases consumers' expected surplus (as the price decreases from $c_E|_{b=\underline{b}} = c(B_I - \theta + \underline{b})$ to $c_E|_{b=b(\underline{\theta})} = c(B_I - \theta + b(\underline{\theta}))$). Therefore, without loss* *of generality, we can set* $\underline{b} = b(\underline{\theta})$*, implying*

$$\pi(b(\underline{\theta}), \underline{\theta}) = -\int_{\underline{\theta}}^{\hat{\theta}} \frac{\partial \pi}{\partial \theta}(b(s), s)\, ds = \pi(\underline{b}, \underline{\theta}).$$

■

This concludes the proof of part (ii). ■

# C An Example Where Bunching Is Not Optimal

We provide here an example where bunching is no longer optimal.

## C.1 Setup

### C.1.1 Demand and Supply Conditions

We will adopt the following specifications:

- *Linear demand:* Letting $p$ denote the market price (i.e., the lowest of the firms' prices), consumer demand is given by:

  $$D(p) = 1 - p.$$

  It follows that consumer surplus is equal to:

  $$S(p) = \frac{(1-p)^2}{2},$$

  and the industry monopoly profit, based on a constant marginal cost $\gamma$, is equal to:

  $$p^m(\gamma) = \frac{1+\gamma}{2}.$$

- *Linear unit cost:* If a firm benefits from a bandwidth $\tilde{B}$, its unit cost is given by:

  $$c\left(\tilde{B}\right) = C - \tilde{B}.$$

### C.1.2 Two types of entrant

We denote the bandwidth initially available to the incumbent by

$$B_I = B$$

and that initially available to the entrant by

$$B_E = B - \tilde{\theta}.$$

Thus, as before, the parameter $\tilde{\theta}$ reflects the handicap of the entrant. We assume that this handicap can take two values:

- With probability $\rho \in (0, 1)$, the handicap is given by $\tilde{\theta} = 0$; that is, the entrant is initially as efficient as the incumbent.
- With probability $\rho' = 1 - \rho$, the handicap is given by $\tilde{\theta} = B$; that is, the entrant has initially no bandwidth.

### C.1.3 Calibration

For the sake of exposition, we further assume that:

- The additional bandwidth is large enough to enable both types of entrant to win the market competition; that is:

  $$\Delta = B + 2\varepsilon,$$

  where $\varepsilon > 0$. With this notation, the relevant values of the critical bandwidth threshold,

  $$\hat{b}(\theta) = \frac{\Delta + \theta}{2},$$

  are equal to:

  $$\hat{b} = \hat{b}(0) = \frac{\Delta}{2} = \frac{B}{2} + \varepsilon,$$
  $$\hat{b}' = \hat{b}(B) = \frac{B + \Delta}{2} = B + \varepsilon.$$

- The cost function is normalized such that

$$C = B + \Delta = 2\left(B + \varepsilon\right).$$

This ensures that all unit costs remain non-negative (the incumbent benefits from a zero unit cost if it obtains all the additional bandwidth, and both unit costs are positive otherwise), and also simplifies some of the exposition.[55]

### C.1.4 Prices

When an entrant of type $\tilde{\theta}$ obtains an additional bandwidth $\tilde{b} \geq \hat{b}(\tilde{\theta})$, the market price $p_{\tilde{\theta}}\left(\tilde{b}\right)$ is equal to the cost of the incumbent:

$$c\left(B_I + \Delta - \tilde{b}\right) = C - \left(B + \Delta - \tilde{b}\right).$$

Using $C = B + \Delta$, this simplifies to

$$p_{\tilde{\theta}}\left(\tilde{b}\right) = \tilde{b}.$$

When instead the incumbent wins the competition (i.e., when $\tilde{b} < \hat{b}\left(\theta\right)$), the market price is determined by the cost of the entrant, and is thus equal to:

$$p_{\tilde{\theta}}\left(\tilde{b}\right) \equiv \begin{cases} B + 2\varepsilon - \tilde{b} & \text{if } \tilde{\theta} = \theta = 0, \\ 2\left(B + \varepsilon\right) - \tilde{b} & \text{if } \tilde{\theta} = \theta' = B. \end{cases}$$

### C.1.5 Profit

When an entrant of type $\tilde{\theta}$ obtains an additional bandwidth $\tilde{b} \in [\hat{b}(\tilde{\theta}), 1]$, it wins the product-market competition and obtains a profit equal to:

$$\begin{aligned} \pi(\tilde{b}, \tilde{\theta}) &\equiv [c(B + \Delta - \tilde{b}) - c(B - \tilde{\theta} + \tilde{b})]D(c(B + \Delta - \tilde{b})) \\ &= 2[\tilde{b} - \hat{b}(\tilde{\theta})](1 - \tilde{b}). \end{aligned}$$

[55]See for instance below the derivation of the market price $p_{\tilde{\theta}}\left(\tilde{b}\right)$ for $\tilde{b} \geq \hat{b}\left(\tilde{\theta}\right)$.

We have:

$$\frac{\partial \pi}{\partial \tilde{\theta}}(\tilde{b}, \tilde{\theta}) = -(1 - \tilde{b}) \leq 0,$$

with a strict inequality for $\tilde{b} < 1$, and:

$$\frac{\partial^2 \pi}{\partial \tilde{\theta} \partial \tilde{b}} \pi(\tilde{b}, \tilde{\theta}) = 1 > 0.$$

Finally,

$$\partial_{\tilde{b}} \pi(\tilde{b}, \tilde{\theta}) = 2(1 - \tilde{b}) - 2(\tilde{b} - \hat{b}(\tilde{\theta})) = 4 \left[ \frac{1 + \hat{b}(\tilde{\theta})}{2} - \tilde{b} \right] = 4 \left[ \hat{p}^m(\hat{b}(\tilde{\theta})) - \tilde{b} \right],$$

where

$$\hat{p}^m(\gamma) = \frac{1 + \gamma}{2}$$

denotes the monopoly price based on a unit cost $\gamma$. Hence, $\partial_{\tilde{b}} \pi(\tilde{b}, \tilde{\theta})$ is positive as long as:

$$p_{\tilde{\theta}}(\tilde{b}) = \tilde{b} < \hat{p}^m(\hat{b}(\tilde{\theta})) = \frac{1 + \hat{b}(\tilde{\theta})}{2} = \frac{1 + \frac{\Delta + \tilde{\theta}}{2}}{2} = \frac{1}{2} \left( 1 + B + \varepsilon + \frac{\tilde{\theta}}{2} \right).$$

In particular, when the entrant has no handicap ($\theta = 0$), in order to ensure that the price ($p_\theta(b) = b$) remains below the monopoly level ($\hat{p}^m(\hat{b})$) in the relevant range ($b \in [\hat{b}, \hat{b}']$), we need:

$$\hat{b}' = B + \varepsilon < \hat{p}^m(\hat{b}) = \frac{1 + \frac{B}{2} + \varepsilon}{2}$$
$$\iff B < \frac{2}{3}(1 - \varepsilon). \tag{26}$$

## C.2 Bunching Mechanisms

When considering bunching mechanisms, which allocate the same additional bandwidth $b$ to both types of entrant, without loss of generality we can restrict attention to $b \in [\hat{b}, \hat{b}']$, as any lower value ($b < \hat{b}$) is dominated by $b = \hat{b}$, and any higher value ($b > \hat{b}'$) is dominated by $b = \hat{b}'$. For any value $b$ in that range:

- when the entrant has no handicap, the market price is equal to the cost of the incumbent; and

- otherwise, the market price is equal to the cost of the entrant.

Hence, expected consumer surplus is equal to:

$$\begin{aligned} S_B(b) &= \rho S(c(B+\Delta-b)) + \rho' S(c(B-\theta'+b)) \\ &= \rho S(b) + \rho' S(2B+2\varepsilon-b). \end{aligned}$$

This expected surplus is convex in $b$:

$$\begin{aligned} S_B'(b) &= -\rho D(b) + \rho' D(2B+2\varepsilon-b), \\ S_B''(b) &= \rho+\rho' > 0. \end{aligned}$$

It follows that the best bunching mechanism consists of allocating either $\hat{b}$ or $\hat{b}'$ to the entrant; both options are moreover equivalent when:

$$\begin{aligned} & S_B(\hat{b}') = S_B(\hat{b}) \\ \Longleftrightarrow & \rho S(C - s(B+\Delta-\hat{b}')) + \rho' S(C - s(B-\theta'+\hat{b}')) \\ & \quad = \rho S(C - s(B+\Delta-\hat{b})) + \rho' S(C - s(B-\theta'+\hat{b})) \qquad (27) \\ \Longleftrightarrow & \frac{\rho}{\rho'} = \frac{S(C-(B-\theta'+\hat{b}')) - S(C-(B-\theta'+\hat{b}))}{S(C-(B+\Delta-\hat{b})) - S(C-(B+\Delta-\hat{b}'))} \\ \Longleftrightarrow & \frac{\rho}{\rho'} = \frac{1-\varepsilon-\frac{5B}{4}}{1-\varepsilon-\frac{3B}{4}}. \qquad (28) \end{aligned}$$

## C.3 Discriminating mechanisms

We will consider a candidate discriminating mechanism which gives the non-handicapped entrant a bandwidth $b \in [\hat{b}, \hat{b}']$ (in exchange for a transfer $t$), and the handicapped entrant a higher bandwidth $b' > \hat{b}'$ (in exchange for a transfer $t'$).[56] To be incentive-compatible, the mechanism must satisfy:

$$\begin{aligned} \pi(b,\theta) - t &\geq \pi(b',\theta) - t', \\ \pi(b',\theta') - t' &\geq -t. \end{aligned}$$

[56] By inspecting the incentive constraints for all possible cases (where $b \gtreqless \hat{b}$ and $b' \gtreqless \hat{b}'$), it can be checked that the best discriminating mechanism has indeed these features.

Combining these conditions imposes:

$$\pi\left(b,\theta\right)\geq\pi\left(b',\theta\right)-\pi\left(b',\theta'\right). \tag{29}$$

The right-hand side of this inequality decreases as $b'$ increases:

$$\begin{aligned}\frac{d}{db'}\left(\pi\left(b',\theta\right)-\pi\left(b',\theta'\right)\right) &= \frac{\partial\pi}{\partial\tilde{b}}\left(b',\theta\right)-\frac{\partial\pi}{\partial\tilde{b}}\left(b',\theta'\right)\\ &= -\int_{\theta}^{\theta'}\frac{\partial^2\pi}{\partial\tilde{\theta}\partial\tilde{b}}\left(b',x\right)dx\\ &< 0.\end{aligned}$$

Hence, for any given bandwidth, $b\in\left[\hat{b},\hat{b}'\right]$, that an entrant with no handicap would receive, the best value for the bandwidth, $b'$, that a handicapped entrant should receive is the lowest one that is compatible with (29); that is, $b'$ should be chosen such that:

$$\begin{aligned}&\pi\left(b,\theta\right)=\pi\left(b',\theta\right)-\pi\left(b',\theta'\right)\\ &\quad\Longleftrightarrow 2\left(b-\hat{b}\right)\left(1-b\right)=2\left(b'-\hat{b}\right)\left(1-b'\right)-2\left(b'-\hat{b}'\right)\left(1-b'\right)\\ &\quad\Longleftrightarrow b'=\beta\left(b\right)\equiv\left[1-\frac{b-\hat{b}}{\hat{b}'-\hat{b}}\left(1-b\right)\right]=2-b-\frac{2}{B}\left(b-\varepsilon\right)\left(1-b\right)=\frac{B\left(2-b\right)-2\left(b-\varepsilon\right)\left(1-b\right)}{B}.\end{aligned}$$

This optimal value is such that:

$$\beta\left(\hat{b}'\right)=\frac{\hat{b}'\left(\hat{b}'-\hat{b}\right)}{\hat{b}'-\hat{b}}=\hat{b}',$$

and, for $b\in\left[\hat{b},\hat{b}'\right]$:

$$\begin{aligned}\beta'\left(b\right) &= \frac{d}{db}\left(\frac{B\left(2-b\right)-2\left(b-\varepsilon\right)\left(1-b\right)}{B}\right)\\ &= -\frac{2+B+2\varepsilon-4b}{B}\\ &= -\frac{1+\hat{b}-2b}{\hat{b}'-\hat{b}}=-2\frac{\hat{p}^m\left(\hat{b}\right)-b}{\hat{b}'-\hat{b}},\end{aligned}$$

which is negative as long as

$$b < \hat{p}^m\left(\hat{b}\right) = \frac{1+\hat{b}}{2} = \frac{1}{2}\left(1 + \frac{B}{2} + \varepsilon\right).$$

In particular:

$$\beta'\left(\hat{b}'\right) = -2\frac{\hat{p}^m\left(\hat{b}\right) - \hat{b}'}{\hat{b}' - \hat{b}} = \left[-2\frac{\frac{1}{2} + \frac{\Delta}{4} - \frac{\Delta+B}{2}}{\frac{B}{2}}\right]_{\Delta = B+2\varepsilon} = -\frac{2\left(1-\varepsilon\right) - 3B}{B},$$

which is thus negative as long as $B$ satisfies (26). Hence, as long as $B$ satisfies this condition, $\beta\left(b\right)$ is indeed higher than $\hat{b}'$ for $b$ lower than but close to $\hat{b}'$.

Expected consumer surplus is then equal to:

$$\begin{aligned} S_D\left(b\right) &= \rho S\left(c\left(B+\Delta-b\right)\right) + \rho' S\left(c\left(B+\Delta-\beta\left(b\right)\right)\right) \\ &= \rho\frac{\left(1-b\right)^2}{2} + \rho'\frac{\left(1-\beta\left(b\right)\right)^2}{2}. \end{aligned}$$

Therefore:

$$S_D'\left(b\right) = -\rho\left(1-b\right) - \rho'\left(1-\beta\left(b\right)\right)\beta'\left(b\right),$$

Bunching will for instance not be optimal if:

- the probabilities of the two types are such that expected consumer surplus is the same in the situation where both types receive $\hat{b}$ and in the situation where they both receive $\hat{b}'$; and,
- starting from the latter situation, where both types receive $\hat{b}'$, a small reduction in the bandwidth $b$ allocated to the entrant in case of no handicap, together with an increase in the bandwidth $b'$ allocated to the entrant in case of a large handicap, up to $b' = \beta\left(b\right)$, increases expected consumer surplus.

Hence, to exhibit an example where bunching is not optimal, it suffices to find parameters $B$ and $\varepsilon$ such that $S_D'\left(\hat{b}'\right) < 0$ for the probabilities $\rho$ and $\rho'$ that satisfy

(28). As:

$$S'_D\left(\hat{b}'\right) = -\rho\left(1-\hat{b}'\right) - \rho'\left(1-\hat{b}'\right)\beta'\left(\hat{b}'\right)$$
$$= -\rho\left(1-\hat{b}'\right)\left[1+\frac{\rho'}{\rho}\beta'\left(\hat{b}'\right)\right],$$

this amounts to finding parameters $B$ and $\varepsilon$ such that the terms within square brackets is positive, that is:

$$-\beta'\left(\hat{b}'\right) = \frac{2-3B-2\varepsilon}{B} < \frac{\rho}{\rho'} = \frac{1-\varepsilon-\frac{5B}{4}}{1-\varepsilon-\frac{3B}{4}}.$$

This requires:

$$\frac{2-3B-2\varepsilon}{B} < \frac{1-\frac{5B}{4}-\varepsilon}{1-\frac{3B}{4}-\varepsilon}$$
$$\Longleftrightarrow (2-3B-2\varepsilon)\left(1-\frac{3B}{4}-\varepsilon\right) < B\left(1-\frac{5B}{4}-\varepsilon\right)$$
$$\Longleftrightarrow 0 < B\left(1-\frac{5B}{4}-\varepsilon\right) - (2-3B-2\varepsilon)\left(1-\frac{3B}{4}-\varepsilon\right) = -2\left(1-\varepsilon-\frac{7B}{4}\right)(1-\varepsilon-B),$$

which amounts to:

$$\frac{4}{7}(1-\varepsilon) < B < 1-\varepsilon.$$

Combining these conditions with (26), it suffices to choose $B$ and $\varepsilon$ such that:

$$\frac{4}{7}(1-\varepsilon) < B < \frac{2}{3}(1-\varepsilon).$$

## C.4 Numerical example

### C.4.1 Parameter values

For $\varepsilon = 0$, the above conditions boil down to:

$$\frac{4}{7} = \frac{12}{21} < B < \frac{2}{3} = \frac{14}{21}.$$

We will thus consider the case

$$B = \frac{13}{21},$$

and choose $\varepsilon$ "small enough" to satisfy (26), namely, such that:

$$B < \frac{2}{3}(1-\varepsilon) \Longleftrightarrow \varepsilon < 1 - \frac{3B}{2} = \frac{1}{14} \simeq 0.07.$$

We will thus take $\varepsilon = 0.05\,(= 1/20)$. We then have:

$$\begin{aligned}
\hat{b} &= \frac{13}{42} + \frac{1}{20} = \frac{151}{420} \simeq 0.36,\\
\hat{b}' &= \frac{13}{21} + \frac{1}{20} = \frac{281}{420} \simeq 0.67,\\
\Delta &= \frac{13}{21} + \frac{1}{10} = \frac{151}{210} \simeq 0.72,\\
\beta(b) &= \left[\frac{B(2-b) - 2(b-\varepsilon)(1-b)}{B}\right]_{B=\frac{13}{21},\varepsilon=\frac{1}{20}} = \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2,\\
p^m\left(\hat{b}\right) &= \left[\frac{1+\hat{b}}{2}\right]_{\hat{b}=13/42+1/20} = \frac{571}{840} \simeq 0.68,\\
p^m\left(\hat{b}'\right) &= \left[\frac{1+\hat{b}'}{2}\right]_{\hat{b}'=13/21+1/20} = \frac{701}{840} \simeq 0.83,\\
\rho &= \frac{37}{139} \simeq 0.27,\\
\rho' &= \frac{102}{139} \simeq 0.73,
\end{aligned}$$

and:

$$-\beta'\left(\hat{b}'\right) = \frac{9}{130} \simeq 0.06 < \frac{\rho}{\rho'} = \frac{37}{102} \simeq 0.36.$$

The function $\beta(b)$ is depicted by the following figure (for $b \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$):

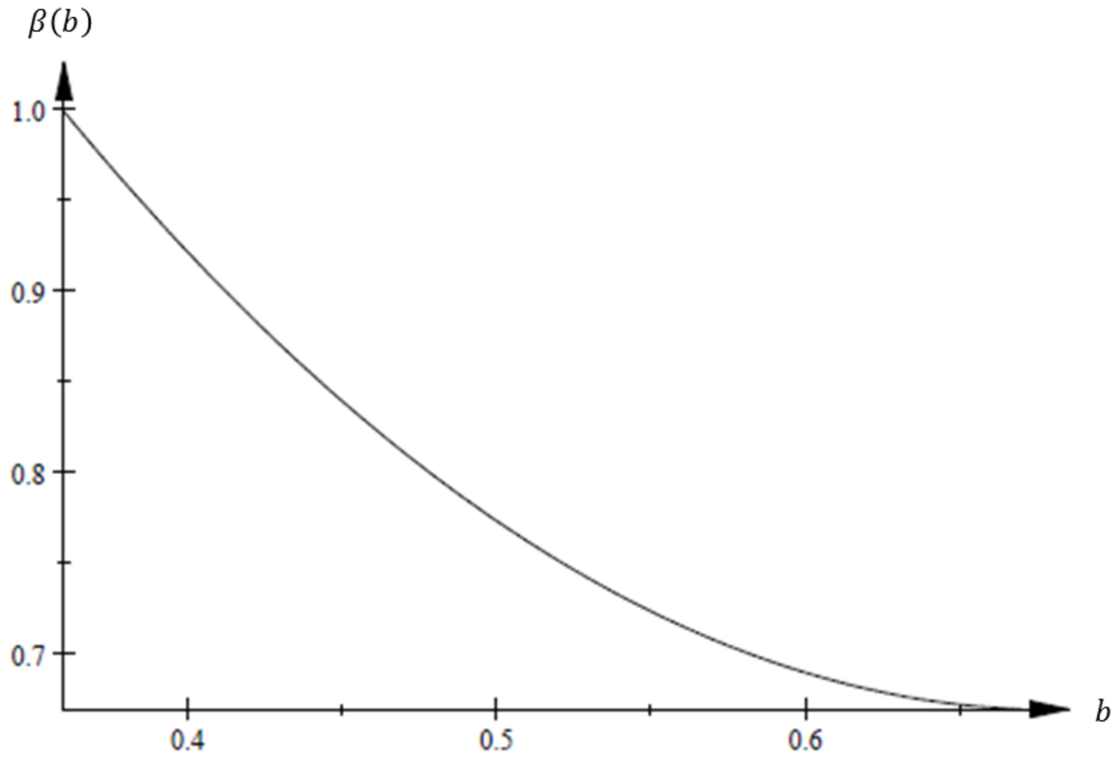


Figure 3: Bandwidth for handicapped entrant

In particular, we have:

$$\beta(b) \leq \Delta \Longleftrightarrow \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2 \leq \frac{151}{210}$$
$$\Longleftrightarrow b \geq \bar{b} = \frac{571}{840} - \frac{\sqrt{3}\sqrt{3667}}{840} \simeq 0.55.$$

It can be checked that the above conditions are satisfied in this example; in particular:

- Demand is positive (i.e., $b < 1$) in the relevant ranges $b \leq \hat{b}'$ (as $\hat{b}' \simeq 0.67 < 1$) and $b' \leq \Delta$ (as $\Delta \simeq 0.72 < 1$). It follows that $\partial_\theta \pi(b, \theta) < 0$ in the relevant ranges.

- We also have $\partial_b \pi(b,\theta) > 0$ (i.e., $b < \hat{p}^m\left(\hat{b}(\theta)\right)$) in these ranges:

$$\begin{aligned} b \leq \hat{b}' &= \frac{281}{420} \simeq 0.67 < p^m\left(\hat{b}\right) = \frac{571}{840} \simeq 0.68, \\ b' \leq \Delta &= \frac{151}{210} \simeq 0.72 \leq p^m\left(\hat{b}'\right) = \frac{701}{840} \simeq 0.83. \end{aligned}$$

### C.4.2 Prices

In case of bunching, for $b = b' \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$, the price is equal to $b$ if the entrant faces no handicap, and it is otherwise equal to:

$$\begin{aligned} p_{\theta'}(b') &= [C - (B - \theta' + b)]_{\theta'=B, \Delta=B+2\varepsilon, C=2(B+\varepsilon)} \\ &= \frac{281}{210} - b. \end{aligned}$$

In case of discrimination, for $b \in \left[\bar{b}, \hat{b}'\right] \simeq [0.55, 0.67]$ and $b' = \beta(b) = \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2$, the price is the same as in the previous scenario (i.e., it is equal to $b$) if the entrant faces no handicap, and it is otherwise equal to:

$$p_{\theta'}(b') = \beta(b) = \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2.$$

The following figure depicts the price in case of handicap, in the two scenarios: bunching (thin line, for $b \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$) and discrimination (bold curve, for $b \in \left[\bar{b}, \hat{b}'\right] \simeq$

[0.55, 0.67]):

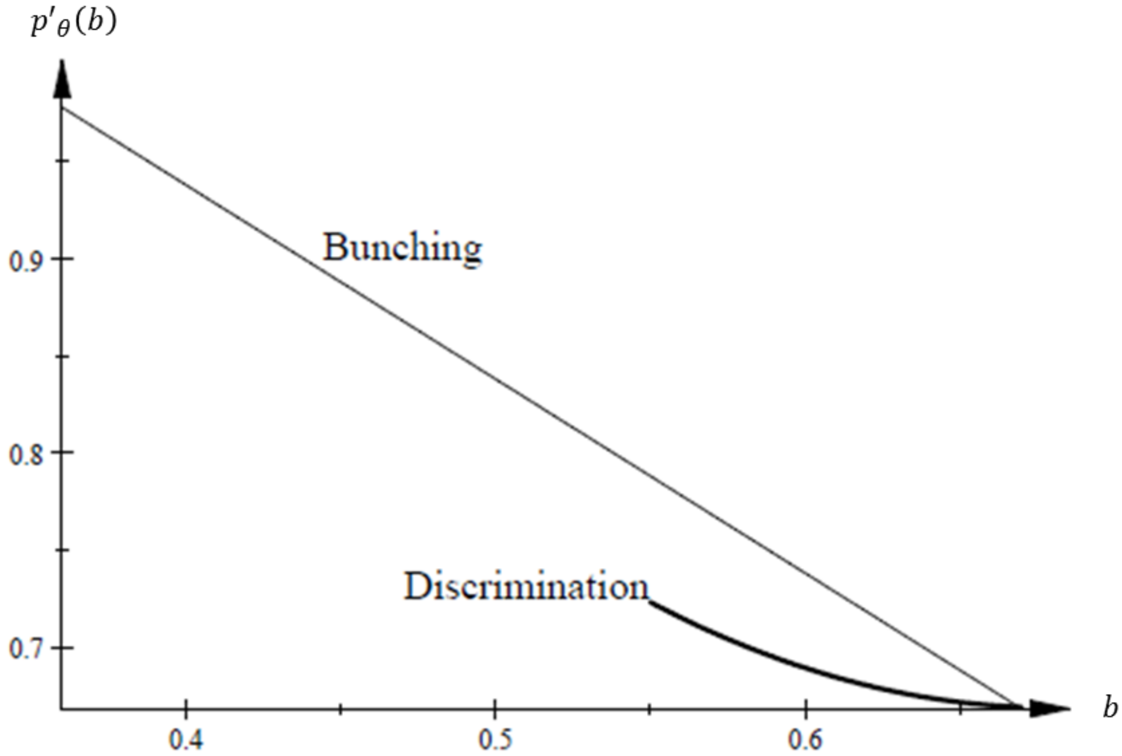


Figure 4: Market price when the entrant is handicapped

The figure confirms that the price $p_{\theta'}(b)$ is lower in the discriminating scenario than in the bunching scenario.

### C.4.3 Consumer surplus

$$p_{\theta'}(b') = \begin{cases} \frac{281}{210} - b & \text{if} \quad b' = b, \\ \frac{281}{130} - \frac{571}{130}b + \frac{42}{13}b^2 & \text{if} \quad b' = \beta(b). \end{cases}$$

Building on the above analysis, and using

$$\rho = \frac{37}{139} \simeq 0.27 \text{ and } \rho' = \frac{102}{139} \simeq 0.73,$$

consumers' expected surplus is given by:

- If $b' = b$ ("Bunching"), then for $b = b' \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$:

$$\begin{aligned} S_B(b) &= \rho \frac{(1-b)^2}{2} + \rho' \frac{(1-p_{\theta'}(b))^2}{2} \\ &= \frac{37}{139}\frac{(1-b)^2}{2} + \frac{102}{139}\frac{\left(1-\left(\frac{281}{210}-b\right)\right)^2}{2} \\ &= \frac{1}{2}b^2 - \frac{18}{35}b + \frac{2573}{14\,700}. \end{aligned}$$

From the above analysis, this expected consumer is maximal for $b = \hat{b}$ and $b = \hat{b}'$, where it is equal to:

$$S_B\left(\hat{b}\right) = S_B\left(\hat{b}'\right) = \frac{19321}{352800} \simeq 0.05.$$

- If $b' = \beta(b)$ ("Discriminating"), then for $b \in \left[\bar{b}, \hat{b}'\right] \simeq [0.55, 0.67]$ and $b' \in \left[\hat{b}', \Delta\right] \simeq [0.67, 0.72]$):

$$\begin{aligned} S_D(b) &= \rho \frac{(1-b)^2}{2} + \rho' \frac{(1-\beta(b))^2}{2} \\ &= \frac{37}{139}\frac{(1-b)^2}{2} + \frac{102}{139}\frac{\left(1-\left(\frac{281}{130}-\frac{571}{130}b+\frac{42}{13}b^2\right)\right)^2}{2} \\ &= \frac{8996400b^2 - 6468840b + 1475501}{2349100}(1-b)^2. \end{aligned}$$

The following figure depicts expected consumer surplus in the bunching scenario (thin curve, for $b \in \left[\hat{b}, \hat{b}'\right] \simeq [0.36, 0.67]$) and the discriminating scenario (bold curve, for $b \in \left[\bar{b}, \hat{b}'\right] \simeq [0.55, 0.67]$); it shows that discriminating is indeed optimal:

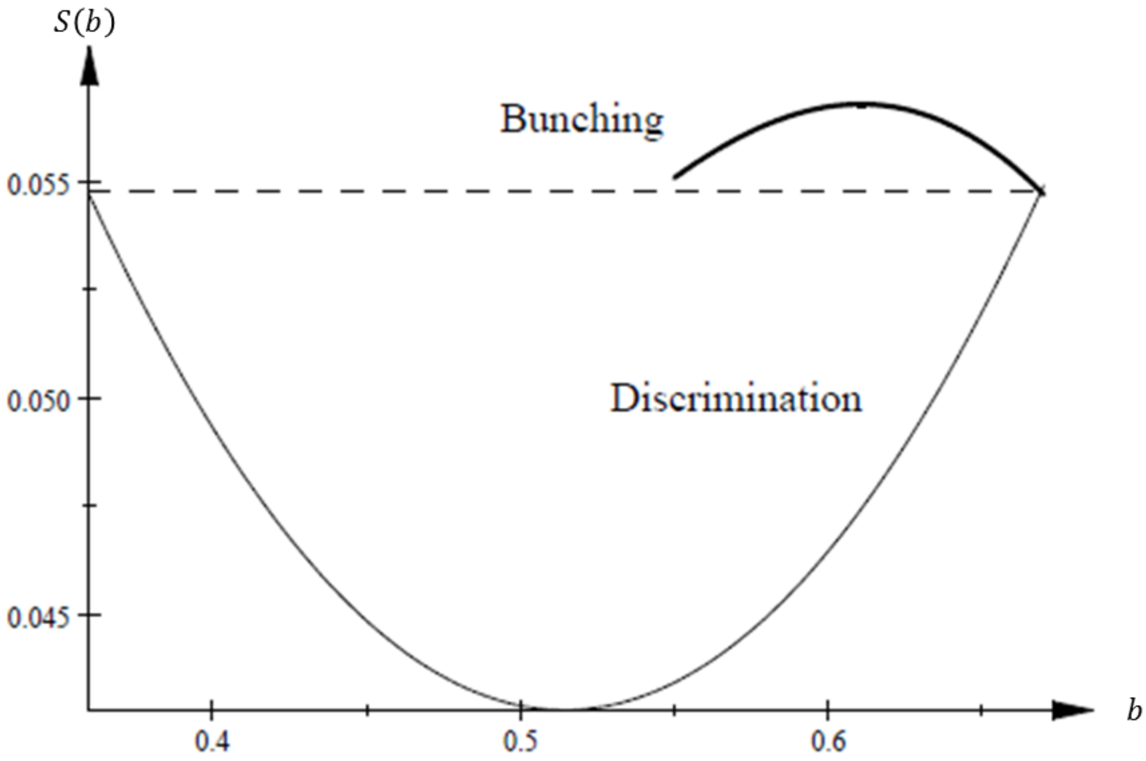


Figure 5: Consumer surplus

To determine the socially optimal mechanism, it suffices to maximize $S_D(b)$, which yields:

$$S'_S(b) = \frac{359856}{23491}b^3 - \frac{3669246}{117455}b^2 + \frac{1800737}{90350}b - \frac{4709921}{1174550} = 0,$$

leading to

$$b^* = \frac{291}{560} + \frac{\sqrt{51}\sqrt{1189\,211}}{85\,680} \simeq 0.610\,54.$$