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"On the Heston Model with Stochastic Volatility: Analytic Solutions and Complete Markets"

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On the Heston Model with Stochastic Volatility: Analytic Solutions and Complete Markets*

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Abstract

We study the Heston model for pricing European options on stocks with stochastic volatility. This is a Black-Scholes-type equation whose spatial domain for the logarithmic stock price $x \in \mathbb{R}$ and the variance $v \in (0, \infty)$ is the half-plane $\mathbb{H} = \mathbb{R} \times (0, \infty)$. The *volatility* is then given by $\sqrt{v}$. The diffusion equation for the price of the European call option $p = p(x, v, t)$ at time $t \leq T$ is parabolic and degenerates at the boundary $\partial\mathbb{H} = \mathbb{R} \times \{0\}$ as $v \to 0+$. The goal is to hedge with this option against volatility fluctuations, i.e., the function $v \mapsto p(x, v, t) : (0, \infty) \to \mathbb{R}$ and its (local) inverse are of particular interest. We prove that $\frac{\partial p}{\partial v}(x, v, t) \neq 0$ holds almost everywhere in $\mathbb{H} \times (-\infty, T)$ by establishing the analyticity of p. To this end, we are able to show that the Black-Scholes-type operator, which appears in the diffusion equation, generates a holomorphic C^0-semigroup in a suitable weighted L^2-space over $\mathbb{H}$. We show that the C^0-semigroup solution can be extended to a holomorphic function in a complex domain, by establishing some new a priori weighted L^2-estimates over certain complex "shifts" of $\mathbb{H}$ for the unique holomorphic extension. These estimates depend only on the weighted L^2-norm of the terminal data over $\mathbb{H}$.

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Key words: Heston model; stochastic volatility; Black-Scholes equation; European call option; degenerate parabolic equation; terminal value problem; holomorphic extension; analytic solution

1 Introduction

For several decades, simple market models have been very important and useful products of numerous mathematical studies of financial markets. Several of them have become very popular and are extensively used by the financial industry (F. BLACK and M. SCHOLES [6], S. L. HESTON [23], and J.-P. FOUQUE, G. PAPANICOLAOU, and K. R. SIRCAR [17] to mention only a few). These models are usually concerned with asset pricing in a volatile market under clearly specified rules that are supposed to guarantee "fair pricing" (e.g., arbitrage-free prices in T. BJÖRK [5]).

Assets are typically represented by stocks, securities (e.g., bonds), and their derivatives (such as options on stocks and similar contracts). An important role of a derivative is to assess, reduce or eliminate the volatile behavior of a particular asset (or an entire portfolio). A common way to achieve this objective is to add a derivative on the volatile asset to the portfolio containing this asset. This procedure, called hedging, is closely connected with the problem of ***market completion*** (M. ROMANO and N. TOUZI [42], M. H. A. DAVIS [10]). There have been a number of successful attempts to obtain a market completion by (call or put) options on stocks. The pricing of such options involves various kinds of the Black-Scholes-type equations. These attempts are typically based on probabilistic, analytic, and numerical techniques, some of them including even explicit formulas, cf. Y. ACHDOU and O. PIRONNEAU [1, Chapt. 2]. The basic principle behind all Black-Scholes-type models is that the model must be ***arbitrage-free***, that is, any *arbitrage opportunity* must be excluded which is possible only if the option price is a stochastic process that is a martingale (T. BJÖRK [5]). Îto's formula then yields an equivalent linear parabolic equation which will be the object of our investigation, cf. M. H. A. DAVIS [10]. Throughout our present work we study the ***Heston model*** of pricing for *European call options on stocks* with ***stochastic volatility*** (S. L. HESTON [23]) by abstract analytic methods coming from partial differential equations (PDEs, for short) and functional analysis.

In our simple market, described by the ***Heston stochastic volatility model*** (Heston model, for short), market completion by a European call option on the stock has the following meaning: The basic quantities are the *maturity time* T (called also the *exercise time*), $0 < T < \infty$, at which the stock option matures; the *real time* t, $-\infty < t \leq T$; the *time to maturity* $\tau = T - t \geq 0$, $0 \leq \tau < \infty$; the *spot price* of stock $S = S_t > 0$ at time $t \leq T$; the (stochastic) *variance* of the stock market $V = V_t > 0$ at time $t \leq T$; $\sqrt{V}$ is associated with the (stochastic) *volatility* of the stock market; the *strike price* (exercise price) $K \equiv \text{const} > 0$ of the stock option at maturity, a European call or put option; a given (nonnegative) *payoff function* $\hat{h}(S,V) = (S-K)^+$ at time $t = T$ (i.e., $\tau = 0$) for a European call option; and the (call or put) *option price* $U = U(S,V,t) > 0$ at time t, given the stock price S and the variance V. In the derivation of S. L. HESTON's model [23], which is a system of two stochastic differential equations for the pair (X_t, V_t), Îto's formula yields a diffusion equation for the unknown option price $U = U(S, V, t) > 0$ at time t which depends only on the stock price S_t and the variance V_t at time t. This allows us to view the *relative logarithmic stock price* $x = \ln(S_t/K) \in \mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$, and the variance $v = V_t \in (0, \infty)$ as a pair of independent variables in the open half-plane $\mathbb{H} \stackrel{\text{def}}{=} \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2$. Consequently, the option price $p = p(x, v, t) = U(S, V, t)$ is a function of $(x, t) \in \mathbb{R} \times (-\infty, T]$ and $v \in (0, +\infty)$ with the terminal value at maturity time $t = T$ given by

$$p(x, v, T) = (S - K)^+ = K\,(\mathrm{e}^x - 1)^+ \quad \text{for } (x, v) \in \mathbb{H}\,. \tag{1.1}$$

The option price $p = p(x, v, t) \equiv p_\tau(x, v)$, where $\tau = T - t \geq 0$, is (uniquely) determined by a unique, risk neutral martingale measure ([10, 42]), which yields a stochastic process $(p_\tau)_{\tau \geq 0}$. Applying Îto's formula to this process, one concludes that, equivalently to the probabilistic expectation formula for $p(x, v, t)$, this option price can be calculated directly from a partial differential equation of parabolic type with the terminal value condition (1.1). Thus, given the (relative logarithmic) stock price $x \in \mathbb{R}$ at a fixed time $t \in (-\infty, T]$, the function $\tilde{p}_{x,t} : v \mapsto p(x, v, t)$ yields the (unique) option price for every $v \in (0, +\infty)$. According to I. BAJEUX-BESNAINOU and J.-CH. ROCHET [3, p. 12], the characteristic property of a complete market is that $\tilde{p}_{x,t} : (0, +\infty) \to \mathbb{R}_+$ is injective (i.e., one-to-one), which means that any particular option value $p = \tilde{p}_{x,t}(v)$ cannot be attained at two different values of the variance $v \in (0, +\infty)$. We take advantage of this property to give an alternative definition of a ***complete market*** using differential calculus rather than probability theory, see our Definition 5.3 in Section 5. This is a purely mathematical problem that we solve in this article for the Heston model, with a help from [3, Sect. 5] and the work by M. H. A. DAVIS and J. OBŁÓJ [11]; see Section 5 below, Theorem 5.2.

There are several other stochastic volatility models, see, e.g., those listed in [17, Table 2.1, p. 42] and those treated in [17, 27, 36, 43, 48], that are already known to allow or may allow market completion by a European call option. However, the rigorous proofs of market completeness (and their methods) vary from model to model; cf. T. BJÖRK [5]. Some of them are more probabilistic (R. M. ANDERSON and R. C. RAIMONDO [2] with *"endogenous completeness"* of a diffusion driven equilibrium market, I. BAJEUX-BESNAINOU and J.-CH. ROCHET [3], J. HUGONNIER, S. MALAMUD, and E. TRUBOWITZ [25], D. KRAMKOV and S. PREDOIU [31], and M. ROMANO and N. TOUZI [42]), others more analytic (PDEs), e.g., in M. H. A. DAVIS [10], M. H. A. DAVIS and J. OBŁÓJ [11], and P. TAKÁČ [46].

In the derivation of S. L. HESTON's model [23], Îto's formula yields the following diffusion equation

$$\left(\frac{\partial}{\partial t} + \mathbf{A}\right) U(S, V, t) = 0 \quad \text{for } S > 0,\ V > 0,\ t < T\,. \tag{1.2}$$

We call $\mathbf{A}$ the *Black-Scholes-Îto operator* for the Heston model; it is defined by

$$
\begin{aligned}
&(\mathbf{A}U)(S,V,t) \stackrel{\text{def}}{=} \\
&V \cdot \left(\frac{1}{2}\, S^2\, \frac{\partial^2 U}{\partial S^2}(S,V,t) + \rho\sigma\, S\, \frac{\partial^2 U}{\partial S\, \partial V}(S,V,t) + \frac{1}{2}\, \sigma^2\, \frac{\partial^2 U}{\partial V^2}(S,V,t) \right) \\
&+ (r-q)\, S\, \frac{\partial U}{\partial S}(S,V,t) + [\kappa(\theta - V) - \lambda(S,V,t)]\, \frac{\partial U}{\partial V}(S,V,t) - r\, U(S,V,t) \\
&\quad \text{for } S > 0,\ V > 0, \text{ and } t < T,
\end{aligned} \tag{1.3}
$$

with the following additional quantities (constants) as given data: the *risk free rate of interest* $r \in \mathbb{R}$; the *dividend yield* $q \in \mathbb{R}$; the *instantaneous drift* of the stock price returns $r - q \equiv -q_r \in \mathbb{R}$; the *volatility* $\sigma > 0$ of the stochastic volatility $\sqrt{V}$; the *correlation* $\rho \in (-1,1)$ between the Brownian motions for the stock pricing and the volatility; the *rate of mean reversion* $\kappa > 0$ of the stochastic volatility $\sqrt{V}$; the *long term variance* $\theta > 0$ (called also *long-run variance* or *long-run mean level*) of the *stochastic variance* V; the *price of volatility risk* $\lambda(S,V,t) \geq 0$, in [23] chosen to be linear, $\lambda(S,V,t) \equiv \lambda V$ with a constant $\lambda \equiv \text{const} \geq 0$.

We assume a constant *risk free rate of interest* r and a constant *dividend yield* q; hence, $r - q = -q_r$ is the *instantaneous drift of the stock price returns*. All three quantities, r, q, and q_r, may take any real values; but, typically, one has $0 < r \leq q < \infty$ whence also $q_r \geq 0$. We refer the reader to the monograph by J. C. HULL [26, Chapt. 26, pp. 599–607] and to S. L. HESTON's original article [23] for further description of all these quantities.

The diffusion equation (1.2) is supplemented first by the following *dynamic boundary condition* as $V \to 0+$,

$$
\left(\frac{\partial}{\partial t} + \mathbf{B} \right) U(S,0,t) = 0 \qquad \text{for } S > 0,\ t < T\,. \tag{1.4}
$$

The *boundary operator* $\mathbf{B}$ is the trace of the Black-Scholes-Îto operator $\mathbf{A}$ as $V \to 0+$; it corresponds to the Black-Scholes operator with zero volatility:

$$
\begin{aligned}
&(\mathbf{B}U)(S,0,t) \stackrel{\text{def}}{=} \\
&(r-q)\, S\, \frac{\partial U}{\partial S}(S,0,t) + \kappa\theta\, \frac{\partial U}{\partial V}(S,0,t) - r\, U(S,0,t) \\
&\quad \text{for } S > 0,\ V = 0, \text{ and } -\infty < t < T.
\end{aligned} \tag{1.5}
$$

The original *Heston boundary conditions* in [23],

$$
\begin{cases}
U(0,V,t) = 0 & \text{for } V > 0; \\
\lim\limits_{S\to\infty} \dfrac{\partial}{\partial S}\, (U(S,V,t) - S) = 0 & \text{for } V > 0; \\
\lim\limits_{V\to\infty} (U(S,V,t) - S) = 0 & \text{for } S > 0,
\end{cases} \tag{1.6}
$$

at all times $t \in (-\infty, T)$, seem to be "economically" motivated. Mathematically, one may attempt to motivate them by the asymptotic behavior of the solution $U_{\mathrm{BS}}(S,t) \equiv U_{\mathrm{BS}}(S,V_0,t)$ to the Black-Scholes equation, for $S > 0$ and $t \le T$, where the variance $V_0 = \sigma_0^2 > 0$ is a given constant determined from the constant volatility $\sigma_0 > 0$. What we mean are the following *boundary conditions*,

$$\begin{cases} U_{\mathrm{BS}}(0,V,t) = 0 & \text{for } V > 0; \\ \lim_{S\to\infty} \frac{\partial}{\partial S}\left(U_{\mathrm{BS}}(S,V,t) - S\right) = 0 & \text{for } V > 0; \\ \lim_{V\to\infty} \left(U_{\mathrm{BS}}(S,V,t) - S\right) = 0 & \text{for } S > 0, \end{cases} \tag{1.7}$$

at all times $t \in (-\infty, T)$. Roughly speaking, the difference $U(S,V,t) - U_{\mathrm{BS}}(S,V,t)$ becomes asymptotically small near the boundary. The terminal condition as $t \to T-$ for both solutions, U and U_{BS}, is the *payoff function* $\hat{h}(S,V) = (S-K)^+$ for $S > 0$,

$$U(S,V,T) = U_{\mathrm{BS}}(S,V,T) = (S-K)^+ .$$

The solution $U_{\mathrm{BS}}(S,t)$ of the Black-Scholes equation has been calculated explicitly in the original article by F. Black and M. Scholes [6]; see also J.-P. Fouque, G. Papanicolaou, and K. R. Sircar [17, §1.3.4, p. 16].

Finally, the diffusion equation (1.2) is supplemented also by the following *terminal condition* as $t \to T-$, which is given by the payoff function $\hat{h}(S,V) = (S-K)^+$,

$$U(S,V,T) = (S-K)^+ \quad \text{for } S > 0,\ V > 0 . \tag{1.8}$$

The terminal-boundary value problem for eq. (1.2) with the boundary conditions (1.4) and (1.6), as it stands, poses a *mathematically* challenging problem, in particular, due to the degeneracies in the diffusion part of the operator $\mathbf{A}$: Some or all of the coefficients of the second partial derivatives tend to zero as $S \to 0+$ and/or $V \to 0+$, making the diffusion effects disappear on the boundary $\{(S,0) : S > 0\}$, cf. eq. (1.5).

This article is organized as follows. We begin with a rigorous mathematical formulation of the Heston model in Section 2. We make use of weighted Lebesgue and Sobolev spaces originally introduced in P. Daskalopoulos and P. M. N. Feehan [8] and [9, Sect. 2, p. 5048] and P. M. N. Feehan and C. A. Pop [15]. An extension of the problem from the real to a complex domain is formulated in Section 3. Our main results, Proposition 4.1 and Theorem 4.2, are stated in Section 4. Before giving the proofs of these two results, in Section 5 we present an application of them to S. L. Heston's model [23] for *European call options* in Mathematical Finance. There we also provide an affirmative answer (Theorem 5.2) to the problem of *market completeness* as described in M. H. A. Davis and J. Obłój [11]. Our contribution to market completeness is also an alternative definition for a market to be complete (Definition 5.3) which is based on classical concepts of differential calculus (I. Bajeux-Besnainou and J.-Ch. Rochet [3, p. 12]) rather than on probability theory. In addition, we discuss the important *Feller condition* in Remark 5.4 and also mention another application to a related model in Remark 5.5. The proofs of our main results from Section 4 are gradually developed in Sections 6 through 8 and completed in Section 9. Finally, Appendix A contains some technical asymptotic results for functions from our weighted Sobolev spaces, whereas Appendix B is concerned with the density of certain analytic functions in these spaces.

2 Formulation of the mathematical problem

In this section we introduce S. L. Heston's model [23, Sect. 1, pp. 328–332] and formulate the associated Cauchy problem as an evolutionary equation of (degenerate) parabolic type.

2.1 Heston's stochastic volatility model

We consider the Heston model given under the *risk neutral measure* via equations (1) – (4) in [23, pp. 328–329]. The model is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, where $\mathbb{P}$ is the risk neutral probability measure, and the filtration satisfies the usual conditions. Recalling that S_t denotes the *stock price* and V_t the (stochastic) *variance* of the stock market at (the real) time $t \geq 0$, the unknown pair $(S_t, V_t)_{t\geqslant 0}$ satisfies the following system of *stochastic* differential equations,

$$\text{(2.1)} \qquad \begin{cases} \dfrac{\mathrm{d}S_t}{S_t} = -\, q_r \,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t\,, \\ \mathrm{d}V_t = \kappa\,(\theta - V_t)\,\mathrm{d}t + \sigma\,\sqrt{V_t}\;\mathrm{d}Z_t\,, \end{cases}$$

where $(W_t)_{t\geqslant 0}$ and $(Z_t)_{t\geqslant 0}$ are two Brownian motions with the *correlation coefficient* $\rho \in (-1, 1)$, a constant given by $\mathrm{d}\langle W, Z\rangle_t = \rho\,\mathrm{d}t$. This is the original Heston system in [23].

If $X_t = \ln(S_t/K)$ denotes the (natural) logarithm of the *scaled stock price* S_t/K at time $t \geq 0$, relative to the strike price $K > 0$, then the pair $(X_t, V_t)_{t\geqslant 0}$ satisfies the following system of stochastic differential equations,

$$\text{(2.2)} \qquad \begin{cases} \mathrm{d}X_t = -\left(q_r + \tfrac{1}{2}V_t\right)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t\,, \\ \mathrm{d}V_t = \kappa\,(\theta - V_t)\,\mathrm{d}t + \sigma\,\sqrt{V_t}\;\mathrm{d}Z_t\,. \end{cases}$$

Following [11, Sect. 4], let us consider a European call option written in this market with *payoff* $\hat{h}(S_T, V_T) \equiv \hat{h}(S_T) \geq 0$ at maturity $T > 0$, where $\hat{h}(S) = (S - K)^+$ for all $S > 0$. As usual, for $x \in \mathbb{R}$ we abbreviate $x^+ \stackrel{\text{def}}{=} \max\{x,\, 0\}$ and $x^- \stackrel{\text{def}}{=} \max\{-x,\, 0\}$. We set $h(X, V) \equiv h(X) = K\,(\mathrm{e}^X - 1)^+$ for all $X = \ln(S/K) \in \mathbb{R}$, so that $h(X) = \hat{h}(S) = \hat{h}(K\mathrm{e}^X)$ for $X \in \mathbb{R}$. Hence, if the instant values $(X_t, V_t) = (x, v) \in \mathbb{H}$ are known at time $t \in (0, T)$, where $\mathbb{H} = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2$, the *arbitrage-free price* A_t^h of the European call option at this time is given by the expectation formula (with respect to the risk neutral probability measure $\mathbb{P}$)

$$
\begin{aligned}
A_t^h(x,v) &= \mathrm{e}^{-r(T-t)}\, \mathbb{E}_{\mathbb{P}} \left[\hat{h}(S_T) \mid \mathcal{F}_t \right] = \mathrm{e}^{-r(T-t)}\, \mathbb{E}_{\mathbb{P}} \left[h(X_T) \mid \mathcal{F}_t \right] \\
&= \mathrm{e}^{-r(T-t)}\, \mathbb{E}_{\mathbb{P}} \left[h(X_T) \mid X_t = x,\ V_t = v \right] .
\end{aligned} \tag{2.3}
$$

It is justified in [11] and [46] that $A_t^h = p(X_t, V_t, t)$ where p solves the (*terminal* value) Cauchy problem

$$
\begin{cases}
\dfrac{\partial p}{\partial t} + \mathcal{G}_t\, p - rp = 0\,, & (x,v,t) \in \mathbb{H} \times (0,T)\,; \\
\qquad p(x,v,T) = h(x)\,, & (x,v) \in \mathbb{H}\,,
\end{cases} \tag{2.4}
$$

with $\mathcal{G}_t$ being the (time-independent) infinitesimal generator of the time-homogeneous Markov process (X_t, V_t); cf. A. Friedman [19, Chapt. 6] or B. Øksendal [39, Chapt. 8]. Indeed, first, eq. (1.2) is derived from eqs. (2.2) and (2.3) by Îto's formula, then the diffusion equation (2.4) is obtained from eq. (1.2) using

$$
\begin{gathered}
S = K\mathrm{e}^x\,, \quad \frac{\mathrm{d}S}{\mathrm{d}x} = S\,, \quad V = v\,, \\
p(x,v,t) = U(S,V,t)\,, \quad \frac{\partial p}{\partial x}(x,\xi,t) = S\, \frac{\partial U}{\partial S}(S,v,t)\,, \\
\begin{aligned}
\frac{\partial^2 p}{\partial x^2}(x,\xi,t) &= S\, \frac{\partial U}{\partial S}(S,v,t) + S^2\, \frac{\partial^2 U}{\partial S^2}(S,v,t) \\
&= \frac{\partial p}{\partial x}(x,\xi,t) + S^2\, \frac{\partial^2 U}{\partial S^2}(S,v,t)\,.
\end{aligned}
\end{gathered}
$$

Hence, the function $\overline{p} :\ (x,v,t) \mapsto p(x,v,T-t)$ verifies a linear Cauchy problem of the following type, with the notation $\mathbf{x} = (x_1, x_2) \equiv (x,v) \in \mathbb{H}$,

$$
\begin{cases}
\dfrac{\partial \overline{p}}{\partial t} - \displaystyle\sum_{i,j=1}^{2} a_{ij}(\mathbf{x},t)\, \frac{\partial^2 \overline{p}}{\partial x_i\, \partial x_j} - \sum_{j=1}^{2} b_j(\mathbf{x},t)\, \frac{\partial \overline{p}}{\partial x_j} - c(\mathbf{x},t)\, \overline{p} \\
\qquad\qquad = f(\mathbf{x},t) \quad \text{for } (\mathbf{x},t) \in \mathbb{H} \times (0,T)\,; \\
\qquad \overline{p}(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{H}\,,
\end{cases} \tag{2.5}
$$

with the function $f(\mathbf{x},t) \equiv 0$ on the right-hand side, the initial data $u_0(\mathbf{x}) = u_0(x,v) = p(x,v,T) = h(x)$ at $t = 0$, and the coefficients

$$
\begin{aligned}
a(x,v,t) &= \frac{v}{2} \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}\,, \\
b(x,v,t) &= \begin{pmatrix} -\, q_r - \frac{1}{2}v \\ \kappa\,(\theta - v) - \lambda(x,v,T-t) \end{pmatrix} \in \mathbb{R}^2\,, \quad c(x,v,t) \ = -r \in \mathbb{R}\,,
\end{aligned}
$$

where the variable $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ has been replaced by $(x, v) \in \mathbb{H} \subset \mathbb{R}^2$. We have also replaced the meaning of the temporal variable t as real time ($t \leq T$) by the *time to maturity* t ($t \geq 0$), so that the real time has become $\tau = T - t$. According to S. L. HESTON [23, eq. (6), p. 329], the unspecified term $\lambda(x, v, T-t)$ in the vector $b(x, v, t)$ represents the ***price of volatility risk*** and is specifically chosen to be $\lambda(x, v, T-t) \equiv \lambda v$ with a constant $\lambda \geq 0$.

Next, we eliminate the constants $r \in \mathbb{R}$ and $\lambda \geq 0$, respectively, from eq. (2.5) by substituting

$$p^*(x, v, t) \stackrel{\text{def}}{=} \mathrm{e}^{-r(T-t)}\, U(S, V, T-t) \quad \text{for} \quad \overline{p}(x, v, t)\,, \tag{2.6}$$

which is the *reduced option price*, and replacing κ by $\kappa^* = \kappa + \lambda > 0$ and θ by $\theta^* = \frac{\kappa\theta}{\kappa+\lambda} > 0$. Hence, we may set $r = \lambda = 0$. Finally, we introduce also the re-scaled variance $\xi = v/\sigma > 0$ for $v \in (0, \infty)$ and abbreviate $\theta_\sigma \stackrel{\text{def}}{=} \theta/\sigma \in \mathbb{R}$. These substitutions will have a simplifying effect on our calculations later. Eq. (2.5) then yields the following initial value problem for the unknown function $u(x, \xi, t) = p^*(x, \sigma\xi, t)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}u = f(x, \xi, t) & \text{in } \mathbb{H} \times (0, T)\,; \\ u(x, \xi, 0) = u_0(x, \xi) & \text{for } (x, \xi) \in \mathbb{H}\,, \end{cases} \tag{2.7}$$

with the function $f(x, \xi, t) \equiv 0$ on the right-hand side and the initial data $u_0(x, \xi) \equiv h(x)$ at $t = 0$, where the (autonomous linear) ***Heston operator*** $\mathcal{A}$, derived from eq. (2.5), takes the following form,

$$\begin{aligned} (\mathcal{A}u)(x, \xi) \stackrel{\text{def}}{=} &- \frac{1}{2}\,\sigma\xi \cdot \left(\frac{\partial^2 u}{\partial x^2}(x, \xi) + 2\rho\, \frac{\partial^2 u}{\partial x\, \partial \xi}(x, \xi) + \frac{\partial^2 u}{\partial \xi^2}(x, \xi) \right) \\ &+ \left(q_r + \tfrac{1}{2}\sigma\xi\right) \cdot \frac{\partial u}{\partial x}(x, \xi) - \kappa(\theta_\sigma - \xi) \cdot \frac{\partial u}{\partial \xi}(x, \xi) \\ \equiv\ &- \frac{1}{2}\,\sigma\xi \cdot (u_{xx} + 2\rho\, u_{x\xi} + u_{\xi\xi}) \\ &+ \left(q_r + \tfrac{1}{2}\sigma\xi\right) \cdot u_x - \kappa(\theta_\sigma - \xi) \cdot u_\xi \quad \text{for } (x, \xi) \in \mathbb{H}. \end{aligned} \tag{2.8}$$

Recall $\theta_\sigma = \theta/\sigma$. We prefer to use the following asymmetric "divergence" form of $\mathcal{A}$,

$$\begin{aligned} (\mathcal{A}u)(x, \xi) = &- \frac{1}{2}\,\sigma\xi \cdot \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x}(x, \xi) + 2\rho\, \frac{\partial u}{\partial \xi}(x, \xi) \right) + \frac{\partial^2 u}{\partial \xi^2}(x, \xi) \right] \\ &+ \left(q_r + \tfrac{1}{2}\sigma\xi\right) \cdot \frac{\partial u}{\partial x}(x, \xi) - \kappa(\theta_\sigma - \xi) \cdot \frac{\partial u}{\partial \xi}(x, \xi) \\ \equiv\ &- \frac{1}{2}\,\sigma\xi \cdot \left[(u_x + 2\rho\, u_\xi)_x + u_{\xi\xi} \right] \\ &+ \left(q_r + \tfrac{1}{2}\sigma\xi\right) \cdot u_x - \kappa(\theta_\sigma - \xi) \cdot u_\xi \quad \text{for } (x, \xi) \in \mathbb{H}. \end{aligned} \tag{2.9}$$

The ***boundary operator*** defined in eq. (1.5) transforms the left-hand side of eq. (1.4) into the following (logarithmic) form on the boundary $\partial\mathbb{H} = \mathbb{R} \times \{0\}$ of $\mathbb{H}$:

$$\begin{aligned} &\mathrm{e}^{-r\tau} \left(\frac{\partial}{\partial\tau} + \mathbf{B} \right) U(S,0,\tau) \Big|_{\tau=T-t} = - \left(\frac{\partial}{\partial t} + \mathcal{B} \right) u(x,0,t) \\ &= - \frac{\partial u}{\partial t}(x,0,t) - q_r \frac{\partial u}{\partial x}(x,0,t) + \kappa\theta_\sigma \frac{\partial u}{\partial \xi}(x,0,t) \\ &\quad \text{for } x \in \mathbb{R} \text{ and } 0 < t < \infty. \end{aligned} \tag{2.10}$$

The remaining ***boundary conditions*** (1.6) become

$$\begin{cases} u(-\infty,\xi,t) \stackrel{\text{def}}{=} \lim_{x\to-\infty} \left(u(x,\xi,t) - K\mathrm{e}^{x-r(T-t)} \right) = 0 & \text{for } \xi > 0; \\ \lim_{x\to+\infty} \left[\mathrm{e}^{-x} \cdot \frac{\partial}{\partial x} \left(u(x,\xi,t) - K\mathrm{e}^{x-r(T-t)} \right) \right] = 0 & \text{for } \xi > 0; \\ \lim_{\xi\to\infty} \left(u(x,\xi,t) - K\mathrm{e}^{x-r(T-t)} \right) = 0 & \text{for } x \in \mathbb{R}, \end{cases} \tag{2.11}$$

at all times $t \in (0,\infty)$.

In the next paragraph we give a definition of $\mathcal{A}$ as a densely defined, closed linear operator in a Hilbert space.

2.2 Weak formulation in a weighted L^2-space

Now we formulate the initial-boundary value problem for eq. (1.2) with the boundary conditions (1.4) and (1.6) in a weighted L^2-space. In the context of the Heston model, similar weighted Lebesgue and Sobolev spaces were used earlier in P. DASKALOPOULOS and P. M. N. FEEHAN [8] and [9, Sect. 2, p. 5048] and P. M. N. FEEHAN and C. A. POP [15]. To this end, we wish to consider the Heston operator $\mathcal{A}$, defined in eq. (2.9) above, as a densely defined, closed linear operator in the weighted Lebesgue space $H \equiv L^2(\mathbb{H};\mathfrak{w})$, where the weight $\mathfrak{w} : \mathbb{H} \to (0,\infty)$ is defined by

$$\mathfrak{w}(x,\xi) \stackrel{\text{def}}{=} \xi^{\beta-1}\,\mathrm{e}^{-\gamma|x|-\mu\xi} \quad \text{for } (x,\xi) \in \mathbb{H}, \tag{2.12}$$

and $H = L^2(\mathbb{H};\mathfrak{w})$ is the *complex* Hilbert space endowed with the inner product

$$(u,w)_H \equiv (u,w)_{L^2(\mathbb{H};\mathfrak{w})} \stackrel{\text{def}}{=} \int_{\mathbb{H}} u\,\bar{w} \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \quad \text{for } u,w \in H\,. \tag{2.13}$$

Here, $\beta,\gamma,\mu \in (0,\infty)$ are suitable positive constants that will be specified later, in Section 6 (see also Appendix A). However, it is already clear that if we want that the weight $\mathfrak{w}(x,\xi)$ tends to zero as $\xi \to 0+$, we have to assume $\beta > 1$. Similarly, if we want that the initial condition $u_0(x,\xi) = K(\mathrm{e}^x - 1)^+$ for $(x,\xi) \in \mathbb{H}$ belongs to H, we must require $\gamma > 2$.

We prove in Section 6, §6.1, that the sesquilinear form associated to $\mathcal{A}$,

$$(u, w) \mapsto (\mathcal{A}u, w)_H \equiv (\mathcal{A}u, w)_{L^2(\mathbb{H};\mathfrak{w})} ,$$

is *bounded* on $V \times V$, where V denotes the *complex* Hilbert space $H^1(\mathbb{H}; \mathfrak{w})$ endowed with the inner product

$$\begin{aligned}(u, w)_V \equiv (u, w)_{H^1(\mathbb{H};\mathfrak{w})} &\stackrel{\text{def}}{=} \int_{\mathbb{H}} (u_x \, \bar{w}_x + u_\xi \, \bar{w}_\xi) \cdot \xi \cdot \mathfrak{w}(x, \xi) \, \mathrm{d}x \, \mathrm{d}\xi \\ &+ \int_{\mathbb{H}} u \, \bar{w} \cdot \mathfrak{w}(x, \xi) \, \mathrm{d}x \, \mathrm{d}\xi \quad \text{for } u, w \in H^1(\mathbb{H}; \mathfrak{w}) .\end{aligned} \tag{2.14}$$

In particular, by Lemmas A.2 and A.3 in the Appendix (Appendix A), every function $u \in V = H^1(\mathbb{H}; \mathfrak{w})$ satisfies also the following (natural) *zero boundary conditions*,

$$\xi^\beta \cdot \int_{-\infty}^{+\infty} |u(x, \xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \longrightarrow 0 \quad \text{as } \xi \to 0+ , \tag{2.15}$$

$$\xi^\beta \, \mathrm{e}^{-\mu\xi} \cdot \int_{-\infty}^{+\infty} |u(x, \xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \longrightarrow 0 \quad \text{as } \xi \to \infty , \tag{2.16}$$

and

$$\mathrm{e}^{-\gamma|x|} \cdot \int_0^\infty |u(x, \xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \longrightarrow 0 \quad \text{as } x \to \pm\infty . \tag{2.17}$$

(We are no longer using the letter V for variance; it has been replaced by the re-scaled variance $\xi = v/\sigma > 0$.) The following additional *vanishing boundary conditions* are determined by our particular *realization* of the Heston operator $\mathcal{A}$ with the domain $V = H^1(\mathbb{H}; \mathfrak{w})$, cf. (2.20) below:

$$\begin{cases} \xi^\beta \cdot \displaystyle\int_{-\infty}^{+\infty} u_\xi(x, \xi) \cdot \bar{w}(x, \xi) \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \longrightarrow 0 \quad \text{as } \xi \to 0+ ; \\ \xi^\beta \, \mathrm{e}^{-\mu\xi} \cdot \displaystyle\int_{-\infty}^{+\infty} u_\xi(x, \xi) \cdot \bar{w}(x, \xi) \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \longrightarrow 0 \quad \text{as } \xi \to \infty , \end{cases} \tag{2.18}$$

$$\mathrm{e}^{-\gamma|x|} \cdot \int_0^\infty (u_x + 2\rho \, u_\xi) \, \bar{w}(x, \xi) \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \longrightarrow 0 \quad \text{as } x \to \pm\infty , \tag{2.19}$$

for every function $w \in V$. The validity of these boundary conditions on the boundary $\partial\mathbb{H} = \mathbb{R} \times \{0\}$ of the half-plane $\mathbb{H} = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2$ (i.e., as $\xi \to 0+$) and as $\xi \to \infty$ is discussed below, in §2.4. They guarantee that $\mathcal{A}$ is a closed, densely defined linear operator in the Hilbert space H which possesses a unique extension to a bounded linear operator $V \to V'$, denoted by $\mathcal{A} : V \to V'$ again, with the property that there is a constant $c \in \mathbb{R}$ such that $\mathcal{A} + c\,I$ is *coercive* on V. Consequently, every function $v \in V$ from the domain $\mathcal{D}(\mathcal{A}) \subset H$ of $\mathcal{A}$, $\mathcal{D}(\mathcal{A}) = \{v \in V : \mathcal{A}v \in H\}$, must satisfy not only (2.15), (2.16), and (2.17) (thanks to $v \in V$), but also the boundary conditions (2.18) and

(2.19) (owing to $v \in \mathcal{D}(\mathcal{A})$). A detailed discussion of all boundary conditions is provided in §2.4 below. The coercivity of $\mathcal{A} + c\,I$ on V will be proved in Section 6, §6.2.

The sesquilinear form $(u, w) \mapsto (\mathcal{A}u, w)_H$ is used in the *Hilbert space definition* of the linear operator $\mathcal{A}$ by the following procedure. For any given $u, w \in H^1(\mathbb{H}; \mathfrak{w}) \cap W^{2,\infty}(\mathbb{H})$, we use eq. (2.9) to calculate the inner product

$$
\begin{aligned}
&(\mathcal{A}u, w)_H \equiv (\mathcal{A}u, w)_{L^2(\mathbb{H};\mathfrak{w})} = \\
&\frac{\sigma}{2} \int_{\mathbb{H}} [(u_x + 2\rho\, u_\xi) \cdot \bar{w}_x + u_\xi \cdot \bar{w}_\xi] \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&+ \frac{\sigma}{2} \int_{\mathbb{H}} \left[(u_x + 2\rho\, u_\xi)\, \bar{w} \cdot \xi \cdot \partial_x \mathfrak{w}(x,\xi) + u_\xi \cdot \bar{w} \cdot \partial_\xi \big(\xi \cdot \mathfrak{w}(x,\xi) \big) \right] \mathrm{d}x\, \mathrm{d}\xi \\
&- \frac{\sigma}{2} \int_0^\infty (u_x + 2\rho\, u_\xi)\, \bar{w} \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}\xi \,\Big|_{x=-\infty}^{x=+\infty} \\
&- \frac{\sigma}{2} \int_{-\infty}^{+\infty} u_\xi \cdot \bar{w} \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x \,\Big|_{\xi=0}^{\xi=\infty} \\
&\qquad - \int_{\mathbb{H}} \left[- \left(q_r + \tfrac{1}{2}\sigma\xi \right) u_x + \kappa(\theta_\sigma - \xi)\, u_\xi \right] \cdot \bar{w} \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
&= \frac{\sigma}{2} \int_{\mathbb{H}} (u_x \cdot \bar{w}_x + 2\rho\, u_\xi \cdot \bar{w}_x + u_\xi \cdot \bar{w}_\xi) \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&+ \frac{\sigma}{2} \int_{\mathbb{H}} \left[-\gamma\, \operatorname{sign} x \cdot (u_x + 2\rho\, u_\xi)\, \bar{w} \cdot \xi + (\beta - \mu\xi)\, u_\xi \cdot \bar{w} \right] \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&\qquad - \frac{\sigma}{2} \left[\lim_{x \to +\infty} \left(\mathrm{e}^{-\gamma|x|} \cdot \int_0^\infty (u_x + 2\rho\, u_\xi)\, \bar{w} \cdot \xi^\beta\, \mathrm{e}^{-\mu\xi}\, \mathrm{d}\xi \right) \right. \\
&\qquad\qquad \left. - \lim_{x \to -\infty} \left(\mathrm{e}^{-\gamma|x|} \cdot \int_0^\infty (u_x + 2\rho\, u_\xi)\, \bar{w} \cdot \xi^\beta\, \mathrm{e}^{-\mu\xi}\, \mathrm{d}\xi \right) \right] \\
&+ \frac{\sigma}{2} \left[\lim_{\xi \to 0+} \left(\xi^\beta \cdot \int_{-\infty}^{+\infty} u_\xi \cdot \bar{w} \cdot \mathrm{e}^{-\gamma|x|}\, \mathrm{d}x \right) - \lim_{\xi \to \infty} \left(\xi^\beta\, \mathrm{e}^{-\mu\xi} \cdot \int_{-\infty}^{+\infty} u_\xi \cdot \bar{w} \cdot \mathrm{e}^{-\gamma|x|}\, \mathrm{d}x \right) \right] \\
&\qquad - \int_{\mathbb{H}} (-q_r\, u_x + \kappa\theta_\sigma\, u_\xi) \cdot \bar{w} \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&\qquad + \int_{\mathbb{H}} \left(\tfrac{1}{2}\sigma\, u_x + \kappa\, u_\xi \right) \bar{w} \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi\,,
\end{aligned}
$$

where we now impose the vanishing boundary conditions (2.18) and (2.19).

Hence, the sesquilinear form (2.20) becomes

$$
\begin{aligned}
(\mathcal{A}u, w)_H &= \frac{\sigma}{2} \int_{\mathbb{H}} (u_x \cdot \bar{w}_x + 2\rho\, u_\xi \cdot \bar{w}_x + u_\xi \cdot \bar{w}_\xi) \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&\quad + \frac{\sigma}{2} \int_{\mathbb{H}} (1 - \gamma\, \operatorname{sign} x)\, u_x \cdot \bar{w} \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&\quad + \int_{\mathbb{H}} \left(\kappa - \gamma\rho\sigma\, \operatorname{sign} x - \tfrac{1}{2}\mu\sigma \right) u_\xi \cdot \bar{w} \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&+ q_r \int_{\mathbb{H}} u_x \cdot \bar{w} \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi + \left(\tfrac{1}{2}\beta\sigma - \kappa\theta_\sigma \right) \int_{\mathbb{H}} u_\xi \cdot \bar{w} \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi\,.
\end{aligned} \tag{2.21}
$$

All integrals on the right-hand side converge absolutely for any pair $u, w \in V$; see the proof of our Proposition 6.1 below. In what follows we use the last formula, eq. (2.21), to define the sesquilinear form (2.20) in $V \times V$. Of course, in the calculations above we have assumed the boundary conditions in (2.18) and (2.19).

We make use of the *Gel'fand triple* $V \hookrightarrow H = H' \hookrightarrow V'$, i.e., we first identify the Hilbert space H with its dual space H', by the Riesz representation theorem, then use the imbedding $V \hookrightarrow H$, which is dense and continuous, to construct its adjoint mapping $H' \hookrightarrow V'$, a dense and continuous imbedding of H' into the dual space V' of V as well. The (complex) inner product on H induces a sesquilinear duality between V and V'; we keep the notation $(\,\cdot\,,\,\cdot\,)_H$ also for this duality.

2.3 The Cauchy problem in the real domain

Let us return to the initial value problem (2.7). The letter T stands for an arbitrary (finite) upper bound on time t. The latter, t, can still be regarded as time to maturity.

Definition 2.1 Let $0 < T < \infty$, $f \in L^2((0,T) \to V')$, and $u_0 \in H$. A function $u : \mathbb{H} \times [0,T] \to \mathbb{R}$ is called a *weak solution* to the initial value problem (2.7) if it has the following properties:

(i) the mapping $t \mapsto u(t) \equiv u(\,\cdot\,,\,\cdot\,,t) : [0,T] \to H$ is a continuous function, i.e., $u \in C([0,T] \to H)$;

(ii) the initial value $u(0) = u_0$ in H;

(iii) the mapping $t \mapsto u(t) : (0,T) \to V$ is a Bôchner square-integrable function, i.e., $u \in L^2((0,T) \to V)$; and

(iv) for every function

$$\phi \in L^2((0,T) \to V) \cap W^{1,2}((0,T) \to V') \hookrightarrow C([0,T] \to H)\,,$$

the following equation holds,

$$\begin{aligned} &(u(T),\phi(T))_H - \int_0^T \left(u(t), \tfrac{\partial\phi}{\partial t}(t)\right)_H \,\mathrm{d}t + \int_0^T (\mathcal{A}u(t),\phi(t))_H \,\mathrm{d}t \\ &\quad = (u_0,\phi(0))_H + \int_0^T (f(t),\phi(t))_H \,\mathrm{d}t\,. \end{aligned} \tag{2.22}$$

The following remarks are in order:

First, our definition of a weak solution is equivalent with that given in L. C. EVANS [12, §7.1], p. 352. We are particularly interested in the solution with the initial value

$u_0(x,\xi) = K\,(\mathrm{e}^x - 1)^+$ for $(x,\xi) \in \mathbb{H}$, cf. eq. (1.8). Clearly, we have $u_0 \in H$ if and only if $\gamma > 2$, $\beta > 0$, and $\mu > 0$.

$W^{1,2}((0,T) \to V')$ denotes the Sobolev space of all functions $\phi \in L^2((0,T) \to V')$ that possess a distributional time-derivative $\phi' \in L^2((0,T) \to V')$. The norm is defined in the usual way; cf. L. C. EVANS [12, §5.9]. The properties of $V \equiv H^1(\mathbb{H};\mathfrak{w})$ justify the notation $V' = H^{-1}(\mathbb{H};\mathfrak{w})$.

The continuity of the imbedding

$$L^2((0,T) \to V) \cap W^{1,2}((0,T) \to V') \hookrightarrow C([0,T] \to H)$$

is proved, e.g., in L. C. EVANS [12, §5.9], Theorem 3 on p. 287.

2.4 The Heston operator and boundary conditions

We have seen in our definition of the sesquilinear form (2.21) in paragraph §2.2 that the boundary conditions (2.18) and (2.19) are necessary for performing integration by parts to obtain the sesquilinear form (2.21). They should be valid for every weak solution $u : \mathbb{H} \times [0,T] \to \mathbb{R}$ of the initial value problem (2.7) at a.e. time $t \in (0,T)$, and for every test function $w \in V$. A natural way to satisfy these conditions is to estimate the absolute value of the integrals from above by Cauchy's inequality and then impose or verify the following boundary conditions,

$$(2.23)\qquad \begin{cases} \xi^{\beta} \cdot \displaystyle\int_{-\infty}^{+\infty} |u_{\xi}(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|}\,\mathrm{d}x \le \mathrm{const} < \infty & \text{as } \xi \to 0+\,; \\ \xi^{\beta}\,\mathrm{e}^{-\mu\xi} \cdot \displaystyle\int_{-\infty}^{+\infty} |u_{\xi}(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|}\,\mathrm{d}x \le \mathrm{const} < \infty & \text{as } \xi \to \infty+\,, \end{cases}$$

$$(2.24)\qquad \mathrm{e}^{-\gamma|x|} \cdot \int_0^{\infty} |u_x + 2\rho\,u_{\xi}|^2 \cdot \xi^{\beta}\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi \le \mathrm{const} < \infty \quad \text{as } x \to \pm\infty\,,$$

together with (2.15), (2.16), i.e.,

$$(2.25)\qquad \begin{cases} \xi^{\beta} \cdot \displaystyle\int_{-\infty}^{+\infty} |w(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|}\,\mathrm{d}x \longrightarrow 0 & \text{as } \xi \to 0+\,; \\ \xi^{\beta}\,\mathrm{e}^{-\mu\xi} \cdot \displaystyle\int_{-\infty}^{+\infty} |w(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|}\,\mathrm{d}x \longrightarrow 0 & \text{as } \xi \to \infty\,, \end{cases}$$

and (2.17) for w in place of u. In other words, we have

- (2.23) and (2.25) $\Rightarrow$ (2.18) whereas (2.24) and (2.17) $\Rightarrow$ (2.19).

Indeed, by Lemma A.2, the latter boundary conditions, (2.25), are satisfied for every test function $w \in V$. Similarly, (2.17) holds by Lemma A.3. We stress that only the boundary conditions in (2.23) and (2.24) are *imposed*; they do *not* follow from $u \in V$.

Two of these boundary conditions on the boundary $\partial\mathbb{H} = \mathbb{R} \times \{0\}$ of the half-plane $\mathbb{H} = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2$ limit from above the growth of the solution $u(x, \xi)$ at an arbitrarily low volatility level $\sqrt{\xi}$, i.e., as the variance $\xi \to 0+$.

From now on, we use exclusively formula (2.21) to define the linear operator $\mathcal{A} : V \to V'$ that appears in the sesquilinear form (2.20) obtained directly for the Heston operator (2.9). This means that we no longer need the boundary conditions in (2.23) and (2.24) (or in (2.18) and (2.19)) imposed on $u \in V$.

We refer the reader to the recent work by P. M. N. Feehan [13], Appendix B, §B.1, pp. 57–58, for numerous interesting properties of $\mathcal{A}$.

Remark 2.2 (Coercivity conditions.) It is important to remark at this stage of our investigation of the Heston operator $\mathcal{A}$ that, in order to ensure the coercivity of $\mathcal{A} + c\,I$ on V, one has to assume the well-known ***Feller condition*** ([16, 20]),

$$\tfrac{1}{2}\sigma^2 - \kappa\theta < 0\,. \tag{2.26}$$

However, *Feller's condition* (2.26) is *not sufficient* for obtaining the desired coercivity. We need to guarantee also

$$c_1' = \tfrac{1}{2}\sigma \left[\left(\frac{\kappa}{\sigma} - \gamma\,|\rho| \right)^2 - \gamma(1+\gamma) \right] \geq 0\,;$$

cf. ineq. (6.15) in the proof of Proposition 6.2 below. That is, we need to assume

$$\kappa \geq \sigma \left(\gamma\,|\rho| + \sqrt{\gamma(1+\gamma)} \right) \quad (> \sigma\gamma(|\rho| + 1))\,. \tag{2.27}$$

The last inequality is an additional condition to *Feller's condition*, $\frac{1}{2}\sigma^2 - \kappa\theta < 0$, both of them requiring the rate of mean reversion $\kappa > 0$ of the stochastic volatility in system (2.1) to be sufficiently large. This additional condition is caused by the fact that W. Feller [16] considers only an analogous problem in one space dimension ($\xi \in \mathbb{R}_+$), so that the solution $u = u(\xi)$ is independent from $x \in \mathbb{R}$. In particular, if the initial condition $u_0 = u(\,\cdot\,,\,\cdot\,,0) \in H$ for $u(x,\xi,t)$ permits us to take $\gamma > 0$ arbitrarily small, then inequality (2.27) is easily satisfied, provided *Feller's condition* $\frac{1}{2}\sigma^2 - \kappa\theta < 0$ is satisfied. However, if we wish to accommodate also initial conditions of type $u_0(x,\xi) = K\,(\mathrm{e}^x - 1)^+$ for $(x,\xi) \in \mathbb{H}$, then we are forced to take $\gamma > 2$ to ensure that $u_0 \in H$. □

We will see in Section 4 that the initial value problem (2.7) has a unique weak solution $u : \mathbb{H} \times [0,T] \to \mathbb{R}$. Recall that, by eq. (1.8), we are particularly interested in the solution with the initial value $u_0(x,\xi) = K\,(\mathrm{e}^x - 1)^+$ for $(x,\xi) \in \mathbb{H}$. We are not able to show that even this particular solution satisfies Heston's boundary conditions (1.4) and (2.11). However, the asymptotic boundary conditions in (2.11) are taken into account by the choice of function spaces H and V. Heston's boundary operator (2.10) assumes the existence of traces of certain functions of (x,ξ) as $\xi \to 0+$ which have to satisfy a partial differential equation derived from (1.4). In conditions (2.17) and (2.25) we assume only that some of the functions in the boundary operator (2.10) do not blow up too fast as $\xi \to 0+$.

3 The complex domain: Preliminaries and notation

We complexify the real space-time domain $\mathbb{H} \times (0, \infty)$ as follows:

We denote by

$$\mathfrak{X}^{(r)} \stackrel{\text{def}}{=} \mathbb{R} + \mathrm{i}(-r, r) \subset \mathbb{C} \tag{3.1}$$

the *complex strip* of width $2r$, $r \in (0, \infty)$, which consists of all (complex) numbers $z = x + \mathrm{i}y \in \mathbb{C}$ whose imaginary part, $y = \mathfrak{Im}\, z$, is bounded by $|y| < r$, while the real part, $x = \mathfrak{Re}\, z$, may take any value $x \in \mathbb{R}$ (see Figure 1). This is the complexification of the variable $x \in \mathbb{R}$. The remaining two independent variables, $\xi, t \in (0, \infty)$, will be complexified by angular domains with the vertex at zero. We denote by

$$\Delta_\vartheta \stackrel{\text{def}}{=} \{\zeta = \varrho \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} : \ \varrho > 0 \text{ and } \theta \in (-\vartheta, \vartheta)\} \tag{3.2}$$

the *complex angle* of angular width 2ϑ, $\vartheta \in (0, \pi/2)$ (Figure 2). Notice that the standard logarithm $\zeta \mapsto z = \log \zeta$ is a conformal mapping from the angle Δ_ϑ onto the strip $\mathfrak{X}^{(\vartheta)}$. Now, given any $\vartheta_\xi, \vartheta_t \in (0, \pi/2)$, we complexify ξ as $\zeta = \xi + \mathrm{i}\eta \in \Delta_{\vartheta_v}$, so that $\xi = \mathfrak{Re}\, \zeta > 0$, and t as $t = \alpha + \mathrm{i}\tau \in \Delta_{\vartheta_t}$, whence $\alpha = \mathfrak{Re}\, t > 0$, thus stressing that we allow for complex time $t \in \Delta_{\vartheta_t}$ in accordance with the usual notation for holomorphic C^0-semigroups. The half-plane $\mathbb{H} = \mathbb{R} \times (0, \infty)$ is naturally imbedded into the complex domain

$$\mathfrak{Y}^{(r)} \stackrel{\text{def}}{=} \mathfrak{X}^{(r)} \times \Delta_{\arctan r} \subset \mathbb{C}^2 , \quad r \in (0, \infty) . \tag{3.3}$$

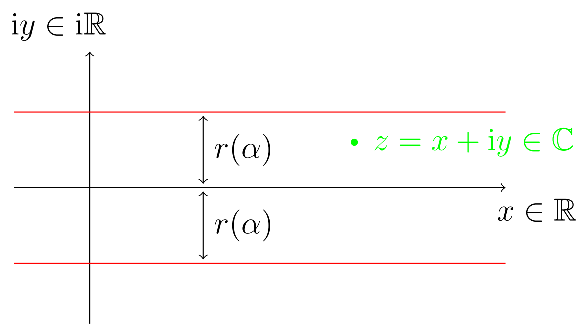

Figure 1. Strip $\mathfrak{X}^{(r)} = \mathbb{R} + \mathrm{i}(-r, r))$ for $r = r(\alpha)$, $\alpha > 0$.

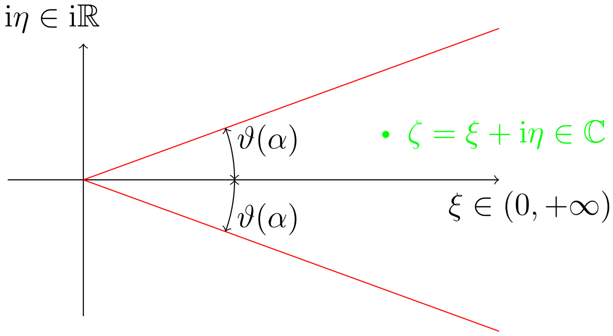

Figure 2. Angle Δ_ϑ.

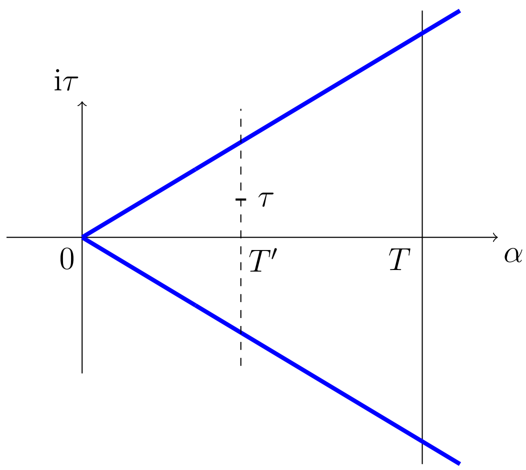

Figure 3. $\Sigma^{(\alpha)}(\nu_0)$.

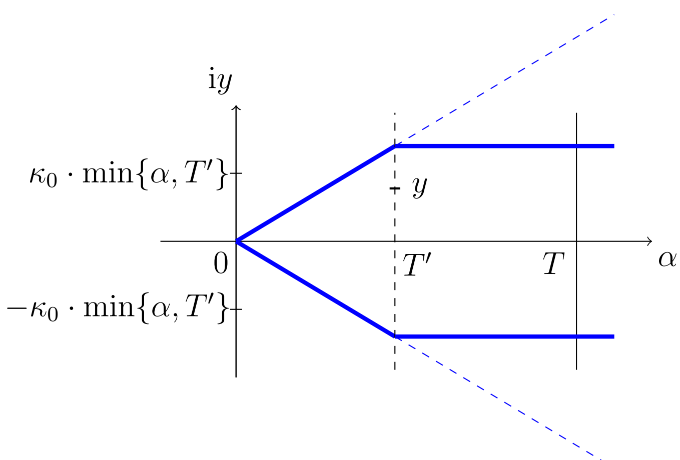

Figure 4. $\Gamma_T^{(T')}(\kappa_0, \nu_0)$.

In order to give a plausible lower estimate on the space-time domain of holomorphy (i.e., the domain of complex analyticity) of a weak solution u to the homogeneous initial value problem (2.7) with $f \equiv 0$, we introduce a few more subsets of $\mathbb{C}^2 \times \mathbb{C}$ (cf. P. Takáč et al. [45, p. 428] or P. Takáč [46, pp. 58–59]):

The two constants $\kappa_0, \nu_0 \in (0,\infty)$ used below will be specified later (in Theorem 4.2); $0 \leq \alpha < \infty$ is an arbitrary number. First, we set

$$
\begin{aligned}
\text{(3.4)} \qquad \mathfrak{Y}^{(\kappa_0\alpha)} &= \mathfrak{X}^{(\kappa_0\alpha)} \times \Delta_{\arctan(\kappa_0\alpha)} \\
&= \big\{ (z,\zeta) = (x + \mathrm{i}y, \xi + \mathrm{i}\eta) \in \mathbb{C}^2 : \\
&\qquad |y| < \kappa_0\alpha \ \text{ and } \ |\arctan(\eta/\xi)| < \kappa_0\alpha,\ \xi > 0 \big\}\,, \\
\text{(3.5)} \qquad \Sigma^{(\alpha)}(\nu_0) &= \{ t = \alpha + \mathrm{i}\tau \in \mathbb{C} :\ \nu_0|\tau| < \alpha \} = \alpha + \mathrm{i}\left(-\nu_0^{-1}\alpha\,, \nu_0^{-1}\alpha \right)
\end{aligned}
$$

(Figure 3), and for $0 < T' \leq T \leq \infty$, we introduce the following complex parabolic domain,

$$
\text{(3.6)} \qquad \Gamma_T^{(T')}(\kappa_0,\nu_0) = \bigcup_{\alpha\in(0,T)} \left[\mathfrak{Y}^{(\kappa_0\cdot\min\{\alpha,T'\})} \times \Sigma^{(\alpha)}(\nu_0) \right] \subset \mathbb{C}^2 \times \mathbb{C}
$$

(Figure 4). Additional properties of this domain will be presented later, in Section 8, eq. (8.1).

In order to get a better picture of the domain $\Gamma_T^{(T')}(\kappa_0,\nu_0) \subset \mathbb{C}^2 \times \mathbb{C}$, it is worth to notice that the mapping $(z,\zeta,t) \longmapsto (z, \log\zeta, \log t)$ maps $\Gamma_T^{(T')}(\kappa_0,\nu_0)$ diffeomorphically onto the set of all complex triples

$$
\begin{aligned}
&(z,\zeta',t') = (x+\mathrm{i}y, \xi'+\mathrm{i}\eta', \alpha'+\mathrm{i}\tau') \equiv (x,\xi',\alpha') + \mathrm{i}(y,\eta',\tau') \in \mathbb{C}^2\times\mathbb{C} \simeq \mathbb{R}^3\times\mathbb{R}^3\,, \\
&\qquad \text{such that} \quad 0 < \alpha = \mathfrak{Re}\, t = \mathrm{e}^{\alpha'}\cdot\cos\tau' < T \ \text{ together with} \\
&|y| < \kappa_0\alpha\,,\ |\eta'| < \arctan(\kappa_0\alpha)\,, \ \text{ and } \ |\tau'| < \arctan(1/\nu_0)\,.
\end{aligned}
$$

In particular, there is no restriction on x and ξ' in the plane $(x,\xi') \in \mathbb{R}^2$, while $\alpha' = \log|t| \in \mathbb{R}$. These claims follow from simple calculations using $\zeta = \mathrm{e}^{\xi'}\cdot\mathrm{e}^{\mathrm{i}\eta'}$ and $t = \mathrm{e}^{\alpha'}\cdot\mathrm{e}^{\mathrm{i}\tau'}$.

4 Main result

Our main result, Theorem 4.2, gives the analyticity (more precisely, a holomorphic extension to a complex domain) of a unique weak solution to the homogeneous initial value problem (2.7) with $f \equiv 0$ in $\mathbb{H} \times (0,T)$. Such a weak solution exists and is unique by the following classical result (Proposition 4.1) that summarizes a pair of standard theorems for abstract parabolic problems due to J.-L. Lions [37, Chapt. IV], Théorème 1.1 (§1, p. 46) and Théorème 2.1 (§2, p. 52). For alternative proofs, see also e.g. L. C. Evans [12, Chapt. 7, §1.2(c)], Theorems 3 and 4, pp. 356–358, J.-L. Lions [38, Chapt. III, §1.2], Theorem 1.2 (p. 102) and remarks thereafter (p. 103), A. Friedman [18], Chapt. 10, Theorem 17, p. 316, or H. Tanabe [47, Chapt. 5, §5.5], Theorem 5.5.1, p. 150.

Proposition 4.1 *Let ρ, σ, θ, q_r, and γ, be given constants in $\mathbb{R}$, $\rho \in (-1,1)$, $\sigma > 0$, $\theta > 0$, and $\gamma > 0$. Assume that $\kappa \in \mathbb{R}$ is sufficiently large, such that both inequalities, (2.26) (Feller's condition) and (2.27) are satisfied. Next, let us choose $\beta \in \mathbb{R}$ such that $1 < \beta \le 2\kappa\theta/\sigma^2$. Set $\mu = (\kappa/\sigma) - \gamma\,|\rho|$ (> 0). Let $0 < T < \infty$, $f \in L^2((0,T) \to V')$, and $u_0 \in H$ be arbitrary. Then the initial value problem (2.7) (with $u_0 \in H$) possesses a unique weak solution*

$$u \in C([0,T] \to H) \cap L^2((0,T) \to V)$$

in the sense of Definition 2.1. *Moreover, this solution satisfies also $u \in W^{1,2}((0,T) \to V')$ and there exists a constant $C \equiv C(T) \in (0,\infty)$, independent from f and u_0, such that*

$$\begin{aligned} \sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 \,\mathrm{d}t + \int_0^T \left\| \tfrac{\partial u}{\partial t}(t) \right\|_{V'}^2 \mathrm{d}t \\ \le C \left(\|u_0\|_H^2 + \int_0^T \|f(t)\|_{V'}^2 \,\mathrm{d}t \right) . \end{aligned} \tag{4.1}$$

Finally, if $u_0 : \mathbb{H} \to \mathbb{R}$ defined by $u_0(x,\xi) = K\,(\mathrm{e}^x - 1)^+$, for $(x,\xi) \in \mathbb{H}$, should belong to H, one needs to take $\gamma > 2$.

The *proof* of this proposition is given towards the end of Section 6. All what we have to do in this proof is to verify the *boundedness* and *coercivity* hypotheses for the sesquilinear form (2.21) in $V \times V$ which are assumed in J.-L. LIONS [37, Chapt. IV, §1], inequalities (1.1) (p. 43) and (1.9) (p. 46), respectively.

Our main result is the following theorem which provides an analytic extension of the weak solution u to the initial value problem (2.7) from the real domain $\mathbb{H} \times [0,T]$ to a complex domain $\Gamma_T^{(T')}(\kappa_0,\nu_0)$ defined in (3.6).

Theorem 4.2 *Let ρ, σ, θ, q_r, and γ, be given constants in $\mathbb{R}$, $\rho \in (-1,1)$, $\sigma > 0$, $\theta > 0$, and $\gamma > 0$. Assume that β, γ, κ, and μ are chosen as specified in* Proposition 4.1 *above. Then the constants $\kappa_0, \nu_0 \in (0,\infty)$ and $T' \in (0,T]$ can be chosen sufficiently small and such that the (unique) weak solution*

$$u \in C([0,T] \to H) \cap L^2((0,T) \to V)$$

of the homogeneous initial value problem (2.7) (with $f \equiv 0$ and $u_0 \in H$) possesses a unique holomorphic extension

$$\tilde{u} : \Gamma_T^{(T')}(\kappa_0,\nu_0) \to \mathbb{C}$$

to the complex domain $\Gamma_T^{(T')}(\kappa_0,\nu_0) \subset \mathbb{C}^3$ with the following properties: There are some constants $C_0, c_0 \in \mathbb{R}_+$ such that

$$\int_0^\infty \int_{-\infty}^{+\infty} |\tilde{u}\,(x + \mathrm{i}y,\, \xi(1 + \mathrm{i}\omega),\, \alpha + \mathrm{i}\tau)|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \le C_0\,\mathrm{e}^{c_0\alpha} \cdot \|u_0\|_H^2 \tag{4.2}$$

for every $\alpha \in (0,T]$ *and for all* $y,\omega,\tau \in \mathbb{R}$ *satisfying*

$$\max\{|y|,\, |\arctan\omega|\} < \kappa_0 \cdot \min\{\alpha,\, T'\} \quad \text{and} \quad \nu_0|\tau| < \alpha\,. \tag{4.3}$$

Consequently, for any $T_0 \in (0,T']$, *the domain* $\Gamma_T^{(T')}(\kappa_0,\nu_0)$ *contains the Cartesian product*

$$\mathfrak{X}^{(\kappa_0 T_0)} \times \Delta_{\kappa_0 T_0} \times \left[(T_0,T) + \mathrm{i}\left(-\tfrac{T_0}{\nu_0},\, \tfrac{T_0}{\nu_0} \right) \right]$$

and the estimate in (4.2) *is valid for every* $\alpha \in [T_0,T]$ *and for all* $y,\omega,\tau \in \mathbb{R}$ *such that, independently from* α,

$$\max\{|y|,\, |\arctan\omega|\} < \kappa_0 T_0 \quad \text{and} \quad \nu_0|\tau| < T_0\,. \tag{4.4}$$

The *proof* of this theorem takes advantage of results from Sections 7 and 8, and Appendix B. It is formally completed at the end of Section 9.

5 An application to Mathematical Finance

This section is concerned with an application of our main result, Theorem 4.2 (Section 4), to S. L. HESTON's *stochastic volatility* model [23] for *European call options* described in Section 2. Our goal will be to provide an affirmative answer to the problem of *market completeness* in Mathematical Finance as described in M. H. A. DAVIS and J. OBŁÓJ [11]. We recall that the model is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, where $\mathbb{P}$ is the risk neutral probability measure. Since an equivalent martingale measure $\mathbb{P}^*$ is *not* unique, the market is *incomplete*. The reader is referred to M. H. A. DAVIS [10], J. C. HULL [26], J. HULL and A. WHITE [27], A. L. LEWIS [36], E. M. STEIN and J. C. STEIN [43], and J. B. WIGGINS [48] for additional important work on this subject. We closely follow the approach in [11, Sect. 3] labeled *"martingale model"* for market completeness. Two more interesting papers on market completeness, written and circulated independently and simultaneously, deserve to be mentioned: J. HUGONNIER, S. MALAMUD, and E. TRUBOWITZ [25] and F. RIEDEL and F. HERZBERG [41]. They are based on the existence of an Arrow-Debreu equilibrium and its implementation as a Radner equilibrium. It is shown or assumed that in this setup, allocation and prices are analytic functions of the state and time variables. The remaining arguments taking advantage of analytic entries in the parabolic problem are similar to ours; cf. [41, §2.3, p. 403].

An extensive account of various stochastic volatility models for European call options and possible market completion by such options is given in P. TAKÁČ [46, Sect. 8, pp. 74–83]. Therefore, we restrict the discussion below to the HESTON model [23, Sect. 1] which seems to be very popular. An important basic feature of this model is the explicit form of its solution [23, pp. 330–331], eqs. (10) – (18). We apply our main analyticity result, Theorem 4.2, to the Heston model. Another frequently used stochastic volatility model is the so-called *"3/2 model"* investigated in S. L. Heston [24], P. Carr and J. Sun [7], A. Itkin and P. Carr [28], and in the monographs by J. Baldeaux and E. Platen [4] and A. L. Lewis [36]. After a suitable transformation of variables, it seems to be possible to treat the 3/2 model by mathematical tools similar to those we use in our present work.

We will answer the question of ***market completeness*** by investigating some qualitative properties (such as analyticity) of the (unique) weak solution

$$u \in C([0,T] \to H) \cap L^2((0,T) \to V)$$

to the initial value problem (2.7) obtained in our Theorem 4.2. Let us recall the Heston operator $\mathcal{A}$ defined in formula (2.8).

The coefficients of the linear operator $\mathcal{A}$ are independent of time t and $x \in \mathbb{R}$, and their dependence on $\xi \in (0,\infty)$ is very simple (linear). As a natural consequence, the domain $\Gamma_T^{(T')}(\kappa_0, \nu_0)$ of the holomorphic extension $\tilde{u}$ of the weak solution u obtained in our Theorem 4.2 is simpler than in the corresponding result obtained in P. Takáč [46, Theorem 3.3, pp. 58–59] for uniformly elliptic operators with variable analytic coefficients.

Remark 5.1 It seems to be likely that one may allow both, the correlation coefficient $\rho \equiv \rho(x,\xi,t)$ and the volatility of volatility $\sigma \equiv \sigma(x,\xi,t)$ to depend on the variables x, ξ, and t, provided this dependence is analytic, with all partial derivatives bounded, and both functions ρ and σ bounded below and above by some positive constants.

Last but not least, we would like to mention that negative values of the correlation coefficient $\rho \in (-1,1)$ are *not* unusual in a volatile market: asset prices tend to decrease when volatility increases ([17, p. 41]).

□

The market completion by a European call option has been obtained in M. H. A. Davis and J. Obłój [11, Proposition 5.1, p. 56] based on the *validity* of a more general analyticity result [11, Theorem 4.1, p. 54]. However, the main hypothesis in this theorem is the *analyticity* of the solution $\overline{p}(x,v,t) = p(x,v,T-t)$ of the parabolic problem (2.5) in the domain $\mathbb{H} \times (0,T)$. (Warning: We use the symbol $\overline{p}$ to denote the function $(x,v,t) \mapsto p(x,v,T-t)$, not the complex conjugate of p.) Of course, the initial condition $h(x) = K\,(\mathrm{e}^x - 1)^+$, $x \in \mathbb{R}$, is *not* analytic. Nevertheless, in our Theorem 4.2 we have established the analyticity result missing in [11] (Theorem 4.1, p. 54). Consequently, all conclusions in [11] on market completion, that are based on the validity of Theorem 4.1 ([11, p. 54]), are valid for the Heston model. In Heston's model with a European call option, the notion of a *complete market* is rigorously defined in [11, Definition 3.1, p. 52] as follows (in probabilistic and measure-theoretic terms): *Every contingent claim can be replicated by a self-financing trading strategy in the stock and bond* (contingent claims can be perfectly hedged against risks). This is the case for Heston's model, by Corollary 4.2 (p. 54) and Proposition 5.1 (p. 56) in [11]. We now briefly sketch how the analyticity of the solution $u(x,\xi,t)$ in $\mathbb{H}\times(0,T)$ facilitates market completion. We keep the notation $u(x,\xi,t)$ for a weak solution to problem (2.7) which is the specific form of problem (2.5) for Heston's model. The relation between the solution $\overline{p}(x,v,t) = p(x,v,T-t)$ of the parabolic problem (2.5) and the weak solution $u(x,\xi,t)$ to the initial value problem (2.7) is obvious, i.e., $\overline{p}(x,v,t) = u(x,\xi,t) = u(x,v/\sigma,t)$, by means of the substitutions $v=\sigma\xi$ with the new independent variable $\xi\in\mathbb{R}_+$ and $\theta_\sigma = \theta/\sigma\in\mathbb{R}$, and by replacing the constants κ and θ, respectively, by $\kappa^* = \kappa+\lambda > 0$ and $\theta^* = \frac{\kappa\theta}{\kappa+\lambda} > 0$. Hence, we may set $r=\lambda=0$ in eq. (2.5). Conversely, let $p:\mathbb{H}\times(0,T)\to\mathbb{R} : (x,v,t)\mapsto p(x,v,t)$ denote the unique solution of the (*terminal* value) Cauchy problem (2.4). We set $u(x,\xi,t) = p(x,\sigma\xi,T-t)$ for all $(x,\xi)\in\mathbb{H}$ and $t\in(0,T)$, so that $u:[0,T]\to H$ is the (unique) weak solution of the initial value problem (2.7) used in Section 4, Theorem 4.2. By the main result of this article, Theorem 4.2, function $u:\mathbb{H}\times(0,T)\to\mathbb{R}$ can be (uniquely) extended to a holomorphic function in the domain $\Gamma_T^{(T')}(\kappa_0,\nu_0)\subset\mathbb{C}^2\times\mathbb{C}$. Consequently, the Jacobian matrix

$$G(x,\xi,t) = \begin{pmatrix} 1\,, & 0 \\ \frac{\partial u}{\partial x}(x,\xi,t)\,, & \frac{\partial u}{\partial \xi}(x,\xi,t) \end{pmatrix}$$

of the mapping $(x,\xi)\mapsto(x,u(x,\xi,t)) : \mathbb{H}\subset\mathbb{R}^2\to\mathbb{R}^2$ possesses determinant $\det G(x,\xi,t) = \frac{\partial u}{\partial\xi}(x,\xi,t)$ with a holomorphic extension to $\Gamma_T^{(T')}(\kappa_0,\nu_0)$. The determinant $\det G$ being (real) analytic in all of $\mathbb{H}\times(0,T)$, its set of zeros is either Lebesgue negligible (i.e., of zero Lebesgue measure) or else it is the whole domain $\mathbb{H}\times(0,T)$ (cf. S. G. KRANTZ and H. R. PARKS [33, p. 83]). Hence, it suffices to examine $\det G$ in an arbitrarily small neighborhood of a single "central" point.

Finally, we can apply Proposition 5.1 (and its proof) from [11, p. 56] to conclude that a European call option in Heston's model (2.1) *completes the market*:

Theorem 5.2 *Assume that* $\kappa > 0$ *is sufficiently large, such that at least the* Feller condition (2.26) *is satisfied; cf.* Proposition 4.1. *Assume that the payoff function* $h(x) = \hat{h}(K\mathrm{e}^x)$ *is* not *affine, that is,* $h''(x) = 0$ *does not hold for every* $x\in\mathbb{R}$. *Then the stochastic volatility model* (2.1) *with a European call option yields a* complete market.

Under quite different sufficient conditions, a related result on market completeness is established in M. ROMANO and N. TOUZI [42, Theorem 3.1, p. 406]: *A single European call option completes the market when there is stochastic volatility driven by one extra Brownian motion* (under some additional assumptions; see [42, pp. 404–407]). The inequality $\det G(x,\xi,t) = \frac{\partial u}{\partial\xi}(x,\xi,t)\neq 0$ (more precisely, $\frac{\partial u}{\partial\xi}(x,\xi,t) > 0$) plays also there a decisive role. An earlier result in P. TAKÁČ [46, Theorem 8.5, p. 82] covers an alternative stochastic volatility model from J.-P. FOUQUE, G. PAPANICOLAOU, and K. R. SIRCAR [17, §2.5, p. 47], eqs. (2.18) – (2.19). The parabolic partial differential operator (i.e., the Îto operator) in this model is uniformly parabolic and, consequently, mathematically entirely different from the degenerate Îto operator in the Heston model. Our main analyticity result, Theorem 4.2 (Section 4), is specialized to cover HESTON's model and, consequently, does not seem to be directly applicable to the stochastic volatility models in [17, 27, 36, 43, 48].

Based on the result in Theorem 5.2 above, combined with those in I. BAJEUX-BESNAINOU and J.-CH. ROCHET [3, p. 12], we suggest the following (alternative) ***analytic*** definition of a *complete market*, at least in the case of Heston's model:

Definition 5.3 There is a set $N \subset \mathbb{H} \times (0,\infty) \subset \mathbb{R}^2 \times \mathbb{R}$ of zero Lebesgue measure such that the mapping $\pi_t : (x,v) \mapsto (x, \overline{p}(x,v,t)) : \mathbb{H} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeomorphism at every point $(x_0, v_0, t) \in [\mathbb{H} \times (0,\infty)] \setminus N$.

Equivalently, for every $t \in (0,\infty)$, the set $N_t = \{(x,v) \in \mathbb{H} : (x,v,t) \in N\} \subset \mathbb{R}^2$ has zero Lebesgue measure and at the point $(x_0, v_0) \in \mathbb{H} \setminus N_t$, the Jacobian matrix

$$J(x_0, v_0, t) = \left(\begin{array}{cc} 1\,, & 0 \\ \frac{\partial \overline{p}}{\partial x}(x,v,t)\,, & \frac{\partial \overline{p}}{\partial v}(x,v,t) \end{array} \right) \Bigg|_{(x,v)=(x_0,v_0)}$$

of the mapping π_t is regular which means that $\det J(x_0,v_0,t) = \frac{\partial \overline{p}}{\partial v}(x,v,t)\big|_{(x,v)=(x_0,v_0)} \neq 0$.

The property $\frac{\partial \overline{p}}{\partial v}(x_0,v_0,t) \neq 0$ allows us to apply the local *implicit function theorem* to conclude that, by fixing (x_0, t), we obtain an open neighborhood $(v_0 - \delta, v_0 + \delta)$ of $v_0 \in (0,\infty)$ $(0 < \delta < \infty$ small enough) such that either $\frac{\partial \overline{p}}{\partial v}(x_0, \cdot\,, t) > 0$ (which is the case in [3, 42]), or else $\frac{\partial \overline{p}}{\partial v}(x_0, \cdot\,, t) < 0$ holds throughout $(v_0 - \delta, v_0 + \delta)$. Hence, the function $\overline{p}(x_0, \cdot\,, t) : (v_0 - \delta, v_0 + \delta) \to \mathbb{R}$ is either strictly monotone increasing or else strictly monotone decreasing. This means that, in a small (open) neighborhood of v_0, one can perfectly hedge against small volatility fluctuations, expressed through the variance $v = (\text{volatility})^2$ satisfying $|v - v_0| < \delta$, by a European call option $\overline{p}(x_0, v, t)$ priced near the value of $\overline{p}(x_0, v_0, t)$. Merely the local *implicit function theorem* has to be envoked.

Remark 5.4 (i) We stress that our Theorem 4.2 (Section 4) allows to consider any payoff function $h \in H$, $h(x,v) \equiv h(x) = \hat{h}(K\mathrm{e}^x)$ for $x \in \mathbb{R}$, in particular. This is a significant advantage over the corresponding result in P. TAKÁČ [46, Theorem 3.3, p. 59] which allows only for a payoff function $h \in L^2(\mathbb{R})$. The hypothesis that the payoff function $h : \mathbb{R} \to \mathbb{R}$ is *not* affine is technical and comes from the proof of Proposition 5.1 in [11, Eq. (5.2), p. 57]. It excludes a solution $u(x,\xi,t)$ with the partial derivative $\frac{\partial u}{\partial x}(x,\xi,t) \equiv \mathrm{const}(\xi,t) \in \mathbb{R}$ independent from $x \in \mathbb{R}$.

(ii) The *Feller condition* (2.26) (cf. [16, 20]) is needed to guarantee the unique solvability and well-posedness of the initial value problem (2.7). This condition was discovered in W. FELLER [16] for the corresponding parabolic problem in the variables $(\xi, t) \in (0, \infty)^2$ only. If this condition is violated, a suitable boundary condition on the behavior of the solution $u(\xi, t)$ needs to be imposed as $\xi \to 0+$. FELLER's result [16] explains why we are able to prove the well-posedness of problem (2.7) with practically *no* boundary boundary conditions as $\xi \to 0+$ or $\xi \to \infty$, except for (2.23) and (2.25) and the requirement that $u(\,\cdot\,, \cdot\,, t) \in H$ together with (2.24) and (2.17) for every $t \in [0, T]$. Notice that the last three conditions are easily satisfied by a regular solution, thanks to $\beta > 1$ and $\gamma > 2$. Our additional *condition on the size* of $\kappa > 0$, i.e., κ large enough, comes from the facts that we have to deal with a solution $u(x, \xi, t)$ depending also on the additional space variable $x \in \mathbb{R}$ and our underlying function space H is the Hilbert space $H = L^2(\mathbb{H}; \mathfrak{w})$ with a special weight $\mathfrak{w}(x, \xi)$.

Remark 5.5 The *"3/2 stochastic volatility model"* [4, 7, 24, 28, 36] mentioned at the beginning of this section requires some major changes in technical details used in our present work, although we believe that similar mathematical tools can still be applied. For instance, the weight function $\mathfrak{w}(x, \xi)$ defined in (2.12) and the sesquilinear form $(\mathcal{A}u, w)_H$ defined in (2.21) will have to be changed significantly.

6 The Heston operator in the real domain

At the end of this section **we prove** Proposition 4.1 by verifying the boundedness and coercivity hypotheses (in §6.1 and §6.2, respectively) for the sesquilinear form (2.21) in $V \times V$ assumed in J.-L. LIONS [37, Chapt. IV, §1], inequalities (1.1) (p. 43) and (1.9) (p. 46), respectively.

Our boundedness and coercivity results for the Heston operator $\mathcal{A} : V \to V'$ make use of five lemmas stated and proved in the Appendix (Appendix A). Recall that $\beta > 0$, $\gamma > 0$, and $\mu > 0$ are constants in the weight $\mathfrak{w}(x, \xi)$ which is defined in eq. (2.12).

6.1 Boundedness of the Heston operator

In this paragraph we verify the boundedness of the sesquilinear form (2.21) in $V \times V$. This property is equivalent to $\mathcal{A}$ being *bounded* as a linear operator from V to V'.

Proposition 6.1 (Boundedness.) *Let β, γ, μ, ρ, σ, θ, q_r, and κ be given constants in $\mathbb{R}$, $\beta > 1$, $\gamma > 0$, $\mu > 0$, $-1 < \rho < 1$, $\sigma > 0$, and $\theta > 0$. Then there exists a constant $C \in (0, \infty)$, such that, for all pairs $u, w \in V$, we have*

$$|(\mathcal{A}u, w)_H| \le C \cdot \|u\|_V \cdot \|w\|_V \,. \tag{6.1}$$

Proof. For any given $u, w \in V$, we apply Cauchy's inequality to the right-hand side of eq. (2.21) to estimate the inner product

$$
\begin{aligned}
&|(\mathcal{A}u, w)_H| \leq \\
&\frac{\sigma}{2}\int_{\mathbb{H}} \left[(|u_x| + 2|\rho|\,|u_\xi|)\cdot|\bar{w}_x| + |u_\xi|\cdot|\bar{w}_\xi|\right]\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&+\frac{1}{2}\int_{\mathbb{H}} \left[(1+\gamma)\sigma\,|u_x| + (|2\kappa-\mu\sigma| + 2\gamma\rho\sigma)\ |u_\xi|\right]\cdot|\bar{w}|\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&+\int_{\mathbb{H}} \left(|q_r|\,|u_x| + \left|\tfrac{1}{2}\beta\sigma - \kappa\theta_\sigma\right|\,|u_\xi|\right)\cdot|\bar{w}|\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,.
\end{aligned}
$$

(We abbreviate $\theta_\sigma \stackrel{\text{def}}{=} \theta/\sigma \in \mathbb{R}$.)

With the abbreviations of the five integrals below,

$$
\begin{aligned}
A_1 &= \int_{\mathbb{H}} (|u_x| + 2|\rho|\,|u_\xi|)^2\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,, \\
B_1 &= \int_{\mathbb{H}} |w_x|^2\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,, \\
A_2 &= \int_{\mathbb{H}} |u_\xi|^2\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,, \quad B_2 = \int_{\mathbb{H}} |w_\xi|^2\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,, \\
J &= \int_{\mathbb{H}} (|u_x| + |u_\xi|)^2\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq 2\int_{\mathbb{H}} (|u_x|^2 + |u_\xi|^2)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,,
\end{aligned}
$$

we thus obtain

$$
\begin{aligned}
&|(\mathcal{A}u, w)_H| \leq \frac{\sigma}{2}\left[(A_1B_1)^{1/2} + (A_2B_2)^{1/2}\right] \\
&+\frac{1}{2}\cdot\max\left\{(1+\gamma)\sigma,\ |2\kappa-\mu\sigma| + 2\gamma\rho\sigma\right\}\cdot J^{1/2}\left(\int_{\mathbb{H}} |w|^2\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\right)^{1/2} \\
&+\ \max\left\{|q_r|,\ \left|\tfrac{1}{2}\beta\sigma - \kappa\theta_\sigma\right|\right\}\cdot J^{1/2}\left(\int_{\mathbb{H}} \frac{|w(x,\xi)|^2}{\xi}\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\right)^{1/2}\,.
\end{aligned}
$$

With the help of these abbreviations and the Cauchy-type elementary inequality

$$
(A_1B_1)^{1/2} + (A_2B_2)^{1/2} \leq (A_1 + A_2)^{1/2}\cdot(B_1 + B_2)^{1/2}\,,
$$

which is equivalent with $\left[(A_1B_2)^{1/2} - (A_2B_1)^{1/2}\right]^2 \geq 0\,,$
the last inequality above yields

$$
\begin{aligned}
|(\mathcal{A}u, w)_H| &\leq \frac{\sigma}{2}\,(A_1 + A_2)^{1/2}\cdot(B_1 + B_2)^{1/2} \\
&+ M_1\left(\int_{\mathbb{H}} \left(|u_x|^2 + |u_\xi|^2\right)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\right)^{1/2} \\
&\quad\times\left[\int_{\mathbb{H}} \left(\left|\frac{w(x,\xi)}{\xi}\right|^2 + |w|^2\right)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\right]^{1/2}\,,
\end{aligned}
$$

with the constant

$$M_1 \stackrel{\mathrm{def}}{=} 2 \cdot \max\left\{ \tfrac{1}{2}(1+\gamma)\sigma,\ \left|\kappa - \tfrac{1}{2}\mu\sigma\right| + \gamma\rho\sigma,\ |q_r|,\ \left|\tfrac{1}{2}\beta\sigma - \kappa\theta_\sigma\right| \right\} > 0\,.$$

With the help of the Cauchy inequality

$$4|\rho|\,|u_x| \cdot |u_\xi| \le 4|u_x|^2 + |\rho|^2\,|u_\xi|^2\,,$$

whence

$$\begin{aligned}(|u_x| + 2|\rho|\,|u_\xi|)^2 + |u_\xi|^2 &= |u_x|^2 + 4|\rho|\,|u_x| \cdot |u_\xi| + (1+4|\rho|^2)\,|u_\xi|^2 \\ &\le 5|u_x|^2 + (1+5\rho^2)|u_\xi|^2 \le 6\left(|u_x|^2 + |u_\xi|^2\right)\,,\end{aligned}$$

by $|\rho| < 1$, this inequality yields

$$A_1 + A_2 \le 6 \int_{\mathbb{H}} \left(|u_x|^2 + |u_\xi|^2\right) \cdot \xi \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi$$

and, consequently, also

$$\begin{aligned}|(\mathcal{A}u, w)_H| \le{}& \left(\int_{\mathbb{H}} \left(|u_x|^2 + |u_\xi|^2\right) \cdot \xi \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \right)^{1/2} \\ &\times \left\{ \frac{\sigma}{2}\sqrt{6} \left(\int_{\mathbb{H}} \left(|w_x|^2 + |w_\xi|^2\right) \cdot \xi \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \right)^{1/2} \right. \\ &\left. + M_1 \left[\int_{\mathbb{H}} \left(\left| \frac{w(x,\xi)}{\xi} \right|^2 + |w|^2 \right) \cdot \xi \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \right]^{1/2} \right\}.\end{aligned}$$

Applying the Sobolev and Hardy inequalities (A.11) and (A.16) to this estimate we deduce that there exists a constant $C \in (0,\infty)$, such that the estimate in (6.1) holds for all pairs $u, w \in V$. Here, we recall that, by Remark A.6, the norm $\|w\|_V^\sharp$ defined in the Hilbert space V by eq. (A.20) is equivalent with the original norm $\|w\|_V$ defined by eq. (2.14).

Proposition 6.1 is proved. ■

6.2 Coercivity in the real domain

We wish to investigate the Heston operator $\mathcal{A}$ as a densely defined, closed linear operator in the weighted Lebesgue space $H = L^2(\mathbb{H}; \mathfrak{w})$.

We investigate the *coercivity* of the linear operator $\mathcal{A}$ in $V = H^1(\mathbb{H}; \mathfrak{w})$. In fact, we will show that the coercivity property holds for $\mathcal{A} + \frac{1}{2}c_2'\,I$ in place of $\mathcal{A}$, where $c_2' > 0$ is a suitable constant (large enough) specified at the end of this paragraph. As a trivial consequence, the linear operator $-\left(\mathcal{A} + \frac{1}{2}c_2'\,I\right)$ is *dissipative* in H. For establishing the coercivity, hypotheses (2.26) and (2.27) described in Remark 2.2 are crucial.

We use the sesquilinear form from eq. (2.21) to verify the coercivity of the linear operator $\mathcal{A}$ in the Hilbert space V:

$$
\begin{aligned}
& 2\cdot\mathfrak{Re}(\mathcal{A}u,u)_H = J_1+J_2+\cdots+J_5 \equiv \\
& \sigma\int_{\mathbb{H}}\left[u_x\cdot\bar{u}_x+\rho\,(u_\xi\cdot\bar{u}_x+u_x\cdot\bar{u}_\xi)+u_\xi\cdot\bar{u}_\xi\right]\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
& +\frac{\sigma}{2}\int_{\mathbb{H}}(1-\gamma\ \operatorname{sign}x)\,(u_x\cdot\bar{u}+\bar{u}_x\cdot u)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
& +\int_{\mathbb{H}}\left(\kappa-\gamma\rho\sigma\ \operatorname{sign}x-\tfrac{1}{2}\mu\sigma\right)(u_\xi\cdot\bar{u}+\bar{u}_\xi\cdot u)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
& +q_r\int_{\mathbb{H}}(u_x\cdot\bar{u}+\bar{u}_x\cdot u)\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
& +\left(\tfrac{1}{2}\beta\sigma-\kappa\theta_\sigma\right)\int_{\mathbb{H}}(u_\xi\cdot\bar{u}+\bar{u}_\xi\cdot u)\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,.
\end{aligned}
\tag{6.2}
$$

All integrals on the right-hand side converge absolutely for any $u\in V$, by the proof of Proposition 6.1 above.

Proposition 6.2 (Coercivity.) *Let ρ, σ, θ, q_r, and γ be given constants in $\mathbb{R}$, $\rho\in(-1,1)$, $\sigma>0$, $\theta>0$, and $\gamma>0$. Assume that β, γ, κ, and μ are chosen as specified in* Proposition 4.1. *Then there exists a constant $c_2'\in(0,\infty)$ such that the following* Gårding inequality

$$
2\cdot\mathfrak{Re}(\mathcal{A}u,u)_H \ge \sigma\,(1-|\rho|)\cdot\|u\|_V^2 - c_2'\cdot\|u\|_H^2 \tag{6.3}
$$

is valid for all $u\in V$.

Proof. Let us consider eq. (6.2) with an arbitrary $u\in V$. The first integral on the right-hand side of eq. (6.2) is estimated from below by Cauchy's inequality

$$
u_\xi\cdot\bar{u}_x+u_x\cdot\bar{u}_\xi = 2\cdot\mathfrak{Re}(u_\xi\cdot\bar{u}_x)\le 2|u_\xi|\cdot|\bar{u}_x|\le |u_x|^2+|u_\xi|^2,
$$

$$
\begin{aligned}
\frac{J_1}{\sigma} &\equiv \int_{\mathbb{H}}\left[u_x\cdot\bar{u}_x+\rho\,(u_\xi\cdot\bar{u}_x+u_x\cdot\bar{u}_\xi)+u_\xi\cdot\bar{u}_\xi\right]\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&\ge \int_{\mathbb{H}}\left[|u_x|^2-|\rho|\,(|u_x|^2+|u_\xi|^2)+|u_\xi|^2\right]\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (1-|\rho|)\int_{\mathbb{H}}(|u_x|^2+|u_\xi|^2)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= (1-|\rho|)\left(\|u\|_V^2-\|u\|_H^2\right)\,.
\end{aligned}
\tag{6.4}
$$

The second integral in eq. (6.2), J_2, consists of two different parts that we treat by integration-by-parts as follows, using the following simple formulas,

$$
\begin{aligned}
&\frac{\partial}{\partial x}\,\mathfrak{w}(x,\xi) = \,-\,\gamma\,\xi^{\beta-1}\,\mathrm{e}^{-\gamma|x|-\mu\xi}\cdot\operatorname{sign}x = \,-\,\gamma\cdot\operatorname{sign}x\cdot\mathfrak{w}(x,\xi)\,,\\
&\frac{\partial}{\partial \xi}\,\mathfrak{w}(x,\xi) = (\beta-1)\,\xi^{\beta-2}\,\mathrm{e}^{-\gamma|x|-\mu\xi} - \mu\,\xi^{\beta-1}\,\mathrm{e}^{-\gamma|x|-\mu\xi}\\
&= (\beta-1-\mu\xi)\,\xi^{\beta-2}\,\mathrm{e}^{-\gamma|x|-\mu\xi} = \left(\frac{\beta-1}{\xi}-\mu\right)\cdot\mathfrak{w}(x,\xi)\,,\\
&\frac{\partial}{\partial \xi}\,(\xi\cdot\mathfrak{w}(x,\xi)) = \frac{\partial}{\partial \xi}\left(\xi^{\beta}\,\mathrm{e}^{-\gamma|x|-\mu\xi}\right)\\
&= \beta\cdot\xi^{\beta-1}\,\mathrm{e}^{-\gamma|x|-\mu\xi} - \mu\,\xi^{\beta}\,\mathrm{e}^{-\gamma|x|-\mu\xi} = (\beta-\mu\xi)\cdot\mathfrak{w}(x,\xi)\,.
\end{aligned}
$$

Consequently, the first part of the integral in $2J_2/\sigma$ in eq. (6.2), becomes

$$
\begin{aligned}
&\int_{\mathbb{R}}(u_x\,\bar{u}+\bar{u}_x\,u)\cdot\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x = \int_{\mathbb{R}}(|u|^2)_x\cdot\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x\\
&= |u(x,\xi)|^2\cdot\mathrm{e}^{-\gamma|x|}\Big|_{x=-\infty}^{x=+\infty} + \gamma\int_{\mathbb{R}}|u(x,\xi)|^2\cdot\operatorname{sign}x\cdot\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x\\
&= \gamma\int_{\mathbb{R}}|u(x,\xi)|^2\cdot\operatorname{sign}x\cdot\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x
\end{aligned}
$$

for almost every $\xi\in(0,\infty)$, with a help from Lemma A.3. Integrating this equality with respect to $\xi\in(0,\infty)$ and the measure $\xi^{\beta}\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi$, we arrive at

$$
\begin{aligned}
&\int_{\mathbb{H}}(u_x\,\bar{u}+\bar{u}_x\,u)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\\
&= \gamma\int_{\mathbb{H}}|u(x,\xi)|^2\cdot\operatorname{sign}x\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,.
\end{aligned}
\tag{6.5}
$$

Recall that $\mathfrak{w}(x,\xi)=\xi^{\beta-1}\,\mathrm{e}^{-\gamma|x|-\mu\xi}$. Similarly, we get

$$
\begin{aligned}
&\int_{\mathbb{R}}(u_x\,\bar{u}+\bar{u}_x\,u)\cdot\operatorname{sign}x\cdot\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x\\
&= \,-\int_{-\infty}^{0}(u_x\,\bar{u}+u\,\bar{u}_x)\,\mathrm{e}^{\gamma x}\,\mathrm{d}x + \int_{0}^{\infty}(u_x\,\bar{u}+u\,\bar{u}_x)\,\mathrm{e}^{-\gamma x}\,\mathrm{d}x\\
&= \,-\int_{-\infty}^{0}(|u|^2)_x\cdot\mathrm{e}^{\gamma x}\,\mathrm{d}x + \int_{0}^{\infty}(|u|^2)_x\cdot\mathrm{e}^{-\gamma x}\,\mathrm{d}x\\
&= \,-\,|u(x,\xi)|^2\,\mathrm{e}^{\gamma x}\Big|_{-\infty}^{0} + \gamma\int_{-\infty}^{0}|u(x,\xi)|^2\,\mathrm{e}^{\gamma x}\,\mathrm{d}x\\
&\qquad + |u(x,\xi)|^2\,\mathrm{e}^{-\gamma x}\Big|_{0}^{\infty} + \gamma\int_{0}^{\infty}|u(x,\xi)|^2\,\mathrm{e}^{-\gamma x}\,\mathrm{d}x\\
&= \,-\,2|u(0,\xi)|^2 + \gamma\int_{-\infty}^{\infty}|u(x,\xi)|^2\,\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x\,.
\end{aligned}
$$

Integrating this equality with respect to $\xi \in (0, \infty)$ and the measure $\xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi$, we arrive at

$$
\begin{aligned}
& \int_{\mathbb{H}} (u_x \, \bar{u} + u \, \bar{u}_x) \cdot \operatorname{sign} x \cdot \xi \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \\
& = \; - 2 \int_0^\infty |u(0,\xi)|^2 \, \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi + \gamma \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \,.
\end{aligned} \tag{6.6}
$$

Finally, we combine the identities in (6.5) and (6.6) to obtain

$$
\begin{aligned}
& \frac{2J_2}{\sigma} \equiv \int_{\mathbb{H}} (1 - \gamma \; \operatorname{sign} x) \, (u_x \cdot \bar{u} + \bar{u}_x \cdot u) \cdot \xi \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \\
& = 2\gamma \int_0^\infty |u(0,\xi)|^2 \, \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi - \gamma^2 \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \\
& + \gamma \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \operatorname{sign} x \cdot \xi \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \,.
\end{aligned} \tag{6.7}
$$

In order to treat the third integral in eq. (6.2), we need to calculate

$$
\begin{aligned}
& \int_0^\infty (u_\xi \cdot \bar{u} + \bar{u}_\xi \cdot u) \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi = \int_0^\infty (|u|^2)_\xi \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \\
& = |u(x,\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \Big|_{\xi=0}^{\xi=\infty} - \int_0^\infty |u(x,\xi)|^2 \cdot (\beta - \mu\xi) \, \xi^{\beta-1} \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \,.
\end{aligned}
$$

Integrating first this equality with respect to $x \in (-\infty, \infty)$ and the measure $\mathrm{e}^{-\gamma|x|} \, \mathrm{d}x$, then applying the vanishing trace results (2.15) and (2.16), we arrive at

$$
\begin{aligned}
J_3 & \equiv \int_{\mathbb{H}} \left(\kappa - \gamma\rho\sigma \; \operatorname{sign} x - \tfrac{1}{2}\mu\sigma \right) (u_\xi \cdot \bar{u} + \bar{u}_\xi \cdot u) \cdot \xi \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \\
& = \; - \left(\kappa - \tfrac{1}{2}\mu\sigma \right) \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot (\beta - \mu\xi) \, \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \\
& + \gamma\rho\sigma \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \operatorname{sign} x \cdot (\beta - \mu\xi) \, \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \,.
\end{aligned} \tag{6.8}
$$

The fourth integral in eq. (6.2) is treated analogously to the second one. It suffices to replace β by $\beta - 1$ in the equality (6.5) which then yields

$$
\begin{aligned}
\frac{J_4}{q_r} & \equiv \int_{\mathbb{H}} (u_x \, \bar{u} + \bar{u}_x \, u) \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \\
& = \gamma \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \operatorname{sign} x \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \,.
\end{aligned} \tag{6.9}
$$

Finally, the last integral in eq. (6.2) is treated analogously to the third one,

$$
\begin{aligned}
\frac{J_5}{\frac{1}{2}\beta\sigma - \kappa\theta_\sigma} & \equiv \int_{\mathbb{H}} (u_\xi \cdot \bar{u} + \bar{u}_\xi \cdot u) \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \\
& = \; - \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \left(\frac{\beta - 1}{\xi} - \mu \right) \cdot \mathfrak{w}(x,\xi) \, \mathrm{d}x \, \mathrm{d}\xi \,.
\end{aligned} \tag{6.10}
$$

We collect the second through fifth integrals, cf. eq. (6.2),

$$
\begin{aligned}
J_2 + \ldots J_5 &= \gamma\sigma \int_0^\infty |u(0,\xi)|^2\, \xi^\beta\, \mathrm{e}^{-\mu\xi}\, \mathrm{d}\xi \\
&+ \left[-\tfrac{1}{2}\sigma\gamma^2 + \mu\left(\kappa - \tfrac{1}{2}\mu\sigma\right)\right] \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&+ \left[\tfrac{1}{2}\sigma\gamma - \mu\gamma\rho\sigma\right] \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \operatorname{sign} x \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&+ \left[-\beta\left(\kappa - \tfrac{1}{2}\mu\sigma\right) + \mu\left(\tfrac{1}{2}\beta\sigma - \kappa\theta_\sigma\right)\right] \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&+ \left[\beta\gamma\rho\sigma + \gamma q_r\right] \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \operatorname{sign} x \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
&- (\beta - 1)\left(\tfrac{1}{2}\beta\sigma - \kappa\theta_\sigma\right) \int_{\mathbb{H}} \frac{|u(x,\xi)|^2}{\xi} \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \,,
\end{aligned}
$$

whence

$$
\begin{aligned}
& J_2 + \ldots J_5 \geq \\
& \left\{ \left[\mu\kappa - \tfrac{1}{2}\sigma(\gamma^2 + \mu^2)\right] - \sigma\gamma \left|\tfrac{1}{2} - \mu\rho\right| \right\} \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
& + \left\{ \left[\beta\mu\sigma - \kappa(\beta + \mu\theta_\sigma)\right] - \gamma\left|\beta\rho\sigma + q_r\right| \right\} \|u\|_H^2 \\
& + (\beta - 1)\left(\kappa\theta_\sigma - \tfrac{1}{2}\beta\sigma\right) \int_{\mathbb{H}} \frac{|u(x,\xi)|^2}{\xi} \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \\
& \equiv c_1 \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi + c_2 \cdot \|u\|_H^2 \\
& + c_3 \int_{\mathbb{H}} \frac{|u(x,\xi)|^2}{\xi} \cdot \mathfrak{w}(x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi \,,
\end{aligned}
\tag{6.11}
$$

where the constants

$$
\begin{aligned}
c_1 &\stackrel{\text{def}}{=} \left[\mu\kappa - \tfrac{1}{2}\sigma(\gamma^2 + \mu^2)\right] - \sigma\gamma\left|\tfrac{1}{2} - \mu\rho\right| \,, \\
c_2 &\stackrel{\text{def}}{=} \left[\beta\mu\sigma - \kappa(\beta + \mu\theta_\sigma)\right] - \gamma\left|\beta\rho\sigma + q_r\right| \,, \\
c_3 &\stackrel{\text{def}}{=} (\beta - 1)\left(\kappa\theta_\sigma - \tfrac{1}{2}\beta\sigma\right) \,,
\end{aligned}
$$

are estimated from below as follows:

$$
c_1 \geq c_1' \stackrel{\text{def}}{=} \mu\kappa - \tfrac{1}{2}\sigma(\gamma^2 + \mu^2) - \sigma\gamma\left(\tfrac{1}{2} + \mu\,|\rho|\right) \,, \tag{6.12}
$$

$$
c_2 > -\infty \,, \tag{6.13}
$$

$$
c_3 = \frac{\beta - 1}{\sigma}\left(\kappa\theta - \tfrac{1}{2}\beta\sigma^2\right) \geq 0 \,. \tag{6.14}
$$

The constant $c_3 \in \mathbb{R}$ is nonnegative thanks to *Feller's condition*, $\frac{1}{2}\sigma^2 - \kappa\theta < 0$, provided we choose $\beta \in \mathbb{R}$ such that $1 < \beta \leq 2\kappa\theta/\sigma^2$. The sign of the constant c_2 does not matter as it stands as a coefficient with the norm $\|u\|_H$. Finally, in order to guarantee $c_1' \geq 0$, we first choose $\mu > 0$ such that this value of μ maximizes the function

$$\begin{aligned}\mu \mapsto c_1' \equiv c_1'(\mu) &= \mu\kappa - \tfrac{1}{2}\sigma(\gamma^2 + \mu^2) - \sigma\gamma\left(\tfrac{1}{2} + \mu\,|\rho|\right) \\ &= \tfrac{1}{2}\sigma\left[-\left(\mu - \frac{\kappa}{\sigma} + \gamma\,|\rho|\right)^2 + \left(\frac{\kappa}{\sigma} - \gamma\,|\rho|\right)^2 - \gamma(1+\gamma)\right],\end{aligned}$$

that is, $\mu = (\kappa/\sigma) - \gamma\,|\rho|$, provided $\kappa > \sigma\gamma|\rho|$. With this value of μ, we have to satisfy

$$c_1' = \tfrac{1}{2}\sigma\left[\left(\frac{\kappa}{\sigma} - \gamma\,|\rho|\right)^2 - \gamma(1+\gamma)\right] \geq 0\,,$$

that is, ineq. (2.27).

Finally, applying inequalities (6.12), (6.13), and (6.14) to the right-hand side of eq. (6.11), and inequality (6.4) to eq. (6.2), we obtain

$$\begin{aligned}&2\cdot \mathfrak{Re}(\mathcal{A}u,u)_H \geq \sigma\,(1-|\rho|)\left(\|u\|_V^2 - \|u\|_H^2\right) \\ &+ c_1' \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi + c_2\,\|u\|_H^2 \\ &+ c_3 \int_{\mathbb{H}} \frac{|u(x,\xi)|^2}{\xi} \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \geq \sigma\,(1-|\rho|)\,\|u\|_V^2 - c_2'\,\|u\|_H^2\,,\end{aligned} \tag{6.15}$$

where $c_2' = \sigma\,(1-|\rho|) + |c_2| > 0$ is a constant.

Consequently, the linear operator $\mathcal{A} + \frac{1}{2}c_2'\,I$ is *coercive* in V and $-\left(\mathcal{A} + \frac{1}{2}c_2'\,I\right)$ is *dissipative* in H. More precisely, ineq. (6.15), when combined with our definitions of equivalent norms in $V = H^1(\mathbb{H};\mathfrak{w})$, yields the *Gårding inequality* in (6.3).

The proof of Proposition 6.2 is complete. ■

Remark 6.3 (Feller's condition.) *Feller's condition* $\frac{1}{2}\sigma^2 - \kappa\theta < 0$ and our choice of $\beta \in \mathbb{R}$ such that $1 < \beta \leq 2\kappa\theta/\sigma^2$ guarantee $c_3 \geq 0$ in the proof of Proposition 6.2 above. In addition, to guarantee also

$$c_1' = \tfrac{1}{2}\sigma\left[\left(\frac{\kappa}{\sigma} - \gamma\,|\rho|\right)^2 - \gamma(1+\gamma)\right] \geq 0\,,$$

we need to assume ineq. (2.27). □

Proof of **Proposition 4.1.** In Propositions 6.1 and 6.2 above we have verified the boundedness and coercivity hypotheses for the linear operator $\mathcal{A} : V \to V'$ required in J.-L. LIONS [37, Chapt. IV], Théorème 1.1 (§1, p. 46) and Théorème 2.1 (§2, p. 52). Consequently, these well-known results from [37, Chapt. IV] yield the desired conclusion of Proposition 4.1 on the existence and uniqueness of a weak solution to the initial value problem (2.7). Finally, the energy estimate (4.1) can be found in L. C. EVANS [12, Chapt. 7, §1.2(b)], Theorem 2, p. 354. ■

7 The Heston operator in the complex domain

In the first paragraph of this section, §7.1, we apply the classical theory of *sectorial operators* as *infinitesimal generators* of *holomorphic semigroups* of bounded linear operators in the complex Hilbert space $H = L^2(\mathbb{H};\mathfrak{w})$. This theory provides a (unique) holomorphic extension of the unique weak solution $u : \mathbb{H} \times [0,T] \to \mathbb{R}$ of the initial value problem (2.7) with $f \equiv 0$, obtained in Proposition 4.1, to the complex domain $\mathbb{H} \times \Delta_{\vartheta'}$ that is holomorphic in the time variable $t \in \Delta_{\vartheta'}$. To obtain a holomorphic extension of u to the complex domain $\mathfrak{V}^{(r)} = \mathfrak{X}^{(r)} \times \Delta_{\arctan r} \subset \mathbb{C}^2$ in the space variables (x,ξ), that has been defined in eq. (3.3) for $r \in (0,\infty)$, we first replace the (possibly nonsmooth) initial data $u_0 \in H$ by an entire function $u_{0,n} : \mathbb{C}^2 \to \mathbb{C}$; $n = 1,2,3,\dots$, constructed in §7.2, such that $u_{0,n}|_{\mathbb{H}} \in H$, ineq. (7.6) is valid, and the sequence $\|u_{0,n}|_{\mathbb{H}} - u_0\|_H \to 0$ as $n \to \infty$. Given such initial data $u_0|_{\mathbb{H}} \in H$, where $u_0 : \mathbb{C}^2 \to \mathbb{C}$ is an entire function satisfying ineq. (7.6), the main result of the entire section, Proposition 7.1 proved in §7.2, provides a (unique) holomorphic extension of the solution u to the complex domain $\mathfrak{X}^{(r)} \times \Delta_{\arctan r} \times \Delta_{\vartheta'} \subset \mathbb{C}^3$; hence, in all its variables (x,ξ,t), provided the initial values (at $t = 0$) are holomorphic in the complex domain $\mathfrak{V}^{(r)} = \mathfrak{X}^{(r)} \times \Delta_{\arctan r} \subset \mathbb{C}^2$. The case of general initial data $u_0 \in H$ will be postponed until Section 9 where we let the analytic initial data $u_{0,n}|_{\mathbb{H}}$ converge to arbitrary initial data u_0 in H as $n \to \infty$. Finally, the convergence of the (unique) holomorphic extensions to a smaller domain

$$\Gamma_T^{(T')}(\kappa_0,\nu_0) \subset \mathfrak{V}^{(r)} \times \Delta_{\vartheta'}$$

of the corresponding weak solutions $u_n : \mathbb{H} \times [0,T] \to \mathbb{R}$ of the initial value problem (2.7) with $f \equiv 0$ and the initial data $u_{0,n}|_{\mathbb{H}} \in H$, obtained in Proposition 4.1, to a holomorphic function $u : \Gamma_T^{(T')}(\kappa_0,\nu_0) : \mathbb{C}$ will be established in the next section (Section 8). This argument will help us to complete the proof of our main result (Theorem 4.2).

Next, we define a few function spaces for functions on $\mathfrak{V}^{(r)} \subset \mathbb{C}^2$. We denote by $\mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})$ the Banach space of all complex-valued, Lebesgue measurable functions $u : \mathfrak{V}^{(r)} \to \mathbb{C}$, such that, for each pair $y,\omega \in \mathbb{R}$ with $|y| < r$ and $|\omega| < r$, the following integral converges,

$$\int_0^\infty \int_{-\infty}^{+\infty} |u(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega))|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi < \infty\,, \tag{7.1}$$

and the norm

$$\|u\|_{\mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})} \stackrel{\mathrm{def}}{=} \operatorname*{ess\,sup}_{|y|<r,\ |\omega|<r} \left(\int_0^\infty \int_{-\infty}^{+\infty} |u(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega))|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \right)^{1/2} < \infty\,. \tag{7.2}$$

It is well-known that $\mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})$ is a vector space and $\|\cdot\|_{\mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})}$ defines a norm on it; cf. P. Takáč [46, Sect. 5]. It is easy to verify that $\mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})$ is a Banach space. We denote by $\mathcal{H}^2(\mathfrak{V}^{(r)})$ the Hardy space of all holomorphic functions $u : \mathfrak{V}^{(r)} \to \mathbb{C}$ such that $u \in \mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})$. It is well-known that $\mathcal{H}^2(\mathfrak{V}^{(r)})$ is a closed vector subspace of $\mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})$. We refer to E. M. STEIN and G. WEISS [44, Chapt. III] for basic theory of Hardy spaces; the most relevant results about $H^2(\mathfrak{V}^{(r)})$ can be found in [44, Chapt. III], §2, pp. 91–101, and §6.12, pp. 127–128.

The problem of analyticity (holomorphic extension) of a weak solution to the homogeneous Cauchy problem (2.7) (with $f \equiv 0$) can be split into two parts, *analyticity in time* and *analyticity in space*; see §7.1 and §7.2 below, respectively. Since the partial differential operator $\mathcal{A} : V \to V'$ in eq. (2.7) is independent from time t, analyticity in the time variable t follows from the well-known theory of analytic C^0-semigroups as described below.

7.1 Analyticity in the complex time variable t

Our results from the previous section (Section 6) on the boundedness and coercivity of the linear operator $\mathcal{A} : V \to V'$ in eq. (2.7) show that $\mathcal{A}$ is a *sectorial operator* in the complex Hilbert space H. More precisely, the linear operator $-\left(\mathcal{A} + \frac{1}{2}c_2'\, I\right)$ in H possesses a bounded inverse, by the Lax-Milgram theorem, and ineq. (6.3) implies that there are constants $\vartheta \in (0, \pi/2)$ and $M_\vartheta \in (0, \infty)$, such that

$$\text{(7.3)} \qquad \| \left(\lambda I + \tfrac{1}{2}c_2' + \mathcal{A}\right)^{-1} \|_{\mathcal{L}(H \to H)} \leq M_\vartheta / |\lambda| \\ \text{holds for all } \lambda = \varrho \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \varrho > 0 \text{ and } \theta \in \left(-\tfrac{1}{2}\pi - \vartheta,\, \tfrac{1}{2}\pi + \vartheta\right) .$$

Consequently, $-\left(\mathcal{A} + \frac{1}{2}c_2'\, I\right)$ is the *infinitesimal generator* of a *holomorphic semigroup* of uniformly bounded linear operators $\left\{\mathrm{e}^{-c_2' t/2}\, \mathrm{e}^{-t\mathcal{A}} : t \in \mathbb{R}_+\right\}$ in H, i.e.,

$$\text{(7.4)} \qquad \|\mathrm{e}^{-t\mathcal{A}}\|_{\mathcal{L}(H \to H)} \leq M'_{\vartheta'}\, \mathrm{e}^{(c_2'/2) \cdot \Re\mathfrak{e}\, t} \quad \text{holds for all } t \in \Delta_{\vartheta'} ,$$

where $\vartheta' \in (0, \vartheta)$ is arbitrary and $M'_{\vartheta'} \in (0, \infty)$ is a suitable constant depending on ϑ'; see, e.g., Theorem 5.7.2 in H. TANABE [47], §5.7, p. 161, combined with [47, Theorem 5.7.6], §5.7.4, p. 179. This means that the strongly continuous mapping $t \mapsto \mathrm{e}^{-c_2' t/2}\, \mathrm{e}^{-t\mathcal{A}}$ of $\mathbb{R}_+$ into the Banach algebra of all bounded linear operators on H (endowed with the operator norm $\|\cdot\|_{\mathcal{L}(H \to H)}$) can be extended uniquely to a holomorphic mapping in a *complex angle* $\Delta_{\vartheta'}$ of angular width $2\vartheta'$, defined in (3.2), $\vartheta' \in (0, \pi/2)$ small enough, $0 < \vartheta' < \vartheta < \pi/2$.

Hence, the unique weak solution $u : \mathbb{H} \times [0, T] \to \mathbb{R}$ of the initial value problem (2.7) with $f \equiv 0$, obtained in Proposition 4.1, extends uniquely to the complex domain $\mathbb{H} \times \Delta_{\vartheta'}$ and is holomorphic in the time variable $t \in \Delta_{\vartheta'}$. Furthermore, by ineq. (7.4) above, the following estimate holds for any initial condition $u_0 \in H$,

$$\text{(7.5)} \qquad \|u(\,\cdot\,, \,\cdot\,, t)\|_H = \|\mathrm{e}^{-t\mathcal{A}} u_0\|_H \leq M'_{\vartheta'}\, \mathrm{e}^{(c_2'/2) \cdot \Re\mathfrak{e}\, t}\, \|u_0\|_H \quad \text{for all } t \in \Delta_{\vartheta'} .$$

7.2 The Cauchy problem in the complex domain

Given an initial condition $u_0 \in H$, in the Appendix (Appendix B) there is a sequence of entire functions $u_{0,n} : \mathbb{C}^2 \to \mathbb{C}$; $n = 1, 2, 3, \dots$, with $u_{0,n}|_{\mathbb{H}} \in H$, constructed such that

$$\|u_{0,n}|_{\mathbb{H}} - u_0\|_H \longrightarrow 0 \quad \text{as } n \to \infty .$$

An important property of each function $u_{0,n} : \mathbb{C}^2 \to \mathbb{C}$ is the following decay inequality: Given any numbers $r \in (0, \infty)$ and $\vartheta \in (0, \pi/2)$, for each $n = 1, 2, 3, \dots$, there exists a constant $A_n \equiv A_n(r, \vartheta) \in (0, \infty)$ such that

$$|u_{0,n}(x + \mathrm{i}y, \xi + \mathrm{i}\eta)| \le A_n\, \mathrm{e}^{-(x^2+\xi)/4} \quad \text{whenever } z = x + \mathrm{i}y \in \mathfrak{X}^{(r)} \text{ and } \zeta = \xi + \mathrm{i}\eta \in \Delta_\vartheta , \tag{7.6}$$

where the right-hand side is in $H = L^2(\mathbb{H}; \mathfrak{w})$.

To begin with, let us fix an arbitrary index $n \in \mathbb{N}$; $\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$, for which we abbreviate $u_0 \equiv u_{0,n}$ with $u_0|_{\mathbb{H}} \in H$. Hence, throughout this paragraph we assume that either $u_0 : \mathbb{C}^2 \to \mathbb{C}$ is an entire function or at least $u_0 : \mathfrak{X}^{(r)} \times \Delta_\vartheta \to \mathbb{C}$ is a holomorphic function that satisfies an analogue of (7.6) with a constant $A_0 \equiv A_0(r, \vartheta) \in (0, \infty)$:

$$|u_0(x + \mathrm{i}y, \xi + \mathrm{i}\eta)| \le A_0\, \mathrm{e}^{-(x^2+\xi)/4} \quad \text{whenever } z = x + \mathrm{i}y \in \mathfrak{X}^{(r)} \text{ and } \zeta = \xi + \mathrm{i}\eta \in \Delta_\vartheta . \tag{7.7}$$

To simplify our hypotheses and notation, we take $r \in (0, \infty)$ arbitrary and $\vartheta = \arctan r \in (0, \pi/2)$, so that $\mathfrak{X}^{(r)} \times \Delta_\vartheta = \mathfrak{V}^{(r)} \subset \mathbb{C}^2$ is the complex domain $\mathfrak{V}^{(r)} = \mathfrak{X}^{(r)} \times \Delta_{\arctan r} \subset \mathbb{C}^2$ that has been defined in eq. (3.3). The general case of $u_0 \in H$ will be treated in the next section (Section 8).

We formulate the corresponding analyticity result for such an initial condition u_0 as the following special case of Theorem 4.2:

Proposition 7.1 *Let ρ, σ, θ, q_r, and γ be given constants in $\mathbb{R}$, $\rho \in (-1, 1)$, $\sigma > 0$, $\theta > 0$, and $\gamma > 0$. Assume that β, γ, κ, and μ are chosen as specified in* Proposition 4.1. *Finally, let us assume that $u_0 : \mathfrak{V}^{(r)} \to \mathbb{C}$ is a holomorphic function that satisfies a bound similar to (7.7),*

$$|u_0(x + \mathrm{i}y, \xi + \mathrm{i}\eta)| \le A_0\, \mathrm{e}^{-(x^2+\xi)/4} \quad \textit{whenever } z = x + \mathrm{i}y \in \mathfrak{X}^{(r)} \textit{ and } \zeta = \xi + \mathrm{i}\eta \in \Delta_{\arctan r} , \tag{7.8}$$

where $r \in (0, \infty)$ is some number and $A_0 \equiv A_0(r) \in (0, \infty)$ is a constant.

Then the (unique) weak solution

$$u \in C([0, T] \to H) \cap L^2((0, T) \to V)$$

of the homogeneous initial value problem (2.7) *(with $f \equiv 0$ and this u_0) possesses a unique holomorphic extension $\tilde{u} : \mathfrak{Y}^{(r')} \times \Delta_{\vartheta'} \to \mathbb{C}$ to the complex domain $\mathfrak{Y}^{(r')} \times \Delta_{\vartheta'} \subset \mathbb{C}^3$, where $r' \in (0, r]$ and $\vartheta' \in (0, \pi/2)$ are some constants. Furthermore, there are additional constants $C_0, c_0 \in \mathbb{R}_+$ such that*

$$\begin{aligned}(7.9)\qquad &\int_0^\infty \int_{-\infty}^{+\infty} |\tilde{u}\,(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, t)|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\le C_0\, \mathrm{e}^{c_0 \cdot \Re\mathfrak{e}\, t} \cdot \int_0^\infty \int_{-\infty}^{+\infty} |u_0\,(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega))|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\end{aligned}$$

for every $t \in \Delta_{\vartheta'}$ and for all $y, \omega \in \mathbb{R}$ such that $|y| < r'$ and $|\omega| < r'$.

Before giving the *proof* of this proposition, we make a few important remarks: The proof hinges upon the fact that if the holomorphic extension $\tilde{u} : \mathfrak{Y}^{(r')} \times \Delta_{\vartheta'} \to \mathbb{C}$ of a weak solution

$$u \in C([0,T] \to H) \cap L^2((0,T) \to V)$$

of the homogeneous initial value problem (2.7) exists, then it must satisfy the following initial value problem with complex partial derivatives:

$$(7.10)\qquad \begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + (\tilde{\mathcal{A}}\tilde{u})(z,\zeta,t) = 0 & \text{in } \mathfrak{Y}^{(r')} \times \Delta_{\vartheta'}\,; \\ \qquad\qquad \tilde{u}(z,\zeta,0) = u_0(z,\zeta) & \text{for } (z,\zeta) \in \mathfrak{Y}^{(r')}\,, \end{cases}$$

where the complex partial differential operator $\tilde{\mathcal{A}}$ is given by

$$\begin{aligned}(7.11)\qquad (\tilde{\mathcal{A}}\tilde{u})(z,\zeta) &= -\frac{1}{2}\sigma\zeta \cdot \left[\frac{\partial}{\partial z}\left(\frac{\partial \tilde{u}}{\partial z}(z,\zeta) + 2\rho\frac{\partial \tilde{u}}{\partial \zeta}(z,\zeta)\right) + \frac{\partial^2 \tilde{u}}{\partial \zeta^2}(z,\zeta)\right] \\ &\quad + \left(q_r + \tfrac{1}{2}\sigma\zeta\right) \cdot \frac{\partial \tilde{u}}{\partial z}(z,\zeta) - \kappa(\theta_\sigma - \zeta) \cdot \frac{\partial \tilde{u}}{\partial \zeta}(z,\zeta) \\ &\equiv -\frac{1}{2}\sigma\zeta \cdot \left[(\tilde{u}_z + 2\rho\,\tilde{u}_\zeta)_z + \tilde{u}_{\zeta\zeta}\right] + \left(q_r + \tfrac{1}{2}\sigma\zeta\right) \cdot \tilde{u}_z - \kappa(\theta_\sigma - \zeta) \cdot \tilde{u}_\zeta \\ &\quad \text{for } (z,\zeta) \in \mathfrak{Y}^{(r')} = \mathfrak{X}^{(r')} \times \Delta_{\arctan r'}\,.\end{aligned}$$

This operator has been obtained from the Heston operator (2.9) by the natural complexification of the variables x and ξ as $z = x + \mathrm{i}y$ and $\zeta = \xi + \mathrm{i}\eta$, respectively, with the imaginary parts $y, \eta \in \mathbb{R}$. However, to establish the conclusion of Proposition 7.1, we need to choose the imaginary parts $y, \eta \in \mathbb{R}$ such that $|y| < r'$ and $\eta = \xi\omega$ with $|\omega| < r'$, where y and ω are fixed, while x and ξ are the independent variables, $(x,\xi) \in \mathbb{H}$. Hence, we have to investigate the function

$$\begin{aligned}(7.12)\qquad &v : (x,\xi,t) \longmapsto v(x,\xi,t) \equiv v^{(\mathrm{i}\omega+\omega^*)}_{(\mathrm{i}y+z^*)}(x,\xi,t) \\ &\stackrel{\text{def}}{=} \tilde{u}\Big(x + \mathrm{i}y + z^*,\, \xi(1+\mathrm{i}\omega+\omega^*),\, t\Big) : \mathbb{H} \times \Delta_{\vartheta'} \to \mathbb{C}\end{aligned}$$

with the complexified space variables

$$z + z^* = x + \mathrm{i}y + z^* = x + x^* + \mathrm{i}(y + y^*)\,, \qquad (7.13)$$
$$\zeta + \zeta^* = \xi(1 + \mathrm{i}\omega) + \zeta^* = \xi(1 + \mathrm{i}\omega + \omega^*)\,.$$

Here, $z^*, \omega^* \in \mathbb{C}$ are complex numbers with sufficiently small absolute values, such that

$$\mathrm{i}y + z^* \in \mathfrak{X}^{(r')} \quad \text{and} \quad 1 + \mathrm{i}\omega + \omega^* \in \Delta_{\arctan r'}\,, \qquad (7.14)$$

which guarantees that the argument of the function $\tilde{u}$ in eq. (7.12) above stays in $\mathfrak{V}^{(r')} \times \Delta_{\vartheta'}$ for all $(x, \xi, t) \in \mathbb{H} \times \Delta_{\vartheta'}$. Small complex perturbations $(z^*, \omega^*) \in \mathbb{C}^2$ are needed to calculate partial derivatives of the function $\tilde{u}(z, \zeta, t)$ with respect to the real and imaginary parts of its arguments $(z, \zeta) \in \mathfrak{V}^{(r')}$. The complex differentiability (yielding the holomorphy) with respect to the time variable $t \in \Delta_{\vartheta'}$ has been treated in the previous paragraph (§7.1).

A simple application of the chain rule,

$$\frac{\partial v}{\partial x}(x, \xi, t) = \frac{\partial \tilde{u}}{\partial z}\,(z + z^*,\, \zeta + \zeta^*,\, t) \quad \text{and} \quad \frac{\partial v}{\partial \xi} = (1 + \mathrm{i}\omega + \omega^*)\,\frac{\partial \tilde{u}}{\partial \zeta}\,,$$

shows that the function $v : \mathbb{H} \times \Delta_{\vartheta'} \to \mathbb{C}$ defined in eq. (7.12) must be a weak solution to the following initial value problem with real partial derivatives:

$$(7.15) \qquad \begin{cases} \dfrac{\partial v}{\partial t} + \left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} v\right)(x, \xi, t) = 0 & \text{in } \mathbb{H} \times \Delta_{\vartheta'}\,; \\ v(x, \xi, 0) = u_0\left(x + \mathrm{i}y + z^*,\, \xi(1 + \mathrm{i}\omega + \omega^*)\right) & \text{for } (x, \xi) \in \mathbb{H}\,, \end{cases}$$

where the real partial differential operator $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}$ is given by

$$\begin{aligned}
&\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} v\right)(x, \xi) = \\
&- \frac{1}{2}\,(1 + \mathrm{i}\omega + \omega^*)\sigma\xi \cdot \left[\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x}(x, \xi) + \frac{2\rho}{1 + \mathrm{i}\omega + \omega^*} \cdot \frac{\partial v}{\partial \xi}(x, \xi) \right) \right. \\
&\qquad\qquad \left. + \frac{1}{(1 + \mathrm{i}\omega + \omega^*)^2} \cdot \frac{\partial^2 v}{\partial \xi^2}(x, \xi) \right] \\
&+ \left[q_r + \tfrac{1}{2}(1 + \mathrm{i}\omega + \omega^*)\sigma\xi \right] \cdot \frac{\partial v}{\partial x}(x, \xi) \\
&- \frac{\kappa}{1 + \mathrm{i}\omega + \omega^*}\,[\theta_\sigma - (1 + \mathrm{i}\omega + \omega^*)\xi] \cdot \frac{\partial v}{\partial \xi}(x, \xi) \\
&\equiv\; - \frac{1}{2}\,\sigma\xi \cdot \left[((1 + \mathrm{i}\omega + \omega^*) v_x + 2\rho\, v_\xi)_x + (1 + \mathrm{i}\omega + \omega^*)^{-1} v_{\xi\xi} \right] \\
&\quad + \left[q_r + \tfrac{1}{2}(1 + \mathrm{i}\omega + \omega^*)\sigma\xi \right] \cdot v_x - \kappa \left[(1 + \mathrm{i}\omega + \omega^*)^{-1}\theta_\sigma - \xi \right] \cdot v_\xi \\
&\quad \text{for } (x, \xi) \in \mathbb{H}.
\end{aligned}$$

Consequently, recalling the definition of $\mathcal{A}$ in eq. (2.9), we have

$$\begin{aligned} &\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}v\right)(x,\xi) = (\mathcal{A}v)(x,\xi) \\ &- \frac{\sigma}{2}\,(\mathrm{i}\omega+\omega^*)\xi\cdot\left(v_{xx} - (1+\mathrm{i}\omega+\omega^*)^{-1}v_{\xi\xi}\right) \\ &+ \frac{\sigma}{2}\,(\mathrm{i}\omega+\omega^*)\xi\cdot v_x + \frac{\mathrm{i}\omega+\omega^*}{1+\mathrm{i}\omega+\omega^*}\,\kappa\theta_\sigma\cdot v_\xi \quad \text{for } (x,\xi)\in\mathbb{H}. \end{aligned} \tag{7.16}$$

It is important to note that the linear operator $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}: V\to V'$ does *not* depend on $y\in\mathbb{R}$ or $z^*\in\mathbb{C}$. However, it does depend on $\omega\in\mathbb{R}$ and $\omega^*\in\mathbb{C}$; more precisely, it depends on the sum $\mathrm{i}\omega+\omega^*$.

To derive the sesquilinear form associated to $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}$,

$$(v,w)\mapsto \left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}v,w\right)_H\,, \tag{7.17}$$

we apply the same methods as for obtaining eq. (2.21) associated to $\mathcal{A}$. We thus arrive at

$$\begin{aligned} &\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}v,w\right)_H = (\mathcal{A}v,w)_H \\ &+ \frac{\sigma}{2}\,(\mathrm{i}\omega+\omega^*)\int_{\mathbb{H}}\left(v_x\cdot\bar{w}_x - (1+\mathrm{i}\omega+\omega^*)^{-1}v_\xi\cdot\bar{w}_\xi\right)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &- \frac{\sigma}{2}\,(\mathrm{i}\omega+\omega^*)\int_{\mathbb{H}}\big[\gamma\ \mathrm{sign}\,x\cdot v_x\,\bar{w}\cdot\xi \\ &\qquad\qquad + (1+\mathrm{i}\omega+\omega^*)^{-1}(\beta-\mu\xi)\,v_\xi\cdot\bar{w}\big]\,\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &+ \frac{\sigma}{2}\,(\mathrm{i}\omega+\omega^*)\int_{\mathbb{H}} v_x\,\bar{w}\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &+ \frac{\mathrm{i}\omega+\omega^*}{1+\mathrm{i}\omega+\omega^*}\,\kappa\theta_\sigma\int_{\mathbb{H}} v_\xi\,\bar{w}\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,, \end{aligned}$$

where we have taken advantage of the vanishing boundary conditions (2.18) and (2.19) with the pair of functions (v,w) in place of (u,w), while performing integration-by-parts on the second summand on the right-hand side of eq. (7.16); cf. also eqs. (2.15), (2.16), and (2.17).

Finally, the sesquilinear form (7.17) becomes

$$\begin{aligned} &\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}v,w\right)_H = (\mathcal{A}v,w)_H \\ &+ \frac{\sigma}{2}\,(\mathrm{i}\omega+\omega^*)\int_{\mathbb{H}}\left(v_x\cdot\bar{w}_x - (1+\mathrm{i}\omega+\omega^*)^{-1}v_\xi\cdot\bar{w}_\xi\right)\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &+ \frac{\sigma}{2}\,(\mathrm{i}\omega+\omega^*)\int_{\mathbb{H}}(1-\gamma\ \mathrm{sign}\,x)\,v_x\cdot\bar{w}\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &+ \frac{\sigma}{2}\cdot\frac{\mathrm{i}\omega+\omega^*}{1+\mathrm{i}\omega+\omega^*}\,\mu\int_{\mathbb{H}} v_\xi\cdot\bar{w}\cdot\xi\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &- \frac{\mathrm{i}\omega+\omega^*}{1+\mathrm{i}\omega+\omega^*}\,\left(\tfrac{1}{2}\beta\sigma-\kappa\theta_\sigma\right)\int_{\mathbb{H}} v_\xi\cdot\bar{w}\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,. \end{aligned} \tag{7.18}$$

All integrals on the right-hand side converge absolutely for any pair $u, w \in V$, in analogy with eq. (2.21). In what follows we use the last formula, eq. (7.18), to define the sesquilinear form (7.17) in $V \times V$.

The following two results, respectively, are analogues of Propositions 6.1 and 6.2 with similar proofs. Here, the sesquilinear form from eq. (7.18) replaces that from (2.21). We use the former to verify the boundedness and coercivity of the linear operator $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} : V \to V'$ in the Hilbert space $V = H^1(\mathbb{H}; \mathfrak{w})$. The details of these proofs are left to an interested reader.

Proposition 7.2 (Boundedness.) *Let β, γ, μ, ρ, σ, θ, q_r, and κ be given constants in $\mathbb{R}$, $\beta > 1$, $\gamma > 0$, $\mu > 0$, $-1 < \rho < 1$, $\sigma > 0$, and $\theta > 0$. Then, given any number $r \in (0, \infty)$, there exists a constant $C^* \in (0, \infty)$, such that, for all numbers $\omega \in (-r, r)$ and $\omega^* \in \mathbb{C}$ with $|\omega^*| \le 1/2$, and for all pairs $u, w \in V$, we have*

$$\left|\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}u, w\right)_H\right| \le C^* \cdot \|u\|_V \cdot \|w\|_V . \tag{7.19}$$

In our next proposition, the number $r \in (0, \infty)$ has to be sufficiently small, unlike in the analogous Proposition 6.2 where it is arbitrary.

Proposition 7.3 (Coercivity.) *Let ρ, σ, θ, q_r, and γ be given constants in $\mathbb{R}$, $\rho \in (-1, 1)$, $\sigma > 0$, $\theta > 0$, and $\gamma > 0$. Assume that β, γ, κ, and μ are chosen as specified in* Proposition 4.1. *Then there exist constants $r \in \left(0, \frac{1}{2}\right]$ and $c_2'' \in (0, \infty)$ such that the following* Gårding inequality

$$2 \cdot \Re\mathfrak{e}\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}u, u\right)_H \ge \frac{\sigma}{2}(1 - |\rho|) \cdot \|u\|_V^2 - c_2'' \cdot \|u\|_H^2 \tag{7.20}$$

is valid for all $\omega \in (-r, r)$ and $\omega^* \in \mathbb{C}$ with $|\omega^*| \le r$, and for all $u \in V$.

Now we are ready to prove Proposition 7.1.

Proof of **Proposition 7.1.** It is obvious that we must find a method how to solve the initial value problem (7.15) with a conclusion similar to that provided in paragraph §7.1 for the initial value problem (2.7) with $f \equiv 0$, thanks to Propositions 6.1 and 6.2 for the linear operator $\mathcal{A} : V \to V'$. Notice that the initial condition in problem (7.15) reads

$$v(x, \xi, 0) = v_0(x, \xi) \stackrel{\text{def}}{=} u_0\left(x + \mathrm{i}y + z^*, \, \xi(1 + \mathrm{i}\omega + \omega^*)\right) \quad \text{for } (x, \xi) \in \mathbb{H} . \tag{7.21}$$

Thus, we must first adapt these two propositions to the linear operator $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} : V \to V'$ for any fixed numbers $y, \omega \in \mathbb{R}$ with $|y| < r'$ and $|\omega| < r'$, and for any fixed complex numbers $z^*, \omega^* \in \mathbb{C}$ with sufficiently small absolute values, such that (7.14) holds. It suffices to do this for some $r' \in (0, r]$ small enough. Hence, the couple $(z + z^*, \, \zeta + \zeta^*)$

from eq. (7.13) that appears also as the argument of the function u_0 in eq. (7.21) above stays in $\mathfrak{Y}^{(r')} \subset \mathfrak{Y}^{(r)}$ for all $(x,\xi) \in \mathbb{H}$, thanks to $0 < r' \leq r$.

In analogy with Propositions 6.1 and 6.2 (boundedness and coercivity, respectively) for the operator $\mathcal{A} : V \to V'$, Propositions 7.2 and 7.3 (Appendix A) for the operator $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} : V \to V$ guarantee that $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}$ is a *sectorial operator* in the Hilbert space H, provided $|\omega| < r'$ and $|\omega^*|$ is small enough. Hence, $-\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}$ is the *infinitesimal generator* of a *holomorphic semigroup* of bounded linear operators $\left\{ \mathrm{e}^{-t\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}} : t \in \mathbb{R}_+ \right\}$ in H, i.e.,

$$\| \mathrm{e}^{-t\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}} \|_{\mathcal{L}(H\to H)} \leq M''_{\vartheta''} \, \mathrm{e}^{(c''_2/2)\cdot \Re\mathfrak{e}\, t} \quad \text{holds for all } t \in \Delta_{\vartheta''} , \tag{7.22}$$

where $\vartheta'' \in (0,\vartheta)$ is arbitrary and $M''_{\vartheta''}, c''_2 \in (0,\infty)$ are suitable constants depending on ϑ'', but independent from the particular choice of $\omega \in \mathbb{R}$ or $\omega^* \in \mathbb{C}$ such that $|\omega| < r'$ and $|\omega^*|$ is small enough. This semigroup provides the (unique) holomorphic extension $v : \Delta_{\vartheta''} \to H$ of the (unique) weak solution

$$v \equiv v^{(\mathrm{i}\omega+\omega^*)}_{(\mathrm{i}y+z^*)} \in C([0,T] \to H) \cap L^2((0,T) \to V)$$

to the initial value problem (7.15). The uniqueness guarantees that this solution depends on the fixed data $y, \omega \in \mathbb{R}$ and $z^*, \omega^* \in \mathbb{C}$ only through the sums $\mathrm{i}y + z^*$ and $\mathrm{i}\omega + \omega^*$, as so do the operator $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}$ (which, in fact, is independent from y and z^*) and the initial condition (7.21). Indeed, let $y_j, \omega_j \in \mathbb{R}$ and $z^*_j, \omega^*_j \in \mathbb{C}$ satisfy (7.14) for both $j = 1,2$, i.e.,

$$\mathrm{i}y_j + z^*_j \in \mathfrak{X}^{(r')} \quad \text{and} \quad 1 + \mathrm{i}\omega_j + \omega^*_j \in \Delta_{\arctan r'} . \tag{7.23}$$

Consider the corresponding (unique) weak solution

$$v^{(j)} \equiv v^{(\mathrm{i}\omega_j+\omega^*_j)}_{(\mathrm{i}y_j+z^*_j)} \in C([0,T] \to H) \cap L^2((0,T) \to V)$$

to the initial value problem (7.15) together with its (unique) holomorphic extension $v^{(j)} : \Delta_{\vartheta''} \to H$; $j = 1,2$. The initial condition (7.21) is given by

$$v^{(j)}(x,\xi,0) = v^{(j)}_0(x,\xi) \stackrel{\text{def}}{=} u_0 \left(x + \mathrm{i}y_j + z^*_j,\, \xi(1 + \mathrm{i}\omega_j + \omega^*_j) \right) \quad \text{for } (x,\xi) \in \mathbb{H} . \tag{7.24}$$

Consequently, if

$$\mathrm{i}y_1 + z^*_1 = \mathrm{i}y_2 + z^*_2 \quad \text{and} \quad \mathrm{i}\omega_1 + \omega^*_1 = \mathrm{i}\omega_2 + \omega^*_2 ,$$

then $v^{(1)}_0 = v^{(2)}_0$ in H and, therefore, the uniqueness for problem (7.15) forces $v^{(1)}(x,\xi,t) \equiv v^{(2)}(x,\xi,t)$ for $(x,\xi,t) \in \mathbb{H} \times \Delta_{\vartheta''}$. This uniqueness result allows us to give the following (correct) definition of a function $\tilde{u} : \mathfrak{Y}^{(r')} \times \Delta_{\vartheta''} \to \mathbb{C}$ by the formula

$$\tilde{u}\Big(x + \mathrm{i}y + z^*,\, \xi(1 + \mathrm{i}\omega + \omega^*),\, t \Big) \stackrel{\text{def}}{=} v^{(\mathrm{i}\omega+\omega^*)}_{(\mathrm{i}y+z^*)}(x,\xi,t) \quad \text{for all } (x,\xi) \in \mathbb{H} \text{ and for all } t \in \Delta_{\vartheta''} . \tag{7.25}$$

Notice that it suffices to take $z^* = \omega^* = 0$ and arbitrary numbers $y, \omega \in \mathbb{R}$ with $|y| < r'$ and $|\omega| < r'$ to define $\tilde{u}$.

The function

$$t \mapsto v_{(\mathrm{i}y+z^*)}^{(\mathrm{i}\omega+\omega^*)}(x,\xi,t) : \Delta_{\vartheta''} \to \mathbb{C}$$

being holomorphic, by §7.1, it is obvious that also $\tilde{u} : \mathfrak{V}^{(r')} \times \Delta_{\vartheta''} \to \mathbb{C}$ is holomorphic in the time variable $t \in \Delta_{\vartheta''}$. Furthermore, the estimate in (7.9) follows immediately from inequality (7.22) by taking $C_0 = M''_{\vartheta''} > 0$ and $c_0 = c''_2/2 > 0$.

Taking advantage of the differentiability of the coefficients of the partial differential operator $\mathcal{A}^{(\mathrm{i}\omega+\omega^*)}$ in eq. (7.16), we observe that if the initial data $u_0 \in \mathcal{L}^{2,\infty}(\mathfrak{V}^{(r)})$ are C^∞-smooth (in the real-variable sense) then also the (unique) solution $\tilde{u}(\,\cdot\,,\,\cdot\,t) : \mathfrak{V}^{(r')} \to \mathbb{C}$ to the initial value problem (7.15) is C^∞-smooth in $\mathbb{H}$, by Theorem 19 and Corollary (to Theorem 19) in A. FRIEDMAN [18, Chapt. 10], on p. 321 and p. 322, respectively.

Now we take advantage of the holomorphic data v_0 in the initial condition (7.21) with respect to the small complex parameters $(z^*, \omega^*) \in \mathbb{C}^2$ in order to show that, for each fixed $t \in \Delta_{\vartheta'}$, the function $\tilde{u}(\,\cdot\,,\,\cdot\,t) : \mathfrak{V}^{(r')} \to \mathbb{C}$ is holomorphic. To this end we first realize that the initial data v_0 in (7.21), which depend on the real parameters $x^* = \mathfrak{Re}\, z^*$, $y^* = \mathfrak{Im}\, z^*$, $\alpha^* = \mathfrak{Re}\, \omega^*$, and $\beta^* = \mathfrak{Im}\, \omega^*$, are continuously differentiable (i.e., C^1-smooth in the real-variable sense) with respect to these parameters. We wish to prove that the same is true of each function $v_{(\mathrm{i}y+z^*)}^{(\mathrm{i}\omega+\omega^*)}$ with respect to the parameters $x^*, y^*, \alpha^*, \beta^* \in \mathbb{R}$.

In order to be able to apply well-known results from D. HENRY [22, Chapt. 3, §4] on the continuous dependence and differentiability of the solution $v_{(\mathrm{i}y+z^*)}^{(\mathrm{i}\omega+\omega^*)}$ with respect to parameters, we rewrite the initial value problem (7.15) equivalently as

$$(7.26)\qquad \begin{cases} \dfrac{\partial w}{\partial t} + \left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} w\right)(x,\xi,t) = -\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} v_0\right)(x,\xi) & \text{in } \mathbb{H} \times \Delta_{\vartheta'}\,; \\ w(x,\xi,0) = 0 & \text{for } (x,\xi) \in \mathbb{H}\,, \end{cases}$$

where

$$(7.27)\qquad \begin{gathered} w(x,\xi,t) \equiv w_{(\mathrm{i}y+z^*)}^{(\mathrm{i}\omega+\omega^*)}(x,\xi,t) \stackrel{\text{def}}{=} v_{(\mathrm{i}y+z^*)}^{(\mathrm{i}\omega+\omega^*)}(x,\xi,t) - v_0(x,\xi,t) \equiv \\ \tilde{u}\Big(x + \mathrm{i}y + z^*,\, \xi(1 + \mathrm{i}\omega + \omega^*),\, t\Big) - u_0\Big(x + \mathrm{i}y + z^*,\, \xi(1 + \mathrm{i}\omega + \omega^*)\Big) \end{gathered}$$

is the new unknown function of $(x,\xi,t) \in \mathbb{H} \times \Delta_{\vartheta'}$. It is easy to see that the function

$$-\left(\mathcal{A}^{(\mathrm{i}\omega+\omega^*)} v_0\right)(x,\xi) = -(\tilde{A}u_0)\Big(x + \mathrm{i}y + z^*,\, \xi(1 + \mathrm{i}\omega + \omega^*)\Big)$$

of $(z^*, \omega^*) \in \mathbb{C}$ is holomorphic, for $|z^*|$ and $|\zeta^*|$ small enough; hence, C^1-smooth with respect to the real parameters $x^* = \mathfrak{Re}\, z^*$, $y^* = \mathfrak{Im}\, z^*$, $\alpha^* = \mathfrak{Re}\, \omega^*$, and $\beta^* = \mathfrak{Im}\, \omega^*$. By HENRY's theorem [22, Theorem 3.4.4, pp. 64–65], the unknown function $w_{(\mathrm{i}y+z^*)}^{(\mathrm{i}\omega+\omega^*)}(x,\xi,t)$

possesses the same C^1-smoothness property, for every fixed $t \in \Delta_{\vartheta'}$. Next, we apply the Cauchy-Riemann operators

$$\frac{\partial}{\partial \bar{z}^*} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x^*} + \mathrm{i}\,\frac{\partial}{\partial y^*}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{\omega}^*} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial \alpha^*} + \mathrm{i}\,\frac{\partial}{\partial \beta^*}\right)$$

to both sides of eq. (7.26) (differentiation with respect to parameters), thus concluding that both derivatives,

$$\frac{\partial}{\partial \bar{z}^*} w^{(\mathrm{i}\omega+\omega^*)}_{(\mathrm{i}y+z^*)}(x,\xi,t) \quad \text{and} \quad \frac{\partial}{\partial \bar{\omega}^*} w^{(\mathrm{i}\omega+\omega^*)}_{(\mathrm{i}y+z^*)}(x,\xi,t)\,,$$

are the (unique) weak solutions of the initial value problem (7.26) with the zero initial data. Thus, both derivatives must vanish identically for all $(z^*, \omega^*) \in \mathbb{C}$ with $|z^*|$ and $|\zeta^*|$ small enough. Consequently, the difference $\tilde{u}(\,\cdot\,,\,\cdot\,t) - u_0 : \mathfrak{V}^{(r')} \to \mathbb{C}$ is holomorphic, and so is the function $\tilde{u}(\,\cdot\,,\,\cdot\,t) : \mathfrak{V}^{(r')} \to \mathbb{C}$, as claimed. D. Henry provides an alternative proof of analyticity in his [22, Corollary 3.4.5, p. 65] that employs an analytic implicit function theorem via Lemmas 3.4.2 and 3.4.3 in [22, pp. 63–64].

To complete our proof of Proposition 7.1, we apply the classical Hartogs's theorem on separate analyticity (see, e.g., S. G. Krantz [32, Theorem 1.2.5, p. 32] and remarks around) to conclude that the function $\tilde{u} : \mathfrak{V}^{(r)} \times \Delta_{\vartheta''} \to \mathbb{C}$, defined by the formula in eq. (7.25), is holomorphic not only separately in the variables $(z,\zeta) \in \mathfrak{V}^{(r')}$ and $t \in \Delta_{\vartheta''}$, but also jointly in (z,ζ,t) in its entire domain. ■

8 L^2-bounds in the complex domain

In order to give a plausible lower estimate on the space-time domain of holomorphy (i.e., the domain of complex analyticity) of a weak solution u to the homogeneous initial value problem (2.7) with $f \equiv 0$, we introduce a few more subsets of $\mathbb{C}^2 \times \mathbb{C}$ (cf. P. Takáč et al. [45, p. 428] or P. Takáč [46, pp. 58–59]):

The two constants $\kappa_0, \nu_0 \in (0,\infty)$ used below will be specified later (in the proof of Theorem 4.2); $0 \le \alpha < \infty$ is an arbitrary number. First, we recall the definitions of the complex sets $\mathfrak{V}^{(\kappa_0\alpha)} \subset \mathbb{C}^2$, $\Sigma^{(\alpha)}(\nu_0) \subset \mathbb{C}$, and $\Gamma_T^{(T')}(\kappa_0,\nu_0) \subset \mathbb{C}^2 \times \mathbb{C}$ given in Section 3, eqs. (3.4), (3.5), and (3.6), respectively.

Let us introduce the function $\chi(s) \stackrel{\text{def}}{=} \min\{s,\,1\}$ for $s \in \mathbb{R}_+ \stackrel{\text{def}}{=} [0,\infty)$; hence, it's derivative is given by $\chi'(s) = 1$ for $0 \le s \le 1$ and $\chi'(s) = 0$ for $1 < s < \infty$. Since the x-section of $\Gamma_T^{(T')}(\kappa_0,\nu_0)$ is independent from $x \in \mathbb{R}$, if $\kappa_0 T' < \pi/2$, setting

$$\begin{aligned} \hat{\Gamma}_T^{(T')}(\kappa_0,\nu_0) \stackrel{\text{def}}{=} \Big\{ & (y,\zeta,t) = (y, \xi + \mathrm{i}\eta, \alpha + \mathrm{i}\tau) \in \mathbb{R} \times \mathbb{C} \times \mathbb{C} : \\ & 0 < \alpha < T \ \text{ together with } \ |y| < \kappa_0 T' \chi\big(\tfrac{\alpha}{T'}\big)\,,\ \xi > 0\,, \\ & |\arctan(\eta/\xi)| < \kappa_0 T' \chi\big(\tfrac{\alpha}{T'}\big)\,, \ \text{ and } \ \nu_0 |\tau| < T' \chi\big(\tfrac{\alpha}{T'}\big) \Big\}\,, \end{aligned} \tag{8.1}$$

we may identify $\Gamma_T^{(T')}(\kappa_0, \nu_0) \simeq \mathbb{R} \times \hat{\Gamma}_T^{(T')}(\kappa_0, \nu_0)$.

The most important part of the proof of Theorem 4.2 is the a priori estimate in (4.2). It is proved in the following proposition. An example of a holomorphic extension $\tilde{u} : \mathfrak{V}^{(r)} \times \Delta_{\vartheta'} \to \mathbb{C}$ to a complex domain containing $\Gamma_T^{(T')}(\kappa_0, \nu_0) \subset \mathbb{C}^3$ is given in Proposition 7.1, provided κ_0, ν_0^{-1}, and $T' \in (0, T]$ are small enough.

Proposition 8.1 *Let ρ, σ, θ, q_r, and γ be given constants in $\mathbb{R}$, $\rho \in (-1, 1)$, $\sigma > 0$, $\theta > 0$, and $\gamma > 0$. Assume that β, γ, κ, and μ are chosen as specified in* Proposition 4.1. *Then, given any numbers $r \in (0, \infty)$ and $\vartheta' \in (0, \pi/2)$, the constants $\kappa_0, \nu_0^{-1} \in (0, \infty)$ and $T' \in (0, T]$ can be chosen sufficiently small, such that*

$$\Gamma_T^{(T')}(\kappa_0, \nu_0) \subset \mathfrak{V}^{(r)} \times \Delta_{\vartheta'}$$

and there exist some constants $C_0, c_0 \in \mathbb{R}_+$ with the following property:

If $u_0 : \mathfrak{V}^{(r)} \to \mathbb{C}$ is a holomorphic function that satisfies the bound (7.8) *in* Proposition 7.1 *and if $\tilde{u} : \mathfrak{V}^{(r)} \times \Delta_{\vartheta'} \to \mathbb{C}$ is the holomorphic extension of the (unique) weak solution*

$$u \in C([0, T] \to H) \cap L^2((0, T) \to V)$$

of the homogeneous initial value problem (2.7) *(with $f \equiv 0$ and this u_0) that has been obtained in* Proposition 7.1*, then the estimate in* (4.2) *holds with the constants $C_0 = 1$ and $c_0 = c_2' \in \mathbb{R}_+$ from* Proposition 6.2*, for every $\alpha \in (0, T]$ and for all $y, \omega, \tau \in \mathbb{R}$ satisfying* (4.3)*, depending on α. depending on α.*

Before giving the *proof* of this proposition, we first observe that the holomorphic extension $\tilde{u}(z, \zeta, t)$ must be unique, by uniqueness of the holomorphic extension in each of the variables $z, \zeta, t \in \mathbb{C}$. Consequently, the remarks following the statement of Proposition 7.1 apply also in the setting of our Proposition 8.1. The holomorphic extension $\tilde{u} : \Gamma_T^{(T')}(\kappa_0, \nu_0) \to \mathbb{C}$ of a weak solution

$$u \in C([0, T] \to H) \cap L^2((0, T) \to V)$$

of the homogeneous initial value problem (2.7) must satisfy the following initial value problem with complex partial derivatives; cf. (7.10):

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + (\tilde{\mathcal{A}} \tilde{u})(z, \zeta, t) = 0 & \text{in } \Gamma_T^{(T')}(\kappa_0, \nu_0)\,; \\ \tilde{u}(z, \zeta, 0) = u_0(z, \zeta) & \text{for } (z, \zeta) = (x, \xi) \in \mathbb{H}\,, \end{cases} \tag{8.2}$$

where the complex partial differential operator $\tilde{\mathcal{A}}$ is given by eq. (7.11) and $\tilde{u} \in \mathcal{H}^2(\mathfrak{V}^{(r)})$.

Proof of **Proposition 8.1.** In order to establish the estimate in (4.2), we need to control the behavior of the holomorphic extension $\tilde{u}(z,\zeta,t)$ of the solution $u(x,\xi,t)$ at every point

$$(z,\zeta,t) = (x+\mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha+\mathrm{i}\tau) \in \Gamma_T^{(T')}(\kappa_0,\nu_0)$$

by the initial condition $u_0 : \mathbb{H} \to \mathbb{C}$ defined only at points $(x,\xi,0) \in \mathbb{H}\times\{0\} = \mathbb{R}\times(0,\infty)\times\{0\}$. Given any such two points, $(x,\xi,0)$ and (z,ζ,t), we connect them by the following piecewise linear path parametrized by the real time $s \in [0,\, \mathfrak{Re}\, t]$, i.e., by $0 \le s \le \alpha$:

Given any point

$$(z,\zeta,t) = (x+\mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha+\mathrm{i}\tau) \in \Gamma_T^{(T')}(\kappa_0,\nu_0)\,,$$

we set

$$y_0 = \frac{T'}{\min\{\alpha,\, T'\}}\, y\,, \quad \omega_0 = \tan\left(\frac{T'}{\min\{\alpha,\, T'\}}\, \arctan\omega\right)\,, \quad \text{and} \quad \phi = \frac{\tau}{\alpha}\,.$$

Thus, conditions (4.3) are equivalent with

$$\max\{|y_0|,\, |\arctan\omega_0|\} < \kappa_0 T' \quad \text{and} \quad |\phi| < \nu_0^{-1}\,. \tag{8.3}$$

Fixing $(y_0,\omega_0,\phi) \in \mathbb{R}^3$ as in (8.3) above, we recall $\chi(s) \stackrel{\mathrm{def}}{=} \min\{s,\, 1\}$ for $s \in \mathbb{R}_+ \stackrel{\mathrm{def}}{=} [0,\infty)$ and define the path

$$\begin{aligned} &\varsigma \equiv \varsigma_{x,\xi} : \; [0,T] \to \{(x,\xi,0)\} \cup \Gamma_T^{(T')}(\kappa_0,\nu_0) : \\ &s \longmapsto \Big(x+\mathrm{i}\chi(s/T')y_0,\, \xi\,(1+\mathrm{i}\chi(s/T')\omega_0)\,,\, (1+\mathrm{i}\phi)s\Big)\,. \\ &\;= (x,\xi,s) + \mathrm{i}\big(\chi(s/T')y_0,\, \chi(s/T')\omega_0,\, \phi s\big)\,. \end{aligned} \tag{8.4}$$

The numbers $y,\omega,\phi \in \mathbb{R}$ are related to (z,ζ,t) by $\phi = \frac{\tau}{\alpha}$, $y = \mathfrak{Im}\, z$, and $\omega = \frac{\mathfrak{Im}\,\zeta}{\mathfrak{Re}\,\zeta}$. For $s=0$ and $s=\alpha=\mathfrak{Re}\, t$ we get the points $(x,\xi,0)$ and (z,ζ,t), respectively.

Next, we define the function $v : \mathbb{H}\times[0,T] \to \mathbb{C}$ by the values of $\tilde{u}$ on the image of the path ς,

$$v(x,\xi,s) \stackrel{\mathrm{def}}{=} \tilde{u}\Big(x+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)y_0,\, \xi\,\big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\big)\,,\, (1+\mathrm{i}\phi)s\Big)\,, \quad (x,\xi,s) \in \mathbb{H}\times[0,T]\,. \tag{8.5}$$

We calculate

$$\frac{\partial v}{\partial s}(x,\xi,s) = (1+\mathrm{i}\phi)\,\frac{\partial\tilde{u}}{\partial t} + \frac{\mathrm{i}}{T'}\cdot\chi'\big(\tfrac{s}{T'}\big)\left(\frac{\partial\tilde{u}}{\partial z}\, y_0 + \frac{\partial\tilde{u}}{\partial\zeta}\,\xi\omega_0\right)\,, \tag{8.6}$$

$$\frac{\partial v}{\partial x}(x,\xi,s) = \frac{\partial\tilde{u}}{\partial z}\,, \tag{8.7}$$

$$\frac{\partial v}{\partial\xi}(x,\xi,s) = \big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\big)\,\frac{\partial\tilde{u}}{\partial\zeta}\,. \tag{8.8}$$

We prefer to use the complex form (7.11) of the (time-independent) Heston operator (2.9). Hence, according to the initial value problem (8.2),

$$v \in C([0,T] \to H) \cap L^2((0,T) \to V)$$

is a weak solution of the following initial value problem,

$$\text{(8.9)} \qquad \begin{cases} \dfrac{\partial v}{\partial s} + (\hat{\mathcal{A}}(s)v)(x,\xi,s) = 0 & \text{in } \mathbb{H} \times (0,T)\,; \\ v(x,\xi,0) = u_0(x,\xi) & \text{for } (x,\xi) \in \mathbb{H}\,, \end{cases}$$

where the (time-dependent) partial differential operator $\hat{\mathcal{A}}(s)$ is given by

$$\begin{aligned}
&(\hat{\mathcal{A}}(s)v)(x,\xi) \overset{\text{def}}{=} (1+\mathrm{i}\phi)\,(\tilde{\mathcal{A}}\tilde{u})(z,\zeta) - \frac{\mathrm{i}}{T'}\cdot\chi'\big(\tfrac{s}{T'}\big)\left(\frac{\partial\tilde{u}}{\partial z}\,y_0 + \frac{\partial\tilde{u}}{\partial\zeta}\,\xi\omega_0\right) \\
&= -\frac{1}{2}\,(1+\mathrm{i}\phi)\sigma\xi\cdot\Bigg[\Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)\,\frac{\partial^2 v}{\partial x^2} + 2\rho\,\frac{\partial^2 v}{\partial x\,\partial\xi}(x,\xi) \\
&\qquad\qquad + \Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)^{-1}\,\frac{\partial^2 v}{\partial\xi^2}(x,\xi)\Bigg] \\
&+ (1+\mathrm{i}\phi)\,\Big[q_r + \tfrac{1}{2}\,\Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)\,\sigma\xi\Big]\,\frac{\partial v}{\partial x}(x,\xi) \\
&- (1+\mathrm{i}\phi)\kappa\left[\theta_\sigma\,\Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)^{-1} - \xi\right]\frac{\partial v}{\partial\xi}(x,\xi) \\
&- \frac{\mathrm{i}}{T'}\cdot\chi'\big(\tfrac{s}{T'}\big)\left[y_0\,\frac{\partial v}{\partial x} + \Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)^{-1}\xi\omega_0\,\frac{\partial v}{\partial\xi}\right] \\
&= (1+\mathrm{i}\phi)\cdot(\mathcal{A}v)(x,\xi) \\
&- \frac{\mathrm{i}}{2}\,(1+\mathrm{i}\phi)\sigma\xi\cdot\chi\big(\tfrac{s}{T'}\big)\omega_0\left[\frac{\partial^2 v}{\partial x^2} - \Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)^{-1}\frac{\partial^2 v}{\partial\xi^2}\right] \\
&+ \frac{\mathrm{i}}{2}\,(1+\mathrm{i}\phi)\cdot\chi\big(\tfrac{s}{T'}\big)\omega_0\left[\sigma\xi\,\frac{\partial v}{\partial x}(x,\xi) + 2\kappa\theta_\sigma\,\Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)^{-1}\frac{\partial v}{\partial\xi}(x,\xi)\right] \\
&- \frac{\mathrm{i}}{T'}\cdot\chi'\big(\tfrac{s}{T'}\big)\left[y_0\,\frac{\partial v}{\partial x} + \Big(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\Big)^{-1}\xi\omega_0\,\frac{\partial v}{\partial\xi}\right]
\end{aligned}$$

which yields the following formula,

$$\text{(8.10)} \qquad \begin{aligned}
&(\hat{\mathcal{A}}(s)v)(x,\xi) = (1+\mathrm{i}\phi)\cdot(\mathcal{A}v)(x,\xi) \\
&- \mathrm{i}\cdot\frac{y_0}{T'}\cdot(\mathcal{L}_1(s)v)(x,\xi) - \mathrm{i}\cdot\frac{\omega_0}{T'}\cdot(\mathcal{L}_2(s)v)(x,\xi) \\
&+ \frac{\mathrm{i}}{2}\,(1+\mathrm{i}\phi)\sigma\,\omega_0\cdot(\mathcal{L}_3(s)v)(x,\xi) + \mathrm{i}(1+\mathrm{i}\phi)\kappa\theta_\sigma\,\omega_0\cdot(\mathcal{L}_4(s)v)(x,\xi)\,,
\end{aligned}$$

where we have abbreviated

$$(8.11)\qquad (\mathcal{L}_1(s)v)(x,\xi) \stackrel{\text{def}}{=} \chi'\big(\tfrac{s}{T'}\big)\cdot \frac{\partial v}{\partial x}(x,\xi)\,,$$

$$(8.12)\qquad (\mathcal{L}_2(s)v)(x,\xi) \stackrel{\text{def}}{=} \chi'\big(\tfrac{s}{T'}\big)\,\left(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\right)^{-1}\xi\,\frac{\partial v}{\partial \xi}(x,\xi)\,,$$

$$(8.13)\qquad (\mathcal{L}_3(s)v)(x,\xi) \stackrel{\text{def}}{=} -\,\chi\big(\tfrac{s}{T'}\big)\xi\left[\frac{\partial^2 v}{\partial x^2} - \left(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\right)^{-1}\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial v}{\partial x}\right]\,, \quad \text{and}$$

$$(8.14)\qquad (\mathcal{L}_4(s)v)(x,\xi) \stackrel{\text{def}}{=} \chi\big(\tfrac{s}{T'}\big)\,\left(1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\right)^{-1}\frac{\partial v}{\partial \xi} \quad \text{for } (x,\xi)\in\mathbb{H}\,.$$

We insert eq. (8.10) into (8.9), thus arriving at

$$(8.15)\qquad \begin{aligned} &\frac{\partial v}{\partial s}(x,\xi,s) = -(1+\mathrm{i}\phi)\cdot(\mathcal{A}v)(x,\xi)\\ &+\mathrm{i}\cdot\frac{y_0}{T'}\cdot(\mathcal{L}_1(s)v)(x,\xi) + \mathrm{i}\cdot\frac{\omega_0}{T'}\cdot(\mathcal{L}_2(s)v)(x,\xi)\\ &-\frac{\mathrm{i}}{2}(1+\mathrm{i}\phi)\sigma\,\omega_0\cdot(\mathcal{L}_3(s)v)(x,\xi) - \mathrm{i}(1+\mathrm{i}\phi)\kappa\theta_\sigma\,\omega_0\cdot(\mathcal{L}_4(s)v)(x,\xi)\end{aligned}$$

for $(x,\xi,s)\in\mathbb{H}\times(0,T)$.

In Propositions 6.1 and 6.2 above we have verified the boundedness and coercivity hypotheses for the linear operator $\mathcal{A}\colon V\to V'$ defined by sesquilinear form in eq. (2.21). Estimates analogous to those used in the proof of Proposition 6.1 show that all linear operators $\mathcal{L}_j(s)\colon V\to V'$; $j=1,2,3,4$, are uniformly bounded for $s\in[0,T]$ and $\omega_0\in\mathbb{R}$, i.e., there is a constant $L\in(0,\infty)$ such that

$$(8.16)\qquad \left|(\mathcal{L}_j(s)v,w)_H\right| \le L\cdot\|v\|_V\,\|w\|_V \quad \text{holds for all } v,w\in V$$

and for all $s\in[0,T]$ and all $\omega_0\in\mathbb{R}$; $j=1,2,3,4$. Here, we have used the definition of $\chi(s)=\min\{s,\,1\}$ and $\left|1+\mathrm{i}\chi\big(\tfrac{s}{T'}\big)\omega_0\right|\ge 1$.

In order to obtain the upper bound (4.2) for the integral on the left-hand side,

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^{+\infty}\left|\tilde{u}\left(x+\mathrm{i}y,\,\xi(1+\mathrm{i}\omega),\,\alpha+\mathrm{i}\tau\right)\right|^2\cdot\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\\ &=\int_0^\infty\int_{-\infty}^{+\infty}|v(x,\xi,s)|^2\mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi = \|v(\,\cdot\,,\,\cdot\,,s)\|_H^2\,,\end{aligned}$$

cf. eq. (8.5), we first take the time derivative of the second integral above, then apply eq. (8.15):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\,\|v(\,\cdot\,,\,\cdot\,,s)\|_H^2 &= \int_{\mathbb{H}} \left(\frac{\partial v}{\partial s}\,\bar{v} + v\,\frac{\partial \bar{v}}{\partial s} \right) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= -\int_{\mathbb{H}} \Big((\mathcal{A}v)(x,\xi)\,\bar{v} + v\,\overline{(\mathcal{A}v)(x,\xi)} \Big) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&- \mathrm{i}\phi \int_{\mathbb{H}} \Big((\mathcal{A}v)(x,\xi)\,\bar{v} - v\,\overline{(\mathcal{A}v)(x,\xi)} \Big) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&+ \mathrm{i}\,\frac{y_0}{T'} \int_{\mathbb{H}} \Big((\mathcal{L}_1(s)v)(x,\xi)\,\bar{v} - v\,\overline{(\mathcal{L}_1(s)v)(x,\xi)} \Big) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&+ \mathrm{i}\,\frac{\omega_0}{T'} \int_{\mathbb{H}} \Big((\mathcal{L}_2(s)v)(x,\xi)\,\bar{v} - v\,\overline{(\mathcal{L}_2(s)v)(x,\xi)} \Big) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&- \frac{\mathrm{i}}{2}\,\sigma\omega_0 \int_{\mathbb{H}} \Big((1+\mathrm{i}\phi)(\mathcal{L}_3(s)v)(x,\xi)\,\bar{v} - (1-\mathrm{i}\phi)v\,\overline{(\mathcal{L}_3(s)v)(x,\xi)} \Big) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&- \mathrm{i}\kappa\theta_\sigma\omega_0 \int_{\mathbb{H}} \Big((1+\mathrm{i}\phi)(\mathcal{L}_4(s)v)(x,\xi)\,\bar{v} - (1-\mathrm{i}\phi)v\,\overline{(\mathcal{L}_4(s)v)(x,\xi)} \Big) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,.
\end{aligned}$$

We estimate the integrals on the right-hand side above as follows. First, we take advantage of the coercivity of $\mathcal{A}:\ V \to V'$ expressed in terms of the *Gårding inequality* (6.3). Second, we employ the boundedness of $\mathcal{A}$, i.e., ineq. (6.1). Third, we employ the boundedness of $\mathcal{L}_j(s)$, i.e., ineq. (8.16). Consequently, we arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\,\|v(\,\cdot\,,\,\cdot\,,s)\|_H^2 &= \int_{\mathbb{H}} \left(\frac{\partial v}{\partial s}\,\bar{v} + v\,\frac{\partial \bar{v}}{\partial s} \right) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\
&\le -\sigma\,(1-|\rho|)\cdot\|v\|_V^2 + c_2'\cdot\|v\|_H^2 \\
&+ 2C|\phi|\,\|v\|_V^2 + 2L\,\frac{|y_0|}{T'}\,\|v\|_V^2 + 2L\,\frac{|\omega_0|}{T'}\,\|v\|_V^2 \\
&+ L\,|1+\mathrm{i}\phi|\,\sigma|\omega_0|\,\|v\|_V^2 + 2L\,|1+\mathrm{i}\phi|\,\kappa\theta_\sigma\,|\omega_0|\,\|v\|_V^2\,.
\end{aligned} \tag{8.17}$$

To estimate the coefficients on the right-hand side above, we recall the conditions on $(y_0,\omega_0,\phi)\in\mathbb{R}^3$ required in (8.3). In order to estimate the ratio ω_0/T' in a simple way, let us take the constants $\kappa_0\in(0,\infty)$ and $T'\in(0,T]$ small enough, such that $\kappa_0 T'\le\pi/4$. The function $x\mapsto x^{-1}\tan x$ being strictly monotone increasing on $(0,\infty)$, with the limit equal to 1 as $x\to 0+$, we employ condtition (8.3) to obtain

$$\frac{|\omega_0|}{T'} < \frac{\kappa_0}{\kappa_0 T'}\cdot\tan(\kappa_0 T') \le \kappa_0\cdot\frac{\tan(\pi/4)}{\pi/4} = \frac{4\kappa_0}{\pi} < 2\kappa_0\,.$$

Then ineq. (8.17) yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\,\|v(\,\cdot\,,\,\cdot\,,s)\|_H^2 &\le -\sigma\,(1-|\rho|)\cdot\|v\|_V^2 + c_2'\cdot\|v\|_H^2 \\
&+ \left(2C\nu_0^{-1} + 2L\kappa_0 + 4L\kappa_0\right)\|v\|_V^2 \\
&+ \left(L(1+\nu_0^{-1})\sigma\cdot 2\kappa_0 T' + 2L(1+\nu_0^{-1})\kappa\theta_\sigma\cdot 2\kappa_0 T'\right)\|v\|_V^2 \\
&= -\sigma\,(1-|\rho|)\cdot\|v\|_V^2 + c_2'\cdot\|v\|_H^2 + \tilde{C}\,\|v\|_V^2\,,
\end{aligned} \tag{8.18}$$

where $\tilde{C} \in (0,\infty)$ is a constant,

$$\begin{aligned}\tilde{C} &\stackrel{\text{def}}{=} \left(2C\nu_0^{-1} + 2L\kappa_0 + 4L\kappa_0\right) \\ &\quad + \left(L(1+\nu_0^{-1})\sigma \cdot 2\kappa_0 T' + 2L(1+\nu_0^{-1})\kappa\theta_\sigma \cdot 2\kappa_0 T'\right) \\ &= 2C\nu_0^{-1} + 6L\kappa_0 + 2L(1+\nu_0^{-1})(\sigma + 2\kappa\theta_\sigma)\cdot \kappa_0 T' .\end{aligned}$$

Here, the constants $\kappa_0, \nu_0^{-1} \in (0,\infty)$ and $T' \in (0,T]$ can be chosen sufficiently small, such that

$$\Gamma_T^{(T')}(\kappa_0,\nu_0) \subset \mathfrak{V}^{(r)} \times \Delta_{\vartheta'}$$

holds together with $0 < \tilde{C} \leq \sigma\,(1-|\rho|)$.

Then ineq. (8.18) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \|v(\,\cdot\,,\,\cdot\,,s)\|_H^2 \leq c_2' \cdot \|v\|_H^2 \quad \text{for } s \in (0,T)\,.$$

The desired inequality (4.2) now follows by taking $C_0 = 1$, $c_0 = c_2'$, and $s = \alpha$.

The proof of Proposition 8.1 is complete. ∎

9 End of the proof of the main result

In this section we finally finish the proof of Theorem 4.2. We will make use of the holomorphic approximation and the a priori estimates established in the previous two sections, Sections 7 and 8.

For a given function $u_0 \in H = L^2(\mathbb{H};\mathfrak{w})$, a sequence of entire (holomorphic) functions

$$\tilde{u}_{0,n} : \mathbb{C}^2 \to \mathbb{C}\,; \quad n = 1,2,2,\dots\,,$$

is constructed in Appendix B (§ B.2), whose restrictions to the complex domain $\mathfrak{X}^{(r)} \times \Delta_{\vartheta_v}$ belong to $H^2(\mathfrak{X}^{(r)} \times \Delta_{\vartheta_v})$ and satify

$$\|\tilde{u}_{0,n}|_{\mathbb{H}} - u_0\|_H \longrightarrow 0 \quad \text{as } n \to \infty\,;$$

cf. § B.2, properties (i), (ii), and (iii). In paragraph §7.2, for every fixed $n = 1,2,3,\dots$, we have used the function $\tilde{u}_{0,n}$ as the initial data for the initial value problem (7.10),

$$\text{(9.1)} \qquad \begin{cases} \dfrac{\partial \tilde{u}_n}{\partial t} + \tilde{\mathcal{A}}\tilde{u}_n = 0 \quad \text{for } (x,\xi,s) \in \mathbb{H} \times (0,T)\,; \\ \tilde{u}_n\big(x+\mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, 0\big) = \tilde{u}_{0,n}\big(x+\mathrm{i}y,\, \xi(1+\mathrm{i}\omega)\big) \quad \text{for } (x,\xi) \in \mathbb{H}\,. \end{cases}$$

Recall that $\tilde{\mathcal{A}}$ stands for the natural complexification of the Heston operator $\mathcal{A}$ defined in eq. (7.11). More precisely, this initial value problem has been solved by general theory of holomorphic semigroups for fixed values of $y, \omega \in \mathbb{R}$ such that $|y| < r$ and $|\arctan \omega| < \vartheta_v$. In paragraph §7.1 we have proved that the unique weak solution

$$t \longmapsto \big[(x,\xi) \mapsto \tilde{u}_n\big(x+\mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, t\big)\big] : [0,T] \to H$$

to problem (9.1) possesses a holomorphic extension with respect to time t to an angle Δ_{ϑ_t}, for some $\vartheta_t \in (0, \pi/2)$. Furthermore, in paragraph §7.2 (Proposition 7.1) we have proved that, for every $t \in \Delta_{\vartheta_t}$, the solution $\tilde{u}_n(\,\cdot\,,\,\cdot\,,t) : \mathfrak{X}^{(r)} \times \Delta_{\vartheta_v} \longrightarrow \mathbb{C}$ is a holomorphic function that belongs to $H^2(\mathfrak{X}^{(r)} \times \Delta_{\vartheta_v})$. Consequently, the function $\tilde{u}_n : \mathfrak{X}^{(r)} \times \Delta_{\vartheta_v} \times \Delta_{\vartheta_t} \longrightarrow \mathbb{C}$ is holomorphic in all its variables.

Now let us recall the time-dependent path ς from (8.4),

$$\begin{aligned} &\varsigma \equiv \varsigma_{x,\xi} : [0,T] \to \{(x,\xi,0)\} \cup \Gamma_T^{(T')}(\kappa_0,\nu_0) : \\ &s \longmapsto \Big(x + \mathrm{i}\chi(s/T')y_0,\, \xi\,(1+\mathrm{i}\chi(s/T')\omega_0)\,,\, (1+\mathrm{i}\phi)s\Big)\,. \\ &= (x,\xi,s) + \mathrm{i}\big(\chi(s/T')y_0,\, \chi(s/T')\omega_0,\, \phi s\big)\,, \end{aligned}$$

where the numbers $y_0, \omega_0, \phi \in \mathbb{R}$ obey conditions (8.3),

$$\max\{|y_0|,\, |\arctan \omega_0|\} < \kappa_0 T' \quad \text{and} \quad |\phi| < \nu_0^{-1}\,,$$

with some constants $\kappa_0, \nu_0^{-1} \in (0,\infty)$ and $T' \in (0,T]$ small enough, such that also

$$\kappa_0 T' \le \min\{r,\, \vartheta_v\} \quad \text{and} \quad \nu_0^{-1} \le \tan \vartheta_t\,.$$

Here, $0 < \vartheta_v, \vartheta_t < \pi/2$ are some given numbers. In the previous section (Section 8), Proposition 8.1, we have shown that along this path, $\varsigma \equiv \varsigma_{x,\xi}$, whose value at each $s \in [0,T]$ is viewed as a function of the pair $(x,\xi) \in \mathbb{H}$, the H-norm of the function $(x,\xi) \longmapsto v_n(x,\xi,s) : \mathbb{H} \times [0,T] \to \mathbb{C}$, defined by (8.5),

$$v_n(x,\xi,s) \stackrel{\text{def}}{=} \tilde{u}_n\Big(x + \mathrm{i}\chi(\tfrac{s}{T'})y_0,\, \xi\,\big(1+\mathrm{i}\chi(\tfrac{s}{T'})\omega_0\big)\,,\, (1+\mathrm{i}\phi)s\Big)\,, \qquad (x,\xi,s) \in \mathbb{H} \times [0,T]\,,$$

is uniformly bounded with the bound depending solely on the norm $\|\tilde{u}_{0,n}|_{\mathbb{H}}\|_H$, the time interval length $T > 0$, and the constant $c_2' > 0$ in inequality (6.3).

Next, we take advantage of the fact that we treat homogeneous *linear* parabolic problems, (2.7) (with $f \equiv 0$) in the real domain $\mathbb{H} \times (0,T)$, and its natural complexification (7.10) in the complex domain $\mathfrak{Y}^{(r')} \times \Delta_{\vartheta'}$. Consequently, given any indices $m, n \in \mathbb{N}$, the difference $\tilde{u}_n - \tilde{u}_m : \mathfrak{Y}^{(r')} \times \Delta_{\vartheta'} \to \mathbb{C}$ is a holomorphic function that obeys the parabolic equation in problem (7.10). Hence, we may apply our crucial a priori estimate (4.2) in Proposition 8.1 to the difference $\tilde{u}_n - \tilde{u}_m$, thus obtaining

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^{+\infty} |\tilde{u}_n\,(x+\mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha + \mathrm{i}\tau) \\ &\qquad - \tilde{u}_m\,(x+\mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha+\mathrm{i}\tau)|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\qquad \le \mathrm{e}^{c_2'\alpha} \cdot \int_0^\infty \int_{-\infty}^{+\infty} |\tilde{u}_n(x,\xi,0) - \tilde{u}_m(x,\xi,0)|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\qquad = \mathrm{e}^{c_2'\alpha} \cdot \|u_{0,n} - u_{0,m}\|_H^2 \end{aligned} \tag{9.2}$$

for every $\alpha \in (0,T]$ and for all $y, \omega, \tau \in \mathbb{R}$ satisfying conditions (4.3),

$$\max\{|y|,\, |\arctan \omega|\} < \kappa_0 \cdot \min\{\alpha,\, T'\} \quad \text{and} \quad \nu_0 |\tau| < \alpha\,,$$

depending on α.

It follows from $\tilde{u}_{0,n}|_{\mathbb{H}} \to u_0$ in H as $n \to \infty$, that $\{\tilde{u}_{0,n}|_{\mathbb{H}}\}_{n=1}^{\infty}$ is a Cauchy sequence in H. By ineq. (9.2), also the functions

$$w_n(x,\xi) \stackrel{\text{def}}{=} \tilde{u}_n\Big(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha + \mathrm{i}\tau\Big)\,, \quad (x,\xi) \in \mathbb{H}\,, \tag{9.3}$$

form a Cauchy sequence $\{w_n\}_{n=1}^{\infty}$ in H, *uniformly* for all choices of $\alpha + \mathrm{i}\tau \in \mathbb{C}$ and $y, \omega \in \mathbb{R}$ satisfying $0 < \alpha \leq T$ and conditions (4.3), that is to say, for

$$\max\{|y|,\, |\arctan \omega|\} < \kappa_0 \cdot \min\{\alpha,\, T'\} \quad \text{and} \quad \nu_0 |\tau| < \alpha \leq T\,. \tag{9.4}$$

Such numbers $\alpha + \mathrm{i}\tau \in \mathbb{C}$ and $y, \omega \in \mathbb{R}$ being fixed, let $w \stackrel{\text{def}}{=} \lim_{n\to\infty} w_n$ be the limit in H of this Cauchy sequence. In analogy with eq. (9.3), we set

$$\tilde{u}\Big(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha + \mathrm{i}\tau\Big) \stackrel{\text{def}}{=} w(x,\xi)\,, \quad (x,\xi) \in \mathbb{H}\,. \tag{9.5}$$

Then $\tilde{u} : \Gamma_T^{(T')}(\kappa_0, \nu_0) \to \mathbb{C}$ is a complex-valued, Lebesgue measurable function that satisfies the following inequality, by letting $m \to \infty$ in ineq. (9.2),

$$\begin{aligned} &\int_0^{\infty} \int_{-\infty}^{+\infty} \big|\tilde{u}_n\left(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha + \mathrm{i}\tau\right) \\ &\qquad - \tilde{u}\left(x + \mathrm{i}y,\, \xi(1+\mathrm{i}\omega),\, \alpha + \mathrm{i}\tau\right)\big|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\qquad \leq \mathrm{e}^{c_2'\alpha} \cdot \int_0^{\infty} \int_{-\infty}^{+\infty} |\tilde{u}_n(x,\xi,0) - u_0(x,\xi)|^2 \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\qquad = \mathrm{e}^{c_2'\alpha} \cdot \|u_{0,n} - u_0\|_H^2 \end{aligned} \tag{9.6}$$

for all choices of $\alpha + \mathrm{i}\tau \in \mathbb{C}$ and $y, \omega \in \mathbb{R}$ satisfying conditions (9.4) above.

A trivial consequence of (9.6) and (9.4) is that the sequence of functions $\tilde{u}_n : \Gamma_T^{(T')}(\kappa_0, \nu_0) \to \mathbb{C}$; $n = 1, 2, 3, \dots$, converges in the complex domain $\Gamma_T^{(T')}(\kappa_0, \nu_0)$ to the function $\tilde{u} : \Gamma_T^{(T')}(\kappa_0, \nu_0) \to \mathbb{C}$ locally in the L^2-topology. Since $\tilde{u}_n$ is holomorphic in $\Gamma_T^{(T')}(\kappa_0, \nu_0)$, it can be expressed by the Cauchy integral formula for polydiscs (S. G. KRANTZ [32], Theorem 1.2.2 (p. 24), or F. JOHN [29], Chapt. 3, Sect. 3(c), eq. (3.22c), p. 71). From this formula we deduce by standard limiting arguments using ineq. (9.6) that also the limit function $\tilde{u}$ is expressed by the same Cauchy integral formula for polydiscs. It follows that also $\tilde{u}$ is holomorphic in $\Gamma_T^{(T')}(\kappa_0, \nu_0)$, as desired. Obviously, Proposition 8.1 guarantees that $\tilde{u}$ satisfies ineq. (4.2).

To derive the relation of $\tilde{u}$ to problem (2.7) (with $f \equiv 0$) in the real domain $\mathbb{H} \times (0,T)$, let us take $y = \omega = \tau = 0$ in ineq. (9.6). Letting $n \to \infty$ we observe that the function

$$\hat{u} : (x,\xi,t) \longmapsto \tilde{u}(x,\xi,t) : \mathbb{H} \times (0,T) \to \mathbb{C} \tag{9.7}$$

is a weak solution to the Cauchy problem (2.7) (with $f \equiv 0$). However, the initial value problem (2.7) (with $f \equiv 0$) possesses a unique weak solution

$$u \in C([0,T] \to H) \cap L^2((0,T) \to V)\,,$$

by a pair of standard theorems for abstract parabolic problems due to J.-L. Lions [37, Chapt. IV], Théorème 1.1 (§1, p. 46) and Théorème 2.1 (§2, p. 52) (for alternative proofs, see also e.g. L. C. Evans [12, Chapt. 7, §1.2(c)], Theorems 3 and 4, pp. 356–358, J.-L. Lions [38, Chapt. III, §1.2], Theorem 1.2 (p. 102) and remarks thereafter (p. 103), A. Friedman [18], Chapt. 10, Theorem 17, p. 316, or H. Tanabe [47, Chapt. 5, §5.5], Theorem 5.5.1, p. 150).

Hence, we have $\hat{u} = u$ in $\mathbb{H} \times (0,T)$, thus proving that $\tilde{u} : \Gamma_T^{(T')}(\kappa_0, \nu_0) \to \mathbb{C}$ is a holomorphic extension of u.

The proof of Theorem 4.2 is complete. ■

A Appendix: Trace, Sobolev's, and Hardy's inequalities

Our boundedness and coercivity results for the Heston operator $\mathcal{A}\colon V \to V'$ make use of the following five lemmas: Recall that $V = H^1(\mathbb{H};\mathfrak{w})$ and $\beta > 0$, $\gamma > 0$, and $\mu > 0$ are constants in the weight $\mathfrak{w}(x,\xi)$ which is defined in eq. (2.12).

Lemma A.1 (A pointwise trace inequality.) *Let $\beta > 0$, $\gamma > 0$, and $\mu > 0$. Then the following inequality holds for every function $u \in V$ and at almost every point $x \in \mathbb{R}$,*

$$\frac{\partial}{\partial \xi}\left(\xi^\beta\, \mathrm{e}^{-\mu\xi}\, |u(x,\xi)|^2\right) \le \frac{1}{\mu}\, |u_\xi(x,\xi)|^2 \cdot \xi^\beta\, \mathrm{e}^{-\mu\xi} + \beta\, |u(x,\xi)|^2 \cdot \xi^{\beta-1}\, \mathrm{e}^{-\mu\xi} \tag{A.1}$$

for almost every $\xi \in (0,\infty)$.

Furthermore, for a.e. $x \in \mathbb{R}$ we have the limits

$$\lim_{\xi\to 0+}\left(\xi^\beta \cdot |u(x,\xi)|^2\right) = 0 \quad \textit{and} \tag{A.2}$$

$$\lim_{\xi\to\infty}\left(\xi^\beta\, \mathrm{e}^{-\mu\xi} \cdot |u(x,\xi)|^2\right) = 0\,. \tag{A.3}$$

Proof. The following partial derivatives exist almost everywhere in $\mathbb{H}$; we first calculate

$$\begin{aligned}&\frac{\partial}{\partial \xi}\left(\xi^\beta\, \mathrm{e}^{-\mu\xi} |u(x,\xi)|^2\right)\\ &= (u_\xi\, \bar{u} + u\, \bar{u}_\xi)\cdot \xi^\beta\, \mathrm{e}^{-\mu\xi} + \beta\, |u(x,\xi)|^2 \cdot \xi^{\beta-1}\, \mathrm{e}^{-\mu\xi} - \mu\, |u(x,\xi)|^2 \cdot \xi^\beta\, \mathrm{e}^{-\mu\xi}\,,\end{aligned}$$

then apply the Cauchy inequality

$$u_\xi\,\bar{u} + u\,\bar{u}_\xi = 2\cdot \Re\mathfrak{e}(u_\xi\,\bar{u}) \le 2|u_\xi|\cdot|u| \le \mu^{-1}\,|u_\xi|^2 + \mu\,|u|^2$$

to estimate

$$\frac{\partial}{\partial\xi}\left(\xi^\beta\,\mathrm{e}^{-\mu\xi}\,|u(x,\xi)|^2\right) \le \frac{1}{\mu}\,|u_\xi|^2\cdot\xi^\beta\,\mathrm{e}^{-\mu\xi} + \beta\,|u(x,\xi)|^2\cdot\xi^{\beta-1}\,\mathrm{e}^{-\mu\xi}\,.$$

This proves ineq. (A.1).

Recall that $u \in V$. Integrating the right-hand side of the last inequality with respect to the measure $\mathrm{e}^{-\gamma|x|-\mu\xi}\,\mathrm{d}x\,\mathrm{d}\xi$ over $\mathbb{H} = \mathbb{R}\times(0,\infty)$ we infer that, for a.e. $x\in\mathbb{R}$, both integrals below converge,

$$\text{(A.4)}\qquad \int_0^\infty |u_\xi(x,\xi)|^2\cdot\xi^\beta\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi < \infty \quad\text{and}\quad \int_0^\infty |u(x,\xi)|^2\cdot\xi^{\beta-1}\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi < \infty\,.$$

Let $x\in\mathbb{R}$ be such a point. The right-hand side of ineq. (A.1) is integrable with respect to the Lebesgue measure $\mathrm{d}\xi$ over $(0,\infty)$, and so is the positive part $\phi^+(\xi) = \max\{\phi(\xi),\,0\}$ of the partial derivative

$$\xi \longmapsto \phi(\xi) \stackrel{\text{def}}{=} \frac{\partial}{\partial\xi}\left(\xi^\beta\,\mathrm{e}^{-\mu\xi}\,|u(x,\xi)|^2\right)\,.$$

Thus, the existence of the limit in (A.2),

$$\lim_{\xi\to 0+}\left(\xi^\beta\cdot|u(x,\xi)|^2\right) = L_0(x) \quad\text{ for a.e. }\; x\in\mathbb{R}\,,$$

is deduced from

$$\text{(A.5)}\qquad L_0(x) \stackrel{\text{def}}{=} \liminf_{\xi\to 0+}\left(\xi^\beta\cdot|u(x,\xi)|^2\right)$$

and the following inequality, obtained by integrating ineq. (A.1) and valid for all $0<\xi'<\xi''<\infty$,

$$\text{(A.6)}\qquad \begin{aligned}(\xi'')^\beta\,\mathrm{e}^{-\mu\xi''}\,|u(x,\xi'')|^2 - (\xi')^\beta\,\mathrm{e}^{-\mu\xi'}|u(x,\xi')|^2 &\stackrel{\text{def}}{=} \left[\xi^\beta\,\mathrm{e}^{-\mu\xi}\,|u(x,\xi)|^2\right]_{\xi=\xi'}^{\xi=\xi''}\\ \le \frac{1}{\mu}\int_{\xi'}^{\xi''}|u_\xi(x,\xi)|^2\cdot\xi^\beta\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi &+ \beta\int_{\xi'}^{\xi''}|u(x,\xi)|^2\cdot\xi^{\beta-1}\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi\,.\end{aligned}$$

By similar reasoning, one derives the existence of the limit in (A.3),

$$\lim_{\xi\to\infty}\left(\xi^\beta\,\mathrm{e}^{-\mu\xi}\cdot|u(x,\xi)|^2\right) = L_\infty(x) \quad\text{ for a.e. }\; x\in\mathbb{R}\,,$$

from

$$\text{(A.7)}\qquad L_\infty(x) \stackrel{\text{def}}{=} \liminf_{\xi\to\infty}\left(\xi^\beta\,\mathrm{e}^{-\mu\xi}\cdot|u(x,\xi)|^2\right)\,.$$

Finally, both limits, $L_0(x)$ and $L_\infty(x)$, are nonnegative and finite, by the integrability properties of $u_\xi(x,\,\cdot\,)$ and $u(x,\,\cdot\,)$ stated in (A.4). Moreover, the second integral in (A.4) forces $L_0(x) = L_\infty(x) = 0$, thanks to $\int_0^\delta \xi^{-1}\,\mathrm{d}\xi = \int_{1/\delta}^\infty \xi^{-1}\,\mathrm{d}\xi = \infty$ for any $\delta>0$. ■

Lemma A.1 has the following global analogue with a similar proof.

Lemma A.2 (A trace inequality.) *Let $\beta > 0$, $\gamma > 0$, and $\mu > 0$. Then the following inequality holds for every function $u \in V$,*

$$
\begin{aligned}
\text{(A.8)} \qquad & \frac{\partial}{\partial \xi} \left(\xi^{\beta} \, \mathrm{e}^{-\mu\xi} \int_{\mathbb{R}} |u(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \right) \\
& \leq \frac{1}{\mu} \int_{\mathbb{R}} |u_\xi(x,\xi)|^2 \cdot \xi^{\beta} \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x + \beta \int_{\mathbb{R}} |u(x,\xi)|^2 \cdot \xi^{\beta-1} \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x
\end{aligned}
$$

for almost every $\xi \in (0,\infty)$.

Furthermore, the limits in (2.15) and (2.16) are valid.

Proof. We integrate both sides of ineq. (A.1) with respect to the measure $\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x$ over $\mathbb{R}$ to obtain ineq. (A.8).

Since $u \in V$, the right-hand side of ineq. (A.8) is integrable with respect to the Lebesgue measure $\mathrm{d}\xi$ over $(0,\infty)$, and so is the positive part $\phi^+(\xi) = \max\{\phi(\xi), 0\}$ of the partial derivative

$$
\xi \longmapsto \phi(\xi) \stackrel{\text{def}}{=} \frac{\partial}{\partial \xi} \left(\xi^{\beta} \, \mathrm{e}^{-\mu\xi} \int_{\mathbb{R}} |u(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \right) .
$$

Thus, the existence of the limit in (2.15),

$$
\lim_{\xi \to 0+} \left(\xi^{\beta} \cdot \int_{-\infty}^{+\infty} |u(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \right) = L_0 \,,
$$

is deduced from

$$
\text{(A.9)} \qquad L_0 \stackrel{\text{def}}{=} \liminf_{\xi \to 0+} \left(\xi^{\beta} \cdot \int_{-\infty}^{+\infty} |u(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \right)
$$

and the following inequality, obtained by integrating ineq. (A.8) and valid for all $0 < \xi' < \xi'' < \infty$, cf. (A.6):

$$
\begin{aligned}
& (\xi'')^{\beta} \, \mathrm{e}^{-\mu\xi''} \int_{\mathbb{R}} |u(x,\xi'')|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x - (\xi')^{\beta} \, \mathrm{e}^{-\mu\xi'} \int_{\mathbb{R}} |u(x,\xi')|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \\
& \stackrel{\text{def}}{=} \left[\xi^{\beta} \, \mathrm{e}^{-\mu\xi} \int_{\mathbb{R}} |u(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \right]_{\xi=\xi'}^{\xi=\xi''} \\
& \leq \frac{1}{\mu} \int_{\xi'}^{\xi''} \int_{\mathbb{R}} |u_\xi(x,\xi)|^2 \cdot \xi^{\beta} \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \\
& + \beta \int_{\xi'}^{\xi''} \int_{\mathbb{R}} |u(x,\xi)|^2 \cdot \xi^{\beta-1} \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \,.
\end{aligned}
$$

By similar reasoning, one derives the existence of the limit in (2.16),

$$
\lim_{\xi \to \infty} \left(\xi^{\beta} \, \mathrm{e}^{-\mu\xi} \cdot \int_{-\infty}^{+\infty} |u(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \right) = L_\infty \,,
$$

from

$$L_\infty \stackrel{\text{def}}{=} \liminf_{\xi\to\infty} \left(\xi^\beta \, \mathrm{e}^{-\mu\xi} \cdot \int_{-\infty}^{+\infty} |u(x,\xi)|^2 \cdot \mathrm{e}^{-\gamma|x|} \, \mathrm{d}x \right) . \tag{A.10}$$

Again, as in our proof of Lemma A.1 above, both limits, L_0 and L_∞, are nonnegative and finite, by the integrability properties of $u \in V$. Moreover, $u \in H$ forces $L_0 = L_\infty = 0$, thanks to $\int_0^\delta \xi^{-1} \, \mathrm{d}\xi = \int_{1/\delta}^\infty \xi^{-1} \, \mathrm{d}\xi = \infty$ for any $\delta > 0$. ■

Our second trace result, Lemma A.3 below, is a simple analogue in the x-direction of Lemma A.2 above. Its proof is analogous to that of Lemma A.2 and is left to the reader; cf. A. Kufner [34].

Lemma A.3 (Another trace inequality.) *Let $\beta > 0$, $\gamma > 0$, and $\mu > 0$. Then the limits in (2.17) hold for every function $u \in V$.*

We take advantage of the trace results in Lemmas A.1 and A.2 to derive the following embedding lemma.

Lemma A.4 (A Sobolev-type inequality.) *Let $\beta > 0$, $\gamma > 0$, and $\mu > 0$. Then the following* Sobolev-type inequality *holds for every function $u \in V$,*

$$\begin{aligned} \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \le \left(\frac{2}{\mu}\right)^2 \int_{\mathbb{H}} |u_\xi(x,\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \\ + \frac{2\beta}{\mu} \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi^{\beta-1} \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \, . \end{aligned} \tag{A.11}$$

Proof. It suffices to verify the following inequality:

$$\begin{aligned} \int_0^\infty |u(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \le \left(\frac{2}{\mu}\right)^2 \int_0^\infty |u_\xi(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \\ + \frac{2\beta}{\mu} \int_0^\infty |u(\xi)|^2 \cdot \xi^{\beta-1} \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \end{aligned} \tag{A.12}$$

holds for an arbitrary function $u \in W^{1,2}_{\mathrm{loc}}(0,\infty)$ such that

$$\int_0^\infty |u_\xi(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi < \infty \quad \text{and} \tag{A.13}$$

$$\lim_{\xi\to 0+} \left(\xi^\beta \cdot |u(\xi)|^2\right) = \lim_{\xi\to\infty} \left(\xi^\beta \, \mathrm{e}^{-\mu\xi} \cdot |u(\xi)|^2\right) = 0 \, . \tag{A.14}$$

The boundary conditions in (A.14) are justified by Lemma A.1.

Indeed, we begin with the identities

$$
\begin{aligned}
\mu \int_0^\infty |u(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi &= \; - \int_0^\infty |u(\xi)|^2 \cdot \xi^\beta \, (\mathrm{e}^{-\mu\xi})_\xi \, \mathrm{d}\xi \\
&= \; - |u(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \Big|_{\xi=0}^{\xi=\infty} + \int_0^\infty \left(|u(\xi)|^2 \cdot \xi^\beta \right)_\xi \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \\
&= \int_0^\infty (|u(\xi)|^2)_\xi \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi + \beta \int_0^\infty |u(\xi)|^2 \cdot \xi^{\beta-1} \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \\
&= \int_0^\infty (u_\xi \, \bar{u} + u \, \bar{u}_\xi) \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi + \beta \int_0^\infty |u(\xi)|^2 \cdot \xi^{\beta-1} \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \,,
\end{aligned}
\tag{A.15}
$$

by the zero trace conditions (A.14). We apply Cauchy's inequality,

$$
u_\xi \, \bar{u} + u \, \bar{u}_\xi = 2 \cdot \mathfrak{Re}(u_\xi \, \bar{u}) \le 2 \cdot |u_\xi \, \bar{u}| \le \tfrac{2}{\mu} \, |u_\xi|^2 + \tfrac{\mu}{2} \, |u|^2 \,,
$$

to the integral

$$
\begin{aligned}
&\int_0^\infty (u_\xi \, \bar{u} + u \, \bar{u}_\xi) \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \\
&\le \frac{2}{\mu} \int_0^\infty |u_\xi(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi + \frac{\mu}{2} \int_0^\infty |u(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \,.
\end{aligned}
$$

We estimate the last line in (A.15) by this inequality, thus arriving at

$$
\begin{aligned}
&\mu \int_0^\infty |u(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \\
&\le \frac{2}{\mu} \int_0^\infty |u_\xi(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi + \frac{\mu}{2} \int_0^\infty |u(\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \,, \\
&+ \beta \int_0^\infty |u(\xi)|^2 \cdot \xi^{\beta-1} \, \mathrm{e}^{-\mu\xi} \, \mathrm{d}\xi \,.
\end{aligned}
$$

The desired inequality (A.12) follows.

Finally, we integrate ineq. (A.12) with u replaced by $\tilde{u} \equiv u(x, \cdot\,) \in W^{1,2}_{\mathrm{loc}}(0,\infty)$ (for almost every fixed $x \in \mathbb{R}$) with respect to the measure $\mathrm{e}^{-\gamma|x|} \, \mathrm{d}x$ over $\mathbb{R}$ to obtain ineq. (A.11). ■

Now we are ready to prove the following Hardy inequality.

Lemma A.5 (A Hardy-type inequality.) *Let $\beta > 1$, $\gamma > 0$, and $\mu > 0$. Then the following* Hardy-type inequality *holds for every function $u \in V$,*

$$
\begin{aligned}
\int_{\mathbb{H}} \left| \frac{u(x,\xi)}{\xi} \right|^2 \cdot \xi^\beta \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \le \frac{8}{(\beta-1)^2} \int_{\mathbb{H}} |u_\xi(x,\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \\
+ \frac{2\mu^2}{(\beta-1)^2} \int_{\mathbb{H}} |u(x,\xi)|^2 \cdot \xi^\beta \, \mathrm{e}^{-\gamma|x|-\mu\xi} \, \mathrm{d}x \, \mathrm{d}\xi \,.
\end{aligned}
\tag{A.16}
$$

Proof. It suffices to verify the following inequality:

$$\text{(A.17)}\qquad \begin{aligned}\int_0^\infty \left|\frac{u(\xi)}{\xi}\right|^2 \cdot \xi^\beta \cdot \mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi \le \frac{8}{(\beta-1)^2}\int_0^\infty |u_\xi(\xi)|^2\cdot\xi^\beta\cdot\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi \\ + \frac{2\mu^2}{(\beta-1)^2}\int_0^\infty |u(\xi)|^2\cdot\xi^\beta\cdot\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi\end{aligned}$$

holds for an arbitrary function $u \in W^{1,2}_{\mathrm{loc}}(0,\infty)$ such that

$$\text{(A.18)}\qquad \int_0^\infty |u_\xi(\xi)|^2\cdot\xi^\beta\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi<\infty \quad\text{and}\quad \int_0^\infty |u(\xi)|^2\cdot\xi^\beta\,\mathrm{e}^{-\mu\xi}\,\mathrm{d}\xi<\infty\,.$$

The integrability hypotheses in (A.18) are valid for u replaced by the restricted function $\tilde{u} \equiv u(x,\,\cdot\,) \in W^{1,2}_{\mathrm{loc}}(0,\infty)$ for a.e. fixed $x \in \mathbb{R}$; the first one by $u \in V$ and the second one by the previous lemma, Lemma A.4.

Inequality (A.17) is obtained easily from the standard weighted Hardy inequality ([21, Theorem 330, pp. 245–246]),

$$\text{(A.19)}\qquad \int_0^\infty \left|\frac{f(\xi)}{\xi}\right|^2\cdot\xi^\beta\,\mathrm{d}\xi \le \left(\frac{2}{\beta-1}\right)^2\int_0^\infty\left|\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|^2\cdot\xi^\beta\,\mathrm{d}\xi\,,$$

where $\beta > 1$ and $f \in W^{1,2}_{\mathrm{loc}}(0,\infty)$ satisfies $\lim_{\xi\to\infty} f(\xi) = 0$, as follows: We first replace the function f by the product $f(\xi) = u(x,\xi)\cdot\mathrm{e}^{-\mu\xi/2}$, then estimate the partial derivative

$$\begin{aligned} f'(\xi) &= \frac{\partial}{\partial\xi}\left(u(x,\xi)\cdot\mathrm{e}^{-\mu\xi/2}\right) = u_\xi(x,\xi)\cdot\mathrm{e}^{-\mu\xi/2} - \frac{\mu}{2}\,u(x,\xi)\cdot\mathrm{e}^{-\mu\xi/2} \\ &= \Big(u_\xi(x,\xi) + \frac{\mu}{2}\,u(x,\xi)\Big)\cdot\mathrm{e}^{-\mu\xi/2}\end{aligned}$$

by

$$|f'(\xi)|^2 = \left|\frac{\partial}{\partial\xi}\left(u(x,\xi\cdot\mathrm{e}^{-\mu\xi/2}\right)\right|^2 \le 2\left[|u_\xi(x,\xi)|^2 + \left(\frac{\mu}{2}\right)^2|u(x,\xi)|^2\right]\cdot\mathrm{e}^{-\mu\xi}$$

and insert it into ineq. (A.19), thus arriving at ineq. (A.17). Here, the hypothesis $f \in W^{1,2}_{\mathrm{loc}}(0,\infty)$ is satisfied, thanks to $u \in V$, whence even $\int_0^\infty |f'(\xi)|^2\cdot\xi^\beta\,\mathrm{d}\xi < \infty$, with a help from (A.18). Hypothesis $\lim_{\xi\to\infty} f(\xi) = 0$ follows from the trace result (A.3) in Lemma A.1.

The proof is completed by integrating ineq. (A.17) with u replaced by $\tilde{u} \equiv u(x,\,\cdot\,) \in W^{1,2}_{\mathrm{loc}}(0,\infty)$ (for a.e. $x \in \mathbb{R}$) with respect to the measure $\mathrm{e}^{-\gamma|x|}\,\mathrm{d}x$ over $\mathbb{R}$ to obtain ineq. (A.16). ∎

Recall that any function $u \in V = H^1(\mathbb{H};\mathfrak{w})$ satisfies the hypotheses of Lemmas A.4 and A.5 above.

Remark A.6 Owing to the Sobolev- and Hardy-type inequalities (A.11) and (A.16) proved in Lemmas A.4 and A.5, with $1 < \beta < \infty$, the following inner product defines an *equivalent norm* on the Hilbert space V:

$$(u,w)_V^\sharp \stackrel{\mathrm{def}}{=} (u,w)_V + (u,w)_V^\flat \quad \text{for } u,w \in V\,, \tag{A.20}$$

where

$$\begin{aligned} (u,w)_V^\flat &\stackrel{\mathrm{def}}{=} \int_{\mathbb{H}} \frac{u(x,\xi)}{\xi} \cdot \frac{\bar{w}(x,\xi)}{\xi} \cdot \xi \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &\quad + \int_{\mathbb{H}} u\,\bar{w} \cdot \xi \cdot \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \int_{\mathbb{H}} u\,\bar{w} \left(\xi + \frac{1}{\xi}\right) \mathfrak{w}(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \quad \text{for } u,w \in V\,. \end{aligned} \tag{A.21}$$

This fact is used in paragraphs §6.1 and §6.2. □

B Appendix: Density of entire functions in $H = L^2(\mathbb{H};\mathfrak{w})$

As we have already suggested in paragraph §7.2, we wish to approximate an arbitrary initial condition $u_0 \in H = L^2(\mathbb{H};\mathfrak{w})$ by a sequence of entire functions, $u_{0,n} : \mathbb{C}^2 \to \mathbb{C}$; $n = 1,2,3,\dots$, such that their restrictions $u_{0,n}|_{\mathbb{H}}$ to $\mathbb{H} = \mathbb{R} \times (0,\infty)$ satisfy

$$\|u_{0,n}|_{\mathbb{H}} - u_0\|_H \longrightarrow 0 \quad \text{as } n \to \infty\,.$$

Below, we construct rather simple *entire* (holomorphic) functions $u_{0,n} : \mathbb{C}^2 \to \mathbb{C}$; $n = 1,2,3,\dots$, with this property, by using standard results about *Hermite* and *Laguerre* functions. The reader is referred to the monographs by A. N. Kolmogorov and S. V. Fomin [30, Chapt. VII, §3.7, pp. 395–396] and N. N. Lebedev [35, Chapt. 4], §4.9, pp. 60–61 and §4.17, pp. 76–78, for details and proofs.

B.1 Hermite and Laguerre functions in the complex domain

In our approximation procedure below, we first take advantage of the (complex) Hilbert space $H = L^2(\mathbb{H};\mathfrak{w})$ being the *tensor product* of the Hilbert spaces $\mathfrak{H}_1 = L^2(\mathbb{R};\mathfrak{w}_1)$ and $\mathfrak{H}_2 = L^2(\mathbb{R}_+;\mathfrak{w}_2)$, with the weights

$$\mathfrak{w}_1(x) \stackrel{\mathrm{def}}{=} \mathrm{e}^{-\gamma|x|} \quad \text{and} \quad \mathfrak{w}_2(\xi) \stackrel{\mathrm{def}}{=} \xi^{\beta-1}\,\mathrm{e}^{-\mu\xi} \quad \text{for } (x,\xi) \in \mathbb{H}, \tag{B.1}$$

i.e., $H = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, as defined in M. Reed and B. Simon [40, Chapt. II, §4], pp. 49–54. All general properties of a tensor product of two Hilbert spaces that we use below can be found there. Thus, both $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are weighted Lebesgue L^2-spaces with the weighted Lebesgue measures $\mathfrak{w}_1(x)\,\mathrm{d}x$ and $\mathfrak{w}_2(x)\,\mathrm{d}\xi$, respectively.

In order to keep our approximation procedure simple, we take advantage of the density of the weighted Lebesgue L^2-spaces as follows: $L^2(\mathbb{H})$ is densely and continuously imbedded into H, $L^2(\mathbb{R})$ into $\mathfrak{H}_1$, and $L^2(\mathbb{R}_+)$ into $\mathfrak{H}_2$. This claim is an easy consequence of the fact that all weights, $\mathfrak{w}(x,\xi) = \mathfrak{w}_1(x)\cdot\mathfrak{w}_2(\xi)$, $\mathfrak{w}_1(x)$, and $\mathfrak{w}_2(\xi)$ are bounded.

We use a standard approximation method in $\mathfrak{H}_1$ by *Hermite functions*, $h(x) = p(x)\,\exp\left(-\frac{1}{2}x^2\right)$, where $p(x)$ is a polynomial obtained by a linear combination of *Hermite polynomials* $H_n(x)$; $n = 0,1,2,\dots$. We refer to N. N. Lebedev [35, §4.9, pp. 60–61] for a common definition of Hermite polynomials and their basic properties. In particular, $H_n(x)$ is a polynomial of degree $n \geq 0$ and the Hermite functions

$$h_n(x) = H_n(x)\,\exp\left(-\tfrac{1}{2}x^2\right) \quad \text{of } x\in\mathbb{R}\,; \quad n = 0,1,2,\dots\,,$$

form an orthonormal basis in $L^2(\mathbb{R})$, by N. N. Lebedev [35, §4.13, pp. 65–66]. Furthermore, an arbitrary linear combination of these functions, $h(x) = p(x)\,\exp\left(-\frac{1}{2}x^2\right)$, where $p(x)$ is a polynomial, can be extended uniquely to an entire function $\tilde{h}(z) = p(z)\,\exp\left(-\frac{1}{2}z^2\right)$ of the complex variable $z = x + \mathrm{i}y \in \mathbb{C}$. Finally, given any $r > 0$ and $\delta > 0$, there is a constant $C_{r,\delta,p} \in (0,\infty)$, depending only on r, δ, and the polynomial p, such that the following inequalities hold for all $z = x + \mathrm{i}y, z^* \in \mathbb{C}$ with $|y| \leq r$ and $|z^*| \leq \delta$:

$$\begin{aligned}
&|\tilde{h}(x+\mathrm{i}y+z^*)| = |p(x+\mathrm{i}y+z^*)|\cdot\exp\left(-\tfrac{1}{2}\cdot\mathfrak{Re}[(x+\mathrm{i}y+z^*)^2]\right)\\
&= |p(x+\mathrm{i}y+z^*)|\cdot\exp\left(-\tfrac{1}{2}\cdot\mathfrak{Re}\left[(x+\mathrm{i}y)^2 + 2\,(x+\mathrm{i}y)z^* + (z^*)^2\right]\right)\\
&\leq |p(x+\mathrm{i}y+z^*)|\cdot\exp\left(-\tfrac{1}{2}\cdot\left[x^2 - y^2 - 2\,(|x|+|y|)\,|z^*| - |z^*|^2\right]\right)\\
&\leq C_{r,\delta,p}\cdot\exp\left(-\tfrac{1}{2}\,x^2 + 2\delta\,|x|\right)\,.
\end{aligned} \tag{B.2}$$

Consequently, the square of the $L^2(\mathbb{R})$-norm of the function $x \mapsto \tilde{h}(x+\mathrm{i}y+z^*) : \mathbb{R}\to\mathbb{C}$ is uniformly bounded, provided $|y| \leq r$ and $|z^*| \leq \delta$ are satisfied:

$$\int_{-\infty}^{+\infty} |\tilde{h}(x+\mathrm{i}y+z^*)|^2\,\mathrm{d}x \leq C_{r,\delta,p}^2 \cdot \int_{-\infty}^{+\infty} \exp\left(-x^2 + 4\delta\,|x|\right)\,\mathrm{d}x \equiv \mathrm{const}_{r,\delta,p}^2 < \infty\,.$$

A Hermite polynomial based expansion has already been applied to Black-Scholes and Merton type models for European option prices, e.g., in the recent work by D. Xiu [49].

Analogously, in $\mathfrak{H}_2$ we use *Laguerre functions*, $\ell(\xi) = q(\xi)\,\exp\left(-\frac{1}{2}\xi\right)$, where $q(\xi)$ is a polynomial obtained by a linear combination of *Laguerre polynomials* $L_n(\xi)$; $n = 0,1,2,\dots$. We refer to N. N. Lebedev [35, §4.17, pp. 76–78] for a common definition of Laguerre polynomials and their basic properties. In particular, $L_n(\xi)$ is a polynomial of degree $n \geq 0$ and the Laguerre functions

$$\ell_n(\xi) = L_n(\xi)\,\exp\left(-\tfrac{1}{2}\xi\right) \quad \text{of } \xi\in\mathbb{R}_+\,; \quad n = 0,1,2,\dots\,,$$

form an orthonormal basis in $L^2(\mathbb{R}_+)$, by N. N. Lebedev [35, §4.21, pp. 83–84]. Furthermore, an arbitrary linear combination of these functions, $\ell(\xi) = q(\xi)\, \exp\left(-\frac{1}{2}\xi\right)$, where $q(\xi)$ is a polynomial, can be extended uniquely to an entire function $\tilde{\ell}(\zeta) = q(\zeta)\, \exp\left(-\frac{1}{2}\zeta\right)$ of the complex variable $\zeta = \xi(1+\mathrm{i}\omega) \in \mathbb{C}$. Finally, given any $\vartheta_v > 0$ and $\delta > 0$, there is a constant $C_{\vartheta_v,\delta,q} \in (0,\infty)$, depending only on ϑ_v, δ, and the polynomial q, such that the following inequalities hold for all $\zeta = \xi(1+\mathrm{i}\omega), \zeta^* \in \mathbb{C}$ with $\xi \in \mathbb{R}_+$, $|\arctan\omega| \le \vartheta_v$, and $|\zeta^*| \le \delta$:

$$\begin{aligned}\text{(B.3)}\qquad |\tilde{\ell}(\xi(1+\mathrm{i}\omega)+\zeta^*)| &= |q(\xi(1+\mathrm{i}\omega)+\zeta^*)| \cdot \exp\left(-\tfrac{1}{2}\cdot \mathfrak{Re}[\xi(1+\mathrm{i}\omega)+\zeta^*]\right) \\ &\le |q(\xi(1+\mathrm{i}\omega)+\zeta^*)| \cdot \exp\left(-\tfrac{1}{2}\cdot(\xi - |\zeta^*|)\right) \le C_{\vartheta_v,\delta,q}\cdot \exp\left(-\tfrac{1}{4}\,\xi\right)\,.\end{aligned}$$

Consequently, the square of the $L^2(\mathbb{R}_+)$-norm of the function $\xi \mapsto \tilde{\ell}(\xi(1+\mathrm{i}\omega)+\zeta^*) : \mathbb{R}_+ \to \mathbb{C}$ is uniformly bounded, provided $|\arctan\omega| \le \vartheta_v$ and $|\zeta^*| \le \delta$ are satisfied:

$$\int_0^{+\infty} |\tilde{\ell}(\xi(1+\mathrm{i}\omega)+\zeta^*)|^2\, \mathrm{d}\xi \le C^2_{\vartheta_v,\delta,q} \cdot \int_0^{+\infty} \exp\left(-\tfrac{1}{2}\,\xi\right)\, \mathrm{d}\xi = 2\, C^2_{\vartheta_v,\delta,q} < \infty\,.$$

Summarizing the properties of the Hermite and Laguerre functions, we observe that the product functions

$$e_{mn}(x,\xi) \stackrel{\text{def}}{=} h_m(x)\, \ell_n(\xi) \quad \text{of } (x,\xi) \in \mathbb{H}; \quad m,n = 0,1,2,\dots,$$

form an orthonormal basis in $L^2(\mathbb{H})$ ([40, Chapt. II, §4]).

B.2 Approximation of the initial conditions (Galërkin's method)

We have just shown that, given any initial condition $u_0 \in H = L^2(\mathbb{H};\mathfrak{w})$, there is a sequence of *entire* (holomorphic) functions

$$u_{0,n}(z,\zeta) = P_n(z,\zeta)\, \exp\left(-\frac{1}{2}\,(z^2+\zeta)\right), \quad (z,\zeta) \in \mathbb{C}^2; \quad n = 1,2,3,\dots,$$

with the restrictions $u_{0,n}|_{\mathbb{H}}$ in the tensor product $L^2(\mathbb{H}) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+) \hookrightarrow H = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, such that:

(i) $P_n : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial with complex coefficients.

(ii) The restrictions $u_{0,n}|_{\mathbb{H}}$ of $u_{0,n}$ to $\mathbb{H} = \mathbb{R} \times (0,\infty)$ satisfy

$$\|u_{0,n}|_{\mathbb{H}} - u_0\|_H \longrightarrow 0 \quad \text{as } n \to \infty\,.$$

(iii) There is a constant $K_n \equiv K_{P_n} \in (0,\infty)$, depending on P_n, r, and ϑ_v, $0 < r < \infty$ and $0 < \vartheta_v < \pi/2$, but independent from $y, \omega \in \mathbb{R}$ in $z = x + \mathrm{i}y$, $\zeta = \xi(1+\mathrm{i}\omega) \in \mathbb{C}$ and $z^*, \zeta^* \in \mathbb{C}$ with $|y| < r$, $|\arctan\omega| < \vartheta_v$, and $\max\{|z^*|,\, |\zeta^*|\} < \delta$, such that

$$\int_{\mathbb{H}} |u_{0,n}\,(x + \mathrm{i}y + z^*,\, \xi(1+\mathrm{i}\omega) + \zeta^*)|^2 \,\mathrm{d}x\,\mathrm{d}\xi \leq K_n \equiv \text{const} < \infty$$
$$\text{whenever } \ |y| < r, \quad |\arctan\omega| < \vartheta_v, \quad \text{and } \ \max\{|z^*|,\, |\zeta^*|\} < \delta\,.$$

An analogous estimate remains valid in the weighted Lebesgue space H if the standard Lebesgue measure $\mathrm{d}x\,\mathrm{d}v$ is replaced by the weighted Lebesgue measure $\mathfrak{w}(x,v)\,\mathrm{d}x\,\mathrm{d}v$, thanks to $0 < \mathfrak{w}(x,v) \leq \text{const} < \infty$.

Notice that the estimate in (iii) above follows from

$$\begin{aligned}
&\int_{\mathbb{H}} |u_{0,n}\,(x + \mathrm{i}y + z^*,\, \xi(1+\mathrm{i}\omega) + \zeta^*)|^2 \,\mathrm{d}x\,\mathrm{d}\xi \\
&= \int_0^\infty \int_{-\infty}^\infty |P_n\,(x + \mathrm{i}y + z^*,\, \xi(1+\mathrm{i}\omega) + \zeta^*)|^2 \\
&\qquad\qquad \times \exp\bigl(-\Re\mathfrak{e}[(x+\mathrm{i}y+z^*)^2 + \xi(1+\mathrm{i}\omega) + \zeta^*]\bigr)\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq \int_0^\infty \int_{-\infty}^\infty |P_n\,(x + \mathrm{i}y + z^*,\, \xi(1+\mathrm{i}\omega) + \zeta^*)|^2 \cdot \exp\bigl(-(x^2 - y^2) - \xi\bigr) \\
&\qquad\qquad \times \exp\bigl(2|x+\mathrm{i}y|\cdot|z^*| + |z^*|^2 + |\zeta^*|\bigr)\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq \int_0^\infty \int_{-\infty}^\infty |P_n\,(x + \mathrm{i}y + z^*,\, \xi(1+\mathrm{i}\omega) + \zeta^*)|^2 \cdot \exp\bigl(-x^2 - \xi\bigr) \\
&\qquad\qquad \times \exp\bigl(r^2 + 2(|x|+r)\delta + \delta^2 + \delta\bigr)\,\mathrm{d}x\,\mathrm{d}\xi \\
&\leq K_n \equiv \text{const} < \infty \\
&\quad \text{whenever } \ |y| < r, \quad |\arctan\omega| < \vartheta_v, \quad \text{and } \ \max\{|z^*|,\, |\zeta^*|\} < \delta\,.
\end{aligned} \tag{B.4}$$

As an obvious consequence of properties (i), (ii), and (iii) we obtain that $u_{0,n} : \mathfrak{X}^{(r)} \times \Delta_{\vartheta_v} \to \mathbb{C}$ is a holomorphic function in both its variables (z,ζ) and belongs to the Hardy space $H^2(\mathfrak{X}^{(r)} \times \Delta_{\vartheta_v})$.

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