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On Competitive Nonlinear Pricing*

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Abstract

We study a discriminatory limit-order book in which uninformed market makers compete in nonlinear tariffs to serve an informed insider. Our model allows for general nonparametric specifications of preferences and for arbitrary discrete distributions for the insider's private information. We show that adverse selection severely restricts possible equilibrium outcomes: in any pure-strategy equilibrium, tariffs must be linear and at most one type may trade, leading to an extreme form of market breakdown. As a result, such equilibria only exist under exceptional circumstances. The Bertrand-like logic underlying these results markedly differs from Cournot-like analyses of the limit-order book that postulate a continuum of types. We argue that these contrasting outcomes can be reconciled when one considers ε-equilibria of either the game with a large number of market makers or the game with a large number of insider types. Mixed-strategy equilibria, by contrast, lead to a new class of equilibrium predictions that calls for further analysis.

Keywords: Adverse Selection, Competing Mechanisms, Limit-Order Book.
JEL Classification: D43, D82, D86.

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1 Introduction

Important financial markets, such as EURONEXT or NASDAQ, rely on a discriminatory limit-order book to balance supply and demand. This book gathers the limit orders posted by market makers.[1] Any upcoming order is then matched with the best offers in the book. Pricing is discriminatory, in that each market maker gets paid at the price he has quoted for a given volume of shares.[2] This paper contributes to the study of price formation on such discriminatory markets.

An important obstacle to trade on such markets is that market makers may face an insider with superior information about the fundamental value of the traded asset. This makes them reluctant to sell, as they suspect that the fundamental value is likely to be high when the asset is in high demand. One way to alleviate this adverse-selection phenomenon is for market makers to post collections of limit orders or, equivalently, tariffs that are convex in the quantity traded. The insider then hits the resulting limit-order book with a market order that reflects her private information, paying a higher price at the margin for higher quantities.[3] The problem of price formation thus amounts to characterizing the tariffs posted by competing market makers in anticipation of the insider's trading strategy.

In a well-known article, Glosten (1994) proposed a candidate nonlinear tariff, meant to describe the limit-order book as a whole, and that can be interpreted as a marginal version of Akerlof (1970) pricing. Namely, this tariff specifies that any additional share beyond a quantity Q can be bought at a price equal to the expected value of the asset, conditional on the event that the insider buys at least Q shares. Because demand typically increases when the insider has more favorable information, this tariff is convex and yields zero expected profit to the market makers. Glosten (1994) additionally argues that this tariff is the only one robust to entry by an uninformed market maker. As acknowledged in Glosten (1998), however, a natural question is whether this tariff can be sustained in an equilibrium of a competitive game with strategic market makers.

This issue was first addressed in Biais, Martimort, and Rochet (2000) in a model in which a risk-averse insider with private but imperfect information about the fundamental value of an asset may trade for informational or hedging purposes. When the insider's marginal valuation for the asset—an aggregate of the insider's informational and hedging motivations to trade—is continuously distributed, they exhibit a unique pure-strategy equilibrium with strictly convex tariffs, which may be interpreted as infinite collections of infinitesimal limit orders. The equilibrium outcome is reminiscent of Cournot competition: with a finite number of market makers, the equilibrium is symmetric and each market maker earns a positive

[1]A limit order allows to trade at a specified price any quantity up to a specified limit.

[2]By contrast, uniform limit-order books compute a single price that balances supply and demand and that applies to all trades that are matched. The appropriate modeling tool is supply-function equilibria, see, for instance, Wilson (1979), Grossman (1981), Klemperer and Meyer (1989), Kyle (1989), and Vives (2011).

[3]This differs from uniform-price auctions, in which both informed and uninformed traders simultaneously post supply functions. Note also the difference with Treasury-bill auctions, for which it is typically assumed that the bidders are the holders of private information (Wilson (1979), Back and Zender (1993)).

expected profit. In the limit when the number of market makers grows large, the equilibrium aggregate tariff converges to the Glosten (1994) tariff. Back and Baruch (2013) complement this study by focusing on a slightly different game in which market makers are restricted to post convex tariffs from the outset. Focusing on symmetric equilibria with strictly convex tariffs, they identify the same equilibrium tariff as Biais, Martimort, and Rochet (2000). These results have been interpreted as a strategic foundation to Glosten's (1994) original approach, in the spirit of Cournotian foundations of competitive equilibrium (Parlour and Seppi (2008), Vayanos and Wang (2011)).

However, the existence of such equilibria is not always guaranteed. In a clarifying note, Biais, Martimort, and Rochet (2013) acknowledge that their existence result requires several restrictions on the distribution of the insider's marginal valuation for the asset and on the expected value of the asset conditional on this marginal valuation. Back and Baruch (2013) provide an alternative set of sufficient conditions and emphasize that existence obtains only if the adverse selection problem is severe enough. This contrasts with the generality of Glosten's (1994) construction and thus raises the question of how robust the findings of this literature are to specifications of the model.

In this paper, we address this question by setting up a general model of the discriminatory limit-order book in which a privately informed insider can trade with several market makers. Our objective in proposing and studying this model is to uncover the common strategic structure of many trade situations, without making specific assumptions on, for instance, the source of the gains from trade. In line with this objective, a novel feature of our analysis is to allow for general nonparametric specifications of preferences. In particular, some of our results remain valid when the insider's preferences exhibit wealth effects, or when the market makers have strictly convex order-handling costs. We also depart from the existing literature by assuming that the insider's private information, or type, can take an arbitrary but finite number of values. Beyond discreteness, we impose no restriction on the distribution of types. Market makers compete to serve the insider by posting tariffs. To capture the functioning of a discriminatory limit-order book, we focus on equilibria in which market makers post convex tariffs on the equilibrium path. As for deviations, we consider two scenarios. In a first version of the game, the *arbitrary-tariff game*, market makers can post arbitrary tariffs, as in Biais, Martimort, and Rochet (2000). This represents a situation in which side trades can take place outside the book. In a second version of the game, the *convex-tariff game*, market makers can only post convex tariffs, as in Back and Baruch (2013). This represents a situation in which all trades must take place through the book.

The properties of equilibria in our discrete-type setting are strikingly different from those obtained in the continuous-type models studied so far in the literature. First, any pure-strategy equilibrium with convex tariffs actually features linear tariffs, at odds with the equilibria with strictly convex tariffs that emerges in continuous-type models. Second, such linear equilibria essentially exist only in the knife-edge Bertrand case where there is no adverse selection and market makers have a constant unit cost of serving demand. In all other cases, pure-strategy equilibria typically fail to exist when there are sufficiently many types. Indeed, any candidate equilibrium is such that either a single insider type trades in equilibrium or, if several insider types trade, all do so at the same marginal cost for the market makers, a property that is increasingly difficult to satisfy when the number of types gets large. These results hold irrespective of the distribution of types as long as it remains discrete, both in the arbitrary-tariff game and, in the absence of wealth effects, in the convex-tariff game.

To formally establish these results, we proceed throughout by necessary conditions. That is, we fix a pure-strategy equilibrium with convex tariffs of either game, assuming that such an equilibrium exists, and we investigate its properties. The logical structure of our argument can be described as follows.

1. We first notice that, fixing the convex tariffs posted in equilibrium by all but one market makers, the preferences of the insider over the trades she can conduct with the remaining market maker are fairly regular. In particular, the corresponding indirect utility functions for the different insider types satisfy weak quasiconcavity and weak single-crossing properties. This suggests that the best response of the market maker in question can be analyzed using standard mechanism-design techniques (in the arbitrary-tariff game) or standard price-theory arguments (in the convex-tariff game).

2. The weak single-crossing property does not ensure that the quantity traded by the insider with a given market maker is nondecreasing in her type. Still, the insider always has a best response such that higher types trade no less than lower types with this market maker. We show that this implies that the latter can break ties in his favor as long as this property is satisfied. Applying this tie-breaking result to all market makers simultaneously then has the striking consequence that any pure-strategy equilibrium with nondecreasing individual quantities must feature linear pricing. This linear-pricing result requires very little structure on the market makers' preferences and hence holds quite generally.

3. The previous step of the argument prompts us to first focus on equilibria with linear tariffs and nondecreasing individual quantities. The main result is that, except when there is no adverse selection and market makers have a constant and type-independent unit cost of serving demand, such equilibria exhibit an extreme form of market breakdown. For instance, if the cost of serving demand is strictly increasing in the insider's type, only the highest type can trade because each market maker has an incentive to reduce his supply at the equilibrium price by posting an appropriate limit order. It follows that, under this pure adverse-selection scenario, equilibria with linear tariffs and nondecreasing individual quantities exist only under exceptional circumstances. Indeed, all types except the highest one must not be willing to trade at the equilibrium price, which is unlikely when some types have preferences close to those of the highest one. This establishes our main results for the special case of pure-strategy equilibria with nondecreasing individual quantities.

4. Up to this point, our argument has focused on a subclass of equilibria, hence leaving open the possibility that equilibria that do not feature nondecreasing individual quantities may exhibit very different properties. To complete the argument, we show that the restriction to nondecreasing individual quantities is actually innocuous. Specifically, we prove that, for a large class of preferences for the market makers, any pure-strategy equilibrium with convex tariffs can be turned into an other equilibrium with the same tariffs and the same expected profits for the market makers, but now with nondecreasing individual quantities. Key to this result is that, for a given profile of convex tariffs, allocations with nondecreasing individual quantities achieve efficient risk-sharing between market makers.

The upshot of this analysis is, therefore, that the structure of pure-strategy equilibria in arbitrary discrete-type models, when they exist, is strikingly different from that arising in the continuous-type models of Biais, Martimort, and Rochet (2000) and Back and Baruch (2013). It is also different from that of the Glosten (1994) allocation, except in the degenerate case where only the highest insider type trades in this allocation. We provide for the convex-tariff game necessary and sufficient conditions for the existence of pure-strategy equilibria; these conditions fail to hold when one attempts to approximate a continuous type set by an increasing sequence of discrete type sets. It should be noted, incidentally, that the resulting lack of lower hemicontinuity of the pure-strategy-equilibrium correspondence is intrinsically tied to the nonexclusivity of competition. Indeed, in exclusive-competition models of adverse selection, such as Rothschild and Stiglitz's (1976), lower hemicontinuity is vacuously satisfied: as in our model, necessary conditions for the existence of pure-strategy equilibria become increasingly restrictive when one increases the number of types, but in the continuous-type limit no pure-strategy equilibrium exists either (Riley (1985, 2001)).

To overcome this tension between discrete- and continuous-type models, we explore three alternative avenues of research, corresponding to different ways of relaxing the equilibrium concept. As in Glosten (1994), Biais, Martimort, and Rochet (2000), and Back and Baruch (2013), we focus on the case where the insider has quasilinear utility and the market makers have linear costs.

We first examine what happens when the number K of market makers grows large, holding fixed the number of types. We show that there exists an ε-equilibrium of either game, with ε of the order of $1/K^2$, that implements the Glosten (1994) allocation. The intuition is that, if $K-1$ market makers each contribute to provide a fraction $1/K$ of the Glosten (1994) tariff, the resulting aggregate tariff is almost entry-proof from the perspective of the remaining market maker. Once again, this stands in sharp contrast with exclusive-competition models, in which a pure-strategy equilibrium fails to exist precisely when the Rothschild and Stiglitz (1976) allocation is not entry-proof, independently of the number of competing firms. By contrast, the Glosten (1994) allocation is always entry-proof, which allows one to construct approximate equilibria when there are many market makers.

We then explore the dual scenario in which the number I of types grows large, so as to approximate a continuous type set, holding fixed the number of market makers. We show that there exists an ε-equilibrium of either game, with ε of the order of $1/I$, that implements the Biais, Martimort, and Rochet (2000) allocation. The intuition is that, if $K-1$ market makers each post the strictly convex tariff that arises in the symmetric equilibrium they characterize, the same tariff is an approximate best response for the remaining market maker in the discretized model when the number of types is large enough. Mathematically, this is because the equilibrium payoff in the continuous-type model can be, up to terms of order of $1/I$, approximated in a Riemann sense by corresponding payoffs in the discretized model. Thus, whereas no sequence of exact equilibria of discrete-type models converges to the Biais, Martimort, and Rochet (2000) equilibrium, convergence obtains when one extends attention to approximate equilibria.

We finally offer a preliminary exploration of mixed-strategy equilibria. Whereas a general characterization of such equilibria seems hard to obtain,[4] we explicitly construct a mixed-strategy equilibrium of the two-type convex-tariff game in the case where the necessary and sufficient conditions for the existence of a pure-strategy equilibrium do not hold. This equilibrium is in the spirit of equilibria of Bertrand–Edgeworth oligopoly (Allen and Hellwig (1986)), with the added difficulties that market makers face adverse selection and that the insider can strategically combine their offers. In equilibrium, each market maker posts a random limit order that serves the demand of the low-cost type at a price between the average cost of serving both types and the cost of serving the high-cost type; on top of this, he offers to sell any quantity at the high cost. Posting a limit order with a price close to the average cost allows to sell a larger quantity to the low-cost type; however, doing so also increases the losses incurred with the high-cost type. By contrast, posting a limit order with a price close to the high cost limits the latter losses, but also reduces the quantity traded with the low-cost type. The uncertainty about the price charged by the other market makers induces a perfect balance between the desire to raise the price in order to avoid making losses with the high-cost type and the desire to lower the price in order to attract the low-cost type. Unlike in Biais, Martimort, and Rochet (2000), each market maker earns zero expected profit in equilibrium. Interestingly, the resulting distribution over aggregate equilibrium outcomes does not necessarily converge to the Glosten (1994) allocation when the number of market makers goes to infinity.

The paper is organized as follows. Section 2 describes the model. Section 3 states our main results. Section 4 establishes that pure-strategy equilibria with convex tariffs and nondecreasing individual quantities feature linear pricing. Section 5 shows that such equilibria only exist in exceptional cases when there is adverse selection or when market makers have strictly convex costs. Section 6 extends these results to all equilibria with

[4]It should be noted, in this respect, that a complete characterization of mixed-strategy equilibria is not available even for the conceptually simpler exclusive-competition model. The most comprehensive results in this context are provided by Farinha Luz (2016), who shows that the two-type insurance model of Rothschild and Stiglitz (1976) admits a unique symmetric equilibrium.

convex tariffs, completing the proof of our main results. Section 7 offers necessary and sufficient conditions for the existence of a pure-strategy equilibrium. Section 8 discusses the interpretation of our results and their relationship to the literature. Section 9 discusses alternative avenues of research, relaxing the equilibrium concept. Section 10 concludes. Proofs not given in the text can be found in the Appendix.

2 The Model

Our model features a privately informed insider who can purchase nonnegative amounts of an asset from several market makers. Shares are homogeneous, so that the insider only cares about her aggregate trade. Unless otherwise stated, we allow for general nonparametric payoff functions and arbitrary discrete distributions for the insider's type.

2.1 The Insider

The insider (she) is privately informed of her preferences. Her type i can take a finite number $I \geq 1$ of values with positive probabilities m_i such that $\sum_i m_i = 1$. Each insider type only cares about the aggregate quantity $Q \geq 0$ she purchases from the market makers and the aggregate transfer T she makes in return.[5] Type i's preferences over aggregate quantity-transfer bundles $(Q, T) \in \mathbb{R}_+ \times \mathbb{R}$ are represented by a utility function $U_i(Q, T)$ that is continuous and strictly quasiconcave in (Q, T) and strictly decreasing in T. The following strict single-crossing property is the main determinant of the insider's behavior.

Assumption SC-$\boldsymbol{U}$ *For all $i < i'$, $Q < Q'$, T, and T',*

$$U_i(Q, T) \leq U_i(Q', T') \ \textit{ implies } \ U_{i'}(Q, T) < U_{i'}(Q', T').$$

In words, a higher type is more eager to increase her purchases than lower types are. As an illustration, consider the demand of type i at price p,

$$D_i(p) \equiv \arg\max\{U_i(Q, pQ) : Q \in \mathbb{R}_+ \cup \{\infty\}\}.$$

The continuity and strict quasiconcavity of U_i ensure that $D_i(p)$ is uniquely defined and continuous in p. Moreover, Assumption SC-U implies that, for each p, $D_i(p)$ is nondecreasing in i. To avoid discussing knife-edge cases involving kinks, we strengthen this property by requiring that demand be strictly increasing in the insider's type as long as it remains finite.

Assumption ID-$\boldsymbol{U}$ *For all $i < i'$ and $p \in \mathbb{R}$,*

$$0 < D_i(p) < \infty \ \textit{ implies } \ D_i(p) < D_{i'}(p).$$

[5]In the limit-order-book interpretation of our model, we thus focus on the ask side of the book, in line with Back and Baruch (2013) and Biais, Martimort, and Rochet (2013).

A sufficient condition for Assumptions SC-U and ID-U to hold is that the marginal rate of substitution $MRS_i(Q,T)$ of shares for transfers be well defined and strictly increasing in i for all (Q,T). Assumptions SC-U and ID-U are maintained throughout the paper.

Some of our results are valid for such general utility functions for the insider, allowing for risk aversion and wealth effects (Theorem 1). Others rely on quasilinearity (Theorem 2), though not on any particular parametrization of the insider's utility function. The corresponding assumption is as follows.

Assumption QL-U *The insider has quasilinear utility $U_i(Q,T) = u_i(Q) - T$, where $u_i(Q)$ is differentiable and strictly concave in Q.*

Under this additional assumption, Assumptions SC-U and ID-U only require that the derivative $u_i'(Q)$ be increasing in i for all Q. For instance, in Biais, Martimort, and Rochet (2000), $U_i(Q,T) = \theta_i Q - (\alpha\sigma^2/2)Q^2 - T$, reflecting that the insider has CARA utility with absolute risk-aversion parameter α and faces residual Gaussian risk with variance σ^2. Assumptions SC-U and ID-U then hold if θ_i is increasing in i. In this case, as in Glosten (1994) and Back and Baruch (2013), the insider's demand function is independent of her wealth, so that Assumption QL-U holds.

2.2 The Market Makers

There are $K \geq 2$ market makers. Each market maker (he) only cares about the quantity $q \geq 0$ he provides the insider with and the transfer t he receives in return. Such pair (q,t) we call a *trade*. Market maker k's preferences over trades with type i are represented by a profit function $v_i^k(q,t)$ that is continuous and weakly quasiconcave in (q,t) and strictly increasing in t. Note that this profit can depend on the insider's type, a common-value case that has received a lot of attention in the market-microstructure literature (Glosten and Milgrom (1985), Kyle (1985), Glosten (1994)). This contrasts with the private-value case, in which the market maker's profit does not depend on i. To allow for both cases, we only require that each market maker weakly prefer to sell lower quantities to higher types.

Assumption SC-v *For all k, $i < i'$, $q < q'$, t, and t',*

$$v_i^k(q,t) \geq v_i^k(q',t') \text{ implies } v_{i'}^k(q,t) \geq v_{i'}^k(q',t').$$

According to Assumptions SC-U and SC-v, an insider with a higher type is willing to buy more shares but faces market makers who are weakly more reluctant to sell. Our model thus typically features adverse selection, with private values as a limiting case.

The assumptions we impose at this stage on the market makers' profit functions are very general, allowing for risk aversion and inventory costs (Stoll (1978), Ho and Stoll (1981, 1983)). Whereas some of our results are valid for such general profit functions, others require more structure, typically in the form of symmetry and quasilinearity (Theorems 1–2). First, one may follow Glosten (1994), Biais, Martimort, and Rochet (2000), and Back and Baruch (2013), and assume that market makers have identical linear profit functions.

Assumption L-v *For each i, each market maker k earns a profit $v_i^k(q,t) = t - c_i q$ when he trades (q,t) with type i, where c_i is the unit cost of serving type i.*

Here, the market makers are assumed to be risk-neutral and c_i may be thought of as the liquidation value of the asset when the insider is of type i. Assumption SC-v then amounts to imposing that $c_{i'} \geq c_i$ when $i' > i$. Alternatively, one may follow Roll (1984) and assume that each market maker incurs a strictly increasing and strictly convex order-handling cost when selling shares.

Assumption C-v *For each i, each market maker k earns a profit $v_i^k(q,t) = t - c_i(q)$ when he trades (q,t) with type i, where the cost $c_i(q)$ is strictly convex in q, with $c_i(0) = 0$.*

Assumption SC-v then amounts to imposing that $\partial^- c_{i'}(q') \geq \partial^+ c_i(q)$ when $i' > i$ and $q' > q$.[6] Assumption C-v generalizes Roll (1984) by allowing for both order-handling and adverse-selection costs.

We shall state our main results for the symmetric case where the market makers' profit functions satisfy Assumption L-v or Assumption C-v. We will, however, indicate in the course of the formal analysis to which extent some our results can be extended to more general, possibly asymmetric profit functions.

2.3 Timing and Strategies

The game unfolds as follows:

1. The market makers $k = 1, \ldots, K$ simultaneously post tariffs t^k. Each tariff t^k is defined over a domain $A^k \subset \mathbb{R}_+$ that contains 0 and is such that $t^k(0) = 0$.

2. After privately learning her type, the insider purchases a quantity $q^k \in A^k$ from each market maker k, for which she pays in total $\sum_k t^k(q^k)$.

A pure strategy s for the insider maps any tariff profile $(t^1, \ldots, t^K)$ and any type i into a quantity profile $(q^1, \ldots, q^K)$. To ensure that type i's problem

$$\max \left\{ U_i \left(\sum_k q^k, \sum_k t^k(q^k) \right) : (q^1, \ldots, q^K) \in A^1 \times \cdots \times A^K \right\} \tag{1}$$

always has a solution, we require that the domains A^k be compact and that each tariff t^k be lower semicontinuous over A^k. These requirements are light enough to allow market makers

[6]For any convex function f defined over a convex subset of $\mathbb{R}$, we use the notation $\partial f(x)$, $\partial^- f(x)$, and $\partial^+ f(x)$ to denote the subdifferential of f at x, the minimum element of $\partial f(x)$, and the maximum element of $\partial f(x)$, respectively. Hence $\partial f(x) = [\partial^- f(x), \partial^+ f(x)]$.

to offer menus $\mu \equiv \{(0,0),\dots,(q_i,t_i),\dots\}$ containing a finite number of trades, including the null trade $(0,0)$.

We call the above game the *arbitrary-tariff game*. In this game, market makers can post arbitrary tariffs, as in Biais, Martimort, and Rochet (2000) and Attar, Mariotti, and Salanié (2011, 2014). It is also interesting to study the *convex-tariff game*, in which market makers can only post convex tariffs, as in Back and Baruch (2013). That is, it is required of any admissible strategy for market maker k that the domain A^k be a compact interval containing 0 and that the tariff t^k be convex over A^k. Any such tariff can then be interpreted as a collection of limit orders posted by market maker k.

2.4 Equilibria with Convex Tariffs

We focus until Section 9 on pure-strategy perfect-Bayesian equilibria $(t^1,\dots,t^K,s)$ with convex tariffs t^k. This last restriction is hardwired in the market makers' strategy spaces in the convex-tariff game, while it is a constraint on equilibrium strategies in the arbitrary-tariff game. The focus on convex tariffs intends to describe an idealized discriminatory limit-order book in which market makers post limit orders, or collections of limit orders. Such instruments are known to have nice efficiency properties under complete information.[7] It is thus natural to ask whether they perform as well under adverse selection. In line with this interpretation, characterizing equilibria with convex tariffs of the arbitrary-tariff game amounts to studying the robustness of the book to side trades that may take place in the dark, outside the book (Theorem 1); by contrast, characterizing equilibria of the convex-tariff game amounts to studying the inherent stability of the book (Theorem 2). We perform the latter exercise under stronger assumptions than the former, so that the two sets of results are not nested.

The focus on equilibria with convex tariffs also ensures that, on the equilibrium path, the insider's preferences over collections of individual trades are well behaved, as we now show. Recall first that convexity of tariffs is preserved under aggregation. In particular, the minimum aggregate transfer the insider has to make in return for an aggregate quantity Q,

$$T(Q) \equiv \min\left\{\sum_k t^k(q^k) : q^k \in A^k \text{ for all } k \text{ and } \sum_k q^k = Q\right\}, \tag{2}$$

is convex in Q in equilibrium.[8] As a consequence, and because the utility functions U_i are strictly quasiconcave, any type i has a uniquely determined aggregate equilibrium demand Q_i, which is nondecreasing in i under Assumption SC-U. Similarly, if the insider wishes to trade an aggregate quantity $Q^{-k} \in \sum_{k'\neq k} A^{k'}$ with the market makers other than k, the

[7] Biais, Foucault, and Salanié (1998) show in the single-type case that equilibria of the game with convex tariffs exist and are efficient (see also Dubey (1982)). A difference with our setting, though, is that they assume that the insider's demand for shares is perfectly inelastic.

[8] Formally, T is the *infimal convolution* of the individual tariffs t^k posted by the market makers (see Rockafellar (1970, Theorem 5.4)). Note that $\sum_k A^k = [0, \sum_k \max A^k]$ when the tariffs t^k are convex.

minimum transfer she has to make in return is

$$T^{-k}(Q^{-k}) \equiv \min\left\{\sum_{k'\neq k} t^{k'}(q^{k'}) : q^{k'} \in A^{k'} \text{ for all } k' \neq k \text{ and } \sum_{k'\neq k} q^{k'} = Q^{-k}\right\}$$

and once more the aggregate tariff T^{-k} is convex in equilibrium. In turn, each type i evaluates any trade (q, t) she may make with market maker k through the indirect utility function

$$z_i^{-k}(q,t) \equiv \max\left\{U_i(q + Q^{-k}, t + T^{-k}(Q^{-k})) : Q^{-k} \in \sum_{k'\neq k} A^{k'}\right\}. \tag{3}$$

Observe that the maximum in (3) is always attained and that $z_i^{-k}(q, t)$ is strictly decreasing in t and continuous in (q, t).[9] The convexity of the tariff T^{-k} and the quasiconcavity of the utility function U_i imply that $z_i^{-k}(q, t)$ is weakly quasiconcave in (q, t). Moreover, the convexity of the tariffs T^{-k} and Assumption SC-U together imply that the family of functions z_i^{-k} satisfy the following weak single-crossing property.

Property SC-z *For all k, $i < i'$, $q \leq q'$, t, and t',*

$$z_i^{-k}(q,t) \leq z_i^{-k}(q',t') \quad \textit{implies} \quad z_{i'}^{-k}(q,t) \leq z_{i'}^{-k}(q',t'), \tag{4}$$

$$z_i^{-k}(q,t) < z_i^{-k}(q',t') \quad \textit{implies} \quad z_{i'}^{-k}(q,t) < z_{i'}^{-k}(q',t'). \tag{5}$$

Our focus on equilibria with convex tariffs thus ensures that the indirect utility functions z_i^{-k} satisfy regularity properties that they inherit from the primitive utility functions U_i.[10] It should be noted, however, that the functions z_i^{-k} satisfy quasiconcavity and single-crossing only in a weak sense, unlike the functions U_i. For instance, if all market makers offer to sell any quantity at the same unit price p, then some insider type may be indifferent between different trades with any given market maker. Our analysis will pay a particular attention to such ties.

3 The Main Results

Our central theorems state necessary conditions for equilibria in convex tariffs.

Theorem 1 *Suppose that the arbitrary-tariff game has a pure-strategy equilibrium with convex tariffs. Then, in any such equilibrium,*

(i) If market makers have linear costs (Assumption L-v), all trades take place at a constant unit price equal to the highest cost c_I. Each type i trades $D_i(c_I)$ and all types who trade have the same unit cost $c_i = c_I$.

[9] The last statement follows from Berge's maximum theorem (Aliprantis and Border (2006, Theorem 17.31)).

[10] This contrasts with the analysis in Attar, Mariotti, and Salanié (2011), where the presence of a capacity constraint and the absence of restrictions on equilibrium menus could result in indirect utility functions that were discontinuous and did not satisfy any single-crossing property.

(ii) *If market makers have strictly convex costs (Assumption C-v), all trades take place at a constant unit price p. Only type I may trade. If* $D_I(p) > 0$, *then* $p \in \partial c_I(D_I(p)/K)$ *and all market makers trade the same quantity* $D_I(p)/K$ *with type I.*

Theorem 2 *Suppose that the convex-tariff game has a pure-strategy equilibrium. Then, in any such equilibrium, statement (i) in Theorem 1 holds if the insider has quasilinear utility (Assumption QL-U).*

We prove these two theorems in Sections 4–6. To do so, we presuppose the existence of an equilibrium $(t^1, \ldots, t^K, s)$ with convex tariffs of either game and we investigate its properties. In the arbitrary-tariff game, this equilibrium should be robust to deviations by market makers to arbitrary tariffs, whereas in the convex-tariff game, it should only be robust to deviations by market makers to convex tariffs.

4 The Linear-Pricing Result

In this section, we prove the linear-pricing result for equilibria in which the quantity q_i^k traded by the insider with any market maker k is nondecreasing in her type i. Such equilibria we call *equilibria with nondecreasing individual quantities.* This first step is motivated by the fact that, under Assumption SC-U, aggregate quantities traded in equilibrium cannot decrease with the insider's type. Section 6 extends the linear-pricing result to all equilibria, showing that the restriction to nondecreasing individual quantities involves no loss of generality.

4.1 The Arbitrary-Tariff Game

We first consider the arbitrary-tariff game, in line with Biais, Martimort, and Rochet (2000). We start with a tie-breaking lemma that gives a lower bound for each market maker's equilibrium expected profit, given the tariffs posted by his opponents. We then use this lemma to establish our linear-pricing result.

4.1.1 How the Market Makers Can Break Ties

Consider an equilibrium $(t^1, \ldots, t^K, s)$ with convex tariffs of the arbitrary-tariff game, and suppose that market maker k deviates to a menu $\{(0, 0), \ldots, (q_i, t_i), \ldots\}$ designed so that type i selects the alternative (q_i, t_i). For this to be the case, it must be that the following incentive-compatibility and individual-rationality constraints hold for any types i and i':

$$z_i^{-k}(q_i, t_i) \geq z_i^{-k}(q_{i'}, t_{i'}), \tag{6}$$

$$z_i^{-k}(q_i, t_i) \geq z_i^{-k}(0, 0). \tag{7}$$

These constraints are formulated in terms of the insider's indirect utility functions, which are endogenous objects. Fortunately, under Property SC-z, we only need to consider a subset of these constraints. Specifically, we focus on the *downward local constraints*

$$z_i^{-k}(q_i, t_i) \geq z_i^{-k}(q_{i-1}, t_{i-1}) \tag{8}$$

for all i, where $(q_0, t_0) \equiv (0, 0)$ by convention to handle the individual-rationality constraint of type 1. Clearly, these constraints are not sufficient to ensure that each type i will choose to trade (q_i, t_i) after market maker k's deviation. Indeed, local upward incentive constraints need not hold. More importantly, a given type may be indifferent between two trades, thus creating some ties. Nevertheless, as we shall now see, as long as he sticks to menus with nondecreasing quantities, market maker k can secure the expected profit he would obtain if he could break such ties in his favor. Define

$$V^k(t^{-k}) \equiv \sup \left\{ \sum_i m_i v_i^k(q_i, t_i) \right\} \tag{9}$$

over all menus $\{(0, 0), \ldots, (q_i, t_i), \ldots\}$ that satisfy (8) for all i and that have nondecreasing quantities, that is, $q_{i+1} \geq q_i$ for all $i < I$.

Lemma 1 *In any pure-strategy equilibrium $(t^1, \ldots, t^K, s)$ with convex tariffs of the arbitrary-tariff game, market maker k's expected profit is at least $V^k(t^{-k})$.*

The idea here is that, from any menu verifying the constraints in (9), one can play both with transfers (which can be increased if (8) does not bind) and with quantities (so as to avoid cycles of binding incentive-compatibility constraints) to build another menu with no lower payoffs verifying (6)–(7). In the absence of cycles, transfers can then be slightly perturbed to make these constraints strict inequalities, which ensures that the insider has a unique best response. It should be noted that this result only relies on Assumptions SC-U and SC-v. In particular, market makers need not have symmetric, quasilinear, or even quasiconcave profit functions.

4.1.2 Equilibria with Nondecreasing Individual Quantities

The above tie-breaking lemma suggests that we first focus on equilibria with nondecreasing individual quantities, that is, $q_{i+1}^k \geq q_i^k$ for all k and $i < I$. Suppose, therefore, that such an equilibrium exists. The equilibrium trades of market maker k then verify all the constraints in (9). An immediate consequence of Lemma 1 is thus that these trades must be solution to (9). Because the functions z_i^{-k} are strictly decreasing in transfers and weakly quasiconcave, it follows in turn that all downward local constraints (8) must bind. This standard result turns out to be very demanding when equilibrium tariffs are convex. Indeed, consider an insider type who exhausts aggregate supply at some price p. When facing a given market maker, this type never wants to mimic another type who does not exhaust this market maker's supply at price p, because she would end up paying too much to get her aggregate quantity. In these circumstances, one may wonder how to build a chain of binding downward local constraints that goes all the way down to the null trade.

Let us make this point more formally. Because equilibrium tariffs are convex, one can define $\overline{s}^k(p)$ as the quantity supplied at price p by market maker k and $\overline{S}(p)$ as the aggregate quantity offered by the market makers at this price.[11] Now, suppose that there exist i and p such that the quantity Q_i traded by type i is no less than $\overline{S}(p)$ and that the latter aggregate supply is positive:

$$Q_i \geq \overline{S}(p) > 0.$$

For this value of p, consider the smallest such i. Because type i has convex preferences and exhausts aggregate supply at price p, any of her best responses must be such that she trades at least $\overline{s}^k(p)$ with each market maker k. Because the downward local constraints for type i must bind for all k, two cases may arise:

(i) Either $i > 1$. Then, i being the smallest type such that $Q_i \geq \overline{S}(p)$, one must have $Q_{i-1} < \overline{S}(p)$ and thus $q^k_{i-1} < \overline{s}^k(p)$ for some market maker k. But then the individual-rationality constraint (8) of type i cannot bind for this market maker, a contradiction.

(ii) Or $i = 1$. But then, because at least one market maker k offers $\overline{s}^k(p) > 0$ at price p, the individual-rationality constraint (8) of type 1 cannot bind for this market maker, once again a contradiction.

This shows that, for any price p at which aggregate supply is positive, all types must trade an aggregate quantity below this level: $Q_i < \overline{S}(p)$ for all i if $\overline{S}(p) > 0$. Because $\overline{S}$ is right-continuous, we can safely consider the infimum of the set of such prices; call it again p. At price p, either aggregate supply is zero and there is no trade; or aggregate supply is positive and the insider faces a linear tariff with slope p. Because Q_i is then strictly less than $\overline{S}(p)$ for all i, each type i must trade $D_i(p)$ in the aggregate. We, therefore, have established the following result.

Proposition 1 *In any pure-strategy equilibrium with convex tariffs and nondecreasing individual quantities of the arbitrary-tariff game, there exists a price p such that all trades take place at unit price p and each type i purchases $D_i(p)$ in the aggregate.*

The upshot of Proposition 1 is that the possibility of side trades leads to linear pricing. This shows the disciplining role of competition in our model: although market makers can post arbitrary tariffs, they all end up trading at the same price. The role of binding downward local constraints is graphically clear, as illustrated in Figure 1: when such a constraint binds for type i and market maker k, market maker k's equilibrium tariff must be linear

[11] Note that p is a marginal, or limit price. When market maker k posts a convex tariff t^k, his supply correspondence is the inverse of the subdifferential of t^k (Biais, Martimort, and Rochet (2000, Definition 2)): for each $p \in \mathbb{R}$, the supply of market maker k at the marginal price p is the set $\{q : p \in \partial t^k(q)\}$. This set is a nonempty compact interval with lower and upper bounds $\underline{s}^k(p)$ and $\overline{s}^k(p)$ that are nondecreasing in p. When this interval is nontrivial, t^k is affine over it, with slope p. We let $\underline{S}(p) \equiv \sum_k \underline{s}^k(p)$ and $\overline{S}(p) \equiv \sum_k \overline{s}^k(p)$. Observe finally that $\overline{s}^k$ is right-continuous for all k and that $\overline{S}$ inherits this property.

over $[q_{i-1}^k, q_i^k]$, because z_i^{-k} represents convex preferences. But considering a market maker in isolation is not enough: as the above argument makes clear, it is because all type i's downward local constraints must simultaneously bind for all market makers that the linear-pricing result holds.

This result is quite general: as pointed out in our discussion of Lemma 1, we need not postulate that the market makers have symmetric, quasilinear, or even quasiconcave profit functions. This result also markedly differs from those obtained in the continuous-type case by Biais, Martimort, and Rochet (2000), who show that an equilibrium with strictly convex tariffs and nondecreasing individual quantities exists under certain conditions on players' valuations and distribution functions.

4.2 The Convex-Tariff Game

The above analysis relies on the market makers' ability to post arbitrary tariffs, including finite menus of trades. One may thus wonder whether this does not give them too much freedom to deviate, and ultimately drives the linear-pricing result. To examine this question, we now consider the convex-tariff game, in line with Back and Baruch (2013). We conduct the analysis under two additional assumptions. First, we assume that each insider type has quasilinear utility (Assumption QL-U). Second, we assume that market makers have linear costs (Assumption L-v). These assumptions are not without loss of generality as they exclude wealth effects and insurance considerations. Yet they are general enough to encompass prominent examples studied in the literature, such as the CARA-Gaussian example studied in Back and Baruch (2013, Example 1).

Focusing on convex tariffs has two main advantages. First, it allows us to rely on simple tools such as supply functions and first-order conditions, the properties of which are well known under convexity assumptions. This contrasts with using arbitrary menus, with their cohort of incentive-compatibility constraints, and makes for more intuitive proofs. (Some of our arguments are in fact quite direct when considering figures.) Second, compared to the arbitrary-tariff game, we reduce the set of deviations available to the market makers. This can a priori only enlarge the set of equilibria. In spite of this, we shall derive for the convex-tariff game a linear-pricing result similar to Proposition 1. The structure of the argument parallels that in Section 4.1: we first establish a tie-breaking lemma, which we then use to establish our linear-pricing result.

4.2.1 How the Market Makers Can Break Ties

We first reformulate Lemma 1. Consider an equilibrium $(t^1, \ldots, t^K, s)$. Suppose that market maker k deviates to a convex tariff t with domain A. For type i to select the quantity q_i in this tariff, it must be that

$$q_i \in \arg\max \{z_i^{-k}(q, t(q)) : q \in A\}. \tag{10}$$

This constraint is not sufficient to ensure that type i will choose to purchase q_i from market maker k after the deviation. Indeed, type i may be indifferent between two quantities in the tariff t, thus creating some ties. Nevertheless, as we shall now see, as long as he sticks to nondecreasing quantities, market maker k can secure the expected profit he would obtain if he could break such ties in his favor. Define

$$V^k_{\mathrm{co}}(t^{-k}) \equiv \sup \left\{ \sum_i m_i v^k_i(q_i, t(q_i)) \right\} \tag{11}$$

over all convex tariffs t and all families of quantities q_i that satisfy (10) for all i and that are nondecreasing, that is, $q_{i+1} \geq q_i$ for all $i < I$.

Lemma 2 *Suppose that the insider has quasilinear utility and market makers have linear costs. Then, in any pure-strategy equilibrium* $(t^1, \ldots, t^K, s)$ *of the convex-tariff game, market maker k's expected profit is at least* $V^k_{\mathrm{co}}(t^{-k})$.

When the insider's preferences are quasilinear, only the slope of the tariff t matters for q_i to be a best response of type i. As illustrated in Figure 2, one can therefore replace the tariff t by a piecewise linear tariff inducing the same best response for the insider and yielding market maker k an expected profit at least equal to that he obtained by posting t. Moreover, consider a segment of this piecewise linear tariff with slope p and the set of types who trade on this segment. If there exists a quantity $\overline{q}$ on this segment such that market maker k prefers all types who trade above $\overline{q}$ to trade $\overline{q}$, then he can raise his profits by truncating this segment at $\overline{q}$, as illustrated in Figure 3. Indeed, this reduces the quantities traded by those types, with transfers that are as least as high. Finally, market maker k can slightly reduce the slope p. This ensures that all the relevant insider types buy the maximum quantity $\overline{q}$ at price p. Proceeding in this way for each segment of his tariff, market maker k can secure the announced expected profit.

4.2.2 Equilibria with Nondecreasing Individual Quantities

Lemma 2 implies that, in any equilibrium with nondecreasing individual quantities, market makers post piecewise linear tariffs that can be interpreted as finite collections of limit orders. Another feature of such an equilibrium that follows from Lemma 2 is that, if there is a kink in the aggregate tariff, there exists at least one insider type who trades exactly at this kink. In other words, this type exhausts the aggregate supply $\overline{S}(p)$ at some price p for which $\overline{S}(p) > 0$, which means that she exhausts the supply $\overline{s}^k(p)$ of each market maker k at price p. This implies that each market maker offering some trades at price p is indispensable for this type to reach her equilibrium payoff. However, it can be shown using Bertrand-type arguments that the tariff resulting from the aggregation of all market makers' tariffs shares with the Glosten (1994) tariff the property that any increase in quantity must be priced at the corresponding increase in costs, which implies zero expected profit by construction. As one can hardly be indispensable and at the same time earn zero expected profit, we get that, if some insider type were to exhaust the aggregate supply $\overline{S}(p)$ at some price p for which $\overline{S}(p) > 0$, at least one of the market makers could slightly increase his tariff in a profitable way. That is, the following result holds.

Proposition 2 *Suppose that the insider has quasilinear utility and market makers have linear costs. Then, in any pure-strategy equilibrium with nondecreasing individual quantities of the convex-tariff game, there exists a price p such that all trades take place at unit price p and each type i purchases $D_i(p)$ in the aggregate.*

In particular, the Glosten (1994) allocation cannot be implemented in an equilibrium with nondecreasing individual quantities unless only type I trades in that allocation.

5 Market Breakdown

We now determine equilibrium prices and quantities in the candidate equilibria with linear tariffs and nondecreasing individual quantities characterized in Propositions 1–2. We show that, both in the arbitrary-tariff game and in the convex-tariff game, such equilibria when they exist typically exhibit an extreme form of market breakdown; besides, equilibria only exist under exceptional circumstances.[12]

5.1 Linear Costs

The case where market makers have linear costs (Assumption L-v) is easily handled, thanks to two arguments. First, the standard Bertrand undercutting argument implies that market makers must make zero expected profit: otherwise, because the functions D_i are continuous, any market maker k could claim almost all profits for himself by charging a uniform unit price slightly below the equilibrium price p. This implies that, if trade takes place in equilibrium, the price p cannot lie above the highest possible cost c_I. Second, in equilibrium p cannot lie below c_I either. Indeed, if it did, then market makers would want to limit the quantities they sell to type I, which they can do by posting a limit order at the equilibrium price with a well-chosen maximum quantity. Formally, in the arbitrary-tariff game, any market maker k could deviate to a menu that would allow types $i < I$ to purchase the equilibrium quantity q_i^k at unit price p, whereas type I would be asked to purchase only q_{I-1}^k at unit price p. Such an offer is incentive-compatible and individually rational, with nondecreasing quantities. Similarly, in the convex-tariff game, any market maker k could deviate to a limit order $t(q) = p \min\{q, q_{I-1}^k\}$. A best response for any type $i < I$ is then to purchase q_i^k as before, whereas a best response for type I is to purchase q_{I-1}^k, preserving nondecreasing quantities. In either case, it follows from Lemmas 1–2 that the variation in market maker

[12]In the terminology of Hendren (2014), we either have an instance of *equilibrium of market unravelling*, or an instance of *unravelling of market equilibrium.*

k's expected profit is at most zero,

$$m_I(p - c_I)(q^k_{I-1} - q^k_I) \leq 0.$$

Summing on k yields $m_I(p - c_I)[D_{I-1}(p) - D_I(p)] \leq 0$, which, under Assumption ID-U, implies that $p \geq c_I$ if $D_I(p) > 0$. Because aggregate expected profits are zero, we get that $p = c_i = c_I$ for any type i who trades. Hence the following result.

Proposition 3 *Suppose that market makers have linear costs, and consider an equilibrium with linear tariffs and nondecreasing individual quantities of either the arbitrary-tariff game or the convex-tariff game. If trade takes place, then the equilibrium price is equal to the highest cost c_I and all types who trade have the same unit cost $c_i = c_I$.*

This result highlights a tension between zero expected profits in the aggregate and the high equilibrium price c_I. In the pure private-value case where the cost c_i is independent of the insider's type i, this tension disappears and we obtain the usual Bertrand result, leading to an efficient outcome. By contrast, in the pure common-value case where the cost c_i is strictly increasing in the insider's type i, only the highest type I can trade in equilibrium, whereas all types $i < I$ must be excluded from trade. This market breakdown due to adverse selection is much more dramatic than in Akerlof (1970) or Rothschild and Stiglitz (1976), as at most one type trades in equilibrium no matter the distribution of types. Moreover, the conditions for the existence of an equilibrium become very restrictive: one must have $D_i(c_I) = 0$ for all $i < I$ if an equilibrium is to exist at all.

5.2 General Profit Functions

Due to the simplicity of our setting under linear tariffs, we can extend the above analysis to the case of more general convex preferences for the market makers. This allows us to encompass the case where market makers are risk-neutral with respect to transfers but have strictly convex order-handling costs, as in Roll (1984), or more general cases allowing for risk aversion, as in Stoll (1978) and Ho and Stoll (1981, 1983). Again, our argument is twofold.

5.2.1 Limit Orders as Best Responses

The first argument is a characterization result that does not depend on the game under study and may, therefore, be of some independent interest. Consider a situation in which all trades take place at some price p and suppose that the demands $D_i(p)$ are bounded. A natural deviation for any market maker k consists in posting a limit order at a price $p' < p$, with a maximum quantity $\overline{q}$. As he offers the best price, he trades a quantity $\min\{D_i(p'), \overline{q}\}$ with each type i. Because demand functions and profit functions are continuous, by making p' go to p, market maker k can claim the profits associated to the quantities

$$\min\{D_i(p), \overline{q}\}$$

for all i, where $\overline{q}$ remains to be chosen. For future reference, we call such quantities *limit-order quantities at price p*. On the other hand, and as suggested by Lemmas 1–2, one may also want to characterize market maker k's most preferred trades at price p, assuming that he sticks to nondecreasing quantities. These trades solve

$$\sup\left\{\sum_i m_i v_i^k(q_i, pq_i)\right\} \tag{12}$$

under the feasibility constraints

$$0 \leq q_i \leq D_i(p) \tag{13}$$

for all i, and the constraint that quantities be nondecreasing, that is, $q_{i+1} \geq q_i$ for all $i < I$. Under our assumptions, the mappings $q \mapsto v_i^k(q, pq)$ are continuous and weakly quasiconcave for all i and, according to Assumption SC-v, they satisfy a single-crossing property. The following result characterizes the solutions to problem (12)–(13).

Lemma 3 *Let p be such that the demands $D_i(p)$ are bounded. Then problem (12)–(13) has a solution with limit-order quantities at price p. Moreover, if the mapping $q \mapsto v_i^k(q, pq)$ is strictly quasiconcave, then all solutions to (12)–(13) are limit-order quantities at price p.*

The proof relies on a very simple reasoning: if the price is high enough to convince a market maker to supply a positive quantity to high types, then from Assumption SC-v market maker k will want to provide the highest possible quantities to lower types.[13] The result itself is a neat characterization of limit orders: they are the optimal tool to use under linear pricing when a market maker faces adverse selection.

5.2.2 Equilibria

Our second argument relies on equilibrium considerations. Note first that, in an equilibrium with linear tariffs at price p and nondecreasing individual quantities, each market maker k's expected profit cannot lie above the expected profit from his most preferred trades at price p. Because, by Lemma 3, this expected profit can be approximated by a well-chosen limit order at a price arbitrarily close to p, it follows that, in such an equilibrium, the expected profit of market maker k is equal to the value of (12)–(13). Therefore, the quantities sold by market maker k in equilibrium must be solutions to (12)–(13). Moreover, when his profit functions are strictly quasiconcave, such solutions must be limit-order quantities.

For simplicity, assume, moreover, that market makers have identical profit functions and strictly convex costs (Assumption C-v). Then all problems (12)–(13) are identical. By strict

[13] The proof given in the Appendix also allows for a continuous set of types. Concerning the generality of the result, notice that the ordering of the demands $D_i(p)$ does not play any role. One could as well allow for arbitrary bounded values, provided that the nondecreasing quantities constraint is replaced by the constraint that individual quantities be comonotonic with aggregate demand, that is, $D_i(p) \leq D_{i'}(p)$ implies $q_i \leq q_{i'}$.

convexity, they admit a single, common solution, which must be a family of limit-order quantities. Each market maker k thus trades in equilibrium the quantities $\min\{D_i(p), \overline{q}\}$, for some well-chosen $\overline{q}$. But as any type i cannot trade more than $D_i(p)$, it must be that each market maker k sells the same quantity $\overline{q}$ to all types of the insider who trade and, therefore, that the aggregate demand of all types who trade is the same. Because, by Assumption ID-U, $D_I(p) > D_{I-1}(p)$ if trade takes place, the following result holds.

Proposition 4 *Suppose that market makers have strictly convex costs, and consider an equilibrium with linear tariffs and nondecreasing individual quantities of either the arbitrary-tariff game or the convex-tariff game. Then only type I can trade and, if $D_I(p) > 0$, then $p \in \partial c_I(D_I(p)/K)$ and all market makers trade the same quantity $D_I(p)/K$ with type I only.*

Proposition 4 is stated for the case where Assumption C-v holds (Roll (1984)). The result, however, readily extends to the case to the case where the market makers have identical profit functions v_i such that the mappings $q \mapsto v_i(q, pq)$ are strictly concave. This is for instance the case if market makers are risk-averse, as when $v_i(q,t) \equiv v(t - c_i q)$ for some strictly concave von Neumann-Morgenstern utility function v (Stoll (1978), Ho and Stoll (1981, 1983)).

When there is a single insider type, that is, when $I = 1$, Proposition 4 states that any equilibrium is competitive in the usual sense: (i) the insider purchases her optimal demand $D_1(p)$ at price p; (ii) the market makers maximize their profit $v_1(q, pq)$ at price p; (iii) the equilibrium price p equalizes the insider's demand and the sum of the market makers' supplies. Equilibrium outcomes are hence first-best efficient.

With multiple insider types, the unique candidate equilibrium outcome remains that which would prevail in an economy populated by type I only. A necessary condition for equilibrium is thus that all types $i < I$ demand a zero quantity at the equilibrium price p. The market breakdown effect is thus similar to the one characterized by Proposition 3 when market makers have linear costs and the conditions for the existence of an equilibrium are very restrictive in this case as well.

A novel insight of Proposition 4 is that the result that no trade may take place except perhaps at the top of the insider's type distribution now holds whether or not the environment features common values. To illustrate this point, consider for instance the case of strictly convex costs (Assumption C-v) and suppose that the cost function is the same for each type, $c_i(q) \equiv c(q)$ for all i and q, whereas demands $D_i(p)$ are strictly increasing in i. As a market maker's profit $t - c(q)$ on a given trade (q, t) does not depend on the identity of the insider, we are in a private-value setting, so that only risk sharing matters. Still, oligopolistic competition threatens the existence of equilibria: each market maker would like to reduce his maximum supply if the equilibrium price were too low; but a high equilibrium price strengthens competition to attract lower types. Thus competition is strong enough to imply that, in equilibrium, at most one type can trade.

6 Other Equilibrium Outcomes

We finally show that focusing on equilibria with nondecreasing individual quantities involves no loss of generality: one can turn any equilibrium with convex tariffs into an equilibrium with the same tariffs and the same payoffs, but now with nondecreasing individual quantities. This result holds both in the arbitrary-tariff game and in the convex-tariff game. The proof is in fact very general and only relies on a property specifying that, in a certain sense, allocations with nondecreasing quantities are efficient.

To understand why, notice that market makers have to choose tariffs before demand realizes. How risk is collectively shared then becomes a central question. Given a profile $(t^1, \ldots, t^K)$ of convex tariffs, recall that each type has a uniquely determined aggregate trade (Q_i, T_i). Define a feasible allocation $(q_1^1, \ldots, q_1^K, \ldots, q_I^1, \ldots, q_I^K)$ as an allocation that satisfies

$$\sum_k q_i^k = Q_i \quad \text{and} \quad \sum_k t^k(q_i^k) = T_i \tag{14}$$

for all i; in other words, this allocation describes a best response of the insider to the tariffs $(t^1, \ldots, t^K)$. Define an efficient risk-sharing allocation as a feasible allocation that is not Pareto-dominated by any other feasible allocation from the market makers' perspective: there is no other feasible allocation that yields as high an expected profit to each market maker, and a strictly higher expected profit to at least one market maker. Our result relies on the following property.

Property P *For any profile of convex tariffs* $(t^1, \ldots, t^K)$, *there exists an efficient risk-sharing allocation with nondecreasing individual quantities.*

This property is reminiscent of the usual risk-sharing result (Borch (1962)): efficiency requires that any increase in the aggregate quantity to be shared should translate into an increase in the individual shares of market makers. Nevertheless, notice that, in our setting, the market makers' payoff functions are state-dependent, because they directly depend on the insider's type. Moreover, the convexity of the tariffs $(t^1, \ldots, t^K)$ may make the payoffs $v_i^k(q, t^k(q))$ nonconcave in q. To bypass these difficulties, we have to impose more restrictions on the market maker's profit functions than in the previous sections. Notable special cases are Assumptions L-v and C-v used in Theorems 1–2.[14]

Lemma 4 *Suppose that all market makers have the same profit function, given by*

$$v_i^k(q, t) = t - c_i(q),$$

where for each i the cost $c_i(q)$ is convex in q. Then Property P is satisfied.

[14] One can more generally show that Lemma 4 holds for market makers with heterogenous cost functions c_i^k, the derivatives of which satisfy $c_i^{k\prime} = f_i \circ a^k$, where f_i is strictly increasing and a^k is nondecreasing. This in particular allows to handle the case of market makers with heterogeneous inventories, in which one has $c_i^k(q) = c_i(q - I^k)$ for some given inventories I^k.

We can now turn to the study of an arbitrary equilibrium $(t^1, \ldots, t^K, s)$ of either the arbitrary-tariff game or the convex-tariff game. Let v^k be the equilibrium expected profit of market maker k. Depending on the game under study, Lemma 1 and Lemma 2 offer lower bounds $V^k(t^{-k})$ and $V^k_{\text{co}}(t^{-k})$ for this expected profit, respectively. We can build another lower bound by imposing in problem (9) and problem (11) the additional constraint that the transfers to market maker k be computed using the equilibrium tariff t^k; this defines $\underline{V}^k(t^1, \ldots, t^K)$. We therefore have

$$v^k \geq \underline{V}^k(t^1, \ldots, t^K) \tag{15}$$

for all k. On the other hand, if Property P is satisfied, we know that, given the tariffs $(t^1, \ldots, t^K)$, there exists an efficient risk-sharing allocation $(q^1_1, \ldots, q^K_1, \ldots, q^1_I, \ldots, q^K_I)$ with nondecreasing individual quantities. In particular, for each k, the quantities $(q^k_1, \ldots, q^k_I)$ satisfy the constraints in the problem that defines $\underline{V}^k(t^1, \ldots, t^K)$. This implies that, for each k, we have

$$\underline{V}^k(t^1, \ldots, t^K) \geq \sum_i m_i v^k_i(q^k_i, t^k(q^k_i)). \tag{16}$$

Chaining inequalities (15)–(16), we get that each market maker k's equilibrium expected profit lies above his expected profit from the allocation $(q^1_1, \ldots, q^K_1, \ldots, q^1_I, \ldots, q^K_I)$. As the latter is a Pareto optimum, this is impossible unless all inequalities are equalities. Hence, for each k, we have

$$v^k = \sum_i m_i v^k_i(q^k_i, t^k(q^k_i)). \tag{17}$$

We now build an equilibrium that implements the efficient risk-sharing allocation $(q^1_1, \ldots, q^K_1, \ldots, q^1_I, \ldots, q^K_I)$. Let us define s^* as the insider's strategy that selects this allocation if the tariff profile $(t^1, \ldots, t^K)$ is posted; otherwise, s^* selects the same quantities as s. We claim that $(t^1, \ldots, t^K, s^*)$ forms an equilibrium. Indeed, the insider plays a best response to any tariff profile. Moreover, in the initial equilibrium $(t^1, \ldots, t^K, s)$, no market maker has a profitable deviation. Hence, for each k and for any tariff $\hat{t}^k \neq t^k$,[15] we have

$$v^k \geq \sum_i m_i v^k_i(s^k_i(\hat{t}^k, t^{-k}), \hat{t}^k(s^k_i(\hat{t}^k, t^{-k}))).$$

But, from (17) and the definition of s^*, this can be rewritten as

$$\sum_i m_i v^k_i(s^{*k}_i(t^k, t^{-k}), t^k(s^{*k}_i(t^k, t^{-k}))) \geq \sum_i m_i v^k_i(s^{*k}_i(\hat{t}^k, t^{-k}), \hat{t}^k(s^{*k}_i(\hat{t}^k, t^{-k}))),$$

which expresses that market maker k has no profitable deviation when the other market maker post the tariffs t^{-k} and the insider plays her best response s^*. Hence the following result, which holds both in the arbitrary-tariff game and in the convex-tariff game.

[15] In the game with convex tariffs, $\hat{t}^k$ must additionally be convex.

Proposition 5 *Suppose Property P is satisfied and let* $(t^1, \ldots, t^K, s)$ *be an equilibrium with convex tariffs. Then there exists a strategy* s^* *for the insider such that* $(t^1, \ldots, t^K, s^*)$ *is an equilibrium with nondecreasing individual quantities that yields the same expected profit to each market maker.*

As noted above, Property P is satisfied under the assumptions of Theorems 1–2. By Proposition 5, any equilibrium can thus be turned into an equilibrium with the same tariffs and nondecreasing individual quantities. A direct implication of Propositions 1–2 is then that all equilibria must involve linear pricing and Theorems 1–2 follow as immediate consequences of Propositions 3–4. We have also learnt that equilibria, when they exist, support efficient risk-sharing allocations among market makers.

7 Necessary and Sufficient Conditions for Equilibrium

We now provide necessary and sufficient conditions for the existence of a pure-strategy equilibrium. To simplify the exposition, we focus on the convex-tariff game, under the same assumptions on preferences as in Theorem 2, and we dispense with the requirement that market makers' tariffs be defined over a compact domain.[16] We moreover consider the pure common-value case where the cost c_i is strictly increasing in the insider's type i, so that only the highest type I can trade in equilibrium. For each i, let $\bar{c}_i \equiv \mathbf{E}[c_{\tilde{i}} \mid \tilde{i} \geq i] = \sum_{i' \geq i} m_{i'} c_{i'} / \sum_{i' \geq i} m_{i'}$ be the upper-tail expectation of the market-makers' cost conditional on the insider's type being at least i. The following result then holds.

Theorem 3 *Suppose that the insider has quasilinear utility (Assumption QL-U) and market makers have linear costs (Assumption L-v), and that higher types are strictly more costly to serve. Then the convex-tariff game has a pure-strategy equilibrium if and only if*

$$u_i'(0) \leq \bar{c}_i \tag{18}$$

for all $i < I$. *An equilibrium can then be supported by at least two market makers posting the linear tariff*

$$t(q) \equiv c_I q \tag{19}$$

for all $q \geq 0$, *while the other market makers, if any, stay inactive.*

To see that conditions (18) are necessary for an equilibrium to exist, observe that if $u_i'(0) > \bar{c}_i$ for some $i < I$, then it is possible for any market maker to deviate and attract type i with a limit-order $\hat{t}(q) = (\bar{c}_i + \varepsilon) \min\{q, \hat{q}_i\}$, where ε and $\hat{q}_i$ are positive and small enough as to induce type i to trade $\hat{q}_i$ with the deviating market maker. Types higher than

[16] The corresponding technical analysis that we thereby omit can be conducted along the same lines as in the two-type model of Attar, Mariotti, and Salanié (2014, Theorem 2).

i are also attracted by this trade, and, because $\varepsilon > 0$, the resulting expected profit for the deviating market maker is positive; this is a fortiori true if some types $i' < i$ are attracted as well. Hence the tariff $\hat{t}$ is a profitable deviation, as each market maker, according to Theorem 2, must earn zero expected profit in any candidate equilibrium.

To see that, under conditions (18), an equilibrium can be supported by at least two market makers posting the tariff (19), let us suppose that one market maker deviates by posting a convex tariff $\hat{t}$. Following this deviation, one may clearly assume that the insider only trades with the deviating market maker and one market maker posting the tariff t. Denote by $\hat{q}_i$ and q_i the quantities traded by type i with these two market makers, respectively, and by $(\hat{Q}_i, \hat{T}_i) \equiv (\hat{q}_i + q_i, \hat{t}(\hat{q}_i) + t(q_i))$ her aggregate trade. Because the tariff $\hat{t}$ is convex, one can further impose that the insider trades nondecreasing individual quantities with the nondeviating market maker. As a result,

$$t(q_i) - t(q_{i-1}) - \overline{c}_i(q_i - q_{i-1}) = (c_I - \overline{c}_i)(q_i - q_{i-1}) \geq 0 \tag{20}$$

for all i, where $(q_0, t_0) \equiv (0, 0)$ by convention. Now, integrating by parts the deviating market maker's expected profit yields

$$\sum_i m_i[\hat{t}(\hat{q}_i) - c_i\hat{q}_i] = \sum_i \overline{m}_i[\hat{t}(\hat{q}_i) - \hat{t}(\hat{q}_{i-1}) - \overline{c}_i(\hat{q}_i - \hat{q}_{i-1})], \tag{21}$$

where $\overline{m}_i \equiv \sum_{i' \geq i} m_{i'}$ for all i and $(\hat{q}_0, \hat{t}_0) \equiv (0, 0)$ by convention. That is, his expected profit can be computed by integrating the expected profit from successive layers of quantities. The profit from the last layer, $\hat{t}(\hat{q}_I) - \hat{t}(\hat{q}_{I-1}) - \overline{c}_I(\hat{q}_I - \hat{q}_{I-1})$, cannot be positive as type I has the option to buy any quantity at price $c_I = \overline{c}_I$ from a nondeviating market maker offering the tariff t. Suppose then, by way of contradiction, that the profit from a lower layer is positive,

$$\hat{t}(\hat{q}_i) - \hat{t}(\hat{q}_{i-1}) - \overline{c}_i(\hat{q}_i - \hat{q}_{i-1}) > 0 \tag{22}$$

for some $i < I$. Summing (20) and (22) yields

$$\hat{T}_i - \hat{T}_{i-1} - \overline{c}_i(\hat{Q}_i - \hat{Q}_{i-1}) > 0.$$

Observing that this implies $\hat{Q}_i > \hat{Q}_{i-1}$. Taking advantage of (18), we then get that

$$\frac{\hat{T}_i - \hat{T}_{i-1}}{\hat{Q}_i - \hat{Q}_{i-1}} > u_i'(0) \geq u_i'(\hat{Q}_{i-1})$$

by concavity of u_i. But this is impossible, for type i would then be strictly better off trading $(\hat{Q}_{i-1}, \hat{T}_{i-1})$ than $(\hat{Q}_i, \hat{T}_i)$ in the aggregate, a contradiction. It follows that the deviating market maker's profit from each layer of quantity is nonpositive, and thus that there is no profitable deviation. This concludes the proof of Theorem 3.

Conditions (18) are clearly very restrictive: in an equilibrium in which type I trades, one must have $u_i'(0) \leq \overline{c}_i$ for all $i < I$ and $u_I'(0) > c_I = \overline{c}_I$. That is, type I must have preferences that are sufficiently different from those of types $i < I$. In particular, these conditions are increasingly difficult to meet when one increases the number of types, as when one attempts to approximate a continuous type set by an increasing sequence of discrete type sets.

The proof of Theorem 3 crucially exploits the fact that higher types are strictly more costly to serve. It is important, by contrast, to observe that a pure strategy equilibrium always exists in the pure private-value case in which c_i is constant in i. To see this point, assume that at least two market makers stand ready to sell any positive quantity at a unit price equal to the constant marginal cost, while the remaining market makers, if any, only offer the null trade. A standard Bertrand argument then guarantees that, given these tariffs, no market maker can unilaterally deviate and profitably attract at least one type of the insider. The resulting equilibrium allocation is first-best efficient.

8 Discussion

In this section, we put our main results in perspective and relate them to the literature.

1. The model we use is standard—one may even say canonical—and can be seen as the adverse-selection extension of the Bertrand-competition model; the restriction to equilibria with convex tariffs is motivated by our focus on discriminatory pricing in a limit-order book. We allow for arbitrary finite distributions of types for the insider and for a rich set of convex preferences for the insider and the market makers. The strict convexity of the insider's preferences implies that the aggregate quantity of the asset she is ready to purchase continuously responds to variations in prices. This may reflect that she trades the asset partly for hedging purposes, as in Glosten (1989), Biais, Martimort, and Rochet (2000), and Back and Baruch (2013), and partly for informational purposes. News traders, that is, insiders who are perfectly informed of the liquidation value of the asset and trade only on this information, as in Dennert (1993) or Baruch and Glosten (2016), are a limiting case of our analysis. Finally, the model is fully strategic, in that it does not rely on noise traders who are insensitive to prices, unlike much of the market-microstructure literature.

2. An important insight of our analysis is that any candidate equilibrium must satisfy a strong Bertrand property: in both the arbitrary-tariff game and in the convex-tariff game, no market maker is indispensable for providing any type with her equilibrium trades. The reason is that, otherwise, a market maker would have an incentive to raise his price on the marginal trade that he makes with some type. We use standard mechanism-design techniques (Lemma 1) or standard price-theory arguments (Lemma 2) to show that he can do so without reducing her expected profit with the other types. Discreteness of the set of types is crucial for this logic. Indeed, in models with a continuum of types, Biais, Martimort, and Rochet (2000) and Back and Baruch (2013) show how to construct an equilibrium in which all market makers post the same, strictly convex tariff and thus are indispensable as each type has a unique best response. Although strictly convex tariffs are not consistent with equilibrium in the discrete-type case—as the consideration of the one-type case readily shows—they can be sustained in the continuous-type case because a local change in the tariff affects the behavior of all neighboring types, unlike in the discrete-type case. To illustrate this point, suppose that, over some interval of quantities, a market maker deviates by proposing, instead of the relevant portion of his strictly convex equilibrium tariff, the corresponding chord. This would increase the deviator's profit if the insider's behavior remained the same. But such a change increases (decreases) the marginal price for relatively low-cost (high-cost) types who used to trade in this interval and thus, under common values or strictly convex costs, trades change in an unfavorable way. This last effect is reinforced when the insider simultaneously trades with several market makers: any increase in the quantity purchased from a single market maker is compensated by a reduction in the quantity she purchases from the others. The equilibrium in Biais, Martimort, and Rochet (2000) and Back and Baruch (2013) strikes a delicate balance between these two effects and, as in any Cournot-like equilibrium, the elasticity of demand for each type comes to play a crucial role. This is why their construction requires complex and quite restrictive joint conditions on the distribution of the insider's type and on the expected value of the asset conditional on her type. By contrast, our results hold for general discrete-type environments and do not rely on such conditions.

3. A key feature of the candidate equilibria of our model is that market makers want to hedge against the adverse-selection risk or, when they have strictly convex costs, against the high-demand risk. A strictly convex tariff would perform this role by making high-cost and, therefore, high-demand types trade at a higher marginal price than low-cost and, therefore, low-demand types. However, whereas such tariffs naturally arise in the continuous-type environments of Glosten (1994), Biais, Martimort, and Rochet (2000), or Back and Baruch (2013), they are ruled out in our discrete-type environment as any equilibrium must feature linear pricing. Simpler tariffs such as limit orders then play a key role. We have shown that, in a situation in which all market makers but one post linear tariffs, using a well-chosen limit order is the best way for the remaining market maker to limit his exposure to the adverse-selection and the high-demand risks.[17] However, limit orders are consistent with equilibrium only under exceptional circumstances. This is because the equilibrium price must be high enough to convince market makers to serve high-cost types. But such a high price means that each market maker would like to serve all the demand emanating from low-cost types, which is inconsistent with equilibrium unless these types do not wish to trade at that price. This confirms and extends in a radical way earlier results obtained by Attar, Mariotti, and Salanié (2014), who show in the two-type case that at most one type trades in any pure-strategy equilibrium of the arbitrary-tariff game.[18]

[17]This truncation argument is general and also applies in a candidate linear-price equilibrium of a model with a continuum of types. It also implies that marginal profits in an equilibrium in strictly convex tariffs must be nonnegative at the upper end of the distribution of types.

[18]That is, even allowing for equilibria with nonconvex tariffs. This might be relevant for the analysis of competition on less regulated markets, such as over-the-counter-markets, in which trading is bilateral and fully nonexclusive.

4. Our results show that the existence of pure-strategy equilibria for the discriminatory limit-order book is problematic in common-value environments. In particular, such equilibria typically fail to exist when there are sufficiently many types, as when one approximates the continuous sets of types postulated by Biais, Martimort, and Rochet (2000) or Back and Baruch (2013); that is, given the positive existence results derived by these authors, the pure-strategy-equilibrium correspondence fails to be lower hemicontinuous when one moves from discrete-type models to continuous-type models. A novel insight of our analysis is that the market also may break down or an equilibrium may fail to exist altogether even in private-value environments, as long as a market maker's marginal cost is not constant in the quantity of the asset that he trades with the insider.[19]

9 Relaxing the Equilibrium Concept

Our main results suggest that requiring a pure-strategy-equilibrium foundation for the limit-order book may be too demanding. In this section, we explore two alternative avenues of research that relax the equilibrium concept in different ways. We first focus on ε-equilibria, both in a game with a fixed number of insider types but with a growing population of market makers, and in a game with a fixed number of market maker but with a growing number of insider types. We then turn to mixed-strategy equilibria, and explicitly show how to construct such an equilibrium for a particular specification of the convex-tariff game.

9.1 ε-Equilibria

9.1.1 The Glosten (1994) Allocation as an Approximate Equilibrium Outcome

As pointed out in the Introduction, a natural candidate for describing the discriminatory limit-order book as a whole is the Glosten (1994) tariff, foreshadowed by early contributions of Jaynes (1978) and Hellwig (1988). This tariff, a marginal version of Akerlof (1970), is by construction robust to entry.[20] However, according to our analysis, it is not strategically stable, because some market maker providing part of it would have an incentive to deviate and take advantage of his competitors' tariffs. The reason for this is that, in this aggregate tariff, market makers trading with low-cost insiders are indispensable for providing these types with their equilibrium trades. Yet these market makers make zero expected profits, which does not square with their being indispensable.[21] A natural question is how much profits they forego by not playing a best response. The answer turns out to depend on the

[19]By contrast, as pointed out in Section 7, a pure-strategy equilibrium always exists under private values when market makers have identical linear costs. This result obtains irrespective of the distribution of types, whether it is discrete or continuous.

[20]This is shown by Glosten (1994) in a model in which the insider has quasilinear utility and types are continuously distributed. Attar, Mariotti, and Salanié (2016a) provide a simple argument that dispenses with quasilinearity in the two-type case, and Attar, Mariotti, and Salanié (2016b) provide a general result for arbitrary distributions of types under weak conditions on the insider's preferences.

[21]We exploited this logic in the proof of Proposition 2.

market structure, that is, on the number of market makers.

To illustrate this point, suppose for simplicity that $I = 2$, that the insider has quasilinear utility (Assumption QL-U), and that the market makers have linear costs (Assumption L-v), with $c_1 < c_2$. Using the notation of Section 7, denote by $\overline{c}_1 \equiv m_1 c_1 + m_2 c_2$ the average cost and suppose, furthermore, that

$$0 < (u_1')^{-1}(\overline{c}_1) < (u_2')^{-1}(c_2) < \infty.$$

Let $Q_1^* \equiv (u_1')^{-1}(\overline{c}_1)$ and $Q_2^* \equiv (u_2')^{-1}(c_2)$. In this context, the Glosten (1994) tariff is the piecewise-linear tariff defined by

$$T^G(q) \equiv 1_{\{q \leq Q_1^*\}} \overline{c}_1 q + 1_{\{q > Q_1^*\}} [\overline{c}_1 Q_1^* + c_2(q - Q_1^*)].$$

When facing T^G, type 1 trades the aggregate quantity Q_1^* and type 2 trades the aggregate quantity Q_2^* at marginal prices $\overline{c}_1$ and c_2, respectively.

Consider the following implementation of T^G: suppose each market maker offers to sell any quantity in $[0, Q_1/K]$ at unit price $\overline{c}_1$ and then to sell any additional quantity at unit price c_2, which amounts to the tariff $T^G(Kq)/K$. Clearly, these convex tariffs aggregate into T^G. Each market maker k is indispensable for providing T^G and, therefore, has a profitable deviation. The question is how much he can gain by deviating. Because the market makers other than k post convex tariffs, Property SC-z is satisfied. As a result, we can assume that, following a deviation by market maker k, the insider selects a best response in which she trades nondecreasing individual quantities with him.

To get an upper bound on market maker k's expected gain from deviating, there is no loss of generality in letting him offer a menu consisting of the null trade and of a well-chosen pair of trades $(\hat{q}_1, \hat{t}_1)$ and $(\hat{q}_2, \hat{t}_2)$ targeted at types 1 and 2, respectively, and resulting, in analogy with (21), in an expected profit

$$\hat{t}_1 - \overline{c}_1 \hat{q}_1 + m_2[\hat{t}_2 - \hat{t}_1 - c_2(\hat{q}_2 - \hat{q}_1)]. \tag{23}$$

From the above observation, we can suppose that $\hat{q}_1 \leq \hat{q}_2$. Because type 2 has the option to buy the nonnegative marginal quantity $\hat{q}_2 - \hat{q}_1$ at a unit price at most equal to c_2, one must have $\hat{t}_2 - \hat{t}_1 \leq c_2(\hat{q}_2 - \hat{q}_1)$ and, therefore, the expected profit (23) is bounded above by $\hat{t}_1 - \overline{c}_1 \hat{q}_1$. Hence we may as well assume that market maker k offers a single trade $(\hat{q}, \hat{t})$ distinct from the null trade and which both types accept to make. A necessary condition for type 1 to accept the trade $(\hat{q}, \hat{t})$ is that $z_1^{-k}(\hat{q}, \hat{t}) \geq z_1^{-k}(0, 0)$, which implies $z_2^{-k}(\hat{q}, \hat{t}) \geq z_2^{-k}(0, 0)$ by Property SC-z. Thus an upper bound for (23) is

$$\max\{\hat{t} - \overline{c}_1 \hat{q} : z_1^{-k}(\hat{q}, \hat{t}) \geq z_1^{-k}(0, 0)\}.$$

As the aggregate quantity $(K-1)\, Q_1^*/K$ remains available for trade at unit price $\overline{c}_1$ following market maker k's deviation, an even more generous upper bound is

$$\max\left\{\hat{t} - \overline{c}_1 \hat{q} : z_1^{-k}(\hat{q}, \hat{t}) \geq u_1\left((K-1)\, \frac{Q_1^*}{K}\right) - \overline{c}_1 (K-1)\, \frac{Q_1^*}{K}\right\}. \tag{24}$$

Any optimal trade $(\hat{q}, \hat{t})$ solution to (24) is such that the constraint in (24) is binding and type 1 ends up purchasing Q_1^* in the aggregate. A particular solution is such that

$$\hat{q} = \frac{Q_1^*}{K} \text{ and } \hat{t} = u_1(Q_1^*) - u_1\left((K-1)\frac{Q_1^*}{K}\right).$$

As $u_1'(Q_1^*) = \bar{c}_1$, and assuming that u_1 is twice differentiable at Q_1^*, a Taylor–Young expansion yields the following approximation for (24):

$$\hat{t} - \bar{c}_1\hat{q} = \frac{|u_1''(Q_1^*)|}{2}\left(\frac{Q_1^*}{K}\right)^2 + o\left(\frac{1}{K^2}\right),$$

which implies that the maximum expected gain from deviating vanishes at rate $1/K^2$ as the number K of market makers goes to infinity. One can thus rationalize the Glosten (1994) aggregate allocation as an $O(1/K^2)$-equilibrium outcome when there are many market makers. This reconciles our findings with those of Biais, Martimort, and Rochet (2000), who show in the continuous-type case that the equilibrium aggregate tariff converges to the Glosten (1994) tariff in the competitive limit.

9.1.2 The Biais, Martimort, and Rochet (2000) Allocation as an Approximate Equilibrium Outcome

In the last section, we examined what happens when one lets the number of market makers grow large, holding the number of types fixed. In this section, we explore the dual scenario in which one lets the number of types grow large, so as to approximate a continuous type set, holding the number of market makers fixed. In the continuous-type limit, Biais, Martimort, and Rochet (2000) have shown that a symmetric pure-strategy equilibrium in which market makers post a strictly convex tariff exists under certain assumptions on primitives. However, according to our analysis, this tariff is not part of an equilibrium of any discretized version of the game: each market maker has a better response than posting it. A natural question is how much profits they forego by not playing a best response. The answer turns out to depend on the richness of the type set, that is, on the number of types.

To illustrate this point, let us first recall how Biais, Martimort, and Rochet (2000) proceed to solve the game with arbitrary tariffs. Let the insider's type θ be distributed over some bounded interval $[\underline{\theta}, \overline{\theta}]$ according to a distribution F with positive and continuous density f. Type θ's utility function is $U(Q, T, \theta) = u(Q, \theta) - T$, while the market makers' cost of serving type θ is $v(\theta)$.[22] Now, select a market maker k, and suppose that all the other market makers post a strictly convex tariff t. Thanks to symmetry, the resulting aggregate tariff is given by $T^{-k}(Q) \equiv (K-1)t(Q/(K-1))$. Thus market k faces an insider whose indirect utility from trading (q, t) is $z^{-k}(q, t, \theta) = \zeta^{-k}(q, \theta) - t$, where

$$\zeta^{-k}(q, \theta) \equiv \max\{u(q + Q^{-k}, \theta) - T^{-k}(Q^{-k}) : Q^{-k} \geq 0\}. \quad (25)$$

[22]Biais, Martimort, and Rochet (2000) more specifically assume that $u(Q, \theta) = \theta Q - (\alpha\sigma^2/2)Q^2$. We stick to the general notation to ease the exposition.

Market maker k's problem is then to find a menu of trades $\{(\chi(\theta), \tau(\theta)) : \theta \in [\underline{\theta}, \overline{\theta}]\}$ that maximizes his expected profit

$$\int_{\underline{\theta}}^{\overline{\theta}} [\tau(\theta) - v(\theta)\chi(\theta)] f(\theta)\, \mathrm{d}\theta$$

under the incentive-compatibility constraints

$$\zeta^{-k}(\chi(\theta), \theta) - \tau(\theta) \geq \zeta^{-k}(q(\theta'), \theta) - \tau(\theta')$$

and the participation constraints

$$\zeta^{-k}(\chi(\theta), \theta) - \tau(\theta) \geq \zeta^{-k}(\theta, 0).$$

for all $(\theta, \theta') \in [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}]$. Using standard techniques, we obtain that a relaxed version of this problem consists in the pointwise maximization of

$$-\zeta^{-k}(0, \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left[\zeta^{-k}(\chi(\theta), \theta) - v(\theta)\chi(\theta) - \frac{1 - F(\theta)}{f(\theta)}\, \zeta_{\theta}^{-k}(\chi(\theta), \theta)\right] f(\theta)\, \mathrm{d}\theta. \qquad (26)$$

Under certain assumptions on primitives, Biais, Martimort, and Rochet (2000) show that there exists a tariff t that induces the indirect utility function (25), and such that the solution χ to the relaxed problem (26) can be implemented by t. In these circumstances, there exists a symmetric equilibrium in which each market maker posts the tariff t.

We can approximate the above construction in our discrete-type setting. Let us choose I discrete types $\theta_1, \ldots, \theta_I$ in $[\underline{\theta}, \overline{\theta}]$, regularly spaced for simplicity, so that

$$\theta_i = \underline{\theta} + \frac{i - 1}{I}\, (\overline{\theta} - \underline{\theta}),$$

with respective probabilities $m_1, \ldots, m_I$ given by

$$m_i = F(\theta_{i+1}) - F(\theta_i)$$

where $\theta_{I+1} \equiv \overline{\theta}$ by convention. As above, suppose that all market makers but k post the tariff t. Therefore, the insider's indirect utility function remains given by (25). In the discretized version of the game, the relaxed problem still provides an upper bound to the expected profit of market maker k, and its objective now writes as

$$-\zeta^{-k}(0, \underline{\theta}) + \sum_{i=1}^{I} m_i \left[\zeta^{-k}(\chi(\theta_i), \theta_i) - v(\theta_i)\chi(\theta_i) - \frac{\sum_{i' > i} m_{i'}}{m_i} [\zeta^{-k}(\chi(\theta_i), \theta_{i+1}) - \zeta^{-k}(\chi(\theta_i), \theta_i)]\right]. \qquad (27)$$

Compared to (26), the first term in (27) is the same, and the sum is shown to approximate the integral thanks to the following equalities:

$$m_i = \frac{\overline{\theta} - \underline{\theta}}{I}\, f(\theta_i) + o\left(\frac{1}{I}\right),$$

$$\sum_{i'>i} m_{i'} = 1 - F(\theta_i),$$

$$\zeta^{-k}(\chi(\theta_i), \theta_{i+1}) - \zeta^{-k}(\chi(\theta_i), \theta_i) = \frac{\overline{\theta} - \underline{\theta}}{I} \zeta_\theta^{-k}(\chi(\theta_i), \theta_i) + o\left(\frac{1}{I}\right)$$

for all i. The sum in (27) then equals

$$\frac{\overline{\theta} - \underline{\theta}}{I} \sum_{i=1}^{I} f(\theta_i)\left[\zeta^{-k}(\chi(\theta_i), \theta_i) - v(\theta_i)\chi(\theta_i) - \frac{1 - F(\theta_i)}{f(\theta_i)} \zeta_\theta^{-k}(\chi(\theta_i), \theta_i)\right] + O\left(\frac{1}{I}\right).$$

which is the Riemann approximation of the integral in (26), with precision of the order of $1/I$. This shows that, in the discretized version of the game, if all market makers but k post the tariff t, the maximum expected gain for k from deviating from t vanishes at rate $1/I$ as the number I of types goes to infinity. One can thus rationalize the Biais, Martimort, and Rochet (2000) aggregate allocation as an $O(1/I)$-equilibrium outcome when one approximates the continuous type set they postulate by an increasing sequence of discrete type sets.

9.2 Mixed-Strategy Equilibria

An alternative avenue of research is to consider mixed-strategy equilibria. One can adapt the arguments of Carmona and Fajardo (2009) to show that the convex game admits a mixed-strategy equilibrium.[23] Yet, as pointed out in the Introduction, characterizing such equilibria appears to be a difficult task. Dennert (1993) and Baruch and Glosten (2016) construct mixed-strategy equilibria for related games, but they make the extreme assumption that the insider is a perfectly informed news trader. As a result, in equilibrium, she totally empties the book when the price is below the liquidation value of the asset, and does not trade otherwise. By contrast, our analysis incorporates other motives to trade, such as hedging, that make the insider's demand for the asset a continuous function of prices. A related point is that, because their models do not feature gains from trade, Dennert (1993) and Baruch and Glosten (2016) resort to noise traders to ensure that trade takes place in the mixed-strategy equilibria they characterize. Our model, by contrast, is fully strategic.

The goal of this section is to construct a mixed-strategy equilibrium of the convex-tariff game in the case where the necessary and sufficient conditions for the existence of a pure-strategy equilibrium delineated in Theorem 3 do not hold. We suppose for simplicity that $I = K = 2$, that the insider has quasilinear utility (Assumption QL-U), and that the market makers have linear costs (Assumption L-v), with $c_1 < c_2$. Denoting by $\overline{c}_1 \equiv m_1 c_1 + m_2 c_2$ the average cost, we furthermore impose the following restriction on demand functions.

Assumption D *$D_2(p) - D_1(p) \geq D_1(\overline{c}_1)$ for all $p < c_2$.*

Assumption D holds as long as the two insider types have sufficiently different willingness

[23] In the game with arbitrary tariffs, however, it is unclear that there exists a mixed-strategy equilibrium in which market makers only randomize over convex tariffs, as required by the rules of the limit-order book.

to trade.[24] Observe that, unlike in the necessary and sufficient conditions (18) for the existence of a pure-strategy equilibrium, we may now have $D_1(\bar{c}_1) = (u_1')^{-1}(\bar{c}_1) > 0$.

In our candidate mixed-strategy equilibrium, each market maker posts a random limit order $\tilde{t}(q) = \tilde{p} \min\{q, D_1(\tilde{p})\}$, where $\tilde{p}$ is a random variable with absolutely continuous cumulative distribution function F over the support $[\bar{c}_1, c_2]$. In addition, each market maker stands ready to sell any positive quantity at the unit price c_2. The argument then consists of two steps. First, we derive a closed-form expression for F ensuring that no market maker can earn a positive expected profit by deviating to an arbitrary convex tariff. Second, we observe that any realization of the random limit order $\tilde{t}$ makes zero expected profit when the other market maker plays as prescribed. This achieves the construction of a mixed-strategy equilibrium in which each market maker earns zero expected profit.

Let then $t(q)$ be an arbitrary convex tariff posted, say, by market maker 2, and let $p(q) \equiv t'(q)$, which is well defined except perhaps on a countable set of q's. Choose now a given $q > 0$, and compute the expected profit earned by market maker 2 when selling an infinitesimal quantity $\mathrm{d}q$ at the price $p(q)$. One can without loss of generality restrict attention to those quantities q such that $p(q) < c_2$: indeed, given the strategy of market maker 1, whenever $p(q) \geq c_2$, it is optimal for both types of the insider not to trade with market maker 2.

Letting $p \in [\bar{c}_1, c_2]$ be a realization of the random price $\tilde{p}$ posted by market maker 1, one has $p(q) \leq p$ with probability $1 - F(p(q))$. In this case, $p(q)$ is the best price on the market and, given Assumption L-U, type 1 accepts to buy the additional quantity $\mathrm{d}q$ if $q < D_1(p(q))$. It is straightforward to check that market maker 2 is unable to sell $\mathrm{d}q$ to type 1 whenever $p < p(q)$, in which case she is able to fully satisfy her demand $D_1(p)$ by trading with market maker 1. Letting $\mathrm{d}q$ go to zero, we get that the expected profit earned by market maker 2 when trading $(p(q), q)$ with type 1 is

$$\tilde{v}_1^2(p(q), q) = [p(q) - c_1][1 - F(p(q))]1_{\{q < D_1(p(q))\}}.$$

If $p(q) \leq p$, type 2 also accepts to buy $\mathrm{d}q$ if $q < D_2(p(q))$. However, she might also be willing to trade this quantity with market maker 2 when $p < p(q)$ provided that $D_2(p(q)) - D_1(p) > q$: type 2 then buys the amount $\mathrm{d}q$ to satisfy part of her residual demand $D_2(p(q)) - D_1(p)$. Overall, the expected profit earned by market maker 2 when trading $(p(q), q)$ with type 2 is

$$\tilde{v}_2^2(p(q), q) = [p(q) - c_2]\left\{[1 - F(p(q))]1_{\{q < D_2(p(q))\}} + \int_{\{p < p(q)\}} 1_{\{q < D_2(p(q)) - D_1(p)\}} \, \mathrm{d}F(p)\right\}.$$

We now construct a distribution F such that the expected profit

$$\int [m_1 \tilde{v}_1^2(p(q), q) + m_2 \tilde{v}_2^2(p(q), q)] \, \mathrm{d}q$$

that market maker 2 earns by posting the tariff $t(q)$ is nonpositive in every continuation

[24] For the quadratic preferences considered by Biais, Martimort and Rochet (2000), Assumption D holds whenever $\theta_2 - \theta_1 \geq \theta_1 - c$.

equilibrium. To do so, we compute an upper bound by maximizing pointwise the function $m_1\tilde{v}_1^2(p',q) + m_2\tilde{v}_2^2(p',q)$ with respect to $p' < c_2$, for any given q. By doing so, we only neglect the requirement that the marginal price $p(q)$ be nondecreasing in q, as implied by the convexity of $t(q)$.

Suppose first that p' is such that $q \geq D_1(p')$. In this case, $\tilde{v}_1^2(p',q) = 0$, and $\tilde{v}_2^2(p',q) \leq 0$ as $p' \leq c_2$, implying that $m_1\tilde{v}_1^2(p',q) + m_2\tilde{v}_2^2(p',q)$. Consider next the case $q < D_1(p')$. Then, according to Assumption SC-U, we also have $q < D_2(p')$. Using the fact that $p \geq \bar{c}_1$, we get

$$D_1(p) \leq D_1(\bar{c}_1) \leq D_2(p') - D_1(p'),$$

where the last inequality follows from Assumption D. Taken together, these inequalities imply that market maker 2's profit can be rewritten as

$$m_1(p' - c_1)[1 - F(p')] + m_2(p' - c_2).$$

This is equal to zero if $\bar{c}_1 \leq p' < c_2$ and strictly negative if $p' < \bar{c}_1$ whenever the distribution F is given by

$$F(p) = \frac{p - \bar{c}_1}{m_1(p - c_1)} \tag{28}$$

over the support $[\bar{c}_1, c_2]$. Observe that F is strictly increasing over this interval, with $F(\bar{c}_1) = 0$ and $F(c_2) = 1$, as requested. This shows that, if market maker 1 randomizes according to the distribution (28), market maker 2 cannot earn a positive expected profit by deviating to an arbitrary convex tariff.

To complete the argument, let the tariff $t(q)$ be the limit order $p' \min\{q, D_1(p')\}$ for any given p' in the support of F. Proceeding as above, it is straightforward to check that

$$\int [m_1\tilde{v}_1^2(p',q) + m_2\tilde{v}_2^2(p',q)] \, \mathrm{d}q = 0,$$

which in turn guarantees that the proposed mixed strategies support an equilibrium with zero expected profit.

The proof shows how the uncertainty generated by the mixed strategy played by market maker 1 prevents market maker 2 from profitably attracting type 1. In particular, equilibrium strategies are such that, for any realization of uncertainty, the two types trade at different unit prices, with type 1 purchasing her optimal demand at the lowest of the quoted prices, and type 2 purchasing her first-best quantity $D_2(c_2)$. The latter trade, however, typically takes place at a unit price higher than in the Glosten (1994) tariff.

Three properties of this mixed-strategy equilibrium are worth emphasizing.

1. First, as already mentioned, Assumption D allows for $D_1(\bar{c}_1) > 0$. Yet the mixed-strategy equilibrium constructed above boils down to the pure-strategy equilibrium constructed in Theorem 3 in the limiting case where condition (18) holds, so that $D_1(\bar{c}_1) = (u_1')^{-1}(\bar{c}_1) = 0$. In that case, all the limit orders $p' \min\{q, D_1(p')\}$, $p' \in [\bar{c}_1, c_2]$ are degenerate and all trades take place at unit price c_2.

2. Second, each market maker earns zero expected profit in equilibrium. This contrasts with the positive-profit equilibria constructed by Biais, Martimort, and Rochet (2000) and Back and Baruch (2013) in models with a continuum of types. This also contrasts with the mixed-strategy equilibria of Bertrand–Edgeworth oligopoply described by Allen and Hellwig (1986), though the equilibrium strategies are similar. In Bertrand–Edgeworth oligopoly, positive profits emerge in equilibrium because firms have strictly convex costs. By contrast, the threat of adverse selection in our model drives expected profits down to zero in spite of the linearity of costs.

3. We have supposed for simplicity that $K = 2$, but our argument straightforwardly extends to $K > 2$. One can, for instance, assume that market makers 1 and 2 play as prescribed, while the remaining market makers only offer the null trade. In this case, given the random limit orders posted by market makers 1 and 2, any inactive market maker is facing an increased uncertainty over the relevant distribution of market prices. From his viewpoint, this makes it harder to attract type 1 and prevents any profitable deviation. Interestingly, this shows that the Glosten (1994) allocation does not necessarily emerge when the number of market makers goes to infinity.

10 Conclusion

In this article, we have studied the impact of adverse selection in a canonical model of price competition in which uninformed market makers compete in tariffs to serve an insider whose private information has an arbitrary discrete distribution. Our results markedly differ from those so far obtained in the literature, assuming continuous type distributions. Indeed, pure-strategy equilibria in our model feature linear pricing, while the market maker's ability to restrict the quantity they offer at the equilibrium price leads to an extreme form of market breakdown. This, in turn, implies that pure-strategy equilibria typically fail to exist in sufficiently fine discretizations of existing continuous-type models, for which pure-strategy equilibria with strictly convex tariffs are known to exist under certain parametric assumptions on preferences and information. This tension between discrete and continuous models can be reduced by considering ε-equilibria of games with a large number of market makers or with a large number of insider types. Still, we also provide a robust example of a mixed-strategy equilibrium with zero expected profits that bears no apparent relationship with existing equilibrium candidates. This finding suggests that further investigations of mixed-strategy equilibria are in order to reach a full understanding of the consequences of adverse selection for competitive markets.

Appendix

Proof That the Functions z_i^{-k} Are Weakly Quasiconcave. Fix a type i and a market maker k. For the sake of clarity, we hereafter omit the indices i and k in this proof. Let (q,t) and (q',t') be two trades, and let Q^- and $Q^{-\prime}$ be the associated solutions to (3). For each $\lambda \in [0,1]$, $z^-(\lambda q + (1-\lambda)q', \lambda t + (1-\lambda)t')$ is at least

$$U(\lambda q + (1-\lambda)q' + \lambda Q^- + (1-\lambda)Q^{-\prime}, \lambda t + (1-\lambda)t' + T^-(\lambda Q^- + (1-\lambda)Q^{-\prime}))$$

because $\lambda Q^- + (1-\lambda)Q^{-\prime}$ is an admissible candidate in (3). Because T^- is convex and U is decreasing in transfers, this lower bound is itself at least

$$U(\lambda(q+Q^-) + (1-\lambda)(q'+Q^{-\prime}), \lambda[t + T^-(q+Q^-)] + (1-\lambda)[t' + T^-(q'+Q^{-\prime})]),$$

and because U is quasiconcave this expression is at least

$$\min\{U(q+Q^-, t+T^-(Q^-)), U(q'+Q^{-\prime}, t'+T^-(Q^{-\prime}))\},$$

which is $\min\{z^-(q,t), z^-(q',t')\}$ by construction. The result follows. ■

Proof of Property SC-*z*. Fix some k, $q < q'$, t, and t'. Let $\mathcal{T}(Q) \equiv t + T^{-k}(Q-q)$, defined for $Q \geq q$. Similarly, let $\mathcal{T}'(Q) \equiv t' + T^{-k}(Q-q')$, defined for $Q \geq q'$. According to (3), for each i, computing $z_i^{-k}(q,t)$ amounts to maximizing $U_i(Q, \mathcal{T}(Q))$ with respect to $Q \geq q$. Let $\mathcal{Q}_i \geq q$ be the solution to this problem; it is unique as U_i is strictly quasiconcave and strictly decreasing in aggregate transfers, and $\mathcal{T}(Q)$ is convex in Q. Similarly, computing $z_i^{-k}(q',t')$ amounts to maximizing $U_i(Q, \mathcal{T}'(Q))$ with respect to $Q \geq q'$. Let $\mathcal{Q}_i' \geq q'$ be the unique solution to this problem. The proof consists of two steps.

Step 1 We first prove (5). Suppose that

$$z_i^{-k}(q,t) < z_i^{-k}(q',t')$$

for some $i < I$ and let $i' > i$. Because $\mathcal{Q}_{i'} \geq q$ is an admissible candidate in the problem that defines $z_i^{-k}(q,t)$, we must have

$$U_i(\mathcal{Q}_{i'}, \mathcal{T}(\mathcal{Q}_{i'})) \leq z_i^{-k}(q,t) < z_i^{-k}(q',t') = U_i(\mathcal{Q}_i', \mathcal{T}'(\mathcal{Q}_i')). \tag{29}$$

Two cases may now arise:

(i) Suppose first that $\mathcal{Q}_{i'} < \mathcal{Q}_i'$. Then

$$z_{i'}^{-k}(q,t) = U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}(\mathcal{Q}_{i'})) < U_{i'}(\mathcal{Q}_i', \mathcal{T}'(\mathcal{Q}_i')) \leq z_{i'}^{-k}(q',t'),$$

where the first inequality follows from (29), Assumption SC-U, and the assumptions that $i < i'$ and $\mathcal{Q}_{i'} < \mathcal{Q}_i'$, and the second inequality follows from the fact that $\mathcal{Q}_i' \geq q'$ is an admissible candidate in the problem that defines $z_{i'}^{-k}(q',t')$. This shows (5).

(ii) Suppose next that $\mathcal{Q}_{i'} \geq \mathcal{Q}_i'$. Because $\mathcal{Q}_i' \geq q' > q$ is an admissible candidate in the problem that defines $z_i^{-k}(q,t)$, we have

$$U_i(\mathcal{Q}'_i, \mathcal{T}(\mathcal{Q}'_i)) \le z_i^{-k}(q,t) < z_i^{-k}(q',t') = U_i(\mathcal{Q}'_i, \mathcal{T}'(\mathcal{Q}'_i)),$$

which shows $\mathcal{T}'(\mathcal{Q}'_i) < \mathcal{T}(\mathcal{Q}'_i)$. Moreover, because $q < q'$ and T^{-k} is convex, $\mathcal{T}'(Q) - \mathcal{T}(Q)$ is nonincreasing in Q for $Q \ge q'$. Because $\mathcal{Q}_{i'} \ge \mathcal{Q}'_i$, this shows $\mathcal{T}'(\mathcal{Q}_{i'}) < \mathcal{T}(\mathcal{Q}_{i'})$. Now, as $\mathcal{Q}_{i'} \ge \mathcal{Q}'_i \ge q'$, $\mathcal{Q}_{i'}$ is an admissible candidate in the problem that defines $z_{i'}^{-k}(q',t')$ and thus

$$U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}'(\mathcal{Q}_{i'})) \le z_{i'}^{-k}(q',t').$$

Hence, as $\mathcal{T}'(\mathcal{Q}_{i'}) < \mathcal{T}(\mathcal{Q}_{i'})$, we directly obtain

$$z_{i'}^{-k}(q,t) = U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}(\mathcal{Q}_{i'})) < U_{i'}(\mathcal{Q}_{i'}, \mathcal{T}'(\mathcal{Q}_{i'})) \le z_{i'}^{-k}(q',t').$$

This shows (5).

Step 2 The proof of (4) follows from (5) by continuity. Indeed, suppose that $z_i^{-k}(q,t) = z_i^{-k}(q',t')$ for some $i < I$ and let $i' > i$. Then, for each $\varepsilon > 0$, $z_i^{-k}(q,t+\varepsilon) < z_i^{-k}(q',t')$ and thus $z_{i'}^{-k}(q,t+\varepsilon) < z_{i'}^{-k}(q',t')$ from (5) as $i < i'$ and $q < q'$. Because $z_{i'}^{-k}$ is continuous, one can take limits as ε goes to zero to obtain (4). The result follows. ■

Proof of Lemma 1. Fix a market maker k. For the sake of clarity, we hereafter omit the index k in this proof. Fix a menu $\mu^* = \{(0,0), \ldots, (q_i^*, t_i^*), \ldots\}$ with nondecreasing quantities that satisfies (8) for all i. The proof consists of two steps.

Step 1 First, we establish that there exists a menu $\mu = \{(0,0), \ldots, (q_i, t_i), \ldots\}$ with nondecreasing quantities that satisfies the following properties:

(a) $\sum_i m_i v_i(q_i, t_i) \ge \sum_i m_i v_i(q_i^*, t_i^*)$.

(b) For each $i \ge 1$, $z_i^-(q_i, t_i) \ge z_i^-(q_{i-1}, t_{i-1})$.

(c) For each $i \ge 2$, if $q_i > q_{i-1}$, then $z_{i-1}^-(q_{i-1}, t_{i-1}) > z_{i-1}^-(q_i, t_i)$.

Notice that (b) is identical to (8), whereas (c) is a strict version of the upward local incentive-compatibility constraints. We proceed by contradiction and assume that there is no menu that satisfies (a), (b), and (c). Nevertheless, the set of menus with nondecreasing quantities that satisfy (a) and (b) is nonempty, as it contains μ^*. Therefore, one can select in this set a menu μ that maximizes the index $i' \ge 2$ of the first violation of (c). For this index i', we have $q_{i'} > q_{i'-1}$.

One can even impose that the menu μ satisfy (b) as an equality at $i = i'$. Indeed, if (b) is a strict inequality at i', one can increase $t_{i'}$ until reaching an equality: this is feasible because $z_{i'}^-$ is weakly quasiconcave and strictly decreasing in t. This change in $t_{i'}$ defines a new menu that still satisfies (a), (b) for all i (with an equality at $i = i'$), and (c) for all $i < i'$; but, according to our definition of μ, (c) is violated at $i = i'$. With a slight abuse of notation, we call this menu $\mu = \{(0,0), \ldots, (q_i, t_i), \ldots\}$ again.

Now, because (b) holds as an equality at i' and because $q_{i'} > q_{i'-1}$, from the contraposition of (5) in property SC-z we get $z^-_{i'-1}(q_{i'-1}, t_{i'-1}) \geq z^-_{i'-1}(q_{i'}, t_{i'})$. Recall, however, that (c) is violated at i'. The only remaining possibility is thus that this inequality is in fact an equality. So (b) and (c) are equalities at i' and we face a cycle of binding incentive constraints that we now eliminate by pooling both types on the same quantity. Two cases may arise:

(i) Suppose first that $v_{i'}(q_{i'}, t_{i'}) \leq v_{i'}(q_{i'-1}, t_{i'-1})$. Then one can build a new menu μ' from μ by allocating $(q_{i'-1}, t_{i'-1})$ to types $i'-1$ and i'. (a) is relaxed by construction. (b) and (c) are unaffected for $i < i'$ and trivially hold at $i = i'$ as types $i'-1$ and i' are pooled on the same trade. Finally, (b) also holds for $i > i'$, because, by Property SC-z, the downward incentive-compatibility constraints are satisfied as soon as the downward local incentive-compatibility are satisfied. But then any violation of (c) for the new menu μ' would have to take place for types strictly above i', in contradiction with our definition of μ.

(ii) So it must be that $v_{i'}(q_{i'}, t_{i'}) > v_{i'}(q_{i'-1}, t_{i'-1})$. Then one can build a new menu μ' from μ by allocating $(q_{i'}, t_{i'})$ to types $i'-1$ and i'. (a) is relaxed because, as $q_{i'} > q_{i'-1}$, we can apply the contraposition of SC-v to obtain $v_{i'-1}(q_{i'}, t_{i'}) > v_{i'-1}(q_{i'-1}, t_{i'-1})$. (b) and (c) are unaffected for $i < i'-1$ and trivially hold at $i = i'$ as types $i'-1$ and i' are pooled on the same trade. (b) is unaffected for $i > i'$. At $i = i'-1$, because (c) was an equality at $i = i'$ for the menu μ, the change from μ to μ' does not affect type $i'-1$'s payoff and so (b) holds at $i'-1$. There remains to check that (c) holds at $i = i'-1$ (in the case $i' \geq 3$). As (c) at i' is an equality in the menu μ, the contraposition of (5) in SC-z implies that

$$z^-_{i'-2}(q_{i'-1}, t_{i'-1}) \geq z^-_{i'-2}(q_{i'}, t_{i'}).$$

We also know that (c) holds for the menu μ at $i = i'-1$ and hence

$$z^-_{i'-2}(q_{i'-2}, t_{i'-2}) > z^-_{i'-2}(q_{i'-1}, t_{i'-1}).$$

These inequalities together imply that (c) holds at $i = i'-1$. But once more we get a contradiction, as μ' verifies (a), (b), and (c) for $i \leq i'$.

Step 2 In Step 1, we established that, for any menu μ^* with nondecreasing quantities that satisfies (8), there exists a menu μ with nondecreasing quantities that yields market maker k at least as much expected profit as μ^* and that satisfies properties (b) and (c). By continuity of the functions z^-_i, one can then slightly reduce each transfer in the menu μ to get a menu μ' so that both (b) and (c) now hold as strict inequalities. Hence the local incentive-compatibility and type 1's individual-rationality constraint in μ' are slack. Property SC-z together with the fact that quantities in the menu μ' are nondecreasing then ensure that, when faced with μ', the insider has a unique best response. As the reduction in transfers in μ' relative to μ is arbitrarily small, we get that market maker k can approximate his expected profit in μ and, a fortiori, his expected profit in μ^*. The result follows. ■

Proof of Lemma 2. We begin with some preliminary remarks on the insider's best response when facing an arbitrary profile of convex tariffs $(t^1, \ldots, t^K)$.

Step 0 Recall that, given an arbitrary profile $(t^1, \ldots, t^K)$ of convex tariffs, the aggregate demand Q_i of type i is uniquely defined and nondecreasing in i. Given Q_i, type i's utility-maximization problem (1) reduces to minimizing her total payment for Q_i, $T(Q_i)$, as defined by problem (2). This is a convex problem, so that, by the Kuhn–Tucker theorem, one can associate to any of its solutions $(q_i^1, \ldots, q_i^K)$ a Lagrange multiplier p_i such that $p_i \in \partial t^k(q_i^k)$ for all k. If there were two different solutions $(q_i^1, \ldots, q_i^K)$ and $(q_i'^1, \ldots, q_i'^K)$ to (1) with different multipliers $p_i < p_i'$, then, because each tariff is convex, one would obtain $q_i^k \leq q_i'^k$ for all k and because both solutions must sum to the same Q_i they would be identical, a contradiction. This shows that two different solutions must share the same p_i. Thus one can associate to each type i a price p_i such that, whatever the solution $(q_i^1, \ldots, q_i^K)$ to (2), one has $p_i \in \partial t^k(q_i^k)$ for all k. Finally, we can without loss of generality adopt the convention that p_i is nondecreasing in i. Indeed, if $p_i > p_{i+1}$ for some $i < I$, then, because $p_i \in \partial t^k(q_i^k)$ and $p_{i+1} \in \partial t^k(q_{i+1}^k)$ for all k, one has $q_i^k \geq q_{i+1}^k$ for all k. Because these quantities sum to Q_i and Q_{i+1}, respectively, and because $Q_i \leq Q_{i+1}$, it actually follows that $q_i^k = q_{i+1}^k$ for all k. Hence $p_i \in \partial t^k(q_{i+1}^k)$ for all k and one can replace p_{i+1} by p_i. Given this convention, $\underline{s}^k(p_i)$ and $\overline{s}^k(p_i)$ are nondecreasing in i for all k.

Now, suppose that $(t^1, \ldots, t^K)$ are equilibrium tariffs and that market maker k deviates to some convex tariff t. Let $(q_1, \ldots, q_I)$ be a nondecreasing family of quantities such that (10) holds for all i. We know from Property SC-z that such a family exists. Letting $p_i \in \partial t(q_i)$ be a Lagrange multiplier for type i's problem of minimizing her total payment, one may according to Step 0 impose that p_i be nondecreasing in i. In the present quasilinear setting with differentiable strictly concave utility functions u_i, we actually have that each type i purchases $D_i(p_i) = (u_i')^{-1}(p_i)$ in the aggregate, which uniquely pins down the value of p_i. The proof consists of four steps.

Step 1 Letting $\boldsymbol{p} \equiv (p_1, \ldots, p_I)$ and $\boldsymbol{q} \equiv (q_1, \ldots, q_I)$, construct the piecewise linear tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ such that $t_{\boldsymbol{p},\boldsymbol{q}}(0) = 0$ and

$$t_{\boldsymbol{p},\boldsymbol{q}}(q) = t_{\boldsymbol{p},\boldsymbol{q}}(q_{i-1}) + p_i(q - q_{i-1})$$

for all i and $q \in (q_{i-1}, q_i]$, where $q_0 \equiv 0$ by convention. Because the price and quantity families $(p_1, \ldots, p_I)$ and $(q_1, \ldots, q_I)$ are nondecreasing, the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ is convex. It is straightforward to check that $t_{\boldsymbol{p},\boldsymbol{q}}(q_i) \geq t(q_i)$ for all i.[25] Moreover, because $p_i = \partial^- t_{\boldsymbol{p},\boldsymbol{q}}(q_i)$, it remains a best response for any type i to purchase q_i from market maker k if the tariffs $(t_{\boldsymbol{p},\boldsymbol{q}}, t^{-k})$ are posted. In fact, under quasilinearity, $t_{\boldsymbol{p},\boldsymbol{q}}$ is the highest convex tariff with the property that the family $(q_1, \ldots, q_I)$ is a best response of the insider to this tariff, given the equilibrium tariffs t^{-k} of the market makers other than k (see Figure 2).

Step 2 According to Step 1, we can henceforth consider that market maker k deviates to the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$. As in Footnote 11, define the interval $[\underline{s}^k(p_i), \overline{s}^k(p_i)] \equiv \{q : p_i \in \partial t_{\boldsymbol{p},\boldsymbol{q}}(q)\}$ for any type i. Define also a family $(\overline{q}_1, \ldots, \overline{q}_I)$ as follows:

[25] An important observation is that one will have $t_{\boldsymbol{p},\boldsymbol{q}}(q_i) > t(q_i)$ for some i if and only if t is not itself of the form $t_{\boldsymbol{p},\boldsymbol{q}}$ given a nondecreasing family of quantities $(q_1, \ldots, q_I)$ such that the constraints (10) hold for all i. Thus t must be of the form $t_{\boldsymbol{p},\boldsymbol{q}}$ at a solution of the problem defining $V_{\text{co}}^k(t^{-k})$ and at least one type must trade at any kink of t.

(i) If $\underline{s}^k(p_i) < \overline{s}^k(p_i)$ and if $I_i^+ \equiv \{i' : p_{i'} = p_i > c_{i'}\} \neq \emptyset$, set $\overline{q}_i \equiv \max\{q_{i'} : i' \in I_i^+\}$.

(ii) Otherwise, set $\overline{q}_i \equiv \underline{s}^k(p_i)$.

Observe that the family $(\overline{q}_1, \ldots, \overline{q}_I)$ is nondecreasing. Intuitively, there is a single value of $\overline{q}$ for each value of p in $\{p_1, \ldots, p_I\}$: below $\overline{q}$, one finds all the types with $p > c_i$ who trade at price p and for which market maker k would like to increase trade. Above $\overline{q}$, the opposite holds: because $p \leq c_i$, market maker k would like to reduce the quantity he trades with these types.

Step 3 One way for market maker k to achieve these objectives is to reduce the slope of the tariff $t_{\boldsymbol{p},\boldsymbol{q}}$ on quantities between $\underline{s}^k(p_i)$ and $\overline{q}_i$ and to increase it between $\overline{q}_i$ and $\overline{s}^k(p_i)$. Consider accordingly a small positive ε and let $\hat{t} \equiv t_{\boldsymbol{p}-\varepsilon\mathbf{1}_I,\overline{\boldsymbol{q}}}$, where $\mathbf{1}_I \equiv (1, \ldots, 1) \in \mathbb{R}^I$ and $\overline{\boldsymbol{q}} \equiv (\overline{q}_1, \ldots, \overline{q}_I)$. Note that, for any type i, we have $\partial^-\hat{t}(\overline{q}_i) \leq p_i - \varepsilon < p_i < \partial^+\hat{t}(\overline{q}_i)$, so that slopes were changed in the right directions (see Figure 3). Let $(\hat{q}_1, \ldots, \hat{q}_I)$ be any best response of the insider to the tariff $\hat{t}$, given the equilibrium tariffs t^{-k} of the market makers other than k. Given the definition of $\overline{q}_i$, two cases must be distinguished:

(i) If $p_i > c_i$, then $\underline{s}^k(p_i) \leq q_i \leq \overline{q}_i$. Then, because for each $q \leq q_i$ the tariff $\hat{t}$ satisfies

$$\partial^-\hat{t}(q) \leq \partial^-\hat{t}(\overline{q}_i) \leq p_i - \varepsilon < p_i$$

and type i has quasilinear preferences, one must have $\hat{q}_i \geq q_i$.

(ii) If $p_i \leq c_i$, then $\overline{q}_i \leq q_i \leq \overline{s}^k(p_i)$. Then, because for each $q \geq q_i$ the tariff $\hat{t}$ satisfies

$$\partial^+\hat{t}(q) \geq \partial^+\hat{t}(\overline{q}_i) > p_i$$

and type i has quasilinear preferences, one must have $\hat{q}_i \leq q_i$.

Step 4 Finally, for all q and ε, we have $\hat{t}(q) = t_{\boldsymbol{p}-\varepsilon\mathbf{1}_I,\overline{\boldsymbol{q}}}(q) \geq t_{\boldsymbol{p},\boldsymbol{q}}(q) - O(\varepsilon)$ (see Figure 3). Thus, for any best response $(\hat{q}_1, \ldots, \hat{q}_I)$ of the insider to the tariff $\hat{t}$, given the equilibrium tariffs t^{-k} of the market makers other than k, we have

$$\begin{aligned}\sum_i m_i[\hat{t}(\hat{q}_i) - c_i\hat{q}_i] &\geq \sum_i m_i[t_{\boldsymbol{p},\boldsymbol{q}}(\hat{q}_i) - c_i\hat{q}_i] - O(\varepsilon)\\ &\geq \sum_i m_i[t_{\boldsymbol{p},\boldsymbol{q}}(q_i) - c_i q_i] - O(\varepsilon)\\ &\geq \sum_i m_i[t(q_i) - c_i q_i] - O(\varepsilon),\end{aligned}$$

where the second inequality follows from the fact that $\hat{q}_i \leq q_i$ if $p_i \leq c_i$ and $\hat{q}_i \geq q_i$ if $p_i > c_i$ by Step 3, and the third inequality follows from Step 1. Hence, by posting the tariff $\hat{t}$, market maker k can secure an expected profit within $O(\varepsilon)$ of $\sum_i m_i[t(q_i) - c_i q_i]$, where ε is arbitrarily small. The result follows. ■

Proof of Proposition 2. As a preliminary remark, observe that, if $(t^1, \ldots, t^K, s)$ is an equilibrium with nondecreasing individual quantities, then, from Lemma 2, each market maker k must earn an expected profit $V_{\text{co}}^k(t^{-k})$. Thus the tariff t^k is a solution to the problem that defines $V_{\text{co}}^k(t^{-k})$. According to Footnote 25, this implies that the tariff t^k is piecewise linear and that at least one type trades at any kink of t^k. Recall from Footnote 11 that $\underline{s}^k(p) = \inf\{q : p \in \partial t^k(q)\}$, $\overline{s}^k(p) = \sup\{q : p \in \partial t^k(q)\}$, and $\overline{S}(p) = \sum_k \overline{s}^k(p)$ for all k and p. Similarly let $\underline{S}(p) \equiv \sum_k \underline{s}^k(p)$ for all p. The proof consists of four steps.

Step 1 For each $Q \geq 0$, define $T(Q)$ as in (2) to be the minimal aggregate transfer the insider has to make in return for the aggregate quantity Q and let $p \equiv \partial^- T(Q_I)$ be the highest price at which trade takes place. Because any market maker k who supplies quantities beyond $\underline{s}^k(p)$ at price p to types i such that $u_i'(Q_i) \geq p$ must have an incentive to do so, one must have

$$\sum_{\{i\,:\,u_i'(Q_i)\geq p\}} m_i(p - c_i)[q_i^k - \underline{s}^k(p)] \geq 0 \tag{30}$$

for all k. We now show that, for each k, (30) holds as an equality. To this end, note that any market maker k could deviate by posting a tariff equal to t^k up to $\underline{s}^k(p)$, and then offering to sell any additional quantity between $\underline{s}^k(p)$ and $\overline{S}(p)$ at price p. A best response for the insider is to continue purchasing the equilibrium quantity q_i^k from market maker k if $u_i'(Q_i) < p$ and to purchase Q_i from market maker k if $u_i'(Q_i) \geq p$. One can thus apply Lemma 2 to conclude that

$$\begin{aligned}\sum_{\{i\,:\,u_i'(Q_i)\geq p\}} m_i(p - c_i)[Q_i - \underline{s}^k(p)] &\leq \sum_{\{i\,:\,u_i'(Q_i)\geq p\}} m_i(p - c_i)[q_i^k - \underline{s}^k(p)] \\ &\leq \sum_{\{i\,:\,u_i'(Q_i)\geq p\}} m_i(p - c_i)[Q_i - \underline{S}(p)] \end{aligned}\tag{31}$$

for all k, where the second inequality follows from the inequalities (30). Summing the inequalities (31) over k yields

$$(K-1)\underline{S}(p) \sum_{\{i\,:\,u_i'(Q_i)\geq p\}} m_i(p - c_i) \leq 0$$

and this inequality is strict as soon as (30) is strict for some k. If this were the case, then, as $K > 1$, we would have $\sum_{\{i\,:\,u_i'(Q_i)\geq p\}} m_i(p - c_i) < 0$, which, because $Q_i \geq \underline{S}(p)$ for all i such that $u_i'(Q_i) \geq p$ and because Q_i and c_i are nondecreasing in i, would contradict the fact that $\sum_{\{i\,:\,u_i'(Q_i)\geq p\}} m_i(p - c_i)[Q_i - \underline{S}(p)] > 0$ when (30) holds for all k with at least one strict inequality. It follows that all the inequalities (30) are in fact equalities, as claimed.

Step 2 From now on, suppose by way of contradiction that some trades take place at a price strictly lower than p and let p' be the highest such price, that is, $p' \equiv \partial^- T(\underline{S}(p))$ and $\overline{s}^k(p') = \underline{s}^k(p)$ for all k. We follow the same procedure as in Step 1. First, because any market maker k that supplies quantities beyond $\underline{s}^k(p')$ at price p' to types i such that $p > u_i'(Q_i) \geq p'$ must have an incentive to do so and because, according to Step 1, he does not make any additional profit trading at price p in equilibrium, one must have

$$\sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p' - c_i)[\min\{q_i^k, \overline{s}^k(p')\} - \underline{s}^k(p')] \geq 0 \tag{32}$$

for all k. Second, in analogy with (30), we show that, for each k, (32) holds as an equality. To this end, note that any market maker k could deviate by posting a tariff equal to t^k up to $\underline{s}^k(p')$, and then offering to sell any additional quantity between $\underline{s}^k(p')$ and $\overline{S}(p') = \underline{S}(p)$ at price p'. A best response for the insider is to continue purchasing the equilibrium quantity q_i^k from market maker k if $p' > u_i'(Q_i)$, to purchase Q_i from market maker k if $p > u_i'(Q_i) \geq p'$, and to purchase $\overline{S}(p')$ from market maker k if $u_i'(Q_i) \geq p$. Because, according to Step 1, market maker k does not make any additional profit trading at price p in equilibrium, one can thus apply Lemma 2 to conclude that, in analogy with (31),

$$\sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p'-c_i)[\min\{Q_i,\overline{S}(p')\} - \underline{s}^k(p')]$$

$$\leq \sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p'-c_i)[\min\{q_i^k,\overline{s}^k(p')\} - \underline{s}^k(p')]$$

$$\leq \sum_{\{i\,:\,u_i'(Q_i)\geq p'\}} m_i(p'-c_i)[\min\{Q_i,\overline{S}(p')\} - \underline{S}(p')].$$

One can then proceed as in Step 1 to show that the inequalities (32) are in fact equalities, as claimed.

Step 3 The upshot of Steps 1–2 is that, if trades take place at prices p and p', no market maker can make additional profits on these trades. We now show that this leads to a contradiction, thereby establishing that all trades must take place at price p. Note that according to our preliminary remark, there exists at least one type who exhausts supply at price p', that is, who purchases $\overline{s}^k(p')$ from each market maker k and thus has a unique best response to the equilibrium tariffs $(t^1,\ldots,t^K)$. Let i_0 be the lowest such type; all types $i_0,\ldots,I$ then exhaust supply at price p'. It follows from Step 2 that $p' \leq \mathbf{E}[c_{\tilde{i}}\,|\,\tilde{i}\geq i_0] = \sum_{i\geq i_0} m_i c_i / \sum_{i\geq i_0} m_i$. This must hold as an equality, for, otherwise, some market maker k would have an incentive to offer less than $\overline{s}^k(p')$ at price p'. Now, either $i_0 = 1$ and, for each k, $q_{i_0-1}^k \equiv 0$ by convention, or $i_0 > 1$ and, by definition of i_0, there exists some k such that

$$\max\{\underline{s}^k(p'), q_{i_0-1}^k\} < \overline{s}^k(p').$$

Take any such k and let $\underline{q}^k \equiv \max\{\underline{s}^k(p'), q_{i_0-1}^k\}$, so that $t^k(\overline{s}^k(p')) = t^k(\underline{q}^k) + p'[\overline{s}^k(p') - \underline{q}^k]$. Because type i_0 has a unique best response to $(t^1,\ldots,t^K)$, there exists $\varepsilon > 0$ such that

$$z_{i_0}^{-k}(\overline{s}^k(p'), t^k(\overline{s}^k(p')) + \varepsilon[\overline{s}^k(p') - \underline{q}^k]) > z_{i_0}^{-k}(q, t^k(q)) \tag{33}$$

for all $q \leq \underline{q}^k$. Define

$$\overline{q}^k \equiv \max\{\arg\max\{z_{i_0}^{-k}(q, t^k(\underline{q}^k) + (p'+\varepsilon)[q - \underline{q}^k]) : q \in [\underline{q}^k, \overline{s}^k(p')]\}\}. \tag{34}$$

Then, because

$$t^k(\overline{s}^k(p')) + \varepsilon[\overline{s}^k(p') - \underline{q}^k] = t^k(\underline{q}^k) + (p'+\varepsilon)[\overline{s}^k(p') - \underline{q}^k],$$

it follows from (33) that $\underline{q}^k < \overline{q}^k \leq \overline{s}^k(p')$. Market maker k could deviate by posting a tariff equal to t^k up to $\underline{q}^k$, and then offering to sell any additional quantity between $\underline{q}^k$ and $\overline{q}^k$

at price $p' + \varepsilon$. A best response for the insider is to continue purchasing the equilibrium quantity q_i^k from market maker k if $i \leq i_0 - 1$ and, according to (34) along with the single-crossing property (4), to purchase the quantity $\overline{q}^k$ from market maker k if $i \geq i_0$. Because, according to Step 1, market maker k does not make any additional profit trading at price p in equilibrium, one can thus apply Lemma 2 to conclude that

$$\sum_{i \geq i_0} m_i(p' + \varepsilon - c_i)(\overline{q}^k - \underline{q}^k) \leq 0,$$

which, because $\varepsilon > 0$ and $\overline{q}^k > \underline{q}^k$, contradicts the above-noted fact that $p' = \mathbf{E}[c_{\tilde{i}} \,|\, \tilde{i} \geq i_0]$. Hence all trades must take place at price p, as claimed.

Step 4 At price p, either aggregate supply is zero and there is no trade, or aggregate supply is positive and the insider faces a linear tariff with slope p. To complete the proof, we must show that, in the latter case, each type i can purchase her unconstrained demand $D_i(p)$ at price p. Indeed, otherwise, some type would exhaust supply at price p and would thus have a unique best response to the equilibrium tariffs $(t^1, \ldots, t^K)$. Let i_0 be the lowest such type; all types $i_0, \ldots, I$ would then exhaust supply at price p. Arguing as in Step 3, we get that this leads to a contradiction. Hence each type can freely choose her preferred quantity $D_i(p)$ at price p. Hence the result. ■

Proof of Lemma 3. Define $\nu_i^k(q) \equiv v_i^k(q, pq)$ for all q. In this proof, we more generally assume that the insider's type i is distributed over some subset $\mathcal{I}$ of $\mathbb{R}$. The corresponding distribution $\mathbf{m}$ may be discrete, continuous, or mixed. We also assume that the appropriate generalization of SC-v holds and that $\sup\{D_i(p) : i \in \mathcal{I}\} < \infty$. Now, observe that, if the nondecreasing quantities $(q_i)_{i \in \mathcal{I}}$ satisfy the constraints in (12)–(13), so do the quantities $(\min\{q_i, \overline{q}\})_{i \in \mathcal{I}}$ for all $\overline{q}$. Hence we can restrict our quest for a solution to (12)–(13) to the set of nondecreasing quantities $(q_i)_{i \in \mathcal{I}}$ such that (13) holds for each i and such that

$$\int \nu_i^k(\overline{q}) 1_{\{q_i \geq \overline{q}\}} \, \mathbf{m}(\mathrm{d}i) \leq \int \nu_i^k(q_i) 1_{\{q_i \geq \overline{q}\}} \, \mathbf{m}(\mathrm{d}i) \tag{35}$$

for all $\overline{q} \in [0, \|\tilde{q}\|_\infty)$, where $\|\tilde{q}\|_\infty \equiv \inf\{q : \mathbf{m}[\{i \in \mathcal{I} : q_i \leq q\}] = 1\}$ is the essential supremum of the quantities $(q_i)_{i \in \mathcal{I}}$. Note that this set is nonempty as it contains $(0)_{i \in \mathcal{I}}$. We now show that any $(q_i)_{i \in \mathcal{I}}$ in this set yields market maker k an expected profit at most equal to that provided by $(\min\{D_i(p), \|\tilde{q}\|_\infty)_{i \in \mathcal{I}}$. This is obvious if $\|\tilde{q}\|_\infty = 0$. If $\|\tilde{q}\|_\infty > 0$, then, for each $\varepsilon \in (0, \|\tilde{q}\|_\infty]$, applying (35) for $\overline{q} = \|\tilde{q}\|_\infty - \varepsilon$ yields that there exists a type i' such that $q_{i'} \geq \|\tilde{q}\|_\infty - \varepsilon$ and

$$\nu_{i'}^k(\|\tilde{q}\|_\infty - \varepsilon) \leq \nu_{i'}^k(q_{i'}).$$

Applying the contraposition of SC-v yields[26]

$$\nu_i^k(\|\tilde{q}\|_\infty - \varepsilon) \leq \nu_i^k(q_{i'})$$

[26] Strictly speaking, the contraposition of SC-v gives only that $v_{i'}^k(q', t') > v_{i'}^k(q, t)$ implies $v_i^k(q', t') > v_i^k(q, t)$. But because the profit functions are continuous and monotonic in transfers, one can easily show as in Step 2 of the proof to Property SC-z that $v_{i'}^k(q', t') \geq v_{i'}^k(q, t)$ implies $v_i^k(q', t') \geq v_i^k(q, t)$, which is the implication we use here.

for any type $i \leq i'$. Because the quantities $(q_i)_{i \in \mathcal{I}}$ are nondecreasing, this actually holds for any type i such that $q_i < \|\tilde{q}\|_\infty - \varepsilon$. As the functions ν_i^k are weakly quasiconcave, it follows that, for any type i such that $q_i < \|\tilde{q}\|_\infty - \varepsilon$, ν_i^k is nondecreasing over $[0, \|\tilde{q}\|_\infty - \varepsilon]$. Because this is true for all $\varepsilon > 0$, we have shown that, for any type i such that $q_i < \|\tilde{q}\|_\infty$, the function ν_i^k is nondecreasing over $[0, \|\tilde{q}\|_\infty]$. Hence market maker k could choose quantities equal to $(\min\{D_i(p), \|\tilde{q}\|_\infty\})_{i \in \mathcal{I}}$ without reducing his expected profit relative to $(q_i)_{i \in \mathcal{I}}$, as claimed. This implies that problem (12)–(13) reduces to

$$\sup\left\{\int \nu_i^k(\min\{D_i(p), \overline{q}\})\, \mathbf{m}(\mathrm{d}i) : \overline{q} \in [0, \sup\{D_i(p) : i \in \mathcal{I}\}]\right\},$$

which has a solution as the objective function is continuous in $\overline{q}$, owing to the fact that the functions $(\nu_i^k)_{i \in \mathcal{I}}$ are continuous along with Lebesgue's dominated convergence theorem. Hence (12)–(13) has a solution with limit-order quantities at price p. Finally, if the mappings ν_i^k are strictly quasiconcave, any solution to (12)–(13) is of this form. ■

Proof of Lemma 4. Consider a profile $(t^1, \ldots, t^K)$ of convex tariffs. Recall that the resulting optimal aggregate trade (Q_i, T_i) is uniquely determined for each type i and that one can associate to each type i a price p_i as in Step 0 of the proof of Lemma 2. To find an efficient risk-sharing allocation, one may first solve for each i

$$\max\left\{\sum_k v_i^k(q_i^k, t^k(q_i^k)) : (q_i^1, \ldots, q_i^K) \in A^1 \times \cdots \times A^K\right\}$$

subject to (14). Because the market makers have identical quasilinear profit functions, this problem can be rewritten as:

$$\min\left\{\sum_k c_i(q_i^k) : (q_i^1, \ldots, q_i^K) \in A^1 \times \cdots \times A^K\right\}$$

subject to

$$\sum_k q_i^k = Q_i$$

and

$$\underline{s}^k(p_i) \leq q_i^k \leq \overline{s}^k(p_i)$$

for all k, where the latter constraints ensure that the vector of trades $(q_i^1, \ldots, q_i^K)$ is indeed a best response of type i. We want to show that this family of problems indexed by i admits a family of solutions with nondecreasing individual quantities.

To do so, notice first that each of these problems is well behaved, with a nonempty compact set of solutions. Hence there exists a family of solutions $(q_1^1, \ldots, q_1^K, \ldots, q_I^1, \ldots, q_I^K)$ that minimizes the following measure of violations

$$\sum_k \sum_{i<I} \max\{q_i^k - q_{i+1}^k, 0\}. \tag{36}$$

Let us proceed by contradiction and suppose that this minimum is positive. Then, at the minimum, one has

$$q_i^k > q_{i+1}^k \tag{37}$$

for some k and $i < I$. Given that $\underline{s}^k(p_i)$ and $\overline{s}^k(p_i)$ are nondecreasing in i, this implies that

$$\underline{s}^k(p_i) \leq \underline{s}^k(p_{i+1}) \leq q_{i+1}^k < q_i^k \leq \overline{s}^k(p_i) \leq \overline{s}^k(p_{i+1}). \tag{38}$$

Because the intervals $[\underline{s}^k(p_i), \overline{s}^k(p_i)]$ and $[\underline{s}^k(p_{i+1}), \overline{s}^k(p_{i+1})]$ have a nontrivial intersection, it must be that $p_i = p_{i+1}$. Therefore, for any market maker k' we have $\underline{s}^{k'}(p_i) = \underline{s}^{k'}(p_{i+1})$ and $\overline{s}^{k'}(p_i) = \overline{s}^{k'}(p_{i+1})$. Moreover, because $q_i^k > q_{i+1}^k$ and $Q_i \leq Q_{i+1}$, we know that there exists $k' \neq k$ such that

$$q_i^{k'} < q_{i+1}^{k'}. \tag{39}$$

Using the equalities we have just shown, this implies that

$$\underline{s}^{k'}(p_i) = \underline{s}^{k'}(p_{i+1}) \leq q_i^{k'} < q_{i+1}^{k'} \leq \overline{s}^{k'}(p_i) = \overline{s}^{k'}(p_{i+1}). \tag{40}$$

Given (38) and (40), one can slightly reduce q_i^k and increase $q_i^{k'}$ by the same positive amount ε, so that all constraints are still verified. Such a transformation reduces the criterion (36), so that the resulting trade cannot be a solution to the problem for type i. Hence

$$c_i(q_i^k - \varepsilon) + c_i(q_i^{k'} + \varepsilon) > c_i(q_i^k) + c_i(q_i^{k'}).$$

By convexity, this implies $q_i^k \leq q_i^{k'}$. Alternatively, one can slightly increase q_{i+1}^k and reduce $q_{i+1}^{k'}$ by the same positive amount ε. We similarly obtain

$$c_{i+1}(q_{i+1}^k + \varepsilon) + c_{i+1}(q_{i+1}^{k'} - \varepsilon) > c_{i+1}(q_{i+1}^k) + c_{i+1}(q_{i+1}^{k'}),$$

which implies $q_{i+1}^k \geq q_{i+1}^{k'}$. But it is easily seen that these last two inequalities together with (37) and (39) yield a contradiction. The result follows. ■

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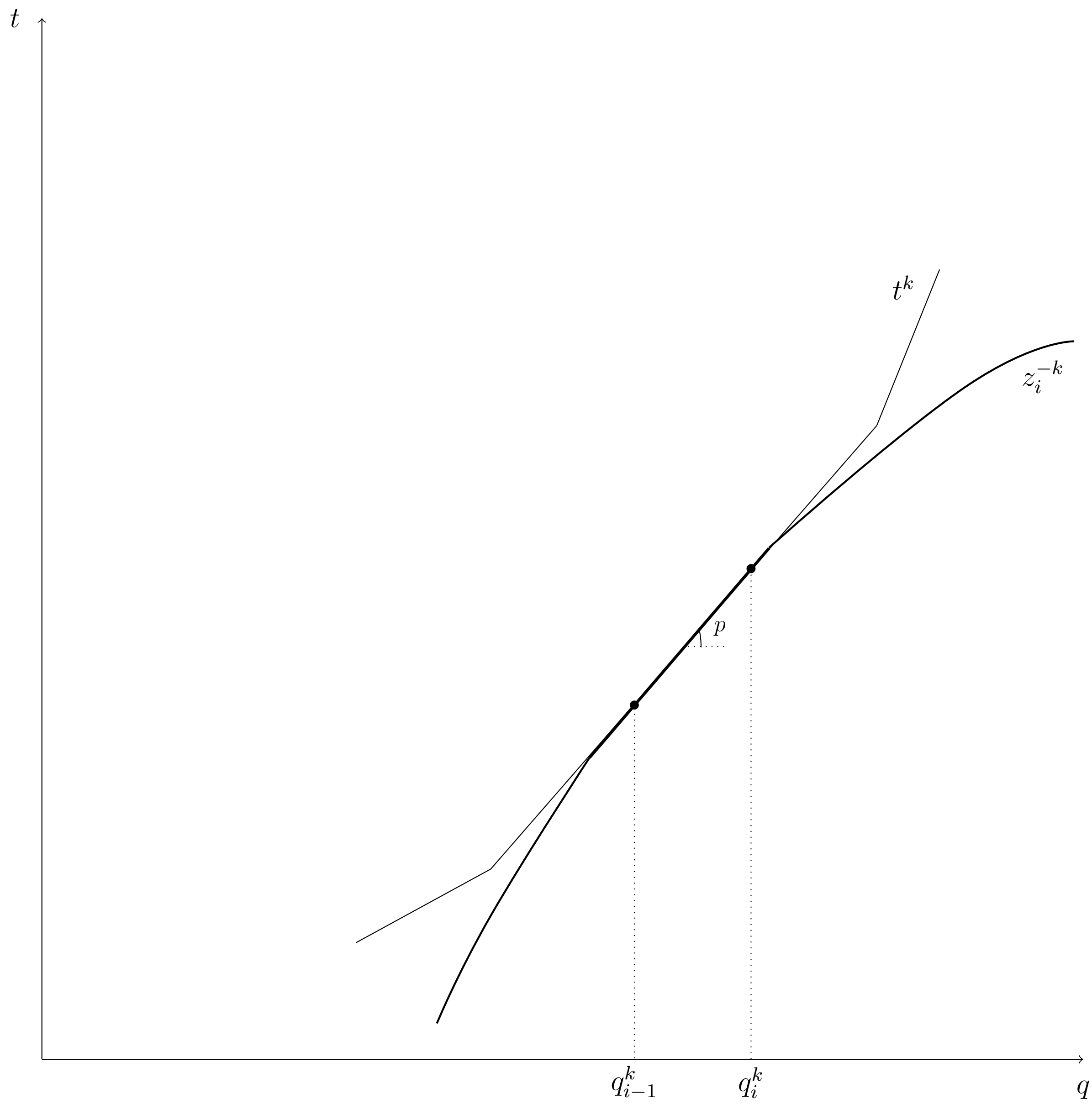

Figure 1 Binding downward local constraints and linearity.

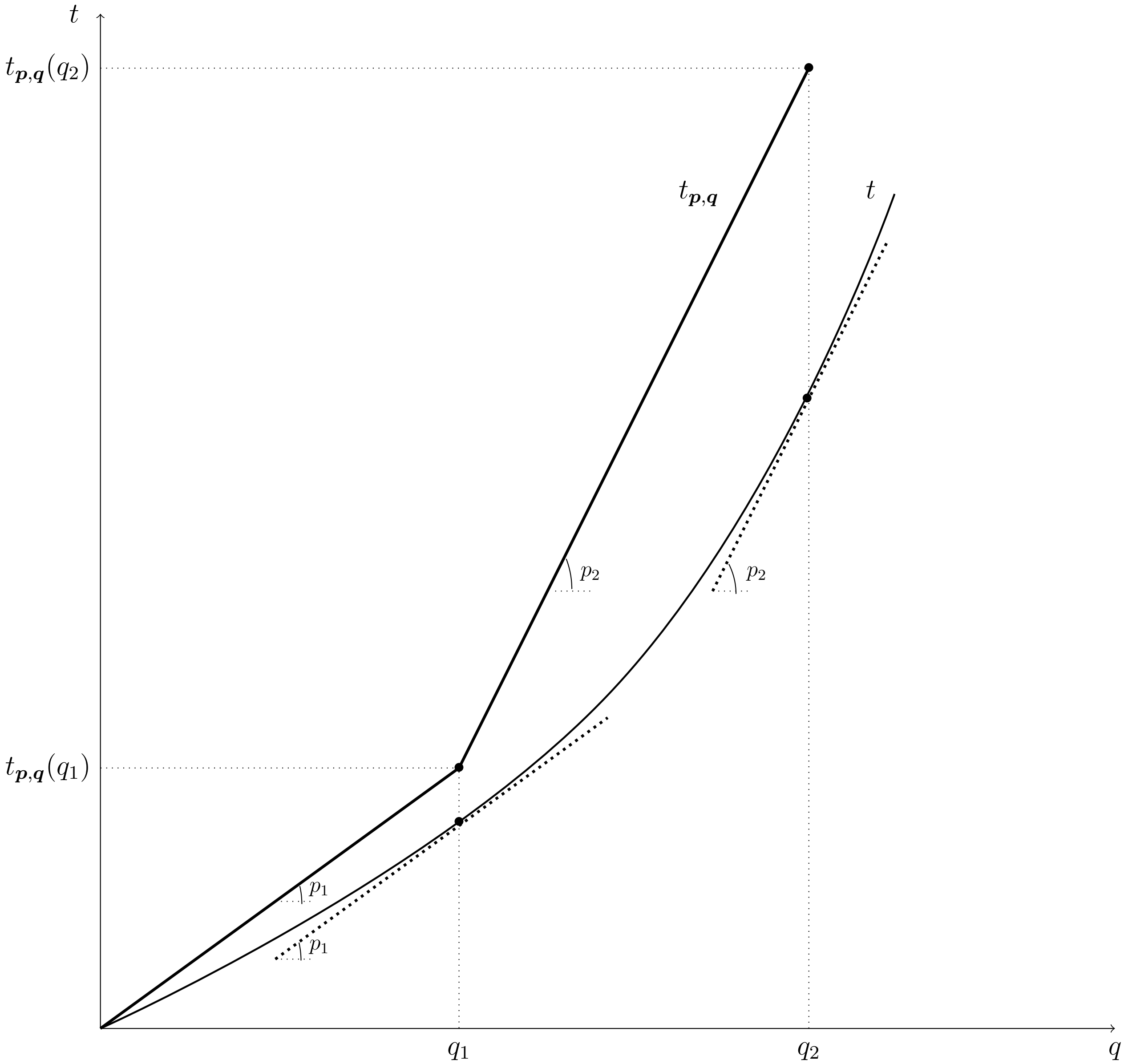

Figure 2 The $t_{\boldsymbol{p},\boldsymbol{q}}$ schedule in the case $I = 2$.

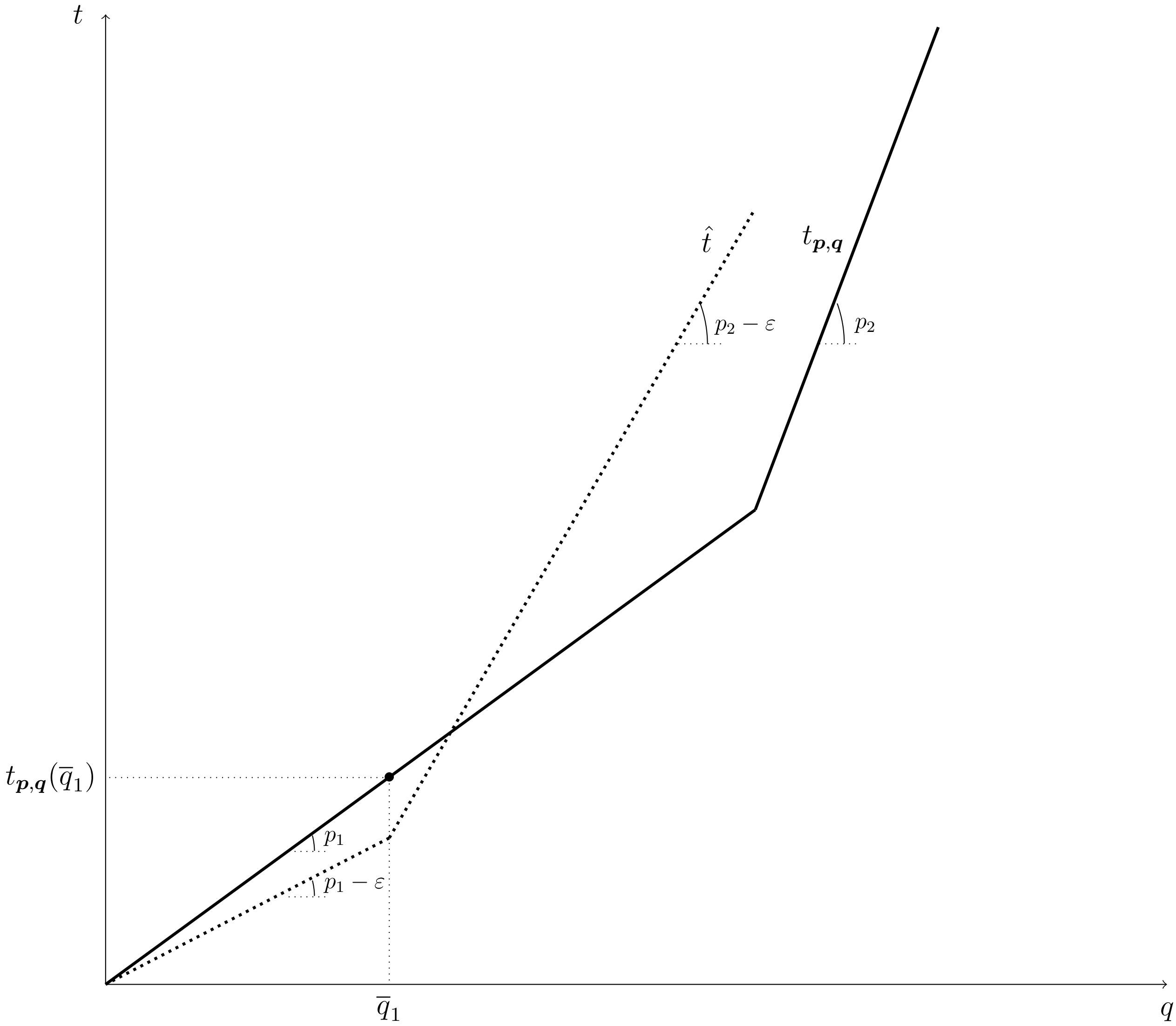

Figure 3 The $\hat{t}$ schedule in the case $I = 2$.