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"Oil Extraction and Price Dynamics"

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Oil Extraction and Price Dynamics

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Abstract

We use the dynamic production function identities and an empirical model of oil prices based only on oil extraction data to analyze the dynamics of oil prices as we transition into the contraction phase of oil extraction. We explore the implications with respect to several common scenarios.

1 Introduction

This paper is an attempt to understand the economic dynamics of peak oil extraction, or more generally peak resource extraction. We use economic production function theory and empirical data to analyze the price dynamics of oil extraction.

There is considerable variation in estimates of future oil extraction without climate mitigation policies in place. On the low side are the Association for the Study of Peak Oil and Gas (ASPO) and Uppsala Global Energy Systems (UGES). Total and the International Energy Agency (IEA) give somewhat higher estimates. Exxon, BP and the Energy Information Administration (EIA) give high estimates. The International Institute for Applied System Analysis (IIASA) on which the International Panel on Climate Change (IPCC) basis its climate change scenarios gives relatively high estimates. Hotelling's rule has a very poor record predicting both future extraction and future prices [11]. Other models used to predict future oil extraction such as Hubbert linearization have been criticized because they do not take prices into account. In fact the quantity of oil reserves is very much a function of price [9], the higher the price the larger the reserves because high prices permit the use of expensive extraction technologies to recover more oil. In [39] the authors give three scenarios for China's future oil extraction rate and Ultimate Recoverable Resource (URR) based on three different price scenarios. Thus a robust model predicting future prices would be a valuable asset to those predicting future extraction which is

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important for economic reasons and for concerns about greenhouse gas emissions. Viewing previous price estimates of the EIA [33] one sees that they lack a robust price predicting model as any volatility in prices leads to very poor price estimates.

Many empirical studies [10, 28, 24, 13, 29, 2, 18, 12] show that food and energy are key quantities to consider when evaluating economic production. The "Cost Share Theorem" from neoclassic equilibrium theory [2, Appendix A] says that cost share is proportional to the elasticity or scaling factors of the variables in the production function equation. Because the cost share of labor is almost 10 times the cost share of energy in the economy, many economists discount the idea that energy is important for economic production. In Section 2, we prove the dynamic production function identities, (2.8) and (2.9), which we then use to show that cost share dynamics are a more robust indication of the importance of a quantity in an economy rather than its size (because fewer assumptions are required). In particular important items have either slowly increasing or decreasing cost shares in a growing economy and decreasining cost shares in a contracting economy. We differ from other works on economic production functions such as [32, 2, 18] in that we make very few assumptions on the structure of the economy or the nature of the economic production function. We assume only some regularity of the production function to obtain our results. Our results are thus much more general and can be applied to a larger set of economies, both past and present.

We present an empirical model for the price of oil at year t as a function of extraction at year t, year $t-1$, and year $t-2$. We hope that understanding this model will enable better future price prediction aiding those predicting future oil extraction.

We find that the price of oil during the contraction phase of oil extraction will be high but decreasing as extraction rates decrease creating negative feedback loops accelerating the contraction because of contraction of economic production. Higher price extraction will be unprofitable reducing extraction rates which will result in lower prices resulting in less extraction, etc. During the expansion phase of oil extraction, high prices increased the extraction rate and low prices increased the market for oil. During the contraction phase, our data suggests that high prices will decrease the market for oil and low prices will decrease the extraction rate.

2 Economic Production Function Theory

Economic production functions not only indicate what macro economic quantities one should measure to detect motors of economic growth, but give indications of the dynamics of the economy when production of these economic motors peaks and goes into decline.

2.1 Definitions and Hypotheses

Let $Y(t)$ be a measure of economic production expressed in currency where t denotes time. We make no assumptions about the structure of the economy other than the existence of a well developed monetary system to determine the distribution of wealth. Let $q(t) \in \mathbb{R}^d$ be the measurable quantities in the economy (quantities with a price). We will make use of the following assumptions:

(H1)

$$\overline{Y}(t) = \overline{Y}(q(t), t) \tag{2.1}$$

is a locally $C^1(\mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R})$ with $q(t) = (q_1(t), \ldots q_d(t)) \in \mathbb{R}^d$. Prices are locally $C^1(\mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R})$ functions of quantities.

(H2) Over short periods of time the economic production function depends more on $q(t)$ than on the time variable, therefore, for short periods of time $\overline{Y}(q(t), t) \approx Y(q(t))$, $q(t) \in \mathbb{R}^d$.

(H3) For short periods of time we have $\dfrac{\partial Y}{\partial q_i} \neq 0$ for $i = 1, \ldots, d$.

Remark 2.1. 1. *The regularity assumptions are not essential to the theory developed below, their purpose is to simplify notation.*

2. *In the real world, $d = d(t)$. Care must be taken when applying results to new quantities or disappearing quantities.*

Assumption (H1) is not a strong assumption, as the very fact that GDP is measurable, means we measure certain quantities and use prices to evaluate the value added by the domestic economy.

Assumption (H2) essentially says that we assume it takes time for the economy to change. Certain quantities are fungible, but we assume it takes a certain amount of time to switch from one item to the other.

Assumption (H3) is not a strong assumption, as we are eliminating only quantities that have a small effect on the economy at any given time.

Because the units of $Y(t)$ are in currency, an equivalent formulation can be to consider $Y(p(t))$ where $p(t) \in \mathbb{R}^d$ is the vector of prices of $q(t)$. We use this formulation in (2.9).

Let $p_i(t)$ be the cost per unit of $q_i(t)$ and let $C_i(t)$ be the *cost share* or *intensity* of q_i in the economy,

$$C_i(t) \stackrel{\text{def}}{=} p_i(t) q_i(t) / Y(t). \tag{2.2}$$

We will denote by $Y_i(t, u) = Y_i(\overline{q}_i(t), u)$, with $\overline{q}_i(t) \in \mathbb{R}^{d-1}$, consisting of q_j, $j \neq i$, the quantity

$$Y_i(t, u) \stackrel{\text{def}}{=} \int_0^u \frac{\partial Y(\overline{q}_i(t), q_i(s))}{\partial q_i} q_i'(s) \mathrm{d}s \tag{2.3}$$

where we make the technical assumption that $Y_i(\overline{q}_i(t), q_i(0)) = 0$ for all $i \in [\![1, d]\!]$ and all $t \in \mathbb{R}_+$. We define the functions $\tilde{p}_i(t,u)$ and $\tilde{C}_i(t,u)$ similarly. We define the index

$$\begin{aligned} I_{Y_i}(t_1,t_2) &= I_{Y_i}(t,t_1,t_2) \\ &= Y_i(t,t_2)/Y_i(t,t_1). \end{aligned} \tag{2.4}$$

We do not indicate the dependance on t in the index and will always assume that $t \in [t_1, t_2]$. We define the indices $I_{\tilde{C}_i}(t_1,t_2)$ and $I_{\tilde{p}_i}(t_1,t_2)$ similarly.

Elasticity, or how quantities scale in the economic production function is very important. Suppose $d = 1$. One can write $Y(t) = Cq^{\alpha(t)}$. If $\alpha(t) \equiv \alpha$, a constant, then the production function is homogeneous of degree α and we call α the elasticity or scaling factor. If $\alpha = 1$ Y is *linear* in q, if $\alpha < 1$, Y is *sublinear* in q, otherwise, Y is *superlinear* in q. Scaling factors are important in many sciences and mathematics. One looks for constant or average scaling empirically by normalizing quantities at a start date, taking logs and performing linear regression.

2.2 Main Results

Theorem 2.1. *Assume (H1), then*

1. *If $\alpha_i(t)$ is the scaling factor of q_i in Y, then $\alpha_i(t) - 1$ is the scaling factor of p_i/C_i.*

2. *Assume that C_i is constant. Then sublinear scaling of Y in q_i occurs if and only if $p_i(q_i)$ is monotone decreasing, linear scaling implies price is independent of q_i, and superlinear scaling occurs if and only if $p_i(q_i)$ is monotone increasing.*

3. *The greater $\dfrac{\partial Y}{\partial q_i}$, the smaller the scarcity rent in the in the sense of* (2.7) *(see discussion below).*

4. *The index of $Y_i(u)$ for any $(t_1,t_2) \in \mathbb{R}^2_+x$ is given by*

$$I_{Y_i}(t_1,t_2) = I_{\tilde{p}_i}(t_1,t_2) I_{\tilde{C}_i}(t_2,t_1) I_{q_i}(t_1,t_2). \tag{2.5}$$

Proof. From (2.2) one immediately obtains

$$p_i(t) = C_i(t)Y(t)/q_i(t). \tag{2.6}$$

Properties (1) and (2) can be read directly from (2.6).

Taking the derivative of (2.6) one obtains the dynamic price identity:

$$\frac{\partial p_i}{\partial q_i} = \frac{\partial C_i}{\partial q_i}\frac{Y}{q_i} + C_i \frac{\dfrac{\partial Y}{\partial q_i} q_i - Y}{q^2}. \tag{2.7}$$

The scarcity rent of a quantity varies inversely to its importance in the economic production function in the following sense. The more important a quantity in the economic production function, the greater the partial derivative of Y with respect to that quantity. But from (2.7), we see that the price is an increasing function of the partial derivative of Y with respect to q, or price decreases as quantity decreases, a negative scarcity . This proves (3). (Of course, for an important quantity one would expect the derivative of the cost share to be strongly negative and dominate the positive term.)

Solving (2.2) for Y, taking the log and then the derivative with respect to q_i, one obtains the first dynamic production function identity:

$$\frac{\partial Y}{\partial q_i} = Y \left(\frac{\frac{\partial p_i}{\partial q_i}}{p_i} - \frac{\frac{\partial C_i}{\partial q_i}}{C_i} + \frac{1}{q_i} \right). \tag{2.8}$$

Taking the derivative with respect to prices, we obtain the second dynamic production function identity:

$$\frac{\partial Y}{\partial p_i} = Y \left(\frac{\frac{\partial q_i}{\partial p_i}}{q_i} - \frac{\frac{\partial C_i}{\partial p_i}}{C_i} + \frac{1}{p_i} \right). \tag{2.9}$$

The sign of the left hand side of (2.8) is the same as the sign in parentheses on the right hand side. To obtain (2.5), we multiply (2.8) by $\frac{\mathrm{d}q}{\mathrm{d}t}/Y$ and integrate from t_1 to t_2 and take the exponential of the resulting equation. $\square$

Remark 2.2. *1. The main point of equations* (2.5), (2.9), *and* (2.8) *is that $Y_i(u)$ is impossible to measure directly because one cannot vary just one quantity in the economy, during periods of growth or contraction many quantities vary simultaneously. In some cases we can draw conclusions on $I_{\tilde{p}_i}$ and $I_{\tilde{C}_i}$ from measurements of I_{p_i} and I_{C_i}. See Section 4.1.*

2. *Equation* (2.7) *should not be considered a precise model because different quantities in the equation move at different speeds. Prices move more quickly than does the reaction of the economy to price changes. It makes more sense to use* (2.7) *with annual data using the average yearly prices than with spot prices.*

3. *Equation* (2.7) *is not in general an accurate way to measure $\frac{\partial Y}{\partial q_i}$ because the term in parentheses can be the difference between two small numbers of similar magnitude and hence numerically unstable. Equation* (2.5) *does provide a method for measuring the contribution of a quantity in $I_Y(t_1, t_2)$ in cases where*

$$\tilde{C}_i(t) \approx C_i(t) \text{ and} \tag{2.10}$$
$$\tilde{p}_i(t) \approx p_i(t) \tag{2.11}$$

since $I_Y(t_1,t_2) = I_{Y_u}\Pi_{j=1}^d I_{Y_j}(t_1,t_2)$. *because of the simultaneous growth of other independent quantities. These equations tell us that in times of economic growth, the cost share of economic motors should either stagnate or decrease and that the cost per unit should increase. In times of economic contraction, the opposite dynamic can be deduced. Economic drags on the economy have the opposite dynamic.*

4. *The variable useful work,* $U \stackrel{\text{def}}{=} eE$, *where* e *is efficiency and* E *is primary energy (or more properly exergy) production was introduced by Ayres and Warr [3] we can write* $Y(U(t),\bar{q}) = Y(e(t)E(t),\bar{q})$ *where* $\bar{q} \in \mathbb{R}^{d-1}$ *are the remaining quantities. Then* (2.6) *becomes* $p_E = C_E Y(eE,\bar{q})/q_E$. *Assuming* Y *to be an increasing function of* U, *one sees that energy efficiency increases the price per unit of energy assuming* $C_E(t)$ *remains constant. We thus have a very simple explanation of the empirically observed Jevons paradox or the rebound effect [15, 4]. Efficiency can be used to lower prices if it is used to decrease cost share. See discussion in Section 4.1.*

2.3 Representative Variables and the Implicit Function Theorem

One might be interested in using standard techniques to determine the nature of $Y(t)$. If one accepts assumptions (H2) and (H3), one can use the implicit function theorem to reduce the number of variables to a small set of representative variables locally. This is very familiar to investors who use economic indicators to gauge the state of the economy. One can judge the quality of the indicator variables by seeing how closely the empirical scaling factors of the variables compare to the general scaling factors of the prices computed from p_i/C_i. By choosing overlapping time periods, empirical results will show whether or not the computed functions can be stitched together for longer periods of time.

2.4 The Cost Share Theorem

Many empirical studies [10, 28, 24, 13, 29, 2, 18, 12, 14] show that food and energy are key quantities to consider when evaluating economic production. The "Cost Share Theorem" from neoclassic equilibrium theory [2, Appendix A] says that for an economy in equilibrium, cost share is proportional to the elasticity or scaling factors of the variables in the production function equation. Because the cost share of labor is almost 10 times the cost share of energy in the economy, many economists discount the idea that energy is important for economic production. Theorem 2.1 shows that the cost share theorem is extremely speculative in either a growing or shrinking economy. We make the following remarks relative to the cost share theorem and Theorem 2.1:

Remark 2.3. *1. Theorem 2.1 shows that in a growing or shrinking economy, the dynamics of the cost share is the important consideration, not the size. For important quantities, the derivative of the cost share is small* *or negative, that is, in a growing economy the cost share of important quantities decreases or remains constant while the cost share of unimportant quantities increases. This makes sense. Important quantities such as oil permit the growth of less important parts of the economy such as art or competitive sports.*

2. *Equilibrium theory studies steady states. The economy is dynamic. A dynamical system might converge to a steady state but one does not expect a dynamical system to be in a steady state while it is changing. We feel that a dynamical system approach to modeling the economy would be more useful as in [22, 23, 5].*

3. *In the above cited articles empirical evidence did not support the Cost Share Theorem. Rather than question the hypotheses on which the theorem is based, the authors attempted to save equilibrium theory by introducing constraints (thus introducing a Lagrange multiplier to explain the importance of energy in the economy). We do not use neoclassic equilibrium theory.*

There is a high probability that the hypotheses on which The Cost Share Theorem is based are not verified. The Cost Share Theorem says that a large cost share corresponds to importance in the economic production function. Theorem 2.1 says that increasing cost share is a sign of unimportance in economic production, however a large cost share frequently translates into political power. This dynamic can induce governments to embrace counter productive policies as during the stagflation period (see Section 3.1), the government might support large sectors of the economy which are unimportant in economic production rather than important areas of the economy with smaller cost shares.

In our view, a small cost share is not a sign of unimportance, in the case of energy production, it is a sign of what might be called ecodiversity.

2.5 An Example

We suppose a very limited economy produces 3 quantities: E, G, and F. We assume prices are adjusted for inflation for the 3 quantities and we normalize all prices to one. The size of this economy is $Y(t) = \sum_{i=1}^{3} p_i q_i = E + G + F$. Now suppose that E is a motor of economic growth in the following sense, when E grows 10% this produces a growth of 5% in both G and F in the next time period. Growth in G has no effect on E or F. However F is a drag on economic growth since a 10% growth in F causes a 5% percent contraction in E and G. We can name our quantities to make the example more realistic. We call E energy production, which permits us to produce more of G and F. Let us call G gold extraction and F fun production. Fun decreases growth in E and G because in fact many people do not like producing energy or mining for gold, so as soon as there is something fun they stop work to enjoy the fun which reduces production of E and G.

Let us assume that $t = n \in \mathbb{N}$ and that the initial conditions are $E_0 = G_0 = F_0 = 1$. Now assume that population growth would cause growth of 10% in each time period, but the interactions occur in the next time period. Thus $E_1 = G_1 = F_1 = 1.1$ but $E_2 = 1.21 - .05 = 1.16$ because of the fun interaction. We have $G_2 = 1.21 - .05 + .05 = 1.21$ and $F_2 = 1.21 + .05 = 1.26$. We see that the cost share C_E has dropped from 1/3 to 0.32, C_G is unchanged and C_F increases from 1/3 to .35, while $I_Y(0,2) = 1.21$. Repeating the experiment with $E_0 = G_0/2 = F_0/2 = 1/2$, we obtain $E_1 = .55$, $G_1 = 1.1$, $F_1 = 1.1$, $E_2 = .575$, $G_2 = 1.21$, $F_2 = 1.26$. With these initial conditions, C_E decreases from .2 to .19, C_G is almost unchanged and C_F increases from .40 to .41. In this case $I_Y(0,2) = 1.22$ so that a smaller initial cost share of the economic motor produces greater overall growth.

Remark 2.4. *Not all quantities that drag on economic growth are fun.*

3 Empirical Results

3.1 Secular Cycles

Turchin and Nefedov [38] empirically identified recurring secular or economic cycles in agrarian societies. Similar phenomena apply to oil and resource production so we recall their findings.

The cycle begins with a period of growth, in population and living standards lasting on the order of a hundred of years. Then comes a period of stagflation in which population density approaches the carrying capacity of the land (one says increased population pressure) lasting on the order of half a century. During the stagflation period peasants leave the countryside for cities, the difference between the elite and the commoners increases, and the price of food rises relative to wages. Population ceases to grow because food production ceases to grow. Initially the elite are somewhat better off in the stagflation period because wages are low and they can employ a larger number of former peasants who have left the countryside. As the stagflation period progresses, the ratio of elite population to working class population rises (the working class have a lower birthrate and a higher mortality rate due to malnutrition and cramped living conditions in cities) creating competition among the elite. Social mobility increases, mostly downward as elites lose their status. The inter elite competition creates fissures which lead to civil war and the final crisis stage lasting a few decades in which population decreases and the state breaks down. There follows an inter-cycle lasting several decades before a new growth period ensues.

Secular cycles have been linked to unsustainable agricultural production systems [31, 10].

3.2 An Empirical Study of Oil Prices

In this section we study empirical data in order to understand what the price dynamics of the contraction phase of oil extraction might be.

We worked with data from BP's 2016 Statistical Review. We used prices and extraction data from the BP's data set:

- Annual crude oil prices in 2015 US dollars per barrel (deflated using the Consumer Price Index for the US) available from 1861 to 2015.
- Annual world oil production in thousand barrels (daily mean) from 1965 to 2015. These data include crude oil, shale oil, oil sands and NGLs (natural gas liquids - the liquid content of natural gas where this is recovered separately). However, these data exclude liquid fuels from other sources such as biomass and derivatives of coal and natural gas.

We used data from [36] to obtain annual world crude oil extraction in thousand metric tons from 1937 to 1970. We used the world average conversion factor of 7.6 barrels per ton to convert this data to barrels per year.

Let $(P_t)_t$ denote the time series of oil prices (in 2015 dollars adjusted for inflation) from year 1861 to year 2015 and $(Q_t)_t$ the time series of quantities of oil extracted (in million barrels daily) from year 1965 to year 2015 for BP data and from 1937 to 1970 for Etemad & Luciani data from [36].

A possible first idea to study the series $(P_t)_{t=1861\ldots2015}$ may be to fit a classical time series model to the data. As the series of prices is clearly not stationary, one can apply a preliminary Box-Cox transformation in order to stabilize the variance. We thus define the log-prices:

$$p_t = \log P_t. \tag{3.1}$$

This transformation is consistent with the assumption of log-normality of the prices often encountered in the literature. One can try to eliminate the trend by differentiating. The lag-1 difference operator ∇ is defined for a series $(x_t)_t$ by

$$\nabla x_t \stackrel{\text{def}}{=} x_t - x_{t-1}.$$

We are then led to study the so-called log-return at year t:

$$r_t = \nabla p_t = p_t - p_{t-1} = \log P_t - \log P_{t-1}. \tag{3.2}$$

It is well-known that the log-returns data $(\nabla p_t)_t$ may at first glance look like a white noise. But, by looking closer, one would observe that the magnitude of the fluctuations still depends on time. In fact, the series $(|r_t|)_t$ and $(r_t^2)_t$ are often strongly autocorrelated. Moreover, the distribution of the log-returns may also present heavy tails. To take into account these two stylized facts, the log-returns are often modeled by nonlinear models such as ARCH/GARCH models or derivative models. In [35] and [37], the authors uses the daily Brent and West Texas Intermediate (WTI) crude oil spot prices to forecast the volatility of crude oil markets using the ARCH/GARCH model class.

The above method uses historical data to predict future oil prices through time-series model. Many other techniques to forecast oil prices have been considered by researchers: financial models, structural models and computational methods using artificial neural networks, see [34] and references therein. In structural models, oil prices are modeled as a function of explanatory variables such as oil consumption, oil extraction and even non-oil variables such as interest rates. However, the use of such models is limited by the need of future values of the explanatory variables to derive predictions of oil prices. Furthermore, the number of possible explanatory variables may be very large.

3.2.1 Price explained by oil extraction

The approach we consider here is structural. We try to derive information on the price from a single explanatory time series: oil extraction. From Figure 1, one sees that Q_t cannot explain P_t because the price P is not uniquely determined by the extracted quantity Q, in other words several prices correspond to the same produced quantity. For this reason, we attempt to use, in addition to Q_t, the lag-1 difference and the lag-2 difference of the series $(Q_t)_t$ at year t:

$$\nabla Q_t \overset{\text{def}}{=} Q_t - Q_{t-1}, \tag{3.3}$$

$$\nabla^2 Q_t \overset{\text{def}}{=} \nabla(Q_t - Q_{t-1}) = Q_t - 2\,Q_{t-1} + Q_{t-2}. \tag{3.4}$$

Note that Q_t, ∇Q_t and $\nabla^2 Q_t$ are linearly independent so their span is the same as that of Q_t, Q_{t-1} and Q_{t-2}. We prefer the former variables to the latter because ∇Q_t and $\nabla^2 Q_t$ are the discreet first and second derivatives of Q_t with step time $h = 1$ making them easier to interpret. From an economic point of view, it is quite natural to postulate that the market for oil a given year is influenced by production in previous years.

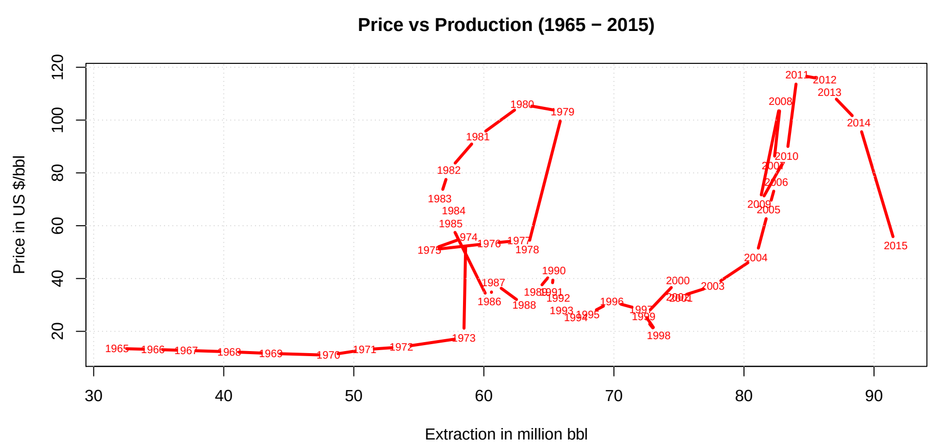

Figure 1: Oil price in function of oil production from 1965 to 2015

We consider the following model:

$$p_t = a + b\,Q_t + c\,\nabla Q_t + d\,\nabla^2 Q_t + \epsilon_t \tag{3.5}$$

where a, b, c, d are coefficients determined by the linear regression and $(\epsilon_t)_t$ is a centered second order stationary process. Equation (3.5) is equivalent to

$$P_t = \exp(a + b\,Q_t + c\,\nabla Q_t + d\,\nabla^2 Q_t + \epsilon_t). \tag{3.6}$$

The dependency of price P_t on these variables is non-linear. As the logarithm function flattens large values, the model takes into account the inelasticity of oil prices. That is, small changes in the supply provoke large changes in price.

3.2.2 Data from 1965 to 2015

The R output for the linear regression with the data starting at year 1965 is given in Figure 2. First, note that we have lost two years because of the lag-2 differences $(\nabla^2 Q_t)_t$ that are only available form year 1967 with the data set starting in 1965. Adjusted R-squared being 0.5544 means that the model explain 55.44% of the variance from the mean taking into account the number of explanatory variables. From the stars in the R output, we obtain four significant coefficients for the model (3.5) that are:

$$\begin{aligned} a &\approx 2.055 \\ b &\approx 0.029 \\ c &\approx -0.204 \\ d &\approx 0.081 \end{aligned}$$

Remark 3.1. *We considered several other models. The regression with the 3-lag difference $\nabla^3 Q_t$ in addition to the other variables gave a coefficient which was not significant. Using $\tilde{p}_t \stackrel{\text{def}}{=} \log(\log P(t))$ marginally improved the fit of the model (with the same variables). In fact taking further logs also marginally improved the fit. We use the above model for simplicity and because, as we will see, the model is not robust over different time periods.*

We have plotted the adjusted prices and the real prices in Figure 3. Moreover, the model allows us to derive a prediction of the price P_{t+1} at year t. Namely, if the residuals are independent, as they are centered, the prediction of ϵ_{t+1} is zero. Then, from the extracted quantity at year $t+1$, we obtain at year t a prediction of the price:

$$\begin{aligned} \hat{P}_{t+1} &= \exp\left(a + b\,Q_{t+1} + c\,\nabla Q_{t+1} + d\,\nabla^2 Q_{t+1}\right) \qquad (3.7) \\ &= \exp\left(a + b\,Q_{t+1} + c\,(Q_{t+1} - Q_t) + d\,(Q_{t+1} - 2\,Q_t + Q_{t-1})\right). \end{aligned}$$

The conclusions we have made above are in fact not justified here because the residuals of the regression are correlated. One can observe the dependence

```
Call:
lm(formula = log(Price67) ~ Quantity67 + DQuantity67 + DDQuantity67)

Residuals:
     Min       1Q   Median       3Q      Max
-0.91420 -0.31041 -0.03264  0.30502  1.07623

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.055439   0.362196   5.675 9.46e-07 ***
Quantity67    0.028599   0.005126   5.579 1.31e-06 ***
DQuantity67  -0.203812   0.043555  -4.679 2.65e-05 ***
DDQuantity67  0.081201   0.036108   2.249   0.0295 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4528 on 45 degrees of freedom
Multiple R-squared:  0.5822,    Adjusted R-squared:  0.5544
F-statistic:  20.9 on 3 and 45 DF,  p-value: 1.249e-08
```

Figure 2: R output for linear regression (data from 1965 to 2015)

of the residuals while plotting the autocorrelation funtion and the partial autocorrelation function with the R commands *acf* and *pacf*. One can also perform the Box-Pierce test, the turning point test and the difference sign test with the R commands *Box.test*, *turning.point.test* and *difference.sign.test*. All these tests reject the hypothesis of a white noise at the level $\alpha = 5\%$. We attempt then to fit an ARMA model to the residuals. As the partial autocorrelation at lag 1 was very high, we have chosen the $AR(1)$ model. In fact, among the models $ARMA(p,q)$ with $p = 0,1,2$ and $q = 0,1,2$, the one that minimizes the AIC and the BIC criteria is the $AR(1)$ model. Finally, we perform a linear regression with a covariance structure of an $AR(1)$ with the R command *gls*. The model is fitted by maximizing the log-likelihood. The R output is given in Figure 8. The coefficients of the model (3.5) are:

$$\begin{aligned} a &\approx 1.591 \\ b &\approx 0.032 \\ c &\approx -0.062 \\ d &\approx 0.018 \end{aligned} \tag{3.8}$$

with an $AR(1)$-noise $(\epsilon_t)_t$ such that:

$$\epsilon_t - \phi\,\epsilon_{t-1} = \eta_t$$

with ϕ estimated by 0.816 and $(\eta_t)_t$ an actual centered white noise with estimated standard deviation 0.474. The *intervals* R-command allows us to obtain

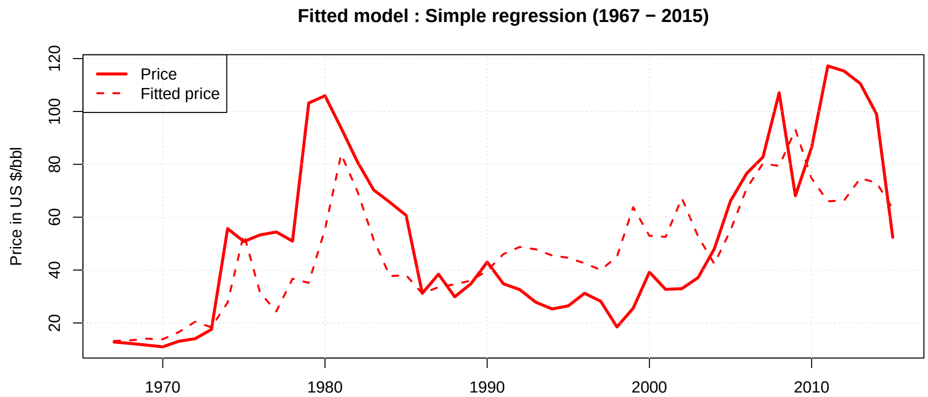

Figure 3: Fitted price from 1967 to 2015

95% confidence intervals for the model coefficients (see Figure 7). The plot of price and fitted price is given in Figure 9.

We tested the stability of the coefficients by trying the regression in different sub-intervals and found that the coefficients were not stable. This means the model cannot be used directly to predict future prices, there is some sort of averaging going on. One must understand why the coefficients differ in different time frames and then analyze how future conditions might effect the coefficients. To this end we split the data into four periods with different growth rate and price characteristics (see Figure 4) and computed the coefficients of the classical linear regression for each period (see Figures 5 and 6).

3.2.3 Data from 1937 to 1970

From Figure 10, one sees that for this period the price is a function of the extracted quantity. A linear regression of the log-price with explanatory variables Q_t and ∇Q_t shows that the coefficient of ∇Q_t is not significant in the R output of Figure 11. The R output for the sub-model of (3.5 with $c = d = 0$ is given in Figure 12. Again the residuals are not i.i.d. So we use the linear model with covariance structure $ARMA(1,1)$ chosen among the $ARMA(p,q)$ models with $p = 0,1,2$ and $q = 0,1,2$ with the Akaike AIC criterion. The R output is given in Figure 13. The model coefficients are

$$\begin{aligned} a &= 2.943 \\ b &= -0.01 \end{aligned}$$

with an $ARMA(1,1)$-noise $(\epsilon_t)_t$ such that :

$$\epsilon_t - \phi\,\epsilon_{t-1} = \eta_t + \theta\,\eta_{t-1}$$

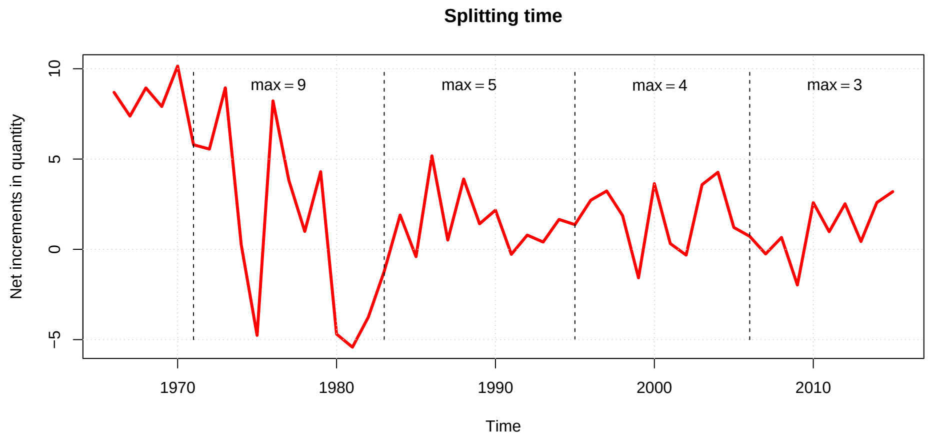

Figure 4: Percent increases in quantity from 1971 to 2015

with (ϕ, θ) estimated by $(0.04, 0.69)$ and $(\eta_t)_t$ an actual centered white noise with estimated standard deviation 0.08. The *intervals* R-command allows us to obtain 95% confidence intervals for the model coefficients (see Figure 14). The plot of price and fitted price is given in Figure 15.

3.2.4 Interpretation of the results

It is well known that oil prices are correlated with economic growth. Indeed, some use estimates of future GWP to obtain estimates of future oil prices. It is not surprising then that the model (3.5) gives a reasonable fit of GWP if one replaces p_t by $y_t \overset{\text{def}}{=} \log Y_t$ where Y_t denotes GWP. We will make some conjectures about this relationship presently.

We begin by examining what the predictive model (3.7) for the period 1967-2015 using the coefficients (3.8).

If extraction is constant for two years then $\nabla Q_t = \nabla^2 Q_t = 0$. In that case the model reduces to

$$p_{\text{basic}}(t) = \exp(1.591 + 0.032 \times Q_t + \epsilon_t) \tag{3.9}$$

with 0 the expected value of ϵ_t. We call this the basic price formula that predicts the price if extraction is constant. Note that the price increases with the quantity, consistent with a positive derivative in (2.9) as $\dfrac{\partial P_t}{\partial Q_t}/P_t = .032 > 0$.

Suppose $\nabla Q_t = \text{constant} \neq 0$. Then $\nabla^2 Q_t = 0$ and the model becomes

$$p(t) = \exp(1.591 + 0.032 \times Q_t - 0.062 \times \nabla Q_t). \tag{3.10}$$

Note that the coefficient of ∇Q_t is about twice as large as the coefficient of Q_t and of opposite sign. Thus ∇Q_t gives a larger price signal than Q_t. The signal goes in the opposite direction of the change but it only lasts for a year. One might understand this as follows: a rise (fall) in extraction causes growth (contraction) in the economy much more next year than this year (as in the example in Section 2.5).

Example 3.1. *If extraction is constant at 88 mbd, the basic price is* $\hat{P}_{basic}(t) \approx$ *\$82/barrel. If* $Q_t = 88$, $Q_{t-1} = 86$, *and* $Q_{t-2} = 84$ *then* $\nabla Q_t = \nabla Q_{t-1} = 2$ *and* $\nabla^2 Q_t = 0$. *Then* $\hat{P}_t \approx$ *\$72/barrel less than the basic price because extraction is rising. If* $Q_{t-1} = 90$ *and* $Q_{t-2} = 92$, *then* $\nabla Q_t = \nabla Q_{t-1} = -2$ *and* $\nabla^2 Q_t = 0$. *Then* $\hat{P}(t) \approx$ *\$93/barrel greater than the basic price because extraction is falling.*

The second derivative is negative at local maximuma and positive at local minimuma so that the second derivative will mollify the price change caused by the first derivative. This explains why peak extraction is frequently associated with low prices. A minimum in extraction will thus be associated with relatively high prices. Economically this factor can be interpreted as follows: it takes two years for the economic growth (contraction) produced by an increase (decrease) in extraction to take hold. The first year it is rather fragile and easily reduced (increased) by a drop (rise) in extraction.

Example 3.2.

1. *If* $Q_t = 88$, $Q_{t-1} = 86$, *and* $Q_{t-2} = 88$, *then extraction reaches a local minimum at* Q_{t-1}. *We compute* $\hat{P}(t) \approx \$78$ *rather than \$72 as in Example 3.1 with the same increase in extraction from 86 mbd.*

2. *If* $Q_t = 88$, $Q_{t-1} = 90$, *and* $Q_{t-2} = 88$, *then extraction reaches a local maximum at* Q_{t-1}. *We compute* $\hat{P}(t) \approx \$86$ *rather than \$93 as in Example 3.1 with the same decrease in extraction from 90 mbd.*

3. *It is interesting to note that if* $Q_t = q_0 \rho^t$ *with* $\rho = (1 + r)$ *and the growth rate* r *in a reasonable range* ($0 < r < .12$) *then* $\hat{p}(t)$ *is an increasing function of* t. *For example if* $Q_t = 80(1.02)^t$, *then* $\hat{p}(3) \approx \$58.3 < \$60.8 \approx \hat{p}(4)$, *thus increasing extraction at a constant rate produces increasing prices. However* $\hat{p}(4)$ *is much lower than the basic price at the same extraction quantity which is \$87.6.*

We believe that the differences in the different periods we studied stem from the importance of oil in the economy, that is the size of the partial derivative of economic production with respect to oil varied during the different periods. From 1937 to 1970, oil extraction increased regularly and so was never an impediment to economic growth. In the 1970's, oil shocks created irregular supplies and constraints in oil supplies reduced economic growth. Mathematically this is expressed by a larger partial derivative of economic production with respect to oil extraction levels. this period, as in the period from 2005 to 2014. This leads us to the following conjectures in accord with (2.8) and our empirical results:

Conjecture 3.1. *The principle signal of scarcity of oil production is that* $\dfrac{\partial p}{\partial q} > 0$.

Conjecture 3.2. *Volatility in the supply of oil leads to increased price dependence on the discreet first and second derivatives of supply.*

The standard scarcity rent view, that lower quantities leads to higher prices can be explained by temporary dependence on the first and second discreet derivatives of extraction levels. After the short term spikes, the price stabilizes with economic production adjusting to the new level of extraction. In other words, a shortage of important items in economic production ultimately leads to lower prices because economic production contracts reducing demand. This can be read from Equation (2.6).

In [1], the authors note that $\frac{\partial p}{\partial q} > 0$ is unstable. When extraction levels are increasing, $\frac{\partial p}{\partial q} > 0$ encourages many actors to enter the extraction business which commonly leads to sharp increases in extraction levels and a price collapse. This occurred in the late 1970's and early 1980's and again from 2000 to 2013 when capital expenses increased at roughly 11% per year [17, 26]. If extraction levels are decreasing then $\frac{\partial p}{\partial q} > 0$ leads to bankruptcies of higher priced extractors which leads to sharply decreasing extraction levels.

4 Scenarios

The success of our analysis using the dynamic production function identities should be judged by its ability to predict the dynamics of future scenarios.

Throughout this section we make the following hypothesis:

(H4) For most of the contraction phase of oil extraction, $\frac{\partial Y}{\partial q_i} > 0$ and is relatively large.

Hypothesis (H4) and (2.8) give us the following bound on price increases:

$$I_{\tilde{p}_i}(t_1, t_2) < I_{\tilde{C}_i}(t_1, t_2)/I_{q_i}(t_1, t_2) \tag{4.1}$$

4.1 Cost Share Dynamics.

In the case of a sharp unexpected change in extraction levels, that is in the case of oil shocks or gluts, one can approximate $I_{\tilde{p}_i}(t_1, t_2)$ and $I_{\tilde{C}_i}(t_1, t_2)$ with $I_{p_i}(t_1, t_2)$ and $I_{C_i}(t_1, t_2)$ respectively. These measures can then be used with $I_Q(t_1, t_2)$ to estimate the effect of an oil shock or glut on the economy.

Observe that consumers choices influence cost share. For example in the 1940's, people routinely heated to 14 degrees C. Suppose lower prices and/or technology reduce the amount of money relative to a salary to heat to 14 degrees. The consumer can decide to either use her disposable income to purchase other items, or heat to 16 degrees C. Likewise when the price of fuel is high, consumers tend to buy more fuel efficient vehicles than when the price is low.

For fundamental quantities such as oil, we thus expect that in a growing economy $I_{\tilde{C}_i} > I_{\tilde{C}_i^{\min}}$, where $I_{tildeC_i^{\min}}$ is the minimum possible cost share. In a shrinking economy we expect that $I_{\tilde{C}_i}$ will be very close to $I^{\min}_{tildeC_i}$. Moreover in a contracting economy, substitutes may be found for oil so that $I_{C_i} < I_{\tilde{C}_i}$. From (2.6), we have, for any times t_1 and t_2,

$$I_{\tilde{p}_i}(t_1, t_2) = I_{Y_i}(t_1, t_2) I_{\tilde{C}_i}(t_1, t_2) / I_{q_i}(t_1, t_2). \tag{4.2}$$

It is clear from (4.2) that during the expansion phase oil production economic growth and consumer choices will support oil prices while during the contraction phase, economic contraction and consumer choices will tend to decrease prices. We believe (4.2) can be used to measure and calibrate Jevons paradox which will not support prices as much in a shrinking economy.

In the preceding argument, the cost share we speak of is not exactly C_i. It is the cost share in consumer budgets. Because a standard effect of stagflation is to reduce wages, consumers with lower wages will be more sensitive to higher prices and change their preferences accordingly.

Rising cost share of energy is not benign. It means a loss of diversity in the economy. It means less money for some non core jobs in the economy which leads to lower wages and fewer opportunities in these areas. This is a principle aspect of stagflation. The cost share of food and energy rises in family budgets and less money can be devoted to other areas of the economy.

From (2.9), we see that $\dfrac{\partial q_i}{\partial p_i} < 0$ (which implies $\dfrac{\partial p_i}{\partial q_i} < 0$) causes a drag on the economy. One can interpret this fact as follows: lower prices send a signal to suppliers that supplies should fall, higher prices send the opposite signal. Thus if the suppliers ignore market signals, this causes the economy to contract. The drag can be compensated if $\dfrac{\partial C_i}{\partial p_i} < 0$ or at least small. In other words, if $\dfrac{\partial q_i}{\partial p_i} < 0$, and $1/p_i$ is small, then in order for $\dfrac{\partial Y}{\partial p_i}$ to be positive, q_i must cause growth in other sectors of the economy. The more efficiently oil is used, the greater the growth of other sectors of the economy.

4.2 Current analysis

Using the vocabulary of Turchin and Nefedov, we put the growth phase of oil extraction from 1865 to 1973. Before 1973, there was strong regular growth in extraction volumes. We then enter a period of volatility in extraction volumes characterized by the price shocks of the 1970's. Although there was a strong political component to the price shocks in the 1970's [19], the period gives valuable lessons on how prices behave when oil supplies are constrained. We begin to see signs of stagflation with respect to oil extraction in the late 1970's: stagnant wages and increased difference between the elites and the working class [27]. Of course the modern economy of the 20th and 21st centuries is much more stratified than that of the agrarian societies studied by Turchin and Nefedov with several classes of workers (blue collar, white collar, etc.) and some classes of workers (white collar) at least initially belong to what can be considered an elite class.

If one considers the oil extracting firms to be a class of elites, one sees inter-elite competition intensifying in two distinct periods. We speak of the price wars of the late 1980's and beginning in late 2014 to the present.

The reasons for decreased pressure on oil prices in the 1990's should be the subject of future research. Perhaps economic growth during that period was driven by the high technology sector less reliant on oil. Perhaps efficiency gains attained during the period of high prices took pressure off of prices. Perhaps stagnant wages reduced demand of the working class.

It is highly probable that the current low price environment (relative to extraction costs) signals the beginning of the contraction phase of oil extraction.

4.3 Analysis of common scenarios

4.3.1 The magic market.

A common scenario is that the contraction phase of oil extraction will cause shortages of oil which will cause prices to increase and the market will find a solution. This scenario contradicts Conjecture 3.1 and implies great efficiency gains will be needed to keep the economy from contracting.

This scenario does not fit with the empirical evidence presented in [25] in which scarcity of essential items is associated with extremely short term thinking and poor general performance.

Empirically we see that current markets are dysfunctional. The current low price environment is attracting record amounts of capital to produce oil at a loss [30, 8]. The market is ignoring price signals which we have noted is a drag on the economy. Furthermore consumers in the U.S. are purchasing inefficient vehicles unaware that many analysts are expecting oil extraction levels to fall. The (non OPEC) oil extraction industry has accumulated a large amount of debt [21] in the last 5 years which will make it difficult to invest in new projects even if prices rise significantly.

For these reasons we find this scenario improbable.

4.3.2 The undulating plateau (prolonged stagflation).

The IEA in their World Energy Outlook frequently refers to an undulating plateau in oil extraction lasting several decades to about 2050. This scenario implies roughly constant extraction levels and increasing prices because extraction costs are increasing. Thus $\frac{\partial q_i}{\partial p_i}$ is close to 0. This means that $|\frac{\partial p_i}{\partial q_i}|$ is large so that there will be high price volatility. Moreover the economy will lose diversity because in general $\frac{\partial C_i}{\partial p_i} > 0$. We believe that the loss of diversity translates into lower wages and increased efficiency in the use of oil to combat rising cost share. These two forces, along with a stagnating economy contribute to downward pressure on prices (see (4.2)). The undulating plateau would thus end when the market price for oil falls below the cost of extraction. Indeed, this description fits the 2005 to 2014 period, and the current low price of oil seems to be the signal that the contraction phase has begun.

4.3.3 The rule of Hotelling.

The rule of Hotelling predicts that the price of oil increases exponentially irrespective of the quantity produced. This implies that in the contractive stage of oil extraction, $\frac{\partial p_i}{\partial q_i} << 0$. To satisfy (H4), $\frac{\partial C_i}{\partial q_i} < 0$. This implies efficiency must grow dramatically. Since efficiency is a bounded quantity, this will not be sustainable.

4.4 Probable scenarios

The oil extraction industry is cyclical with periods of high prices spurring investment followed by periods of low prices in which investment falls. Much oil extraction is not profitable at current prices [20, 7].

According to Rystad Energy oil extraction will decline slightly from 2016 to 2020 [16]. We believe that efficiency gains should make this decrease bearable. After 2020, we anticipate that non-OPEC oil extraction will fall more rapidly because of the cuts in capital expenses that have taken place in the last two years and accumulated debt of the oil extraction industry [21]. This will cause tension between the extraction industry and the financial sector as they grapple with ways to pay off debt and maintain extraction levels (inter-elite competition). In any case whether it is investors or extractors, this will cause increased social mobility, mostly downward which will put downward pressure on oil prices.

Eventually extraction rates will fall fast enough to raise prices at least temporarily, as suggested by the empirical model (3.6), until companies and citizens go bankrupt and demand decreases. We anticipate improved efficiency in oil use will be used to reduce cost share, there will not be a corresponding price rebound.

A Seneca cliff [6] can be imagined if for example a precipitous drop in non OPEC (after 2020) extraction coincides with conflict in the Persian Gulf simultaneously reducing OPEC extraction rates. If extraction rates fall precipitously and remain low for two years, one can expect a price spike followed by a drop in prices which decimates first industries which use oil followed by the oil extraction industry and a recovery will be highly unlikely.

Note that both theoretical considerations from the dynamic production function identities and the empirical model (3.6) indicate that price dynamics will speed the rate of decline in oil extraction. During the growth phase of oil extraction, low prices increased demand and high prices increase supply, during the contraction phase, low prices will diminish supply and high prices will diminish demand.

5 Conclusion

Our analysis and empirical evidence are consistent with oil being a fundamental quantity in economic production. Our analysis indicates that once the contraction period for oil extraction begins, price dynamics will accelerate the decline in extraction rates: extraction rates decline because of a decrease in profitability of the extraction business.

Our empirical model for prices can be used by those studying future extraction rates whose models currently do not consider price parameters.

We believe that the contraction period in oil extraction has begun and that policy makers should be making contingency plans. Strategies for economies facing energy constraints are reviewed in [31].

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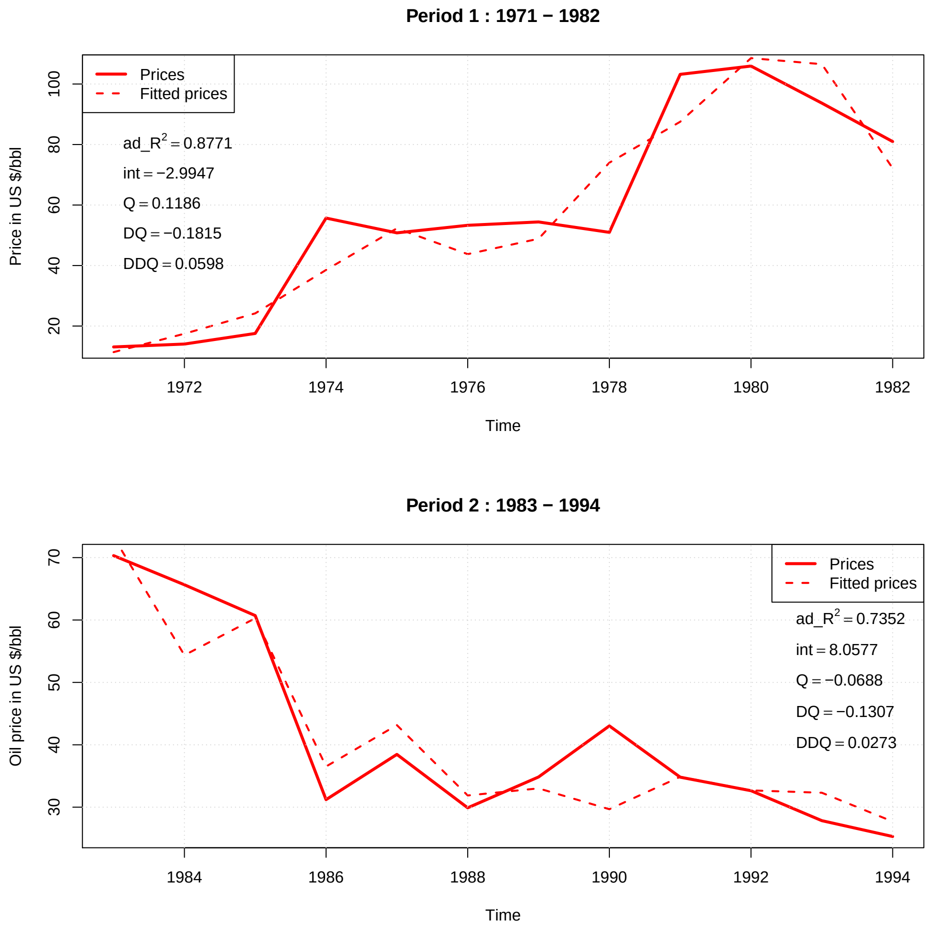
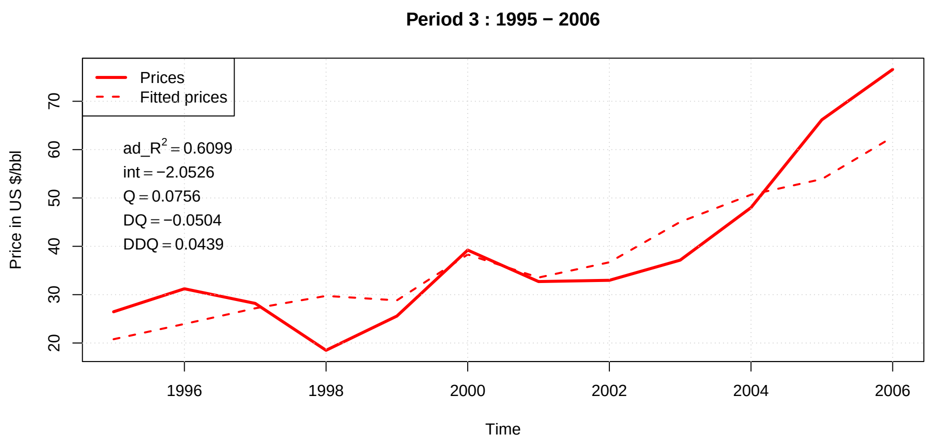


Figure 5: Adjusted price and linear regression coefficients for the first two periods



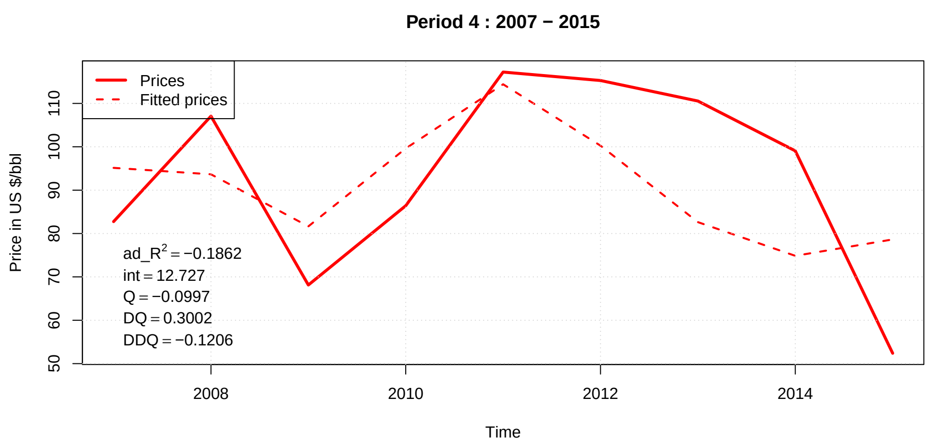


Figure 6: Adjusted price and linear regression coefficients for the last two periods

```
Approximate 95% confidence intervals

 Coefficients:
                    lower        est.      upper
(Intercept)   0.022397944  1.59149406 3.16059019
Quantity67    0.009351441  0.03233695 0.05532245
DQuantity67  -0.136791534 -0.06161179 0.01356795
DDQuantity67 -0.026275104  0.01760809 0.06149129
attr(,"label")
[1] "Coefficients:"

 Correlation structure:
        lower      est.     upper
Phi 0.5850527 0.8157226 0.9242878
attr(,"label")
[1] "Correlation structure:"

 Residual standard error:
    lower      est.     upper
0.3095668 0.4745731 0.7275315
```

Figure 7: Confidence intervals for generalized linear model

```
Generalized least squares fit by maximum likelihood
  Model: log(Price67) ~ Quantity67 + DQuantity67 + DDQuantity67
  Data: NULL
       AIC     BIC    logLik
  25.46087 36.8118 -6.730437

Correlation Structure: AR(1)
 Formula: ~1
 Parameter estimate(s):
      Phi
0.8157226

Coefficients:
                  Value Std.Error    t-value p-value
(Intercept)   1.5914941 0.7790544  2.0428536  0.0470
Quantity67    0.0323369 0.0114123  2.8335228  0.0069
DQuantity67  -0.0616118 0.0373267 -1.6506111  0.1058
DDQuantity67  0.0176081 0.0217880  0.8081572  0.4233

 Correlation:
             (Intr) Qntt67 DQnt67
Quantity67   -0.963
DQuantity67   0.181 -0.253
DDQuantity67 -0.076  0.134 -0.836

Standardized residuals:
        Min          Q1         Med          Q3         Max
-2.00263835 -0.78485751 -0.06177291  0.89327110  2.19086672

Residual standard error: 0.4745731
Degrees of freedom: 49 total; 45 residual
```

Figure 8: R output for generalized linear model

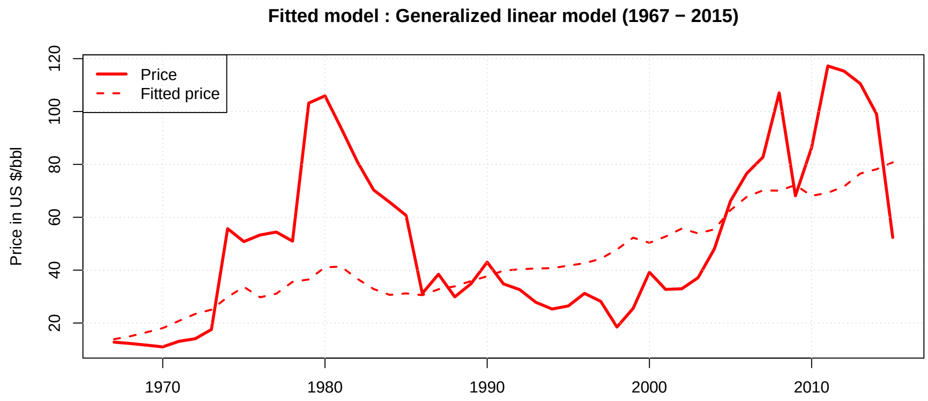


Figure 9: Fitted price from 1967 to 2015

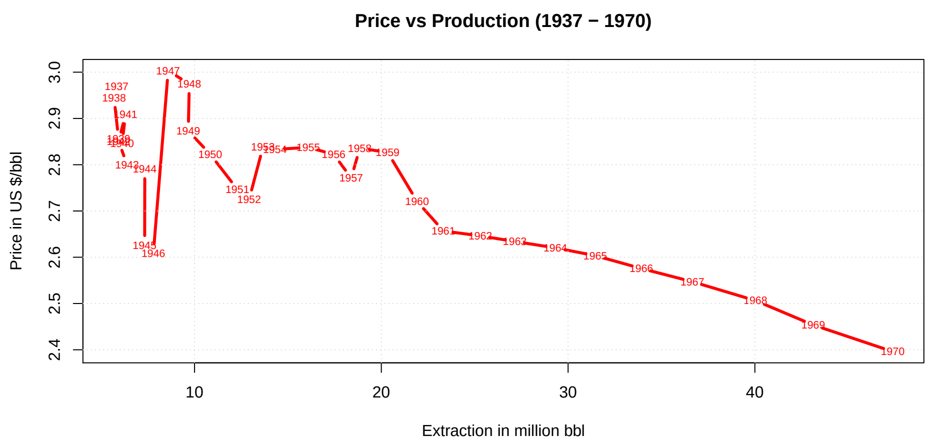


Figure 10: Oil price in function of oil production from 1937 to 1970

```
Call:
lm(formula = log(Price38) ~ Quantity38 + DQuantity38)

Residuals:
      Min        1Q    Median        3Q       Max
-0.242481 -0.029444 -0.004043  0.049495  0.158822

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.932949   0.030064  97.557  < 2e-16 ***
Quantity38  -0.010851   0.003625  -2.994  0.00548 **
DQuantity38  0.005838   0.039986   0.146  0.88489
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.08699 on 30 degrees of freedom
Multiple R-squared:  0.6866,    Adjusted R-squared:  0.6657
F-statistic: 32.86 on 2 and 30 DF,  p-value: 2.767e-08
```

Figure 11: R output for linear regression with Q and ∇Q

```
Call:
lm(formula = log(Price37) ~ Quantity37)

Residuals:
      Min        1Q    Median        3Q       Max
-0.247301 -0.027531 -0.003024  0.045956  0.155702

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938555   0.026636 110.323  < 2e-16 ***
Quantity37  -0.010602   0.001242  -8.536 9.38e-10 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.08592 on 32 degrees of freedom
Multiple R-squared:  0.6948,    Adjusted R-squared:  0.6853
F-statistic: 72.86 on 1 and 32 DF,  p-value: 9.375e-10
```

Figure 12: R output for linear regression with Q

```
Generalized least squares fit by maximum likelihood
  Model: log(Price37) ~ Quantity37
  Data: NULL
        AIC       BIC   logLik
  -76.68285 -69.05105 43.34143

Correlation Structure: ARMA(1,1)
 Formula: ~1
 Parameter estimate(s):
      Phi1     Theta1
0.04192361 0.68672117

Coefficients:
                 Value  Std.Error  t-value p-value
(Intercept)  2.9432219 0.03648706 80.66482       0
Quantity37  -0.0108237 0.00166328 -6.50744       0

 Correlation:
           (Intr)
Quantity37 -0.826

Standardized residuals:
        Min          Q1         Med          Q3         Max
-3.02031726 -0.35963721 -0.02228464  0.53613448  1.84587136

Residual standard error: 0.08285304
Degrees of freedom: 34 total; 32 residual
```

Figure 13: R output for generalized linear regression with Q from 1937 to 1970

```
Approximate 95% confidence intervals

 Coefficients:
                  lower        est.        upper
(Intercept)  2.86890018  2.94322188  3.017543581
Quantity37  -0.01421165 -0.01082367 -0.007435683
attr(,"label")
[1] "Coefficients:"

 Correlation structure:
            lower       est.     upper
Phi1   -0.3509428 0.04192361 0.4222400
Theta1  0.3357870 0.68672117 0.8702507
attr(,"label")
[1] "Correlation structure:"

 Residual standard error:
     lower       est.      upper
0.06233411 0.08285304 0.11012631
```

Figure 14: R output confidence intervals for generalized linear regression with Q from 1937 to 1970

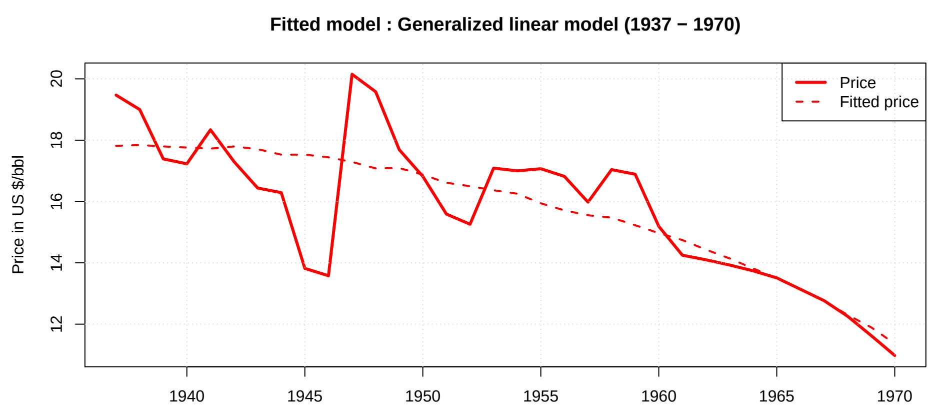


Figure 15: Fitted price from 1937 to 1970