Appendix A
A Generalization of the Harsanyi NTU Value to Games with Incomplete Information ☆

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1. Introduction

The purpose of this supplementary note is to provide detailed computations of the different value allocations presented in the paper.

2. Example 1: A Collective Choice Problem

A value allocation in this example can be uniquely supported by the utility weights $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3^H, \bar{\lambda}_3^L) = (1, 1, 9/10, 1/5)$, which correspond to the supporting normal vector to the incentive efficient frontier in the individually rational zone. For these utility weights, the optimal value of the dual variables is $(\bar{\alpha}_1(L \mid H), \bar{\alpha}_1(H \mid L)) = (0, 0)$. Virtual utilities take the form

$$\begin{aligned} v_i(d, t, \bar{\lambda}, \bar{\alpha}) &= u_i(d, t), \quad \forall t \in \{H, L\}, \ \forall i = 1, 2 \\ v_3(d, H, \bar{\lambda}, \bar{\alpha}) &= u_3(d, H) \\ v_3(d, L, \bar{\lambda}, \bar{\alpha}) &= 2u_3(d, L) \end{aligned}$$

Thus, the payoff matrix in the $(\bar{\lambda}, \bar{\alpha})$-virtual utility game is

(v_1, v_2, v_3)	L	H
$[d_1, d_2, d_3]$	$(0, 0, 0)$	$(0, 0, 0)$
$[d_{12}, d_3]$	$(5, 5, 0)$	$(5, 5, 0)$
$[d_{13}^1, d_2]$	$(0, 0, 10)$	$(0, 0, 10)$
$[d_{13}^3, d_2]$	$(10, 0, -10)$	$(10, 0, 0)$
$[d_{23}^2, d_1]$	$(0, 0, 10)$	$(0, 0, 10)$
$[d_{23}^3, d_1]$	$(0, 10, -10)$	$(0, 10, 0)$

For any (M or H) bargaining solution, μ_N solves the primal problem for $\bar{\lambda}$. Hence, it maximizes the Lagrangian in (2.3). Then, condition (2.5) implies that

$$W_N(\mu_N, t, \bar{\lambda}, \bar{\alpha}) = \max_{d \in D} \sum_{i \in N} v_i(d, t, \bar{\lambda}, \bar{\alpha}) = 10, \quad \forall t \in \{H, L\} \tag{1}$$

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2.1. The M-Value

Myerson's (1984b) rational threats for coalition $S \subset N$ solve

$$\max_{\mu_S \in \mathcal{M}_S} \sum_{t \in \{H,L\}} p(t) \sum_{i \in S} v_i(\mu_S, t, \bar{\lambda}, \bar{\alpha}) \tag{2}$$

Clearly, $W_{\{i\}}(\mu_{\{i\}}, t, \bar{\lambda}, \bar{\alpha}) = 0$ for every $t \in \{H, L\}$ and every $i \in N$. Let us consider the problem faced by coalition $S = \{i, 3\}$ $(i = 1, 2)$. Notice that, for any mechanism $\mu_{\{i,3\}}$, the expected worth for this coalition is

$$\sum_{t \in \{H,L\}} p(t) \sum_{j \in S} v_j(\mu_{\{i,3\}}, t, \lambda, \alpha) = 9[\mu_{\{i,3\}}(d^i_{i3} \mid H) + \mu_{\{i,3\}}(d^3_{i3} \mid H)] + \mu_{\{i,3\}}(d^i_{i3} \mid L) \tag{3}$$

Hence, if we were to maximize (3) subject to $\mu_{\{i,3\}} \in \mathcal{M}_{\{i,3\}}$, an optimal solution should satisfy

$$\mu_{\{i,3\}}(d^i_{i3} \mid H) + \mu_{\{i,3\}}(d^3_{i3} \mid H) = \mu_S(d^i_{i3} \mid L) = 1 \tag{4}$$

Therefore, $W_{\{i,3\}}(\mu_{\{i,3\}}, t, \bar{\lambda}, \bar{\alpha}) = 10$ for all $t \in \{H, L\}$.

Because the members of coalition $\{1, 2\}$ do not have private information, a mechanism for this coalition is simply an element of $\Delta(D_{\{1,2\}}) = \mathcal{M}_{\{1,2\}}$. Then, it is clear that the only rational threat for coalition $\{1, 2\}$ is the mechanism $\mu_{\{1,2\}}(d_{12}) = 1$. Then, $W_{\{1,2\}}(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) = 10$ for every $t \in \{H, L\}$.

Summarizing, the conditionally transferable virtual utility game is described by

$$\begin{aligned} W_i(\mu_i, t, \bar{\lambda}, \bar{\alpha}) &= 0, \quad \forall i \in N,\ \forall t \in \{H, L\} \\ W_S(\mu_S, t, \bar{\lambda}, \bar{\alpha}) &= 10, \quad S \subseteq N, S \neq \{i\}\ (i = 1, 2, 3),\ \forall t \in T \end{aligned}$$

Then, $\phi_i(W(\eta, t, \bar{\lambda}, \bar{\alpha})) = \frac{10}{3}$ for all $i \in N$ and $t \in \{H, L\}$. Therefore, a mechanism $\bar{\mu}_N$ satisfies the warrant equations if and only if

$$U(\bar{\mu}_N) = (U_1(\bar{\mu}_N), U_2(\bar{\mu}_N), U_3(\bar{\mu}_N \mid H), U_3(\bar{\mu}_N \mid L))) = \left(\tfrac{10}{3}, \tfrac{10}{3}, \tfrac{10}{3}, \tfrac{5}{3}\right)$$

This allocation is achieved by the incentive efficient mechanism

$$\bar{\mu}_N([d_{12}, d_3] \mid t) = \tfrac{2}{3},\ \bar{\mu}_N([d^2_{23}, d_1] \mid t) = \bar{\mu}_N([d^1_{13}, d_2] \mid t) = \tfrac{1}{6}, \quad \forall t \in \{H, L\}.$$

The allocation $U(\bar{\mu}_N)$ is feasible and by construction it solves the primal for $\bar{\lambda}$. Thus it is the unique (non-degenerated) M-value of this game.

2.2. Coalitionally Incentive Compatible M-Value

Let us consider the game with transferable virtual utility when coalitional threats are required to be incentive compatible. Clearly, rational threats do not change for coalition $\{1, 2\}$, since its members do not face any informational problem.

Now we wonder if there is an incentive compatible mechanism for $\{i, 3\}$ satisfying (4). Incentive constraints for $\{i, 3\}$ are

$$\mu_{\{i,3\}}(d^i_{i3} \mid H) \geq \mu_{\{i,3\}}(d^i_{i3} \mid L) \tag{5}$$

$$\mu_{\{i,3\}}(d^i_{i3} \mid L) - \mu_{\{i,3\}}(d^3_{i3} \mid L) \geq \mu_{\{i,3\}}(d^i_{i3} \mid H) - \mu_{\{i,3\}}(d^3_{i3} \mid H) \tag{6}$$

Note that (6) holds for any mechanism satisfying (4). Then (4) and (5) implies $\mu_S(d^i_{i3} \mid H) = 1$. Then the unique incentive compatible rational threat for $\{i, 3\}$ is the mechanism

$$\mu_{\{i,3\}}(d^i_{i3} \mid t) = 1, \quad \forall t \in \{H, L\}$$

Hence, $W_{\{i,3\}}(\mu_{\{i,3\}}, t, \bar{\lambda}, \bar{\alpha}) = 10$ for all $t \in T$. Therefore, the transferable virtual utility game does not change when we require coalitional threats to be incentive compatible. As a consequence, the value allocation continues to be the same. Imposing incentive compatibility on the mechanisms μ_S does not always change the worth of the coalition, because the virtual worth of all coalitions is computed using virtual utilities as determined by the vector $(\bar{\lambda}, \bar{\alpha})$ specified for the grand coalition.

2.3. The H-Value

We compute now our cooperative solution concept. We proceed recursively. When $S = \{i\}$ $(i \in N)$, $W_{\{i\}}(\mu_{\{i\}}, t, \bar{\lambda}, \bar{\alpha}) = 0$ for every $t \in \{H, L\}$. Then, optimal egalitarian threats for coalition $S \subset N$ solve

$$\begin{aligned} \max_{\mu_S \in \mathcal{M}_S} \quad & \sum_{t \in \{H,L\}} p(t) \sum_{i \in S} v_i(\mu_S, t, \bar{\lambda}, \bar{\alpha}) \\ \text{s.t.} \quad & \sum_{t_{-i} \in T_{-i}} p(t_{-i}) v_i(\mu_S, t, \bar{\lambda}, \bar{\alpha}) = \sum_{t_{-i} \in T_{-i}} p(t_{-i}) v_j(\mu_S, t, \bar{\lambda}, \bar{\alpha}), \quad \forall t_i, \ \forall i, j \in S \end{aligned} \tag{7}$$

For coalition $S = \{1, 2\}$, problem (7) reduces to

$$\begin{aligned} \max_{\mu_{\{1,2\}} \in \Delta(D_{\{1,2\}})} \quad & \sum_{t \in \{H,L\}} p(t) \sum_{i \in S} v_i(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) \\ \text{s.t.} \quad & \sum_{t \in \{H,L\}} p(t) v_1(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) = \sum_{t \in \{H,L\}} p(t) v_2(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) \end{aligned}$$

Clearly the unique solution to this optimization problem is the mechanism $\mu_{\{1,2\}}(d_{12}) = 1$. Then, $W_{\{1,2\}}(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) = 10$ for every $t \in \{H, L\}$.

Now consider coalition $S = \{i, 3\}$ $(i = 1, 2)$. Problem (7) reduces to

$$\begin{aligned} \max_{\mu_{\{i,3\}} \in \mathcal{M}_{\{i,3\}}} \quad & \sum_{t \in \{H,L\}} p(t) \sum_{i \in S} v_i(\mu_{\{i,3\}}, t, \bar{\lambda}, \bar{\alpha}) \\ \text{s.t.} \quad & v_i(\mu_{\{i,3\}}, t, \bar{\lambda}, \bar{\alpha}) = v_3(\mu_{\{i,3\}}, t, \bar{\lambda}, \bar{\alpha}), \quad \forall t \in \{H, L\} \end{aligned}$$

Notice that

$$v_i(\mu_{\{i,3\}}, H, \bar{\lambda}, \bar{\alpha}) = v_3(\mu_{\{i,3\}}, H, \bar{\lambda}, \bar{\alpha}) \iff \mu_{\{i,3\}}(d^i_{i3} \mid H) = \mu_{\{i,3\}}(d^3_{i3} \mid H) \tag{8}$$

$$v_i(\mu_{\{i,3\}}, L, \bar{\lambda}, \bar{\alpha}) = v_3(\mu_{\{i,3\}}, L, \bar{\lambda}, \bar{\alpha}) \iff \mu_{\{i,3\}}(d^i_{i3} \mid L) = 2\mu_{\{i,3\}}(d^3_{i3} \mid L) \tag{9}$$

The unique mechanism for $\{i, 3\}$ satisfying (4), (8) and (9) is

$$\mu_{\{i,3\}}(d^i_{i3} \mid H) = \mu_{\{i,3\}}(d^3_{i3} \mid H) = \tfrac{1}{2}, \quad \mu_{\{i,3\}}(d^3_{i3} \mid L) = 1 - \mu_{\{i,3\}}(d^i_{i3} \mid L) = \tfrac{1}{3}$$

It is thus the unique optimal egalitarian threat for $\{i,3\}$. We conclude that $W_{\{i,3\}}(\mu_{\{i,3\}}, H, \bar{\lambda}, \bar{\alpha}) = 10$ and $W_{\{i,3\}}(\mu_{\{i,3\}}, L, \bar{\lambda}, \bar{\alpha}) = \frac{20}{3}$.

Therefore, the Shapley value allocations of the $(\bar{\lambda}, \bar{\alpha})$-virtual utility game are

$$\phi(W(\eta, H, \bar{\lambda}, \bar{\alpha})) = \left(\tfrac{10}{3}, \tfrac{10}{3}, \tfrac{10}{3}\right) \quad \text{and} \quad \phi(W(\eta, L, \bar{\lambda}, \bar{\alpha})) = \left(\tfrac{35}{9}, \tfrac{35}{9}, \tfrac{20}{9}\right)$$

A mechanism μ_N^* thus satisfies the warrant equations if and only if

$$U(\mu_N^*) = (U_1(\mu_N^*), U_2(\mu_N^*), U_3(\mu_N^* \mid H), U_3(\mu_N^* \mid L))) = \left(\tfrac{61}{18}, \tfrac{61}{18}, \tfrac{60}{18}, \tfrac{20}{18}\right)$$

This allocation is achieved by the incentive efficient mechanism

$$\mu_N^*([d_{13}^3, d_2^0] \mid H) = \mu_N^*([d_{23}^3, d_1^0] \mid H) = \mu_N^*([d_{23}^2, d_1^0] \mid H) = \tfrac{1}{3},$$

$$\mu_N^*([d_{12}, d_3^0] \mid L) = 1 - \mu_N^*([d_{23}^2, d_1^0] \mid L) = \tfrac{7}{9}$$

The allocation $U(\mu_N^*)$ is feasible and by construction it solves the primal for $\bar{\lambda}$. Thus it is the unique H-value of this game.

2.4. *Coalitionally Incentive Compatible H-Value*

We require now coalitional threats to be incentive compatible. $W_{\{i\}}(\mu_{\{i\}}, t, \bar{\lambda}, \bar{\alpha}) = 0$ ($i \in N$) for every $t \in \{H, L\}$. Coalition $\{1,2\}$ does not face any informational problem, then $\mu_{\{1,2\}}(d_{12}) = 1$ is an incentive compatible optimal egalitarian threat for $\{1,2\}$. We conclude that $W_{\{1,2\}}(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) = 10$ for every $t \in \{H, L\}$.

Consider a coalition $\{i,3\}$ ($i = 1,2$). When incentive constraints are not imposed, the unique optimal egalitarian threat for this coalition violates the constraint asserting that 3_H must not be tempted to report 3_L; thus (5) is the only binding incentive constraint. Then an incentive compatible optimal egalitarian threat for $\{i,3\}$ satisfies (5) (as equality) together with the balanced contributions conditions (8) and (9). The unique mechanism for $\{i,3\}$ satisfying these requirements is

$$\mu_{\{i,3\}}(d_{i3}^i \mid H) = \mu_{\{i,3\}}(d_{i3}^3 \mid H) = \mu_{\{i,3\}}(d_{i3}^i \mid L) = \tfrac{1}{2}, \quad \mu_{\{i,3\}}(d_{i3}^3 \mid L) = \mu_{\{i,3\}}([d_i, d_3] \mid L) = \tfrac{1}{4}$$

Therefore, it is the unique incentive compatible optimal egalitarian threat. The virtual worths for coalition $\{i,3\}$ are now $W_{\{i,3\}}(\mu_{\{i,3\}}, H, \bar{\lambda}, \bar{\alpha}) = 10$ and $W_{\{i,3\}}(\mu_{\{i,3\}}, L, \bar{\lambda}, \bar{\alpha}) = 5$.

Therefore, the Shapley value allocations of the $(\bar{\lambda}, \bar{\alpha})$-virtual utility game are

$$\phi(W(\eta, H, \bar{\lambda}, \bar{\alpha})) = \left(\tfrac{10}{3}, \tfrac{10}{3}, \tfrac{10}{3}\right) \quad \text{and} \quad \phi(W(\eta, L, \bar{\lambda}, \bar{\alpha})) = \left(\tfrac{25}{6}, \tfrac{25}{6}, \tfrac{10}{6}\right)$$

A mechanism μ_N^* thus satisfies the warrant equations if and only if

$$U(\mu_N^*) = (U_1(\mu_N^*), U_2(\mu_N^*), U_3(\mu_N^* \mid H), U_3(\mu_N^* \mid L))) = \left(\tfrac{41}{12}, \tfrac{41}{12}, \tfrac{40}{12}, \tfrac{10}{12}\right)$$

This allocation is achieved by the incentive efficient mechanism

$$\mu_N^*([d_{13}^3, d_2^0] \mid H) = \mu_N^*([d_{23}^3, d_1^0] \mid H) = \mu_N^*([d_{23}^2, d_1^0] \mid H) = \tfrac{1}{3},$$

$$\mu_N^*([d_{12}, d_3^0] \mid L) = 1 - \mu_N^*([d_{23}^2, d_1^0] \mid L) = \tfrac{5}{6}$$

The allocation $U(\mu_N^*)$ is feasible and by construction it solves the primal for $\bar{\lambda}$. Thus it is the unique coalitionally incentive compatible H-value of this game.

3. Example 2: de Clippel's Example

In this game the incentive efficient frontier coincides with an hyperplane with slope $\bar{\lambda} = (\bar{\lambda}_1^H, \bar{\lambda}_1^L, \bar{\lambda}_2, \bar{\lambda}_3) = (4/5, 1/5, 1, 1)$. Hence, a value allocation can only be supported by the utility weights $\bar{\lambda}$. The unique solution of the dual problem for $\bar{\lambda}$ is $(\bar{\alpha}_1(L \mid H), \bar{\alpha}_1(H \mid L)) = (0, 0)$. Virtual utilities thus coincide with real utilities in every state.

A bargaining solution, μ_N, must solve the primal problem for $\bar{\lambda}$. Hence, it maximizes the Lagrangian in (2.3). Then, condition (2.5) implies that $W_N(\mu_N, H, \bar{\lambda}, \bar{\alpha}) = 90$ and $W_N(\mu_N, L, \bar{\lambda}, \bar{\alpha}) = 30$.

3.1. The H-Value

For any coalition $S = \{i\}$ $(i \in N)$, $W_{\{i\}}(\mu_{\{i\}}, t, \bar{\lambda}, \bar{\alpha}) = 0$ for every $t \in \{H, L\}$. Then, optimal egalitarian threats for a two-person coalition solve (7). Clearly, for any coalition $S \subset N$ different from $\{1, 2\}$, we have that $W_S(\mu_S, t, \bar{\lambda}, \bar{\alpha}) = 0$ for all $t \in \{H, L\}$. An optimal egalitarian threat for $\{1, 2\}$ solves

$$\begin{aligned} \max_{\mu_{\{1,2\}} \in \mathcal{M}_{\{1,2\}}} \quad & \sum_{t \in T_1} p(t) \sum_{i \in S} v_i(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) \\ \text{s.t.} \quad & v_1(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) = v_2(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}), \quad \forall t \in \{H, L\} \end{aligned}$$

The expected (virtual) worth for this coalition is

$$\begin{aligned} &\sum_{t \in \{H,L\}} p(t) \sum_{j \in S} v_j(\mu_{\{1,2\}}, t, \bar{\lambda}, \bar{\alpha}) \\ &\qquad = 72[\mu_{\{1,2\}}(d_{12}^1 \mid H) + \mu\{1,2\}(d_{12}^2 \mid H)] + 6[\mu\{1,2\}(d_{12}^1 \mid L) + \mu\{1,2\}(d_{12}^2 \mid L)] \qquad (10) \end{aligned}$$

On the other hand,

$$v_1(\mu_{\{1,2\}}, H, \bar{\lambda}, \bar{\alpha}) = v_2(\mu_{\{1,2\}}, H, \bar{\lambda}, \bar{\alpha}) \iff \mu_{\{1,2\}}(d_{12}^1 \mid H) = \mu_{\{1,2\}}(d_{12}^2 \mid H) \qquad (11)$$

$$v_1(\mu_{\{1,2\}}, L, \bar{\lambda}, \bar{\alpha}) = v_2(\mu_{\{1,2\}}, L, \bar{\lambda}, \bar{\alpha}) \iff \mu_{\{1,2\}}(d_{12}^1 \mid L) = 5\mu_{\{1,2\}}(d_{12}^2 \mid L) \qquad (12)$$

If we were to maximize (10) subject only to $\mu_{\{1,2\}} \in \mathcal{M}_{\{1,2\}}$, an optimal solution must be of the form

$$\mu_{\{1,2\}}(d_{12}^1 \mid t) + \mu_{\{1,2\}}(d_{12}^2 \mid t) = 1, \quad \forall t \in \{H, L\} \qquad (13)$$

The unique mechanism satisfying (11)-(13) is

$$\mu_{\{1,2\}}(d_{12}^1 \mid H) = \mu_{\{1,2\}}(d_{12}^2 \mid H) = \tfrac{1}{2}, \quad \mu_{\{1,2\}}(d_{12}^2 \mid L) = 1 - \mu_{\{1,2\}}(d_{12}^1 \mid L) = \tfrac{1}{6},$$

thus it is the unique optimal egalitarian threat for $\{1, 2\}$. Therefore, $W_{\{1,2\}}(\mu_{\{1,2\}}, H, \bar{\lambda}, \bar{\alpha}) = 90$ and $W_{\{1,2\}}(\mu_{\{1,2\}}, L, \bar{\lambda}, \bar{\alpha}) = 30$. We conclude that the Shapley value allocations of the $(\bar{\lambda}, \bar{\alpha})$-virtual utility game are

$$\phi(W(\eta, H, \bar{\lambda}, \bar{\alpha})) = (45, 45, 0) \quad \text{and} \quad \phi(W(\eta, L, \bar{\lambda}, \bar{\alpha})) = (15, 15, 0)$$

A mechanism μ_N^* thus satisfies the warrant equations if and only if it achieves the utility allocation in (3.2). Hence, (3.2) is the unique H-value.

3.2. Coalitionally Incentive Compatible H-Value

We now require coalitional threats to be incentive compatible. Clearly $W_S(\mu_S, t, \bar{\lambda}, \bar{\alpha})$ does not change for any coalition $S \subset N$ different from $\{1,2\}$. Incentive constraints for coalition $\{1,2\}$ are

$$\mu_{\{1,2\}}(d_{12}^1 \mid H) \geq \mu_{\{1,2\}}(d_{12}^1 \mid L) \quad (14)$$

$$\mu_{\{1,2\}}(d_{12}^1 \mid L) - 2\mu_{\{1,2\}}(d_{12}^2 \mid L) \geq \mu_{\{1,2\}}(d_{12}^1 \mid H) - 2\mu_{\{1,2\}}(d_{12}^2 \mid H) \quad (15)$$

When incentive constraints are not imposed, the unique optimal egalitarian threat for $\{1,2\}$ violates the constraint asserting that 1_H must not be tempted to report 1_L; thus (14) is the only binding incentive constraint. Then an incentive compatible optimal egalitarian threat for $\{1,2\}$ satisfies (14) (as equality) together with the balanced contributions conditions (11) and (12). Thus, the unique incentive compatible optimal egalitarian threat is the mechanism

$$\mu_{\{1,2\}}(d_{12}^1 \mid H) = \mu_{\{1,2\}}(d_{12}^2 \mid H) = \tfrac{1}{2},$$

$$\mu_{\{1,2\}}(d_{12}^1 \mid L) = \tfrac{1}{2}, \quad \mu_{\{1,2\}}(d_{12}^2 \mid L) = \tfrac{1}{10}, \quad \mu_{\{1,2\}}([d_1, d_2] \mid L) = \tfrac{2}{5}$$

Hence, the virtual worths for coalition $\{1,2\}$ are now $W_{\{1,2\}}(\mu_{\{1,2\}}, H, \bar{\lambda}, \bar{\alpha}) = 90$ and $W_{\{1,2\}}(\mu_{\{1,2\}}, L, \bar{\lambda}, \bar{\alpha}) = 18$. The Shapley value allocations corresponding to the $(\bar{\lambda}, \bar{\alpha})$-virtual utility game are

$$\phi(W(\eta, H, \bar{\lambda}, \bar{\alpha})) = (45, 45, 0) \quad \text{and} \quad \phi(W(\eta, L, \bar{\lambda}, \bar{\alpha})) = (13, 13, 4)$$

A mechanism μ_N^* thus satisfies the warrant equations if and only if

$$U(\mu_N^*) = (U_1(\mu_N^* \mid H), U_1(\mu_N^* \mid L), U_2(\mu_N^*), U_3(\mu_N^*)) = (45, 13, 38.6, 0.8)$$

This allocation is achieved by the incentive efficient mechanism

$$\mu_N^*([d_{12}^1, d_3] \mid H) = \mu_N^*([d_{12}^2, d_1] \mid H) = \tfrac{1}{2},$$

$$\mu_N^*([d_{12}^1, d_3] \mid L) = \mu_N^*(d_{23} \mid L) = \tfrac{13}{30} \quad \mu_N^*(d_{32} \mid L) = \tfrac{2}{15}$$

Hence, $U(\mu_N^*)$ is feasible, and thus it is the unique coalitionally incentive compatible H-value of this game.