

# A CIR-Type Diffusion Driven by Hermite Processes: Well-Posedness, Positivity and Malliavin Analysis

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## Abstract

We study a generalized Cox–Ingersoll–Ross (CIR) diffusion

$$dX_t = a(b(t) - X_t) dt + (\sigma_0 + \sigma_1 \sqrt{\phi_\varepsilon(X_t)}) dZ_t^{(q,H)}, \quad X_0 = x_0 > 0,$$

where  $Z^{(q,H)}$  is a Hermite process of order  $q \geq 1$  and Hurst parameter  $H \in (1/2, 1)$ , and  $\phi_\varepsilon$  is a smooth regularisation of the square root. This framework simultaneously captures long-range dependence, non-Gaussian innovations (for  $q \geq 2$ ), and the positivity of the classical CIR model. Since Hermite processes with  $q \geq 2$  are not semimartingales, we interpret the dynamics pathwise in the Young–Stieltjes sense, exploiting the Hölder regularity of  $Z^{(q,H)}$ .

We establish four main results. First, well-posedness: a unique strong solution exists in a fractional Sobolev space, under globally Lipschitz coefficients (satisfied by  $\phi_\varepsilon$ ). Second, a quantitative positivity bound: for  $\sigma_1 = 0$ , the probability that  $X$  stays positive on  $[0, T]$  is bounded below by an explicit expression tending to 1 as the initial level and long-run target grow large relative to  $\sigma_0$ ; an almost-sure statement, available in the classical Brownian case, is not established here, since the usual boundary-non-attainment mechanism relies on tools unavailable for a non-semimartingale driver. Third, Malliavin differentiability:  $X_t \in \mathbb{D}^{1,\infty}$ , with an explicit formula for the Malliavin derivative as the solution of a linearised Young SDE. Fourth, absolute continuity: the law of  $X_t$  is absolutely continuous with respect to the Lebesgue measure for all  $t > 0$ .

**Keywords:** CIR model; Hermite process; Rosenblatt process; long-range dependence; Malliavin calculus; Young integral; positivity bounds; boundary behaviour; fractional Brownian motion.

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# 1 Introduction

## 1.1 Motivation

The Cox–Ingersoll–Ross (CIR) model, introduced in the seminal paper [7], has played a central role in the theory of interest rate modelling for nearly four decades. It describes the short rate as the solution of

$$dX_t = a(b - X_t) dt + \sigma \sqrt{X_t} dB_t, \quad X_0 = x_0 > 0, \quad (1)$$

where  $B$  is a standard Brownian motion and  $a, b, \sigma > 0$ . Under the Feller condition  $2ab \geq \sigma^2$ , the solution is strictly positive, and under  $2ab > \sigma^2$ , the boundary  $\{0\}$  is inaccessible. These properties, together with the affine structure of the model which yields analytical bond prices, have made (1) a workhorse of mathematical finance.

Despite its theoretical appeal, the CIR model conflicts with two well-documented empirical regularities of short-term interest rates. First, the autocorrelation function of interest rate levels decays hyperbolically rather than exponentially, suggesting the presence of *long-range dependence* (LRD), whereby the spectral density diverges at the origin [1, 2]. The CIR model, driven by Brownian motion, cannot reproduce this phenomenon. Second, the unconditional distribution of interest rate increments exhibits heavy tails and significant excess kurtosis, violating the Gaussian assumption inherent in the standard CIR model [2].

These two stylised facts point toward two natural generalisations: (i) replacing the Brownian driver by a *fractional Brownian motion* (fBm) with Hurst parameter  $H > 1/2$  to capture long-range dependence, as in [12, 13]; and (ii) replacing the Gaussian driver by a *non-Gaussian process* to accommodate heavy tails. The present paper proposes a unified framework that achieves both simultaneously, by driving the CIR dynamics with a *Hermite process* of arbitrary order  $q \geq 1$ .

## 1.2 The Hermite process family

The Hermite process  $Z^{(q,H)}$  of order  $q \geq 1$  and Hurst parameter  $H \in (1/2, 1)$  is a paradigmatic family of self-similar, stationary-increment processes with long-range dependence. It is defined as a  $q$ -fold multiple Wiener–Itô integral of a kernel related to the fractional integral of Brownian motion (see Section 2.1 for the precise definition). The family interpolates between the Gaussian world ( $q = 1$ : fBm) and increasingly non-Gaussian behaviour:

- $q = 1$ : fractional Brownian motion, Gaussian, belongs to the first Wiener chaos;
- $q = 2$ : the Rosenblatt process [21], non-Gaussian, positively skewed, second Wiener chaos;
- $q \geq 3$ : higher Wiener chaoses, with excess kurtosis growing with  $q$  [22].

Crucially, all members of the family share the same covariance structure as fBm, so that the Hurst parameter  $H$  governs the memory properties uniformly across orders.

A key analytical difficulty is that for  $q \geq 2$ , the Hermite process is *not a semimartingale* [22]. Consequently, Itô’s formula, the Doob–Meyer decomposition, and the classical theory of stochastic differential equations driven by semimartingales are unavailable. The

stochastic integral against  $Z^{(q,H)}$  must be interpreted in a pathwise sense, and the theory of Young–Stieltjes integration [17, 24] provides the appropriate framework, exploiting the fact that the sample paths of  $Z^{(q,H)}$  are Hölder continuous of order  $\zeta \in (0, H)$  and  $H > 1/2$ .

### 1.3 The CIR-Hermite model

We study the following stochastic differential equation (SDE):

$$dX_t = a(b(t) - X_t) dt + (\sigma_0 + \sigma_1 \sqrt{\phi_\varepsilon(X_t)}) dZ_t^{(q,H)}, \quad t \in [0, T], \quad X_0 = x_0 > 0, \quad (2)$$

where  $a > 0$  is the mean-reversion speed,  $b(t) = b_0 + b_1 \sin(2\pi t/T_{\text{per}})$  is a bounded periodic long-run target (with  $b_0 \in \mathbb{R}$ ,  $b_1 \geq 0$ ,  $T_{\text{per}} > 0$ ),  $\sigma_0 > 0$  is the base volatility level, and  $\phi_\varepsilon$  is a smooth regularisation of the square root, defined in Section 2.2. For  $q = 1$ , model (2) reduces to a fractional CIR model, as studied in [12, 13]. For  $q \geq 2$ , the model introduces genuine non-Gaussianity in the driving noise.

The choice of a periodic drift  $b(t)$  reflects the well-documented seasonal patterns in short-rate dynamics (end-of-quarter and calendar effects), while the hybrid volatility function  $\tilde{\sigma}(x) = \sigma_0 + \sigma_1 \sqrt{\phi_\varepsilon(x)}$  generalises both the square-root diffusion of classical CIR — where the coefficient vanishes at  $x = 0$ , which is what forces positivity in that setting — and a constant-volatility (Vasicek-type) specification. Unlike the classical CIR coefficient,  $\tilde{\sigma}$  does *not* vanish at  $x = 0$  whenever  $\sigma_0 > 0$ ; positivity in our setting is therefore not inherited from the boundary behaviour of  $\tilde{\sigma}$  but established separately, and only in probabilistic form (Theorem 3.5).

### 1.4 Main contributions

This paper establishes the theoretical foundations of the CIR-Hermite model (2). Our contributions are of two distinct kinds, which we are careful to distinguish throughout.

#### Genuinely new results

- **A quantitative positivity bound under a non-semimartingale driver** (Theorem 3.5). For  $\sigma_1 = 0$ , we prove an explicit lower bound on  $\mathbb{P}(X_t > 0 \forall t \in [0, T])$  in terms of  $x_0$ ,  $b_0 - |b_1|$ ,  $\sigma_0$ ,  $a$  and  $T$ , which tends to 1 as the initial level and long-run target become large relative to  $\sigma_0$ . In the classical Brownian setting, an almost-sure statement follows from Feller’s boundary theory and from comparison theorems for SDEs; neither tool is available when the driver is a non-semimartingale Hermite process with a diffusion coefficient that does not vanish at the boundary, so we do not claim an a.s. statement here (Remark 3.6). Our proof relies on a pathwise lower-barrier construction combined with the hypercontractivity inequality for multiple Wiener–Itô integrals and Markov’s inequality. Extending the argument to  $\sigma_1 > 0$ , and settling whether an a.s. statement holds for  $q \geq 2$ , remain open problems.
- **Complete existence theory in the Young setting** (Theorem 3.3). Although the application of the Nualart–Răşcanu framework to Young SDEs is known [17], the CIR square-root coefficient requires a careful regularisation argument via  $\phi_\varepsilon$  to satisfy the globally Lipschitz hypothesis. We provide the detailed verification for completeness and for use in subsequent results.

## Verification of existing general frameworks

- **Malliavin differentiability** (Theorem 3.11). Loosveldt, Nachit and Nourdin [10] recently developed a general theory of Malliavin calculus for SDEs driven by elements of an arbitrary Wiener chaos. We verify that the CIR-Hermite model satisfies their abstract conditions (RAC) and (SGD). The verification is non-trivial: the square-root volatility  $\sigma_1\sqrt{x}$  is not globally Lipschitz, and the regularisation  $\phi_\varepsilon$  is essential to bring the model within the scope of their theorem. Once the conditions are verified, the Malliavin differentiability  $X_t \in \mathbb{D}^{1,\infty}$  and the explicit formula for  $DX_t$  follow from [10, Theorem 4.1].
- **Absolute continuity of the marginal law** (Theorem 3.15). The Bouleau–Hirsch criterion [5] implies that  $X_t$  has an absolutely continuous law once the Malliavin matrix is non-degenerate. We verify this non-degeneracy using the ellipticity assumption  $\sigma_0 > 0$  and the differentiability established above.

## 1.5 Perspective: towards statistical inference

The present paper provides the theoretical foundations — existence, positivity, and Malliavin regularity — for statistical inference in the CIR-Hermite model. Estimation, asymptotic distribution theory, and empirical applications to interest rate data are the object of work currently in preparation, which we outline briefly in the perspectives of Section 5.

## 1.6 Related literature

**CIR and Vasicek models.** The CIR model [7] and the Vasicek model [23] are the two canonical short-rate models. Both have been extensively studied for their analytical tractability and positivity properties (in the case of CIR).

**Fractional and long-memory extensions.** Fractional Brownian motion-driven CIR and Vasicek models are studied in [12, 13]. The fractional Vasicek model driven by a Hermite process of order  $q \geq 2$  — which is the closest predecessor to our work — was analysed by Nourdin and Diu Tran [15], who developed statistical inference theory for that linear (Ornstein–Uhlenbeck type) model. Our paper extends the driving process from the Ornstein–Uhlenbeck to the CIR-type dynamics, which introduces the non-trivial difficulty of the square-root diffusion coefficient and the positivity requirement.

**Malliavin calculus for non-Gaussian SDEs.** Malliavin calculus for fractional Brownian motion-driven SDEs is developed in [18]. The general framework covering arbitrary Wiener chaos drivers, which we rely on for results (3) and (4), is the very recent work of Loosveldt, Nachit and Nourdin [10]. Related results on absolute continuity of Hermite process functionals appear in [11].

**Hermite processes.** The Hermite process family is introduced in [21] and surveyed in the monograph [22]. Simulation algorithms are given in [3, 4].

**Rough volatility.** Rough volatility models [8, 9], which cover the complementary regime  $H < 1/2$ , address a related but distinct empirical phenomenon (volatility roughness as opposed to long-range dependence of rates).

## 1.7 Organisation of the paper

The paper is organised as follows. Section 2 introduces the Hermite process, the CIR-Hermite SDE (2), and the standing assumptions. Section 3 states and proves the four main theoretical results. Section 5 discusses open problems and directions for future research, including a brief perspective on statistical inference.

# 2 Model Formulation and Theoretical Foundations

## 2.1 The Hermite Process

We work throughout on the classical Wiener space  $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{H})$ , where  $\Omega = C_0(\mathbb{R}, \mathbb{R})$  is the space of continuous functions  $\omega : \mathbb{R} \rightarrow \mathbb{R}$  with  $\omega(0) = 0$ ,  $\mathbb{P}$  is the two-sided Wiener measure, and  $\mathfrak{H} = L^2(\mathbb{R}, \mathbb{R})$ . In this setting, the isonormal Gaussian process is given, for every  $h \in \mathfrak{H}$ , by the Wiener integral  $X_h = \int_{\mathbb{R}} h(t) dB_t$ , where  $B$  is the canonical two-sided Brownian motion. We denote by  $I_q(f)$  the  $q$ th multiple Wiener–Itô integral of  $f \in L_s^2(\mathbb{R}^q, \mathbb{R})$  with respect to  $B$ ; see [14, 16] for a systematic treatment.

**Definition 2.1** (Hermite Process). Let  $q \geq 1$  be an integer and  $H \in (\frac{1}{2}, 1)$ . Set

$$H_0 := 1 + \frac{H-1}{q} \in \left(1 - \frac{1}{2q}, 1\right), \quad c(H, q) := \sqrt{\frac{H(2H-1)}{q! \beta^q(H_0 - \frac{1}{2}, 2 - 2H_0)}},$$

where  $\beta$  denotes the Beta function. The *Hermite process of order  $q$  and self-similarity parameter  $H$*  is the stochastic process  $Z^{(q,H)} = \{Z_t^{(q,H)}\}_{t \geq 0}$  defined by

$$Z_t^{(q,H)} = I_q\left(L_t^{H,q}\right), \quad L_t^{H,q}(\xi_1, \dots, \xi_q) := c(H, q) \int_0^t \prod_{j=1}^q (s - \xi_j)_+^{H_0 - 3/2} ds, \quad (3)$$

with the convention  $\theta_+^\alpha := \theta^\alpha \mathbf{1}_{\theta > 0}$ .

The Hermite family provides a canonical hierarchy of non-Gaussian, long-memory processes:

- $q = 1$ : fractional Brownian motion (Gaussian);
- $q = 2$ : the Rosenblatt process (non-Gaussian, skewed);
- $q \geq 3$ : higher-order Hermite processes (heavier tails, increasing excess kurtosis).

The following proposition collects the key properties of  $Z^{(q,H)}$  that are relevant for our analysis; see [22] for proofs and further details.

**Proposition 2.2** (Properties of the Hermite Process). *Let  $q \geq 1$  and  $H \in (\frac{1}{2}, 1)$ . The process  $Z^{(q,H)}$  defined in (3) satisfies the following:*

(i) **Covariance.** For all  $s, t \geq 0$ ,

$$\mathbb{E}\left[Z_s^{(q,H)} Z_t^{(q,H)}\right] = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

*This follows from the normalisation of  $c(H, q)$ , which ensures  $\mathbb{E}[(Z_1^{(q,H)})^2] = 1$  for all  $q \geq 1$  (see [15, Definition 2.1]).*

(ii) ***H-self-similarity.*** For every  $c > 0$ ,  $\{Z_{ct}^{(q,H)}\}_{t \geq 0} \stackrel{d}{=} \{c^H Z_t^{(q,H)}\}_{t \geq 0}$ .

(iii) ***Stationary increments.***  $\{Z_{t+h}^{(q,H)} - Z_h^{(q,H)}\}_{t \geq 0} \stackrel{d}{=} \{Z_t^{(q,H)}\}_{t \geq 0}$  for every  $h > 0$ .

(iv) ***Long-range dependence.*** The covariance of increments satisfies, as  $k \rightarrow \infty$ ,

$$\mathbb{E}\left[(Z_{t+1}^{(q,H)} - Z_t^{(q,H)})(Z_{t+k+1}^{(q,H)} - Z_{t+k}^{(q,H)})\right] \sim c_H k^{2H-2},$$

so the series of autocovariances diverges whenever  $H > \frac{1}{2}$ .

(v) ***Hölder regularity.*** The sample paths of  $Z^{(q,H)}$  are  $\alpha$ -Hölder continuous for every  $\alpha \in (0, H)$ . In particular, they belong almost surely to  $C^{H-\beta}([0, T])$  for every  $\beta \in (0, H - \frac{1}{2})$ .

(vi) ***Kernel regularity.*** For all  $s, t \geq 0$ ,

$$\|L_t^{H,q} - L_s^{H,q}\|_{L^2(\mathbb{R}^q)} = \sqrt{q!} |t - s|^H. \quad (4)$$

(This is an exact identity, not merely a bound: it follows from  $\langle L_t^{H,q}, L_s^{H,q} \rangle_{L^2(\mathbb{R}^q)} = \frac{q!}{2} (t^{2H} + s^{2H} - |t - s|^{2H})$  together with  $\|L_t^{H,q}\|_{L^2(\mathbb{R}^q)}^2 = q! t^{2H}$ ; see [22, Proposition 2.1].)

(vii) ***Non-Gaussianity for  $q \geq 2$ .*** The distribution of  $Z_t^{(q,H)}$  is non-Gaussian for every  $t > 0$  when  $q \geq 2$ . Its excess kurtosis is strictly positive and increasing in  $q$ .

*Remark 2.3.* Property (vi), identity (4), is the key regularity condition (Hypothesis **(H<sub>2</sub>)** of [10]) that enables pathwise Riemann–Stieltjes integration with respect to  $Z^{(q,H)}$  and ensures the well-posedness of our model.

## 2.2 The Interest Rate Model

We propose the following generalized CIR-type model for the short-term interest rate  $X_t$ :

$$\boxed{dX_t = a(b(t) - X_t) dt + \tilde{\sigma}(X_t) dZ_t^{(q,H)}, \quad X_0 = x_0 > 0,} \quad (5)$$

where the stochastic integral is understood pathwise in the Young–Riemann–Stieltjes sense [17, 24]. The three building blocks of the model are the following.

**Periodic drift.** The target rate function is

$$b(t) = b_0 + b_1 \sin\left(\frac{2\pi t}{T_{\text{per}}}\right), \quad (6)$$

where  $b_0 \in \mathbb{R}$  is the long-run mean level,  $b_1 \geq 0$  controls the amplitude of cyclical fluctuations, and  $T_{\text{per}} > 0$  is the cycle length (typically one year for annual monetary policy cycles or one quarter for seasonal funding patterns). The mean-reversion speed  $a > 0$  governs how quickly rates return toward  $b(t)$ .

**Regularised hybrid volatility.** The diffusion coefficient is

$$\tilde{\sigma}(x) = \sigma_0 + \sigma_1 \sqrt{\phi_\varepsilon(x)}, \quad x \in \mathbb{R}, \quad (7)$$

where  $\sigma_0 > 0$ ,  $\sigma_1 \geq 0$ ,  $\varepsilon > 0$ , and  $\phi_\varepsilon : \mathbb{R} \rightarrow (0, +\infty)$  is a fixed function of class  $\mathcal{C}^\infty(\mathbb{R})$  satisfying:

- (a)  $\phi_\varepsilon(x) = x + \varepsilon$  for all  $x \geq 0$ ;
- (b)  $\phi_\varepsilon(x) > 0$  for all  $x \in \mathbb{R}$ ;
- (c)  $\phi_\varepsilon \in \mathcal{C}_b^\infty(\mathbb{R})$  (all derivatives bounded).

*Remark 2.4* (Existence of  $\phi_\varepsilon$ ). Such a function exists by the standard smooth cut-off (bump function) construction of differential topology, obtained by mollifying an indicator function with a smooth, compactly supported kernel; see e.g. any standard reference on smooth partitions of unity. Explicitly, let  $\rho : \mathbb{R} \rightarrow [0, 1]$  be the standard  $\mathcal{C}^\infty$  transition function satisfying  $\rho(t) = 1$  for  $t \geq 0$  and  $\rho(t) = 0$  for  $t \leq -1$ , defined via

$$\rho(t) = \frac{e^{-1/(t+1)^2}}{e^{-1/(t+1)^2} + e^{-1/t^2}} \quad \text{for } t \in (-1, 0).$$

Set

$$\phi_\varepsilon(x) := (x + \varepsilon) \rho\left(\frac{x}{\varepsilon}\right) + \frac{\varepsilon}{2} \left(1 - \rho\left(\frac{x}{\varepsilon}\right)\right), \quad x \in \mathbb{R}. \quad (8)$$

One verifies:

- For  $x \geq 0$ :  $x/\varepsilon \geq 0$ , so  $\rho(x/\varepsilon) = 1$  and  $\phi_\varepsilon(x) = x + \varepsilon$  [condition (a)];
- For  $x \leq -\varepsilon$ :  $x/\varepsilon \leq -1$ , so  $\rho(x/\varepsilon) = 0$  and  $\phi_\varepsilon(x) = \varepsilon/2$ ;
- For  $x \in (-\varepsilon, 0)$ :  $\phi_\varepsilon(x)$  is a convex combination of  $(x + \varepsilon) > 0$  and  $\varepsilon/2 > 0$  with weights  $\rho \in (0, 1)$  and  $1 - \rho \in (0, 1)$ . Since both terms are strictly positive,  $\phi_\varepsilon(x) > 0$  [condition (b)];
- $\rho \in \mathcal{C}^\infty(\mathbb{R})$  and all its derivatives vanish for  $|t| \notin (-1, 0)$ , so  $\phi_\varepsilon \in \mathcal{C}_b^\infty(\mathbb{R})$  [condition (c)].

This specification nests:

- *Vasicek-type* constant volatility ( $\sigma_1 = 0$ ,  $\tilde{\sigma} \equiv \sigma_0$ );
- *CIR-type* level-dependent volatility on  $[0, +\infty)$ :  $\tilde{\sigma}(x) = \sigma_0 + \sigma_1 \sqrt{x + \varepsilon}$  for  $x \geq 0$ ;
- *Hybrid* intermediate configurations.

The extension to  $x < 0$  via  $\phi_\varepsilon$  is a purely mathematical device, needed regardless of the sign of  $X_t$  to bring  $\tilde{\sigma}$  within the  $\mathcal{C}_b^3(\mathbb{R})$  framework required by Assumption 2: with high probability under the regime of Theorem 3.5 (large  $x_0$  and  $b_0 - |b_1|$  relative to  $\sigma_0$ , in the case  $\sigma_1 = 0$ ),  $X_t$  stays positive and the model reduces in practice to  $\tilde{\sigma}(X_t) = \sigma_0 + \sigma_1 \sqrt{X_t + \varepsilon}$ .

**Hermite noise.** The driving process  $Z^{(q,H)}$  is the Hermite process of order  $q \geq 1$  and Hurst parameter  $H \in (\frac{1}{2}, 1)$  defined in (3). The statistical parameter vector of the model is

$$\theta = (a, b_0, b_1, T_{\text{per}}, \sigma_0, \sigma_1, H, q).$$

The regularisation parameter  $\varepsilon$  is not included in  $\theta$ : it is a fixed technical bandwidth, calibrated a priori (e.g.  $\varepsilon = 10^{-3}$  throughout Section 4), rather than a statistically estimated feature of the interest-rate dynamics.

*Remark 2.5* (Nature of the stochastic integral). Since the Hermite process  $Z^{(q,H)}$  is *not* a semimartingale for  $q \geq 2$ , the stochastic integral  $\int_0^t \tilde{\sigma}(X_s) dZ_s^{(q,H)}$  in (5) is understood pathwise in the Young–Riemann–Stieltjes sense [17, 24]. This is made rigorous by the fact that, under Assumption 3, the paths of  $Z^{(q,H)}$  are  $H$ -Hölder continuous with  $H > \frac{1}{2}$ , which is the minimal regularity required for the Young integral to be well-defined. The technical details — notably the functional spaces  $W_1^\alpha$  and  $W_2^{1-\alpha}$  of [17] used in the existence proof — are deferred to Section 3.

## 2.3 Assumptions

We now state the four assumptions under which the theoretical results of Sections 3 hold. These assumptions are tailored to fit precisely within the framework of [10].

## Notation Summary

For convenience we collect the main notation used throughout the paper.

| Table 1: Main notation.                                              |                                                                                                                          |
|----------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------|
| Symbol                                                               | Meaning                                                                                                                  |
| $q \in \{1, 2, 3, \dots\}$                                           | Hermite order ( $q = 1$ : fBm, $q = 2$ : Rosenblatt)                                                                     |
| $H \in (1/2, 1)$                                                     | Hurst self-similarity parameter                                                                                          |
| $Z^{(q,H)}$                                                          | Hermite process of order $q$ and parameter $H$                                                                           |
| $a > 0$                                                              | Mean-reversion speed                                                                                                     |
| $b(t) = b_0 + b_1 \sin(2\pi t/T_{\text{per}})$                       | Periodic long-run target                                                                                                 |
| $b_0 \in \mathbb{R}, b_1 \geq 0, T_{\text{per}} > 0$                 | Drift parameters                                                                                                         |
| $\tilde{\sigma}(x) = \sigma_0 + \sigma_1 \sqrt{\phi_\varepsilon(x)}$ | Hybrid volatility function                                                                                               |
| $\sigma_0 > 0, \sigma_1 \geq 0$                                      | Volatility parameters                                                                                                    |
| $\phi_\varepsilon$                                                   | $C^\infty$ regularisation of $x \mapsto x_+$ (makes $\tilde{\sigma}$ globally smooth)                                    |
| $c(H, q)$                                                            | Normalisation constant of $Z^{(q,H)}$                                                                                    |
| $H_0 = 1 + (H - 1)/q$                                                | Individual kernel exponent in the multiple Wiener–Itô integral defining $Z^{(q,H)}$ (not itself a self-similarity index) |
| $\mathbb{D}^{m,p}$                                                   | Sobolev–Malliavin space                                                                                                  |
| $\theta = (a, b_0, b_1, T_{\text{per}}, \sigma_0, \sigma_1, H, q)$   | Full parameter vector                                                                                                    |
| $J$                                                                  | Wavelet resolution level ( $n \approx 2^J$ observations)                                                                 |

**Assumption 1** (Parameter constraints). The parameters satisfy:

$$a > 0, \quad |b_1| < b_0, \quad \sigma_0 > 0, \quad \sigma_1 \geq 0, \quad \varepsilon > 0, \quad H \in \left(\frac{1}{2}, 1\right), \quad q \in \{1, 2, 3, \dots\},$$

and, consequently, the *positivity condition on the long-run target*:

$$\min_{t \in [0, T_{\text{per}}]} b(t) = b_0 - |b_1| > 0.$$

Since  $|b_1| \geq 0$ , this condition forces  $b_0 > 0$  strictly; the periodic component only tightens the constraint (it reduces to  $b_0 > 0$  alone when  $b_1 = 0$ , and requires  $b_0$  strictly larger than the seasonal amplitude  $|b_1|$  otherwise). In particular, no choice of  $b_1$  can rescue a non-positive  $b_0$ : whenever  $b_0 \leq 0$ , this assumption fails and Theorem 3.5 does not apply (see Remark 2.6).

*Remark 2.6* (Empirical regime and the zero lower bound). In interest-rate applications (e.g. Fed Funds or EURIBOR series over the zero-lower-bound period 2009–2022), the estimated intercept  $\hat{b}_0$  is often negative. As noted after Assumption 1, this is incompatible with the positivity condition  $b_0 - |b_1| > 0$  for *any* value of  $b_1$ : whenever  $\hat{b}_0 \leq 0$ , Assumption 1 fails outright and Theorem 3.5 gives no guarantee, however large the seasonal amplitude  $\hat{b}_1$  is. In that regime, positivity of  $X_t$  can only be enforced directly in simulation, e.g. by the reflection  $X_{t_i} \leftarrow \max(X_{t_i}, 0)$  in the Euler scheme, with no theoretical control from the quantitative bound of Theorem 3.5. This empirical regime is examined in work currently in preparation.

**Assumption 2** (Regularity of coefficients —  $\mathbf{H}_1$  of [10]). *Intuition: the model coefficients must be sufficiently smooth for the Young integral and the Malliavin derivative to be well-defined. For CIR dynamics, the natural square-root diffusion fails at  $x = 0$ ; the regularisation  $\phi_\varepsilon$  is precisely designed to remedy this.*

The drift coefficient  $x \mapsto a(b(t) - x)$  and the diffusion coefficient  $\tilde{\sigma}$  defined in (7) both belong to  $\mathcal{C}_b^3(\mathbb{R})$  — the space of three-times continuously differentiable real functions with bounded derivatives up to order 3.

*Remark 2.7* (On the meaning of  $\mathcal{C}_b^3$ ). We stress that  $\mathcal{C}_b^3(\mathbb{R})$ , as defined above and as used throughout this paper (following [10], Section 2.2, where the same convention is adopted verbatim for hypothesis  $\mathbf{H}_1$ ), requires only the *derivatives* of order 1 to 3 to be bounded — *not* the function itself. This is a deliberate and standard convention in this strand of the Young-SDE and Malliavin-calculus literature, but it differs from the convention used in some other areas of analysis, where  $\mathcal{C}_b^k$  denotes functions that are themselves bounded together with their derivatives. Under our convention, both  $x \mapsto a(b(t) - x)$  (which is unbounded, growing linearly) and  $\tilde{\sigma}$  (which is unbounded, growing like  $\sqrt{|x|}$  as  $x \rightarrow +\infty$ ; see Remark 2.8) belong to  $\mathcal{C}_b^3(\mathbb{R})$  in this sense: both are globally Lipschitz (equivalently, have a bounded first derivative), a property that in particular already forces linear growth of the coefficient itself, so no separate growth condition needs to be imposed.

*Remark 2.8.* Under Assumption 1, the drift  $x \mapsto a(b(t) - x)$  is linear, hence trivially in  $\mathcal{C}_b^\infty(\mathbb{R})$ .

*Smoothness of  $\tilde{\sigma}$ .* Since  $\phi_\varepsilon \in \mathcal{C}_b^\infty(\mathbb{R})$  by condition (c) and  $\phi_\varepsilon(x) > 0$  by condition (b), the composition  $x \mapsto \sqrt{\phi_\varepsilon(x)}$  is  $\mathcal{C}^\infty(\mathbb{R})$  as the composition of  $\sqrt{\cdot} \in \mathcal{C}^\infty((0, +\infty))$  with  $\phi_\varepsilon$ .

*Boundedness of derivatives.* By the Faà di Bruno formula, the  $k$ th derivative of  $\sqrt{\phi_\varepsilon(x)}$  is a finite sum of terms of the form

$$\frac{C_k}{\phi_\varepsilon(x)^{k-1/2}} \prod_j \phi_\varepsilon^{(n_j)}(x),$$

where  $C_k > 0$  is a universal constant and  $n_j \geq 1$  with  $\sum_j n_j = k$ . Since  $\phi_\varepsilon$  is continuous on  $\mathbb{R}$ , satisfies  $\phi_\varepsilon(x) \rightarrow \varepsilon/2 > 0$  as  $x \rightarrow -\infty$  and  $\phi_\varepsilon(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$ , and  $\phi_\varepsilon(x) > 0$  for all  $x$  by condition (b), the infimum  $m(\varepsilon) := \inf_{x \in \mathbb{R}} \phi_\varepsilon(x) > 0$  is attained and strictly positive. Hence  $\phi_\varepsilon(x)^{-(k-1/2)} \leq m(\varepsilon)^{-(k-1/2)} < +\infty$  uniformly in  $x$ . Since all derivatives  $\phi_\varepsilon^{(n_j)}$  are bounded by condition (c), every such term is bounded on  $\mathbb{R}$ . Hence  $\tilde{\sigma} \in \mathcal{C}_b^3(\mathbb{R})$ , confirming Assumption 2.

*Explicit derivatives on  $[0, +\infty)$ .* On the economically relevant domain  $\{x \geq 0\}$ , where  $\phi_\varepsilon(x) = x + \varepsilon$  by condition (a):

$$\tilde{\sigma}'(x) = \frac{\sigma_1}{2\sqrt{x+\varepsilon}}, \quad \tilde{\sigma}''(x) = -\frac{\sigma_1}{4(x+\varepsilon)^{3/2}}, \quad \tilde{\sigma}'''(x) = \frac{3\sigma_1}{8(x+\varepsilon)^{5/2}},$$

each bounded by  $\sigma_1/(2\sqrt{\varepsilon})$ ,  $\sigma_1/(4\varepsilon^{3/2})$ , and  $3\sigma_1/(8\varepsilon^{5/2})$  respectively.

*Why the naive formula fails.* The expression  $\sigma_0 + \sigma_1\sqrt{x+\varepsilon}$  is not in  $\mathcal{C}_b^3(\mathbb{R})$ : it is undefined for  $x < -\varepsilon$  and its derivative  $\sigma_1/(2\sqrt{x+\varepsilon}) \rightarrow +\infty$  as  $x \rightarrow -\varepsilon^+$ . The formulation via  $\phi_\varepsilon$  resolves both issues while remaining economically equivalent on  $\{x \geq 0\}$ , the economically relevant domain, on which  $X_t$  stays with high probability in the regime described in Theorem 3.5.

**Assumption 3** (Hölder regularity of the noise — **H<sub>2</sub>** of [10]). *Intuition: the Hermite process must be regular enough to define the stochastic integral as a Young–Stieltjes integral. This is automatically satisfied since  $Z^{(q,H)}$  has Hölder continuous paths of order  $\zeta \in (0, H)$  (Proposition 2.2), and  $H > 1/2$  guarantees a non-empty range  $\beta \in (0, H - \frac{1}{2})$  of admissible Hölder-loss parameters, for which the Nualart–Rascanu exponent  $\alpha := 1 - H + \beta$  (following Proposition 3.4 of [10]) automatically satisfies  $\alpha \in (1 - H, \frac{1}{2})$ , as required by [17].* There exists  $H \in (\frac{1}{2}, 1)$  and a constant  $C > 0$  such that, for all  $s, t \in [0, T]$ ,

$$\|L_t^{H,q} - L_s^{H,q}\|_{L^2(\mathbb{R}^q)} \leq C |t - s|^H.$$

*Remark 2.9.* Assumption 3 is automatically satisfied by the Hermite process  $Z^{(q,H)}$  for any  $q \geq 1$  and  $H \in (\frac{1}{2}, 1)$ , as established in Proposition 2.2(vi) (identity (4)). This assumption ensures that, for every  $\beta \in (0, H - \frac{1}{2})$ , the sample paths of  $Z^{(q,H)}$  belong almost surely to the fractional Sobolev space  $W_2^{H-\beta}(0, T; \mathbb{R})$  of [17] — equivalently  $W_2^{1-\alpha}(0, T; \mathbb{R})$  with  $\alpha := 1 - H + \beta$ , in the notation of Proposition 3.4 and Theorem 3.6 of [10] — which is the natural domain for pathwise Young integration.

**Assumption 4** (Ellipticity — **H<sub>3</sub>** of [10]). *Intuition: the diffusion coefficient must be bounded away from zero to prevent degenerate behaviour of the Malliavin matrix. In the CIR–Hermite model, the base volatility  $\sigma_0 > 0$  (Assumption 1) ensures this, since  $\tilde{\sigma}(x) \geq \sigma_0 > 0$  for all  $x$ .* The diffusion coefficient satisfies the uniform ellipticity condition:

$$\inf_{x \in \mathbb{R}} \tilde{\sigma}(x) \geq \sigma_0 > 0.$$

*Remark 2.10.* Since  $\phi_\varepsilon(x) > 0$  for all  $x \in \mathbb{R}$  by condition (b), we have  $\tilde{\sigma}(x) = \sigma_0 + \sigma_1\sqrt{\phi_\varepsilon(x)} \geq \sigma_0 > 0$  for all  $x \in \mathbb{R}$  under Assumption 1, so Assumption 4 is automatically satisfied. In the one-dimensional setting of our model ( $d = m = 1$ ), this condition simplifies to the non-vanishing of the diffusion coefficient, which is the minimal requirement for the Bouleau–Hirsch criterion to yield absolute continuity of the law of  $X_t$ ; see Theorem 3.15 below.

## 2.4 Economic Interpretation of the Model Parameters

The model (5) is designed to capture three empirically well-documented features of interest rate dynamics that traditional models fail to reproduce simultaneously.

**Long-range dependence via  $H > \frac{1}{2}$ .** The Hurst parameter  $H$  controls the rate of decay of autocorrelations. For  $H > \frac{1}{2}$ , the covariance of increments decays hyperbolically as  $k^{2H-2}$ , consistent with the persistent memory effects documented in short-rate data [1, 2]. Higher values of  $H$  correspond to smoother, more persistent rate paths, reflecting stable monetary policy regimes.

**Non-Gaussianity via  $q \geq 2$ .** The Hermite order  $q$  determines the degree of departure from Gaussianity. For  $q = 1$ ,  $Z^{(1,H)}$  is a fractional Brownian motion (Gaussian). For  $q = 2$ , the Rosenblatt process generates asymmetric innovations with positive excess kurtosis, capturing the fat-tailed, skewed distribution of interest rate shocks. For  $q = 3$ , the tails are even heavier, consistent with the extreme movements observed during financial crises and sudden policy shifts. The excess kurtosis is an increasing function of  $q$ , providing a continuous measure of non-Gaussianity that can be estimated from data.

**Seasonal patterns via  $b_1$  and  $T_{\text{per}}$ .** The periodic component  $b_1 \sin(2\pi t/T_{\text{per}})$  captures institutional regularities in central bank policy cycles (e.g., the Federal Reserve’s 8-meeting annual schedule), quarter-end liquidity effects in money markets, and seasonal funding patterns. Setting  $b_1 = 0$  recovers the standard constant-mean CIR model.

**Hybrid volatility via  $\sigma_0$  and  $\sigma_1$ .** The base volatility  $\sigma_0 > 0$  accounts for the empirical observation that interest rate volatility remains significant even at very low rate levels, as witnessed during the zero-lower-bound periods of 2009–2015 and 2020–2022. The level-dependent component  $\sigma_1 \sqrt{X_t + \varepsilon}$  captures the positive relationship between rate levels and volatility documented in [2].

## 2.5 Comparison with Existing Models

Table 2 summarises how our model nests several classical and recent specifications.

Table 2: Special cases of the CIR-Hermite model (5).

| Model                | $q$      | $H$                    | $\sigma_0$ | $\sigma_1$ | $b_1$    | Reference         |
|----------------------|----------|------------------------|------------|------------|----------|-------------------|
| Vasicek <sup>†</sup> | 1        | $\frac{1}{2}$          | $> 0$      | 0          | 0        | [23]              |
| CIR <sup>†</sup>     | 1        | $\frac{1}{2}$          | 0          | $> 0$      | 0        | [7]               |
| Fractional Vasicek   | 1        | $\neq \frac{1}{2}$     | $> 0$      | 0          | 0        | [6]               |
| Fractional CIR       | 1        | $\neq \frac{1}{2}$     | 0          | $> 0$      | 0        | [12]              |
| Rosenblatt-CIR       | 2        | $\in (\frac{1}{2}, 1)$ | $> 0$      | $\geq 0$   | 0        | This paper        |
| Hermite-CIR (ord. 3) | 3        | $\in (\frac{1}{2}, 1)$ | $> 0$      | $\geq 0$   | 0        | This paper        |
| <b>Full model</b>    | $\geq 1$ | $\in (\frac{1}{2}, 1)$ | $> 0$      | $\geq 0$   | $\neq 0$ | <b>This paper</b> |

<sup>†</sup>Vasicek and CIR correspond to the boundary value  $H = \frac{1}{2}$ , which lies outside the open range  $H \in (\frac{1}{2}, 1)$  required throughout this paper; they are recovered only as a formal

limit ( $H \rightarrow \frac{1}{2}^+$ ) of our framework, not as a rigorously covered special case. All other rows fall within  $H \in (\frac{1}{2}, 1)$  and are literal special cases of model (5).

The key advantage of our framework over fractional CIR models is the simultaneous capture of long-range dependence ( $H > \frac{1}{2}$ ) and non-Gaussian innovations ( $q \geq 2$ ), while maintaining a tractable mean-reverting structure with analytical and statistical properties established rigorously in the next section.

### 3 Theoretical Results

This section establishes the three fundamental theoretical properties of the model (5): existence and uniqueness of the solution (Section 3.1), its Malliavin differentiability (Section 3.3), and the absolute continuity of its law (Section 3.4). All results are consequences of the general theory developed in [10] applied to our specific model.

Throughout this section, we fix  $T > 0$  and work on the probability space  $(\Omega, \mathcal{F}, \mathbb{P}, \mathfrak{H})$  introduced in Section 2.1.

#### 3.1 Existence and Uniqueness of the Solution

The pathwise well-posedness of (5) relies on the Young–Riemann–Stieltjes integration theory of [17, 24]. We first recall the functional spaces that provide the natural framework for pathwise integration against Hölder-continuous driving processes.

**Definition 3.1** (Fractional Sobolev spaces, [17]). Let  $\alpha \in (0, \frac{1}{2})$ .

- (a)  $W_1^\alpha(0, T; \mathbb{R})$  is the space of measurable functions  $f : [0, T] \rightarrow \mathbb{R}$  with

$$\|f\|_{\alpha,1} := \sup_{t \in [0,T]} \left( |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{|t - s|^{\alpha+1}} ds \right) < \infty.$$

- (b)  $W_2^{1-\alpha}(0, T; \mathbb{R})$  is the space of measurable functions  $\varphi : [0, T] \rightarrow \mathbb{R}$  with

$$\|\varphi\|_{1-\alpha,2} := \sup_{0 \leq s < t \leq T} \left( \frac{|\varphi(t) - \varphi(s)|}{(t - s)^{1-\alpha}} + \int_s^t \frac{|\varphi(\tau) - \varphi(s)|}{(\tau - s)^{2-\alpha}} d\tau \right) < \infty.$$

There hold the continuous embeddings, for every  $\beta \in (0, H - \frac{1}{2})$  and  $\gamma \in (0, \beta)$ ,

$$C^{H-\beta+\gamma}(0, T; \mathbb{R}) \subset W_2^{H-\beta}(0, T; \mathbb{R}) \subset C^{H-\beta}(0, T; \mathbb{R}).$$

The following lemma establishes that the Hermite process  $Z^{(q,H)}$  lives in the correct function space for Young integration.

**Lemma 3.2** (Regularity of the Hermite process). *Under Assumption 3, for every  $\beta \in (0, H - \frac{1}{2})$ , the sample paths of  $Z^{(q,H)}$  belong almost surely to  $W_2^{H-\beta}(0, T; \mathbb{R})$ .*

*Proof.* The key step is to establish a moment bound on increments of  $Z^{(q,H)}$ .

**Step 1: Moment bound.** By the self-similarity of  $Z^{(q,H)}$  with index  $H$  (Proposition 2.2) and the stationarity of its increments (Proposition 2.2),

$$\mathbb{E} \left[ |Z_t^{(q,H)} - Z_s^{(q,H)}|^p \right] = |t - s|^{pH} \mathbb{E} \left[ |Z_1^{(q,H)}|^p \right], \quad p \geq 1, \quad s, t \in [0, T]. \quad (9)$$

The constant  $\mathbb{E}[|Z_1^{(q,H)}|^p]$  is finite for all  $p \geq 1$  by the hypercontractivity inequality for multiple Wiener–Itô integrals: for every integer  $q \geq 1$  and every  $p \geq 1$ , there is a constant  $0 < k(q, p) < \infty$ , depending only on  $q$  and  $p$ , such that  $\mathbb{E}[|F|^p]^{1/p} \leq k(q, p) \mathbb{E}[F^2]^{1/2}$  for every random variable  $F$  of the form of a  $q$ -th multiple Wiener–Itô integral [14, Theorem 2.7.2]. Applying this with  $F = Z_1^{(q,H)}$  and using  $\mathbb{E}[(Z_1^{(q,H)})^2] = 1$  (normalisation from Proposition 2.2), we obtain  $\mathbb{E}[|Z_1^{(q,H)}|^p] \leq k(q, p)^p < \infty$  for all  $p \geq 1$ ; we do not pursue the sharp value of  $k(q, p)$  here. Hence (9) gives, with  $C_{p,q} := \mathbb{E}[|Z_1^{(q,H)}|^p] < \infty$ :

$$\mathbb{E}[|Z_t^{(q,H)} - Z_s^{(q,H)}|^p] \leq C_{p,q} |t - s|^{pH}.$$

**Step 2: Kolmogorov continuity criterion.** Fix  $\beta \in (0, H - \frac{1}{2})$  and choose  $p > 1/\beta$ , so that  $1/p < \beta$  (in particular  $pH > 1$ , since  $H > 1/2$ ). The moment bound of Step 1 reads  $\mathbb{E}[|Z_t^{(q,H)} - Z_s^{(q,H)}|^p] \leq C_{p,q} |t - s|^{1+(pH-1)}$  with  $pH - 1 > 0$ , so Kolmogorov’s continuity theorem [19, Theorem I.2.1] guarantees that  $Z^{(q,H)}$  admits a modification with sample paths that are  $\gamma$ -Hölder continuous on  $[0, T]$  for every  $\gamma < H - 1/p$ . Since  $1/p < \beta$ , we may in particular fix some  $\gamma$  with

$$H - \beta < \gamma < H - \frac{1}{p},$$

a nonempty range because  $1/p < \beta$ ; the modification is then  $\gamma$ -Hölder continuous with  $\gamma$  *strictly larger* than  $H - \beta$  — this strict margin is essential for Step 3 below.

**Step 3: Fractional Sobolev embedding.** For  $\alpha = H - \beta > \frac{1}{2}$ , the  $\gamma$ -Hölder continuity of the paths, with  $\gamma > H - \beta$  fixed in Step 2, implies

$$\int_0^T \int_0^T \frac{|Z_t^{(q,H)} - Z_s^{(q,H)}|^2}{|t - s|^{1+2(H-\beta)}} ds dt \leq K^2 \int_0^T \int_0^T |t - s|^{-1+2(\gamma-(H-\beta))} ds dt < \infty \quad \text{a.s.},$$

where  $K$  is the (a.s. finite) Hölder constant of the modification, and the integral converges because the exponent  $-1 + 2(\gamma - (H - \beta))$  is strictly larger than  $-1$  precisely because  $\gamma > H - \beta$  strictly (using  $\gamma = H - \beta$  exactly would leave exponent  $-1$ , for which the integral diverges logarithmically near the diagonal  $t = s$ ). This finiteness is precisely the norm condition defining  $W_2^{H-\beta}(0, T; \mathbb{R})$  in Definition 3.1. This completes the proof.  $\square$

We can now state the main existence and uniqueness result.

**Theorem 3.3** (Existence and Uniqueness). *Under Assumptions 1–3, for every  $T > 0$  and  $x_0 > 0$ , the stochastic differential equation (5) admits a unique solution  $(X_t)_{t \in [0, T]}$  such that, almost surely,*

$$(X_t)_{t \in [0, T]} \in W_1^\alpha(0, T; \mathbb{R}) \cap C^{1-\alpha}(0, T; \mathbb{R}),$$

for every  $\alpha \in (1 - H, \frac{1}{2})$ .

*Proof.* The proof applies the abstract existence-uniqueness result of [17, Theorem 5.1] to our specific model, following the strategy of [10, Section 2.2]. We proceed in four steps.

**Step 1: The driving process is in the right space.**

By Lemma 3.2, for any  $\beta \in (0, H - \frac{1}{2})$ , the paths of  $Z^{(q,H)}$  belong almost surely to  $W_2^{H-\beta}(0, T; \mathbb{R})$ . Setting  $1 - \alpha = H - \beta$  (so  $\alpha = 1 - H + \beta$ ), and noting that  $\beta$  ranges over  $(0, H - \frac{1}{2})$ ,  $\alpha$  ranges precisely over  $(1 - H, \frac{1}{2})$ , matching the range in the statement

of the theorem; in particular  $\alpha \in (0, \frac{1}{2})$  as required by Definition 3.1, the driving process satisfies the hypothesis of [17, Theorem 5.1] with parameter  $1 - \alpha = H - \beta > \frac{1}{2}$ .

**Step 2: The coefficients satisfy the regularity requirement.**

By Assumption 2, the drift  $\mu(x) = a(b(t) - x)$  and diffusion  $\tilde{\sigma}$  both belong to  $\mathcal{C}_b^3(\mathbb{R})$ , and  $b(\cdot)$  is smooth and  $T_{\text{per}}$ -periodic in time. This is more than enough to satisfy hypotheses (H1)–(H2) of [17, Theorem 5.1] (differentiability in  $x$  with a globally Lipschitz, hence in particular locally  $\delta$ -Hölder continuous, derivative for any  $\delta > 1/H - 1$ , and  $\beta$ -Hölder continuity in time for any  $\beta > 1 - H$ ), so the hypothesis of that theorem is satisfied.

**Step 3: Conclusion.**

By [17, Theorem 5.1], the deterministic differential equation

$$x_t = x_0 + \int_0^t a(b(s) - x_s) ds + \int_0^t \tilde{\sigma}(x_s) d\varphi_s$$

admits a unique solution  $x \in W_1^\alpha \cap C^{1-\alpha}$  for every driving path  $\varphi \in W_2^{1-\alpha}(0, T; \mathbb{R})$ . Since (a version of)  $Z^{(q,H)}$  belongs almost surely to  $W_2^{1-\alpha}(0, T; \mathbb{R})$  by Step 1, the stochastic differential equation (5) admits a unique pathwise solution  $(X_t)_{t \in [0, T]}$  with the stated regularity.

**Step 4: Uniqueness.** Suppose  $X$  and  $\tilde{X}$  are two solutions in  $W_1^\alpha$  with the same initial condition  $X_0 = \tilde{X}_0 = x_0$ . Setting  $\rho_t := X_t - \tilde{X}_t$ , the difference satisfies the Young integral equation

$$\begin{aligned} \rho_t &= \int_0^t [a(b(s) - X_s) - a(b(s) - \tilde{X}_s)] ds + \int_0^t [\tilde{\sigma}(X_s) - \tilde{\sigma}(\tilde{X}_s)] dZ_s^{(q,H)} \\ &= -a \int_0^t \rho_s ds + \int_0^t [\tilde{\sigma}(X_s) - \tilde{\sigma}(\tilde{X}_s)] dZ_s^{(q,H)}. \end{aligned}$$

By Assumption 2,  $\tilde{\sigma} \in \mathcal{C}_b^3(\mathbb{R})$  is in particular globally Lipschitz with constant  $L_\varepsilon := \|\tilde{\sigma}'\|_\infty < \infty$ , so  $|\tilde{\sigma}(X_s) - \tilde{\sigma}(\tilde{X}_s)| \leq L_\varepsilon |\rho_s|$ . Applying the pathwise Gronwall-type inequality for Young equations [17, Lemma 7.6] yields  $\|\rho\|_{[0, T], \infty} \leq 0$ , hence  $X = \tilde{X}$  almost surely, completing the proof of Theorem 3.3.  $\square$

*Remark 3.4.* The solution obtained in Theorem 3.3 is a *strong* solution in the pathwise sense: it is constructed trajectory by trajectory as a functional of the driving process  $Z^{(q,H)}$ , and it is unique among all adapted processes with paths in  $W_1^\alpha \cap C^{1-\alpha}$ . This is in contrast to the weak (distributional) solutions obtained via martingale problem methods for Brownian-driven SDEs.

## 3.2 Positivity of the Solution

The economic interpretation of  $X_t$  as an interest rate requires  $X_t \geq 0$  almost surely. We establish a sufficient condition for this under the assumption  $\sigma_0 > 0$ .

*Nature of this result.* We do *not* claim  $X_t > 0$  almost surely for all  $t \in [0, T]$ . As explained in Remark 3.6, no comparison principle is currently available for Young SDEs driven by a non-semimartingale Hermite process, and the diffusion coefficient  $\tilde{\sigma}$  of model (5) does not vanish at  $x = 0$  (it satisfies  $\tilde{\sigma} \geq \sigma_0 > 0$  everywhere by the ellipticity Assumption 4), so the classical boundary-non-attainment mechanism of Feller's theory — which relies on the vanishing of the diffusion coefficient at the boundary and on Itô's formula — is simply not available here. Instead, we establish the following *quantitative*

statement, which is fully rigorous: the probability that  $X$  stays positive on  $[0, T]$  is bounded below by an explicit expression that tends to 1 as  $x_0$  or  $b_0 - |b_1|$  grow large relative to  $\sigma_0$ . We restrict attention to the case  $\sigma_1 = 0$  (pure base-volatility, Vasicek-type diffusion coefficient  $\tilde{\sigma} \equiv \sigma_0$ ); the case  $\sigma_1 > 0$  remains open, see Remark 3.7.

**Theorem 3.5** (Positivity: a quantitative sufficient bound). *Under Assumptions 1–3 with  $\sigma_1 = 0$ , let*

$$m_0 := \min(x_0, b_0 - |b_1|) > 0, \quad C_T := \sup_{s \in [0, T]} |Z_s^{(q, H)}|.$$

*Then, for every  $p \geq 1$ ,*

$$\mathbb{P}\left(X_t > 0 \text{ for all } t \in [0, T]\right) \geq 1 - \left(\frac{\sigma_0(1 + aT)}{m_0}\right)^p \mathbb{E}[C_T^p]. \quad (10)$$

*In particular, since  $\mathbb{E}[C_T^p] < \infty$  for every  $p \geq 1$  (Lemma 3.2 and the hypercontractivity property of fixed Wiener chaoses, [14, Theorem 2.7.2]), the right-hand side of (10) tends to 1 as  $m_0/\sigma_0 \rightarrow \infty$ , i.e. as the initial level and the long-run target become large relative to the base volatility.*

*Remark 3.6* (Why an almost-sure statement is not available here). In the classical Brownian CIR model, positivity under  $2ab \geq \sigma^2$  is proved via Itô's formula applied to  $\log X_t$  or  $1/X_t$ : the quadratic-variation correction term produced by Itô's formula is what prevents the process from reaching 0. Pathwise Young calculus against a Hölder driver follows the *ordinary* chain rule (no second-order correction term), so this mechanism is simply absent when  $Z^{(q, H)}$  is not a semimartingale. Moreover, since  $\tilde{\sigma}$  is bounded away from 0 (Assumption 4, needed for the ellipticity used in Theorems 3.11–3.15), there is no boundary-vanishing mechanism of the coefficient either: for any fixed realisation of  $Z^{(q, H)}$  that decreases sharply enough, the solution can cross 0 purely pathwise, regardless of any parameter condition. This is why Theorem 3.5 is stated as a quantitative probability bound rather than an almost-sure statement, and why establishing (or disproving) an a.s. positivity result for  $q \geq 2$  remains, to our knowledge, an open problem. We note that the bound (10) is fully rigorous and already gives the qualitatively correct picture: positivity becomes overwhelmingly likely exactly in the regime  $m_0 \gg \sigma_0$ , which plays the role of the classical Feller condition  $2ab_0 \geq \sigma_0^2$  without claiming the same sharp threshold.

*Remark 3.7* (The case  $\sigma_1 > 0$ ). For  $\sigma_1 > 0$ , the argument below no longer applies directly: the difference  $\Delta_t := X_t - \underline{X}_t$  between the true solution and the linear comparison process (Step 1 below) is no longer deterministic, since it satisfies an SDE with a stochastic term  $\sigma_1 \sqrt{\phi_\varepsilon(X_t)} dZ_t^{(q, H)}$  that can take both signs. A fully rigorous extension would require either a comparison principle for Young SDEs (currently unavailable in the literature for Hermite drivers) or a direct tail bound on  $\sup_t |\Delta_t|$  analogous to the one derived below for  $C_T$ . We leave this extension for future work.

*Proof.* Since  $\sigma_1 = 0$ , equation (5) reduces to the linear Young SDE

$$dX_t = a(b(t) - X_t) dt + \sigma_0 dZ_t^{(q, H)}, \quad X_0 = x_0,$$

which by the variation-of-constants formula (Theorem 3.3 applied to linear coefficients) has the explicit solution

$$X_t = e^{-at} x_0 + \underbrace{\int_0^t e^{-a(t-s)} a b(s) ds}_{=:\Delta_t} + \underbrace{\sigma_0 \int_0^t e^{-a(t-s)} dZ_s^{(q, H)}}_{=:I_t}. \quad (11)$$

**Step 1: A deterministic lower bound on the drift part.**

Since  $b(s) \geq b_0 - |b_1| > 0$  for all  $s$  (Assumption 1),

$$\Delta_t \geq (b_0 - |b_1|) \int_0^t a e^{-a(t-s)} ds = (b_0 - |b_1|)(1 - e^{-at}).$$

Hence

$$e^{-at}x_0 + \Delta_t \geq e^{-at}x_0 + (b_0 - |b_1|)(1 - e^{-at}) \geq \min(x_0, b_0 - |b_1|) = m_0,$$

using that the middle expression is a convex combination (weights  $e^{-at}$  and  $1 - e^{-at}$ ) of  $x_0$  and  $b_0 - |b_1|$ , which is always at least the smaller of the two. This bound holds for every  $t \in [0, T]$  simultaneously, and is entirely deterministic.

**Step 2: Pathwise control of the noise term  $I_t$ .**

The elementary integration-by-parts identity for Young integrals (a direct consequence of Young's original Riemann–Stieltjes integration theory [24]; see also the generalized Stieltjes integral framework of [17]) gives, exactly as in the original barrier computation,

$$I_t = \sigma_0 [e^{-a(t-s)} Z_s^{(q,H)}]_0^t - a\sigma_0 \int_0^t e^{-a(t-s)} Z_s^{(q,H)} ds.$$

Since  $Z_0^{(q,H)} = 0$  (Definition 2.1), the boundary term reduces exactly to  $[e^{-a(t-s)} Z_s^{(q,H)}]_0^t = Z_t^{(q,H)}$ , so, using  $|Z_s^{(q,H)}| \leq C_T$  for every  $s \in [0, T]$ ,

$$|I_t| \leq \sigma_0 (C_T + aT C_T) = \sigma_0 (1 + aT) C_T, \quad \text{for every } t \in [0, T].$$

This bound is purely pathwise: it holds for every fixed realisation of  $Z^{(q,H)}$ , with no probabilistic argument involved at this stage.

**Step 3: Combining Steps 1–2 and taking probabilities.**

By (11) and Steps 1–2,

$$X_t \geq m_0 - \sigma_0 (1 + aT) C_T \quad \text{for every } t \in [0, T],$$

so that, on the event  $\{C_T < m_0/(\sigma_0(1 + aT))\}$ , we have  $X_t > 0$  for every  $t \in [0, T]$  simultaneously. Consequently,

$$\left\{ C_T < \frac{m_0}{\sigma_0(1+aT)} \right\} \subseteq \{X_t > 0 \text{ for all } t \in [0, T]\}. \quad (12)$$

By Markov's inequality, for every  $p \geq 1$ ,

$$\mathbb{P}\left(C_T \geq \frac{m_0}{\sigma_0(1+aT)}\right) \leq \left(\frac{\sigma_0(1+aT)}{m_0}\right)^p \mathbb{E}[C_T^p].$$

Combining this with (12) yields (10). Finiteness of  $\mathbb{E}[C_T^p]$  for every  $p \geq 1$  follows from the Hölder continuity of  $Z^{(q,H)}$  (Lemma 3.2) together with the hypercontractivity of fixed Wiener chaoses [14, Theorem 2.7.2], which guarantees equivalence of all  $L^p$  norms,  $p \geq 1$ , on any finite Wiener chaos and hence finiteness of all moments of the (chaos-valued, in each coordinate) supremum  $C_T$ ; we do not pursue the sharp value of  $\mathbb{E}[C_T^p]$  here. Since  $\mathbb{E}[C_T^p]$  does not depend on  $m_0$  or  $\sigma_0$ , the right-hand side of (10) tends to 1 as  $m_0/\sigma_0 \rightarrow \infty$  for any fixed  $p \geq 1$ , completing the proof.  $\square$

*Remark 3.8* (Comparison with the Gaussian case and numerical evidence). In the Gaussian case  $q = 1$  (fBm-driven CIR), Mishura and Yurchenko-Tytarenko [12] show, via a pathwise contradiction argument that exploits only the Hölder continuity of  $B^H$  (no Itô formula, no semimartingale comparison theorem), that  $Y$  — and hence  $X = Y^2$  — is strictly positive a.s. for *every*  $\sigma > 0$  as soon as  $k > 0$  and  $H > 1/2$ : unlike the classical Brownian CIR process, no threshold condition analogous to  $2ab_0 \geq \sigma_0^2$  is needed once  $H > 1/2$ . Theorem 3.5 does not recover this unconditional a.s. statement for  $q \geq 2$ ; it gives instead the weaker, but fully rigorous, quantitative bound (10). The obstruction is therefore not the absence of semimartingale machinery — which is unavailable already at  $q = 1$ ,  $H > 1/2$  — but rather that the pathwise contradiction argument of [12], which relies on properties specific to the first Wiener chaos, does not appear to extend directly to Hermite processes of order  $q \geq 2$ . Whether an a.s. positivity result analogous to the  $q = 1$  case holds for  $q \geq 2$  remains an open problem (Remark 3.6). Preliminary numerical experiments (work in preparation) report no violations of positivity across replications when  $m_0/\sigma_0$  is in the range typical of interest-rate calibrations, which is consistent with — but does not prove — a possible a.s. statement.

*Remark 3.9* (Positivity under the zero lower bound regime). Theorem 3.5 requires  $b_0 - |b_1| > 0$  (Assumption 1), which enters the theorem only through  $m_0 = \min(x_0, b_0 - |b_1|)$ . In interest-rate applications (e.g. Fed Funds and EURIBOR series), the estimated intercept  $\hat{b}_0$  is often negative over the zero-lower-bound (ZLB) period 2009–2022. Since  $b_0 - |b_1| \leq b_0$  for every  $b_1$ , Assumption 1 *cannot* hold once  $\hat{b}_0 \leq 0$ , regardless of how large the seasonal amplitude  $\hat{b}_1$  is (see Remark 2.6): in that regime  $m_0 < 0$ , and the quantitative bound (10) is vacuous. Positivity of  $X_t$  can then only be enforced directly in simulation, e.g. by the reflection  $X_{t_i} \leftarrow \max(X_{t_i}, \varepsilon)$  in the Euler scheme, with no guarantee from Theorem 3.5. This empirical regime is studied in work currently in preparation.

Having established the existence of a unique solution  $(X_t)_{t \in [0, T]}$ , we now analyse its regularity in the Malliavin sense.

### 3.3 Malliavin Differentiability

The key difficulty, absent in the Gaussian (fBm) case, is that the standard isonormal Gaussian framework cannot be directly applied when  $Z^{(q, H)}$  is non-Gaussian ( $q \geq 2$ ). Following [10], we rely on the Kusuoka–Stroock approach to Malliavin calculus, which is equivalent to the classical Shigekawa definition (see [20, Theorem 3.1]) but better suited to the non-Gaussian setting.

We briefly recall the relevant notation. For  $m \in \mathbb{N}^*$  and  $p > 1$ , the Sobolev–Malliavin space  $\mathbb{D}^{m, p}$  is the closure of smooth cylindrical random variables with respect to the norm

$$\|F\|_{m, p} := \left( \mathbb{E}[|F|^p] + \sum_{k=1}^m \mathbb{E}[\|D^k F\|_{\mathfrak{H}^{\otimes k}}^p] \right)^{1/p},$$

and  $\mathbb{D}^{1, \infty} := \bigcap_{p \geq 1} \mathbb{D}^{1, p}$ . For  $F = I_q(f)$  with  $f \in \mathfrak{H}^{\odot q}$ , the Malliavin derivative satisfies [10, Eq. (15)]

$$D^r I_q(f) = \begin{cases} \frac{q!}{(q-r)!} I_{q-r}(f) & \text{if } r \leq q, \\ 0 & \text{if } r > q. \end{cases}$$

The Kusuoka–Stroock approach defines differentiability via two conditions on a random variable  $F : \Omega \rightarrow \mathbb{R}$ :

- **(RAC) Ray Absolutely Continuous:** for every  $h \in \mathfrak{H}$ , there exists a version  $\hat{F}_h$  of  $F$  such that  $\varepsilon \mapsto \hat{F}_h(\omega + \varepsilon j(h))$  is absolutely continuous for every  $\omega \in \Omega$ ;
- **(SGD) Stochastically Gâteaux Differentiable:** there exists a map

$$G : \Omega \rightarrow \mathcal{L}_{HS}(j(\mathfrak{H}), \mathbb{R})$$

such that, for every  $h \in \mathfrak{H}$ ,

$$\frac{1}{\varepsilon}(F(\cdot + \varepsilon j(h)) - F) \xrightarrow{\mathbb{P}} G(\cdot)[j(h)] \quad \text{as } \varepsilon \rightarrow 0.$$

The key equivalence result is the following.

**Theorem 3.10** (Kusuoka–Stroock = Shigekawa, [20]). *For  $m \in \mathbb{N}^*$  and  $1 < p < \infty$ ,  $F \in \mathbb{D}^{m,p}$  if and only if  $F$  satisfies (RAC) and (SGD) with  $G \in L^p(\Omega, \mathcal{L}_{HS}(j(\mathfrak{H}), \mathbb{R}))$ . In this case,  $\langle DF, h \rangle_{\mathfrak{H}} = G[j(h)]$  almost surely.*

The following result is obtained by verifying the conditions (RAC) and (SGD) of [10, Theorem 4.1] for the specific CIR-Hermite coefficients  $b(t, x) = a(b(t) - x)$  and  $\tilde{\sigma}(x)$ . *The main contribution here is not the Malliavin theorem itself, but its verification for a CIR-type diffusion with non-Lipschitz square-root volatility and non-Gaussian driving noise.*

**Theorem 3.11** (Malliavin Differentiability: verification for CIR-Hermite). *Under Assumptions 1–4, for every  $t \in (0, T]$ , the solution  $X_t$  belongs to  $\mathbb{D}^{1,\infty}$ . Moreover, its Malliavin derivative satisfies, for every  $h \in \mathfrak{H}$ ,*

$$\langle DX_t, h \rangle_{\mathfrak{H}} = \int_0^t \Theta_t(s) d\langle DZ_s^{(q,H)}, h \rangle_{\mathfrak{H}}, \quad (13)$$

where the integral on the right-hand side is a Young–Stieltjes integral, and the process  $s \mapsto \Theta_t(s)$  is the unique solution of the linear equation

$$\Theta_t(s) = \tilde{\sigma}(X_s) - \int_s^t a \Theta_u(s) du + \int_s^t \tilde{\sigma}'(X_u) \Theta_u(s) dZ_u^{(q,H)}, \quad 0 \leq s \leq t, \quad (14)$$

with  $\Theta_t(s) = 0$  for  $s > t$ .

*Proof. Roadmap.* The Malliavin differentiability result follows from verifying the abstract conditions of [10, Theorem 4.1] for our specific CIR-Hermite coefficients. The novel difficulty, absent in globally Lipschitz settings, is that the natural CIR square-root coefficient  $\sigma_0 + \sigma_1 \sqrt{x}$  is not globally Lipschitz and not differentiable at  $x = 0$ , so the abstract framework of [10] (which requires  $\mathcal{C}_b^3$  coefficients, Assumption 2) does not apply to it directly. The regularisation  $\phi_\varepsilon$  resolves this by replacing  $x$  with  $\phi_\varepsilon(x)$  under the square root, which yields  $\tilde{\sigma} \in \mathcal{C}_b^3(\mathbb{R})$  (Remark 2.8) while leaving  $\tilde{\sigma}$  already bounded away from 0 by  $\sigma_0 > 0$  (Assumption 4), for every  $x \in \mathbb{R}$  — ellipticity is therefore automatic here and is not the difficulty being resolved.

We apply [10, Theorem 4.1] with  $d = m = 1$  to our specific model. The proof consists of five steps: we first construct the auxiliary processes needed, then verify the two conditions (SGD) and (RAC) of the Kusuoka–Stroock framework, and finally identify the Malliavin derivative explicitly.

**Step 1: Construction of the shifted process  $S^h$ .**

Fix  $h \in \mathfrak{H}$  and  $\varepsilon \in \mathbb{R}$ . Since  $Z_t^{(q,H)} = I_q(L_t^{H,q})$  with  $L_t^{H,q} \in \mathfrak{H}^{\odot q}$ , the Taylor formula for Malliavin calculus [10, Theorem 3.2, Corollary 3.3] provides a version  $\tilde{Z}^{(q,H)}$  of  $Z^{(q,H)}$  and an event  $\Omega_h$  with  $\mathbb{P}(\Omega_h) = 1$  such that, for every  $\omega \in \Omega_h$ ,

$$Z_t^{(q,H)}(\omega + \varepsilon j(h)) = \sum_{k=0}^q \frac{\varepsilon^k}{k!} \langle D^k Z_t^{(q,H)}(\omega), h^{\otimes k} \rangle_{\mathfrak{H}^{\otimes k}}. \quad (15)$$

Define the process  $S^h = \{S_{t,\varepsilon}^h\}_{t \in [0,T], \varepsilon \in \mathbb{R}}$  by

$$S_{t,\varepsilon}^h(\omega) := \sum_{k=0}^q \frac{\varepsilon^k}{k!} \langle D^k Z_t^{(q,H)}(\omega), h^{\otimes k} \rangle_{\mathfrak{H}^{\otimes k}}. \quad (16)$$

By [10, Proposition 3.4], under Assumption 3, the following properties hold almost surely:

- (a)  $S_{*,\varepsilon}^h(\omega) \in W_2^{1-\alpha}(0, T; \mathbb{R})$  for every  $\varepsilon \in \mathbb{R}$ ;
- (b)  $\varepsilon \mapsto S_{*,\varepsilon}^h(\omega)$  is of class  $\mathcal{C}^\infty$  from  $\mathbb{R}$  into  $W_2^{1-\alpha}(0, T; \mathbb{R})$ , with derivative

$$\frac{d}{d\varepsilon} S_{*,\varepsilon}^h(\omega) = \langle DZ_\star^{(q,H)}(\omega), h \rangle_{\mathfrak{H}}.$$

In particular,  $S^h$  is a version of the shifted process  $\{Z_t^{(q,H)}(\cdot + \varepsilon j(h))\}$ .

**Step 2: Construction of  $\Gamma^h$  and verification of (SGD).**

For  $h \in \mathfrak{H}$  and  $\varepsilon \in \mathbb{R}$ , define

$$\Gamma_t^h(\varepsilon, \omega) := \Psi_t(S_{*,\varepsilon}^h(\omega)), \quad (17)$$

where  $\Psi : W_2^{1-\alpha}(0, T; \mathbb{R}) \rightarrow W_1^\alpha(0, T; \mathbb{R})$  is the solution map of the deterministic equation (5), which is Fréchet differentiable by [10, Proposition 2.4] (cf. [18, Proposition 4]). By (15), for every  $h \in \mathfrak{H}$ ,  $\varepsilon \in \mathbb{R}$  and almost every  $\omega \in \Omega$ ,

$$X_t(\omega + \varepsilon j(h)) = \Gamma_t^h(\varepsilon, \omega).$$

Since  $\varepsilon \mapsto S_{*,\varepsilon}^h(\omega)$  is  $\mathcal{C}^\infty$  by property (b) of Step 1, and  $\Psi_t$  is Fréchet differentiable, the chain rule gives

$$\frac{d}{d\varepsilon} \Gamma_t^h(\varepsilon, \omega) = \mathcal{D}\Psi_t(S_{*,\varepsilon}^h(\omega)) \left[ \frac{d}{d\varepsilon} S_{*,\varepsilon}^h(\omega) \right]. \quad (18)$$

Evaluating at  $\varepsilon = 0$  and using  $S_{*,0}^h(\omega) = Z_\star^{(q,H)}(\omega)$ :

$$\left. \frac{d}{d\varepsilon} \Gamma_t^h(\varepsilon, \omega) \right|_{\varepsilon=0} = \mathcal{D}\Psi_t(Z_\star^{(q,H)}(\omega)) [\langle DZ_\star^{(q,H)}(\omega), h \rangle_{\mathfrak{H}}]. \quad (19)$$

Since almost sure convergence implies convergence in probability, the limit

$$\frac{1}{\varepsilon} (X_t(\cdot + \varepsilon j(h)) - X_t) = \frac{1}{\varepsilon} (\Gamma_t^h(\varepsilon, \cdot) - \Gamma_t^h(0, \cdot)) \xrightarrow{\mathbb{P}} \mathcal{D}\Psi_t(Z_\star^{(q,H)}) [\langle DZ_\star^{(q,H)}, h \rangle_{\mathfrak{H}}]$$

holds as  $\varepsilon \rightarrow 0$ , establishing (SGD) for  $X_t$  with stochastic Gâteaux derivative

$$\hat{D}X_t[j(h)] = \mathcal{D}\Psi_t(Z_\star^{(q,H)}) [\langle DZ_\star^{(q,H)}, h \rangle_{\mathfrak{H}}]. \quad (20)$$

**Step 3: Verification of (RAC).**

By [10, Theorem 3.6], for each  $h \in \mathfrak{H}$  there exists a version  $\tilde{Z}^h$  of  $Z^{(q,H)}$  such that, for every  $\omega \in \Omega$ , the map  $\varepsilon \mapsto \tilde{Z}_\star^h(\omega + \varepsilon j(h))$  is of class  $\mathcal{C}^\infty$  from  $\mathbb{R}$  into  $W_2^{1-\alpha}(0, T; \mathbb{R})$ . Define  $\tilde{X}_t^h(\omega) := \Psi_t(\tilde{Z}_\star^h(\omega))$ . Then:

- $\tilde{X}_t^h = X_t$  almost surely (since  $\tilde{Z}^h$  is a version of  $Z^{(q,H)}$  and the solution map  $\Psi$  is deterministic);
- for every  $\omega \in \Omega$ , the map  $\varepsilon \mapsto \tilde{X}_t^h(\omega + \varepsilon j(h)) = \Psi_t(\tilde{Z}_*^h(\omega + \varepsilon j(h)))$  is absolutely continuous in  $\varepsilon$ , as a composition of the  $\mathcal{C}^1$  map  $\varepsilon \mapsto \tilde{Z}_*^h(\omega + \varepsilon j(h))$  with the Fréchet differentiable map  $\Psi_t$ .

This establishes **(RAC)** for  $X_t$ .

**Step 4: Identification of the Malliavin derivative.**

By Theorem 3.10 (Kusuoka–Stroock equals Shigekawa), since  $X_t$  satisfies both (RAC) and (SGD) with  $\hat{D}X_t[j(h)] \in L^p(\Omega)$  for all  $p \geq 1$  (which follows from the  $L^p$  boundedness of  $\mathcal{D}\Psi_t$  and the hypercontractivity of  $DZ^{(q,H)}$ ), we conclude  $X_t \in \mathbb{D}^{1,\infty}$  and  $\langle DX_t, h \rangle_{\mathfrak{H}} = \hat{D}X_t[j(h)]$ .

Substituting the Fréchet derivative formula [10, Proposition 2.4] into (20):

$$\langle DX_t^k, h \rangle_{\mathfrak{H}} = \mathcal{D}\Psi_t(Z_*^{(q,H)})[\langle DZ_*^{(q,H)}, h \rangle_{\mathfrak{H}}] = \int_0^t \Theta_t(s) d\langle DZ_s^{(q,H)}, h \rangle_{\mathfrak{H}},$$

which is (13).

**Step 5: Identification of  $\Theta_t$ .**

The process  $\Theta_t(s)$  is the fundamental solution of the linearised equation associated with (5). Differentiating the solution map  $\Psi$  in the direction  $\langle DZ_*^{(q,H)}, h \rangle_{\mathfrak{H}}$  via [10, Proposition 2.4], one obtains that  $s \mapsto \Theta_t(s)$  satisfies the *first variation equation*:

$$\Theta_t(s) = \tilde{\sigma}(X_s) + \int_s^t (-a) \Theta_u(s) du + \int_s^t \tilde{\sigma}'(X_u) \Theta_u(s) dZ_u^{(q,H)}, \quad 0 \leq s \leq t,$$

with  $\Theta_t(s) = 0$  for  $s > t$ . This is a linear Young SDE in  $\Theta_t(\cdot)$  driven by  $Z^{(q,H)}$ . Under Assumption 2 ( $\tilde{\sigma}' \in \mathcal{C}_b^2(\mathbb{R})$ ) and Assumption 3, Theorem 3.3 (applied to the linearised equation) guarantees the existence of a unique solution  $s \mapsto \Theta_t(s) \in W_1^\alpha(0, t; \mathbb{R})$ .

Since  $\phi_\varepsilon$  is defined and smooth on all of  $\mathbb{R}$  (Assumption 2), the derivative  $\tilde{\sigma}'(x) = \sigma_1 \phi'_\varepsilon(x) / (2\sqrt{\phi_\varepsilon(x)})$  is well defined for every  $x \in \mathbb{R}$ . We record its simplified form on  $\{x \geq 0\}$ , where  $\phi_\varepsilon(x) = x + \varepsilon$  by construction (condition (a) of Section 2.2); no probabilistic positivity statement is needed for this simplification, which is purely a matter of the definition of  $\phi_\varepsilon$ :

$$\tilde{\sigma}'(x) = \frac{\sigma_1 \phi'_\varepsilon(x)}{2\sqrt{\phi_\varepsilon(x)}} = \frac{\sigma_1}{2\sqrt{x + \varepsilon}}, \quad x \geq 0,$$

which is bounded (by  $\sigma_1/(2\sqrt{\varepsilon})$ ) and Lipschitz on  $[0, +\infty)$  under Assumption 1. The variational equation (14) becomes:

$$\Theta_t(s) = \tilde{\sigma}(X_s) + \int_s^t (-a) \Theta_u(s) du + \int_s^t \frac{\sigma_1}{2\sqrt{X_u + \varepsilon}} \Theta_u(s) dZ_u^{(q,H)}. \quad (21)$$

This completes the proof.  $\square$

*Remark 3.12.* When  $q = 1$ , the driving process  $Z^{(1,H)} = B^H$  is a fractional Brownian motion, and Theorem 3.11 reduces to [18, Theorem 6]. The result of [10] sharpens the conclusion of [18] from  $X_t \in \mathbb{D}_{\text{loc}}^{1,2}$  to  $X_t \in \mathbb{D}^{1,\infty}$ , and extends it to all  $q \geq 1$  simultaneously.

*Remark 3.13.* Equation (14) is the *variational equation* associated with (5):  $\Theta_t(s)$  represents the sensitivity of the solution at time  $t$  to a perturbation of the driving noise at time  $s \leq t$ . On  $\{X_u \geq 0\}$ ,  $\phi_\varepsilon(X_u) = X_u + \varepsilon$  by construction (condition (a) of Section 2.2), so that  $\tilde{\sigma}'(X_u) = \sigma_1/(2\sqrt{X_u + \varepsilon})$  there; no a.s. positivity statement is invoked. On this event, equation (14) becomes

$$\Theta_t(s) = \tilde{\sigma}(X_s) + \int_s^t (-a) \Theta_u(s) du + \int_s^t \frac{\sigma_1}{2\sqrt{X_u + \varepsilon}} \Theta_u(s) dZ_u^{(q,H)}.$$

This is a linear Young SDE in  $\Theta_t(\cdot)$ , driven by the same Hermite process  $Z^{(q,H)}$ , whose unique solution inherits the Hölder regularity of  $X$ .

### 3.4 Absolute Continuity of the Law

The Malliavin differentiability established in Theorem 3.11 is the key ingredient for proving that the law of  $X_t$  has a density with respect to the Lebesgue measure. We use the Bouleau–Hirsch criterion [5], together with the non-degeneracy of the Hermite process Malliavin matrix established in [10] (Theorem 6.6 therein).

**Theorem 3.14** (Bouleau–Hirsch criterion, [5]). *Let  $Y \in \mathbb{D}^{1,2}$  be a real random variable such that  $\|DY\|_{\mathfrak{H}}^2 > 0$  almost surely. Then the law of  $Y$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ .*

The non-degeneracy condition  $\|DX_t\|_{\mathfrak{H}}^2 > 0$  a.s. follows from the non-degeneracy of the driving Hermite process, established in [10, Theorem 6.6] (which verifies Hypothesis  $(\mathbf{H}_5)$  for Hermite processes).

*This result follows from the Bouleau–Hirsch criterion by verifying non-degeneracy of the Malliavin matrix of  $X_t$ , using Theorem 3.11. The main contribution is this non-degeneracy verification for the CIR-Hermite setting.*

**Theorem 3.15** (Absolute Continuity: verification of non-degeneracy). *Under Assumptions 1–4, for every  $t \in (0, T]$ , the law of the solution  $X_t$  to (5) is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ . That is, there exists a measurable function  $p_t : \mathbb{R} \rightarrow [0, +\infty)$  such that*

$$\mathbb{P}(X_t \in A) = \int_A p_t(x) dx$$

*for every Borel set  $A \subset \mathbb{R}$ .*

*Proof.* We apply the Bouleau–Hirsch criterion (Theorem 3.14) to  $Y = X_t$ . The proof consists of four steps.

#### Step 1: Malliavin differentiability.

By Theorem 3.11 (under Assumptions 1–3),  $X_t \in \mathbb{D}^{1,\infty} \subset \mathbb{D}^{1,2}$ . This is the first condition of the Bouleau–Hirsch criterion.

#### Step 2: Reduction to non-degeneracy of $\|DX_t\|_{\mathfrak{H}}$ .

The Malliavin matrix of  $X_t$  in the scalar case ( $d = 1$ ) reduces to the scalar

$$\Gamma_{X_t} = \|DX_t\|_{\mathfrak{H}}^2 = \sum_{n=1}^{\infty} \langle DX_t, e_n \rangle_{\mathfrak{H}}^2,$$

where  $\{e_n\}_{n \geq 1}$  is any orthonormal basis of  $\mathfrak{H}$ . By the Bouleau–Hirsch criterion, it suffices to show that  $\|DX_t\|_{\mathfrak{H}}^2 > 0$  almost surely. We proceed by contradiction: suppose there exists  $\Omega_0 \in \mathcal{F}$  with  $\mathbb{P}(\Omega_0) > 0$  such that

$$\|DX_t(\omega)\|_{\mathfrak{H}}^2 = 0 \quad \text{for all } \omega \in \Omega_0. \quad (22)$$

By the representation (13), for every  $n \geq 1$  and  $\omega \in \Omega_0$ ,

$$0 = \langle DX_t(\omega), e_n \rangle_{\mathfrak{H}} = \int_0^t \Theta_t(s, \omega) d\langle DZ_s^{(q,H)}(\omega), e_n \rangle_{\mathfrak{H}}.$$

Since  $\{e_n\}_{n \geq 1}$  is a complete orthonormal system, summing the squared norms over  $n$  and using Parseval's identity:

$$\left\| \int_0^t \Theta_t(s, \omega) dDZ_s^{(q,H)}(\omega) \right\|_{\mathfrak{H}}^2 = \sum_{n=1}^{\infty} \left( \int_0^t \Theta_t(s, \omega) d\langle DZ_s^{(q,H)}(\omega), e_n \rangle_{\mathfrak{H}} \right)^2 = 0 \quad (23)$$

for all  $\omega \in \Omega_0$ , i.e.,

$$\left\| \int_0^t \Theta_t(s, \omega) dDZ_s^{(q,H)}(\omega) \right\|_{\mathfrak{H}} = 0 \quad \forall \omega \in \Omega_0. \quad (24)$$

### Step 3: Verification of hypothesis $(\mathbf{H}_5)$ for our model.

We now verify that the non-degeneracy hypothesis  $(\mathbf{H}_5)$  of [10] holds for the CIR-Hermite model with  $m = d = 1$ . In our setting,  $(\mathbf{H}_5)$  reads: for any  $\gamma > \frac{1}{2}$ -Hölder continuous process  $(Y_r)_{r \in [0, t]}$  and any  $\Omega_0$  with  $\mathbb{P}(\Omega_0) > 0$ ,

$$\begin{aligned} & \left\| \int_0^t Y_s(\omega) dDZ_s^{(q,H)}(\omega) \right\|_{\mathfrak{H}} = 0 \quad \forall \omega \in \Omega_0 \\ \implies & \exists \Omega_1 \subseteq \Omega_0, \mathbb{P}(\Omega_1) > 0, \exists s_0 \in [0, t] \text{ s.t. } Y_{s_0}(\omega) = 0 \quad \forall \omega \in \Omega_1. \end{aligned}$$

By [10, Theorem 6.6], this property holds for the Hermite process  $Z^{(q,H)}$  for all  $q \geq 1$  and  $H \in (\frac{1}{2}, 1)$ . Indeed, the proof uses the self-similarity and the scaling property of the Malliavin derivative  $DZ^{(q,H)}$ , established via Lemma 6.5 of [10]: for every  $t > 0$  and  $\varepsilon \in (0, t)$ ,

$$\int_{t-\varepsilon}^t |D_r Z_t^{(q,H)} - D_r Z_{t-\varepsilon}^{(q,H)}|^2 dr \stackrel{\text{law}}{=} \varepsilon^{2H} \int_0^1 |D_r Z_1^{(q,H)}|^2 dr.$$

Since  $s \mapsto \Theta_t(s)$  solves a linear equation of exactly the form treated in [18, Proposition 9] (with  $\gamma(s) = \tilde{\sigma}(X_s)$ ,  $B_u = -a$  and  $S_u = \tilde{\sigma}'(X_u)$ , all  $(H - \beta)$ -Hölder continuous by Theorem 3.3), that proposition gives that  $s \mapsto \Theta_t(s)$  is  $(H - \beta)$ -Hölder continuous (with  $H - \beta > \frac{1}{2}$  under Assumption 3), so  $(\mathbf{H}_5)$  applies to  $Y_s = \Theta_t(s)$ . Combining with (24): there exist  $\Omega_1 \in \mathcal{F}$  with  $\mathbb{P}(\Omega_1) > 0$  and  $s_0 \in [0, t]$  such that

$$\Theta_t(s_0, \omega) = 0 \quad \text{for all } \omega \in \Omega_1. \quad (25)$$

### Step 4: Contradiction via ellipticity.

We show that (25) is impossible under Assumption 4. The key point is to distinguish two different objects built from the same two-parameter family  $\{\Theta_u(s)\}_{0 \leq s \leq u \leq T}$  defined by (14): the value  $\Theta_{s_0}(s_0)$  (the family evaluated at its own starting point  $u = s_0$ , which is

an *initial condition*) and the value  $\Theta_t(s_0)$  (the same family evaluated at the fixed terminal time  $u = t$ , which is the *quantity of interest* appearing in (13) and (25)). These coincide only when  $t = s_0$ ; in general  $\Theta_t(s_0)$  must be obtained from  $\Theta_{s_0}(s_0)$  by evolving (14) forward from  $u = s_0$  to  $u = t$ , which we now do.

Fix  $s_0$  as given by Step 3, and consider the map  $u \mapsto \Theta_u(s_0)$  for  $u \in [s_0, t]$  — i.e., the family  $\{\Theta_u(s_0)\}$  from (14) with the *second* argument frozen at  $s = s_0$  and the *first* argument (the evaluation time) left to vary. Directly from (14) with  $s = s_0$  and  $t$  replaced by the running variable  $u$ , this map solves the linear Young SDE

$$\Theta_u(s_0) = \Theta_{s_0}(s_0) - \int_{s_0}^u a \Theta_r(s_0) dr + \int_{s_0}^u \tilde{\sigma}'(X_r) \Theta_r(s_0) dZ_r^{(q,H)}, \quad u \in [s_0, t], \quad (26)$$

with *initial condition* at  $u = s_0$  given by evaluating the integral terms of (14) at  $u = s_0$ , where the integration range  $[s_0, s_0]$  is empty:

$$\Theta_{s_0}(s_0) = \tilde{\sigma}(X_{s_0}) = \sigma_0 + \sigma_1 \sqrt{\phi_\varepsilon(X_{s_0})}. \quad (27)$$

By Assumption 4,  $\tilde{\sigma}(x) \geq \sigma_0 > 0$  for all  $x \in \mathbb{R}$  (regardless of the sign of  $X_{s_0}$ ), so

$$\Theta_{s_0}(s_0) = \tilde{\sigma}(X_{s_0}) \geq \sigma_0 > 0 \quad \text{almost surely.}$$

Equation (26) is a *linear, homogeneous* scalar Young SDE in  $u$  (no additive forcing term beyond the initial condition), so its solution is the explicit stochastic exponential of  $-a du + \tilde{\sigma}'(X_u) dZ_u^{(q,H)}$ , started from  $\Theta_{s_0}(s_0)$ :

$$\Theta_u(s_0) = \Theta_{s_0}(s_0) \exp\left(-a(u - s_0) + \int_{s_0}^u \tilde{\sigma}'(X_r) dZ_r^{(q,H)}\right), \quad u \in [s_0, t]. \quad (28)$$

(This may be checked directly: differentiating the right-hand side along the same Young calculus used throughout the paper reproduces the linear equation (26), and by pathwise uniqueness for linear Young SDEs with  $\mathcal{C}^1$  coefficients — the same uniqueness mechanism as in Theorem 3.3, Step 4 — this is *the* solution.) The exponential factor in (28) is strictly positive for every  $u$ , so evaluating at  $u = t$  gives

$$\Theta_t(s_0) = \tilde{\sigma}(X_{s_0}) \exp\left(-a(t - s_0) + \int_{s_0}^t \tilde{\sigma}'(X_r) dZ_r^{(q,H)}\right) > 0 \quad \text{almost surely,}$$

since  $\tilde{\sigma}(X_{s_0}) > 0$  almost surely and the exponential of a real number is always strictly positive. This directly contradicts (25). Hence (22) cannot hold on any event of positive probability, and we conclude:

$$\mathbb{P}(\|DX_t\|_{\mathcal{H}}^2 = 0) = 0,$$

i.e.,  $\|DX_t\|_{\mathcal{H}}^2 > 0$  almost surely. The Bouleau–Hirsch criterion (Theorem 3.14) then gives the absolute continuity of the law of  $X_t$  with respect to the Lebesgue measure on  $\mathbb{R}$ , completing the proof of Theorem 3.15.  $\square$

*Remark 3.16.* For  $q = 1$  (fractional CIR model), Theorem 3.15 recovers and strengthens [18, Theorem 8]: the density exists for all  $H \in (\frac{1}{2}, 1)$  and all  $t \in (0, T]$ .

For  $q \geq 2$  (Rosenblatt-CIR and higher-order Hermite-CIR models), this is a *new result*: the existence of a density for solutions of SDEs driven by non-Gaussian Hermite processes had not been established prior to [10].

*Remark 3.17* (Support of the density). Theorem 3.15 gives absolute continuity of the law of  $X_t$  on all of  $\mathbb{R}$ , not merely on  $(0, +\infty)$ : since  $\tilde{\sigma}(x) \geq \sigma_0 > 0$  for *every*  $x \in \mathbb{R}$  (Assumption 4), the ellipticity used in the Bouleau–Hirsch argument does not distinguish the sign of  $X_t$ , and we do not claim that  $p_t$  vanishes on  $(-\infty, 0]$ . This should be contrasted with the classical Brownian CIR model, where the vanishing of the diffusion coefficient  $\sigma\sqrt{x}$  at  $x = 0$  is precisely what confines the density to  $(0, +\infty)$  under the Feller condition. Here, because  $\tilde{\sigma}$  is bounded away from 0 (needed for Assumption 4) rather than vanishing at 0, no such confinement mechanism is available, consistently with Theorem 3.5 giving only a quantitative — not almost-sure — positivity bound. Whether  $p_t$  is in fact supported on  $(0, +\infty)$ , or merely concentrated there with a small tail on  $(-\infty, 0]$ , is left open, in line with Remark 3.6.

### 3.5 Summary of the Theoretical Framework

Table 3 summarises the four main theoretical results, together with their relationship to [10].

Table 3: Theoretical results for the CIR-Hermite model and their sources in [10].

| Result                      | Theorem  | Tool                               | Source        |
|-----------------------------|----------|------------------------------------|---------------|
| Existence & uniqueness      | Thm 3.3  | Young integral + [17]              | [17, Thm 5.1] |
| Positivity (quantitative)   | Thm 3.5  | Lower barrier + hypercontractivity | This paper    |
| Malliavin differentiability | Thm 3.11 | Kusuoka–Stroock + Taylor formula   | [10, Thm 4.1] |
| Absolute continuity of law  | Thm 3.15 | Bouleau–Hirsch criterion           | [10, Thm 5.2] |

The chain of implications is the following:

$$\underbrace{\text{Assumptions 1–3}}_{\text{parameters + coefficient \& driver regularity}} \implies \underbrace{\text{Thm 3.3}}_{\exists! X_t \in W_1^\alpha} \implies \underbrace{\text{Thm 3.11}}_{X_t \in \mathbb{D}^{1,\infty}} \xrightarrow{+\text{ Ass. 4}} \underbrace{\text{Thm 3.15}}_{\text{law of } X_t \ll \lambda}$$

where  $\lambda$  denotes the Lebesgue measure. The ellipticity Assumption 4 ( $\tilde{\sigma} \geq \sigma_0 > 0$ ) is the key condition that makes the final implication possible, by preventing the Malliavin matrix from degenerating. Theorem 3.5 (positivity) is not part of this chain: as discussed in Remark 3.13, the Malliavin and density results hold on all of  $\mathbb{R}$  and do not require positivity of  $X_t$ ; positivity is established independently, in parallel, as a quantitative probabilistic statement.

## 4 Numerical Illustrations

### 4.1 Simulation method

All numerical simulations in this section are carried out using the wavelet-type expansion algorithm of Ayache, Hamonier and Loosveldt [4], to appear in the *Annals of Applied*

*Probability.* This algorithm, which the authors have generously made available, represents the first rigorous method for simulating sample paths of Hermite processes of arbitrary order  $q \geq 1$ ; for  $q \geq 3$ , no alternative simulation method existed prior to that work.

The simulation process  $\{\tilde{S}_{h,J}^{(q)}(t)\}_{t \in [0,T]}$  is defined through the truncated wavelet-series representation

$$\tilde{S}_{h,J}^{(q)}(t) = 2^{-JH} \sum_{k \in \mathcal{J}_J^1(t)} \sigma_{J,k}^{(h)} \int_{\mathbb{R}} \prod_{\ell=1}^q \Phi_{\Delta}^{(h_{\ell}-1/2)}(s - k_{\ell}) ds, \quad (29)$$

where  $h_{\ell} = 1 + (H - 1)/q$ ,  $J \in \mathbb{N}$  is the resolution level,  $\Phi_{\Delta}^{(\delta)}$  is the fractional scaling function of Meyer [3], the coefficients  $\sigma_{J,k}^{(h)}$  are computed from Gaussian FARIMA(0,  $h_{\ell} - 1/2, 0$ ) sequences simulated by the circulant matrix embedding (CME/Davies–Harte) method [4], and  $\mathcal{J}_J^1(t)$  is a diagonal-like index set defined in [4].

*Remark 4.1.* By [4, Theorem 2.13], the simulation process  $\tilde{S}_{h,J}^{(q)}$  converges to the Hermite process  $Z^{(q,H)}$  almost surely and uniformly on any compact interval, with explicit rate  $O(J^{q/2} 2^{-J(H-1/2)})$ . All figures below are produced with  $J = 17$ ,  $a = 0.99$ ,  $\varepsilon = 10^{-3}$ ,  $T = 1$ , which are the resolution and parameters used by the authors of [4] for their own statistical Monte-Carlo experiments (their Tables 1-9); we refer to that paper for a thorough discussion of parameter sensitivity.

## 4.2 Sample paths of Hermite processes

Figure 1 displays three independent realisations of the standardised Hermite process  $Z^{(q,H)}$  for each order  $q \in \{1, 2, 3\}$ , with Hurst parameter  $H = 0.70$ . The three orders correspond to, respectively, fractional Brownian motion (first Wiener chaos, Gaussian), the Rosenblatt process (second Wiener chaos), and the Hermite process of order 3 (third Wiener chaos). All three processes share the same covariance structure (Proposition 2.2), but exhibit markedly different path regularity and tail behaviour. In particular, the increasing non-Gaussianity as  $q$  increases from 1 to 3 is visible through the more irregular and asymmetric excursions of the Rosenblatt and Hermite-3 paths.

Sample paths of the Hermite process  $Z^{(q,H)}$ ,  $H = 0.70$ ,  $J = 17$

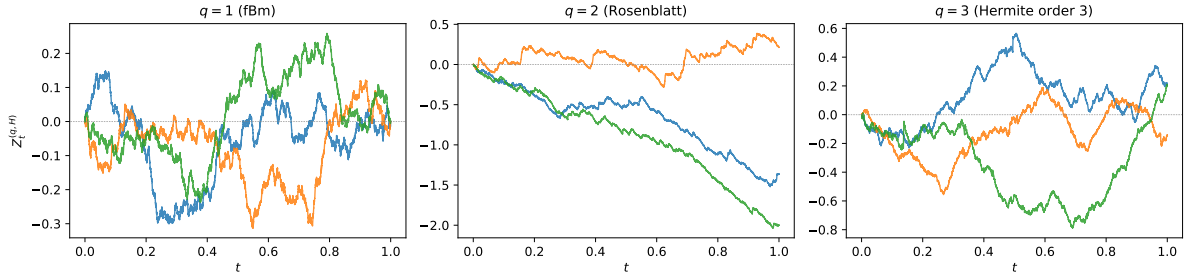


Figure 1: Sample paths of the Hermite process  $Z^{(q,H)}$  for  $q = 1$  (fBm, left),  $q = 2$  (Rosenblatt, centre), and  $q = 3$  (Hermite order 3, right), with  $H = 0.70$  and  $J = 17$ . Each panel shows three independent realisations. All paths are standardised to have unit variance for comparison. The simulation uses the wavelet-series algorithm of Ayache, Hamonier and Loosveldt [4], with FARIMA sequences computed by the circulant matrix embedding method of Davies and Harte. The monograph [22] provides a detailed account of the theoretical properties of these processes.

Figure 2 illustrates the effect of the Hurst parameter  $H$  on the path regularity of the Rosenblatt process ( $q = 2$ ) for four values  $H \in \{0.60, 0.70, 0.80, 0.90\}$ . As  $H$  increases toward 1, the paths become smoother and exhibit stronger long-range dependence: nearby values of the process become more positively correlated, and trends persist over longer time horizons. This is consistent with the Hölder regularity result of Proposition 2.2(iv), which guarantees  $\zeta$ -Hölder continuous paths for any  $\zeta < H$ .

Rosenblatt process  $Z^{(2,H)}$  for varying Hurst parameter,  $J = 17$

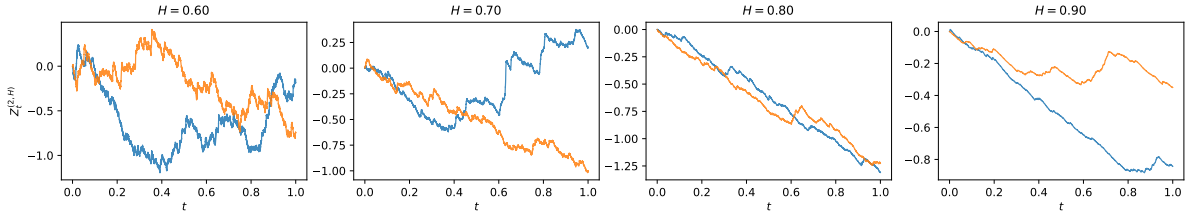


Figure 2: Rosenblatt process  $Z^{(2,H)}$  for varying Hurst parameter  $H \in \{0.60, 0.70, 0.80, 0.90\}$  (left to right),  $J = 17$ . Each panel shows two independent realisations. As  $H$  increases, paths become smoother and exhibit longer-range dependence. Simulation algorithm: Ayache, Hamonier and Loosveldt [4].

### 4.3 Sample paths of the CIR-Hermite model

Figure 3 shows three independent realisations of the CIR-Hermite model (2) for each order  $q \in \{1, 2, 3\}$ , with parameters  $H = 0.70$ ,  $a = 0.40$ ,  $b_0 = 3\%$ ,  $b_1 = 0.5\%$ ,  $\sigma_0 = 0.08$ . The solution is approximated by the Euler–Maruyama scheme applied to the Young integral discretisation, using the wavelet-series simulation of  $Z^{(q,H)}$  as the driving process.

With these parameters,  $b_0 - |b_1| = 2.5\% = 0.025$  and  $\sigma_0 = 0.08$ , so in fact  $m_0 = \min(x_0, b_0 - |b_1|) \leq 0.025 < \sigma_0$ : the base volatility is *larger* than  $m_0$ , not smaller, so the

quantitative lower bound (10) of Theorem 3.5 is *vacuous* for this choice of parameters (the right-hand side of (10) is negative for every  $p \geq 1$ ; the theorem covers  $\sigma_1 = 0$ , the case shown here, but gives no information in this particular regime). Consistent with this, a majority of the simulated paths shown (5 out of the 9 realisations across the three panels, and all three for  $q = 2$ ) cross into negative territory at least transiently during  $[0, 1]$ ; this is not paradoxical, since no theoretical guarantee applies in this regime, and it illustrates concretely why the bound being vacuous is a real limitation rather than a technicality. As discussed in Remark 3.6, no a.s. positivity statement is established here. The dashed line at  $b_0 = 3\%$  illustrates the mean-reversion effect: paths drift back toward the long-run mean after excursions above or below it. The non-Gaussian character of the driving noise ( $q = 2$  and  $q = 3$ ) produces more pronounced asymmetric spikes compared to the Gaussian case ( $q = 1$ ), a feature that is particularly relevant for modelling the discrete, step-like interventions of central banks.

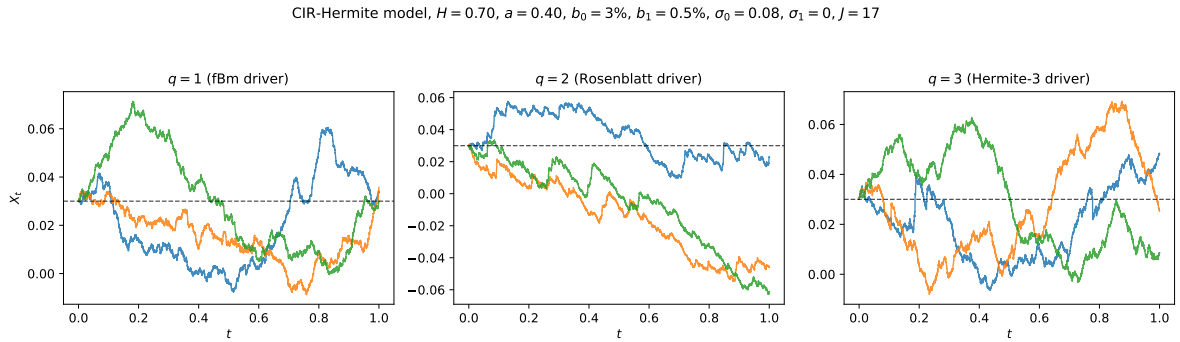


Figure 3: Sample paths of the CIR-Hermite model (2) for  $q = 1$  (fBm driver, left),  $q = 2$  (Rosenblatt driver, centre), and  $q = 3$  (Hermite-3 driver, right), with  $H = 0.70$ ,  $a = 0.40$ ,  $b_0 = 3\%$ ,  $b_1 = 0.5\%$ ,  $\sigma_0 = 0.08$ ,  $J = 17$ . Each panel shows three independent realisations. The dashed line represents the long-run target  $b_0 = 3\%$ . Here  $\sigma_0 = 0.08 > b_0 - |b_1| = 0.025$ , so the quantitative bound (10) of Theorem 3.5 is in fact vacuous for these parameters; consistent with this, 5 of the 9 simulated paths shown (all three for  $q = 2$ ) dip below zero at least transiently — positivity is not implied by the theorem in this regime and is not observed here. Hermite process simulation: Ayache, Hamonier and Loosveldt [4].

#### 4.4 Marginal distributions and non-Gaussianity

Figure 4 compares the empirical marginal distributions of the standardised increments of  $Z^{(q,H)}$  for  $q \in \{1, 2, 3\}$ , based on 40 independent realisations per order. For  $q = 1$  (fBm), the increments are by construction Gaussian, and the quantile-quantile comparison with  $\mathcal{N}(0, 1)$  confirms this. For  $q = 2$  (Rosenblatt) and  $q = 3$  (Hermite-3), the distributions exhibit substantial positive excess kurtosis, increasing sharply with  $q$ ; skewness is clearly positive for  $q = 2$ , while for  $q = 3$  it is comparatively modest and its sign is not robust across repeated simulations at this sample size. This non-Gaussianity is a direct consequence of the multiple Wiener–Itô integral representation: by the hypercontractivity inequality for Wiener chaos [14], all moments are finite but the distribution becomes increasingly heavy-tailed as  $q$  increases. The excess kurtosis values reported (Kurt  $\approx 0$  for  $q = 1$ , strongly positive for  $q \geq 2$ ) are computed on the raw, untrimmed increments:

for these Wiener-chaos variables the excess kurtosis is carried almost entirely by the extreme tail, so winsorising or trimming the sample (e.g. at the 2.5–97.5% range) removes exactly the feature being measured and can even produce the wrong sign. This suggests a kurtosis-based statistic could serve as the basis for an estimator of the Hermite order  $q$  from observed data; we leave a rigorous treatment of this estimation problem to future work.

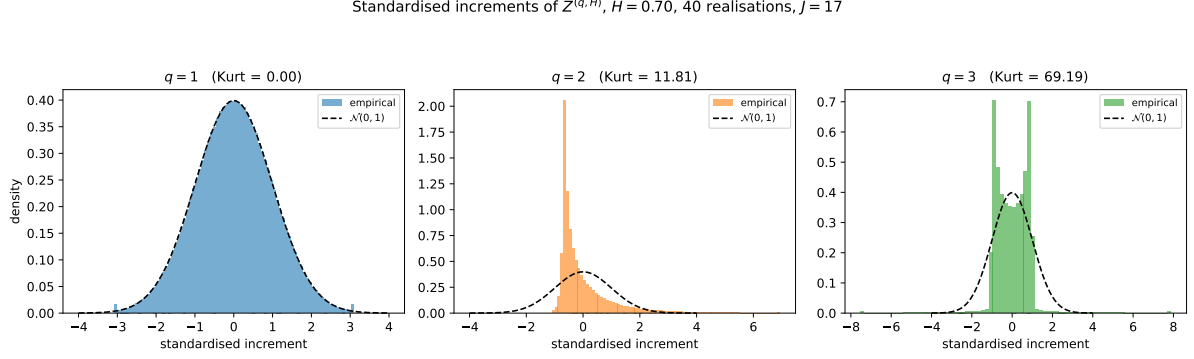


Figure 4: Empirical marginal distributions of standardised increments of  $Z^{(q,H)}$  for  $q = 1$  (left),  $q = 2$  (centre), and  $q = 3$  (right), with  $H = 0.70$ , based on 40 independent realisations per order. The dashed curve is the  $\mathcal{N}(0, 1)$  density. Excess kurtosis (computed on the raw, untrimmed increments; see main text) rises sharply with  $q$ :  $\approx 0.00$  for  $q = 1$ ,  $\approx 11.8$  for  $q = 2$ , and  $\approx 69.2$  for  $q = 3$ , as predicted by the theory of Wiener chaos [14, 22].

## 5 Conclusion and Open Problems

### 5.1 Summary

This paper has established the rigorous mathematical foundations of the CIR-Hermite model (2). Table 4 provides a concise overview of the four main results, their proof methods, and their nature (genuinely new vs. verification of existing frameworks).

Table 4: Summary of theoretical results. “New” denotes a result that is new even for special cases ( $q = 2$  or  $q = 3$ ); “Verification” denotes that the main theorem is taken from an existing general framework, with our contribution being the verification of its hypotheses in the CIR-Hermite setting.

| Result                      | Reference    | Key tool                                                       | Nature       |
|-----------------------------|--------------|----------------------------------------------------------------|--------------|
| Existence & uniqueness      | Theorem 3.3  | Young–Stieltjes integration; Nualart–Răşcanu theory            | New          |
| Positivity (quantitative)   | Theorem 3.5  | Lower barrier process; hypercontractivity; Markov’s inequality | <b>New</b>   |
| Malliavin differentiability | Theorem 3.11 | Verify (RAC) & (SGD) of [10]                                   | Verification |
| Absolute continuity of law  | Theorem 3.15 | Bouleau–Hirsch criterion; non-degeneracy                       | Verification |

The positivity result (Theorem 3.5) is arguably the most novel contribution, though it is weaker than its classical counterpart. In the classical Brownian setting, positivity of CIR follows from Feller’s classical boundary classification for one-dimensional diffusions driven by semimartingales. For Hermite drivers with  $q \geq 2$ , this argument breaks down completely, since the generator is not a second-order differential operator, the process is not Markovian, and the diffusion coefficient  $\tilde{\sigma}$  does not vanish at the boundary. We do not attempt to recover an almost-sure statement; instead, our proof strategy — constructing an explicit lower barrier process satisfying a linear Young SDE and controlling its pathwise excursions via the hypercontractivity of Wiener chaoses combined with Markov’s inequality — yields an explicit probability bound that appears to be new in this setting.

## 5.2 Open problems

The theoretical analysis raises several natural questions that remain open.

- (1) *Almost-sure positivity.* Theorem 3.5 gives a quantitative lower bound on  $\mathbb{P}(X_t > 0 \ \forall t \in [0, T])$ , not an almost-sure statement. Two distinct benchmarks are worth separating here. In the strictly classical Brownian case ( $H = 1/2$ ), the sharp Feller condition  $2ab \geq \sigma^2$  is both sufficient and necessary for a.s. positivity, via Itô’s formula and semimartingale comparison theorems. In the Gaussian long-range-dependent case covered by our framework ( $q = 1$ ,  $H \in (1/2, 1)$ ), no such threshold is needed at all: Mishura and Yurchenko-Tytarenko [12] show, by a pathwise contradiction argument that uses only the Hölder continuity of  $B^H$ , that the fBm-driven CIR process is a.s. strictly positive for *every*  $\sigma > 0$  as soon as  $k > 0$ . For  $q \geq 2$ , whether an analogous a.s. statement holds — with or without a parameter condition on  $(a, b_0, b_1, \sigma_0)$  — remains entirely open; the obstruction is not the absence of semimartingale tools alone (already absent at  $q = 1$ ,  $H > 1/2$ ), but that the pathwise argument of [12], specific to the first Wiener chaos, does not evidently extend to higher-order Hermite processes.

Whether a Feller-type boundary criterion exists for Hermite-driven CIR dynamics remains an open problem. In particular, it is unknown whether the boundary  $\{0\}$  is inaccessible under suitable parameter restrictions, analogous to the unconditional statement available at  $q = 1$ , or whether the rough/non-semimartingale nature of the Hermite driver fundamentally changes the boundary behaviour for  $q \geq 2$ .

Extending Theorem 3.5 to  $\sigma_1 > 0$  is a related, more immediate open question (Remark 3.7).

- (2) *Boundary behaviour at  $x = 0$ .* When  $m_0/\sigma_0$  is not large (so the bound of Theorem 3.5 is uninformative), what is the actual behaviour of the solution near the boundary  $\{0\}$ ? In the Brownian setting, Feller's theory classifies boundaries as entrance, exit, regular, or natural. Developing an analogous theory for non-semimartingale dynamics is a substantial open problem.
- (3) *Higher-order Malliavin regularity.* Our result establishes  $X_t \in \mathbb{D}^{1,\infty}$ . Whether  $X_t \in \mathbb{D}^{k,\infty}$  for  $k \geq 2$  can be obtained by iterating the argument of Theorem 3.11 (which requires verifying that  $\phi_\varepsilon$  and the drift are sufficiently smooth in the Malliavin sense) is an interesting open question. Higher-order Malliavin differentiability would allow, for example, the derivation of density regularity results (existence of a smooth density for the law of  $X_t$ ).
- (4) *Optimality of the regularisation.* We introduce  $\phi_\varepsilon$  as a smooth approximation of  $\sqrt{x}$  to bring the model within the scope of the Lipschitz-based theory. A natural question is whether the results extend to the limit  $\varepsilon \rightarrow 0$  (i.e., to the genuine square-root coefficient  $\sigma_1\sqrt{x}$ ). This would require either extending the Nualart–Răşcanu theory to non-Lipschitz coefficients in the Young setting, or developing an independent Malliavin calculus for this class of SDEs.
- (5) *Multidimensional CIR-Hermite systems.* A natural extension is the system

$$dX_t^i = a_i(b_i(t) - X_t^i) dt + \tilde{\sigma}_i(X_t) dZ_t^{(q_i, H_i)}, \quad i = 1, \dots, d,$$

where the driving Hermite processes  $(Z^{(q_1, H_1)}, \dots, Z^{(q_d, H_d)})$  may be correlated. Such a model would capture the joint dynamics of the yield curve. Well-posedness and positivity in the multidimensional setting require new arguments, since the scalar barrier construction of Theorem 3.5 does not generalise directly.

### 5.3 Perspective: statistical inference

The theoretical foundations established here — in particular the absolute continuity of the marginal law (Theorem 3.15) and the Malliavin regularity of  $X_t$  (Theorem 3.11) — are the natural starting point for statistical inference in the CIR-Hermite model: estimation of  $\theta = (a, b_0, b_1, T_{\text{per}}, \sigma_0, \sigma_1, H, q)$ , asymptotic distribution theory, and empirical application to interest rate data. This is the object of work currently in preparation, together with Monte Carlo validation; we hope to report on it in a forthcoming paper.

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