

An Empirical Framework for Platform Markets*

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Abstract

Quantifying merger effects and conducting other counterfactual analyses is often difficult in platform markets because direct and indirect network effects link demand within and across market sides. We develop a general empirical framework for two-sided markets that incorporates these effects within an otherwise standard empirical IO approach. The framework does not impose a particular demand specification: researchers can estimate flexible demand systems on each side, for example, using BLP-type demand on one side or both. We allow direct and indirect network effects to vary across groups. Standard one-sided merger simulation emerges as a special case when network effects are absent. We provide sufficient conditions for uniqueness in the coordination game within and across market sides and for existence and uniqueness in the firms' pricing game. These conditions are testable. When these conditions hold, the framework supports recovery of marginal costs from firms' first-order conditions and can be used to conduct counterfactual analysis, including merger simulation.

JEL Classification: L13, L40, L82.

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1 Introduction

Network effects have been central to some of the most important competition-policy cases in platform markets in recent years.¹ These cases concern economically significant markets and services that generate substantial consumer surplus (see, e.g., Brynjolfsson et al., 2019; Competition and Markets Authority, 2020). Network effects may be indirect, when participation on one side of the market raises value on the other side, or direct, when participation on one market side affects the value of participating for others on that same market side. Heterogeneity typically matters in markets with network effects because the strength, and sometimes even the sign, of these effects may differ across users and firms. In platform markets, prices, participation, markups, and welfare therefore depend not only on demand elasticities, but also on how network effects interact with the composition of users.

These forces arise across a wide range of industries. In the context of search advertising, a larger user base can improve the ability of advertisers to target users, thereby increasing advertiser demand, while advertisers also care about prominence, relevance, and the composition of users they reach (Allcott et al., 2025; Jeziorski and Segal, 2015; Jeziorski and Moorthy, 2018). In social networks, a larger installed base can make a platform more valuable to users through direct network effects, while a larger number of advertisers can create within-side congestion, as advertisers compete more intensely for limited user attention (Anderson and Peitz, 2023). In online healthcare and other partnership-based platforms, network effects may be participant-specific and interact with bargaining outcomes, so that the value of adding a particular seller depends on that seller’s attributes and on the platform’s existing position (Zhang et al., 2026). In ad-supported streaming and television markets, advertisers value larger and more attractive audiences, while viewers differ in their tolerance for advertising and in the extent to which

¹Examples include the U.S. monopolization case against Google Search (<https://www.justice.gov/atr/case/us-and-plaintiff-states-v-google-llc>), the FTC case against Meta/Facebook (<https://www.ftc.gov/legal-library/browse/cases-proceedings/191-0134-facebook-inc-ftc-v-ftc-v-meta-platforms-inc>), the European Commission’s Google Android decision (<https://eur-lex.europa.eu/legal-content/en/TXT/?uri=CELEX:62018TJ0604>), and recent cases or regulatory proceedings involving Apple’s App Store (<https://eur-lex.europa.eu/eli/C/2025/3348/oj/eng>) and Amazon Marketplace (<https://www.ftc.gov/legal-library/browse/cases-proceedings/1910129-1910130-amazoncom-inc-amazon-ecommerce>). All URLs accessed June 2026.

advertising affects their participation decisions (Wilbur, 2008; Wilbur et al., 2013; Ferrer and Richter, 2025). Similar issues arise in app ecosystems and other digital platforms, where pricing on one side and monetization on the other depend on user composition as well as user levels (Anderson and Bedre-Defolie, 2024; Chandra, 2009).

Many of these markets are oligopolistic, and many are served by multi-product firms competing with one another. This raises a set of standard but difficult-to-answer questions. What determines prices and markups in the presence of direct and indirect network effects? How do firms trade off monetization on one side against participation on the other? What role does heterogeneity play in shaping equilibrium prices, quality, and welfare? And how are merger effects, regulatory interventions, or other counterfactuals altered once the feedback between market sides is taken into account?

These questions are central to modern empirical industrial organization and competition policy. Existing results provide important foundations by clarifying key mechanisms in specific settings, but their direct use in quantitative analysis is limited in richer empirical environments. The theoretical literature on two-sided markets established that price structure, rather than just the overall price level, is a central equilibrium object, and that platform pricing depends on cross-group elasticities and indirect network effects (Rochet and Tirole, 2003, 2006; Armstrong, 2006; Rysman, 2009). Theoretical work also clarified the appropriate scope of two-sided market analysis. Wright (2004) highlighted the pitfalls of applying conventional one-sided pricing logic to two-sided platforms, while Filistrucchi et al. (2013) characterized the conditions under which markets should be treated as genuinely two-sided and when a one-sided approach may nonetheless be adequate for policy analysis. A separate literature on network externalities emphasized coordination, tipping, and multiplicity under direct network effects (Katz and Shapiro, 1985; Farrell and Saloner, 1985). More recently, Peitz and Sato (2026) study a tractable model of asymmetric platform oligopoly with two user groups, logit demand, within- and cross-group network effects, and participation fees charged to both groups.

Structural empirical work has made substantial progress in bringing these mechanisms to data, often by focusing on settings with particular market or institutional features. Existing

models often assume that one side is priced at zero, that only one indirect network effect is present, that firms set quantities, or that demand takes a particularly restrictive form (Rysman, 2004; Kaiser and Wright, 2006; Argentesi and Filistrucchi, 2007; Wilbur, 2008; Fan, 2013; Jeziorski, 2014; Chu and Manchanda, 2016; Song, 2021). These features reflect both the modeling choices needed to make specific applications tractable and the focus of prior work on particular markets or institutional settings. Yet competition policy often requires quantitative analysis of environments in which firms set prices on both sides, network effects operate in both directions, same-side effects also matter, and heterogeneity is first-order important. This creates a need for a more general framework that can accommodate these features jointly.

The technical difficulty is that, in the presence of network effects, demand on one side depends on outcomes on the other side, and potentially on participation on the same side as well. Given platform strategies, users must therefore coordinate on participation levels that are themselves equilibrium objects. This creates a coordination problem on the demand side. Strong enough feedback may generate multiple fixed points, so that quantities are not uniquely determined even for given prices (Weyl, 2010; White and Weyl, 2012). Even when demand is unique given prices, firms internalize these feedback effects when setting prices, which creates a second equilibrium problem at the level of the competitive game. For counterfactual analysis, this is not a minor technical issue. If either the user-side coordination game or the firms' pricing game admits multiple equilibria, price and welfare predictions are not well defined.

This paper develops a general framework for two-sided oligopoly with heterogeneous consumers, group-level direct and indirect network effects, and firms that engage in oligopolistic competition. The framework is designed to remain close to standard empirical IO practice. It does not require a specific functional form for demand. Instead, demand on each side can be modeled flexibly, for example using a BLP-type demand system (Berry et al., 1995) on one side or on both sides, possibly combined with a different specification on the other side, and then embedded in a unified two-sided equilibrium model. Because the framework nests standard demand systems within a two-sided equilibrium model, standard one-sided demand and merger frameworks arise as special cases when network effects are absent.

We first study the coordination game for given prices set by the firms. We provide a sufficient condition for existence and uniqueness of equilibrium demand given platform strategies. The condition has a natural economic interpretation: network effects must be sufficiently weak relative to the heterogeneity that disperses choices across users and groups. This result is useful both conceptually and practically. It clarifies when the feedback loops generated by direct and indirect network effects remain well behaved.

This provides the demand-side discipline needed to recover marginal costs from firms' first-order conditions (Rosse, 1970) and to study the competitive game between firms. To do so, we re-express firms' profit functions using the reduced-form demand functions implied by the model, which depend only on prices once participation has been solved out. In the general model, these reduced-form demands are characterized as the unique fixed point of the coordination game, and their price derivatives are given explicitly. This allows us to derive first- and second-order conditions and the derivatives of best responses with respect to rivals' prices. We then use these objects to study the competitive game between firms. Within one-sided multi-product oligopoly, Nocke and Schutz (2018) exploit the aggregative structure of logit and CES demand to characterize equilibrium pricing and merger effects. We instead derive sufficient conditions for existence and uniqueness of equilibrium in the pricing game under flexible demand, extending familiar best-response arguments from one-sided differentiated-product oligopoly to environments with direct and indirect network effects (Vives, 1999; Mizuno, 2003).

The remainder of the paper is organized as follows. Section 2 presents the general model and some illustrative examples. Section 3 studies existence and uniqueness in the coordination game. Section 4 studies existence and uniqueness in the pricing game between firms. Section 5 concludes.

2 Model structure

2.1 General structure

There are two market sides with J products on each side, so $2 \cdot J$ products in total.² Moreover, there are discrete groups of users on each market side. For example, two online platforms each offer access to G^u different groups of users (where “u” stands for users). These groups could be defined by combinations of demographic characteristics measuring socio-economic status, age, gender, or location. There are G^a groups of advertisers. The groups of advertisers could be defined by the type of product they sell, whether they focus on high quality or low prices, location, the way they sell, and so on. We will say that in this example, there are $J \cdot (G^u + G^a)$ product group pairs, or demand components.

Let $\mathcal{S}$ denote the set of market sides, with $|\mathcal{S}| = 2$. We generally denote elements of this set by lowercase letters. We often use advertisers a in combination with users u , or advertisers a in combination with readers r . For each market side $s \in \mathcal{S}$, let K_s denote the number of demand components on side s . Thus,

$$K = \sum_{s \in \mathcal{S}} K_s$$

is the total number of demand components. All demand components are collected in the vector $q \in \mathbb{R}_+^K$ and ordered by market side, then product, and then group.

Demand is allowed to depend on product characteristics, including prices. In the competitive game between firms that we study in Section 4 below, firms set prices.³ Prices are uniform across groups within a product.⁴ Prices are collected in the column vector p of length $2 \cdot J$. For each market side $s \in \mathcal{S}$, let $P^s \subseteq \mathbb{R}^J$ denote the feasible set of prices on that side, with typical element p^s , and let

$$P = \prod_{s \in \mathcal{S}} P^s \subseteq \mathbb{R}^{2J}$$

²We restrict attention to two sides in order to keep the notation manageable. The arguments below should extend to more than two sides, but doing so would substantially complicate the exposition.

³Instead of prices, firms could also set other product characteristics such as, e.g., the quality of reporting offered to readers as in Fan (2013), or the set of features on an online platform available to users. The framework developed here can be used in such situations as well.

⁴Group-specific prices can be accommodated by redefining each product-group pair as a separate product with its own uniform price.

denote the full feasible set of price vectors. We assume that P is nonempty, compact, and convex, and we allow P to contain negative prices. This is a technical strategy-space assumption used in the analysis of price competition below and is not restrictive in typical empirical applications, provided P is chosen to contain the relevant range of prices.⁵ For the example of an online platform, this means that subscription prices are uniform across user groups for the same subscription plan and that advertising prices do not depend on the type of advertiser, as is common in practice. Negative prices in this context would mean that users are paid for participating.

Demand is defined for all $p \in P$ and $q \in \mathbb{R}_+^K$. For a given product-group pair k , market demand is described by a function

$$q_k : \mathbb{R}^J \times \mathbb{R}_+^K \rightarrow \mathbb{R}_+.$$

It maps the vector of product prices $p^{s(k)}$ on the same market side and the vector of demand components q on both market sides into a scalar demand level:

$$q_k = q_k(p^{s(k)}, q).$$

Thereby, we allow for both *direct* and *indirect* network effects. We call network effects direct if they occur on the same market side (e.g., demand for usage of a platform depends on demand by other users), and indirect if they occur across market sides (e.g., usage depends on advertising).

Stacking these demand components yields the vector-valued demand function

$$q : \mathbb{R}^{2J} \times \mathbb{R}_+^K \rightarrow \mathbb{R}_+^K.$$

We assume throughout that $q = q(p, q)$ is continuously differentiable.

For the fixed-point arguments below, it is useful to work on feasible quantity sets. For each

⁵ P can be interpreted as a large box of admissible prices. Compactness imposes lower and upper bounds on prices, while convexity requires that mixtures of feasible price vectors are feasible. Compactness ensures that firms' best-response problems have solutions under continuous profits, while convexity is convenient for first-order and contraction arguments. The bounds can be chosen to include negative prices when firms may subsidize participation.

market side $s \in \mathcal{S}$, let $Q^s \subseteq \mathbb{R}_+^{K_s}$ denote the feasible set of quantities on that side. Since $|\mathcal{S}| = 2$, we write $-s$ for the other market side. Let

$$Q \equiv \prod_{s \in \mathcal{S}} Q^s$$

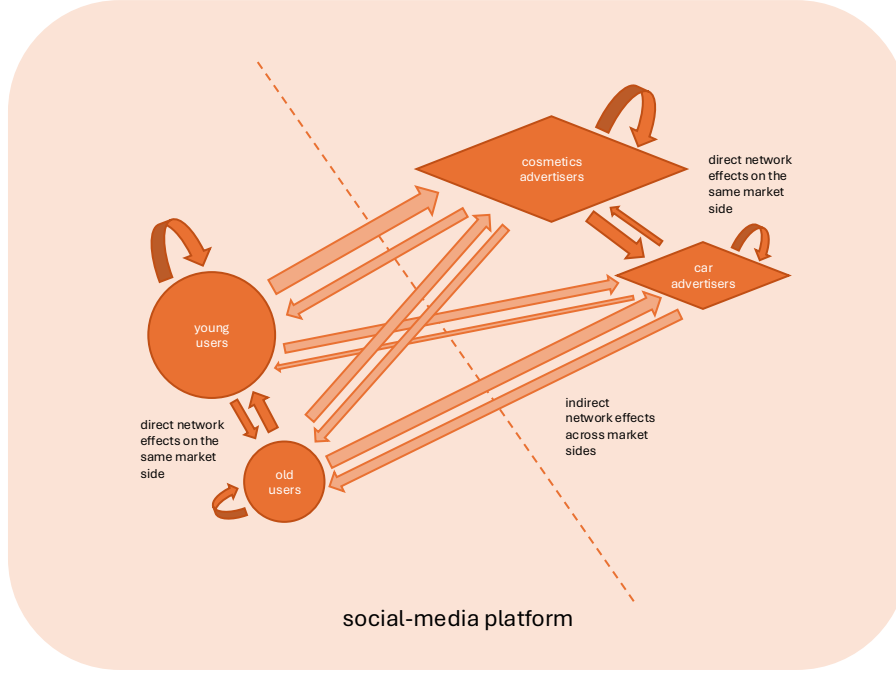
denote the feasible set of all quantity vectors. We assume that each Q^s is nonempty, closed, and convex, and that, for every p^s and $q^{-s} \in Q^{-s}$, the map

$$q^s \mapsto q^s(p^s, q^s, q^{-s})$$

maps Q^s into itself. The notation Q^s only specifies the domain on which the fixed-point arguments are applied. The demand functions themselves remain defined on $\mathbb{R}_+^K$. This technical condition ensures that the fixed-point arguments below are applied on a complete feasible quantity space. It is not restrictive for the demand specifications considered in this paper, and it is satisfied by standard linear, log-linear, and BLP-style demand systems after choosing a feasible quantity set that contains the relevant range of demand.

Figure 1 shows the types of network effects we will take into account. The figure is for one platform. There are 2 groups of users, young and old, and two types of advertisers, for cosmetics and for cars. There are direct network effects within each market side and indirect network effects between market sides. Notice that both direct and indirect network effects are allowed to be heterogeneous. Cosmetics advertisers may care more about young users, while car advertisers may care more about old users. Also direct network effects are allowed to be heterogeneous. Young users may care more about other young users than about old users on the same platform.

The figure shows that feedback loops arise on the platform. Suppose some young users leave the platform. Then, this will have an effect on the same market side, as it may lead to other young users leaving, and likewise some old users. This then probably triggers more users to leave. In addition, advertisers may leave the platform, and in response some other users may join again.



Notes: This figure shows the types of network effects that we take into account. It is drawn for the special case of a single social-media platform. There are two user groups and two groups of advertisers.

Figure 1: Network effects on a social-media platform

2.2 Discussion

In our model, demand depends on prices and quantities. Unlike in some models with network effects, the coordination game and the expectations that play a role in it are not modeled explicitly. This choice is deliberate and in line with most empirical models of two-sided markets (e.g., Rysman, 2004; Kaiser and Wright, 2006; Argentesi and Filistrucchi, 2007; Wilbur, 2008; Fan, 2013; Jeziorski, 2014; Chu and Manchanda, 2016; Song, 2021). It allows us to use the empirical IO toolkit, including demand models like BLP.

Our model is best interpreted as a steady-state representation of a richer dynamic process (Teh and Wright, 2024). This interpretation follows the fixed-point logic used in the literature on network effects. In models of direct network effects, users' willingness to participate depends on the participation of other users, so equilibrium demand must be consistent with the network size that users expect or take as relevant for their choices (Rohlf, 1974; Katz and Shapiro, 1985; Cabral, 1990). In models of indirect network effects and two-sided markets, the same

logic applies across groups: participation on one side affects the attractiveness of the platform to the other side, giving rise to the familiar chicken-and-egg problem of getting both sides on board (Caillaud and Jullien, 2003; Rochet and Tirole, 2003, 2006; Armstrong, 2006). Related work shows that equilibrium predictions can depend on the expectations regime, for example on whether users' expectations respond to platform choices or remain locally fixed (Hagiu and Halaburda, 2014). In the steady state, expectations are met and equal to the realized quantities.

There are two closely related, but conceptually distinct, ways to interpret this fixed-point condition. One is a fulfilled-expectations interpretation: users choose taking as given beliefs about participation, and equilibrium requires those beliefs to be correct. The other is a steady-state interpretation: participation today affects the attractiveness of the platform in the future, and equilibrium demand is the limit point of this adjustment process. In both cases, equilibrium demand solves a fixed-point problem. The difference is that the first interpretation places the fixed point in users' beliefs, while the second places it in an adjustment process for demand itself.

Earlier contributions have emphasized that network effects can generate multiple fulfilled-expectations or adoption equilibria. In this paper, in Sections 3 and 4, we will focus on the case in which the coordination game has a unique equilibrium for given platform strategies. This is a deliberate choice, but it is less restrictive than it may seem. Uniqueness does not rule out strong network effects or tipping-like outcomes; it rules out multiple demand equilibria for the same vector of platform strategies. The unique equilibrium may still be highly concentrated, and small changes in prices or product characteristics may generate large shifts in participation. Thus, the framework can accommodate positive-feedback mechanisms associated with tipping, while focusing on cases in which these mechanisms lead to a unique quantitative prediction. This distinction is important in empirical work, where tipping is often understood as concentration or sharp reallocation generated by network effects rather than as equilibrium multiplicity itself (Katz and Shapiro, 1994; Dubé et al., 2010). When equilibrium demand is unique, the model delivers well-defined counterfactual predictions for mergers, regulation, and other interventions. By contrast, when multiple demand equilibria are possible, counterfactual

analysis becomes conceptually more difficult, since one must also specify how equilibria are selected. Our contribution is to provide sufficient conditions for uniqueness in both the coordination game and the pricing game. These conditions make the restriction operational: they allow researchers to check when the feedback loops generated by direct and indirect network effects are sufficiently well behaved to support unique quantitative predictions.

2.3 Examples

In this section, we will provide three examples of special cases of our model. The first two are used below in Sections 3 and 4 to develop the intuition behind our general results and the higher-level conditions we use for this. They have linear demand systems. This will be particularly useful, because it allows us to explicitly solve for reduced-form demands using substitution or methods from linear algebra. Armstrong (2006) and Kaiser and Wright (2006) provide a micro foundation and derive similar linear demand systems from a Hotelling framework with covered markets, but do not incorporate direct network effects.

These first two examples do not involve groups of users. It will become clear below in Sections 3.4.2 and 4.4.1 why that is. Already in a relatively simple example with 2 platforms and 2 market sides, but without groups, expressions easily become too rich and it is very hard to develop an intuition for this. At the same time, the intuition can, in our view, be developed in relatively simple examples and then carries over to the higher-level conditions that we provide for the general model.

The third example is motivated by empirical practice, where variants of the multinomial logit model, most prominently Berry et al. (1995), are often used. In this example, groups arise naturally when product characteristics are interacted with demographics. The general results developed in Sections 3 and 4 are useful for using this model in a context where demand does not only depend on prices, but also quantities on the same market side and on the other market side. Formally, these demands enter the model as characteristics.

2.3.1 A newspaper with linear demand

In the earlier literature on two-sided markets, newspapers often served as one of the leading examples or the market that was modeled (see, e.g., Rochet and Tirole, 2003, 2006; Armstrong, 2006; Argentesi and Filistrucchi, 2007; Fan, 2013). Here, we study a very simple linear model with one newspaper publisher selling one newspaper to readers and advertising in it to advertisers. Advertisers care about the number q^r of readers. Readers care about the amount q^a of advertising. Prices for one unit of advertising are p^a and prices for one copy of the newspaper are p^r . Demands are linear and given by

$$\begin{aligned} q^a(p^a, q^r) &= \alpha^a - \beta^a p^a + \gamma^a q^r + \theta^a q^a \\ q^r(p^r, q^a) &= \alpha^r - \beta^r p^r + \gamma^r q^a + \theta^r q^r, \end{aligned} \tag{1}$$

with $\beta^a, \beta^r > 0$. This model is more general than previously used models, because we also allow for direct network effects θ^a and θ^r .

Direct network effects on the advertiser side capture advertising congestion or clutter: when more advertisers are active on a platform, they compete for scarce user attention, thereby reducing the effectiveness of each advertisement. This mechanism is closely related to Anderson and Peitz (2023), who model advertising congestion in media markets, and is also supported by empirical evidence on advertising clutter and attention competition, such as Pieters et al. (2002) and Jung and Heo (2021).

A direct readership-side network effect can be motivated by social interactions in news consumption: readers may value a newspaper more when more people in their relevant social group read it, because this increases opportunities for discussion, sharing, and common information. This mechanism is related to the preference externalities documented by George and Waldfogel (2003) in daily newspaper markets and to George and Peukert (2019), who show that social networks affect the demand for news.

2.3.2 Two social media platforms with linear demand

This example is motivated by recent cases, such as the FTC’s monopolization case against Meta, which alleges that Facebook maintained monopoly power in personal social networking services by acquiring Instagram and WhatsApp as emerging competitive threats.⁶ It is also motivated by recent work on competition and network effects in social-media markets, such as Iansiti (2021) and Bursztyn et al. (2025). The advertising side is motivated by work on social-media advertising and advertising-market competition, such as Tucker (2014), Gentzkow et al. (2024), and Katz and Allcott (2025). Indeed, Katz and Allcott (2025) study competition between Facebook and Instagram with targeted advertising and endogenous ad load. These papers motivate the economic forces captured in the example, but they do not use a model with a demand system like the one considered here.⁷ The example shows that the empirical framework developed in this paper can contribute to this literature by representing social-media platform competition through an explicit two-platform demand system in which own and rival quantities enter demand directly.

Consider two social media platforms that are financed by advertising and possibly also by subscription fees. There are direct and indirect network effects. In the context of advertising, indirect network effects arise when advertising demand depends on the number of users. These are the classical indirect network effects studied in the advertising literature (see, e.g., Rochet and Tirole, 2003; Rysman, 2004; Anderson and Coate, 2005; Kaiser and Wright, 2006; Anderson and Gabszewicz, 2006; Armstrong, 2006; Argentesi and Filistrucchi, 2007). Direct network effects on the advertiser side capture advertising congestion or clutter, as discussed in Section 2.3.1 above. Indirect network effects arise on the user side when users care about the amount of

⁶See Federal Trade Commission v. Meta Platforms, Inc., No. 1:20-cv-03590 (D.D.C.), Federal Trade Commission case page: <https://www.ftc.gov/legal-library/browse/cases-proceedings/191-0134-facebook-inc-ftc-v-ftc-v-meta-platforms-inc> (accessed June 2026).

⁷Katz and Allcott (2025) model user demand as being affected by ad load and allow advertisers to value access to users, with user heterogeneity, user overlap, targeting, advertiser substitution, and diminishing returns to advertising determining equilibrium ad load. Unlike the demand system considered here, however, they do not incorporate direct network effects through own- and rival-platform quantities on either the user side or the advertiser side. Moreover, unlike in this paper, platforms choose advertising intensity rather than advertising prices. Therefore, as in quantity-based models of advertising-financed media such as Rysman (2004), advertising quantities are taken as given from the user-demand perspective, whereas in the present framework advertising quantities are determined jointly with user quantities through a simultaneous demand system.

advertising, for example because advertising imposes a nuisance cost or affects the user experience.⁸ Direct network effects on the user side are central to social-media platforms: the value of joining or using a platform may increase with the number of other users with whom one can interact. This mechanism is classical in the economics of network effects; see Katz and Shapiro (1985), Farrell and Saloner (1985), and Shapiro and Varian (1999), and is emphasized in recent work on social-media competition such as Iansiti (2021) and Bursztyn et al. (2025).

Demand for advertising and usage is respectively linear in all prices on the same side and all quantities,

$$\begin{aligned}
q_1^a(p_1^a, p_2^a, q_1^a, q_2^a, q_1^u, q_2^u) &= \alpha_1^a - \beta_{11}^a p_1^a + \beta_{12}^a p_2^a + \gamma_{11}^a q_1^u + \gamma_{12}^a q_2^u + \theta_{11}^a q_1^a + \theta_{12}^a q_2^a \\
q_2^a(p_1^a, p_2^a, q_1^a, q_2^a, q_1^u, q_2^u) &= \alpha_2^a + \beta_{21}^a p_1^a - \beta_{22}^a p_2^a + \gamma_{21}^a q_1^u + \gamma_{22}^a q_2^u + \theta_{21}^a q_1^a + \theta_{22}^a q_2^a \\
q_1^u(p_1^u, p_2^u, q_1^a, q_2^a, q_1^u, q_2^u) &= \alpha_1^u - \beta_{11}^u p_1^u + \beta_{12}^u p_2^u + \gamma_{11}^u q_1^a + \gamma_{12}^u q_2^a + \theta_{11}^u q_1^u + \theta_{12}^u q_2^u \\
q_2^u(p_1^u, p_2^u, q_1^a, q_2^a, q_1^u, q_2^u) &= \alpha_2^u + \beta_{21}^u p_1^u - \beta_{22}^u p_2^u + \gamma_{21}^u q_1^a + \gamma_{22}^u q_2^a + \theta_{21}^u q_1^u + \theta_{22}^u q_2^u.
\end{aligned}$$

The price coefficients β_{jk}^s are positive, $s = a, u$ and $j, k = 1, 2$. The γ_{jk}^s 's are indirect network effects. The θ_{jk}^s 's are direct network effects.

2.3.3 J platforms with logit demand

In empirical practice, logit models are often used, most notably Berry et al. (1995) with random coefficients and demographics. Here, we present a simple version of a logit model to illustrate that this would also be a feasible and natural choice for our empirical framework.^{9,10}

⁸This mechanism is standard in advertising-financed media models. In television markets, Wilbur (2008) provides empirical evidence that viewers are strongly advertising-averse. In print media, however, Kaiser and Song (2009) find little evidence that readers generally dislike advertising; in many magazine segments, readers appear to value advertising. In online and social-media markets, the evidence is more mixed. Zeng et al. (2021) document negative user responses to online advertising, whereas Brynjolfsson et al. (2025) find no significant effect of removing newsfeed ads on users' valuation of Facebook.

⁹Peitz and Sato (2026) study a related model of asymmetric platform oligopoly with two user groups, logit demand, and within- and cross-group network effects. Their logarithmic specification of network effects gives rise to an aggregative-game structure that facilitates the characterization of participation and pricing equilibria. The purpose of the present example is different: it illustrates how logit demand can be embedded in our more general empirical framework, which allows flexible demand systems, multiple groups on each market side, heterogeneous direct and indirect network effects, and multiproduct ownership, and is designed for marginal-cost recovery and counterfactual analysis.

¹⁰It is conceptually straightforward to add random coefficients and demographics. If demographics are added and they are discrete (such as high and low income, or old and young users), then the model generates demand at

There are J platforms. Advertisers think about where to buy the next unit of advertising. The profitability of advertising on platform j is

$$u_{ij}^a = \alpha_j^a - \beta^a p_j^a + \gamma^a q_j^u + \theta^a q_j^a + \varepsilon_{ij}^a$$

and $u_{i0}^a = \varepsilon_{i0}^a$ for not advertising.¹¹

Users choose exactly one platform. The utility from platform j is

$$u_{ij}^u = \alpha_j^u - \beta^u p_j^u + \gamma^u q_j^a + \theta^u q_j^u + \varepsilon_{ij}^u$$

and $u_{i0}^u = \varepsilon_{i0}^u$ from not using any platform. Advertisers consider buying M^a units of advertising. There are M^u potential users.

Assume that ε_{ij}^a and ε_{ij}^u are distributed type I extreme value. Then the corresponding demand functions have the following logit form,

$$q_j^a = M^a \cdot \frac{\exp(\alpha_j^a - \beta^a p_j^a + \gamma^a q_j^u + \theta^a q_j^a)}{1 + \sum_{j'=1}^J \exp(\alpha_{j'}^a - \beta^a p_{j'}^a + \gamma^a q_{j'}^u + \theta^a q_{j'}^a)}$$

and

$$q_j^u = M^u \cdot \frac{\exp(\alpha_j^u - \beta^u p_j^u + \gamma^u q_j^a + \theta^u q_j^u)}{1 + \sum_{j'=1}^J \exp(\alpha_{j'}^u - \beta^u p_{j'}^u + \gamma^u q_{j'}^a + \theta^u q_{j'}^u)}.$$

Note that in the general case, this system cannot easily be solved explicitly for quantity as a function of prices on both sides. The reason is that the system is nonlinear and advertising demand depends on user demand, which depends again on advertising demand, and so on. Argentesi and Filistrucchi (2007) use this model in the context of newspapers, where users are readers, and impose that readers do not care about the amount of advertising. This substantially simplifies the analysis and makes the model tractable, because the “feedback loop” between

the group level.

¹¹In a more general logit model with demographics, this profitability could depend on quantities for all demographics, as illustrated in Figure 1. For instance, the profitability of cosmetics advertisements could depend on the amount of old users and the amount of young users. It could also depend on the number of other cosmetics advertisers and the number of car advertisers on the platform. This means that in the example with two groups on each side, we would have 4 indirect network effects on the advertising side instead of γ^a and four direct network effects instead of θ^a . They would respectively run from each group to each group.

market sides is broken and they can work with the model. Rysman (2004) estimates a similar logit model for the demand for yellow page directories. This is also a two-sided market, because the demand for advertising in those directories depends on the number of users of these directories. He assumes that yellow page directories choose the amount of advertising in the directories. Therefore, there is no feedback loop and he can treat the amount of advertising as given when modeling user demand.

3 Existence and uniqueness in the coordination game

In this section, we develop general results on the existence of equilibrium in the coordination game. We provide sufficient conditions under which there is a unique set of quantities given prices on all market sides.

One could in principle provide high-level conditions on the combination of direct and indirect network effects. However, they are hard to interpret. Our approach is to instead first study uniqueness in the coordination game, separately for each side, for given prices and given quantities on the respective other market side.

We provide a sufficient condition that is similar to the one in the literature on peer effects, where only direct network effects are present. Intuitively, the condition is that direct network effects are not too strong. Subsequently, in a second step, we provide a sufficient condition for uniqueness in the coordination game between sides. Intuitively, it is that the indirect network effects between sides are not too strong.

3.1 General results

By abuse of notation, we use q both for the mapping and its equilibrium value. The equilibrium condition in the coordination game (for given choices p of firms) is therefore

$$q = q(p, q).$$

First, we turn to direct network effects and solve for all the q_j 's taking the q_k 's on the

other market side(s) as given. For example, when there are advertisers and users of a social-media platform and both indirect and direct network effects, like in Section 2.3.2, there is a vector of advertising prices p^a , a vector of subscription fees p^u for users, a vector of advertising quantities q^a , and a vector q^u of user quantities. Each of these vectors has as many elements as there are platforms. Advertising demands are $q^a(p^a, q^a, q^u)$ and user demands are $q^u(p^u, q^a, q^u)$. In the first step, we state testable conditions under which we can solve for unique reduced form demand functions $\tilde{q}^a(p^a, q^u)$ and $\tilde{q}^u(p^u, q^a)$.

For the general case, consider a product k . Collect the prices for products on the same market side in vector p^s , the quantities on the same market side in vector q^s , and the quantities on all other market sides in vector q^{-s} . Then, we have $q_k(p^s, q^s, q^{-s})$ and our aim is to provide a condition so that the reduced-form quantities $\tilde{q}_k(p^s, q^{-s})$ exist. For each side $s \in \mathcal{S}$, we will denote the side-specific reduced-form quantities by $\tilde{q}^s(p^s, q^{-s})$. Collecting these side-specific reduced forms yields the vector $\tilde{q}(p, q)$.

Assumption 1 (direct network effects). *For every market side $s \in \mathcal{S}$, every product-group pair k on side s , every vector of prices p^s , and every vector of quantities $q^{-s} \in Q^{-s}$, the direct within-side network effects are uniformly bounded away from one:*

$$\beta_s(p^s, q^{-s}) \equiv \max_{k \text{ on side } s} \sup_{q^s \in Q^s} \left\{ \sum_{k' \text{ on side } s} \left| \frac{\partial q_k(p^s, q^s, q^{-s})}{\partial q_{k'}} \right| \right\} < 1.$$

The condition requires that, for given prices p^s and quantities q^{-s} on the other market side, the largest row sum of absolute within-side derivatives, uniformly over all within-side quantities $q^s \in Q^s$, is strictly less than one.

Assumption 1 is a familiar sufficient condition for uniqueness in models with direct network effects (see, e.g., Glaeser and Scheinkman, 2003; Bayer and Timmins, 2005; Horst and Scheinkman, 2006). It can be viewed as a nonlinear analogue of standard restrictions in linear peer-effects and network models that limit the strength of endogenous interactions and ensure that the reduced form is uniquely defined (see, e.g., Ballester et al., 2006; Bramoullé et al., 2009).

The following lemma establishes existence and uniqueness of the side-specific reduced-form

quantities, taking quantities on the other market side as given.

Lemma 1 (existence and uniqueness of $\tilde{q}$). *Let Assumption 1 hold. Fix a market side $s \in \mathcal{S}$, prices p^s , and quantities $q^{-s} \in Q^{-s}$ on the other market side. Then there exists a unique vector $\tilde{q}^s(p^s, q^{-s}) \in Q^s$ that solves*

$$q^s = q^s(p^s, q^s, q^{-s}).$$

Equivalently,

$$\tilde{q}^s(p^s, q^{-s}) = q^s(p^s, \tilde{q}^s(p^s, q^{-s}), q^{-s}).$$

Moreover, for any $q_0 \in Q^s$, the sequence defined by

$$q_{n+1} = q^s(p^s, q_n, q^{-s})$$

converges to $\tilde{q}^s(p^s, q^{-s})$.

Proof. Consider changing the quantities q^s on the same market side from vector x to vector y , where $x, y \in Q^s$. Since $q_k(p^s, q^s, q^{-s})$ is continuously differentiable in its second argument, by the fundamental theorem for line integrals, we have

$$q_k(p^s, y, q^{-s}) - q_k(p^s, x, q^{-s}) = \int_x^y \frac{\partial q_k(p^s, q^s, q^{-s})}{\partial q^{s'}} dq^s,$$

with the understanding that this means a line integral along some path from x to y . Because $\frac{\partial q_k(p^s, q^s, q^{-s})}{\partial q^{s'}}$ is the gradient of a scalar function with respect to quantities on the same market side, the integral is path independent. We therefore choose the straight-line path $q^s(t) = x + t(y - x)$, $t \in [0, 1]$, which lies in Q^s by convexity.

We have

$$\begin{aligned}
& |q_k(p^s, y, q^{-s}) - q_k(p^s, x, q^{-s})| \\
&= \left| \int_0^1 \sum_{k' \text{ on side } s} \frac{\partial q_k(p^s, x + t(y-x), q^{-s})}{\partial q_{k'}^s} (y_{k'} - x_{k'}) dt \right| \\
&\leq \int_0^1 \sum_{k' \text{ on side } s} \left| \frac{\partial q_k(p^s, x + t(y-x), q^{-s})}{\partial q_{k'}^s} \right| |y_{k'} - x_{k'}| dt \\
&\leq \int_0^1 \sum_{k' \text{ on side } s} \left| \frac{\partial q_k(p^s, x + t(y-x), q^{-s})}{\partial q_{k'}^s} \right| \|y - x\|_\infty dt \\
&\leq \int_0^1 \beta_s(p^s, q^{-s}) \cdot \|y - x\|_\infty dt \\
&= \beta_s(p^s, q^{-s}) \cdot \|y - x\|_\infty.
\end{aligned}$$

The first equality follows from the line-integral representation using the chosen straight-line path from x to y . The first inequality follows from the triangle inequality. The second uses $|y_{k'} - x_{k'}| \leq \|y - x\|_\infty$ for all k' . The third follows from Assumption 1. The final equality follows from the fact that $\beta_s(p^s, q^{-s}) \cdot \|y - x\|_\infty$ is constant in t .

Taking the maximum over all k on side s , we obtain

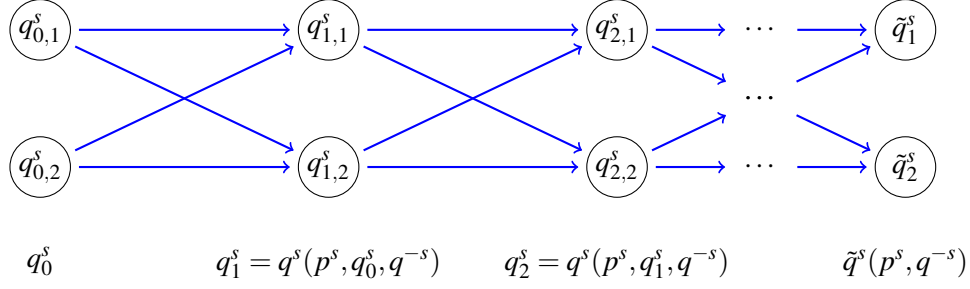
$$\|q^s(p^s, y, q^{-s}) - q^s(p^s, x, q^{-s})\|_\infty \leq \beta_s(p^s, q^{-s}) \|y - x\|_\infty.$$

Since $\beta_s(p^s, q^{-s}) < 1$, the map

$$q^s \mapsto q^s(p^s, q^s, q^{-s})$$

is a contraction on Q^s . Since Q^s is closed in $\mathbb{R}^{K_s}$, it is complete. By the contraction mapping theorem, this map has a unique fixed point in Q^s . This fixed point is $\tilde{q}^s(p^s, q^{-s})$. Moreover, for any $q_0 \in Q^s$, the sequence defined by $q_{n+1} = q^s(p^s, q_n, q^{-s})$ converges to $\tilde{q}^s(p^s, q^{-s})$. This completes the proof. $\square$

Figure 2 illustrates. We start with a vector of initial quantities q_0^s with elements $q_{0,1}^s$ and $q_{0,2}^s$. Primitive demand on market side s depends only on prices on that market side. We iterate on $q^s(p^s, \cdot, q^{-s})$, holding all prices on that market side in the vector p^s and all quantities on the other market side in the vector q^{-s} fixed. Under Assumption 1, quantities converge to $\tilde{q}^s(p^s, q^{-s})$.



Notes: This figure illustrates that, starting from an initial vector of quantities q_0^s on market side s , repeated application of the mapping $q^s(p^s, \cdot, q^{-s})$, for given prices on market side s , p^s , and quantities on the other market side $-s$, q^{-s} , converges to the unique fixed point $\tilde{q}^s(p^s, q^{-s})$. By Lemma 1, this convergence holds provided that Assumption 1 is satisfied. The figure is drawn for the special case in which there are 2 demand components on market side s .

Figure 2: Solving for reduced-form quantities $\tilde{q}^s(p^s, q^{-s})$

Next, we show how we can obtain derivatives $\tilde{B}(p, q) \equiv \frac{\partial \tilde{q}(p, q)}{\partial p'}$ of these reduced-form quantities with respect to prices, and $\tilde{\Gamma}(p, q) \equiv \frac{\partial \tilde{q}(p, q)}{\partial q'}$ of these reduced-form quantities with respect to quantities on the respective other market side.

Define the matrix $B(p, q)$ of price effects with elements $b_{k,k'}(p, q) = \frac{\partial q_k(p^s, q^s, q^{-s})}{\partial p_{k'}}$, the matrix $\Theta(p, q)$ of direct network effects with elements $\theta_{k,k'}(p, q) = \frac{\partial q_k(p^s, q^s, q^{-s})}{\partial q_{k'}^s}$, and the matrix $\Gamma(p, q)$ of indirect network effects with elements $\gamma_{k,k'}(p, q) = \frac{\partial q_k(p^s, q^s, q^{-s})}{\partial q_{k'}^{-s}}$.

$B(p, q)$ is block diagonal with off-diagonal blocks equal to zero. If there are two market sides, it is

$$B(p, q) = \begin{pmatrix} B_{11}(p, q) & 0 \\ 0 & B_{22}(p, q) \end{pmatrix}.$$

$\Theta(p, q)$ is block diagonal with off-diagonal blocks equal to zero. If there are two market sides, it is

$$\Theta(p, q) = \begin{pmatrix} \Theta_{11}(p, q) & 0 \\ 0 & \Theta_{22}(p, q) \end{pmatrix}.$$

$\Gamma(p, q)$ is block diagonal with diagonal blocks equal to zero. If there are two market sides, it is

$$\Gamma(p, q) = \begin{pmatrix} 0 & \Gamma_{12}(p, q) \\ \Gamma_{21}(p, q) & 0 \end{pmatrix}.$$

Lemma 2 (derivatives of $\tilde{q}$). *Let Assumption 1 hold. Then, (i) derivatives of the reduced-form quantities with respect to prices are given by $\tilde{B}(p, q) = (I - \Theta(p, q))^{-1}B(p, q)$, and (ii) derivatives of the reduced-form quantities with respect to quantities on the respective other market sides are given by $\tilde{\Gamma}(p, q) = (I - \Theta(p, q))^{-1}\Gamma(p, q)$.*

Proof. Lemma 1 establishes that the reduced-form quantities $\tilde{q}_k(p^s, q^{-s})$ exist. Stack them into the vector $\tilde{q}(p, q)$. For the two-side block representation below, let q_1 and q_2 denote the subvectors of quantities on market sides 1 and 2, respectively. We first turn to derivatives with respect to quantities. If we hold prices fixed, the reduced-form fixed-point equation can be written, by abuse of notation, as

$$\tilde{q}(p, q) - q = 0,$$

where the q in the argument of $\tilde{q}(p, q)$ denotes quantities on the respective other market side, while the second q denotes the corresponding reduced-form quantities. Its total derivative is

$$\frac{\partial (\tilde{q}(p, q) - q)}{\partial q'} dq = 0. \quad (2)$$

We can rewrite this as

$$\begin{pmatrix} \Theta_{11}(p, q) - I & 0 \\ 0 & \Theta_{22}(p, q) - I \end{pmatrix} \begin{pmatrix} d\tilde{q}_1 \\ d\tilde{q}_2 \end{pmatrix} + \begin{pmatrix} 0 & \Gamma_{12}(p, q) \\ \Gamma_{21}(p, q) & 0 \end{pmatrix} \begin{pmatrix} dq_1 \\ dq_2 \end{pmatrix} = 0.$$

The first term captures changes in the reduced-form quantities on the same market side. The second term captures changes in quantities on the respective other market side, since $\Gamma(p, q)$ has zero diagonal blocks.

Assumption 1 implies that each block $I - \Theta_{ss}(p, q)$ is invertible, and hence the block-

diagonal matrix

$$\begin{pmatrix} I - \Theta_{11}(p, q) & 0 \\ 0 & I - \Theta_{22}(p, q) \end{pmatrix}$$

is invertible. Therefore, we have

$$\begin{pmatrix} d\tilde{q}_1 \\ d\tilde{q}_2 \end{pmatrix} = \begin{pmatrix} I - \Theta_{11}(p, q) & 0 \\ 0 & I - \Theta_{22}(p, q) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Gamma_{12}(p, q) \\ \Gamma_{21}(p, q) & 0 \end{pmatrix} \begin{pmatrix} dq_1 \\ dq_2 \end{pmatrix}.$$

Recall that $\tilde{q}(p, q)$ consists of elements $\tilde{q}_k(p^s, q^{-s})$. Hence, the change in reduced form quantities $\tilde{q}(p, q)$ when the quantity arguments change is

$$\begin{aligned} \frac{\partial \tilde{q}(p, q)}{\partial q'} &= \begin{pmatrix} I - \Theta_{11}(p, q) & 0 \\ 0 & I - \Theta_{22}(p, q) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Gamma_{12}(p, q) \\ \Gamma_{21}(p, q) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (I - \Theta_{11}(p, q))^{-1} \Gamma_{12}(p, q) \\ (I - \Theta_{22}(p, q))^{-1} \Gamma_{21}(p, q) & 0 \end{pmatrix}. \end{aligned} \quad (3)$$

This can also be written more compactly as

$$\frac{\partial \tilde{q}(p, q)}{\partial q'} = (I - \Theta(p, q))^{-1} \Gamma(p, q).$$

Note that the proof also shows in (3) that $\tilde{\Gamma}(p, q)$ is a block matrix with only off-diagonal blocks. We can write this as

$$\tilde{\Gamma}(p, q) = \begin{pmatrix} 0 & \tilde{\Gamma}_{12}(p, q) \\ \tilde{\Gamma}_{21}(p, q) & 0 \end{pmatrix}.$$

Next, we turn to derivatives with respect to prices. Holding quantities on the other market side fixed, by arguments paralleling the ones above, the total derivative of

$$\tilde{q}(p, q) - q = 0$$

is

$$\begin{pmatrix} \Theta_{11}(p, q) - I & 0 \\ 0 & \Theta_{22}(p, q) - I \end{pmatrix} \begin{pmatrix} d\tilde{q}_1 \\ d\tilde{q}_2 \end{pmatrix} + \begin{pmatrix} B_{11}(p, q) & 0 \\ 0 & B_{22}(p, q) \end{pmatrix} \begin{pmatrix} dp_1 \\ dp_2 \end{pmatrix} = 0.$$

Hence,

$$\tilde{B}(p, q) = \frac{\partial \tilde{q}(p, q)}{\partial p'} = (I - \Theta(p, q))^{-1} B(p, q).$$

This proves the result. $\square$

Next, we study existence and uniqueness of the reduced-form quantities given prices only. The following assumption formalizes in which sense we require that network effects are not too strong. To make this explicit, we now write $s(k)$ and $-s(k)$ to denote the same and other market side, respectively, from the perspective of product k . So, e.g., $q^{-s(k)}$ is the vector of quantities on the other market side from the perspective of product k . $q_{k'}^{-s(k)}$ denotes product k' in that vector, and so on.

Assumption 2 (indirect network effects). *For every vector of prices p , the indirect network effects are uniformly bounded away from one:*

$$\alpha(p) \equiv \max_k \sup_{q \in Q} \left\{ \sum_{k''} \left| \sum_{k'} \frac{\partial \tilde{q}_k(p^{s(k)}, q^{-s(k)})}{\partial q_{k'}^{-s(k)}} \cdot \frac{\partial \tilde{q}_{k'}(p^{s(k')}, q^{-s(k')})}{\partial q_{k''}^{-s(k')}} \right| \right\} < 1.$$

In words, the condition bounds the strength of two-step feedback effects across market sides. A change in quantity $q_{k''}$ affects the reduced-form quantity $\tilde{q}_{k'}$ on the other side, which in turn affects $\tilde{q}_k$. The inner sum aggregates these two-step channels from k'' to k . The condition requires the largest row sum of the absolute values of these two-step effects, uniformly over feasible quantities $q \in Q$, to be strictly less than one.

Proposition 1 (existence and uniqueness of $\hat{q}$). *For any vector of prices p , there is a unique quantity vector $\hat{q}(p) \in Q$ if Assumptions 1 and 2 hold. Moreover, for any $q_0 \in Q$, the sequence defined by*

$$q_{n+1} = \tilde{q}(p, \tilde{q}(p, q_n))$$

converges to $\hat{q}(p)$.

Proof. Assumption 2 is related to the matrix

$$\begin{aligned}\tilde{\Gamma}(p, \tilde{q}(p, q))\tilde{\Gamma}(p, q) &= \begin{pmatrix} 0 & \tilde{\Gamma}_{12}(p, \tilde{q}(p, q)) \\ \tilde{\Gamma}_{21}(p, \tilde{q}(p, q)) & 0 \end{pmatrix} \begin{pmatrix} 0 & \tilde{\Gamma}_{12}(p, q) \\ \tilde{\Gamma}_{21}(p, q) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{\Gamma}_{12}(p, \tilde{q}(p, q))\tilde{\Gamma}_{21}(p, q) & 0 \\ 0 & \tilde{\Gamma}_{21}(p, \tilde{q}(p, q))\tilde{\Gamma}_{12}(p, q) \end{pmatrix}.\end{aligned}$$

Its k - k'' -th element is

$$\sum_{k'} \frac{\partial \tilde{q}_k(p^{s(k)}, q^{-s(k)})}{\partial q_{k'}^{-s(k)}} \cdot \frac{\partial \tilde{q}_{k'}(p^{s(k')}, q^{-s(k')})}{\partial q_{k''}^{-s(k')}}.$$

if k and k'' are on the same market side. Otherwise, it is zero. Assumption 2 states that the largest row sum of absolute values of the matrix $\tilde{\Gamma}(p, \tilde{q}(p, q))\tilde{\Gamma}(p, q)$, uniformly over $q \in Q$, is strictly less than one.

Change the quantities q from vector x to vector y , where $x, y \in Q$. Since $\tilde{q}_k(p^{s(k)}, q^{-s(k)})$ is continuously differentiable in its second argument, by the fundamental theorem for line integrals, we have

$$\tilde{q}_k(p, \tilde{q}(p, y)) - \tilde{q}_k(p, \tilde{q}(p, x)) = \int_x^y \sum_{k''} \sum_{k'} \frac{\partial \tilde{q}_k(p^{s(k)}, q^{-s(k)})}{\partial q_{k'}^{-s(k)}} \cdot \frac{\partial \tilde{q}_{k'}(p^{s(k')}, q^{-s(k')})}{\partial q_{k''}^{-s(k')}} dq_{k''},$$

with the understanding that this means a line integral along some path from x to y . Because the integrand is the gradient of the scalar function $\tilde{q}_k(p, \tilde{q}(p, q))$ with respect to q , the integral is path independent. We therefore choose the straight-line path $q(t) = x + t(y - x)$, $t \in [0, 1]$, which lies in Q by convexity.

We have

$$\begin{aligned}
& \left| \tilde{q}_k(p^{s(k)}, \tilde{q}(p, y)) - \tilde{q}_k(p^{s(k)}, \tilde{q}(p, x)) \right| \\
&= \left| \int_0^1 \sum_{k''} \sum_{k'} \frac{\partial \tilde{q}_k(p^{s(k)}, q(t)^{-s(k)})}{\partial q_{k'}^{-s(k)}} \cdot \frac{\partial \tilde{q}_{k'}(p^{s(k')}, q(t)^{-s(k')})}{\partial q_{k''}^{-s(k')}} (y_{k''} - x_{k''}) dt \right| \\
&\leq \int_0^1 \sum_{k''} \left| \sum_{k'} \frac{\partial \tilde{q}_k(p^{s(k)}, q(t)^{-s(k)})}{\partial q_{k'}^{-s(k)}} \cdot \frac{\partial \tilde{q}_{k'}(p^{s(k')}, q(t)^{-s(k')})}{\partial q_{k''}^{-s(k')}} \right| |y_{k''} - x_{k''}| dt \\
&\leq \int_0^1 \sum_{k''} \left| \sum_{k'} \frac{\partial \tilde{q}_k(p^{s(k)}, q(t)^{-s(k)})}{\partial q_{k'}^{-s(k)}} \cdot \frac{\partial \tilde{q}_{k'}(p^{s(k')}, q(t)^{-s(k')})}{\partial q_{k''}^{-s(k')}} \right| \|y - x\|_\infty dt \\
&\leq \int_0^1 \alpha(p) \cdot \|y - x\|_\infty dt \\
&= \alpha(p) \cdot \|y - x\|_\infty.
\end{aligned}$$

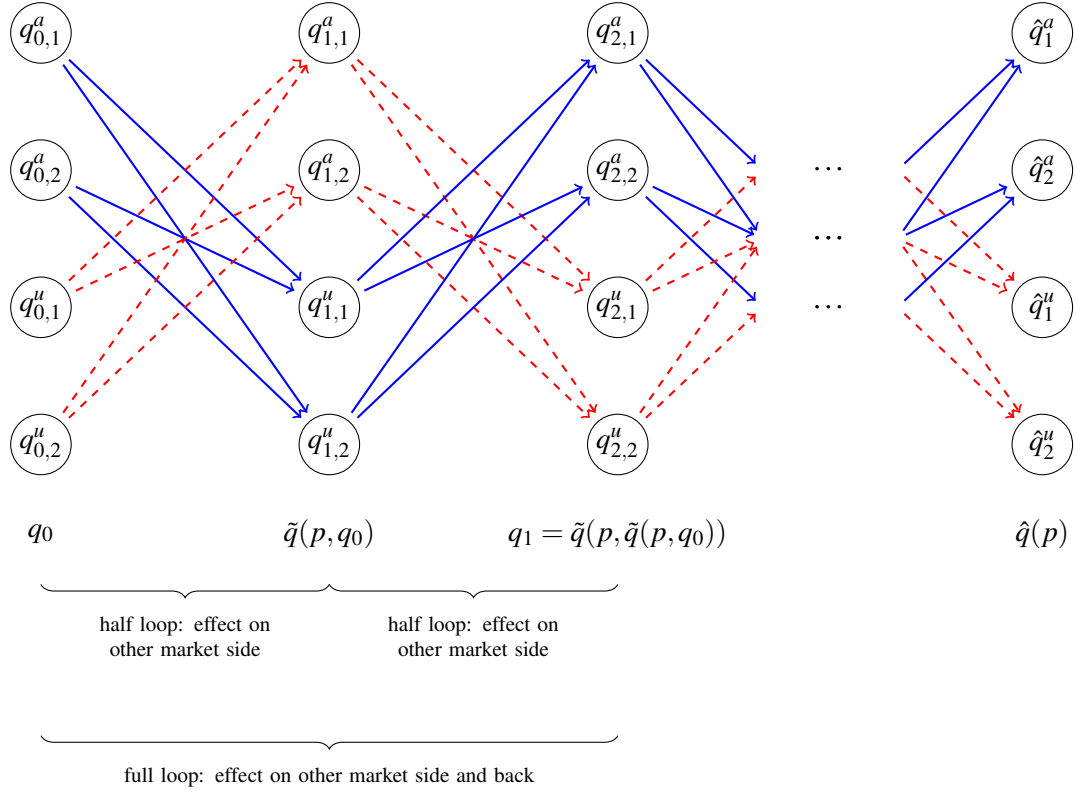
The first equality follows from the line-integral representation using the chosen straight-line path from x to y . The first inequality follows from the triangle inequality. The second uses $|y_{k''} - x_{k''}| \leq \|y - x\|_\infty$ for all k'' . The third follows from Assumption 2. The final equality follows from the fact that $\alpha(p) \cdot \|y - x\|_\infty$ is constant in t .

Taking the maximum over all k , we obtain

$$\|\tilde{q}(p, \tilde{q}(p, y)) - \tilde{q}(p, \tilde{q}(p, x))\|_\infty \leq \alpha(p) \|y - x\|_\infty.$$

Since $\alpha(p) < 1$, the two-step map $q \mapsto \tilde{q}(p, \tilde{q}(p, q))$ is a contraction on $\mathcal{Q}$. Since $\mathcal{Q}$ is complete, the contraction mapping theorem implies that this map has a unique fixed point, denoted by $\hat{q}(p)$. Moreover, for any $q_0 \in \mathcal{Q}$, the sequence defined by $q_{n+1} = \tilde{q}(p, \tilde{q}(p, q_n))$ converges to $\hat{q}(p)$. This proves the result. $\square$

Figure 3 illustrates. We start from a vector q_0 of all quantities on both market sides. In this example, we have market sides a and u , e.g. advertisers and users as in the examples in Sections 3.4.2 and 3.4.3. Then, we iterate on $\tilde{q}(p, \cdot)$, holding all prices fixed. The solid lines start from advertising quantities and the dashed lines start from user quantities. Demands in the first iteration depend only on quantities on the respective other market side: User demand



Notes: The figure illustrates that, starting from an arbitrary initial vector of quantities q_0 , repeated application of the mapping $\tilde{q}(p, \tilde{q}(p, q))$ converges to the unique reduced-form quantities $\hat{q}(p)$. Applying the mapping once gets us from q_0 to $q_1 = \tilde{q}(p, \tilde{q}(p, q_0))$. Part of this is to obtain $\tilde{q}(p, q)$. By Proposition 1, this convergence holds provided that Assumptions 1 and 2 are satisfied. The figure is drawn for the special case in which there are 2 demand components on each market side $s \in \{a, u\}$.

Figure 3: Solving for reduced-form quantities $\hat{q}(p)$

depends on advertising quantities and advertising demand depends on user quantities. This is because reduced form demands $\tilde{q}(p, \cdot)$ depend only on quantities on the respective other market side. For this reason, we also iterate between market sides. The mapping in Proposition 1 is not $\tilde{q}(p, \cdot)$, but $\tilde{q}(p, \tilde{q}(p, \cdot))$. This means that we apply the mapping $\tilde{q}(p, \cdot)$ twice. In the figure, this is denoted as a full loop that starts on one market side, goes to the other, and returns to the market side it started.

In the following lemma, we show how we can obtain derivatives of the reduced-form demand functions $\hat{q}(p)$ with respect to prices.

Lemma 3 (derivatives of $\hat{q}$). *Suppose that Assumptions 1 and 2 hold. The Jacobian matrix that consists of the partial derivatives of the reduced-form demand functions $\hat{q}(p)$ with respect to prices is given by*

$$\frac{\partial \hat{q}(p)}{\partial p'} = (I - \tilde{\Gamma}(p, \hat{q}(p)))^{-1} \tilde{B}(p, \hat{q}(p)).$$

Equivalently, for the two-side block representation,

$$\frac{\partial \hat{q}(p)}{\partial p'} = - \begin{pmatrix} -I & \partial \tilde{q}_1 / \partial q'_2 \\ \partial \tilde{q}_2 / \partial q'_1 & -I \end{pmatrix}^{-1} \begin{pmatrix} \partial \tilde{q}_1 / \partial p'_1 & 0 \\ 0 & \partial \tilde{q}_2 / \partial p'_2 \end{pmatrix},$$

where $\tilde{q}_s$, q_s , and p_s denote the subvectors of reduced-form quantities, quantities, and prices on side $s \in \{1, 2\}$, respectively, and all derivatives are evaluated at $(p, \hat{q}(p))$.

Proof. We apply the implicit function theorem. By Proposition 1, the full reduced-form quantities satisfy $\hat{q}(p) = \tilde{q}(p, \hat{q}(p))$. Equivalently, define $F(p, q) \equiv \tilde{q}(p, q) - q$. Then $\hat{q}(p)$ satisfies $F(p, \hat{q}(p)) = 0$.

The total derivative of $F(p, q) = 0$ is

$$\frac{\partial F(p, q)}{\partial p'} dp + \frac{\partial F(p, q)}{\partial q'} dq = 0.$$

By Lemma 2,

$$\frac{\partial F(p, q)}{\partial p'} = \tilde{B}(p, q)$$

and

$$\frac{\partial F(p, q)}{\partial q'} = \tilde{\Gamma}(p, q) - I.$$

Equivalently,

$$(I - \tilde{\Gamma}(p, \hat{q}(p))) d\hat{q} = \tilde{B}(p, \hat{q}(p)) dp.$$

Assumption 2 implies that $I - \tilde{\Gamma}(p, \hat{q}(p))$ is invertible. Hence,

$$\frac{\partial \hat{q}(p)}{\partial p'} = (I - \tilde{\Gamma}(p, \hat{q}(p)))^{-1} \tilde{B}(p, \hat{q}(p)).$$

The two-side block representation follows from the block forms of $\tilde{B}(p, q)$ and $\tilde{\Gamma}(p, q)$. This proves the result. $\square$

3.2 Discussion

Our approach starts from primitive demands $q(p, q)$ that depend, respectively, on quantities on the same and other market side. We show that under Assumption 1, we can solve for side-specific reduced forms $\tilde{q}(p, q)$ that only depend on quantities on the respective other market side. We then show that under Assumption 2, we can solve for demands $\hat{q}(p)$ that depend on prices only.

As pointed out before, Assumption 1 is a familiar sufficient condition for uniqueness in models with direct network effects. It limits the strength of direct network effects and has an intuitive interpretation: Demand is allowed to spread through the network, but each round of feedback is weaker than the previous one, so the ripple eventually dies out. As a result, demand converges to a unique, stable equilibrium rather than exploding through positive feedback. Assumption 2 is a sufficient condition similar to the one in Assumption 1, but on combinations of indirect network effects. We make it on what we call the “full loop”, where we start on one market side, go to the other market side, and then back to the initial market side. This corresponds to the mapping $\tilde{q}(p, \tilde{q}(p, \cdot))$. One can think of it roughly as being on the product of network effects from both sides. Making the assumption on the product of indirect network effects is desirable, because it can hold when one of the two indirect network effects is big while the other

is small or absent. For that reason, in general, the assumptions one would need to instead make on the single individual indirect network effects would be stronger.

In principle, one could instead make alternative assumptions that allow us to directly solve for $\hat{q}(p)$ by iterating on the primitive demands $q(p, q)$. This would involve making a single higher-level assumption on all network effects, direct and indirect.¹² We prefer to make Assumption 1 on direct network effects and separately Assumption 2 on the product of indirect network effects, because these assumptions are, arguably, easier to interpret and correspond more closely to the two types of assumptions that have been made before in specific models in, respectively, the literature on models with direct network effects and the literature on two-sided markets. If either direct or of the indirect network effects are not present, then the corresponding assumption is trivially satisfied and the other generalizes a specific assumption that has been made before.

3.3 Practical implications

In practice, one would first estimate demands $q^s(p^s, q^s, q^{-s})$ for each market side. Examples of papers that estimate adoption or usage equations in which network size or composition enters directly are Tucker (2008) and Chu and Manchanda (2016). Tucker (2008) estimates how the adoption of an internal video-messaging system by employees depends on the adoption decisions of relevant colleagues. Chu and Manchanda (2016) model whether potential buyers or sellers join Taobao. In their model, participation utilities depend on the installed bases on both the same and the opposite side of the platform, as well as variables such as product variety, buyer quality, and prices. Examples of papers that estimate indirect network effects include both advertising-financed media markets and non-advertising platform markets. In media markets, the main effect typically runs from the audience side to the advertising side: advertisers value platforms with more users, readers, viewers, or listeners, as in Rysman (2004), Kaiser and Wright (2006), Argentesi and Filistrucchi (2007), Wilbur (2008), Fan (2013), Jeziorski (2014),

¹²This assumption would be similar to one of the assumptions we have made. The difference is that it would involve the derivative of all demand components on both market sides with respect to all demand components on both market sides. If one works with a half loop, then it would be similar to Assumption 1. If one works with a full loop, then it would be similar to Assumption 2.

and Song (2021). In non-advertising platform markets, indirect network effects run between different participant groups, for example from buyers to sellers and from sellers to buyers, as in Chu and Manchanda (2016), or between patients and physicians, as in Zhang et al. (2026).

Once the primitive demands have been estimated, Assumptions 1 and 2 can be tested.¹³ In general, primitive demands are nonlinear. Therefore, testing Assumptions 1 and 2 involves calculating derivatives for possible combinations of prices and quantities. In practice, knowledge of the functional form can be used to do this efficiently. For example, in Section 3.4.3 below, we show in detail how we can use properties of the logit model for this.

Given that these assumptions hold, Proposition 1 establishes existence and uniqueness of the vector of reduced form quantities $\hat{q}(p)$. It also establishes that one can compute them by iterating on the mapping $\tilde{q}(p, q)$, as illustrated in Figure 3. For any $q_0 \in \mathbb{R}^K$ the sequence of iterates $q_0, \tilde{q}(p, q_0), \tilde{q}(p, \tilde{q}(p, q_0)), \dots$ converges to $\hat{q}(p)$. This also involves iterating on the mapping $q(p, q)$ in an inner loop to solve for $\tilde{q}(p, q)$, as illustrated in Figure 2.

Thus, the framework provides an operational rather than generally closed-form characterization of $\hat{q}(p)$: the reduced-form quantities are the unique fixed point of the coordination game and can be computed by iteration. Lemma 2 and Lemma 3 show together how one can calculate price effects $\frac{\partial \hat{q}(p)}{\partial p'}$. With these in hand, one can back out marginal cost in the usual way, investigate whether a sufficient condition for uniqueness of equilibrium holds, and can solve for counterfactual equilibria, as we show in Section 4.

Before going there, we first develop the intuition behind the general results by going back to two of the examples from before.

3.4 Back to the examples

We now revisit two examples from Section 2.3 to show what Assumption 1 and 2 mean and what the intuition behind our existence and uniqueness results is. Here, we also touch upon the

¹³One could instead estimate $\tilde{q}^s(p^s, q^{-s})$ directly. In that case, one would have to assume that Assumption 1 holds and one not be able to test it. One could still test Assumption 2. One could also estimate $\hat{q}(p)$ directly. Then, one would assume that both Assumptions 1 and 2 hold, and one could not test either of them. One drawback of estimating $\tilde{q}^s(p^s, q^{-s})$ or $\hat{q}(p)$ directly is that they, unlike primitive demand functions, could change in counterfactual scenarios.

point that these are not necessary for existence and uniqueness, but very natural because they are related to adjustment dynamics.

3.4.1 Back to the example with one newspaper and linear demand

Our point of departure is the setup in Section 2.3.1. Assumption 1 is that $|\theta^a| < 1$ and $|\theta^r| < 1$. It implies

$$\theta^a \neq 1 \text{ and } \theta^r \neq 1. \quad (4)$$

Therefore, we can solve for

$$\begin{aligned} \tilde{q}^a(p^a, q^r) &= \tilde{\alpha}^a - \tilde{\beta}^a p^a + \tilde{\gamma}^a q^r \\ \tilde{q}^r(p^r, q^a) &= \tilde{\alpha}^r - \tilde{\beta}^r p^r + \tilde{\gamma}^r q^a, \end{aligned}$$

where $\tilde{\alpha}^a \equiv \alpha^a / (1 - \theta^a)$, $\tilde{\beta}^a \equiv \beta^a / (1 - \theta^a)$, $\tilde{\gamma}^a \equiv \gamma^a / (1 - \theta^a)$, $\tilde{\alpha}^r \equiv \alpha^r / (1 - \theta^r)$, $\tilde{\beta}^r \equiv \beta^r / (1 - \theta^r)$, and $\tilde{\gamma}^r \equiv \gamma^r / (1 - \theta^r)$.

Assumption 2 is that $|\tilde{\gamma}^a \tilde{\gamma}^r| < 1$. It implies

$$\tilde{\gamma}^a \tilde{\gamma}^r \neq 1. \quad (5)$$

Therefore, we can solve for the reduced-form quantities as functions of prices by substituting one demand equation into the other. This gives

$$\begin{aligned} \hat{q}^a(p^a, p^r) &= \frac{\tilde{\alpha}^a + \tilde{\gamma}^a \tilde{\alpha}^r - \tilde{\beta}^a p^a - \tilde{\gamma}^a \tilde{\beta}^r p^r}{1 - \tilde{\gamma}^a \cdot \tilde{\gamma}^r} \\ \hat{q}^r(p^a, p^r) &= \frac{\tilde{\alpha}^r + \tilde{\gamma}^r \tilde{\alpha}^a - \tilde{\beta}^r p^r - \tilde{\gamma}^r \tilde{\beta}^a p^a}{1 - \tilde{\gamma}^a \cdot \tilde{\gamma}^r}. \end{aligned} \quad (6)$$

Here, we solve for the reduced-form quantities in two steps. In the first step, we solve for the quantities separately for each market side, holding the respective other quantity on the other market side fixed. In the second step, we solve for the reduced-form quantities for both market sides simultaneously. The advantage of proceeding like this is that we arrive at (6). The numerator is the quantity we would get if there was only half a feedback loop: advertising

demand depends on prices and readership demand, which in turn depends on prices (but not again advertising demand; hence, the half loop). Likewise, readership demand depends on readership prices and advertising demand, which in turn depends on prices (but not readership demand; hence, again a half loop). That is as if the feedback loops are interrupted. We arrive at the full quantity by dividing by $1 - \tilde{\gamma}^a \cdot \tilde{\gamma}^r$. So, $1/(1 - \tilde{\gamma}^a \tilde{\gamma}^r)$ is the multiplier, which enters both expressions.¹⁴

We also see from (6) that demand is downward sloping, if, and only if, $\tilde{\gamma}^a \tilde{\gamma}^r < 1$. This is Assumption 2 in the current context.

Moreover, based on (6), under Assumptions 1 and 2, we can re-interpret the reduced-form demand functions as demands for complementary (if $\tilde{\gamma}^a, \tilde{\gamma}^r > 0$) or substitute (if $\tilde{\gamma}^a, \tilde{\gamma}^r < 0$) products. However, a middle case is also possible in which demand on one side depends negatively on the price of the other side but demand on the other side depends positively on the first price ($\tilde{\gamma}^a > 0, \tilde{\gamma}^r < 0$ or $\tilde{\gamma}^a < 0, \tilde{\gamma}^r > 0$). This is the case for instance if advertisers attach a higher value to newspapers with more readers but readers dislike advertising.

The condition (4) $\tilde{\gamma}^a \tilde{\gamma}^r \neq 1$ for existence is implied by Assumption 2, which in this linear context holds whenever $|\tilde{\gamma}^a \tilde{\gamma}^r| < 1$. Formally, a solution to the above system of equations also exists if $|\tilde{\gamma}^a \tilde{\gamma}^r| > 1$. However, this solution is not meaningful in the context of demand, because then, the reduced-form quantities depend positively on their own price.

This shows that the conditions that guarantee existence of the reduced form demands are related to the size of the indirect network effects $\tilde{\gamma}^a$ and $\tilde{\gamma}^r$ after we take the direct network effects into account. In fact, what matters is the product of the two, because this is the feedback

¹⁴Proceeding in two steps comes at a small cost. We can, in principle, also solve for the reduced-form quantities in one step. If we do so, then we only need to require the condition

$$(1 - \theta^a)(1 - \theta^r) - \gamma^a \gamma^r \neq 0,$$

instead of assuming (4) and (5). However, arguably, the latter two conditions are easier to interpret. Moreover, and probably more importantly, we can under the weaker condition only solve for

$$\begin{aligned} \hat{q}^a(p^a, p^r) &= \frac{(1 - \theta^r) \cdot (\alpha^a - \beta^a p^a) + \gamma^a (\alpha^r - \beta^r p^r)}{(1 - \theta^a)(1 - \theta^r) - \gamma^a \gamma^r} \\ \hat{q}^r(p^a, p^r) &= \frac{(1 - \theta^a) \cdot (\alpha^r - \beta^r p^r) + \gamma^r (\alpha^a - \beta^a p^a)}{(1 - \theta^a)(1 - \theta^r) - \gamma^a \gamma^r}, \end{aligned}$$

which is harder to interpret.

factor that propagates through the loop.¹⁵

To see this, re-write (1) as a geometric series. Define $\bar{\alpha}^a \equiv \tilde{\alpha}^a - \tilde{\beta}^a p^a$ and $\bar{\alpha}^r \equiv \tilde{\alpha}^r - \tilde{\beta}^r p^r$. Then,

$$\begin{aligned} q^a &= \bar{\alpha}^a + \tilde{\gamma}^a (\bar{\alpha}^r + \tilde{\gamma}^r (\bar{\alpha}^a + \tilde{\gamma}^a (\bar{\alpha}^r + \tilde{\gamma}^r (\dots)))) \\ &= \bar{\alpha}^a + \tilde{\gamma}^a \tilde{\gamma}^r \bar{\alpha}^a + (\tilde{\gamma}^a \tilde{\gamma}^r)^2 \bar{\alpha}^a + \dots + \tilde{\gamma}^a \bar{\alpha}^r + \tilde{\gamma}^a \tilde{\gamma}^r \tilde{\gamma}^a \bar{\alpha}^r + \tilde{\gamma}^a (\tilde{\gamma}^r \tilde{\gamma}^a)^2 \bar{\alpha}^r + \dots \\ &= (\bar{\alpha}^a + \tilde{\gamma}^a \bar{\alpha}^r) \cdot \left(1 + \tilde{\gamma}^a \tilde{\gamma}^r + (\tilde{\gamma}^a \tilde{\gamma}^r)^2 + \dots\right) \end{aligned}$$

and

$$q^r = (\bar{\alpha}^r + \tilde{\gamma}^r \bar{\alpha}^a) \cdot \left(1 + \tilde{\gamma}^r \tilde{\gamma}^a + (\tilde{\gamma}^r \tilde{\gamma}^a)^2 + \dots\right).$$

Both converge to the reduced form quantities (6) if the absolute value of the so-called common ratio (the factor each term is multiplied by to get the next one), $\tilde{\gamma}^a \tilde{\gamma}^r$, is less than one. This, again, is the condition in Assumption 2.

Writing demands in terms of a geometric series also shows that if one of the two network effects is zero, the reduced-form demand functions always exist because in that case the product of the two network effects is automatically zero. In that case the multiplier is equal to one.¹⁶

3.4.2 Back to the example with two social media platforms and linear demand

The demand system described in Section 2.3.2 can be written in matrix notation as

$$q = \alpha + Bp + \Gamma q + \Theta q,$$

¹⁵One loop consists of advertising demand affecting readership demand and thereby affecting again advertising demand; likewise for the loop originating on the readership side.

¹⁶If, for instance,

$$\begin{aligned} q^a(p^a, q^r) &= \tilde{\alpha}^a - \tilde{\beta}^a p^a + \tilde{\gamma}^a q^r \\ q^r(p^r) &= \tilde{\alpha}^r - \tilde{\beta}^r p^r \end{aligned}$$

so that readers are not affected by advertising, then the reduced form demand functions are

$$\begin{aligned} \hat{q}^a(p^a, p^r) &= (\tilde{\alpha}^a + \tilde{\alpha}^r \tilde{\gamma}^a) - \tilde{\beta}^a p^a - \tilde{\gamma}^a \tilde{\beta}^r p^r \\ \hat{q}^r(p^r) &= \tilde{\alpha}^r - \tilde{\beta}^r p^r, \end{aligned}$$

where it appears evident that reduced form readership demand is not affected by the advertising price (because by assumption advertising quantity does not affect readership demand), while advertising demand is affected by the cover price (since the number of readers affects demand from advertisers).

with vectors

$$q = \begin{pmatrix} q^a \\ q^u \end{pmatrix} = \begin{pmatrix} q_1^a \\ q_2^a \\ q_1^u \\ q_2^u \end{pmatrix}, \alpha = \begin{pmatrix} \alpha^a \\ \alpha^u \end{pmatrix} = \begin{pmatrix} \alpha_1^a \\ \alpha_2^a \\ \alpha_1^u \\ \alpha_2^u \end{pmatrix}, p = \begin{pmatrix} p^a \\ p^u \end{pmatrix} = \begin{pmatrix} p_1^a \\ p_2^a \\ p_1^u \\ p_2^u \end{pmatrix}$$

and block-diagonal matrices

$$B = \begin{pmatrix} B^a & 0 \\ 0 & B^u \end{pmatrix} = \begin{pmatrix} -\beta_{11}^a & \beta_{12}^a & 0 & 0 \\ \beta_{21}^a & -\beta_{22}^a & 0 & 0 \\ 0 & 0 & -\beta_{11}^u & \beta_{12}^u \\ 0 & 0 & \beta_{21}^u & -\beta_{22}^u \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 0 & \Gamma^a \\ \Gamma^r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \gamma_{11}^a & \gamma_{12}^a \\ 0 & 0 & \gamma_{21}^a & \gamma_{22}^a \\ \gamma_{11}^u & \gamma_{12}^u & 0 & 0 \\ \gamma_{21}^u & \gamma_{22}^u & 0 & 0 \end{pmatrix}$$

$$\Theta = \begin{pmatrix} \Theta^a & 0 \\ 0 & \Theta^r \end{pmatrix} = \begin{pmatrix} \theta_{11}^a & \theta_{12}^a & 0 & 0 \\ \theta_{21}^a & \theta_{22}^a & 0 & 0 \\ 0 & 0 & \theta_{11}^u & \theta_{12}^u \\ 0 & 0 & \theta_{21}^u & \theta_{22}^u \end{pmatrix}.$$

We can solve for the quantities in one step as

$$q = (I - \Gamma - \Theta)^{-1}(\alpha + Bp),$$

provided that $I - \Gamma - \Theta$ is invertible. This is a high-level condition that is hard to comprehend even in this simple example. Nonetheless, it is in principle testable.

Another way to write the reduced-form quantities that is closer to the way we proceed in

Section 3.1 is

$$q = (I - \tilde{\Gamma})^{-1}(I - \Theta)^{-1}(\alpha + Bp),$$

where $\tilde{\Gamma} \equiv (I - \Theta)^{-1}\Gamma$. This requires two conditions to hold. First, that $I - \Theta$ is invertible. And second, that $I - \tilde{\Gamma}$ is invertible. They are necessary and sufficient and are related to the sufficient conditions in Assumption 1 and 2, respectively.

Assumption 1 holds if

$$|\theta_{11}^a| + |\theta_{12}^a| < 1$$

$$|\theta_{21}^a| + |\theta_{22}^a| < 1$$

$$|\theta_{11}^u| + |\theta_{12}^u| < 1$$

$$|\theta_{21}^u| + |\theta_{22}^u| < 1.$$

It implies that $\tilde{\Gamma}$, $\tilde{\alpha} \equiv (I - \Theta)^{-1}\alpha$, and $\tilde{B} \equiv (I - \Theta)^{-1}B$ exist.

Write

$$q = (I - \tilde{\Gamma})^{-1}(\tilde{\alpha} + \tilde{B}p).$$

$I - \tilde{\Gamma}$ is invertible if, and only if,

$$\det(I - \tilde{\Gamma}) \neq 0.$$

We have

$$\begin{aligned}
I - \tilde{\Gamma} &= I - (I - \Theta)^{-1} \Gamma \\
&= I - \left[I - \begin{pmatrix} \Theta^a & 0 \\ 0 & \Theta^u \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 & \Gamma^a \\ \Gamma^u & 0 \end{pmatrix} \\
&= I - \begin{pmatrix} I - \Theta^a & 0 \\ 0 & I - \Theta^u \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Gamma^a \\ \Gamma^u & 0 \end{pmatrix} \\
&= I - \begin{pmatrix} (I - \Theta^a)^{-1} & 0 \\ 0 & (I - \Theta^u)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \Gamma^a \\ \Gamma^u & 0 \end{pmatrix} \\
&= I - \begin{pmatrix} 0 & (I - \Theta^a)^{-1} \Gamma^a \\ (I - \Theta^u)^{-1} \Gamma^u & 0 \end{pmatrix}.
\end{aligned}$$

This shows that $I - \tilde{\Gamma}$ is a partitioned matrix and we can write

$$I - \tilde{\Gamma} = I - \begin{pmatrix} 0 & \tilde{\Gamma}^a \\ \tilde{\Gamma}^u & 0 \end{pmatrix},$$

where

$$\tilde{\Gamma}^a \equiv (I - \Theta^a)^{-1} \Gamma^a$$

and

$$\tilde{\Gamma}^u \equiv (I - \Theta^u)^{-1} \Gamma^u.$$

We require the determinant of this partitioned matrix to be nonzero. Interpreting this condition in terms of underlying parameters is, however, challenging in the general case.

$I - \tilde{\Gamma}$ is invertible if, and only if,

$$\det(I - \tilde{\Gamma}) \neq 0.$$

$I - \tilde{\Gamma}$ is a partitioned matrix. Hence, $\det(I - \tilde{\Gamma}) = \det(I) \cdot \det(I - \tilde{\Gamma}^a \tilde{\Gamma}^u) = \det(I - \tilde{\Gamma}^a \tilde{\Gamma}^u) =$

$\det(I - \tilde{\Gamma}^u \tilde{\Gamma}^a) = \det(I) \cdot \det(I - \tilde{\Gamma}^u \tilde{\Gamma}^a)$. So, the condition $\det(I - \tilde{\Gamma}) \neq 0$ is in fact equivalent to

$$\begin{aligned} \det(I - \tilde{\Gamma}^a \tilde{\Gamma}^u) &= \det \left(I - \begin{pmatrix} \tilde{\gamma}_{11}^a \tilde{\gamma}_{11}^u + \tilde{\gamma}_{12}^a \tilde{\gamma}_{21}^u & \tilde{\gamma}_{11}^a \tilde{\gamma}_{12}^u + \tilde{\gamma}_{12}^a \tilde{\gamma}_{22}^u \\ \tilde{\gamma}_{21}^a \tilde{\gamma}_{11}^u + \tilde{\gamma}_{22}^a \tilde{\gamma}_{21}^u & \tilde{\gamma}_{21}^a \tilde{\gamma}_{12}^u + \tilde{\gamma}_{22}^a \tilde{\gamma}_{22}^u \end{pmatrix} \right) \\ &= (1 - \tilde{\gamma}_{11}^a \tilde{\gamma}_{11}^u - \tilde{\gamma}_{12}^a \tilde{\gamma}_{21}^u)(1 - \tilde{\gamma}_{21}^a \tilde{\gamma}_{12}^u - \tilde{\gamma}_{22}^a \tilde{\gamma}_{22}^u) \end{aligned} \quad (7)$$

$$- (\tilde{\gamma}_{11}^a \tilde{\gamma}_{12}^u + \tilde{\gamma}_{12}^a \tilde{\gamma}_{22}^u)(\tilde{\gamma}_{21}^a \tilde{\gamma}_{11}^u + \tilde{\gamma}_{22}^a \tilde{\gamma}_{21}^u) \quad (8)$$

$$\neq 0.$$

Assumption 2 is that

$$\begin{aligned} \sum_{\ell=1}^2 \left| \sum_{k=1}^2 \tilde{\gamma}_{1k}^a \tilde{\gamma}_{k\ell}^u \right| &= |\tilde{\gamma}_{11}^a \tilde{\gamma}_{11}^u + \tilde{\gamma}_{12}^a \tilde{\gamma}_{21}^u| + |\tilde{\gamma}_{11}^a \tilde{\gamma}_{12}^u + \tilde{\gamma}_{12}^a \tilde{\gamma}_{22}^u| < 1 \\ \sum_{\ell=1}^2 \left| \sum_{k=1}^2 \tilde{\gamma}_{2k}^a \tilde{\gamma}_{k\ell}^u \right| &= |\tilde{\gamma}_{21}^a \tilde{\gamma}_{11}^u + \tilde{\gamma}_{22}^a \tilde{\gamma}_{21}^u| + |\tilde{\gamma}_{21}^a \tilde{\gamma}_{12}^u + \tilde{\gamma}_{22}^a \tilde{\gamma}_{22}^u| < 1, \end{aligned}$$

and accordingly for the quantities on the readership side, and implies

$$\begin{aligned} |\tilde{\gamma}_{11}^a \tilde{\gamma}_{11}^u + \tilde{\gamma}_{12}^a \tilde{\gamma}_{21}^u| &< 1 - |\tilde{\gamma}_{11}^a \tilde{\gamma}_{12}^u + \tilde{\gamma}_{12}^a \tilde{\gamma}_{22}^u| \\ |\tilde{\gamma}_{21}^a \tilde{\gamma}_{11}^u + \tilde{\gamma}_{22}^a \tilde{\gamma}_{21}^u| &< 1 - |\tilde{\gamma}_{21}^a \tilde{\gamma}_{12}^u + \tilde{\gamma}_{22}^a \tilde{\gamma}_{22}^u|. \end{aligned}$$

Hence, the first term in parentheses in (7) is bigger than the third and the second is bigger than the fourth, and therefore $\det(I - \tilde{\Gamma}^a \tilde{\Gamma}^u) > 0$. We can thus solve for the reduced-form quantities as functions of prices only.

We can also, as in the previous example in Section 3.4.1, write quantities as a geometric series,

$$q = (I + \tilde{\Gamma} \tilde{\Gamma} + (\tilde{\Gamma} \tilde{\Gamma})^2 + \dots) \cdot (I + \tilde{\Gamma}) \cdot (\tilde{\alpha} + \tilde{B}p).$$

Denote the spectral radius of a matrix by $\rho(\cdot)$. The spectral radius is the largest absolute value of the eigenvalues of that matrix. The series converges to

$$(1 - \tilde{\Gamma} \tilde{\Gamma})^{-1} \cdot (I + \tilde{\Gamma}) \cdot (\tilde{\alpha} + \tilde{B}p) = (1 - \tilde{\Gamma})^{-1} \cdot (\tilde{\alpha} + \tilde{B}p)$$

if $\rho(\tilde{\Gamma}\tilde{\Gamma}) < 1$. Since

$$\tilde{\Gamma}\tilde{\Gamma} = \begin{pmatrix} 0 & \tilde{\Gamma}^a \\ \tilde{\Gamma}^r & 0 \end{pmatrix} \begin{pmatrix} 0 & \tilde{\Gamma}^a \\ \tilde{\Gamma}^r & 0 \end{pmatrix} = \begin{pmatrix} \tilde{\Gamma}^a\tilde{\Gamma}^r & 0 \\ 0 & \tilde{\Gamma}^r\tilde{\Gamma}^a \end{pmatrix},$$

this is equivalent to $\rho(\tilde{\Gamma}^a\tilde{\Gamma}^r) < 1$. A sufficient condition is that the absolute row sums of $\tilde{\Gamma}\tilde{\Gamma}$ are all smaller than one, which is our Assumption 2. This condition is analogous to standard stability conditions in linear systems and input-output models (see e.g. Gandolfo, 1996, p. 117 and Murata, 1977), where convergence is ensured if the spectral radius of the interaction matrix is smaller than one.

3.4.3 Back to the logit example

In Section 2.3.3, we presented an example with J platforms and logit demand on each market side. Here, we show analytical and numerical results for the two-sided coordination problem.

The analytical results pertain to assessing whether Assumption 1 holds. We start with the user side and compute the within-side Jacobian demand function explicitly. Writing $s_j^u = q_j^u(p_1^{u*}, p_2^{u*}, q_1^{a*}, q_2^{a*}, q_1^u, q_2^u)/M^u$ for the market share of platform j implied by the demand function at (q_1^u, q_2^u) , the partial derivatives of demand for platform j with respect to its own quantity and to the rival platform's quantity are

$$\frac{\partial q_j^u}{\partial q_j^u} = M^u \theta^u s_j^u (1 - s_j^u) \quad \text{and} \quad \frac{\partial q_j^u}{\partial q_l^u} = -M^u \theta^u s_j^u s_l^u \quad (l \neq j).$$

For $J = 2$ platforms, the row sum of absolute partial derivatives for row j is therefore $M^u \theta^u s_j^u (1 - s_j^u + s_l^u)$. Since $q_1^u + q_2^u \leq M^u$, or equivalently $s_j^u + s_l^u \leq 1$, the factor $s_j^u (1 - s_j^u + s_l^u)$ is increasing in s_l^u for fixed s_j^u , so its maximum over the constraint set is attained at $s_l^u = 1 - s_j^u$ (when the outside option has zero share), where the row sum equals $2s_j^u (1 - s_j^u)$. Since $2s_j^u (1 - s_j^u) = \frac{1}{2} - 2\left(s_j^u - \frac{1}{2}\right)^2 \leq \frac{1}{2}$, with equality at $s_j^u = \frac{1}{2}$, the global supremum of the row sum of $|\Theta^u|$ over all $(q_1^u, q_2^u) \in \mathcal{Q}^u$, prices p^u , and cross-side quantities q^a equals $M^u \theta^u / 2$. A sufficient condition for Assumption 1 to hold globally on the user side is therefore $M^u \theta^u < 2$. By a similar

argument, the condition is $M^a \theta^a < 2$ for the advertising side.

For each market side s , the analytical bound $M^s |\theta^s|/2$ is the supremum of the row sum of $|\Theta^s|$ over the entire share simplex, attained when both market shares are equal to 0.5. Not every share vector is reachable at admissible prices, however, so this analytical bound is conservative.

To go beyond testing the analytical “worst case” bound, we can determine the set of attainable share vectors. A share vector s^s is attainable if its implied price, for some feasible cross-side vector of quantities q^{-s} , lies in the box P^s of feasible prices for side s . The logit demand system is invertible in own prices: writing $s_0^s \equiv 1 - \sum_{j'} s_{j'}^s$ for the outside-option share, the log-odds inversion gives

$$p_j^s = \frac{1}{\beta^s} \left(\alpha_j^s + \gamma^s q_j^{-s} + \theta^s M^s s_j^s - \ln \frac{s_j^s}{s_0^s} \right),$$

so each share vector and cross-side quantity jointly imply a unique price.

Once the set of attainable share vectors has been determined in that way, we can assess Assumption 1 numerically for each share vector in that set and for all possible quantities on the other market side. This computation is fast, as long as the derivative of market demand on side s with respect to quantities on that side can be computed quickly.

Assumption 2 involves the row sums of $|\tilde{\Gamma}_{12} \tilde{\Gamma}_{21}|$ and $|\tilde{\Gamma}_{21} \tilde{\Gamma}_{12}|$, where $\tilde{\Gamma}_{12} = (I - \Theta_{11})^{-1} \Gamma_{12}$ and $\tilde{\Gamma}_{21} = (I - \Theta_{22})^{-1} \Gamma_{21}$ (notation as in the proof of Proposition 1) are the reduced-form indirect network effects after taking the direct network effects into account. Unlike Assumption 1, whose row-sum maximum $M^s |\theta^s|/2$ is available in closed form, these products admit no comparable analytic bound in the general case.¹⁷ We can verify them numerically. Also here, the computation is fast, as long as derivatives can be computed quickly.

For the logit model, $\tilde{\Gamma}_{12}$ depends on (p, q) only through s^a and $\tilde{\Gamma}_{21}$ only through s^u . Therefore, the supremum of the row sums over $P \times Q$ decouples: s^a and s^u may range independently over the shares each side can reach. Hence, we can assess whether a share s^s is attainable by assessing whether its implied price lies in the admissible box.

¹⁷It may be possible to derive an analytic bound for the special case without direct network effects. This will involve taking the maximum over market shares on both market sides. Therefore, we expect the bound to be more conservative than the analytic bound for Assumption (1).

We next illustrate this using one particular specification of the model. Market sizes, preference parameters, and marginal costs are given by

$$\begin{aligned} M^a &= 1000, \quad M^u = 100,000, \quad mc^a = 10, \quad mc^u = 1, \\ \alpha^a &= 5, \quad \beta^a = 0.05, \quad \gamma^a = 2 \times 10^{-5}, \quad \theta^a = -5 \times 10^{-4}, \\ \alpha^u &= 0.5, \quad \beta^u = 0.3, \quad \gamma^u = -10^{-3}, \quad \theta^u \in \{1.3, 5.0\} \times 10^{-5}. \end{aligned}$$

Advertisers prefer to be among less advertisers because of congestion ($\theta^a < 0$) and value more users ($\gamma^a > 0$). Users prefer more other users on the platform ($\theta^u > 0$) and do not like advertising ($\gamma^u < 0$).

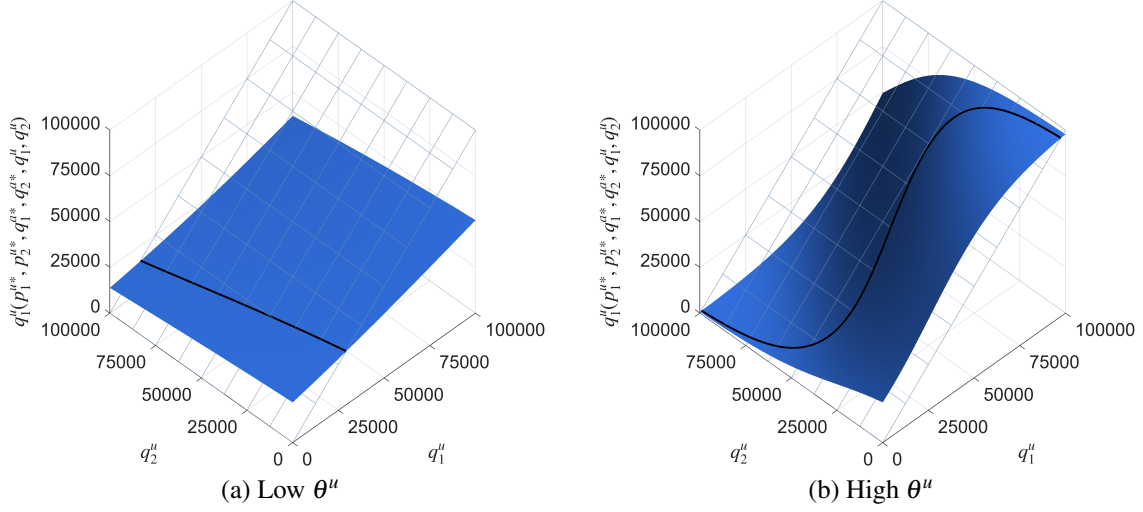
We fix the admissible price set P from the general framework to the box $P = \{p : 0 \leq p_j^s \leq 20mc^s, s \in \{a, u\}, j = 1, 2\}$, with prices non-negative and bounded above by twenty times marginal cost. This box is large relative to the equilibrium prices and will be used below both to verify the assumptions numerically and to solve for the pricing equilibrium.

We present results for two values of θ^u , a low baseline value at which our assumptions hold and there is a unique equilibrium both in the coordination and in the pricing game, and a high value at which our assumptions do not hold and there are multiple equilibria in the coordination game. The equilibrium for the low θ^u case is symmetric. Equilibrium prices are $p_1^{a*} = p_2^{a*} \approx 132.3$ and $p_1^{u*} = p_2^{u*} \approx 4.16$ and quantities are $q_1^{a*} = q_2^{a*} \approx 190$ and $q_1^{u*} = q_2^{u*} \approx 26,192$.

As in Section 3.1, we first assess whether there is a unique set of quantities on the same market side, holding prices and quantities on the other market side constant. For both values of θ^u , we hold the user prices p_1^u and p_2^u and the advertising quantities q_1^a and q_2^a fixed at their values for the unique equilibrium in the low θ^u case.

Figure 4 shows that for both values of θ^u , user demand for platform 1 depends positively on the number of users on that platform and negatively on the number of users on the other platform. The solid line indicates, respectively for each θ^u , the value at which $q_1^u(p_1^{u*}, p_2^{u*}, q_1^{a*}, q_2^{a*}, q_1^u, q_2^u) = q_1^u$. This is the set of fixed points of the function for given prices on the user side and given quantities on the advertiser side.¹⁸

¹⁸The figure for platform 2 is similar because of the symmetry in this example and is therefore omitted. It can be obtained by interchanging the q_1^u and q_2^u axes.



Notes: Each figure shows $q_1^u(p_1^{u*}, p_2^{u*}, q_1^{a*}, q_2^{a*}, q_1^u, q_2^u)$ for two values of θ^u . p^{u*} and q^{a*} are equilibrium prices from the symmetric unique equilibrium for the low θ^u case. The solid line indicates the fixed points $q_1^u(p_1^{u*}, p_2^{u*}, q_1^{a*}, q_2^{a*}, q_1^u, q_2^u) = q_1^u$ for different values of q_2^u .

Figure 4: Dependence of user demand on same-side quantities

Figure 5 shows the sets of fixed points. There is one line for platform 1 where

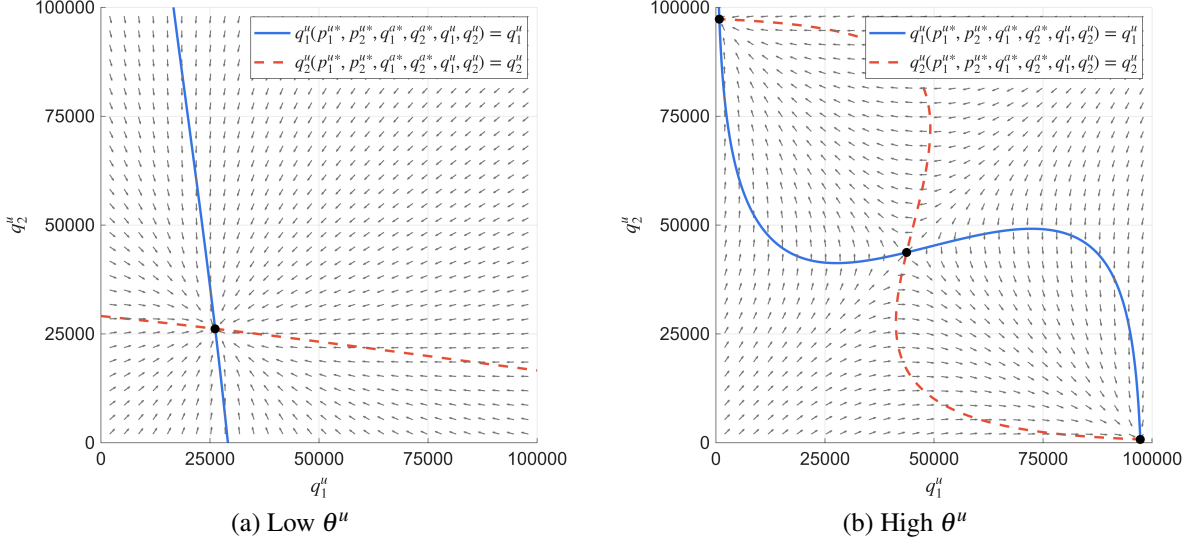
$$q_1^u(p_1^{u*}, p_2^{u*}, q_1^{a*}, q_2^{a*}, q_1^u, q_2^u) = q_1^u$$

and another line for platform 2 where

$$q_2^u(p_1^{u*}, p_2^{u*}, q_1^{a*}, q_2^{a*}, q_1^u, q_2^u) = q_2^u.$$

The intersection points of both lines are overall fixed points, which can be interpreted as equilibria in the user-side coordination game. The grey arrows show the direction in which quantities adjust when they are out of equilibrium, obtained by iterating the demand system; they point toward the stable equilibria. On the left, we see that there is a unique equilibrium. On the right, there are 3 equilibria. We also see that the equilibrium in the middle is not stable. Only arrows on the 45 degree line point toward it, so we move away from it for almost any initial set of quantities.

To assess whether Assumption (1) holds, we first use the analytic bounds derived above.

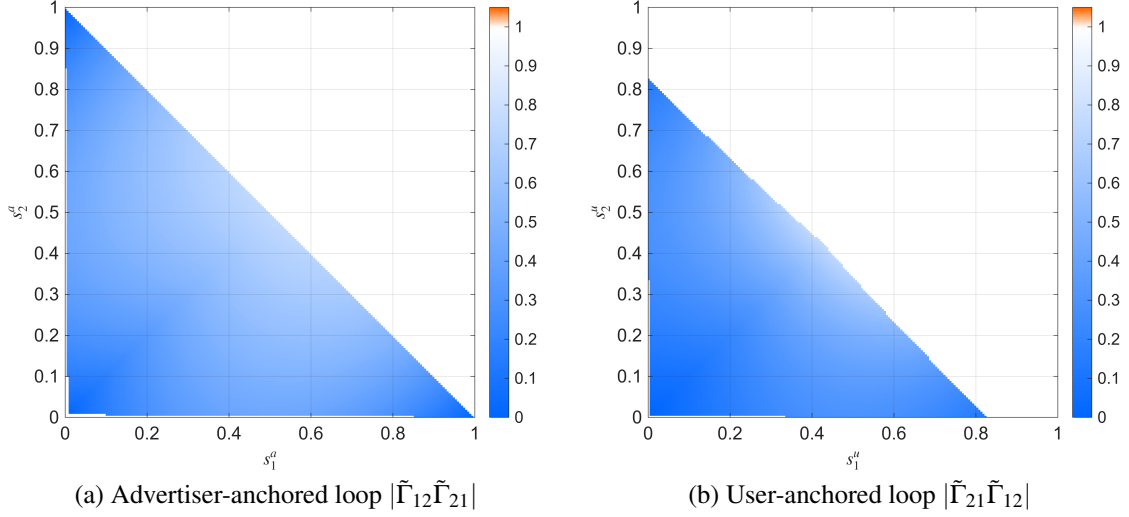


Notes: Each figure shows the fixed points $q_1^u = q_1^u(p_1^{u*}, p_2^{u*}, q_1^{u*}, q_2^{u*}, q_1^u, q_2^u)$ for different values of q_2^u (blue line) and $q_2^u = q_2^u(p_1^{u*}, p_2^{u*}, q_1^{u*}, q_2^{u*}, q_1^u, q_2^u)$ for different values of q_1^u (dashed line). The arrows show the direction in which quantities adjust, obtained by performing one iteration on the quantities and calculating changes.

Figure 5: Phase diagram

For the low specification, $\theta^u = 0.000013$, so that $M^u \theta^u / 2 = 0.65$ is below one. The assumption holds, so we can be sure that there is a unique equilibrium, which is confirmed on the left. For the high specification, $\theta^u = 0.00005$, so that $M^u \theta^u / 2 = 2.5$ exceeds one. The sufficient condition no longer holds, so we cannot be certain that the equilibrium is unique. And indeed, on the right there are three equilibria. On the advertising side, the same analytical bound applies: the supremum of the row sum of $|\Theta^a|$ equals $M^a |\theta^a| / 2$, which for our specification equals 0.25 and is well below one. Assumption 1 therefore holds on the advertising side for all prices and user quantities, so the advertiser coordination game has a unique equilibrium.

We also numerically assess Assumption (1), following the approach described above. For a candidate share vector on one side, inverting the logit share equation pins down the mean utility each platform must deliver to generate its share. Because mean utility is linear in own price and in the platform's cross-side quantity, this yields, for each platform, a closed-form interval of cross-side quantities for which the implied price lies in the admissible price range. A share vector is attainable if both platforms' intervals are non-empty and their minimal required cross-side quantities can jointly be met without exceeding the size of the other side of the market.



Notes: We use the low θ'' value. The two figures report for, respectively, largest row sums of each two-step cross-side feedback loop over the shares each side can reach under an admissible price and some feasible cross-side quantity. See text for details.

Figure 6: Test of Assumption 2

In practice, determining feasible share vectors is fast. With them in hand, one can calculate the maximal row-sum. On the advertising side the gap between the analytical and the numerical approach is small---the worst-case row sum over attainable shares is 0.249 against the analytical bound of 0.250---because attainable advertising shares happen to lie close to 0.5. On the user side, there is a noticeable gap: the attainable maximum is 0.554 against the analytical 0.65.

To assess numerically whether Assumption 2 holds, we sweep a 200×200 grid over each side's two-platform share simplex, with the two sides' grids computed independently, colour the reachable region on each, and, for every reachable share on one side, report the worst-case row sum of the corresponding cross-side loop taken over all reachable shares on the other side.

Figure 6 plots the largest row sum of each two-step cross-side feedback loop over the shares each side can reach under an admissible price and some feasible cross-side quantity. This is done for the low θ'' case. $\tilde{\Gamma}_{12}\tilde{\Gamma}_{21}$ is the advertiser-anchored loop starting on the advertising side, then going to the user side, and then going back to the advertising side. $\tilde{\Gamma}_{21}\tilde{\Gamma}_{12}$ is the mirror loop starting on the user side. Coloured points are reachable; blank regions are shares no admissible price can produce. The row sum stays below one throughout, with supremum 0.76.

The figure shows that both loops stay below one throughout the reachable domain. The supremum over $P \times Q$ is 0.76 for $|\tilde{\Gamma}_{12}\tilde{\Gamma}_{21}|$ and also 0.76 for $|\tilde{\Gamma}_{21}\tilde{\Gamma}_{12}|$. This implies, together with Assumption 1, which also holds, that a reduced form $\hat{q}(p)$ exists at every admissible price vector.

The high θ^u specification behaves differently. Once $M^u\theta^u > 2$ the user-side coordination game itself violates Assumption 1 (its bound is $M^u\theta^u/2 = 2.5$), so $I - \Theta^u$ is nearly singular over part of the attainable set and the reduced-form indirect network effects $\tilde{\Gamma}_{21} = (I - \Theta_{22})^{-1}\Gamma_{21}$ becomes very large there. The two-step feedback then explodes: the maximum attainable row sums are around 447 for $|\tilde{\Gamma}_{12}\tilde{\Gamma}_{21}|$ and around 585 for $|\tilde{\Gamma}_{21}\tilde{\Gamma}_{12}|$, far above one. Intuitively, when same-side coordination is itself fragile, cross-side feedback is unbounded, and neither uniqueness condition can be relied upon — consistent with the multiple coordination equilibria in the right-hand panel of Figure 5.

4 Existence and uniqueness in the competitive game between firms

In this section, we turn to competition between firms. We do so for the general case, but call the market sides a and u . These labels are purely illustrative. Throughout, we refer to them as advertisers and users, following the examples in Sections 2.3.2 and 2.3.3. None of the results depend on this interpretation. The model itself is fully general, allowing for arbitrary groups and both direct and indirect network effects.

4.1 General model

We assume that firms compete in prices. The equilibrium concept is Nash in pure strategies. We write the feasible price space as $P = \prod_f P_f$, where P_f denotes the feasible set of prices chosen by firm f .¹⁹ Best responses are taken over P_f . In such an equilibrium, a firm f takes the prices

¹⁹In footnote 3, we point out that firms could also set prices on the advertising side and the set of features available to users on the user side. Here, we use prices for both market sides because it corresponds most closely to empirical IO practice, as in, e.g., Berry et al. (1995). In Section 5, we point out that one could also think of such

of its competitors as given and maximizes the sum of profits over platforms ℓ in their portfolio $\mathcal{F}_f$,

$$\pi_f = \sum_{\ell \in \mathcal{F}_f} \left\{ (p_\ell^a - mc_\ell^a) \cdot \left(\sum_{g=1}^{G^a} q_{\ell g}^a(p, q) \right) + (p_\ell^u - mc_\ell^u) \cdot \left(\sum_{g=1}^{G^u} q_{\ell g}^u(p, q) \right) \right\}. \quad (9)$$

Here, quantities $q_{\ell g}^a$ and $q_{\ell g}^u$ are functions of prices on the same market side and quantities on the same and on the other market side. Earlier, we have written k for a combination of j and g . From now on, we make a distinction between products j (e.g., a platform) and groups g . There are G^a groups of advertisers and G^u groups of users. Figure 1 above shows an example with one platform with two groups of advertisers and two groups of users.

Recall that $q_{\ell g}^a = q_{\ell g}^a(p^a, q)$ and $q_{\ell g}^u = q_{\ell g}^u(p^u, q)$. Advertising demand on platform j depends on the price of advertising on platform j , so the full vector p^a . It also depends on the number of users in each of the demographic groups, q_{jg}^u for all $g = 1, \dots, G^u$. There are also effects across platforms. Demand for advertising on platform j depends on all the other prices for advertising on other platforms and the numbers of users in each group who are active on all the other platforms. It could also depend on advertising quantities on the same market side, q_{jg}^a for all $g = 1, \dots, G^a$, per our discussion in Section 2.3.1.

To formally define a Nash equilibrium in the context of our model denote the prices set by firm f by p_f . Denote the prices set by all other firms by p_{-f} and explicitly write profits as depending on prices set by f and all its competitors $-f$, $\pi(p_f, p_{-f})$.

Definition 1 (Nash equilibrium). A pure-strategy profile $p^* = (p_f^*, p_{-f}^*)$ is a *Nash equilibrium* if no firm can increase its payoff by unilaterally choosing a different strategy, holding the strategies of all other firms fixed. That is, if

$$\forall f, \forall p_f \in P_f : \pi_f(p_f^*, p_{-f}^*) \geq \pi_f(p_f, p_{-f}^*).$$

We will give conditions so that the second-order conditions hold and there is a unique set of equilibrium prices that satisfy the first-order conditions. Observe that it is generally not possible to find a closed form of the first-order conditions by taking the derivative of (9) with respect

choices as being made in an earlier stage and the stage studied here as the stage determining continuation payoffs. In principle, some prices could also be zero. This could either be imposed, through P_f , or arise endogenously.

to the prices and then explicitly solve for best responses. This is because of the presence of feedback loops, which means that quantities on one market side depend on prices on the same market side and on quantities on the other market side. But quantities on the other market side again depend on quantities on the one market side, which again depend on prices on that first market side, and so on.

Towards this, we write profits as functions of the reduced-form quantities $\hat{q}(p)$. With this, we have $\hat{q}_{jg}^a = \hat{q}_{jg}^a(p^a, p^u)$ and $\hat{q}_{jg}^u = \hat{q}_{jg}^u(p^a, p^u)$, and we can rewrite (9) as

$$\pi_f = \sum_{\ell \in \mathcal{F}_f} \left\{ (p_\ell^a - mc_\ell^a) \cdot \left(\sum_{g=1}^{G^a} \hat{q}_{\ell g}^a(p^a, p^u) \right) + (p_\ell^u - mc_\ell^u) \cdot \left(\sum_{g=1}^{G^u} \hat{q}_{\ell g}^u(p^a, p^u) \right) \right\}.$$

Once, we have rewritten the profit function in terms of the reduced-form quantities, it is straightforward to obtain the first-order conditions. For example, the first-order condition for the advertising price for platform j is given by

$$\frac{\partial \pi_f}{\partial p_j^a} = \left(\sum_{g=1}^{G^a} \hat{q}_{jg}^a(p^a, p^u) \right) + \sum_{\ell \in \mathcal{F}_f} \left\{ (p_\ell^a - mc_\ell^a) \cdot \left(\sum_{g=1}^{G^a} \frac{\partial \hat{q}_{\ell g}^a}{\partial p_\ell^a} \right) + (p_\ell^u - mc_\ell^u) \cdot \left(\sum_{g=1}^{G^u} \frac{\partial \hat{q}_{\ell g}^u}{\partial p_\ell^a} \right) \right\} = 0. \quad (10)$$

It will be convenient to write, from now on,

$$\bar{q}(p) = \begin{pmatrix} \left[\sum_{g=1}^{G^a} \hat{q}_{jg}^a(p^a, p^u) \right]_j \\ \left[\sum_{g=1}^{G^u} \hat{q}_{jg}^u(p^a, p^u) \right]_j \end{pmatrix}$$

for the vector of market demands on both market sides, respectively summed up over all groups.

Also, we write

$$\frac{\partial \bar{q}(p)}{\partial p'} = \begin{pmatrix} \sum_{g=1}^{G^a} \frac{\partial [\hat{q}_{jg}^a / \partial p^{a'}]_j}{\partial p'} & \sum_{g=1}^{G^a} \frac{\partial [\hat{q}_{jg}^a / \partial p^{u'}]_j}{\partial p'} \\ \sum_{g=1}^{G^u} \frac{\partial [\hat{q}_{jg}^u / \partial p^{a'}]_j}{\partial p'} & \sum_{g=1}^{G^u} \frac{\partial [\hat{q}_{jg}^u / \partial p^{u'}]_j}{\partial p'} \end{pmatrix}$$

for the matrix of derivatives of the vector of market demands with respect to all prices. The dimension of this matrix is $2J \times 2J$. $[x_j]_j$ denotes the matrix consisting of the elements x_j ,

stacked on top of one another in the usual way. This gives that, for example, $\left[\hat{q}_{jg}^a/\partial p^{a'}\right]_j$ is the matrix of derivatives of quantities for demographic group g on the advertising side with respect to prices on the advertising side. The summation is then over demographic groups. For instance, the top-left block is the vector consisting of derivatives of all market advertising demands with respect to all advertising prices. We sum over the groups of advertisers because market demand is the sum of demand by each group.

For either market side, denote by ω_f the $J \times 1$ vector indicating which products are owned by firm f . Because firms own the corresponding products on both market sides, the same ownership vector applies on each side. Then, $\Omega = \sum_f \omega_f \omega_f'$ is the usual Nevo (2000, 2001)-type ownership matrix where $\Omega_{ij} = 1$ if product i and j are owned by the same firm, and $\Omega_{ij} = 0$ otherwise. Define the $2J \times 2J$ matrix

$$\bar{Q}(p) \equiv \left(\frac{\partial \bar{q}(p)}{\partial p'} \circ \begin{pmatrix} \Omega & \Omega \\ \Omega & \Omega \end{pmatrix} \right)'.$$

This is the transpose of the Jacobian of aggregate reduced-form demand with respect to prices, multiplied element by element by the corresponding ownership matrix.

Observe that we can write the vector of profits earned by firm f as

$$\pi_f(p_f, p_{-f}) = \omega_f' \{ (p^a - mc^a) \circ \bar{q}^a + (p^u - mc^u) \circ \bar{q}^u \}.$$

The expression in curly brackets is the vector of flow profits, summed over both market sides. It is of dimension $J \times 1$. ω_f' then sums up flow profits for firm f over the products owned by firm f .

The example of the first-order condition above, (10), is the j th row of the system of equations

$$\bar{q}(p) + \bar{Q}(p)(p - mc) = 0. \tag{11}$$

In total, there are $2J$ rows, one for each of the J products and 2 market sides.

Taking the derivative of (11) with respect to prices gives,

$$H(p) \equiv \frac{\partial \bar{q}(p)}{\partial p'} + \bar{Q}(p) + [(p - mc)' \otimes I_{2J}] \frac{\partial \text{vec} \bar{Q}(p)}{\partial p'},$$

where $\text{vec}A$ is the vectorizing operator that stacks the columns A on top of one another. The second-order conditions are that the respective sub-matrices of $H(p)$ for firms f , denoted by $H_f(p)$, are negative definite.

An interior price vector satisfying (11) is a Nash equilibrium if, in addition, each firm's profit is globally concave in its own prices over P_f . Conversely, if the observed quantities and prices satisfy the first-order conditions because they form an equilibrium, the second-order conditions hold, and marginal costs are constant, then, following Rosse (1970), one can solve, as usual, for the vector of marginal costs

$$mc = p + \bar{Q}^{-1}(p) \bar{q}(p).$$

For this, we do not need to assume that the equilibrium is unique. However, a necessary condition is that $\bar{Q}(p)$ is invertible.

A counterfactual equilibrium can be found by solving a new set of first-order conditions like (11) numerically. This involves repeatedly calculating $\bar{q}$, iterating on quantities until convergence following the second part of Proposition 1, and $\bar{Q}(p)$.

In general, there could be multiple equilibria and therefore it is of value for a policy simulation to have a set of sufficient conditions for uniqueness of equilibrium in hand. For each firm f , let p_f denote the subvector of prices chosen by that firm, let p_{-f} collect the prices chosen by all other firms, and let P_f denote firm f 's feasible price set. As usual, we assume $P = \prod_f P_f$, so that firms' feasible price choices are independent across firms. Define the best response function for each firm f ,

$$b_f(p_{-f}) \equiv \arg \max_{p_f \in P_f} \pi_f(p_f, p_{-f}),$$

and denote the vector-valued best response functions of all firms together by $b(p)$.²⁰ Denote

²⁰In general, best responses are correspondences, but here we assume that they are functions. That is, there is a single set of optimal prices for each firm, given the prices set by the competitors.

the j th element of this by $b_j(p)$ and the ℓ th price by p_ℓ .

A sufficient condition for uniqueness is that the stacked best-response mapping $b(p)$ is a contraction. For example, under the sup norm, it suffices that

$$\sup_{p \in P} \max_j \sum_k \left| \frac{\partial b_j(p)}{\partial p_k} \right| < 1. \quad (12)$$

Provided that $b(P) \subseteq P$, this condition also implies existence by the contraction mapping theorem. The condition is stronger than necessary but it guarantees a unique Bertrand-Nash price equilibrium. This use of best-response contractions is in the spirit of standard approaches to uniqueness and stability in oligopoly pricing (Vives, 1999). For general existence results for pure-strategy price equilibria in differentiated-products competition, see Caplin and Nalebuff (1991), and for related uniqueness results in differentiated-products pricing models, see Mizuno (2003).

Next, we discuss how we can test this condition for our framework. Define the selection matrix $S_f = [e'_i]_{i: \tilde{\omega}_{fi}=1}$, where e_i is the i th standard basis vector that has all elements equal to zero and the i th element equal to one, and $\tilde{\omega}_f = (\omega'_f, \omega'_f)'$ is the stacked ownership vector. S_f has as many rows as firm f owns products across market sides. There are $2J$ columns that indicate which products are owned by the firm. For example, if there are 10 products across market sides and the firm owns product 2 and 7, then S_f will have 2 rows. In the first row, the 2nd column will be equal to one and in the second row, the 7th column will be equal to one.

Define the first-order condition map for firm f ,

$$F(p_f, p_{-f}) = S_f \{ \bar{q}(p) + \bar{Q}(p) (p - mc) \}.$$

The derivative of the best response functions with respect to the prices set by the other firms is, by the implicit function theorem,

$$\frac{\partial b_f(p)}{\partial p'_{-f}} = - \left[\frac{\partial F(p_f, p_{-f})}{\partial p'_f} \right]^{-1} \frac{\partial F(p_f, p_{-f})}{\partial p'_{-f}}.$$

This gives

$$\frac{\partial b_f(p)}{\partial p'_{-f}} = -[S_f(\bar{Q}' + \bar{Q})S'_f + C_f]^{-1} \left(\frac{\partial S_f \bar{q}}{\partial p'_{-f}} + S_f \bar{Q} S'_{-f} + C_{-f} \right). \quad (13)$$

Denote by $J_f = 2 \cdot |\mathcal{F}_f|$ the number of products owned by f and by $J_{-f} = 2 \cdot |\mathcal{F}_{-f}|$ the number of products owned by all other firms. Then, C_f is a matrix of dimension $J_f \times J_f$. The ij th element of C_f is given by

$$(C_f)_{ij} = \sum_{k: \bar{\omega}_{fk}=1} \frac{\partial \bar{Q}_{ik}}{\partial p_j} \cdot (p_k - mc_k),$$

where i and j index the platform-side products owned by firm f . C_{-f} is of dimension $J_f \times J_{-f}$ and is given by

$$(C_{-f})_{ij} = \sum_{k: \bar{\omega}_{fk}=1} \frac{\partial \bar{Q}_{ik}}{\partial p_j} \cdot (p_k - mc_k),$$

where i indexes the platform-side products owned by firm f , while j indexes the platform-side products owned by the other firms, collected in $\mathcal{F}_{-f}$.

In practice, one can calculate $\frac{\partial b_f(p)}{\partial p'_{-f}}$ for all firms, and hence for all products, and evaluate numerically whether condition (12) holds.

The following proposition summarizes.

Proposition 2 (existence and uniqueness of Nash equilibrium). *Let Assumptions 1 and 2 hold. Suppose that $b(P) \subseteq P$ and that condition (12) holds. Then a Nash equilibrium in pure strategies exists and is unique. (13) can be used to evaluate this numerically.*

Taken together, the results in this section show that ultimately, the condition for equilibrium existence and uniqueness looks familiar. Our framework and the results in this paper allow us to use this condition in a context with direct and indirect network effects. The network effects affect the first-order conditions through their respective effects on reduced form demand $\hat{q}(p)$.

4.2 Discussion

In order to establish existence and uniqueness of the pricing equilibrium, we work with the row-sum condition (12). Its left-hand side is the largest absolute row sum of the best-reply Jacobian $B(p) = \partial b(p)/\partial p'$, i.e. the matrix sup-norm $\|B(p)\|_\infty = \max_j \sum_k \left| \frac{\partial b_j(p)}{\partial p_k} \right|$. Requiring it to stay below one at every p makes b a sup-norm contraction, which by the contraction mapping theorem yields a unique price equilibrium.

In Section 4.4.1 below, we return to the linear demand example of Section 2.3.2. We show that best replies are affine with a constant Jacobian. Therefore, uniqueness is equivalent to the stacked best-response system being nonsingular, and the commonly used spectral-radius condition is a natural sufficient condition for this.

Both conditions are related. The spectral-radius condition is in turn implied by the row-sum condition (12). The spectral-radius condition is a stability condition and in a linear world, local and global stability coincide. However, in nonlinear systems, the Jacobian $B(p)$ of the best-reply map varies with p . Establishing a contraction requires a genuine matrix norm bounded below one uniformly in p . The spectral radius is not a matrix norm, and therefore $\rho(B(p)) < 1$ at every p does not by itself make b a contraction, so it cannot certify uniqueness in the nonlinear model.

Nonetheless, the spectral radius remains useful for assessing local stability under best-response iteration. It is the local nonlinear analogue of the linear stability condition discussed above: if the spectral radius of the best-response Jacobian evaluated at an equilibrium is smaller than one, the equilibrium is locally asymptotically stable (Vives, 1999; Kuznetsov, 2004).

4.3 Practical considerations

There is one further issue to consider when applying the row-sum condition. Whereas the spectral radius is invariant to the units in which prices are measured, the row sum is not. Consider a fixed rescaling of prices represented by a positive diagonal matrix D , together with the reciprocal rescaling of the corresponding demand coefficients. This leaves demand and the equilibrium

unchanged, while transforming the best-reply Jacobian at every price vector according to

$$B(p) \mapsto D^{-1}B(p)D.$$

The transformation is a diagonal similarity and therefore leaves the eigenvalues, and hence the spectral radius ($\rho(B(p))$), unchanged, while generally changing the row sums. This issue is particularly relevant in the present setting because prices are charged on two different market sides, to different groups of customers, and in different units. For example, advertising prices may be quoted per thousand impressions, whereas user subscription prices are quoted per user per month. The numerical value of the row-sum bound can therefore depend substantially on these otherwise economically innocuous choices of units.

Because a system of units must be fixed across price vectors, we may want to choose D by anchoring the scaling at the equilibrium. Write $B^* = B(p^*)$. Provided that $|B^*|$ has a strictly positive Perron eigenvector v , let $D^* = \text{diag}(v)$, where

$$|B^*|v = \rho(|B^*|)v.$$

Then every row of $D^{*-1}B^*D^*$ has absolute row sum $\rho(|B^*|)$, and

$$\min_{D>0} \|D^{-1}B^*D\|_\infty = \|D^{*-1}B^*D^*\|_\infty = \rho(|B^*|).$$

Thus, D^* delivers the smallest row-sum bound attainable at p^* by a fixed diagonal rescaling.

This is the Collatz-Wielandt characterization of the Perron root (Horn and Johnson, 2013).

Since

$$\|B^*\|_\infty \geq \rho(|B^*|) \geq \rho(B^*),$$

the optimally rescaled row sum at the equilibrium is never larger than the raw row sum. We then hold D^* fixed and evaluate

$$\sup_{p \in P} \|D^{*-1}B(p)D^*\|_\infty$$

over the relevant price region P . A value below one establishes contraction, and hence unique-

ness, on that region. In our numerical application, $B(p)$ varies little over P , so the rescaled row-sum bound remains close to $\rho(|B^*|)$. We report this uniform rescaled bound alongside the corresponding raw row-sum bound in Section 4.4.2.

In practice, the rescaling does not require the pricing equilibrium to be known. Any fixed positive diagonal matrix D represents a valid change of units, and the condition

$$\sup_{p \in P} \|D^{-1}B(p)D\|_{\infty} < 1$$

remains sufficient for uniqueness on the relevant price region P . Knowledge of p^* is useful only for choosing units that make this sufficient condition as sharp as possible. In empirical applications, observed prices are commonly interpreted as equilibrium prices, in which case one can evaluate B at the observed price vector p^{obs} , compute a positive Perron eigenvector of $|B(p^{\text{obs}})|$, and use the resulting diagonal matrix D throughout the analysis.

This procedure does not presuppose that the equilibrium is unique. If several equilibria exist, one may anchor the scaling at any observed or candidate equilibrium. The resulting D is locally optimal at that equilibrium but need not be optimal at the others. Nevertheless, it remains a valid fixed rescaling, and a uniform row-sum bound below one under that scaling still establishes uniqueness and therefore rules out the hypothesized multiplicity.

More generally, any scaling based on a price vector representative of the region P is useful, with better choices being those for which $D^{-1}B(p)D$ has relatively balanced absolute row sums throughout P . If desired, rather than anchoring at a single price vector, one could choose D directly to minimize the worst-case bound

$$\inf D > 0 \sup_{p \in P} \|D^{-1}B(p)D\|_{\infty}.$$

The equilibrium-based Perron scaling is a simple and economically natural approximation to this regional minimax choice when $B(p)$ varies little over P . Finally, observe that this is not always needed. If condition (12) holds without any rescaling, i.e., when D is the identity matrix, then a unique equilibrium is already guaranteed to exist.

4.4 Back to the examples

4.4.1 Back to the linear demand example with two social media platforms

Additional notation and introducing firms For the linear demand example introduced in Section 2.3.2, it follows from the analysis in Section 3.4.2 that, under Assumptions 1 and 2, it has a reduced form

$$q = \hat{\alpha} + \hat{B}p,$$

where $\hat{\alpha} = (I - \Gamma - \Theta)^{-1}\alpha$ and $\hat{B} = (I - \Gamma - \Theta)^{-1}B$. Recall that we ordered products such that

$$q = \begin{pmatrix} q_1^a \\ q_2^a \\ q_1^u \\ q_2^u \end{pmatrix}, \quad p = \begin{pmatrix} p_1^a \\ p_2^a \\ p_1^u \\ p_2^u \end{pmatrix}, \quad \hat{\alpha} = \begin{pmatrix} \hat{\alpha}_1^a \\ \hat{\alpha}_2^a \\ \hat{\alpha}_1^u \\ \hat{\alpha}_2^u \end{pmatrix}.$$

Now consider two firms, indexed by $i \in \{1, 2\}$. Each firm operates on two market sides: an advertising side, denoted by superscript a , and a user side, denoted by superscript u . Firm i chooses two prices,

$$p_i = \begin{pmatrix} p_i^a \\ p_i^u \end{pmatrix}.$$

The rival of firm i is denoted by $j \neq i$. For the derivations that follow, it is useful to reorder the system by firm. Define

$$q_1 = \begin{pmatrix} q_1^a \\ q_1^u \end{pmatrix}, \quad q_2 = \begin{pmatrix} q_2^a \\ q_2^u \end{pmatrix},$$

and

$$\alpha_1 = \begin{pmatrix} \hat{\alpha}_1^a \\ \hat{\alpha}_1^u \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} \hat{\alpha}_2^a \\ \hat{\alpha}_2^u \end{pmatrix}.$$

Then the firm-specific demand systems can be written as

$$q_i = \alpha_i + B_{ii}p_i + B_{ij}p_j.$$

Writing

$$\widehat{B} = \begin{pmatrix} \widehat{b}_{11} & \widehat{b}_{12} & \widehat{b}_{13} & \widehat{b}_{14} \\ \widehat{b}_{21} & \widehat{b}_{22} & \widehat{b}_{23} & \widehat{b}_{24} \\ \widehat{b}_{31} & \widehat{b}_{32} & \widehat{b}_{33} & \widehat{b}_{34} \\ \widehat{b}_{41} & \widehat{b}_{42} & \widehat{b}_{43} & \widehat{b}_{44} \end{pmatrix},$$

the relevant demand blocks, also with the ordering by firm, are

$$\widehat{B}_{11} = \begin{pmatrix} \widehat{b}_{11} & \widehat{b}_{13} \\ \widehat{b}_{31} & \widehat{b}_{33} \end{pmatrix}, \quad \widehat{B}_{12} = \begin{pmatrix} \widehat{b}_{12} & \widehat{b}_{14} \\ \widehat{b}_{32} & \widehat{b}_{34} \end{pmatrix},$$

$$\widehat{B}_{21} = \begin{pmatrix} \widehat{b}_{21} & \widehat{b}_{23} \\ \widehat{b}_{41} & \widehat{b}_{43} \end{pmatrix}, \quad \widehat{B}_{22} = \begin{pmatrix} \widehat{b}_{22} & \widehat{b}_{24} \\ \widehat{b}_{42} & \widehat{b}_{44} \end{pmatrix}.$$

Thus $\widehat{B}_{ii}$ contains the effects of firm i 's own prices on firm i 's own demands, while $\widehat{B}_{ij}$ contains the effects of firm j 's prices on firm i 's demands.

Each firm has constant marginal costs on both sides,

$$mc_i = \begin{pmatrix} mc_i^a \\ mc_i^u \end{pmatrix}.$$

Firm i 's profit is

$$\pi_i(p_i, p_j) = (p_i - mc_i)' q_i.$$

Substituting demand gives

$$\pi_i(p_i, p_j) = (p_i - mc_i)' \left(\alpha_i + \widehat{B}_{ii} p_i + \widehat{B}_{ij} p_j \right).$$

Best response functions Next, we derive the best response functions. Firm i chooses p_i taking p_j as given. The profit-maximization problem is

$$\max_{p_i} \pi_i(p_i, p_j) = \max_{p_i} (p_i - mc_i)' \left(\alpha_i + \widehat{B}_{ii} p_i + \widehat{B}_{ij} p_j \right).$$

The first-order conditions are, in matrix notation,

$$\frac{\partial \pi_i}{\partial p_i} = q_i + \widehat{B}_{ii}'(p_i - mc_i) = 0.$$

Substituting $q_i = \alpha_i + \widehat{B}_{ii}p_i + \widehat{B}_{ij}p_j$ yields

$$\alpha_i + \widehat{B}_{ii}p_i + \widehat{B}_{ij}p_j + \widehat{B}_{ii}'(p_i - mc_i) = 0.$$

Collecting terms with p_i gives

$$(\widehat{B}_{ii} + \widehat{B}_{ii}')p_i = \widehat{B}_{ii}'mc_i - \alpha_i - \widehat{B}_{ij}p_j.$$

Define

$$A_i \equiv \widehat{B}_{ii} + \widehat{B}_{ii}'.$$

If A_i is nonsingular, firm i 's best response is (provided that the second-order conditions hold, see below)

$$BR_i(p_j) = A_i^{-1} \left(\widehat{B}_{ii}'mc_i - \alpha_i - \widehat{B}_{ij}p_j \right).$$

For firm 1, this gives

$$BR_1(p_2) = (\widehat{B}_{11} + \widehat{B}_{11}')^{-1} \left(\widehat{B}_{11}'mc_1 - \alpha_1 - \widehat{B}_{12}p_2 \right).$$

For firm 2,

$$BR_2(p_1) = (\widehat{B}_{22} + \widehat{B}_{22}')^{-1} \left(\widehat{B}_{22}'mc_2 - \alpha_2 - \widehat{B}_{21}p_1 \right).$$

Written in terms of the entries of $\widehat{B}$,

$$A_1 = \widehat{B}_{11} + \widehat{B}_{11}' = \begin{pmatrix} 2\widehat{b}_{11} & \widehat{b}_{13} + \widehat{b}_{31} \\ \widehat{b}_{13} + \widehat{b}_{31} & 2\widehat{b}_{33} \end{pmatrix},$$

and

$$A_2 = \widehat{B}_{22} + \widehat{B}'_{22} = \begin{pmatrix} 2\widehat{b}_{22} & \widehat{b}_{24} + \widehat{b}_{42} \\ \widehat{b}_{24} + \widehat{b}_{42} & 2\widehat{b}_{44} \end{pmatrix}.$$

Existence of best response and equilibrium The first issue is whether each firm has a well-defined best response. Since profits are quadratic in own prices, this follows from own-profit concavity.

The Hessian of firm i 's profit with respect to its own price vector p_i is

$$\frac{\partial^2 \pi_i}{\partial p_i \partial p_i'} = \widehat{B}_{ii} + \widehat{B}'_{ii} = A_i.$$

Thus a sufficient condition for firm i 's problem to be strictly concave in its own prices is

$$A_i = \widehat{B}_{ii} + \widehat{B}'_{ii} \prec 0,$$

where $A_i \prec 0$ denotes that A_i is negative definite. In that case, firm i 's objective is strictly concave in p_i , and the first-order condition characterizes the unique best response.

It will be convenient to write

$$BR_i(p_j) = c_i + D_i p_j,$$

where

$$c_i = A_i^{-1}(\widehat{B}'_{ii} m c_i - \alpha_i), \quad D_i = -A_i^{-1} \widehat{B}_{ij}.$$

Then,

$$BR_1(p_2) = c_1 + D_1 p_2, \quad BR_2(p_1) = c_2 + D_2 p_1.$$

To obtain existence of a Nash equilibrium, recall that firms choose prices from compact and convex strategy sets $P_i \subset \mathbb{R}^2$, and that best responses remain in these sets. Since profits are continuous in all prices and strictly concave in own prices, each best response is nonempty and single-valued. Standard fixed-point arguments then imply existence of a pure-strategy Nash

equilibrium.

Uniqueness of equilibrium Write the two best responses as a single stacked system for each firm,

$$p_1 = c_1 + D_1 p_2, \quad p_2 = c_2 + D_2 p_1.$$

Move all endogenous prices to the left-hand side:

$$p_1 - D_1 p_2 = c_1, \quad -D_2 p_1 + p_2 = c_2.$$

Stacking the two equations gives

$$\begin{pmatrix} I & -D_1 \\ -D_2 & I \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Define

$$\mathcal{D} \equiv \begin{pmatrix} 0 & D_1 \\ D_2 & 0 \end{pmatrix}, \quad C \equiv \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad P \equiv \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

Then the stacked best-response system can be written as

$$P = C + \mathcal{D}P,$$

or

$$(I - \mathcal{D})P = C.$$

Therefore, if $I - \mathcal{D}$ is nonsingular, the equilibrium price vector is unique and is given by

$$P^* = (I - \mathcal{D})^{-1}C.$$

A standard sufficient condition for $I - \mathcal{D}$ to be nonsingular is

$$\rho(\mathcal{D}) < 1,$$

where $\rho(\cdot)$ denotes the spectral radius:

$$\rho(\mathcal{D}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{D}\}.$$

A convenient sufficient condition for this condition is the row-sum norm condition (12) imposed on the stacked best-response matrix. For any square matrix G ,

$$\rho(G) \leq \|G\|_\infty,$$

where

$$\|G\|_\infty = \max_i \sum_j |g_{ij}|.$$

Therefore, if

$$\|\mathcal{D}\|_\infty < 1,$$

then

$$\rho(\mathcal{D}) < 1,$$

and the equilibrium is unique.

Because $\mathcal{D}$ has zero diagonal blocks, this can be written in terms of the two best-response slope matrices as

$$\max\{\|D_1\|_\infty, \|D_2\|_\infty\} < 1.$$

Indeed, if

$$D_1 = \begin{pmatrix} d_{11}^1 & d_{12}^1 \\ d_{21}^1 & d_{22}^1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} d_{11}^2 & d_{12}^2 \\ d_{21}^2 & d_{22}^2 \end{pmatrix},$$

then the four scalar row-sum inequalities are

$$|d_{11}^1| + |d_{12}^1| < 1, \quad |d_{21}^1| + |d_{22}^1| < 1,$$

$$|d_{11}^2| + |d_{12}^2| < 1, \quad |d_{21}^2| + |d_{22}^2| < 1.$$

Substituting the model-specific expressions,

$$D_1 = -A_1^{-1}\widehat{B}_{12}, \quad D_2 = -A_2^{-1}\widehat{B}_{21},$$

a direct sufficient condition is therefore

$$\max \left\{ \left\| A_1^{-1}\widehat{B}_{12} \right\|_{\infty}, \left\| A_2^{-1}\widehat{B}_{21} \right\|_{\infty} \right\} < 1.$$

Substituting $A_i = \widehat{B}_{ii} + \widehat{B}'_{ii}$, this becomes

$$\max \left\{ \left\| (\widehat{B}_{11} + \widehat{B}'_{11})^{-1}\widehat{B}_{12} \right\|_{\infty}, \left\| (\widehat{B}_{22} + \widehat{B}'_{22})^{-1}\widehat{B}_{21} \right\|_{\infty} \right\} < 1.$$

This condition says that each row of the one-step stacked best-response matrix has total absolute feedback less than one. Economically, a change in one firm's price vector induces a best-response adjustment by the rival, and the condition requires these one-step response slopes to be dampened in every price component.

4.4.2 Back to the logit example

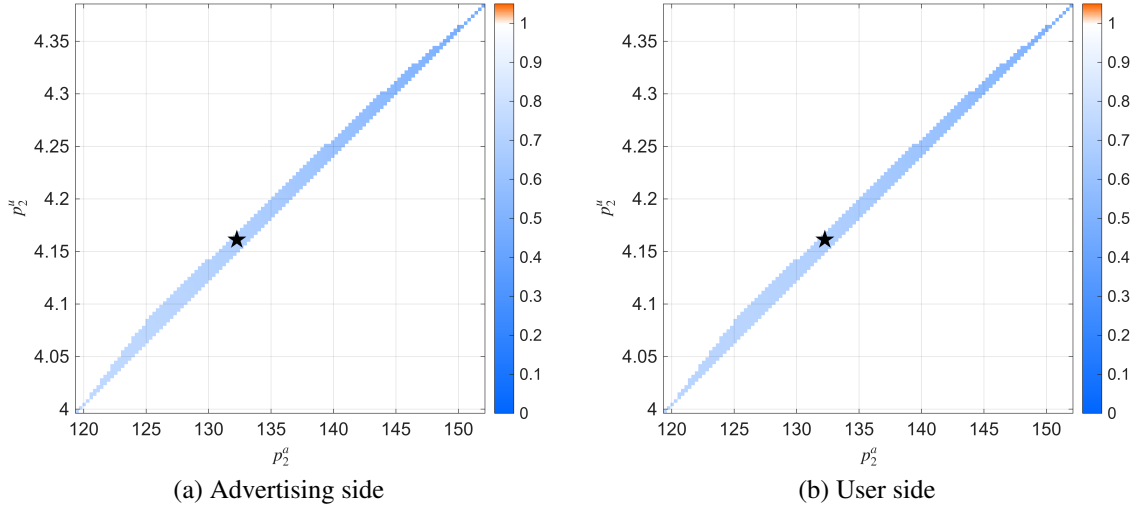
We now verify numerically that the pricing game between the two platforms has a unique equilibrium, restricting attention to the low value of θ'' .²¹ Proposition 2 shows that under Assumptions 1 and 2 the equilibrium is unique if the largest absolute row sum of the stacked best-response Jacobian stays below one, as stated in (12). We evaluate this condition using the analytical best-response derivative in (13). Condition (12) must hold at every price. We implement this by evaluating it on newly constructed sets R_1 and R_2 . These are the sets of prices each firm would ever actually charge. We construct R_f as the convex hull of firm f 's profit-maximizing own price against a grid of the rival's price. We perform up to two rounds of the iterated elimination of never-best responses that defines rationalizable play, stopping as soon as concavity and the row-sum condition both hold on the resulting hull; as detailed below, two

²¹For the high value of θ'' considered above, Assumption 1 fails, the reduced form $\hat{q}(p)$ is no longer single-valued, best responses are not well defined, and the pricing game inherits the multiplicity already present in the coordination game, so we omit that case here.

rounds turn out to be needed here. In the first round, we do this for all prices in the price box the rival could possibly set. For each price, we numerically maximize profits. When doing so, we deliberately do not use the first-order conditions, because we have not yet assessed whether the second-order conditions hold (more on this below). Iterating further would shrink R_f more; in subsequent rounds we grid the rival's price only within the previous round's hull, which is how the second round below refines the first. Every Bertrand-Nash equilibrium is a profile of mutual best responses, so it must lie in $R_1 \times R_2$, a far smaller set than the box, which makes condition (12) restricted to $R_1 \times R_2$ the economically relevant test.

Then, we assess whether the second-order conditions hold. This is important, because the best-response derivative in (13) comes from the implicit function theorem applied to each firm's first-order conditions, valid only where that firm's own pricing problem is strictly concave at the price pair evaluated. As with the row-sum condition, we check concavity where it is needed, on R_1 and R_2 : for firm f , holding the rival's price at each point of its attainable set, we maximize the largest eigenvalue of firm f 's own-price profit Hessian over firm f 's own attainable set. In the first round the worst-case eigenvalue is 1.97 for both firms --- above zero, so concavity does not yet hold everywhere on $R_1 \times R_2$ --- so we shrink each firm's rival-price grid to that round's hull and repeat. In the second round the worst-case eigenvalue is -0.52 for both firms, well below zero, so the implicit-function-theorem derivative used next is valid at every price vector at which condition (12) is evaluated.

This is the concavity we deferred earlier: solving each firm's own first-order condition now finds its best response directly, rather than by numerically maximizing profits. For a given rival price p_{-f} , firm f 's profit is strictly concave over the convex set R_f , so the first-order condition $F(p_f, p_{-f}) = 0$ has at most one root there, and any interior root is the unique global maximizer, firm f 's best response $b_f(p_{-f})$. This is exactly the shortcut we could not take when constructing R_1 and R_2 above, since concavity was not yet known there; now that it is, using the first-order condition is no longer circular. In the figure below, for each rival price $p_2 \in R_2$ we solve $F(p_1, p_2) = 0$ directly for firm 1's own two prices, giving $p_1 = b_1(p_2)$, and evaluate the row-sum condition at the resulting four prices.



Notes: Each figure shows firm 1's best-response row sums based on (12) and (13). We plot the fixed-scaling bound along firm 1's best-response graph as firm 2's prices vary over the attainable set R_2 , using the Perron-optimal units anchored at the equilibrium. See Section 4.3 for additional details. The stars denotes firm 2's equilibrium advertising and user prices, respectively. Because of symmetry, these are also the prices set by firm 1.

Figure 7

Figure 7 plots firm 1's best-response row sum, one panel for each price it sets, over firm 2's attainable price set R_2 , with firm 1 on its best response at each point and firm 2's equilibrium price marked by a star. Specifically, following Section 4.3, we report the fixed-scaling row-sum bound along firm 1's best-response graph over R_2 , using the Perron-optimal units anchored at the equilibrium. The figure shows that condition (12) holds for firm 1. By symmetry, the picture for firm 2 is identical. Condition (12) therefore holds, and the star marks the resulting unique equilibrium.

5 Conclusion

Platform markets pose a central challenge for empirical industrial organization. Participation on one side of the market affects participation on the other, and these feedback effects may differ across user groups, firms, and market sides. As a result, prices, markups, and welfare cannot generally be inferred from one-sided demand elasticities alone. Counterfactual analysis in such markets requires an empirical framework that treats quantities on all sides as jointly determined

in equilibrium.

This paper develops such a framework. We show how flexible demand systems can be embedded in a two-sided oligopoly model with heterogeneous users, direct network effects within sides, and indirect network effects across sides. The approach nests standard differentiated-products models as a special case when network effects are absent, but extends them to environments in which the relevant demand objects are fixed points of a user-side coordination game. We provide sufficient, empirically checkable conditions under which this coordination game has a unique equilibrium. These conditions discipline the feedback loops that arise when participation on one side affects participation on the other and, possibly, participation within the same side.

We then show how these demand-side objects enter firms' pricing incentives. Once reduced-form demand responses to prices are obtained, marginal costs can be recovered from firms' first-order conditions in a way that parallels standard merger analysis, but with the full feedback between market sides incorporated. We also provide sufficient conditions for existence and uniqueness of the firms' pricing equilibrium. This distinction between uniqueness of user participation given prices and uniqueness of firm pricing equilibrium is important: the former is necessary for well-defined demand, while the latter is required for unambiguous counterfactual predictions.

The framework is useful for merger simulation and other policy counterfactuals in platform markets. Ignoring network effects can mismeasure both the level and the structure of competitive effects. A price change on one side may affect profits not only through demand on that side, but also through induced changes in participation on the other side, which then feed back into the original side. These feedbacks can amplify, dampen, or even reverse the predictions obtained from a one-sided analysis. The model makes these channels explicit and provides conditions under which they can be analyzed with the same discipline as in standard empirical IO.

The broader contribution is to provide a bridge between the theory of two-sided markets and the empirical tools used in competition policy. Rather than requiring a specialized demand specification or imposing that one side is priced at zero or that firms set quantities, which could

well be reasonable in some situations, but not all, the framework accommodates flexible demand on both sides and allows network effects to vary across groups. This makes it applicable to a wide range of markets, including media, search, social networks, app ecosystems, and other digital platforms. More generally, the paper shows how familiar tools—demand estimation, cost recovery, and counterfactual simulation—can be extended to environments in which market outcomes are shaped by equilibrium feedback across interdependent user groups.

Finally, we would like to discuss two limitations of this paper. As discussed in [Section 2.2](#), we focus on situations with a unique equilibrium in the coordination game. The main motivation for this is that thereby, the model delivers unique counterfactual predictions, which are useful for policy analysis. This is not to say that we can’t allow for markets that are prone to tipping in our framework. We can allow for tipping, but require that the model delivers a unique prediction. For this, one could add to the model a part that determines what the market tips to, so that the equilibrium is again unique. This could be part of the demand system and empirically motivated. For example, Allcott et al. (2025) find that in the market for online search, which is often described as prone to tipping because usage in the past gives a search engine a head start for producing better search results in the future, Google is genuinely preferred by many users.

A second limitation is that we focus on the interaction between firms and users after earlier choices such as entry, product positioning, platform design, or other long-run investments have been made. This can be seen as a limitation, since these earlier choices are often important and can themselves be modeled as strategic decisions. Indeed, prior work in differentiated-products competition and empirical IO has studied environments in which firms first choose locations, product types, or market participation and then compete in the product market (d’Aspremont et al., 1979; Mazzeo, 2002; Seim, 2006). Related work on two-sided platforms also embeds pricing in richer platform environments with endogenous variety, participation, or commitment choices (Hagiu, 2006, 2009). In principle, such a stage 0 could be added to our framework. Firms could first make entry, positioning, design, or investment decisions, and the coordination and pricing games studied here would then determine continuation payoffs. We leave this extension aside to keep the analysis focused on the two ensuing stages. This focus is natural for

the questions we have in mind. Price setting is central to merger analysis and to many of the policy counterfactuals for which structural empirical IO tools are commonly used. By providing conditions under which demand and pricing outcomes are uniquely defined after earlier choices have been fixed, the framework delivers the continuation model needed for such analyses and for future work that endogenizes the earlier stage.

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