A two-dimensional control problem arising from dynamic contracting theory.[‡]

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Abstract

We study a dynamic corporate finance contracting model in which the firm's profitability fluctuates and is impacted by the unobservable managerial effort. Thereby, we introduce in an agency framework the issue of strategic liquidation. We show that the principal's problem takes the form of a two-dimensional fully degenerate Markov control problem. We prove regularity properties of the value function and derive explicitly the optimal contract that implements full effort. Our regularity results appear in some recent studies but with heuristic proofs that do not clarify the importance of the regularity of the value function at the boundaries.

Keywords: Principal-Agent Problem, two-dimensional control problem, regularity properties

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1 Introduction

Stochastic control models in economics aim at obtaining qualitative properties of value functions and at deriving optimal control policies in order to analyse various economic questions and to propose quite explicit recommendations. To meet this objective, economists always follow a standard route which consists to build value functions in two steps:

i) derive the associated HJB equation whose solution gives a candidate value function, and, as a by-product, a candidate optimal policy, if any,

ii) apply a verification theorem based on Itô's formula which asserts that a *smooth* solution to the HJB equation coincides with the value function.

The key of this approach is to show that the HJB equation admits a solution that is regular enough to apply the Itô's formula needed in the verification theorem. However, it is well-known that the value function of a stochastic control problem is generally a solution to the associated HJB equation in some weak sense such as in the viscosity sense. Yet, the concept of viscosity solution does not give a clear set of conditions to derive regularity results for multi-dimensional problems, even if it has proved to be very efficient to provide numerical approximations of value functions, which forces to argue case by case.

A recent literature gives results to check at hand the regularity of value functions for one-dimensional stochastic control problems: Strulovici and Szydlowski [23] avoid the concept of viscosity solutions and use a shooting method to prove regularity results. Pham [17] shows that the value functions of a class of optimal switching problems are differentiable by means of viscosity solutions. Regarding multi-dimensional control problems, Soner and Shreve [21], Hynd [12], [13] and Hynd and Mawi [14] prove regularity results for elliptic PDEs arising from singular control problems. A common feature of these studies is the uniform ellipticity assumption about the second order differential operator that defines the elliptic PDE. Also in a multi-dimensional framework, Daskalopoulos and Feehan [6] and Lamberton and Terenzi [15] obtain regularity results for the obstacle problem arising from the American option pricing in the Heston model where the second order differential operator degenerates only on the boundary of the domain.

Motivated by economic relevance, two-dimensional fully degenerate stochastic control problems have emerged recently from dynamic contracting in corporate finance.[1] In these new models, existence of derivatives of value functions, regularity properties and existence of optimal controls do not follow from the classical theory or from recent studies and can be very challenging. We study in this paper such a model.

Specifically, we consider a dynamic contracting model in corporate finance in which the firm's profitability fluctuates across time and is impacted by unobservable managerial effort. Thereby, we integrate in an agency framework with hidden action the issue of strategic liquidation studied in the real option literature. We show that the principal's problem takes the form of a two-dimensional fully degenerate Markov control problem. We prove regularity properties of the value function that are instrumental in the construction of the optimal contract that implements full effort, which we derive explicitly. Our mathematical

[1]We review the literature below.

results go beyond our model and complement the heuristic derivation of regularity results made in recent economic studies on dynamic contracting in corporate finance.

Dynamic contracting models in corporate finance are based on the premise that two factors drive the relationship between firm's owners (principal) and firm's managers (agent). First, owners delegate tasks to managers. Second, incentives of managers and those of owners are not fully aligned. The firm's manager may take some actions providing him private benefits and having a negative externality on the firm's cash flows. This impacts firm's owner payoff. Those actions taken by the firm's manager are typically unobservable. Hence, the firm's owner problem is to find the best contract that aligns the interest of the firm's manager with her own. Clearly, the mathematical formulation of the problem depends on the modeling of the cash flows. A common assumption is to model cash flows generated by the firm as the increment of an arithmetic Brownian motion

$$dY_t \;=\; \mu\, dt + \sigma\, dZ_t, \tag{1}$$

where (Z_t) is a standard Brownian motion. The process (Y_t) represents the cumulative cash flows, its increment dY_t the cash flows over a period $[t, t + dt)$, and its drift, the cash flow rate $\mu > 0$, corresponds to the firm's profitability.[2] The attraction of modeling cumulative cash flows as an arithmetic brownian motion comes from the simple form taken by incentive compatibility conditions. The principal problem reduces then to a tractable one-dimensional Markov control problem whose value function has well established regularity properties.[3] In this standard environment, shocks on cash flows are identically, independently distributed. The firm's profitability $\mu > 0$ is constant across time and liquidation is always inefficient. Notably, leaving aside agency issues, the value of the firm at any time t is

$$\mathbb{E}\left(\int_t^{\infty} e^{-r(s-t)} dY_s\right) = \frac{\mu}{r},$$

which clearly shows that the classic dynamics (1) is not appropriate to study optimal exit decision.

We consider the simplest modeling of the cash flows that allows to integrate a strategic liquidation issue in an agency framework. In our model, the firm produces cash flow $X_t\, dt$, where X_t evolves according to a Brownian motion with volatility σ. Specifically, the cumulative cash flows process Y follows now the dynamics

$$dY_t = X_t\, dt, \tag{2}$$

where $X_t = x + \sigma Z_t$. The cash flow rate X_t corresponds to the firm's profitability which fluctuates across time in sharp contrast with the environment defined by (1). Cash flows are serially correlated over time and strategic liquidation is an issue. In particular, leaving aside agency issues, the value of the firm at any time t is $v_0(X_t)$ where

$$v_0(x) = \sup_{\tau} \mathbb{E}\left[\int_0^{\tau} e^{-rs} X_s^x\, ds\right], \tag{3}$$

[2] See the seminal paper of De Marzo and Sannikov [7] and Biais, Mariotti and Rochet [1] for a survey of the literature.

[3] See e.g. Sannikov [20], De Marzo and Sannikov [7] and Strulovici and Szydlowski [23].

which corresponds to a standard exit problem studied in Dixit and Pindyck [9]. We study how moral hazard with unobservable managerial effort impacts this simple strategic liquidation problem. The dynamics of equation (2) defines an environment in which the principal is concerned with the random profitability of the firm (that may induce her to strategically liquidate it) and by the agent's actions on the profitability of the firm. The two concerns are very much interconnected. We show that the principal's problem takes the form of a two-dimensional Markov control problem with state variables, the so-called continuation value of the agent, and the level of the firm's profitability. The associated HJB equation is fully degenerate with discontinuous coefficients. Following the literature, we solve the firm's owner problem in the set of contracts that induce the manager to exert full effort all the time. We establish all the required regularity properties of the associated value function. We point out at each step of our analysis the novelty of our results and explain how they complement recent studies.

Considering a setting in which cash flows are serially correlated across time is key for our analysis. This relates our study to the literature on dynamic contracting with persistent private information initiated by Zhang [26], Strulovici [22] and Williams [25]. In these papers, the principal-agent problem is formulated as a cash diversion problem where principal demands (thruth-telling) reports from the agent. Theses studies are not related to real option theory and to the issue of strategic liquidation. Closer to our study are Faingold and Vasama [10] and Vasama [24] who develop cash diversion models linked to real option theory. Again, our model deals with unobservable managerial effort on firm's profitability and is therefore different. DeMarzo and Sannikov [8] study a dynamic contracting learning model in which the firm's profitability is constant but unknown by the principal and the agent. Our results provide a mathematical ground to these studies that overlooked regularity issues. A last related study is He [11] who proposes an agency model in which the agent controls the size of the firm that follows a geometric brownian motion. Strategic liquidation is not an issue and a scale invariance property allows to write the principal's problem as a one-dimensional control problem. Thus, mathematical issues are very different from our paper. Still, in our model, as in He [11], the cash flow rate fluctuates and is impacted by unobservable managerial effort. It follows that the implementation of the optimal contract is very similar. Specifically, we find that a performance-based grant that involves a stock and a bond implement the optimal contract. The difference with the He's implementation relies on the use of a bond which accounts for the strategic liquidation issue in our model.

The outline of the paper is as follows. Section 2 develops the mathematical model and writes the principal's control problem. Section 3 derives the incentive compatibility conditions and the Markovian representation of the principal's problem. Section 4 contains our main results. We derive regularity properties of the value function of the principal's problem, characterize the optimal contract in the class of contracts inducing effort at any time, and discuss the implementability of the contract. Section 5 summarizes our findings and presents open questions for future research.

2 The model

Principal and agent. We consider a firm that hires a manager to operate a project. The firm's owner, or the principal, has access to unlimited funds and the manager, or agent, is protected by limited liability. The agent and the principal both agree on the same discount rate r. We assume that, at any time t, the project produces observable cash flows if and only if the manager is in charge. In particular, the project is abandoned when the manager is fired and we assume without loss of generality that its scrap value is zero. The cumulative cash flows process (Y_t) and the profitability process (X_t) evolve as

$$dY_t = X_t\, dt \;\text{ and }\; dX_t = -\delta a_t\, dt + \sigma dZ_t^a, \quad X_0 = x \tag{4}$$

where δ and σ are positive constants, (Z_t^a) is a Brownian motion, and $a_t \in [0,1]$ is the agent's unobservable action. The unobservable action $a_t = 0$ is called the effort action, the unobservable action $a_t > 0$ is called the shirking action. Thus, shirking has a negative effect $-\delta a_t$ on profitability. Whenever the agent shirks, he receives a private benefit $Ba_t\, dt$ where B is a positive constant.

Probabilistic model. Formally, we consider the probability space $\Omega = \mathcal{C}([0,\infty),\mathbb{R})$, the set of continuous real functions on $[0,+\infty)$ endowed with the Wiener measure denoted by $\mathbb{P}^0$. Let (Z_t) be a Brownian motion under $\mathbb{P}^0$. We denote $(\mathcal{F}_t)$ the natural filtration generated by (Z_t) completed by the $\mathbb{P}^0$ null sets. Under $\mathbb{P}^0$, we assume that the project's profitability evolves as

$$dX_t = \sigma dZ_t.$$

Thus, $\mathbb{P}^0$ corresponds to the probability distribution of the profitability when the agent chooses at any time the effort action. For any action process $a = (a_t)$ which is assumed to be a $\mathcal{F}_t$ adapted process with values in $[0,1]$, we define

$$\gamma_t^a = \exp\left[\int_0^t -\left(\frac{\delta a_s}{\sigma}\right) dZ_s - \frac{1}{2}\int_0^t \left(\frac{\delta a_s}{\sigma}\right)^2 ds\right].$$

Because the action process (a_t) is bounded, the process (γ_t^a) is an $\mathcal{F}$- martingale. We then define a probability $\mathbb{P}^a$ on Ω such that

$$\frac{d\mathbb{P}^a}{d\mathbb{P}^0}|\mathcal{F}_t = \gamma_t^a.$$

The process (Z_t^a) with

$$Z_t^a = Z_t + \int_0^t \left(\frac{\delta a_s}{\sigma}\right) ds$$

is a Brownian motion under $\mathbb{P}^a$. Therefore, any action process (a_t) induces a probability measure $\mathbb{P}^a$ on Ω for which the dynamics of cash flows is given by Equation (4).

Problem formulation. Following the literature, a contract is a triplet (C, τ_L, a) that specifies nonnegative transfers $C = (C_t)$ (remuneration) from the principal to the agent, a stopping time τ_L at which the project is liquidated and an action process (a_t) that the principal recommends to the agent. The process (C_t) is $\mathcal{F}^X$-adapted, nondecreasing (reflecting agent's limited liability), τ_L is an $\mathcal{F}^X$-stopping time, and, for any action process (a_t), we assume

$$\mathbb{E}^a\left[\int_0^{\tau_L} e^{-rs} dC_s\right] < +\infty. \tag{5}$$

Throughout the paper $(\mathcal{F}_t^X)$ denotes the $\mathbb{P}^a$-augmentation of the filtration generated by $(X_t)_{t\geq 0}$ and $\mathcal{T}^X$ the set of $\mathcal{F}^X$-stopping times.

For a fixed contract $\Gamma = (C, \tau_L, a)$. The agent's expected profit and the principal's expected profit associated to Γ are respectively,

$$V_A(\Gamma) = \mathbb{E}^a\left[\int_0^{\tau_L} e^{-rt}(Ba_t\,dt + dC_t)\right],$$

and

$$V_P(\Gamma) = \mathbb{E}^a\left[\int_0^{\tau_L} e^{-rt}(X_t\,dt - dC_t)\right].$$

An *incentive-compatible* action process $a^*(C,\tau_L) = (a_t^*(C,\tau_L))$ is an agent best reply in term of effort to a given remuneration and liquidation policy (C,τ_L). That is, for any action process a, the action process $a^*(C,\tau_L)$ satisfies

$$\mathbb{E}^a\left[\int_0^{\tau_L} e^{-rt}(Ba_t\,dt + dC_t)\right] \leq \mathbb{E}^{a^*(C,\tau_L)}\left[\int_0^{\tau_L} e^{-rt}(Ba_t^*(C,\tau_L)\,dt + dC_t)\right].$$

We say that a contract (C,τ_L,a) is incentive compatible or (C,τ_L) induces an effort strategy $a^*(C,\tau_L)$ if $a = a^*(C,\tau_L)$. An optimal contract is an incentive compatible contract that maximizes the expected principal's profit at date 0 subject to delivering to the agent a payoff larger than her reservation utility $w_0 > 0$. The principal problem is then to find, if any, an optimal contract. Formally, the principal studies the problem

$$\sup_{C,\tau_L} \mathbb{E}^{a^*(C,\tau_L)}\left[\int_0^{\tau_L} e^{-rt}(X_t\,dt - dC_t)\right] \tag{6}$$

$$\text{s.t } \mathbb{E}^{a^*(C,\tau_L)}\left[\int_0^{\tau_L} e^{-rt}(Ba_t^*(C,\tau_L)\,dt + dC_t)\right] \geq w_0. \tag{7}$$

We refer inequality (7) as the agent's participation constraint.

3 Incentive compatibility and Markov formulation

This section develops in our setting a standard result due to Sannikov [20] generalized by Cvitanic, Possamaï and Touzi [4] and [5]: the continuation value of the agent (defined below) characterizes the incentive compatible actions and allows for a Markov formulation of the principal's problem (6)-(7).

Fix a contract $\Gamma = (C,\tau_L,a)$ and assume for a while that the action process (a_t) is incentive compatible which yields that both players have the same set of information. Let us define the process $W^\Gamma = (W_t^\Gamma)$ as

$$W_t^\Gamma = \mathbb{E}^a\left[\int_t^{\tau_L} e^{-r(s-t)}(Ba_s ds + dC_s)\,|\,\mathcal{F}_t^X\right].$$

The process W^Γ corresponds to the agent's continuation value process associated to a contract Γ. Because C is an increasing process and action process a takes positive values, $W_t^\Gamma \geq 0$ for all $t \leq \tau_L$ while $W_{\tau_L}^\Gamma = 0$ by construction. The following holds.

Lemma 3.1. *The continuation value process W^Γ associated to the incentive compatible contract Γ satisfies under $\mathbb{P}^a$ the dynamics*

$$dW_t^\Gamma = (rW_t^\Gamma - Ba_t)\,dt + \beta_t^\Gamma\,dZ_t^a - dC_t \ \ \textit{for} \ \ t \le \tau_L, \tag{8}$$

where the process $\beta^\Gamma = (\beta_t^\Gamma)$ is $\mathcal{F}^X$ predictable and uniquely defined. It is called hereafter the sensitivity process.

Proof of Lemma 3.1. By assumption (5), the process (U_t) with

$$U_t = e^{-rt}W_t^\Gamma + \int_0^t e^{-rs}(Ba_s ds + dC_s) = \mathbb{E}^a\left[\int_0^{\tau_L} e^{-rs}(Ba_s ds + dC_s)|\mathcal{F}_t^X\right]$$

is a uniformly integrable martingale under $\mathbb{P}^a$. By the martingale Representation theorem, there exists a unique $\mathcal{F}^X$ predictable process (β_t^Γ) such that

$$U_t = Y_0 + \int_0^t e^{-rs}\beta_s^\Gamma\,dZ_s^a,$$

with

$$\mathbb{E}^a\left[\int_0^{\tau_L} e^{-2rs}(\beta_s^\Gamma)^2\,ds\right] < +\infty.$$

Then, Itô's formula, yields (8). □

Thus, any incentive compatible contract $\Gamma = (C, \tau_L, a)$ defines a unique sensitivity process (β_t^Γ) by the representation theorem for Brownian martingale that yields (8). We could interpret Lemma 3.1 in the framework of BSDE as follows: for any given incentive compatible contract $\Gamma = (C, \tau_L, a)$, there exists a unique pair of $\mathcal{F}^X$ adapted process $(W_t^\Gamma, \beta_t^\Gamma)$ such that

$$\begin{cases} W_{\tau_L}^\Gamma &= 0, \\ dW_t^\Gamma &= (rW_t^\Gamma - Ba_t)\,dt + \beta_t^\Gamma\,dZ_t^a - dC_t. \end{cases}$$

However, the question of characterizing incentive-compatible contracts that satisfy the agent's participation constraint (7) remains unanswered. That is, we have to characterize the set $\Gamma(w_0)$ of contracts Γ for which $a = a^*(C, \tau_L)$ and, W_0^Γ is greater than the participation constraint w_0.

To solve this problem, the idea of Sannikov (2008) has been to see the sensitivity process (β_t^Γ) as a control. To this end, let us consider the class of $\mathcal{F}^X$ measurable processes $\beta = (\beta_t)$ such that

$$\mathbb{E}^a\left[\int_0^\infty e^{-2rs}\beta_s^2\,ds\right] < +\infty, \tag{9}$$

and, for any fixed increasing process C, let us consider the process $W^\beta = (W_t^\beta)$ that satisfies the controlled stochastic differential equation under $\mathbb{P}^0$,

$$dW_t^\beta = (rW_t^\beta + h(\beta_t))\,dt + \beta_t\,dZ_t - dC_t \ \text{ and } \ W_0^\beta \ge w_0,$$

with $h(\beta) = \inf_{0\le a\le 1}(\frac{\delta}{\sigma}\beta - B)a$. We would like the process (W_t^β) to play the role of the agent continuation value associated to some incentive compatible contract $\Gamma \in \Gamma(w_0)$. By limited liability, this requires $W_t^\beta \geq 0$ up to the termination date of the contract Γ. Therefore, we introduce

$$\tau_0^\beta(C) = \inf\{t \geq 0\,,\, W_t^\beta = 0\}.$$

Let us recall that we have assumed so far that the action process (a_t) is incentive compatible. The next lemma characterizes incentive compatible contracts as a deterministic function of the control process β.

Lemma 3.2. *For any compensation process (C_t) satisfying (5) and any process (β_t) satisfying (9), the contract $\Gamma = (C, \tau_0^\beta(C), \mathbb{1}_{\beta_t < \frac{\sigma B}{\delta}})$ is incentive compatible and belongs to $\Gamma(w_0)$.*

Proof of Lemma 3.2. The proof follows from a standard application of the martingale optimality principle. For any compensation process (C_t) satisfying (5) and any process (β_t) satisfying (9), the process (R_t) with

$$R_t^a = e^{-rt} W_t^\beta + \int_0^t e^{-rs}(Ba_s ds + dC_s)$$

is a uniformly integrable $\mathbb{P}^a$-supermartingale for every action process (a_t) and a uniformly integrable $\mathbb{P}^{a^*}$-martingale where, for any $t \geq 0$, $a_t^* = \mathbb{1}_{\beta_t < \sigma\lambda}$ with $\lambda = \frac{B}{\delta}$. Therefore

$$\begin{aligned}
\mathbb{E}^a\left[\int_0^{\tau_0^\beta(C)} e^{-rt}(Ba_t dt + dC_t)\right] &\leq R_0^a \\
&= W_0^\beta \\
&= R_0^{a^*} \\
&= \mathbb{E}^{a^*}\left[\int_0^{\tau_0^\beta(C)} e^{-rt}(Ba_t^* dt + dC_t)\right] \qquad (10)
\end{aligned}$$

□

Therefore, the principal's problem is to find a contract $\Gamma = (C, \tau_0^\beta(C), \mathbb{1}_{\beta < \lambda\sigma})$ that maximizes her expected profit at date 0. This leads to the following Markov formulation of problem (6)-(7).

$$V_P(x, w_0) = \max_{w \geq w_0} V_P(x, w) \qquad (11)$$

where

$$V_P(x, w) = \sup_{C,\beta} \mathbb{E}^{a^*}\left[\int_0^{\tau_0^\beta(C)} e^{-rs}(X_s\, ds - \, dC_s)\right]$$

$$\text{with } a^* = (a_t^*) \text{ and } a_t^* = \mathbb{1}_{\beta_t < \sigma\lambda},$$

such that

$$dX_t = -\delta \mathbb{1}_{\beta_t < \sigma\lambda}\, dt + \sigma dZ_t \text{ with } X_0 = x, \qquad (12)$$

$$dW_t = (rW_t - B\mathbb{1}_{\beta_t < \sigma\lambda})\, dt + \beta_t dZ_t - dC_t \text{ with } W_0 = w. \qquad (13)$$

The last result of this section shows that postponing payments is an optimal policy for the principal. Therefore, in the study of the principal's problem, we can simply consider a remuneration scheme with a terminal lump-sum payment.

Lemma 3.3. *It is always optimal for the principal to postpone payments and to pay the agent only at the liquidation time with a lump-sum payment.*

Proof of Lemma 3.3.

First, observe that, from (10), the Principal's value function (11) can be re-written as $V_P(x, w_0) = \max_{w \geq w_0}(v(x, w) - w)$ where

$$v(x,w) = \sup_{C,\beta} \mathbb{E}^{a^*}\left[\int_0^{\tau_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds\right] \quad \text{s.t (12) and (13).} \tag{14}$$

The amount $v(x, w)$ corresponds to the total surplus generated by the project in our moral hazard framework.

Second, note that $\tau_0^\beta(C) = \sigma_0^\beta \wedge \tilde{\tau}_0^\beta(C)$ where for any fixed increasing process (C_t), we have

$$\tilde{\tau}_0^\beta(C) = \inf\{t \geq 0,\, W_{t-}^\beta = 0\},$$

and

$$\sigma_0^\beta = \inf\{t \geq 0,\, (\Delta C)_t = W_{t-}^\beta \text{ and } (\Delta C)_t > 0\}.$$

Third, with no loss of generality, a remuneration process can be written under the form $(C_t)_{t < \tau_0^\beta(C)} + W_{(\tau_0^\beta(C))^-}\mathbb{1}_{t = \tau_0^\beta(C)}$. Therefore, a control policy can be viewed as a pair (C, β) and a stopping time τ at which the Principal pays $W_{\tau-}^\beta$ and liquidate. Thus, we have

$$v(x,w) = \sup_{C,\beta,\tau} \mathbb{E}^{a^*}\left[\int_0^{\tau\wedge\tilde{\tau}_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds\right].$$

Observe that $\tilde{\tau}_0^\beta(0)$ corresponds to the liquidation time when the principal postpones payments up to liquidation. We have

$$\begin{aligned} v(x,w) &\geq \sup_{\beta,\tau} \mathbb{E}^{a^*}\left[\int_0^{\tau\wedge\tilde{\tau}_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds\right], \text{ choosing } C = 0 \qquad (15)\\ &\geq \mathbb{E}^{a^*}\left[\int_0^{(\sigma_0^\beta\wedge\tilde{\tau}_0^\beta(C))\wedge\tilde{\tau}_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds\right], \text{ choosing } \tau = \sigma_0^\beta \wedge \tilde{\tau}_0^\beta(C)\\ &= \mathbb{E}^{a^*}\left[\int_0^{\sigma_0^\beta\wedge\tilde{\tau}_0^\beta(C)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds\right], \text{ observing } \tilde{\tau}_0^\beta(C) \leq \tilde{\tau}_0^\beta(0). \qquad (16)\end{aligned}$$

Taking the supremum over the controls C, β in (16) yields $v(x, w)$. It follows from (15) that

$$v(x,w) = \sup_{\beta,\tau} \mathbb{E}^{a^*}\left[\int_0^{\tau\wedge\tilde{\tau}_0^\beta(0)} e^{-rs}(X_s + B\mathbb{1}_{\beta_s < \lambda\sigma})\, ds\right], \tag{17}$$

which proves that it is optimal to postpone payments. □

We summarize our findings as follows. The principal solves the maximization problem

$$\max_{w \geq w_0}(v(x, w) - w)$$

where

$$v(x,w) = \sup_{\beta,\tau} \mathbb{E}^{a^*}\left[\int_0^{\tau\wedge\tilde{\tau}_0^\beta} e^{-rs}(X_s + B\mathbb{1}_{\beta_s<\lambda\sigma})\,ds\right], \tag{18}$$

with

$$\tilde{\tau}_0^\beta = \inf\{t \geq 0,\, W_{t^-} = 0\};$$

The Markov process (X_t, W_t) is defined by

$$dX_t = -\delta\mathbb{1}_{\beta_t<\sigma\lambda}\,dt + \sigma dZ_t \text{ with } X_0 = x,$$

$$dW_t = (rW_t - B\mathbb{1}_{\beta_t<\sigma\lambda})\,dt + \beta_t dZ_t \text{ with } W_0 = w.$$

The supremum in (18) is taken over the class of $\mathcal{F}^X$-adapted processes β such that

$$\mathbb{E}^{a^*}\left[\int_0^\infty e^{-2rs}\beta_s^2\,ds\right] < +\infty$$

and over stopping time $\tau \in \mathcal{T}^X$.

Solving problem (18) remains very challenging and so far an open question, to the best of our knowledge. The next section restricts the analysis to contracts that are incentive compatible with the full effort action process $a_t = 0$ for every t.

4 Full effort contracts.

We focus on *full-effort* contracts, that is the class of contracts that induces the agent to exert effort at any time. It follows from Lemma 3.2 that the full-effort action process $a = 0$ is incentive compatible if and only if $\beta_t \geq \lambda\sigma$ for all $t \geq 0$. Restricting the analysis to contracts that incentivize the full-effort action leads to re-write problem (18) as follows:

Find a contract $\Gamma = (W_{\tau-}\mathbb{1}_{t=\tau}, \tau\wedge\tilde{\tau}_0^\beta, 0)$ where the pair (τ,β) is solution to

$$v(x,w) = \sup_{\beta\geq\lambda\sigma,\tau} \mathbb{E}^0\left[\int_0^{\tau\wedge\tilde{\tau}_0^\beta} e^{-rs}X_s\,ds\right] \tag{19}$$

such that

$$dX_t = \sigma dZ_t \ \text{ with } \ X_0 = x, \tag{20}$$

$$dW_t = rW_t + \beta_t dZ_t \ \text{ with } \ W_0 = w, \tag{21}$$

where

$$\tilde{\tau}_0^\beta = \inf\{t \geq 0,\, W_{t^-} = 0\}.$$

Problem (19) boils down to a two-dimensional optimal exit decision, hence optimal stopping theory is from now on the key mathematical tool.

Let us consider the sub-solution to problem (19)-(21) where the constraint on the incentives contract is binding (that is when $\beta_t = \lambda\sigma$ for all $t \geq 0$). This yields the two-dimensional constrained optimal stopping problem

$$u(x,w) = \sup_{\tau} \mathbb{E}^0\left[\int_0^{\tau\wedge\tilde{\tau}_0^{\lambda\sigma}} e^{-rs}X_s\,ds\right] \tag{22}$$

such that

$$dX_t = \sigma dZ_t \text{ with } X_0 = x,$$

$$dW_t = rW_t\, dt + \lambda\sigma dZ_t \text{ with } W_0 = w,$$

and

$$\tilde{\tau}_0^{\lambda\sigma} = \inf\{t \geq 0,\, W_{t^-} = 0\}. \tag{23}$$

Observe also that the unconstrained stopping problem

$$v_0(x) = \sup_{\tau} \mathbb{E}^0 \left[\int_0^{\tau} e^{-rs} X_s\, ds \right] \tag{24}$$

corresponds to the firm value in a frictionless world in which there are no asymmetry of information and private benefits. Problem (24) is a standard real option problem that has an explicit solution (see for instance Dixit and Pindyck [9]). We have, for $x \geq x^*$,

$$v_0(x) = \frac{x}{r} - \frac{x^*}{r} e^{\theta(x-x^*)}, \text{ with } \theta = \frac{-\sqrt{2r}}{\sigma} \text{ and } x^* = \frac{1}{\theta}.$$

The threshold x^* is the profitability threshold below which it is optimal to trigger the firm liquidation in the frictionless world. That is, the stopping time

$$\tau^* = \inf\{t \geq 0,\, X_t \leq x^*\} \tag{25}$$

is optimal for (24).

We are ready to state our main result.

Theorem 4.1. *The following holds*

(i) For all $(x, w) \in \mathbb{R} \times \mathbb{R}^+$, $v(x, w) = u(x, w)$. Furthermore, $u(x, w) = v_0(x)$ for all $(x, w) \in \mathbb{R} \times \mathbb{R}^+$ such that $w \geq \lambda(x - x^*)$.

(ii) The contract $((W_{\tau^*_-} \mathbb{1}_{t=\tau^*})_{t\geq 0}, \tau^* \wedge \tilde{\tau}_0^{\lambda\sigma}, 0)$ is a solution to problem (19), (20), (21).

Thus, the principal optimally postpones payments and pays to the agent the amount $W_{\tau^*_-}$ if $\tau^* \leq \tilde{\tau}_0^{\lambda\sigma}$ or nothing if $\tau^* > \tilde{\tau}_0^{\lambda\sigma}$. Either the principal stops at the frictionless threshold τ^*, or stops at $\tilde{\tau}_0^{\lambda\sigma}$ because the agent has limited liability. We will further comment this optimal payment policy at the end of this section where we discuss the implementation of the optimal contract.

The proof of Theorem 4.1 is challenging and requires a series of steps. We comment each step, pointing out the novelty of our results. To ease the reading, we develop in the appendix the most technical arguments.

Proof of Theorem 4.1. To alleviate notations, we write in the sequel τ_0 in place of $\tilde{\tau}_0^{\lambda\sigma}$. We use whenever needed the following notations: (X_t^x) (resp. (W_t^w)) denotes the process (X_t) starting at $X_0 = x$ (resp. the process (W_t) starting at $W_0 = w$), and τ^* the stopping time $\inf\{t \geq 0 : X_t^x = x^*\}$ (resp. τ_0^w, the stopping time $\inf\{t \geq 0 : W_{t^-}^w = 0\}$).

We start with the study of the constrained optimal stopping problem (22). We show the following.

Proposition 4.2. *The exit time $\tau_R = \tau^* \wedge \tau_0$ of the open rectangle $R = (x^*, +\infty) \times (0, +\infty)$ is optimal for (22). That is,*

$$u(x,w) = \mathbb{E}^0\left[\int_0^{\tau_R} e^{-rs} X_s\, ds\right].$$

Moreover, if $w \geq \lambda(x - x^*)$ then, $u(x,w) = v_0(x)$.

Proof of Proposition 4.2. We first show that $u(x,w) = v_0(x)$ for every $w \geq \lambda(x - x^*)$. Note that $(W_t - \lambda X_t)$ is an increasing process up to time τ_0 because $d(W_t - \lambda X_t) = rW_t\, dt \geq 0$, and thus $W_t - w \geq \lambda(X_t - x)$. Therefore, if $w \geq \lambda(x - x^*)$, we have $W_t \geq \lambda(X_t - x^*)$. As a consequence, $\tau_0 \geq \tau^*$ almost surely. Thus, for $w \geq \lambda(x - x^*)$,

$$\begin{aligned} v_0(x) &\geq u(x,w) \\ &\geq \mathbb{E}^0\left[\int_0^{\tau^* \wedge \tau_0} e^{-rs} X_s\, ds\right] \\ &= \mathbb{E}^0\left[\int_0^{\tau^*} e^{-rs} X_s\, ds\right] \\ &= v_0(x). \end{aligned}$$

Second, we show that for all $x > x^*$ and $w > 0$, $u(x,w)$ is strictly positive. Let $0 < w < \lambda(x - x^*)$ and $\varepsilon = \lambda(x - x^*) - w$. Let us introduce the finite stopping time

$$\tau_\varepsilon = \inf\{t \geq 0,\ X_t = x^* + \frac{\varepsilon}{\lambda}\}.$$

Because, $W_t \geq \lambda(X_t - x^*) - \varepsilon$, we have $\tau_0 \geq \tau_\varepsilon$ almost surely. Dynamic programming principle implies

$$\begin{aligned} u(x,w) &\geq \mathbb{E}^0\left[\int_0^{\tau_\varepsilon} e^{-rs} X_s\, ds\right] + \mathbb{E}^0[e^{-r\tau_\varepsilon} u(X_{\tau_\varepsilon}, W_{\tau_\varepsilon})] \\ &\geq \mathbb{E}^0\left[\int_0^{\tau_\varepsilon} e^{-rs} X_s\, ds\right] \\ &= \frac{x}{r} - \frac{x^* + \frac{\varepsilon}{\lambda}}{r} e^{\theta(x - (x^* + \frac{\varepsilon}{\lambda}))} \\ &> 0 \quad \text{because } \varepsilon > 0. \end{aligned}$$

This latter result allows us to conclude the proof. Because $u > 0$ on R, the process

$$M_t = e^{-r(t \wedge \tau_R)} u(X_{t \wedge \tau_R}, W_{t \wedge \tau_R}) + \int_0^{t \wedge \tau_R} e^{-rs} X_s\, ds$$

is a martingale according to optimal stopping theory. Optional sampling theorem gives for all $t \geq 0$,

$$u(x,w) = \mathbb{E}^0\left[e^{-rt \wedge \tau_R} u(X_{t \wedge \tau_R}, W_{t \wedge \tau_R}) + \int_0^{t \wedge \tau_R} e^{-rs} X_s\, ds\right].$$

Note that $\tau_R \leq \tau^*$ which is the hitting time of x^* by a Brownian motion. Therefore, τ_R is almost surely finite. Moreover, $u = 0$ on the boundaries of R. Letting t tend to $+\infty$ gives

$$u(x,w) = \mathbb{E}^0 \left[\int_0^{\tau_R} e^{-rs} X_s \, ds \right].$$

This concludes the proof. □

To prove Theorem 4.1, it remains to show that functions u and v coincide. The road map is as follows. We consider the HJB equation formally associated to the value function v, that is

$$\max(\max_{\beta \geq \lambda\sigma} \mathcal{L}(\beta)v, -v) = 0 \text{ on } \mathbb{R} \times \mathbb{R}_+, \tag{26}$$

with the boundary condition $v(x,0) = 0$ and where $\mathcal{L}(\beta)$ is the fully degenerate differential operator

$$\mathcal{L}(\beta)V \equiv -rV(x,w) + x + rw\frac{\partial V}{\partial w}(x,w) + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}(x,w) + \frac{1}{2}\beta^2\frac{\partial^2 V}{\partial w^2}(x,w) + \sigma\beta\frac{\partial^2 V}{\partial x \partial w}(x,w).$$

We prove that u is a smooth solution to (26). Then, a standard verification argument based on Itô's formula yields $u = v$. The novelty of our analysis is to establish required continuity and smoothness properties of the value function u. The more involved results are about regularity properties of u with respect to w. In particular, we prove that u is locally Lipschitz with respect to w ((assertion ii) of Proposition 4.3 and Lemma 4.4) and we show the existence of $\frac{\partial u}{\partial w}(x,0)$ (Proposition 4.5).

Proposition 4.3. *The value function u is jointly continuous over $[x^*, +\infty) \times [0, \infty)$ and C^∞ over $R = (x^*, +\infty) \times (0, +\infty)$. Furthermore, it satisfies*

$$\max(\mathcal{L}(\lambda\sigma)u, -u) = 0 \tag{27}$$

almost everywhere on $\mathbb{R} \times \mathbb{R}_+$.

Proof of Proposition 4.3. Clearly, $u(x) = 0$ for $x \leq x^*$. To show that u is jointly continuous over $[x^*, +\infty) \times [0, \infty)$, we prove that, for any $(x,w) \in [x^*, +\infty) \times [0, +\infty)$,

i) u is Lipschitz with respect to x, uniformly in w,

ii) u is locally Lipschitz with respect to w.

According to Proposition 4.2, we have for every $x > x_0$,

$$u(x,w) - u(x_0,w) = \mathbb{E}^0 \left[\int_0^{\tau^{*,x} \wedge \tau_0} e^{-rs} X_s^x \, ds \right] - \mathbb{E}^0 \left[\int_0^{\tau^{*,x_0} \wedge \tau_0} e^{-rs} X_s^{x_0} \, ds \right]$$

where $\tau^{*,x} = \inf\{t \geq 0,\, x + \sigma Z_t \leq x^*\}$. Because the stopping time $\tau^{*,x} \wedge \tau_0$ is suboptimal starting from $X_0 = x_0$, we get

$$\begin{aligned} u(x,w) - u(x_0,w) &\leq \mathbb{E}^0 \left[\int_0^{\tau^{*,x} \wedge \tau_0} e^{-rs}(x - x_0) \, ds \right] \\ &\leq \frac{x - x_0}{r}. \end{aligned}$$

Thus, assertion (i) is proven. The proof of assertion (ii) is more involved and relies on the following lemma proved in the appendix.

Lemma 4.4. *For every couple* $(x,w) \in (x^*,+\infty)\times(0,+\infty)$*, there is a constant* C *such that* $u(x,w) \leq C(1+x)w$.

For every $w > w_0$, we have $\tau_0^w \geq \tau_0^{w_0}$ a.s. and thus by Strong Markov Property, we have for all $w > 0$,

$$\begin{aligned} u(x,w) - u(x,w_0) &= \mathbb{E}^0\left[\int_0^{\tau^*\wedge\tau_0^w} e^{-rs}X_s^x\,ds\right] - \mathbb{E}^0\left[\int_0^{\tau^*\wedge\tau_0^{w_0}} e^{-rs}X_s^x\,ds\right] \\ &= \mathbb{E}^0\left[\int_{\tau^*\wedge\tau_0^{w_0}}^{\tau^*\wedge\tau_0^w} e^{-rs}X_s^x\,ds\right] \\ &= \mathbb{E}^0\left[e^{-r\tau_0^{w_0}}\mathbb{1}_{\{\tau_0^{w_0}\leq\tau^*\}}u(X_{\tau_0^{w_0}}^x,(w-w_0)e^{r\tau_0^{w_0}})\right] \\ &\leq C\mathbb{E}^0[(X_{\tau_0^{w_0}}^x+1)\mathbb{1}_{\{\tau_0^{w_0}\leq\tau^*\}}](w-w_0), \end{aligned}$$

where the last inequality follows from Lemma 4.4. Now, observe that for every $t \geq 0$, we have $W_t - w_0 \geq \lambda(X_t^x - x)$ and thus $X_{\tau_0^{w_0}}^x \leq x$ on the set $\{\tau_0^{w_0} \leq \tau^*\}$ which ends the proof of assertion (ii) and, in turn, the proof that u is jointly continuous over $[x^*,+\infty)\times[0,\infty)$.

Now, from optimal stopping theory, the continuous value function u is a viscosity solution to (27) (see for instance, Pham [18], Theorem 4.3.1). We will show that, for any $\epsilon > 0$, u satisfies (27) over $R_\epsilon = (x^*,+\infty)\times(\epsilon,+\infty)$ in the classical sense. To this end, we introduce a deterministic transformation of the process (X_t, W_t) where

$$\begin{cases} dX_t &= \sigma dZ_t, \\ dW_t &= rW_t\,dt + \lambda\sigma dZ_t. \end{cases}$$

Such transformation unveils a parabolic nature of the problem and is similar to the method of characteristics in PDE analysis. Given $(x,w) \in R_\epsilon$, let us define

$$S_t = \lambda X_t - W_t - \lambda x^* \text{ with } S_0 = s = \lambda(x-x^*) - w.$$

We have

$$\begin{cases} dS_t &= -rW_t\,dt, \\ dW_t &= rW_t\,dt + \lambda\sigma dZ_t. \end{cases}$$

Consider the function $\hat{u}(s,w) = u(x^* + \frac{1}{\lambda}(w+s), w)$. The function $\hat{u}$ is jointly continuous because u is jointly continuous. By results on interior regularity for solution to parabolic PDE (see, Krylov [16], Ch 2, Sect. 4, Corollary 3), for any $\epsilon > 0$, the solution on any rectangle $\hat{R}_\epsilon = (0,+\infty)\times(\epsilon,+\infty)$ to

$$rw\frac{\partial f}{\partial s} = rw\frac{\partial f}{\partial w} + \frac{\lambda^2}{2}\frac{\partial^2 f}{\partial w^2} + x^* + \frac{1}{\lambda}(w+s)$$

with boundary condition $f = \hat{u}$ on $\partial\hat{R}_\epsilon$, is $C^\infty(\hat{R}_\epsilon)$ and coincides with $\hat{u}$. Therefore, for any $\epsilon > 0$, $\hat{u}$ is $C^\infty(\hat{R}_\epsilon)$ which, in turn, implies that u is $C^\infty(R_\epsilon)$ and satisfies $\mathcal{L}(\lambda\sigma)u = 0$ on the open set R where $u > 0$ in the classical sense. Thus, u satisfies

$$\max(\mathcal{L}(\lambda\sigma)u, -u) = 0$$

almost everywhere on $\mathbb{R} \times \mathbb{R}_+$. This ends the proof of Proposition 4.3. $\square$

We need additional properties to prove that the value function u is also a smooth solution to (26) almost everywhere on $\mathbb{R} \times \mathbb{R}_+$. Because, $u = 0$ and $\mathcal{L}(\beta)u \leq 0$ for every $\beta \geq \lambda\sigma$ on the set $\{x \leq x^*\}$, it is enough to prove that u satisfies $\max_{\beta \geq \lambda\sigma} \mathcal{L}(\beta)u = 0$ over $R = (x^*, +\infty) \times (0, +\infty)$. We use the following result that we prove in the Appendix.

Proposition 4.5. *For any $x > x^*$, $\dfrac{\partial u}{\partial w}(x, 0)$ exists and is finite. Moreover, for any $(x, w) \in R$, the value function u satisfies*

(i) $\dfrac{\partial u}{\partial w}(x, w) = \mathbb{E}^0 \left[\mathbb{1}_{\tau_0 \leq \tau^*} \dfrac{\partial u}{\partial w}(X_{\tau_0}, 0) \right] \geq 0,$

(ii) $\dfrac{\partial^2 u}{\partial w^2}(x, w) < 0,$

(iii) $\left(\dfrac{\partial^2 u}{\partial x \partial w} + \lambda \dfrac{\partial^2 u}{\partial w^2}\right)(x, w) < 0.$

Proposition 4.5 appears in the literature in different settings, (see for instance, Faingold and Vasama [10], De Marzo and Sannikov [8]). Assertion (ii) corresponds to a concavity property of the value function with respect to the agent's continuation value w. This property is standard in one dimensional agency models.[4] Together with assertion (iii), it implies that choosing $\beta > \lambda\sigma$ is suboptimal in the (two-dimensional) HJB equation (26). In their respective settings, recent contributions provide heuristic justifications of properties (ii) and (iii) based on a stochastic representation for $\frac{\partial v}{\partial w}$, this latter representation is obtained by differentiating the HJB equation associated to the value function v of the principal's problem. In particular, the regularity properties of the value function v over R needed to establish the stochastic representation are not proven. The main forgetting is the proof of the existence and finiteness of $\frac{\partial v}{\partial w}(x, 0)$ that is instrumental in the proof of assertions (i), (ii) and (iii). To the best of our knowledge our paper is the first that offers a complete proof of Proposition 4.5.

The next Proposition follows from Propositions 4.3 and 4.5 and concludes the proof of Theorem 4.1.

Proposition 4.6. *The value function u satisfies the HJB equation* (26)*. Therefore, the two value functions u and v coincide.*

Proof of Proposition 4.6. Because u satisfies the HJB equation (27) almost everywhere on $\mathbb{R} \times \mathbb{R}_+$, it is enough to prove that $\mathcal{L}(\beta)u \leq 0$ everywhere for all $\beta \geq \lambda\sigma$. It is straightforward when $u = 0$. Thus, we have to prove that $\mathcal{L}(\beta)u \leq 0$ on R for all $\beta \geq \lambda\sigma$. By Proposition 4.5, it is immediate that the function $\beta \longrightarrow \mathcal{L}(\beta)u$ takes its maximum over $[\lambda\sigma, +\infty[$ at $\lambda\sigma$. Therefore, u satisfies (26). By a standard verification result, u dominates the value function v. Because the reverse inequality holds by definition, the two value functions u and v coincide. This ends the proof of Proposition 4.6 and Theorem 4.1. $\square$

[4]See Sannikov [20] and De Marzo and Sannikov [7].

To conclude this section, we discuss optimal policies, the implementability of the optimal contract and specify the relationship to the existing literature.

Optimal payment policies. It follows from the proof of Theorem 4.1 that, once the agent's continuation payoff (W_t) reaches $\lambda(X_t - x^*)$, the payment policy $W_{\tau^*_-} 1\!\!1_{t=\tau^*}$ guarantees optimal liquidation of the firm. Another policy that leads to optimal liquidation once W_t reaches $\lambda(X_t - x^*)$ is to pay the agent at continuous rate $rW_t\, dt$ up to the optimal liquidation time τ^*. To see this, assume that the couple (x, w) satisfies the relation $w = \lambda(x - x^*)$ and consider the payment policy $dC_t = rW_t\, dt$. The dynamics

$$dX_t = \sigma dZ_t \;\; X_0 = x,$$

$$dW_t = rW_t + \lambda\sigma dZ_t - dC_t \;\; W_0 = w,$$

imply that $dW_t = \lambda dX_t$, from which we deduce that $W_t = \lambda(X_t - x^*)$. In turn, $\tau_0 = \tau^*$ a.s., that is, the continuation value process W reaches zero at the optimal liquidation time τ^*. Consistent with (10), a direct computation yields that

$$w = \mathbb{E}^0 \left[\int_0^{\tau^*} e^{-rt} rW_s\, ds \right], \tag{28}$$

and, accordingly, the value of the principal satisfies

$$V_P(x, w) = \mathbb{E}^0 \left[\int_0^{\tau^*} e^{-rs}(X_s ds - dC_s) \right] = v_0(x) - w.$$

The above remark shows that paying cash earlier is not costly for the principal provided that the principal can ensure that the profitability process reaches the optimal liquidation threshold x^* before the continuation value of the agent falls to zero.

To summarize, when initial values (x, w) satisfy $w < \lambda(x - x^*)$, deferring payments give incentives to the agent, the continuation value rises and the risk of inefficient liquidation is reduced. Once the agent has established a high performance record, that is when continuation value W_t reaches $\lambda(X_t - x^*)$, any payment policy that leads to liquidation at τ^* is optimal. The same idea is present in He [11] and in DeMarzo and Sannikov [8] where payments to the agent are postponed until inefficient liquidation can be avoided.[5] Note that, as it is usual in dynamic contracting models, the agent may be fired without any compensation at all after a series of bad outcomes leading to inefficient liquidation.[6]

Implementation. When cash flows generated by the firm are modeled as the increment of an arithmetic Brownian motion, the agent's continuation payoff can be linked to actual cash

[5] In DeMarzo and Sannikov [8], cash flows are modeled as the increment of an arithmetic Brownian motion with an unknown drift. Optimal liquidation occurs when beliefs about the constant drift fall to an endogenous threshold. The principal can ensure optimal liquidation once the agent has established a sufficiently high record. In He (2009), liquidation is always inefficient. When the agent establishes a sufficiently high record, the principal can design payments in such a way that the continuation value of the agent becomes proportional to the firm size that follows a geometric brownian motion. Thus, the continuation value of the agent remains always positive and liquidation never occurs.

[6] For instance, in standard models with i.i.d shocks, payouts to the manager take place when the agent's continuation value reaches a payout threshold (see for instance Biais, Mariotti and Rochet [1]).

flows and a combination of long term debt, equity and credit line implements the optimal contract as shown for instance in DeMarzo and Sannikov [7] or Biais, Mariotti, Plantin and Rochet [2].[7] This is not the case in our setting where the firm's profitability fluctuates across time. We share this feature with He [11] whose analysis about the implementation of the optimal contract directly applies to our model.

Precisely, let us again consider that $w = \lambda(x - x^*)$ and let us re-write (28) under the form

$$\begin{aligned} w &= \mathbb{E}^0 \left[\int_0^{\tau^*} e^{-rt} r\lambda(X_t - x^*)\, dt \right] \\ &= r\lambda v_0(x) + \mathbb{E}^0 \left[\int_0^{\tau^*} e^{-rt}(-r\lambda x^*)\, dt \right]. \end{aligned}$$

This latter expression shows that the agent's financial security corresponds to a fraction $r\lambda$ of the firm value *plus* a corporate debt with coupon flow $-r\lambda x^*$.

Following He [11], we design an Incentive Points Plan where the points trace the agent's continuation value W_t to implement an optimal contract. The agent starts with w points. Once W_t hits zero, the agent is fired. When the agent establishes a high record, that is when $W_t = \lambda(X_t - x^*)$, she can redeem these points and get a portion of dividends $r\lambda dt$ onward together with a coupon payment flow $-r\lambda x^* dt$. Thus, a performance-based grant that involves a stock and a bond implement the optimal contract. The difference with the He's implementation simply relies on the presence of the bond in the performance-based grant. This is due to the strategic liquidation issue in our model.

Finally, because of agency frictions, the principal is willing to start the project only if

$$\max_{w \geq w_0} (v(x, w) - w) > 0.$$

An easy computation shows that, if $r\lambda < 1$ then, $v_0(x) > \lambda(x - x^*)$ for x sufficiently large, which shows that the set $\{(x, w) \in R \,|\, v(x, w) - w > 0\}$ is non empty. Therefore, when the agency costs, measured by the parameter λ are not too high, a project with sufficiently high profitability is undertaken.

5 Concluding remarks

In this paper, we consider a dynamic contracting model in corporate finance with moral hazard in which the principal is concerned both with the profitability of the firm that fluctuates across time and with the agent's actions on the profitability. This led us to study a two-dimensional fully degenerate control problem. We derived the Markovian formulation of the principal's problem and proved regularity properties of the associated value function that allow to derive the optimal contract.

We noticed that our regularity results apply to other contracting problems in which regularity issues were overlooked. At a more general level, our work is an attempt to bridge the gap between the continuous time agency model with unobservable managerial effort and the conventional real option literature. Real option theory allows to study a large

[7] This also holds true for DeMarzo and Sannikov [8].

set of problems such as investment opportunities and investment timing, sequential and/or incremental investments, entry and exit strategies. Dealing with these problems in dynamic contracting models with unobservable managerial effort requires a mathematical setting with serially correlated cash flows and leads to challenging two-dimensional control problems. We believe that our model sets a stage for studying more sophisticated real option problems in an agency framework.

We derived the optimal contract of principal's problem in the class of full effort contracts. We know from previous studies that, when cash flows are defined as the increment of an arithmetic Brownian motion, we can find restrictions on the parameters of the model that ensure never inducing shirking is indeed optimal.[8] This remark has been taken as a rationale for restricting attention to full effort contracts in economic applications. We show below that this result does not extend to our setting in which the firm's profitability fluctuates. To see this point, let us consider the HJB equation formally associated to the value function v in (17), that is

$$\max(\max_{\beta}(\mathcal{L}(\beta)V + (B(1 - \frac{\partial V}{\partial w}(x,w)) - \delta\frac{\partial V}{\partial x}(x,w))\mathbb{1}_{\beta<\lambda\sigma}), -V) = 0 \tag{29}$$

and consider (x, w) with $w \geq \lambda(x - x^*)$, so that $u(x, w) = v_0(x)$. It is easy to see that v_0 does not satisfy the HJB equation (29). Indeed, a direct computation yields

$$\max_{\beta}\Big(\mathcal{L}(\beta)v_0(x) + \big(B - \delta\frac{\partial v_0}{\partial x}(x)\big)\mathbb{1}_{\beta<\lambda\sigma}\Big) = \Big(B - \frac{\delta}{r}(1 - e^{\theta(x-x^*)})\Big)\mathbb{1}_{\beta<\lambda\sigma}.$$

The last expression is strictly positive for $\beta < \lambda\sigma$ and for x in a right neighborhood of x^*. It then follows that, incentivizing the agent to exert full effort at any time cannot be optimal. The economic intuition is simple: when the realized profitability is close from the one that triggers liquidation in a frictionless world, incentivizing the agent becomes very costly and taking action $a = 0$ is no longer optimal. Clearly, this situation does not occur in a setting in which the profitability of the firm is constant.

Problem (29) relates to optimal control problems with discontinuous coefficients. Typically, a class of problems about which we know very little. Characterizing the optimal contract in a moral hazard environment with random profitability and identifying when to release pressure on a firm's manager are clearly important economic issues. This and related questions must await for future work.

[8] See DeMarzo and Sannikov [7], Zhu [27].

6 Appendix

Proof of Lemma 4.4. We start with the following observation: for every couple $(x, w) \in (x^*, +\infty) \times (0, +\infty)$, there is some $C > 0$ such that $u(x, w) \leq C(1 + x)$. Indeed,

$$\begin{aligned} u(x, w) &\leq \mathbb{E}^0 \left[\int_0^\infty e^{-rs} |x + \sigma Z_s| \, ds \right] \\ &\leq \frac{x}{r} + \sigma \mathbb{E}^0 \left[\int_0^\infty e^{-rs} |Z_s| \, ds \right] \\ &= \frac{x}{r} + \sigma \sqrt{\frac{2}{\pi}} \mathbb{E}^0 \left[\int_0^\infty e^{-rs} \sqrt{s} \, ds \right] \\ &\leq C(1 + x). \end{aligned}$$

Therefore, Lemma 4.4 holds for $w \geq 1$. Let us now consider $w \in (0, 1)$. We decompose $u(x, w)$ as follows $u(x, w) = u_1(x, w) + u_2(x, w)$, with

$$\begin{aligned} u_1(x, w) &= \mathbb{E}^0 \left[\mathbb{1}_{\{\tau_0^w < \tau_1^w\}} \int_0^{\tau^* \wedge \tau_0^w} e^{-rs} (x + \sigma Z_s) \, ds \right], \\ u_2(x, w) &= \mathbb{E}^0 \left[\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} \int_0^{\tau^* \wedge \tau_0^w} e^{-rs} (x + \sigma Z_s) \, ds \right], \end{aligned}$$

where $\tau_1^w = \inf\{t \geq 0, \, W_t^w = 1\}$. On the event $\{\tau_0^w < \tau_1^w\}$, we have for every $t \leq \tau^* \wedge \tau_0^w < \tau_1^w$ the inequality $X_t \leq \frac{1}{\lambda} + x$. Therefore,

$$u_1(x, w) \leq \left(\frac{1}{\lambda} + x \right) \mathbb{E}^0 \left[\mathbb{1}_{\{\tau_0^w < \tau_1^w\}} \int_0^{\tau_0^w} e^{-rs} \, ds \right].$$

Conditioning the process $(W_t)_{t \in [0, \tau_1^w]}$ on the event $\{\tau_0^w < \tau_1^w\}$ and using Doob h-transform (see Rogers and Williams [19] for a definition) makes $(W_t)_{t \in [0, \tau_1^w]}$ a diffusion absorbed at 0 with generator

$$\tilde{\mathcal{L}} = \frac{\lambda^2 \sigma^2}{2} \frac{\partial^2 .}{\partial w^2} + \left(rw + \lambda \sigma \frac{h'(w)}{h(w)} \right) \frac{\partial .}{\partial w}$$

where

$$h(w) = \mathbb{P}^0(\tau_0^w < \tau_1^w) = \frac{\int_w^1 e^{-rs^2} \, ds}{\int_0^1 e^{-rs^2} \, ds}.$$

Let us denote $\tilde{\tau}_0^w = \inf\{t \geq 0, \, \tilde{W}_t^w = 0\}$ where

$$d\tilde{W}_t = \left(r\tilde{W}_t + \lambda \sigma \frac{h'(\tilde{W}_t)}{h(\tilde{W}_t)} \right) dt + \lambda \sigma dZ_t.$$

We have

$$\begin{aligned} \mathbb{E}^0 \left[\mathbb{1}_{\{\tau_0^w < \tau_1^w\}} \int_0^{\tau_0^w} e^{-rs} \, ds \right] &= \mathbb{P}^0(\tau_0^w < \tau_1^w) \mathbb{E}^0 \left[\int_0^{\tilde{\tau}_0^w} e^{-rs} \, ds \right] \\ &\leq \mathbb{E}^0 \left[\int_0^{\tilde{\tau}_0^w} e^{-rs} \, ds \right] \\ &= \tilde{\phi}(w). \end{aligned}$$

The function $\tilde{\phi}$ satisfies

$$\tilde{\mathcal{L}}\tilde{\phi} - r\tilde{\phi} = 0$$

with $\tilde{\phi}(0) = 0$. A computation shows that $rw + \lambda\sigma\frac{h'(w)}{h(w)} < 0$ and because $\tilde{\phi}$ is nondecreasing, we get that $\tilde{\phi}$ is convex. We deduce that $\tilde{\phi}(w) \leq Cw$ for some positive constant C which implies $u_1(x, w) \leq C(1 + x)w$. Now, we decompose u_2 as follows:

$$\begin{aligned}
u_2(x, w) &= \mathbb{E}^0\left[\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} \int_0^{\tau^* \wedge \tau_1^w} e^{-rs}(x + \sigma Z_s)\, ds\right] + \mathbb{E}^0\left[\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} \int_{\tau^* \wedge \tau_1^w}^{\tau^* \wedge \tau_0^w} e^{-rs}(x + \sigma Z_s)\, ds\right] \\
&= u_3(x, w) + u_4(x, w).
\end{aligned}$$

Because for every $s \leq \tau_1^w$, we have $X_s \leq \frac{1}{\lambda} + x$, it follows that

$$\begin{aligned}
u_3(x, w) &\leq \left(\frac{1}{\lambda} + x\right) \int_0^\infty e^{-rs}\, ds\, \mathbb{P}^0[\tau_0^w > \tau_1^w] \\
&= \frac{\frac{1}{\lambda} + x}{r} \mathbb{P}^0[\tau_0^w > \tau_1^w] \\
&= \frac{\frac{1}{\lambda} + x}{r} (1 - h(w)) \\
&\leq C(1 + x)w.
\end{aligned}$$

Finally, Strong Markov property implies

$$\begin{aligned}
u_4(x, w) &= \mathbb{E}^0\left[\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} e^{-r(\tau^* \wedge \tau_1^w)} u(X_{\tau^* \wedge \tau_1^w}, W_{\tau^* \wedge \tau_1^w})\right] \\
&\leq \mathbb{E}^0\left[\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} e^{-r\tau_1^w} u(X_{\tau_1^w}, 1)\right] \\
&\leq \mathbb{E}^0\left[\mathbb{1}_{\{\tau_0^w > \tau_1^w\}} e^{-r\tau_1^w} C(1 + X_{\tau_1^w})\right] \\
&\leq C(1 + \frac{1}{\lambda} + x)\mathbb{P}^0[\tau_0^w > \tau_1^w] \\
&\leq C(1 + x)w,
\end{aligned}$$

where the first inequality holds because $u(x^*, W_{\tau^*}) = 0$. This ends the proof of Lemma 4.4. □

Proof of Proposition 4.5. We will show that, for w sufficiently small, $u(x, w) = c(x)\, w + o(w)$ where $c(x)$ is a real constant. This will prove that $\frac{\partial u}{\partial w}(x, 0) < +\infty$.
We have

$$u(x, w) \;=\; \mathbb{E}^0\left[\int_0^{\tau_0^w} e^{-rs}(x + \sigma Z_s)\, ds\right] - \mathbb{E}^0\left[\int_{\tau_R}^{\tau_0^w} e^{-rs}(x + \sigma Z_s)\, ds\right] = A - B. \tag{30}$$

First, observe that

$$A = \frac{x}{r}(1 - l(w)) \;+\; \mathbb{E}^0\left[\int_0^{\tau_0^w} e^{-rs} \sigma Z_s\, ds\right], \tag{31}$$

where, we set $l(w) = \mathbb{E}^0[e^{-r\tau_0^w}]$. Following [3], ch. 2, we introduce the function ϕ defined as the non-increasing fundamental solution on $\mathbb{R}$ of

$$\frac{\sigma^2\lambda^2}{2}\phi'' + rw\phi' - r\phi = 0,$$
$$\text{with } \phi(0) = 1, \quad \lim_{w\to+\infty} \phi(w) = 0.$$

The function l coincides with ϕ on $(0,+\infty)$ and in particular l is twice continuously differentiable over $(0,\infty)$ with $l'(0^+)<+\infty$. It follows that, for w small enough, $1-l(w) = -l'(0^+)\, w \,+\, o(w)$.

We now study the second term on the rhs of (31). Let us define the $\mathbb{P}^0$- uniformly integrable martingale

$$N_t = \int_0^t e^{-rs} dZ_s = e^{-rt} Z_t - r \int_0^t e^{-rs} Z_s\, ds.$$

Optional Sampling Theorem gives $\mathbb{E}^0[N_{\tau_0^w \wedge T}] = 0$. Then, letting $T \to +\infty$,

$$\mathbb{E}^0 \left[\int_0^{\tau_0^w} e^{-rs} Z_s\, ds \right] = -\frac{1}{r} \mathbb{E}^0 [e^{-r\tau_0^w} Z_{\tau_0^w}]. \tag{32}$$

Observe that $e^{-rt}W_t = w + \lambda\sigma N_t$ is thus a $\mathbb{P}^0$- uniformly square integrable martingale. Using the dynamics of (W_t), we deduce that

$$we^{-rt} + re^{-rt} \int_0^t W_s\, ds + \lambda\sigma e^{-rt} Z_t$$

is a $\mathbb{P}^0$- uniformly integrable martingale, thus the Optional Sampling Theorem yields

$$\begin{aligned} w = \mathbb{E}^0[e^{-r(T\wedge\tau_0^w)} W_{T\wedge\tau_0^w}] &= w\mathbb{E}^0[e^{-r(T\wedge\tau_0^w)}] + r\mathbb{E}^0 \left[e^{-r(T\wedge\tau_0^w)} \int_0^{T\wedge\tau_0^w} W_s\, ds \right] \\ &+ \lambda\sigma\mathbb{E}^0 \left[e^{-r(T\wedge\tau_0^w)} Z_{T\wedge\tau_0^w} \right]. \end{aligned} \tag{33}$$

Letting T tend to ∞, we obtain

$$w = w\mathbb{E}^0(e^{-r\tau_0^w}) + r\mathbb{E}^0 \left[e^{-r\tau_0^w} \int_0^{\tau_0^w} W_s\, ds \right] + \lambda\sigma\mathbb{E}^0 \left[e^{-r\tau_0^w} Z_{\tau_0^w} \right].$$

Using (32) and (33) one gets

$$\begin{aligned} \mathbb{E}^0 \left(\int_0^{\tau_0^w} e^{-rs} Z_s\, ds \right) &= \frac{1}{\lambda\sigma r} w(\mathbb{E}^0[e^{-r\tau_0^w}) - 1] + \frac{1}{\lambda\sigma} \mathbb{E}^0 \left[e^{-r\tau_0^w} \int_0^{\tau_0^w} W_s\, ds \right] \\ &= \frac{1}{\lambda\sigma} \left(\frac{1}{r} w(l(w)-1) + \mathbb{E}^0 \left[\int_0^\infty e^{-r\tau_0^w} \mathbb{1}_{s \le \tau_0^w} W_s\, ds \right] \right) \\ &= \frac{1}{\lambda\sigma} \left(\frac{1}{r} w(l(w)-1) + \mathbb{E}^0 \left[\int_0^\infty \mathbb{E}^0[e^{-r\tau_0^w} | \mathcal{F}_s] \mathbb{1}_{s \le \tau_0^w} W_s\, ds \right] \right) \\ &= \frac{1}{\lambda\sigma} \left(\frac{1}{r} w(l(w)-1) + \mathbb{E}^0 \left[\int_0^{\tau_0^w} e^{-rs} l(W_s) W_s\, ds \right] \right), \end{aligned}$$

where the last equality follows from the Strong Markov property. Now, it remains to prove that

$$g(w) \equiv \mathbb{E}^0 \left[\int_0^{\tau_0^w} e^{-rs} l(W_s) W_s\, ds \right] \;=\; cw \,+\, o(w),$$

for w small enough. First, we show that the function $w \to wl(w)$ is bounded. Let us consider the real function $k(w) = \frac{\int_w^\infty e^{-\frac{r}{\sigma^2\lambda^2}t^2}\,dt}{\int_0^\infty e^{-\frac{r}{\sigma^2\lambda^2}t^2}\,dt}$ which is the smooth solution to

$$\frac{\sigma^2\lambda^2}{2}k'' + rwk' = 0, \tag{34}$$

$$k(0) = 1, \quad \lim_{w\to+\infty} k(w) = 0. \tag{35}$$

Note that the function $\theta = k - l$ is twice continuously differentiable, bounded over $(0,\infty)$ and satisfies $\theta(0) = \lim_{w\to+\infty}\theta(w) = 0$ together with

$$\frac{\sigma^2\lambda^2}{2}\theta'' + rw\theta' \;=\; -rl \le 0.$$

Then, the process $(\theta(W_t))_t$ is a $\mathbb{P}^0$ bounded supermartingale and thus for every $T > 0$

$$\mathbb{E}^0[\theta(W_{\tau_0^w\wedge T})] \le \theta(w).$$

Letting $T \to +\infty$, we conclude $\theta(w) \ge 0$ because θ is bounded with $\theta(0) = 0$. It follows that $l(w)w \le k(w)w$ over $[0,\infty)$. We observe that $\lim_{w\to\infty} wk(w) = 0$ and thus $\lim_{w\to\infty} wl(w) = 0$. Therefore, the function $w \to wl(w)$ is bounded on $[0,\infty)$ and the function g is well-defined.

Assume for a while the existence of f bounded C^2 solution on $\mathbb{R}$ to the differential equation.

$$\frac{\sigma^2\lambda^2}{2}f'' + rwf' - rf + wl(w) = 0, \quad f(0) = 0. \tag{36}$$

Itô's formula yields for every $T > 0$,

$$\mathbb{E}^0\left[e^{-r(T\wedge\tau_0^w)}f(W_{T\wedge\tau_0^w})\right] = f(w) - \mathbb{E}^0\left(\int_0^{T\wedge\tau_0^w} e^{-rs}l(W_s)W_s\,ds\right). \tag{37}$$

Observe that

$$\begin{aligned}\mathbb{E}^0\left[e^{-r(T\wedge\tau_0^w)}f(W_{T\wedge\tau_0^w})\right] &= \mathbb{E}^0\left[e^{-rT}f(W_T)\mathbb{1}_{T\le\tau_0^w}\right] \qquad \text{because} f(0) = 0\\ &\le ||f||_\infty e^{-rT}.\end{aligned}$$

The monotone convergence theorem gives

$$\lim_{T\to+\infty}\mathbb{E}^0\left[\int_0^{T\wedge\tau_0^w} e^{-rs}l(W_s)W_s\,ds\right] = g(w).$$

Letting T goes to $+\infty$ in (37), we have that g coincides with f on $(0,+\infty)$. Therefore, g is a bounded, twice continuously differentiable function over $(0,\infty)$ and thus $g(w) = g'(0^+)w + o(w)$.

The existence of a bounded solution of (36) comes from the general form of solutions given by the method of variation of constants,

$$f(x) = \alpha x + \beta l(x) + l(x)\int_0^x \frac{u^2 l(u)}{W(u)}\,du - x\int_x^\infty \frac{ul^2(u)}{W(u)}\,du,$$

where $W(x) = \frac{2}{\sigma^2}(l(x) - xl'(x))$ is the Wronskian. Because $l \leq k$ and $W(x) = Ce^{-\frac{r}{\sigma^2\lambda^2}x^2}$ (see [3], ch. 2), the two integrals $\int_0^x \frac{u^2 l(u)}{W(u)}\,du$ and $\int_0^x \frac{ul^2(u)}{W(u)}\,du$ converge. Then, it suffices to choose $\alpha = 0$ and $\beta = -\int_0^\infty \frac{u^2 l(u)}{W(u)}\,du$ to conclude.

Summing up our results, we have obtained

$$\tilde{u}(x,w) \equiv \mathbb{E}^0\left[\int_0^{\tau_0^w} e^{-rs}(x+\sigma Z_s)\,ds\right] = c(x)w + w\eta(w), \quad \lim_{w\to 0}\eta(w) = 0.$$

Moreover, observe that η is a continuous bounded function on $(0,+\infty)$. We now turn to the second term of (30). We will show that, for w sufficiently small,

$$B = \mathbb{E}^0\left[\int_{\tau_R}^{\tau_0^w} e^{-rs}(x+\sigma Z_s)\,ds\right] = Cw + o(w),$$

where C is a constant that may depend on x. Strong Markov Property yields

$$\begin{aligned}
B &= \mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}\int_{\tau^*}^{\tau_0^w} e^{-rs}(x+\sigma Z_s)\,ds\right] \\
&= \mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}\tilde{u}(x^*, W_{\tau^*}^w)\right] \\
&= \mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}\left(CW_{\tau^*} + W_{\tau^*}^w\eta(W_{\tau^*}^w)\right)\right] \\
&= C\mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}W_{\tau^*}^w\right] + \mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}W_{\tau^*}^w\eta(W_{\tau^*}^w)\right] \\
&= Cw + \mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}W_{\tau^*}^w\eta(W_{\tau^*}^w)\right],
\end{aligned}$$

where the last equality used again the Optional Sampling Theorem with the martingale $(e^{-rt}W_t)_t$ because τ^* is almost surely finite. We will end the proof by showing that

$$\mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}W_{\tau^*}^w\eta(W_{\tau^*}^w)\right] = o(w)$$

for sufficiently small w. For every $\varepsilon > 0$, there is some $\delta > 0$ such that $|\eta(w)| \leq \varepsilon$ for all $w < \delta$. Now, fix $\varepsilon > 0$, $w < \delta$ and introduce the stopping time

$$\tau_\delta^w = \inf\{t \geq 0,\ W_t^w = \delta\}.$$

We have

$$\begin{aligned}
\left|\mathbb{E}^0\left[\mathbb{1}_{\{\tau^*<\tau_0^w\}}e^{-r\tau^*}W_{\tau^*}^w\eta(W_{\tau^*}^w)\right]\right| &= \Big|\mathbb{E}^0\left[e^{-r\tau^*}W_{\tau^*}^w\eta(W_{\tau^*}^w)\mathbb{1}_{\{\tau^*<\tau_0^w\}}\mathbb{1}_{\{\tau^*\leq\tau_\delta^w\}}\right] \\
&\quad + \mathbb{E}^0\left[e^{-r\tau^*}W_{\tau^*}^w\eta(W_{\tau^*}^w)\mathbb{1}_{\{\tau^*<\tau_0^w\}}\mathbb{1}_{\{\tau^*>\tau_\delta^w\}}\right]\Big| \\
&\leq \varepsilon\mathbb{E}^0\left[e^{-r\tau^*}W_{\tau^*}^w\mathbb{1}_{\{\tau^*<\tau_0^w\}}\right] + ||\eta||_\infty\mathbb{E}^0\left[e^{-r\tau^*}W_{\tau^*}^w\mathbb{1}_{\{\tau^*<\tau_0^w\}}\mathbb{1}_{\{\tau^*>\tau_\delta^w\}}\right] \\
&\leq \varepsilon w + ||\eta||_\infty\mathbb{E}^0\left[e^{-r\tau_\delta^w}W_{\tau_\delta^w}^w\mathbb{1}_{\{\tau_\delta^w<\tau_0^w\}}\right].
\end{aligned}$$

Proceeding analogously as in the proof of Lemma 4.4, we define on the set $\{\tau_\delta^w < \tau_0^w\}$ the diffusion $(\hat{W}_t)_{t\leq\tau_0}$ absorbed at δ using Doob h-transform and obtain

$$\mathbb{E}^0\left[e^{-r\tau_\delta^w}W_{\tau_\delta^w}\mathbb{1}_{\{\tau_\delta^w<\tau_0^w\}}\right] = \mathbb{P}^0[\tau_\delta^w < \tau_0^w]\mathbb{E}^0\left[e^{-r\hat{\tau}_\delta^w}\hat{W}_{\hat{\tau}_\delta^w}\right].$$

Now, both expressions $\mathbb{P}^0(\tau_\delta^w < \tau_0^w)$ and $\mathbb{E}^0\left[e^{-r\hat{\tau}_\delta^w}\hat{W}_{\hat{\tau}_\delta^w}\right] = \delta\mathbb{E}^0\left[e^{-r\hat{\tau}_\delta^w}\right]$ converges to zero when w converges to zero. Moreover, as in Lemma 4.4, $\mathbb{P}^0(\tau_\delta^w < \tau_0^w) = Cw + o(w)$. This ends the proof that $\frac{\partial u}{\partial w}(x,0)$ exists and is finite. Furthermore, we observe that the function $x \to \frac{\partial u}{\partial w}(x,0)$ is nondecreasing because $x \to u(x,w)$ is nondecreasing.

Proof of assertion (i). We have for $\varepsilon > 0$,

$$u(x,w+\varepsilon) - u(x,w) = \mathbb{E}^0\left[\int_0^{\tau_R^\varepsilon} e^{-rs}X_s\,ds\right] - \mathbb{E}^0\left[\int_0^{\tau_R} e^{-rs}X_s\,ds\right]$$

where $\tau_R^\varepsilon = \inf\{t \geq 0,\ (x+\sigma Z_t, w+\varepsilon+\int_0^t rW_s\,ds + \lambda\sigma Z_t) \notin R\} \geq \tau_R$. Strong Markov property gives for the first term

$$\mathbb{E}^0\left[\int_0^{\tau_R^\varepsilon} e^{-rs}X_s\,ds\right] = \mathbb{E}^0\left[\int_0^{\tau_R} e^{-rs}X_s\,ds\right] + \mathbb{E}^0\left[e^{-r(\tau^*\wedge\tau_0^w)}u(X_{\tau^*\wedge\tau_0^w}, W^{w+\varepsilon}_{\tau^*\wedge\tau_0^w})\right].$$

Using $u(x^*,w) = 0$ for all $w > 0$, we get

$$\frac{1}{\varepsilon}(u(x,w+\varepsilon) - u(x,w)) = \frac{1}{\varepsilon}\mathbb{E}^0\left[e^{-r\tau_0^w}u(X_{\tau_0^w}, W^{w+\varepsilon}_{\tau_0^w})\mathbb{1}_{\tau^*\geq\tau_0^w}\right].$$

Now, observe that $W^{w+\varepsilon}_{\tau_0} = \varepsilon e^{r\tau_0^w}$ and thus

$$\frac{1}{\varepsilon}(u(x,w+\varepsilon) - u(x,w)) = \mathbb{E}^0\left[\frac{u(X_{\tau_0^w}, \varepsilon e^{r\tau_0^w})}{\varepsilon e^{r\tau_0^w}}\mathbb{1}_{\tau^*\geq\tau_0^w}\right] \geq 0. \tag{38}$$

From Proposition 4.5, we know that the random variable $\frac{u(X_{\tau_0^w}, \varepsilon e^{r\tau_0^w})}{\varepsilon e^{r\tau_0^w}}\mathbb{1}_{\tau^*\geq\tau_0^w}$ converges to $\frac{\partial u}{\partial w}(X_{\tau_0^w}, 0)\mathbb{1}_{\tau^*\geq\tau_0^w}$ almost surely when ε tends to zero. Moreover, up to a constant, it is bounded above by $1 + X_{\tau_0^w}$ by Lemma 4.4. Now observe that

$$\lambda X_{\tau_0^w} \leq \lambda x - w$$

on the set $\{\tau^* \geq \tau_0^w\}$ and thus $1 + X_{\tau_0^w}$ is bounded on this set. Then, the dominated convergence Theorem yields assertion (i) by letting ε tend to zero in (38).

Proof of Assertion (ii). First, we will prove that $X^x_{\tau_0^{w_0}} \geq X^x_{\tau_0^{w_1}}$ for any $w_0 \leq w_1$ on the set $\{\tau_0^{w_1} < +\infty\}$. Note that $\tau_0^{w_0} \leq \tau_0^{w_1}$ almost surely for $w_0 \leq w_1$. We integrate $\lambda dX_t = dW_t - rW_t\,dt$ on the interval $(\tau_0^{w_0}, \tau_0^{w_1})$ on the set $\{\tau_0^{w_1} < +\infty\}$. We obtain

$$\lambda(X^x_{\tau_0^{w_1}} - X^x_{\tau_0^{w_0}}) = -W^{w_1}_{\tau_0^{w_0}} - r\int_{\tau_0^{w_0}}^{\tau_0^{w_1}} W_s^{w_1}\,ds \leq 0.$$

According to assertion (i),

$$\begin{aligned}
\frac{\partial u}{\partial w}(x,w_0) &= \mathbb{E}^0\left[\mathbb{1}_{\tau_0^{w_0}\leq\tau^*}\frac{\partial u}{\partial w}(X^x_{\tau_0^{w_0}}, 0)\right] \\
&\geq \mathbb{E}^0\left[\mathbb{1}_{\tau_0^{w_1}\leq\tau^*}\frac{\partial u}{\partial w}(X^x_{\tau_0^{w_0}}, 0)\right] \\
&\geq \mathbb{E}^0\left[\mathbb{1}_{\tau_0^{w_1}\leq\tau^*}\frac{\partial u}{\partial w}(X^x_{\tau_0^{w_1}}, 0)\right] \\
&= \frac{\partial u}{\partial w}(x,w_1),
\end{aligned}$$

where the last inequality comes from the fact $x \to \frac{\partial u}{\partial w}(x,0)$ is nondecreasing. Thus, the function $w \to \frac{\partial u}{\partial w}$ is a decreasing function. Because we know that u is twice continuously differentiable over R, we get assertion (ii).

Proof of assertion (iii). Let us consider f defined as

$$f(x) = \frac{\partial u}{\partial w}(x, \lambda(x-c)) \text{ for } x \geq x^*.$$

To prove assertion (iii), we will show that f is nonincreasing for any c such that $(x, \lambda(x-c))$ is in R. Take $x_0 \leq x_1$ and $w_i = \lambda(x_i - c)$ for $i = 0, 1$. From assertion (ii), we have

$$f(x_0) = \mathbb{E}^0 \left[\mathbb{1}_{\tau_0^{w_0} \leq \tau^{*,x_0}} \frac{\partial u}{\partial w}(X^{x_0}_{\tau_0^{w_0}}, 0) \right].$$

We will show that $\mathbb{1}_{\{\tau_0^{w_0} \leq \tau^{*,x_0}\}} \geq \mathbb{1}_{\{\tau_0^{w_1} \leq \tau^{*,x_1}\}}$ or equivalently that

$$\{\tau_0^{w_0} > \tau^{*,x_0}\} \subset \{\tau_0^{w_1} > \tau^{*,x_1}\}.$$

On the set $\{\tau_0^{w_0} > \tau^{*,x_0}\}$, we have

$$\begin{aligned} X^{x_1}_{\tau^{*,x_0}} &= x^* + x_1 - x_0, \\ W^{w_1}_{\tau^{*,x_0}} &= W^{w_0}_{\tau^{*,x_0}} + (w_1 - w_0)e^{r\tau^{*,x_0}}. \end{aligned}$$

Therefore,

$$\begin{aligned} W^{w_1}_{\tau^{*,x_0}} - \lambda X^{x_1}_{\tau^{*,x_0}} &= W^{w_0}_{\tau^{*,x_0}} + (w_1 - w_0)e^{r\tau^{*,x_0}} - \lambda(x^* + x_1 - x_0) \\ &\geq \lambda(x_1 - x_0)(e^{r\tau^{*,x_0}} - 1) - \lambda x^* \\ &\geq -\lambda x^*, \end{aligned}$$

and thus for all $t \geq \tau^{*,x_0}$, we have $W^{w_1}_t \geq \lambda(X^{x_1}_t - x^*)$. This implies $\tau_0^{w_1} > \tau^{*,x_1}$. Consequently,

$$f(x_0) \geq \mathbb{E}^0 \left[\mathbb{1}_{\tau_0^{w_1} \leq \tau^{*,x_1}} \frac{\partial u}{\partial w}(X^{x_0}_{\tau_0^{w_0}}, 0) \right].$$

Proceeding as previously, on the set $\{\tau_0^{w_1} < +\infty\}$, we have

$$\lambda(X^{x_0}_{\tau_0^{w_0}} - X^{x_1}_{\tau_0^{w_1}}) = r\Big(\int_0^{\tau_0^{w_1}} W^{w_1}_s \, ds - \int_0^{\tau_0^{w_0}} W^{w_0}_s \, ds \Big) \geq 0.$$

Thus,

$$f(x_0) \geq \mathbb{E}^0 \left[\mathbb{1}_{\tau_0^{w_1} \leq \tau^{*,x_1}} \frac{\partial u}{\partial w}(X^{x_1}_{\tau_0^{w_1}}, 0) \right] = f(x_1).$$

□

References

[1] Biais, B., Mariotti T., Rochet J.C.: Dynamic Financial Contracting, Advances in Economics and Econometrics: Tenth World Congress, Cambridge university press, (2013).

[2] Biais, B., Mariotti T., Plantin G., Rochet J.C.: Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications, Review of Economic Studies, 74 (2), 345-390, (2007).

[3] Borodin, A. and Salminen, P. : Handbook on Brownian Motion-Facts and Formulae, Birkhauser, Basel (1996).

[4] Cvitanic, J., Possamaï, D., Touzi, N.: Moral hazard in dynamic risk management. Management Science. 63(10), 3328-3346, (2016).

[5] Cvitanic, J., Possamaï, D., Touzi, N.: Dynamic Programming approach to Principal-agent problems, forthcoming Finance and Stochastics.

[6] Daskalopoulos P., Feehan P.M.N.: $C^{1,1}$ regularity for degenerate elliptic obstacle problems, Journal of Differential Equations 260, (6) 5043-5074, (2016).

[7] DeMarzo, P.M., Sannikov, Y.: Optimal security designand dynamic capital structure in a continuous time agency model. Journal of Finance. 61, 2681-2724, (2006).

[8] DeMarzo, P.M., Sannikov, Y.: Learning, termination, and payout policy in dynamic incentive contracts. Review of Economic Studies, 84(1), 182-236, (2017).

[9] Dixit, A.K., Pindyck, R.S.: Investment Under Uncertainty. Princeton University Press, Princeton, (1994).

[10] Faingold, E., Vasama, S.: Real options and dynamic incentives. Working paper, (2014).

[11] He Z.: Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion, The Review of Financial Studies, 22, 2, 859-892, (2009).

[12] Hynd, R.: Analysis of Hamilton–Jacobi–Bellman equations arising in stochastic singular control. ESAIM: Control, Optimisation and Calculus of Variations, 19(1):112-128, (2013).

[13] Hynd, R.: An eigenvalue problem for a fully nonlinear elliptic equation with gradient con- straint. Calculus of Variations and Partial Differential Equations, 56(2):34, (2017).

[14] Hynd, R., Mawi H.: On Hamilton–Jacobi–Bellman equations with convex gradient constraints. Interfaces and Free Boundaries, 18(3):291-315, (2016).

[15] Lamberton D., Terenzi G.: Variational formulation of American option prices in the Heston Model, arXiv:1711.11311, (2017).

[16] Krylov N.V.: Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, Graduate studies in Mathematics, AMS Providence, (2008).

[17] Pham, H.: On the smooth-fit property for one-dimensional optimal switching problem, Séminaire de Probabilités XL, pp. 187-199, (2007).

[18] Pham, H.: Continuous-time stochastic control and optimization with financial applications. Springer Verlag, (2010).

[19] Rogers L.C.G and Williams R.: Diffusions, Markov Processes and Martingales, Vol 2, Cambridge University Press, (2000).

[20] Sannikov, Y.: A continuous-time version of the principal-agent problem. Review of Economic Studies. 75, 957-984, (2008).

[21] Soner,H. M., Shreve S.E.: Regularity of the value function for a two–dimensional singular stochastic control problem. SIAM Journal on Control and Optimization, 27(4):876-907, (1989).

[22] Strulovici, B.: Contracts, information persistence, and renegotiation. Working paper, Northwestern University, (2011).

[23] Strulovici, B., Szydlowski, M.: On the smoothness of value functions and the existence of optimal strategies in diffusion models. Journal of Economic Theory. 159, 1016-1055, (2015).

[24] Vasama, S.: On Moral Hazard and Persistent Private Information. Bank of Finland Research, Discussion paper No 15/2017, (2017).

[25] Williams, N.: Persistent private information. Econometrica. 79(4), 1233-1275, (2011).

[26] Zhang, Y.: Dynamic contracting with persistent shocks. Journal of Economic Theory, 144, 635-675, (2009).

[27] Zhu, J.Y.: Optimal contracts with shirking. Review of Economic Studies. 0, 1-28, (2012).