Price Caps as Welfare-Enhancing Coopetition

Online Appendix (Not for publication)

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A Nash equilibrium

We establish here the existence of a unique Nash equilibrium in the setting considered in Section 2.1:

Lemma 4 (Nash equilibrium) *In the setting considered in Section 2.1, if in addition*

$$\forall j \in \mathcal{N}, \sum_{i \in \mathcal{N} \setminus \{j\}} |\partial_j R_i(\mathbf{p}_{-i})| < 1,$$

then there exists a unique static Nash equilibrium, which is moreover "stable" under the standard tâtonnement process.

Proof. As it is never optimal for firm i to charge a negative price, and $R_i(\cdot)$ is bounded above by some finite B_i (which obviously must satisfy $B_i > 0$), we have, for $\mathbf{p}_{-i} \in \mathbb{R}_+^{n-1}$:

$$R_i(\mathbf{p}_{-i}) \in \mathcal{C}_i \equiv [0, B_i],$$

where $\mathcal{C}_i$ is a non-empty compact interval of $\mathbb{R}_+$. Note that, by construction, any Nash equilibrium price vector $\mathbf{p}^N = \left(p_i^N\right)_{i \in \mathcal{N}}$ is such that $p_i^N \in \mathcal{C}_i$.

Next, define $\boldsymbol{\phi}(\mathbf{p}) \equiv (\phi_i(\mathbf{p}))_{i \in \mathcal{N}}$, where $\phi_i(\mathbf{p}) = R_i(\mathbf{p}_{-i})$. $\boldsymbol{\phi}$ is a contraction mapping from $\mathcal{C} \equiv \mathcal{C}_1 \times ... \times \mathcal{C}_n$ to $\mathcal{C}$, endowed with the ℓ_1 norm: for any $\mathbf{p} \in \mathcal{C}$, $\boldsymbol{\phi}(\mathbf{p}) \in \mathcal{C}$ and, in addition, for any $\mathbf{p}' \in \mathcal{C}$:

$$\begin{aligned}
\left\|\boldsymbol{\phi}\left(\mathbf{p}'\right)-\boldsymbol{\phi}\left(\mathbf{p}\right)\right\| &= \sum_{i\in\mathcal{N}}\left|\phi_i\left(\mathbf{p}'\right)-\phi_i\left(\mathbf{p}\right)\right| \\
&= \sum_{i\in\mathcal{N}}\left|R_i\left(\mathbf{p}'_{-i}\right)-R_i\left(\mathbf{p}_{-i}\right)\right| \\
&= \sum_{i\in\mathcal{N}}\left|\int_0^1\frac{d}{d\lambda}\left\{R_i\left(\lambda\mathbf{p}'_{-i}+(1-\lambda)\,\mathbf{p}_{-i}\right)\right\}d\lambda\right| \\
&\le \sum_{i\in\mathcal{N}}\int_0^1\{\sum_{j\in\mathcal{N}\backslash\{i\}}\left|\partial_j R_i\left(\lambda\mathbf{p}'_{-i}+(1-\lambda)\,\mathbf{p}_{-i}\right)\right|\left|p'_j-p_j\right|\}d\lambda \\
&= \int_0^1\{\sum_{j\in\mathcal{N}}[\sum_{i\in\mathcal{N}\backslash\{j\}}\left|\partial_j R_i\left(\lambda\mathbf{p}'_{-i}+(1-\lambda)\,\mathbf{p}_{-i}\right)\right|]\left|p'_j-p_j\right|\}d\lambda \\
&\le \sum_{j\in\mathcal{N}}k\left|p'_j-p_j\right| \\
&= k\left\|p'-p\right\|,
\end{aligned}$$

where:

$$k=\max_{\mathbf{p}\in\mathcal{C},j\in\mathcal{N}}\sum_{i\in\mathcal{N}\backslash\{j\}}\left|\partial_j R_i\left(\mathbf{p}_{-i}\right)\right|<1.$$

It follows from the Banach fixed point theorem that $\boldsymbol{\phi}\left(\mathbf{p}\right)$ has a unique fixed point in $\mathcal{C}$, $\mathbf{p}^N$, and that any sequence $\left\{\mathbf{p}_n\right\}_{n\in\mathbb{N}}$ satisfying $\mathbf{p}_{n+1}=\boldsymbol{\phi}\left(\mathbf{x}_n\right)$ converges to this fixed point. Hence, $\mathbf{p}^N$ is the unique Nash equilibrium of the static game, and it is stable under the standard tâtonnement process. ■

B On Assumption A

B.1 A sufficient condition

We show here that Assumption A holds under the following condition:

Assumption A': For any $i\neq j\in\{1,2\}$ and any prices $p_i\in[0,p_i^N)$ and $p_j>p_j^N$:

$$D_j\left(p_j,p_i\right)\partial^2_{11}D_j\left(p_j,p_i\right)<2\left(\partial_1 D_j\left(p_j,p_i\right)\right)^2-C''_j\left(D_j\left(p_j,p_i\right)\right)\left(\partial_1 D_j\left(p_j,p_i\right)\right)^3,$$

and:

$$\begin{aligned}
& D_j\left(p_j,p_i\right)\left[\partial_1 D_j\left(p_j,p_i\right)\partial_2 D_j\left(p_j,p_i\right)-D_j\left(p_j,p_i\right)\partial^2_{12}D_j\left(p_j,p_i\right)\right] \\
< \; & D_i\left(p_i,p_j\right)\left[2\left(\partial_1 D_j\left(p_j,p_i\right)\right)^2-D_j\left(p_j,p_i\right)\partial^2_{11}D_j\left(p_j,p_i\right)\right] \\
& +C''_j\left(D_j\right)\left(\partial_1 D_j\right)^2\left[D_j\left(p_j,p_i\right)\partial_2 D_j\left(p_j,p_i\right)-D_i\left(p_i,p_j\right)\partial_1 D_j\left(p_j,p_i\right)\right].
\end{aligned}$$

The first part of this assumption amounts to say that, for any given price of the other firm, the profit of a given firm is concave with respect to the price of that firm. It is satisfied, for instance, when the cost function is weakly convex (i.e., $C_i''(\cdot) \geq 0$) and the elasticity of the inverse of the residual demand is lower than 2, as is the case for the demand functions usually considered in oligopoly theory – in particular, it holds whenever the residual demand is log-concave (or equivalently, that the elasticity of its inverse is lower than 1), or it is exponential (and thus log-convex) with an elasticity higher than 1.

The second part of the assumption holds, for instance, when the goods are close to being perfect complements.[1]

Firm j's best-response, $R_j(p_i)$, is characterized by the first-order condition:

$$\left[R_j(p_i) - C_j'(D_j(R_j(p_i), p_i))\right] \partial_1 D_j(R_j(p_i), p_i) + D_j(R_j(p_i), p_i) = 0,$$

which yields (dropping the argument $(R_j(p_i), p_i)$):

$$R_j'(p_i) = -\frac{\partial_1 D_j \partial_2 D_j - D_j \partial_{12}^2 D_j - C_j''(D_j)(\partial_1 D_j)^2 \partial_2 D_j}{2(\partial_1 D_j)^2 - D_j \partial_{11}^2 D_j - C_j''(D_j)(\partial_1 D_j)^3},$$

where the denominator of the right-hand side is positive under Assumption A'.

Therefore, Assumption A amounts to (dropping the argument $(R_j(p_i), p_i)$):

$$\begin{aligned} & D_i\left[2(\partial_1 D_j)^2 - D_j \partial_{11}^2 D_j - C_j''(D_j)(\partial_1 D_j)^3\right] \\ & \quad > D_j\left[\partial_1 D_j \partial_2 D_j - D_j \partial_{12}^2 D_j - C_j''(D_j)(\partial_1 D_j)^2 \partial_2 D_j\right], \end{aligned}$$

which follows from Assumption A'.

B.2 A counter-example

We provide here an example where Assumption A does not hold.

B.2.1 Setting

There are two goods 1 and 2, produced at no cost by two different firms 1 and 2, and a unit mass of consumers, indexed by x, where x is uniformly distributed on $[0, 1]$:

[1] Remember that Assumption A holds trivially when prices are strategic complements, as is usually the case for substitutable goods. Hence, considering Assumption A' is useful only when prices are strategic substitutes, which in turn is mostly relevant when the goods are complements. But for perfect complements, demands are of the form $D_i(p_i, p_j) = D(p_1 + p_2)$, and Assumption A' then boils down to

$$\begin{aligned} D(\cdot)\left[(D'(\cdot))^2 - D(\cdot) D''(\cdot)\right] \quad < \quad & D(\cdot)\left[2(D'(\cdot))^2 - D(\cdot) D''(\cdot)\right] \\ & + C_j''(\cdot)(D'(\cdot))^2 \left[D(\cdot) D'(\cdot) - D(\cdot) D'(\cdot)\right], \end{aligned}$$

or $D(\cdot)(D'(\cdot))^2 > 0$, which is trivially satisfied. By continuity, this strict inequality still holds when the above demands are only slightly modified.

- each consumer is willing to buy $1-\lambda$ units of good 1, and the per-unit valuation of consumer x is $v_1(x) = x$; hence, the demand for good 1 is given by:

$$q_1 = D_1(p_1) = \begin{cases} (1-\lambda)(1-p_1) & \text{if} \quad p_1 \le 1, \\ 0 & \text{if} \quad p_1 > 1. \end{cases}$$

- each consumer x is willing to buy λ units of good 2, and the per-unit valuation of consumer x is $v_2(x) = 1 - x + xn_1$, where $n_1 = q_1/(1-\lambda)$ denotes the number of consumers buying good 1; hence, in the relevant range $p_1 \in [0,1]$ (so that $n_1 = 1-p_1$ and $v_2(x) = 1 - xp_1$), the demand for good 2 is given by:

$$q_2 = D_2(p_1, p_2) = \begin{cases} \lambda & \text{if} \quad p_2 \le 1 - p_1, \\ \lambda \dfrac{1-p_2}{p_1} & \text{if} \quad 1 - p_1 \le p_2 \le 1, \\ 0 & \text{if} \quad p_2 \ge 1. \end{cases}$$

B.2.2 Nash equilibrium

The firms' best-responses are as follows:

- firm 1 always charges the monopoly price for its product:

$$R_1(p_2) = p_1^M = \frac{1}{2}.$$

- in the relevant range $p_1 \in [0,1]$, firm 2 never charges a price below $1 - p_1$ (as all consumers are buying at that price) and thus:

$$R_2(p_1) = \begin{cases} 1 - p_1 & \text{if} \quad p_1 \le \frac{1}{2}, \\ \dfrac{1}{2} & \text{if} \quad p_1 \ge \frac{1}{2}. \end{cases}$$

Therefore, in equilibrium, firm 1 charges the monopoly price:

$$p_1^N = p_1^M = \frac{1}{2},$$

leading to:

$$n_1^N = n_1^M = \frac{1}{2},$$

and thus:

$$q_1^N = q_1^M = \frac{1-\lambda}{2}.$$

In response, firm 2 charges :

$$p_2^N = R_2\left(p_1^M\right) = \frac{1}{2},$$

leading to $n_2^N = 1$ (that is, all consumers buy good 2) and:

$$q_2^N = \lambda.$$

Consumers thus obtain a surplus equal to:

$$S^N = (1-\lambda)\frac{1}{8} + \lambda \int_0^1 \left[\left(1 - \frac{x}{2}\right) - \frac{1}{2}\right] dx = \frac{1+\lambda}{8},$$

whereas firms' profit are given by:

$$\begin{aligned} \pi_1^N &= \pi_1^M = \frac{1-\lambda}{4}, \\ \pi_2^N &= \lambda p_2^N = \frac{\lambda}{2}. \end{aligned}$$

Industry profit and total welfare are thus respectively equal to:

$$\begin{aligned} \Pi^N &= \frac{1+\lambda}{4}, \\ W^N &= \Pi^N + S^N = \frac{3}{8}(1+\lambda). \end{aligned}$$

B.2.3 Price caps

Suppose now that a price cap $\bar{p} \in [0, 1/2]$ is imposed on firm 1: that is, firm 1's price must satisfy $p_1 \leq \bar{p}$. As firm 1's profit is quasi-concave in its price and maximal for $p_1 = p_1^M = 1/2 \geq \bar{p}$, in equilibrium firm 1 finds it optimal to charge a price just satisfying the constraint; that is, it charges:

$$\hat{p}_1(\bar{p}) = \bar{p},$$

leading to

$$\hat{n}_1(\bar{p}) = 1 - \bar{p}$$

and

$$\hat{q}_1(\bar{p}) = (1-\lambda)(1-\bar{p}).$$

In response, firm 2 sells to all consumers (that is, $n_2 = 1$ and $q_2 = \lambda$) by charging a price:

$$\hat{p}_2(\bar{p}) = R_2(\bar{p}) = 1 - \bar{p}.$$

Firms' profits are now given by:

$$\begin{aligned} \hat{\pi}_1(\bar{p}) &= (1-\lambda)\,\bar{p}\,(1-\bar{p})\left(\leq \pi_1^N\right), \\ \hat{\pi}_2(\bar{p}) &= \lambda \hat{p}_2(\bar{p}) = \lambda(1-\bar{p}). \end{aligned}$$

If transfers are feasible, the firms will set the price cap so as to maximize the industry profit, equal to:

$$\hat{\Pi}\left(\bar{p}\right)=\left(1-\lambda\right)\bar{p}\left(1-\bar{p}\right)+\lambda\left(1-\bar{p}\right)=\left(1-\bar{p}\right)\left[\lambda+\left(1-\lambda\right)\bar{p}\right],$$

leading to:

$$\bar{p}^{*}=\max\left\{\frac{1}{2}\frac{1-2\lambda}{1-\lambda},0\right\}.$$

In particular, $\bar{p}^{*}=0$ for $\lambda>1/2$; in that case, firm 1 sets its price to 0: $\hat{p}_{1}^{*}=0$, whereas firm 2 extracts all the surplus generated by its good: $\hat{p}_{2}^{*}=1$. As a result:

- Price caps enable the firms to increase their joint profits:[2]

$$\hat{\Pi}^{*}=\left(1-\lambda\right)\times 0+\lambda\times 1=\lambda>\Pi^{N}=\frac{1+\lambda}{4}.$$

- This harms consumers when λ is large enough:

$$\hat{S}^{*}=\left(1-\lambda\right)\times\frac{1}{2}+\lambda\times 0=\frac{1-\lambda}{2},$$

which is lower than $S^{N}=\left(1+\lambda\right)/8$ whenever $\lambda>3/5\,(>1/2)$.[3]

C Multi-product oligopoly under (SC)

We extend here the analysis to multi-product oligopolies where prices are strategic complements (both within and across firms). We show that, under that assumption, price caps cannot generate higher equilibrium prices (regardless of whether goods are complements or substitutes). It follows that price caps can only benefit consumers, and are useful for suppliers of complements, but not for competitors offering substitutes.

C.1 Setting

We consider a multi-product firm oligopoly setting with $n\geq 2$ multi-product firms, indexed by $i\in\mathcal{N}\equiv\{1,...,n\}$, each producing m_i products, indexed by $j\in\mathcal{M}_i\equiv\{1,...,m_i\}$; there are thus in total $m\equiv\sum_{i\in\mathcal{N}}m_i$ prices. Let $C_i\left(\mathbf{q}_i\right)$, where $\mathbf{q}_i=\left(q_i^j\right)_{j\in\mathcal{M}_i}$, denote firm i's cost of producing each good $j\in\mathcal{M}_i$ in quantity q_i^j, and $\mathbf{D}_i\left(\mathbf{p}\right)=\left(D_i^1\left(\mathbf{p}\right),...,D_i^{m_i}\left(\mathbf{p}\right)\right)$ denote the demand for these goods, as a function of the vector of prices $\mathbf{p}=\left(\mathbf{p}_i\right)_{i\in\mathcal{N}}\in\mathbb{R}_+^m$, where $\mathbf{p}_i=\left(p_i^j\right)_{j\in\mathcal{M}_i}\in\mathbb{R}_+^{m_i}$ denotes the vector of firm i's

[2] The inequality holds whenever $\lambda>1/3$, which is implied by $\lambda>1/2$.

[3] It can however be checked that total welfare is increased:

$$\hat{W}^{*}=\left(1-\lambda\right)\times\frac{1}{2}+\lambda\times 1=\frac{1+\lambda}{2}>W^{N}=\frac{3}{8}\left(1+\lambda\right).$$

prices. We will assume that, for $i \in \mathcal{N}$, $\mathbf{D}_i(\cdot)$ and $C_i(\cdot)$ are both C^2 and that, for every $i \in \mathcal{N}$:

- the profit function

$$\pi_i(\mathbf{p}) \equiv \sum_{j \in \mathcal{M}_i} p_i^j D_i^j(\mathbf{p}) - C_i(\mathbf{D}_i(\mathbf{p}))$$

  is strictly quasi-concave in $\mathbf{p}_i$;

- for every $j \in \mathcal{M}_j$, the "product-by-product" best-response function[4]

$$r_i^j\left(\mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}\right) \equiv \arg\max_{p_i^j} \pi_i\left(p_i^j, \mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}\right),$$

  where $\left(\mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}\right)$ denotes the vector of all prices but p_i^j, is well-defined and bounded above.

Remark: Note that we consider here firm i's price decision for one of its products, taking as given not only the other firms' prices, but also firm i's own prices for its other products. Furthermore, as the best-response is bounded, it is interior and thus, given the strict quasi-concavity assumption, uniquely characterized by the first-order condition $\partial_{p_i^j} \pi_i\left(r_i^j\left(\mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}\right), \mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}\right) = 0$.

We still focus on strategic complementarity, both "within firms" and "across firms":

(**SC**) *Strategic complementarity:* for every $i \in \mathcal{N}$ and every $j \in \mathcal{M}_i$, $r_i^j(\cdot)$ strictly increases in p_h^k for any $h \in \mathcal{N}$ and any $k \in \mathcal{M}_h$ such that $(h, k) \neq (i, j)$.

Finally, we assume again that there exists a unique Nash equilibrium, and denote by $\mathbf{p}^N = \left(\mathbf{p}_i^N\right)_{i \in \mathcal{N}}$ the equilibrium prices.

C.2 Price caps

Suppose now that each firm $i \in \mathcal{N}$ faces a price cap $\bar{p}_i^j$ for each product j in $\mathcal{M}_i$. Any resulting equilibrium price vector $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_i)_{i \in \mathcal{N}}$ satisfies $\hat{\mathbf{p}}_i = \bar{\mathbf{R}}_i(\hat{\mathbf{p}}_{-i}; \bar{\mathbf{p}}_i)$ for $i \in \mathcal{N}$, where:

$$\bar{\mathbf{R}}_i(\mathbf{p}_{-i}; \bar{\mathbf{p}}_i) = \left(\bar{R}_i^j(\mathbf{p}_{-i}; \bar{\mathbf{p}}_i)\right)_{j \in \mathcal{M}_i} \equiv \arg\max_{\mathbf{p}_i \leq \bar{\mathbf{p}}_i} \pi_i(\mathbf{p}_i, \mathbf{p}_{-i})$$

denotes firm i's best-response, constrained by the price caps it faces.

We first show that multiple price deviations cannot be more profitable for a firm than isolated ones, so that a firm best responds to its rivals if and only if each of its prices best responds individually to all other prices, including its own. That is, letting

$$\bar{r}_i^j\left(\mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}; \bar{p}_i^j\right) \equiv \arg\max_{p_i^j \leq \bar{p}_i^j} \pi_i\left(p_i^j, \mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}\right)$$

[4] In what follows, $\mathbf{p}^{\mathcal{S}}$ denotes the projection of the vector $\mathbf{p}$ on the subset $\mathcal{S}$; that is: $\mathbf{p}^{\mathcal{S}} = \left(p^j\right)_{j \in \mathcal{S}}$.

denote firm i's constrained product-by-product best-response for good $j \in \mathcal{M}_i$, given all other prices (including its own) and the price cap it faces for that good, we have:

Lemma 5 (constrained best-responses) *For any $i \in \mathcal{N}$, rivals' prices $\mathbf{p}_{-i}$ and price caps $\bar{\mathbf{p}}_i = \left(\bar{p}_i^j\right)_{j \in \mathcal{M}_i}$, firm i's constrained best-response function $\bar{\mathbf{R}}_i\left(\mathbf{p}_{-i}; \bar{\mathbf{p}}_i\right)$ can be characterized as the unique fixed point in $\mathcal{M}_i$ of firm i's constrained product-by-product best-responses; that is:*

$$\hat{\mathbf{p}}_i = \bar{\mathbf{R}}_i\left(\mathbf{p}_{-i}; \bar{\mathbf{p}}_i\right) \Longleftrightarrow \left\{\hat{p}_i^j = \bar{r}_i^j\left(\hat{\mathbf{p}}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}; \bar{p}_i^j\right) \text{ for every } j \in \mathcal{M}_i\right\}.$$

Proof. Consider firm i, for given price caps $\bar{\mathbf{p}}_i = \left(\bar{p}_i^j\right)_{j \in \mathcal{M}_i}$ and given rivals' prices $\mathbf{p}_{-i}$. Obviously, each price in firm i's (constrained) best-response is also a (constrained) product-by-product best-response: $\hat{\mathbf{p}}_i = \bar{\mathbf{R}}_i\left(\mathbf{p}_{-i}; \bar{\mathbf{p}}_i\right)$ implies $p_i^j = \bar{r}_i^j\left(\mathbf{p}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}; \bar{p}_i^j\right)$ for every $i \in \mathcal{N}$ and every $j \in \mathcal{M}_i$; uniqueness then follows from the strict quasi-concavity of the profit function. We now show that, conversely, any fixed point of the (constrained) product-by-product best-responses constitutes a best-response for firm i.

Thus, consider a price vector $\hat{\mathbf{p}}_i = \left(\hat{p}_i^j\right)_{j \in \mathcal{M}_i}$ satisfying $\hat{p}_i^j = \bar{r}_i^j\left(\hat{\mathbf{p}}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}; \bar{p}_i^j\right)$ for every $j \in \mathcal{M}_i$, and suppose that $\hat{\mathbf{p}}_i \neq \check{\mathbf{p}}_i \equiv \bar{\mathbf{R}}_i\left(\mathbf{p}_{-i}; \bar{\mathbf{p}}_i\right)$. By construction, both $\hat{\mathbf{p}}_i \leq \bar{\mathbf{p}}_i$ and $\check{\mathbf{p}}_i \leq \bar{\mathbf{p}}_i$, and so $\varepsilon \check{\mathbf{p}}_i + (1-\varepsilon)\hat{\mathbf{p}}_i \leq \bar{\mathbf{p}}_i$ for any $\varepsilon \in (0,1)$. Furthermore, as $\hat{\mathbf{p}}_i$ consists of product-by-product best responses, for every $j \in \mathcal{M}_i$ either $\partial_{p_i^j}\pi_i\left(\hat{\mathbf{p}}_i, \mathbf{p}_{-i}\right) = 0$ (if $\hat{p}_i^j = r_i^j\left(\hat{\mathbf{p}}_i^{\mathcal{M}_i \setminus \{j\}}, \mathbf{p}_{-i}\right)$, i.e., the price cap $\bar{p}_i^j$ is not binding for $\hat{\mathbf{p}}_i$) or $\partial_{p_i^j}\pi_i\left(\hat{\mathbf{p}}_i, \mathbf{p}_{-i}\right)\left(\check{p}_i^j - \hat{p}_i^j\right) < 0$; furthermore, the latter must hold for some $j \in \mathcal{M}_i$: otherwise, $\hat{\mathbf{p}}_i$ would be firm i's unconstrained best-response (i.e., $\hat{\mathbf{p}}_i = \mathbf{R}_i\left(\mathbf{p}_{-i}\right) \leq \bar{\mathbf{p}}_{-i}$), implying $\check{\mathbf{p}}_i = \mathbf{R}_i\left(\mathbf{p}_{-i}\right) = \hat{\mathbf{p}}_i$, a contradiction. Therefore, for ε positive but small we have:

$$\pi_i\left(\varepsilon \check{\mathbf{p}}_i + (1-\varepsilon)\hat{\mathbf{p}}_i, \mathbf{p}_{-i}\right) - \pi_i\left(\hat{\mathbf{p}}_i, \mathbf{p}_{-i}\right) \simeq \sum_{j \in \mathcal{M}_i} \partial_{p_i^j}\pi_i\left(\hat{\mathbf{p}}_i, \mathbf{p}_{-i}\right)\varepsilon\left(\check{p}_i^j - \hat{p}_i^j\right) < 0.$$

Hence, $\pi_i\left(\hat{\mathbf{p}}_i + \varepsilon\left(\check{\mathbf{p}}_i - \hat{\mathbf{p}}_i\right), \mathbf{p}_{-i}\right) \leq \pi_i\left(\hat{\mathbf{p}}_i, \mathbf{p}_{-i}\right) < \pi_i\left(\check{\mathbf{p}}_i, \mathbf{p}_{-i}\right)$, contradicting the strict quasi-concavity of the profit function π_i.[5] ■

Lemma 5 allows us to treat the present n-player game, where each firm $i \in \mathcal{N}$ has m_i products, as a $m-$player game among single-product firms. Building on this, we now show that price caps cannot be used to raise equilibrium prices:

Proposition 8 (incidence of price caps for multi-product firms under (SC)) *With multi-product firms under* (**SC**) *and for any vector $\bar{\mathbf{p}} = \left(\bar{\mathbf{p}}_i\right)_{i \in \mathcal{N}}$ of price caps, there exists a unique Nash equilibrium, and the equilibrium prices weakly increase with the price cap vector. Therefore: (i) for any vector of price caps $\bar{\mathbf{p}}$, consumers are weakly better off*

[5] We use here the characterizing property that a function $f(x)$ is strictly quasi-concave if and only if, for any $x \neq y$ and $\lambda \in (0,1)$: $f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}$.

under $G_{\bar{\mathbf{p}}}$ than under G_∞; and (ii) in $\mathcal{G}$, it is optimal for the competition authority to allow price caps.

Proof. Fix a vector of price caps $\bar{\mathbf{p}} = (\bar{\mathbf{p}}_i)_{i\in\mathcal{N}}$. From Lemma 5, a price vector $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_i)_{i\in\mathcal{N}} = \left(\left(\hat{p}_i^j\right)_{j\in\mathcal{M}_i}\right)_{i\in\mathcal{N}}$ constitutes a Nash equilibrium for these price caps if and only if it constitutes a Nash equilibrium of the $m-$player game in which each price p_i^j is chosen by a distinct player (subject to the price cap $\bar{p}_i^j$) so as to maximize the profit $\pi_i\left(p_i^j, \mathbf{p}_i^{\mathcal{M}_i\setminus\{j\}}, \mathbf{p}_{-i}\right)$, taking as given firm i's other prices, $\mathbf{p}_i^{\mathcal{M}_i\setminus\{j\}}$, as well as the other firms' prices, $\mathbf{p}_{-i}$. In the absence of price caps, this player's behavior is given by the best-response function $r_i^j\left(\mathbf{p}_i^{\mathcal{M}_i\setminus\{j\}}, \mathbf{p}_{-i}\right)$, which is bounded above by some B_i^j. Without loss of generality, we can thus restrict the price p_i^j to belong to $S_i^j = \left[0, B_i^j\right]$, and can also restrict attention to price caps such that each $\bar{p}_i^j$ belongs to S_i^j (as higher price caps would have no effect). From strict quasi-concavity, when facing the price cap $\bar{p}_i^j$ the constrained best-response can be expressed as:

$$\bar{r}_i^j\left(\mathbf{p}_i^{\mathcal{M}_i\setminus\{j\}}, \mathbf{p}_{-i}; \bar{p}_i^j\right) = \arg\max_{p_i^j\leq\bar{p}_i^j} \pi_i\left(p_i^j, \mathbf{p}_i^{\mathcal{M}_i\setminus\{j\}}, \mathbf{p}_{-i}\right) = \min\left\{r_i^j\left(\mathbf{p}_i^{\mathcal{M}_i\setminus\{j\}}, \mathbf{p}_{-i}\right), \bar{p}_i^j\right\},$$

and is thus increasing in all arguments. Let $\mathbf{S} = \left(\left(S_i^j\right)_{j\in\mathcal{M}_i}\right)_{i\in\mathcal{N}}$ denote the (bounded) relevant set of prices and consider the best-response function $\bar{\mathbf{r}}\left(\mathbf{p}; \bar{\mathbf{p}}\right) : \mathbf{S}\times\mathbf{S}\to\mathbf{S}$, where each price p_i^j is given by the constrained best-response $\bar{r}_i^j\left(\mathbf{p}_i^{\mathcal{M}_i\setminus\{j\}}, \mathbf{p}_{-i}; \bar{p}_i^j\right)$. Knaster-Tarski's Lemma ensures that, for each $\bar{\mathbf{p}}$, there exists a pure-strategy Nash equilibrium of this $m-$player game. To show that this equilibrium is unique, suppose instead that there are two equilibria, $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}'$; one of them, say $\hat{\mathbf{p}}'$, must have a lower price for at least one product. Applying the same argument as in the proof of Proposition 2 then leads to a contradiction.[6]

That the unique equilibrium prices increase with every price cap follows from Theorem 2.5.2 of Topkis (1998). The last conclusion follows from the fact that the unconstrained Nash equilibrium prices is sustainable through high enough price caps (e.g., such that $\bar{p}_i^j \geq B_i^j$ for every $i\in\mathcal{N}$ and every $j\in\mathcal{M}_i$). ∎

Remark: bundling. The above analysis carries over when firms engage in pure or mixed bundling. For instance, consider a case of *pure bundling* in which firm i offers goods $j\in\mathcal{B}_i\subset\mathcal{M}_i$ as a bundle, and only as a bundle. Let p_i^B denote the price charged for the bundle, and $D_i^B\left(p_i^B, \mathbf{p}_i^{\mathcal{M}_i\setminus\mathcal{B}_i}, \mathbf{p}_{-i}\right)$ the demand for the bundle, $D_i^j\left(p_i^B, \mathbf{p}_i^{\mathcal{M}_i\setminus\mathcal{B}_i}, \mathbf{p}_{-i}\right)$ denote the demand for every other good offered by firm i, for $j\in\mathcal{M}_i\setminus\mathcal{B}_i$, and $D_h^k\left(p_i^B, \mathbf{p}_i^{\mathcal{M}_i\setminus\mathcal{B}_i}, \mathbf{p}_{-i}\right)$ the demand for the other firms' products, for $h\in\mathcal{N}\setminus\{i\}$ and $k\in\mathcal{M}_h$; firm i's cost function remains the same as before, with the

[6] See Section C.1.2 of the Appendix, using firm $\hat{\imath}$ for which the difference between $\hat{p}_i$ and $\hat{p}_i$ is the largest, and noting that strict quasi-concavity and $\hat{p}_{\hat{\imath}}' < \hat{p}_{\hat{\imath}}\,(\leq\bar{p}_{\hat{\imath}})$ together imply $\hat{p}_{\hat{\imath}}' = R_{\hat{\imath}}\left(\hat{p}_{-\hat{\imath}}'\right)$ and $\hat{p}_{\hat{\imath}}\leq R_{\hat{\imath}}\left(\hat{p}_{-\hat{\imath}}\right)$.

caveat that each unit of the bundle requires the production of one unit of each good $j \in \mathcal{B}_i$. The previous analysis then remains valid as long as the reaction functions derived from these adjusted demand and cost functions exhibit strategic complementarity. In the case of mixed bundling, where firm i offers the bundle as well as each product $j \in \mathcal{B}_i$ on a stand-alone basis, a similar reasoning applies, interpreting the bundle as an additional good in firm i's product set.

C.3 Firms' incentives

It follows from the above that, again, firms have an incentive to agree on price caps when they offer complements, but not when they offer substitutes:

Corollary 1 **(multi-product firms' incentives to set price caps under** (SC)**)** *Let $\mathcal{P}$ denote the set of prices that are sustainable through price caps.*

(*i*) *If all goods are substitutes, then price caps cannot increase the profit of any firm; that is, $\hat{\mathbf{p}} \in \mathcal{P}$ implies $\pi_i(\hat{\mathbf{p}}) \leq \pi_i(\mathbf{p}^N)$ for every $i \in \mathcal{N}$.*

(*ii*) *If instead all goods are complements, then price caps can be used to increase all firms' profits; that is, there exists $\hat{\mathbf{p}} \in \mathcal{P}$ such that $\pi_i(\hat{\mathbf{p}}) > \pi_i(\mathbf{p}^N)$ for every $i \in \mathcal{N}$.*

Proof. If goods are all substitutes, then for any $\hat{\mathbf{p}} \in \mathcal{P}$ and any $i \in \mathcal{N}$ we have:

$$\pi_i(\hat{\mathbf{p}}) \leq \max_{\mathbf{p}_i} \pi_i(\mathbf{p}_i, \hat{\mathbf{p}}_{-i}) \leq \max_{\mathbf{p}_i} \pi_i(\mathbf{p}_i, \mathbf{p}_{-i}^N) = \pi_i(\mathbf{p}^N),$$

where the first inequality reflects the fact that firm i may be constrained by its price caps $\bar{\mathbf{p}}_i$, and the second inequality stems from the fact that price caps can only sustain prices that are lower than $\mathbf{p}^N$.

If goods are all complements, the reasoning used in the proof of Proposition 4 extends to the case of an oligopoly: starting from the Nash equilibrium prices $\mathbf{p}^N$, reducing all prices by a small amount ε increases all firms' profits, as firms' margins are positive from Lemma 2, and reducing one firm's price has only a second-order effect on the profit of that firm, and a first-order positive effect on the other firms' profits. To conclude the argument, it suffices to show that the new price vector, $\hat{\mathbf{p}}(\varepsilon) = \left(\left(\hat{p}_i^j(\varepsilon)\right)_{j\in\mathcal{M}_i}\right)_{i\in\mathcal{N}}$, where $\hat{p}_i^j(\varepsilon) = p_i^{jN} - \varepsilon$, belongs to $\mathcal{P}$; indeed, we have, for $i \in \mathcal{N}$ and $j \in \mathcal{M}_i$:

$$R_i^j(\hat{\mathbf{p}}_{-i}(\varepsilon)) - p_i^j(\varepsilon) = \int_0^\varepsilon [1 - \sum_{h\in\mathcal{N}\setminus\{i\}} \sum_{k\in\mathcal{M}_h} \partial_{p_h^k} R_i^j(\hat{\mathbf{p}}_{-i}(x))]dx > 0,$$

where the inequality stems from (1). From Propositions 1 and 2, $\hat{\mathbf{p}}(\varepsilon)$ is the unique equilibrium of $G_{\hat{\mathbf{p}}(\varepsilon)}$. ■

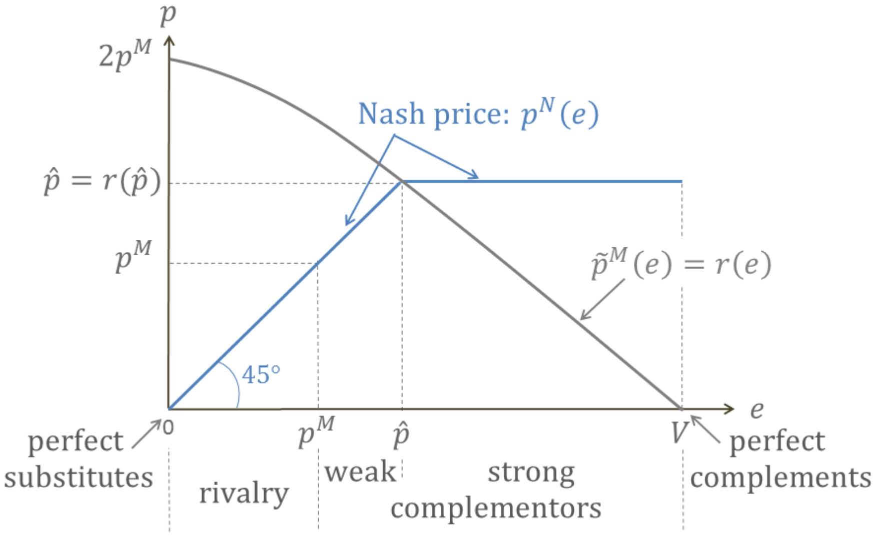

Figure 5: Nash equilibrium in the technology model

D Nash equilibrium in the technology adoption model

Consider, in the technology adoption environment described in Section 4.2, the static game in which the two firms simultaneously set their prices. Without loss of generality we can require prices to belong to the interval $[0, V]$.

From the discussion presented in the text, firm i's best response to firm j setting price $p_j \leq e$ is to set

$$p_i = \min\{e, r(p_j)\},$$

where

$$r(p) = \arg\max_p \{p_i D(p_i + p)\}$$

satisfies

$$-1 < r'(p_j) < 0$$

and has a unique fixed point $\hat{p} > p^M$.

When instead $p_j > e$, then firm i faces no demand if $p_i > p_j$ (as users buy only the lower-priced license), and faces demand $D(p_i + e)$ if $p_i < p_j$. Competition then drives prices down to $p_1 = p_2 = e$. Hence, the Nash equilibrium is unique and such that both firms charge $p^N \equiv \min\{e, \hat{p}\}$.

Figure 5 summarizes this analysis.

E Hotelling with club effects

E.1 Model

Consider the following symmetric duopoly setting, in which for notational simplicity costs are zero (i.e., $C_i(q_i) = 0$ for $i = 1, 2$) and:

- As in the standard Hotelling model, a unit mass of consumers is uniformly distributed along a unit-length segment; the two firms are located at the two ends of the segment, and consumers face a constant transportation cost per unit of distance, which is here normalized to 1.
- Unlike in the Hotelling model, however, consumers enjoy club effects: their gross surplus is $v + \sigma Q$, where $v > 0$, $\sigma \in (0, 1/2)$ reflects the magnitude of these positive externalities and $Q = q_1 + q_2$ denotes the total number of consumers.

For low enough prices, the entire market is covered (i.e., $Q = 1$), and as long as prices are not too asymmetric, each firm faces the classic Hotelling demand given by:

$$D_i^H(p_1, p_2) \equiv \frac{1 - p_i + p_j}{2}.$$

This case arises as long as $|p_1 - p_2| < 1$ (to ensure that the market is shared) and $v + \sigma \geq (1 + p_1 + p_2)/2$ (to ensure that the market is covered). The two goods are then substitutes: $\partial_2 D_i = 1/2 > 0$.

By contrast, for high enough prices, firms are local monopolies and each firm i faces a demand satisfying:

$$q_i = v + \sigma Q - p_i,$$

where now $Q = q_1 + q_2 < 1$. As long as both firms remain active, their demands are then given by:

$$D_i^m(p_1, p_2) \equiv \frac{v - (1 - \sigma) p_i - \sigma p_j}{1 - 2\sigma}.$$

This case arises as long as $Q < 1$ and $q_i \geq 0$, which amounts to $p_1 + p_2 > 2v - (1 - 2\sigma)$ and $(1 - \sigma) p_i + \sigma p_j \leq v$ for $i \neq j \in \{1, 2\}$. The goods are then complements: $\partial_j D_i = -\sigma/(1 - 2\sigma) < 0$.

E.2 Best-responses

We now study firm i's best-response to the price p charged by firm j. For the sake of exposition, we will focus on the range $p \in [0, v]$.[7]

[7] It can be checked that the Nash prices and the monopoly prices lie indeed in this range.

Consider first the case where firm j charges a price $p_j \in [0, v-(1-\sigma)]$, so that it would serve the entire market if firm i were to charge a prohibitive price.[8] In this case, the market remains fully covered whatever price firm i chooses to charge, and it is optimal for firm i to obtain a positive share of that market. Firm i will thus seek to maximize $p_i D_i^H(p_1, p_2)$ and choose to charge:

$$p_i = R^H(p_j) \equiv \arg\max_{p_i} \left\{ p_i D_i^H(p_i, p_j) \right\} = \frac{1+p_j}{2}.$$

Consider now the case where firm j charges a price $p_j \in (v-(1-\sigma), v]$; the market is then covered only if firm i charges a sufficiently low price, namely, if

$$p_i \leq \tilde{p}(p_j) \equiv 2v - (1-2\sigma) - p_j.$$

In this range, firm i seeks to maximize $p_i D_i^H(p_1, p_2)$, which is maximal for $p_i = R^H(p_j)$. If instead firm i chooses to charge a higher price, the firms are local monopolies; furthermore, as $p_j \leq v$, firm j's market share remains positive, whatever price firm i chooses to charge.[9] As it is optimal for firm i to maintain a positive market share as well, the demand will be given by $D_i^m(p_i, p_j)$. In this range, firm i will thus seek to maximize $p_i D_i^m(p_i, p_j)$, which is maximal for:

$$p_i = R^m(p_j) \equiv \arg\max_{p_i} \left\{ p_i D_i^m(p_i, p_j) \right\} = \frac{v - \sigma p_j}{2(1-\sigma)}.$$

Note that:

- The profit functions $p_i D_i^H(p_1, p_2)$ and $p_i D_i^m(p_1, p_2)$ are concave with respect to p_i in their respective relevant ranges.
- $\tilde{p}(v-(1-\sigma)) = v+\sigma > R^m(v-(1-\sigma)) = R^H(v-(1-\sigma)) = (v+\sigma)/2$.
- $\tilde{p}' = -1 < (R^m)' = -\sigma/2(1-\sigma) < 0 < (R^H)' = 1/2$.

It follows that $R^H(p_j) > R^m(p_j)$ in the range $p_j \in (v-(1-\sigma), v]$, and $\tilde{p}(p_j) > R^H(p_j)$ in the beginning of that range. This, in turn, implies that firm i's best response is given by:

$$p_i = R(p_j) \equiv \min\left\{ R^H(p_j), \max\left\{ \tilde{p}(p_j), R^m(p_j) \right\} \right\}.$$

More precisely:

- If $\tilde{p}(v) \geq R^H(v)$, which amounts to $v \geq 3-4\sigma$, then $(R^m(p_j) <) \, R^H(p_j) < \tilde{p}(p_j)$, and thus $R(p_j) = R^H(p_j)$;

[8] To see this, note that under full participation (i.e., $Q=1$), the consumer who the farthest away from firm j is willing to pay $v+\sigma-1 \geq p_j$ for firm j's product.

[9] To see this, note that even if $q_i = 0$, the consumer who's the nearest to firm j is willing to pay at least $v \geq p$ for firm j's product.

- If $R^H(v) > \tilde{p}(v) \geq R^m(v)$, which amounts to $3 - 4\sigma > v \geq 2 - 4\sigma$, then there exists $\check{p} = 4(v + \sigma)/3 - 1$ such that $R^H(\check{p}) = \tilde{p}(\check{p})$, and thus:
  - For $p_j \leq \check{p}$, we have again $(R^m(p_j) <) R^H(p_j) \leq \tilde{p}(p_j)$; hence, $R(p_j) = R^H(p_j)$;
  - For $p_j > \check{p}$, $R^m(p_j) < \tilde{p}(p_j) < R^H(p_j)$; hence, $R(p_j) = \tilde{p}(p_j)$;
- Finally, if $R^m(v) > \tilde{p}(v)$, which amounts to $2 - 4\sigma > v$, then there also exists $\mathring{p} = [(3 - 4\sigma)v - 2(1 - \sigma)(1 - 2\sigma)]/(2 - 3\sigma)$ such that $R^m(\mathring{p}) = \tilde{p}(\mathring{p})$, and thus:
  - For $p_j \leq \check{p}$, we still have $(R^m(p_j) <) R^H(p_j) \leq \tilde{p}(p_j)$; hence, $R(p_j) = R^H(p_j)$;
  - For $\check{p} < p_j < \mathring{p}$, we still have $R^m(p_j) < \tilde{p}(p_j) < R^H(p_j)$; hence, $R(p_j) = \tilde{p}(p_j)$;
  - For $p_j \geq \mathring{p}$, we now have $\tilde{p}(p_j) \leq R^m(p_j) < R^H(p_j)$; hence, $R(p_j) = R^m(p_j)$.

It follows that for low prices (namely, for $p_j < \min\{\check{p}, v\}$), goods are substitutes and prices are strategic complements: $\partial_j D_i = \partial_j D_i^H > 0$ and $R' = (R^H)' > 0$. By contrast, whenever $v < 2 - 4\sigma$ (implying $\mathring{p} < v$), then for high enough prices (namely, for $p_j > \mathring{p}$), goods are complements and prices are strategic substitutes: $\partial_j D_i = \partial_j D_i^m < 0$ and $R' = (R^m)' < 0$.

E.3 Monopoly prices

Conditional on covering the entire market, it is optimal to raise prices until the marginal consumer is indifferent between buying or not: indeed, starting from a situation where $q_1 + q_2 = 1$, and the marginal consumer strictly prefers buying, increasing both prices by the same amount does not affect the firms quantities, q_1 and q_2, but increase both of their margins. Hence, without loss of generality, we can focus on situations such that, for some $q_1 \in [0, 1]$: $q_2 = 1 - q_1$ and $p_i = v + \sigma - q_i$ for $i = 1, 2$. Total profit, as a function of q_1, is then given by:

$$\Pi^H(q_1) = (v + \sigma - q_1)q_1 + [v + \sigma - (1 - q_1)](1 - q_1) = v + \sigma - q_1^2 - (1 - q_1)^2.$$

This profit is concave in q_1 and reaches its maximum for $q_1 = 1/2$, where it is equal to

$$\Pi_H^M \equiv v + \sigma - \frac{1}{2}.$$

Alternatively, the firms may choose to cover only part of the market, in which case they are both local monopolies. For any $Q = q_1 + q_2 \in [0, 1)$ and any $q_1 \in [0, Q]$, the associated quantity for firm 2 is then $q_2 = Q - q_1$ and the associated prices are $p_i = v + \sigma Q - q_i$; the resulting industry profit is therefore given by:

$$\Pi = (v + \sigma Q - q_1)q_1 + [v + \sigma Q - (Q - q_1)](Q - q_1) = (v + \sigma Q)Q - q_1^2 - (Q - q_1)^2.$$

For any $Q \in [0,1)$, it is thus optimal to choose $q_1 = q_2 = Q/2$, which yields an industry profit equal to:

$$\Pi^m(Q) = (v + \sigma Q)\, Q - \frac{Q^2}{2}.$$

This profit is concave in Q and coincides with Π_H^M for $Q = 1$; furthermore, ignoring the constraint $Q \leq 1$, it is maximal for $Q = v/(1-2\sigma)$. Therefore, the monopoly outcome is $q_i^M = q^M = Q^M/2$, where:

$$Q^M = \min\left\{\frac{v}{1-2\sigma}, 1\right\},$$

and the associated prices are $p_1^M = p_2^M = p^M$, where:

$$\begin{aligned} p^M &= v + \sigma Q^M - q^M = v - (1-2\sigma)\, q^M \\ &= \begin{cases} \frac{v}{2} & \text{if} \quad v < 1 - 2\sigma \\ v - \left(\frac{1}{2} - \sigma\right) & \text{if} \quad v \geq 1 - 2\sigma \end{cases} = \max\left\{\frac{v}{2}, v - \left(\frac{1}{2} - \sigma\right)\right\}. \end{aligned}$$

E.4 Nash equilibrium

We first note that, in equilibrium, both firms must obtain a positive market share. Starting from a situation where all consumers are inactive, each firm could profitably attract some consumers by charging a price slightly below v. Furthermore, in a candidate equilibrium in which only one firm attracts consumers, this firm must charge a non-negative price, otherwise it would have an incentive to raise its price in order to avoid making a loss; but then, the other firm could profitably deviate, as charging a price only slightly higher would attract some consumers.

It can also be checked that, when one firm charges $p \leq v$, then the profit of the other firm is globally quasi-concave in the relevant price range $[0, v + \sigma]$. To see this, note that the profit functions $p_i D_i^H(p_i, p_j)$ and $p_i D_i^m(p_i, p_j)$ are both strictly concave in their relevant ranges; the conclusion then follows from the fact that, at the boundary between these two ranges, $D_i^H(p_1, p_2) = D_i^m(p_1, p_2)$ and $\partial_i D_i^H(p_1, p_2) > \partial_i D_i^m(p_1, p_2)$.

Suppose first that, at the Nash equilibrium prices, the market is fully covered *and* the marginal consumer strictly prefers buying (from either firm) to not buying. As both firms must be active, their demands are given by $D_i^H(\cdot)$, and remain so around the Nash prices. Therefore, their best-responses are given by $R^H(\cdot)$. It follows that the Nash equilibrium price is then symmetric, with both firms charging the standard Hotelling price

$$p^H = 1.$$

Conversely, both firms charging p^H is indeed an equilibrium if and only if the consumer that is at equal distance from the two firms (thus facing a transportation cost equal to $1/2$) then strictly prefers to be active, that is, if and only if $p^H + 1/2 < v + \sigma Q|_{Q=1}$, which amounts to:

$$v > \frac{3}{2} - \sigma \left(> 1 = p^H\right).$$

When instead the market is not fully covered at the Nash equilibrium prices, firms' best-responses are given by $R^m(\cdot)$. It follows that the Nash equilibrium price is again symmetric, with both firms charging

$$p^m = \frac{v}{2-\sigma} \left(< v\right).$$

Conversely, both firms charging p^m is indeed an equilibrium if and only if the market is not fully covered at these prices, that is, if and only if $p^m + 1/2 > v + \sigma Q|_{Q=1}$, which amounts to:

$$v < \frac{2-\sigma}{1-\sigma}\left(\frac{1}{2} - \sigma\right).$$

Finally, the Nash equilibrium can also be such that the entire market is "barely" covered, in that the marginal consumer is just indifferent between buying or not. The prices are then such that $p_i^N = v + \sigma - q_i^N$ and satisfy (as $q_1^N + q_2^N = 1$):

$$p_1^N + p_2^N = 2(v+\sigma) - 1. \tag{5}$$

Furthermore, no firm $i = 1, 2$ should benefit from a small deviation; as the market would remain covered if firm i lowers its price, but not so if it increases its price, we must have:

$$\begin{aligned} \partial_i \pi_i (p_1, p_2)\big|_{p_j = p_j^N, p_i = p_i^{N-}} &= \left.\frac{\partial}{\partial p_i}\left\{p_i D_i^H (p_1, p_2)\right\}\right|_{p_j = p_j^N, p_i = p_i^N} = q_i^N - \frac{p_i^N}{2} \geq 0, \\ \partial_i \pi_i (p_1, p_2)\big|_{p_j = p_j^N, p_i = p_i^{N+}} &= \left.\frac{\partial}{\partial p_i}\left\{p_i D_i^m (p_1, p_2)\right\}\right|_{p_j = p_j^N, p_i = p_i^N} = q_i^N - \frac{1-\sigma}{1-2\sigma} p_i^N \leq 0. \end{aligned} \tag{6}$$

Summing-up these conditions for $i = 1, 2$, and using $q_1^N + q_2^N = 1$ and (5), yields:

$$\frac{2-\sigma}{1-\sigma}\left(\frac{1}{2} - \sigma\right) \leq v \leq \frac{3}{2} - \sigma.$$

Figure 1 from Section 2.6 illustrates three possible configurations.

• In the first situation, v is sufficiently high (namely, $v > 3 - 4\sigma \left(> 3/2 - \sigma\right)$) that firms always compete for consumers in the relevant price range $[0, v]$. The goods are thus substitutes $\left(\partial_j D_i = \partial_j D_i^H = 1/2 > 0\right)$, and their prices are strategic complements $\left(R_i' = \left(R^H\right)' = 1/2 > 0\right)$. Furthermore, the monopoly prices lie above the Nash level: $p^M = v + \sigma - 1/2 > p^N = p^H = 1$.

• In the second, intermediate situation, firms compete again for consumers when prices are low, as in the previous situation. However, for higher price levels, firms best-respond to each other so as to maintain full participation; as a result the goods are at the boundary between substitutes and complements[10] and their prices become strategic substitutes ($R_i' = \tilde{p}' = -1 < 0$). While there are multiple Nash equilibria, they all involve the same total price, and the symmetric Nash equilibrium coincides with the monopoly outcome. As firms are symmetric, it is natural to focus on the symmetric Nash equilibrium, which moreover maximizes industry profit: $p^M = p^N = v + \sigma - 1/2$.

• Finally, in the last situation v is sufficiently low (namely, $v < 2 - 4\sigma$) that firms become local monopolies for high enough prices. The goods then become complements ($\partial_j D_i = \partial_j D_i^m = -\sigma/(1-2\sigma) < 0$) and their prices are again strategic substitutes ($R_i' = (R^m)' = -\sigma/2(1-\sigma) < 0$); the monopoly prices then lie below the Nash level: $p^M = v/2 < p^N = v/(2-\sigma)$.

E.5 Price caps

We now study the impact of price caps on the equilibrium prices and profits. As already noted, in the relevant price range each firm's profit function is quasi-concave with respect to the price of that firm. It follows that firms' constrained best responses are of the form $\bar{R}_i(p_j; \bar{p}_i) = \min\{R(p_j), \bar{p}_i\}$. Building on this insight, we now consider the three configurations identified above.

• When v is high enough (namely, $v > 3/2 - \sigma$), the monopoly price lies above the Nash level and, for prices below the Nash level, the goods are substitutes and their prices are strategic complements. It follows that firms have no incentives to adopt price caps, as they can only result into (weakly) lower prices and profits for both firms.

• For intermediate levels of v, firms best-respond to each other so as to maintain full participation. Compared with symmetric Nash equilibrium, which coincides with the monopoly outcome, price caps can only result into lower and more asymmetric prices. Indeed, for any prices $(\hat{p}_1, \hat{p}_2)$ lying below firms' best responses:

- the average is lower than the Nash level: $\hat{p} \equiv (\hat{p}_1 + \hat{p}_2)/2 < p^N$;
- there is asymmetry: $\hat{p}_1 \neq \hat{p}_2$.

It follows that, compared with the symmetric Nash equilibrium without price caps, these price caps can only benefit consumers; to see this, it suffices to decompose the move from (p^N, p^N) to $(\hat{p}_1, \hat{p}_2)$ as:

- a first move from (p^N, p^N) to $(\hat{p}, \hat{p})$, which obviously benefits consumers, as $\hat{p} \leq p^N$;
- an additional move from $(\hat{p}, \hat{p})$ to $(\hat{p}_1, \hat{p}_2)$, which also benefits consumers – keeping the total price constant maintains participation, and among those outcomes

[10] Namely:
$\partial_j D_i(p_i^N, p_j^{N-}) = \partial_j D_i^H(p_i^N, p_J^N) = 1/2 > 0 > \partial_j D_i(p_i^N, p_j^{N+}) = \partial_j D_i^m(p_i^N, p_J^N) = -\sigma/(1-2\sigma)$.

consumers favor asymmetry.[11]

• Finally, when v is low enough (namely, $v < 2 - 4\sigma$), the monopoly price lies below the Nash level and, for prices below the Nash level, the goods are complements and their prices are strategic substitutes. Introducing price caps then lowers the higher of the two equilibrium prices and, while this may be partially compensated by a limited increase in the other price, consumers are always (weakly) better off than in the absence of price caps.[12] Furthermore, firms can use price caps to maintain the monopoly outcome, which, compared with the outcome in the absence of price caps, strictly increases both firms' profits and strictly enhances consumer surplus.

F Complements and substitutes

F.1 Welfare-reducing price caps: an example

We provide here an example with both complements and substitutes, in which the prices of complements exhibit strategic substitutability, in such a way that capping the prices of some of the goods may induce undesirable price increases for other goods – thus violating the spirit of Assumption A.

There are two firms, each producing (costlessly) two goods:

- firm 1 produces goods A_1 and B_1;
- firm 2 produces goods A_2 and B_2.

Let p_{A_1}, p_{A_2}, p_{B_1}, p_{B_2} denote the prices of the four goods.

Consumers are atomless and divided into three groups:

- A mass ε of consumers are only interested in goods A_1 and B_1, which are perfect complements and worth v to them: that is, consumers are willing to buy one unit of both goods as long as $p_{A_1} + p_{B_1} \leq v$.
- A mass ε of consumers are only interested in goods A_2 and B_2, which are perfect complements and worth v to them: that is, consumers are willing to buy one unit of both goods as long as $p_{A_2} + p_{B_2} \leq v$.

[11] Among the prices that satisfy $p_1 + p_2 = 2\hat{p}$, the symmetric outcome ($p_1 = p_2 = \hat{p}$) is the one that maximizes consumer surplus – to see this, note that consumer surplus can be expressed as $\int_0^x tydy + \int_x^1 t(1-y)\,dy = tx^2/2 + t(1-x)^2/2$, where x denotes the location of the marginal consumer that is indifferent between buying or not, and this expression is maximal for $x = 1/2$.

[12] To see this, note that price caps can only reduce the total price (which increases total participation and enhances consumer surplus among symmetric price configurations) and moreover result into asymmetric prices, which, keeping total price constant, generates higher consumer surplus than the symmetric configuration.

- A unit mass of consumers are only interested in goods A_1 and A_2, which are imperfect substitutes for them: the two goods are at the end of a Hotelling segment, along which consumers are uniformly distributed; that is, the demand from these consumers for good A_i is given by, for $i \neq j \in \{1, 2\}$:

$$D_{A_i}\left(p_{A_i}, p_{A_j}\right) = \frac{1}{2} - \frac{p_{A_i} - p_{A_j}}{2t},$$

where t denote the transportation parameter reflecting the degree of differentiation between the two goods, and satisfies $t < v$.

The two firms are therefore competing with goods A_1 and A_2 for consumers of the third group; we will refer to these goods as "competitive". In addition, each firm i offers good B_i as a perfect complement to its competitive good to a distinct group of consumers, over which it has monopoly power; we will refer to goods B_1 and B_2 as "non-competitive".

F.1.1 Nash equilibrium

In the absence of price caps, and as long as the prices of the competitive goods do not exceed v, each firm can charge a total price of v to the consumers interested in its non-competitive good (by charging them $p_{B_i} = v - p_{A_i}$); hence, firm i's profit is given by:

$$\pi_i = p_{A_i} D_{A_i}\left(p_{A_i}, p_{A_j}\right) + \varepsilon v.$$

It follows that the standard Hotelling result prevails: each firm i offers its competitive product at a price equal to t; hence,

$$\begin{aligned} p_{A_1}^N &= p_{A_2}^N = p_A^N = t, \\ p_{B_1}^N &= p_{B_2}^N = p_B^N = v - t. \end{aligned}$$

Each firm earns a profit equal to

$$\pi^N = \frac{t}{2} + \varepsilon v,$$

whereas consumers obtain an aggregate surplus equal to:

$$S^N = 2\int_0^{1/2} \left(V - p_A^N - tx\right) dx + 0 + 0 = V - \frac{5t}{4},$$

where V denotes consumers' value for the competitive good, and is supposed to be large enough to ensure that all the market is always served.

F.1.2 Price caps

Suppose now that the firms face a price cap set to zero on their non-competitive goods: $\bar{p}_{B_1} = \bar{p}_{B2} = 0$. As long as their still compete for consumers of the third group, firm i's profit is now given by:

$$\bar{\pi}_i = p_{A_i} D_{A_i}\left(p_{A_i}, p_{A_j}\right) + \varepsilon p_{A_i} = p_{A_i}\left(\frac{1}{2} - \frac{p_{A_i} - p_{A_j}}{2t}\right) + \varepsilon p_{A_i},$$

leading to:

$$\hat{p}_{A_1} = \hat{p}_{A_2} = \hat{p}_A = (1 + 2\varepsilon)\, t,$$

and:

$$\pi_1 = \pi_2 = \hat{\pi} = (1 + 2\varepsilon)^2 \frac{t}{2} = \pi^N + \varepsilon\left[2t(1+\varepsilon) - v\right].$$

This constitutes indeed an equilibrium as long as:

- Consumers are still buying the non-competitive goods, which requires:

$$v \geq (1 + 2\varepsilon)\, t.$$

- Firms do not prefer to focus on the demand for the non-competitive goods, which requires:

$$\varepsilon v \leq (1 + 2\varepsilon)^2 \frac{t}{2}.$$

As total welfare remains unchanged, in this equilibrium consumers obtain a surplus equal to:

$$\hat{S} = S^N - 2\varepsilon\left[2t(1+\varepsilon) - v\right]. \tag{7}$$

Therefore, if

$$(1 + 2\varepsilon)\, t \leq v < \min\left\{\frac{2 + 2\varepsilon}{1 + 2\varepsilon}, \frac{1 + 2\varepsilon}{2\varepsilon}\right\}(1 + 2\varepsilon)\, t,$$

price caps enable the firms to increase their profits at the expense of consumers. As ε goes to zero, these conditions boil down to to $t \leq v < 2t$ and thus characterize a non-empty set of parameters.

Remark: welfare. Total welfare is here unaffected because total demand is inelastic; making the aggregate demand of the last group of consumers (for whom the goods A_1 and A_2 are substitutes) slightly elastic[13] would yield a reduction in total welfare as well.

Remark: bundling. Allowing the firms to engage in (mixed) bundling would not affect the analysis. In the absence of price caps, each firm i can (and does) extract all the surplus from consumers interested in buying both of its goods by charging them an adequate price on good B_i; hence, offering goods A_i and B_i as a bundle, in addition to offering them on a stand-alone basis, could not increase firm i's profits. When instead a price cap prevents

[13]Following Bénabou and Tirole (2016), a simple way is to introduce outside options, $\tilde{A}_1$ and $\tilde{A}_2$, also located at the two ends of the segment and giving consumers a random value.

firm i from charging a positive price on good B_i, the firm derives all of its profit from selling good A_i (both to consumers interested in buying A_i only, and to those interested in the bundle $A_i - B_i$); offering A_i and B_i as a bundle could not increase this profit, as consumers' arbitrage would prevent firm i from charging more for the bundle than it does for good A_i alone.

F.2 Platforms and apps

We show here that the main insights carry over to a class of situations involving both complements and substitutes. Specifically, we consider a setting in which platforms seek to attract developers for a variety of applications.

F.2.1 Setting

There are n platforms, indexed by $i \in \mathcal{N} \equiv \{1, ..., n\}$. Each platform i charges a price P_i and hosts a continuum of applications, indexed by $x \in [0, 1]$; for each application x, there are $m_{i,x}$ developers, indexed by $j \in \mathcal{M}_{i,x} \equiv \{1, ..., m_{i,x}\}$. The per-user demand for application j is given by $d_{i,x}^j (\mathbf{p}_{i,x})$, where $\mathbf{p}_{i,x} = \left(p_{i,x}^j\right)_{j \in \mathcal{M}_{i,x}} \in \mathbb{R}_+^{m_{i,x}}$ denotes the vector of prices for the application, and $\partial_{p_{i,x}^k} d_{i,x}^j (\mathbf{p}_{i,x}) > 0$ for any $k \in \mathcal{M}_{i,x} \setminus \{j\}$ – that is, developers offer (imperfect) substitutes. Let $s_{i,x} (\mathbf{p}_{i,x})$ denote the consumer net surplus generated by application x on platform i, as a function of the prices $\mathbf{p}_{i,x}$, and

$$S_i (\mathbf{p}_i) = \int_0^1 s_{i,x} (\mathbf{p}_{i,x}) \, dx$$

denote the aggregate net surplus that consumers can derive from the applications running on platform i. Letting $\tilde{P}_i = P_i - S_i (\mathbf{p}_i)$ denote platform i's quality-adjusted price, the demand for that platform is then given by $D_i \left(\tilde{\mathbf{P}}\right)$, where $\tilde{\mathbf{P}} = \left(\tilde{P}_i\right)_{i \in \mathcal{N}}$ and $\partial_{\tilde{P}_h} D_i \left(\tilde{\mathbf{P}}\right) > 0$ for any $h \in \mathcal{N} \setminus \{i\}$.[14] All costs are normalized to zero.

Remark: application multi-homing. For the sake of exposition, we will suppose that applications single-home – that is, each particular app is present on a single platform. However, given our assumption of atomistic apps, the pricing analysis (with or without price caps) does not depend on whether they multi-home or single-home (as long as they can charge platform-specific prices). By contrast, as discussed in Section F.2.4 of this online Appendix, applications' multi-homing decisions may affect their incentives to set price caps.

Remark: complements and substitutes. This setting exhibits substitution among platforms, as well as among the developers of any given application; by contrast, it features complementarity between a platform and its applications, as well as among these applications (and thus, among their developers). Furthermore, the analysis that follows

[14] The demand for application x on platform i is therefore given by $D_i \left(\tilde{\mathbf{P}}\right) d_{i,x}^j (\mathbf{p}_{i,x})$.

applies unchanged if some of the "applications" are actually (atomistic) "components" of a given (non-atomistic) application. For example, for any given (finite or infinite) partition $\mathcal{X} = \{\mathcal{X}_l\}_{l \in \mathcal{L}}$ of $[0,1]$, we could interpret $\mathcal{L}$ as the set of applications running on a given platform (with possibly different sets across platforms); in this interpretation, for any $l \in \mathcal{L}$ and any $x \in \mathcal{X}_l$, the developers in $\mathcal{M}_{i,x}$ are working on component x of application l. Developers then offer substitutes if they work on the same component of an application, and complements if they work on different components or different applications.

We maintain the following assumptions:

- *Applications.* For every $x \in [0,1]$, every $i \in \mathcal{N}$ and every $j \in \mathcal{M}_{i,x}$:
  - The profit function
  $$\pi_{i,x}^{j}\left(\mathbf{p}_{i,x}\right) \equiv p_{i,x}^{j} d_{i,x}\left(\mathbf{p}_{i,x}\right)$$
  is strictly quasi-concave in $p_{i,x}^{j}$;
  - The reaction function
  $$r_{i,x}^{j}\left(\mathbf{p}_{i,x}^{-j}\right) \equiv \arg\max_{p_{i,x}^{j}} \pi_{i,x}^{j}\left(p_{i,x}^{j}, \mathbf{p}_{i,x}^{-j}\right)$$
  is uniquely defined for any prices $\mathbf{p}_{i,x}^{-j}$ of the rival application developers; it is moreover differentiable and bounded above, and satisfies:

  (**sc**) *Strategic complementarity across developers*:
  $$\partial_{p_{i,x}^{h}} r_{i,x}^{j}(\cdot) > 0 \text{ for any } h \in \mathcal{M}_{i,x} \setminus \{j\}.$$
  - Equilibrium: strategic complementarity yields the existence of a fixed point of the function $\mathbf{p}_{i,x} \longrightarrow \mathbf{r}_{i,x}\left(\mathbf{p}_{i,x}\right) \equiv \left(r_{i,x}^{j}\left(\mathbf{p}_{i,x}^{-j}\right)\right)_{j \in \mathcal{M}_{i,x}}$, for every platform i and every application x. For the sake of exposition we assume that this fixed point is unique, and denote it by $\mathbf{p}_{i,x}^{N}$.

- *Platforms.* For every $i \in \mathcal{N}$ and any net surplus $S_i \in \mathbb{R}_+$:
  - The profit function[15]
  $$\Pi_i\left(\tilde{\mathbf{P}}; S_i\right) \equiv \left(\tilde{P}_i + S_i\right) D_i\left(\tilde{\mathbf{P}}\right)$$
  is strictly quasi-concave in $\tilde{P}_i$;

[15] As will become clear, platform i's pricing decision amounts to choosing the quality-adjusted price $\tilde{P}_i = P_i - S_i$; its profit can thus be expressed as $P_i D_i\left(\tilde{\mathbf{P}}\right) = \left(P_i + S_i\right) D_i\left(\tilde{\mathbf{P}}\right)$.

– The reaction function

$$R_i\left(\tilde{\mathbf{P}}_{-i}; S_i\right) = \arg\max_{\tilde{P}_i} \Pi_i\left(\tilde{\mathbf{P}}; S_i\right)$$

is uniquely defined, differentiable and bounded above, and satisfies:

(**SC**) *Strategic complementarity across platforms:*

$$\partial_{\tilde{P}_h} R_i\left(\tilde{\mathbf{P}}_{-i}; S_i\right) > 0 \text{ for any } h \in \mathcal{N} \setminus \{i\}.$$

– Equilibrium: strategic complementarity yields the existence of a fixed point of the function $\tilde{\mathbf{P}} \longrightarrow \mathbf{R}\left(\tilde{\mathbf{P}}, \mathbf{S}\right) \equiv \left(R_i\left(\tilde{\mathbf{P}}_{-i}, S_i\right)\right)_{i\in\mathcal{N}}$, for any $\mathbf{S} = (S_i)_{i\in\mathcal{N}}$; for the sake of exposition, we assume that for $\mathbf{S}^N \equiv \left(S_i\left(\mathbf{p}_i^N\right)\right)_{i\in\mathcal{N}}$ this fixed point is unique, and denote it by $\tilde{\mathbf{P}}^N$.

The timing is as follows:

- Stage 1: platforms and application developers all set their prices simultaneously; all prices are public.
- Stage 2: consumers learn their private benefits for the various platforms and choose which platform to join, if any; they also choose whether to buy the applications developed for the chosen platform.

Remark: Decomposing stage 1 into two distinct stages, where platforms set their prices before application developers do, would not affect the analysis. Likewise, decomposing stage 2 into two distinct stages, where consumers first choose among platforms, before buying the apps, would not affect the analysis either.

F.2.2 Nash equilibrium

As applications are atomistic, a single developer's price has no impact on platform adoption; therefore, for every platform i and every application x, in stage 2 each developer $j \in \mathcal{M}_{i,x}$ seeks to maximize $\pi_{i,x}^j\left(p_{i,x}^j, \mathbf{p}_{i,x}^{-j}\right) D_i$, taking $D_i = D_i\left(\tilde{\mathbf{P}}\right)$ as fixed, and thus chooses $p_{i,x}^j = r_{i,x}^j\left(\mathbf{p}_{i,x}^{-j}\right)$. The above assumptions then imply that the equilibrium prices are uniquely given by $\mathbf{p}_{i,x}^N$. It follows that joining platform $i \in \mathcal{N}$ gives a consumer a net surplus given by $S_i^N \equiv S_i\left(\mathbf{p}_i^N\right)$, where $\mathbf{p}_i^N = \left(\mathbf{p}_{i,x}^N\right)_{x\in[0,1]}$ denotes the vector of equilibrium prices for the applications running on the platform.

Given its rivals' prices, the profit of platform i can be expressed as:

$$P_i\tilde{D}_i(\mathbf{P}) = \left(\tilde{P}_i + S_i^N\right) D_i\left(\tilde{\mathbf{P}}\right) = \Pi_i\left(\tilde{\mathbf{P}}; S_i^N\right).$$

As the platform's price has no incidence on the surplus generated by the applications, we can take the quality-adjusted price as the relevant decision variable. It follows from the above that platform i will choose P_i so as to induce a quality-adjusted price equal to[16] $\tilde{P}_i = R_i\left(\tilde{\mathbf{P}}_{-i}; S_i^N\right)$. The above assumptions then imply that the equilibrium prices are uniquely given by $\mathbf{P}^N = \tilde{\mathbf{P}}^N + \mathbf{S}^N$.

F.2.3 Price caps

Suppose now that each firm faces a price cap. Let $\bar{\mathbf{P}} = \left(\bar{P}\right)_{i\in\mathcal{N}}$ denote the vector of price caps for the platforms; likewise, for every platform i and every application x on that platform, let $\bar{\mathbf{p}}_{i,x} = \left(\bar{p}_{i,x}^j\right)_{j\in\mathcal{M}_{i,x}}$ denote the vector of price caps for that application, and $\bar{\mathbf{p}}_i = \left(\bar{\mathbf{p}}_{i,x}\right)_{x\in[0,1]}$ denote the vector of price caps for all applications running on that platform. The next proposition shows that these price caps can only benefit consumers.

Proposition 16 (platforms and apps) *For any price caps $\bar{P} = \left(\bar{\mathbf{P}}_i\right)_{i\in\mathcal{N}}$ and $\bar{p} = \left(\bar{\mathbf{p}}_i\right)_{i\in\mathcal{N}}$, there exists a unique price-constrained equilibrium, and equilibrium prices weakly increase with the vector of price caps on platform prices and on apps. Therefore: (i) for any vector of price caps $\left(\bar{\mathbf{P}}, \bar{\mathbf{p}}\right)$, consumers are weakly better off under $G_{\left(\bar{\mathbf{P}},\bar{\mathbf{p}}\right)}$ than under G_∞; and (ii) in G, it is optimal for the competition authority to allow price caps.*

Proof. It is straightforward to check that the equilibrium prices of any given application x on any given platform i depend only on $\bar{\mathbf{p}}_{i,x}$, and not on the other price caps. From Proposition 8, there exists a unique Nash equilibrium, and the equilibrium prices weakly increase with $\bar{\mathbf{p}}_{i,x}$. It follows that the equilibrium net surpluses that consumers derive from the applications decrease with $\bar{\mathbf{p}}_{i,x}$ – in particular, they are all (weakly) larger than the Nash levels.

Let $\hat{\mathbf{p}} = \left(\hat{\mathbf{p}}_{i,x}\right)_{i\in\mathcal{N},x\in[0,1]}$ denote a Nash equilibrium sustainable through the applications' price caps $\bar{\mathbf{p}}$, and $\hat{\mathbf{S}} = \left(\hat{S}_i\right)_{i\in\mathcal{N}}$, where

$$\hat{S}_i = \int_0^1 s_{i,x}\left(\hat{\mathbf{p}}_{i,x}\right) dx,$$

denote the associated equilibrium net surpluses. The quality-adjusted prices $\tilde{\mathbf{P}} = \left(\tilde{P}_i\right)_{i\in\mathcal{N}}$ are now equal to:

$$\tilde{P}_i = P_i - \hat{S}_i.$$

Using again these quality-adjusted prices as strategic decision variables, platform i therefore seeks to solve:

$$\max_{P_i \le \bar{P}_i} P_i D_i\left(\tilde{\mathbf{P}}\right) = \max_{\tilde{P}_i \le \bar{P}_i - \hat{S}_i} \Pi_i\left(\tilde{P}_i, \tilde{\mathbf{P}}_{-i}; \hat{S}_i\right).$$

[16] That is, it will choose $P_i = R_i\left(\tilde{\mathbf{P}}_{-i}; S_i^N\right) + S_i^N$.

From quasi-concavity, platform i will thus choose:

$$\tilde{P}_i = \bar{R}_i\left(\tilde{\mathbf{P}}; \hat{S}_i, \bar{P}_i\right) \equiv \min\left\{\bar{P}_i - \hat{S}_i, R_i\left(\tilde{\mathbf{P}}; \hat{S}_i\right)\right\}. \tag{8}$$

This reaction function still exhibits strategic complementarity across platforms: rivals' prices only affect the second term, which satisfies $\partial_{\tilde{P}_k} R_i\left(\tilde{\mathbf{P}}; \hat{S}_i\right) > 0$ for $h \in \mathcal{N}\setminus\{i\}$. Furthermore, the first term in the right-hand side of (8) increases with $\bar{P}_i$, and both terms decrease with $\hat{S}_i$: this is obvious for $\bar{P}_i - \hat{S}_i$, and for $R_i\left(\tilde{\mathbf{P}}; \hat{S}_i\right)$ this follows from

$$\partial^2_{\tilde{P}_i S_i}\Pi_i\left(\tilde{\mathbf{P}}; S_i\right) = \partial_{\tilde{P}_i} D_i\left(\tilde{\mathbf{P}}\right) < 0.$$

The best-response function $\bar{R}_i(\cdot)$ of each platform i is therefore increasing in $\bar{P}_i$ and decreasing in $\hat{S}_i$. Strategic complementarity then yields the result:

- For any equilibrium net surpluses $\hat{\mathbf{S}} = \left(\hat{S}_i\right)_{i\in\mathcal{N}}$ sustainable through the application price caps $\bar{\mathbf{p}}$, and any platform price caps $\bar{\mathbf{P}}$, there exist a unique Nash equilibrium for the platforms' quality-adjusted prices, and the equilibrium quality-adjusted prices weakly increase with $\bar{\mathbf{P}}$.
- As the set of equilibrium net surpluses $\hat{\mathbf{S}} = \left(\hat{S}_i\right)_{i\in\mathcal{N}}$ weakly decrease with $\bar{\mathbf{p}}$, the overall equilibrium quality-adjusted prices weakly increase with both $\bar{\mathbf{P}}$ and $\bar{\mathbf{p}}$.

■

F.2.4 Firms' incentives

Other things equal, introducing caps on platforms' prices increases applications' profits, by increasing the number of platforms' users, but reduces platforms' profits, both by constraining their pricing decisions and by making their rivals more aggressive: for any $\tilde{\mathbf{P}} \leq \tilde{\mathbf{P}}^N$,

$$\max_{\tilde{P}_i \leq \bar{P}_i - S_i^N} \Pi_i\left(\tilde{P}_i, \tilde{\mathbf{P}}_{-i}; S_i^N\right) \leq \max_{\tilde{P}_i} \Pi_i\left(\tilde{P}_i, \tilde{\mathbf{P}}_{-i}; S_i^N\right) \leq \max_{\tilde{P}_i} \Pi_i\left(\tilde{P}_i, \tilde{\mathbf{P}}^N_{-i}; S_i^N\right) = \Pi_i^N,$$

where the second inequality is strict whenever $\tilde{\mathbf{P}} \leq \tilde{\mathbf{P}}^N$.

By contrast, introducing caps on applications' prices can increase not only platforms' profits, by boosting their demands thanks to the greater net surplus that consumers derive from the apps, but it can also benefit the apps, by increasing the number of users. More precisely, consider the introduction of price caps $\bar{\mathbf{p}}_i$ on the applications running on platform i:

- This increases the net surplus S_i generated by these apps, which increases platform i's profit by expanding its demand.

- As noted in the proof of Proposition **16**, $\partial_{S_i} R_i \left(\tilde{\mathbf{P}}_{-i}; S_i\right) \leq 0$; therefore, the increase in the net surplus S_i expands the equilibrium number of users on platform i.

The extent to which this increase in user participation can offset the direct negative impact of the price caps $\bar{\mathbf{p}}_i$ on applications' per-user profitability depends on several factors:

- Consider first the polar case where:[17]

  - there is a single developer for each application, so that

$$p_{i,x}^N = p_{i,x}^M \equiv \arg\max_{p_{i,x}} p_{i,x} d_{i,x} \left(p_{i,x}\right);$$

  - applications single-home – that is, the applications running on platform i run only on that platform.

  In this case, the increase in platform i's user participation can indeed benefit the applications running on that platform. In particular, introducing price caps $\bar{\mathbf{p}}_i$ that are slightly below $\mathbf{p}_i^M$ is likely to have only a second-order effect on applications' per user profit (as these profits are maximal under monopoly), but a first-order effect on platform i's quality-adjusted price, and thus on the number of its users; hence, the applications are likely to benefit from the introduction of such caps.

- The potential benefit of users' greater participation is however diluted for multi-homing applications. For example, if the applications are present on all platforms, then this demand expansion effect arises only if users' aggregate participation is elastic. Otherwise, the number of users would remain unchanged, and the negative effect of price caps on per user profits would prevail.

- This potential benefit of users' greater participation is also lower in case of competition among developers; in particular, introducing a price cap slightly lower than the Nash level would then have a first-order effect on applications' per-user profitability as well as on user participation.

The elasticity of users' aggregate demand for platforms is also a key factor for the profitability of price caps at the industry level. For example, if total platform participation is inelastic, then a platform may still benefit unilaterally from capping the prices of its applications, but this would be at the expense of the other platforms: the industry as a whole would not benefit from introducing price caps.

Consider for instance the following example, where a unit mass of consumers consider joining one of the platforms – and only one: consumers do not derive any benefit from

[17] See for instance the monopoly example studied in the next section.

joining additional platforms, and thus single-home;[18] each user obtains a benefit θ_i from joining platform i, and these private benefits are randomly drawn from a common distribution, with cumulative distribution $F(\cdot)$ and density $f(\cdot)$ over some Θ, with independent drawn across both platforms and consumers. The demand for platform i is then given by:

$$\begin{aligned} D_i\left(\tilde{\mathbf{P}}\right) &= \Pr\left[\theta_i - \tilde{P}_i > \theta_j - \tilde{P}_j \text{ for every } j \in \mathcal{N}\setminus\{i\}\right] \\ &= \int_{\Theta} \prod_{j \in \mathcal{N}\setminus\{i\}} F\left[\theta_i - \tilde{P}_i + \tilde{P}_j\right] f(\theta_i)\, d\theta_i, \end{aligned}$$

and satisfies, for any symmetric prices $\tilde{\mathbf{P}}^s = \left(\tilde{P}^s, ..., \tilde{P}^s\right)$:

$$\begin{aligned} D_i\left(\tilde{\mathbf{P}}^s\right) &= \int_{\Theta} F^{n-1}(\theta) f(\theta)\, d\theta = \frac{1}{n}, \\ \partial_{\tilde{P}_i} D_i\left(\tilde{\mathbf{P}}^s\right) &= -\int_{\Theta} \prod_{j \in \mathcal{N}\setminus\{i\}} F^{n-2}(\theta) f^2(\theta)\, d\theta = -\lambda, \end{aligned}$$

where the constant λ is positive and does not depend on the actual level of the quality-adjusted price $\tilde{P}^s$.

Suppose now symmetric caps on the applications result in the same equilibrium net surplus S for each platform. Any resulting symmetric price equilibrium $\tilde{\mathbf{P}}^s = \left(\tilde{P}^s, ..., \tilde{P}^s\right)$ satisfies the first-order condition:

$$0 = \partial_{\tilde{P}_i} \Pi_i\left(\tilde{\mathbf{P}}^s; S\right) = D_i\left(\tilde{\mathbf{P}}^s\right) + \tilde{P}^s \partial_{\tilde{P}_i} D_i\left(\tilde{\mathbf{P}}^s\right) = \frac{1}{n} - \lambda \tilde{P}^s,$$

and thus:

$$\tilde{P}^s = \frac{1}{\lambda n}.$$

It follows that any increase in the net surplus generated by the applications is entirely passed on to consumers; introducing such price caps would thus benefit consumers at the expense of the applications' profits, without any impact on the profitability of the platforms.

Such a situation generates a prisoners' dilemma: each platform would have an incentive to introduce price caps on its own applications (and would be willing to compensate the applications, in case this negatively affect their profits), but the benefit to that platform would come at the expense of the other platforms and their own applications, so that the industry profit would be reduced as a result.

[18] For example, joining the platform may involve substantial fixed costs (learning how to use it, set-up costs, and so forth), accounted for in the definition of the private benefit θ from single-homing, but making multi-homing undesirable. In some cases, multi-homing may be infeasible (e.g., for broadband Internet access, consumers may choose among alternative suppliers, but only one at a time can operate the local connection to the home).

F.3 Platform & apps: the monopoly case

We consider here a particular case of the setting considered in the previous section, which is used in Section 2.3 of the main text to illustrate the possibility that, even with complements, a monopoly price may lie above the Nash level for one of the products (cf. Figure 6). To see this, we suppose here that there is a single platform as well as a single developer per application. We further assume for simplicity that all applications face the same demand $d(p)$, which is downward-sloping (i.e., $d(p) < 0$). It follows that, in the absence of any price caps, all applications will charge the same price. To be consistent with the notation used in Section 2.3, we will denote the price of the platform by p_1 and that of the applications by p_2.

F.3.1 Complementarity between the platform and the applications

We first check that the platform and the applications are indeed *complements*. Letting

$$s(p_2) \equiv \int_{p_2}^{+\infty} d(p)\, dp$$

denote the additional surplus that platform users derive from applications, the demand for the platform is then given by:[19]

$$D_1(p_1, p_2) \equiv D(p_1 - s(p_2)),$$

where $D(\tilde{p})$ denotes the demand for the platform, as a function of the quality-adjusted price $\tilde{p} = p_1 - s(p_2)$. The demand for each application is thus given by:

$$D_2(p_2, p_1) \equiv D(p_1 - s(p_2))\, d(p_2),$$

and it satisfies:

$$\partial_2 D_1(p_1, p_2) = \partial_1 D_2(p_2, p_1) = D'(p_1 - s(p_2))\, d(p_2) < 0.$$

F.3.2 Best-responses

We now turn to firms' reaction functions, and first note that the applications' best-response is flat (i.e., $R_2(p_1) = 0$); indeed, each application wishes to maximize its per-user profit, which amounts to choosing a price equal to:

$$p_2^N \equiv \arg\max_{p_2} p_2 d(p_2).$$

[19] As all applications are charging the same price p_2, $s(p_2)$ represents both the per-application net surplus, and the total net surplus that consumers derive from the applications.

Note that $p_2^N > 0$: the applications could not obtain any profit by charging a non-positive price, whereas they can secure a positive profit by charging any positive price.[20]

Consider now the platform's best-response to the application price p_2. The profit of the platform can be expressed as:

$$\pi_1(p_1, p_2) \equiv p_1 D_1(p_1, p_2) = p_1 D(p_1 - s(p_2)).$$

Maximizing this profit amounts to

$$\log(\pi_1(p_1, p_2)) = \log(p_1) + L(p_1 - s(p_2)),$$

where $L(\tilde{p}) \equiv \log(D(\tilde{p}))$ denotes the logarithm of the demand for the platform, and:

$$\partial^2_{p_1 p_2} \log(\pi_1(p_1, p_2)) = L''(p_1 - s(p_2))\, d(p_2).$$

Hence, from the platform's standpoint, prices are strategic substitutes (i.e., $R_1'(\cdot) < 0$) whenever the demand for the platform is log-concave (i.e., $L''(\cdot) < 0$).

It follows from the above that, in equilibrium, the applications charge $p_2^N > 0$ whereas the platform charges $p_1^N \equiv R_1\left(p_2^N\right)$, as illustrated by Figure 6.

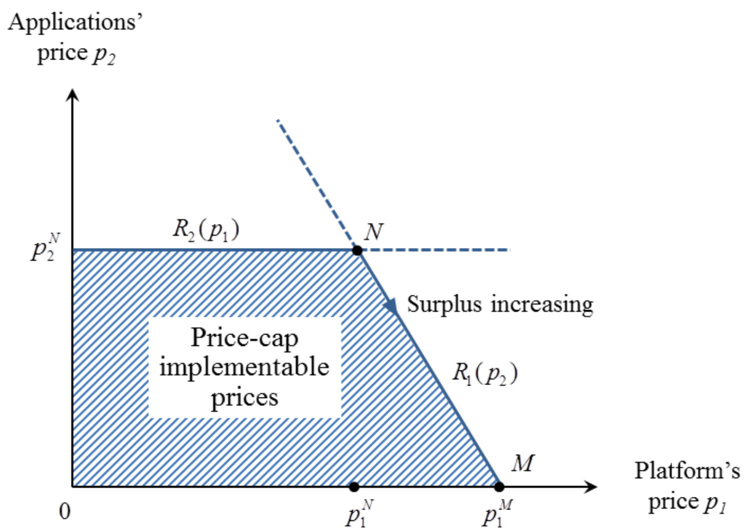

Figure 6: Platform & apps

Finally, we check that Assumption (A) is satisfied. It is obvious for the applications, as $R_2'(p_1) = 0$, and for the platform it follows from the fact that, as noted in Section F.2, the platform's optimal quality-adjusted price is increasing in the net surplus generated by the applications; hence, in response to a decrease in the application price p_2, and thus to an increase in the surplus s, the platform increases its own price – as $R_1'(p_2) < 0$ – but not so as to offset entirely the consumers' benefit from the reduction in the price p_2.

[20] Note that $d(\cdot) \geq 0$ and $d'(\cdot) < 0$ together imply $d(\cdot) > 0$.

F.3.3 Monopoly outcome

We now characterize the monopoly outcome. We start with the observation that, in order to maximize the industry profit, it is optimal to sell the applications at cost. To see this, let us again index the applications by $x \in [0, 1]$; for a given platform price p_1 and given application prices $(p_{2x})_{x\in[0,1]}$, the industry profit can then be expressed as:

$$\Pi\left(p_1, (p_{2x})_x\right) = \left[p_1 + \int_0^1 p_{2x} d\left(p_{2x}\right) dx\right] D\left(p_1 - s\right),$$

where

$$s = \int_0^1 s\left(p_{2x}\right) dx$$

denotes consumers' expected surplus from the applications. Replacing these prices with $\tilde{p}_2 = 0$ and $\tilde{p}_1 = p_1 + s(0) - s$ does not affect the number of platform users (the quality-adjusted price remains equal to $p_1 - s$), and thus the impact on industry profit is equal to:

$$\begin{aligned}
\Delta\Pi &= \left\{s(0) - s - \int_0^1 p_{2x} d\left(p_{2x}\right) dx\right\} D\left(p_1 - s\right) \\
&= \left\{\int_0^1 \left[s(0) - s\left(p_{2x}\right) - p_{2x} d\left(p_{2x}\right)\right] dx\right\} D\left(p_1 - s\right) \\
&> 0,
\end{aligned}$$

where the inequality stems from the fact that the total surplus from the applications, $s(p) + pd(p)$, is maximal under marginal cost pricing, i.e., for $p = 0$.

The monopoly prices are thus $p_2^M = 0$ and

$$p_1^M \equiv \arg\max_{p_1} \left\{\Pi\left(p_1, p_2 = 0\right)\right\} = \arg\max_{p_1} \left\{p_1 D\left(p_1 - s(0)\right)\right\} = R_1(0).$$

The monopoly outcome therefore lies (weakly) below firms' best-responses (more precisely, $p_2^M < R_2\left(p_1^M\right)$ and $p_1^M = R_1\left(p_2^M\right)$). However, as $p_2^M = 0 < p_2^N$, the application price is (strictly) lower than its Nash level $\left(p_2^M < p_2^N\right)$, but the opposite holds for the platform: as $R_1'(\cdot) < 0$, we have:

$$p_1^M = R_1(0) > R_1\left(p_2^N\right) = p_1^N.$$

Figure 6 illustrates these insights.

F.4 Discussion

The two situations considered in Section F.1 and in Sections F.2-F.3 both exhibit a combination of complements and substitutes. In both instances, capping the prices of substitutes would benefit consumers, but is not appealing to the firms.[21] By contrast, capping the prices of (some of the) complements has a more ambiguous effect on firms and consumers: the profit may be reduced on the goods for which a cap is introduced, but it increases for their complements, thanks to an expansion in their demands. Consumers also benefit from the lower prices on the goods for which a cap is introduced, but may face higher prices for their complements.

In the example studied in Section F.1, consumers and firms have indeed perfectly conflicting interests, and introducing price caps on the non-competitive goods can benefit either the consumers or the firms. In essence, price caps benefit the firms (at the expense of consumers) when the spirit of Assumption A is not satisfied, namely, when[22]

$$\frac{\Delta p_A}{\Delta p_B} < -\frac{D_B}{D_A}, \tag{9}$$

where, for $L = A, B$, D_L denotes the aggregate demand for good L (i.e., the sum of the demands for L_1 and L_2), whereas Δp_L denotes the variation in the prices of these goods, following the introduction of a uniform price cap on B_1 and B_2.[23]

By contrast, the spirit of Assumption A is automatically satisfied in the platforms and apps settings: as the applications are atomistic, each individual application price has no influence on platform participation, and so each application seeks to maximize its per-user profit, which does not depend on platforms' prices; hence, platforms' prices have no impact on applications' prices, that is, applications' best-responses are "flat", and Assumption A is trivially satisfied. More generally, we would expect the spirit of Assumption A to hold as long as there are multiple applications, so that their pricing decisions are primarily driven by competition among developers, rather than by the impact of their prices on platform participation (and thus, indirectly, by platforms' prices).

G Non-verifiable quality

G.1 Setting

We consider an oligopoly with $n \geq 2$ single-product firms; each firm $i \in \mathcal{N} \equiv \{1, ..., n\}$ can choose to offer at any price $p_i \in \mathbb{R}_+$ any quality $s_i \in \mathbb{R}_+$ of good i, which it can produce at constant unit cost $c_i(s_i)$; the demand for that good is then given by $D_i(\hat{\mathbf{p}})$, where $\hat{\mathbf{p}} = (\hat{p}_i)_{i \in \mathcal{N}}$ is the vector of net quality-adjusted prices: $\hat{p}_i \equiv p_i - s_i$.

[21] In the example considered in Section F.1, capping the prices of the competitive goods, A_1 and A_2, would reduce the profits derived from the consumers of the third group (for which these goods are substitutes), without any off-setting increase in the profits derived from the other consumers (for which these goods are complements to the non-competitive goods).

[22] Because of the inelasticity of the demand, the condition involves discrete rather than marginal price changes.

[23] In the example, $D_A = 1 + 2\varepsilon$ and $D_B = 2\varepsilon$, whereas $\Delta p_A = 2\varepsilon t$ and $\Delta p_B = t - v$. Condition (9) thus amounts to $v < 2(1 + \varepsilon)t$, the condition under which price caps reduce consumer surplus (see (7)).

We assume that, for $i \in \mathcal{N}$, $D_i(\cdot)$ and $c_i(\cdot)$ are both C^2 and:

- $c_i(0) = 0$, $c_i'(\cdot) > 0$ and $c_i'(s_i) > 1$ for s_i large enough;
- $D_i(\cdot) > 0$, $\partial_i D_i(\cdot) < 0$ (individual demands for a good are positive and downward sloping, as a function of the net price of that good) and $\sum_{j \in \mathcal{N}} \partial_j D_i(\cdot) \leq 0$ (a uniform increase in all net prices reduces individual demands);
- the profit function

  $$\pi_i(\mathbf{p}, \mathbf{s}) \equiv [p_i - c_i(s_i)] D_i(p_1 - s_1, ..., p_n - s_n)$$

  is strictly quasi-concave in (p_i, s_i).

It will be convenient to use net prices rather than prices as strategic variables; we thus define:

$$\hat{\pi}_i(\hat{\mathbf{p}}, s_i) \equiv [\hat{p}_i + s_i - c_i(s_i)] D_i(\hat{\mathbf{p}}).$$

The quasi-concavity of the individual profit functions π_i implies that the profit functions $\hat{\pi}_i$, too, are quasi-concave:

Lemma 6 (Quasi-concavity of $\hat{\pi}$) *For $i \in \mathcal{N}$, the profit function $\hat{\pi}_i(\hat{\mathbf{p}}, \mathbf{s})$ is strictly quasi-concave in $(\hat{p}_i, s_i)$.*

Proof. For $i \in \mathcal{N}$, we have, for any $(\hat{\mathbf{p}}_{-i}, \mathbf{s}_{-i})$, any $(\hat{p}_i', s_i') \neq (\hat{p}_i, s_i)$ and any $\alpha \in (0,1)$ (with $p_i' \equiv \hat{p}_i' + s_i'$ and $p_i \equiv \hat{p}_i + s_i$):

$$\begin{aligned}
& \hat{\pi}_i(\alpha \hat{p}_i' + (1-\alpha)\hat{p}_i, \hat{\mathbf{p}}_{-i}, \alpha s_i' + (1-\alpha) s_i) \\
= \quad & \pi_i(\alpha(\hat{p}_i' + s_i') + (1-\alpha)(\hat{p}_i + s_i), \hat{\mathbf{p}}_{-i}, \alpha s_i' + (1-\alpha) s_i, \mathbf{s}_{-i}) \\
> \quad & \min\{\pi_i(\hat{p}_i' + s_i', \hat{\mathbf{p}}_{-i} + \mathbf{s}_{-i}, s_i', \mathbf{s}_{-i}), \pi_i(\hat{p}_i + s_i, \hat{\mathbf{p}}_{-i} + \mathbf{s}_{-i}, s_i, \mathbf{s}_{-i})\} \\
= \quad & \min\{\hat{\pi}_i(\hat{p}_i', \hat{\mathbf{p}}_{-i}, s_i'), \hat{\pi}_i(\hat{p}_i, \hat{\mathbf{p}}_{-i}, s_i)\},
\end{aligned}$$

where the equalities stem from the definitions of $\pi_i(\cdot)$ and $\hat{\pi}_i(\cdot)$, whereas the inequality follows from the strict quasi-concavity of the profit function $\pi_i(\cdot)$ with respect to (p_i, s_i). ∎

These assumptions imply that maximizing its profit $\hat{\pi}_i(\hat{\mathbf{p}}, s_i)$ leads each firm $i \in \mathcal{N}$ to choose

$$s_i = s_i^M \equiv c_i'^{-1}(1)$$

and a unique best-response

$$\hat{R}_i(\hat{\mathbf{p}}_{-i}) \equiv \arg\max_{\hat{p}_i} \hat{\pi}_i\left(\hat{p}_i, \hat{\mathbf{p}}_{-i}, s_i^M\right).$$

We assume that this best-response is well-defined, C^1, and bounded above.

As the profit of one firm depends on other firms' decisions only through their net prices, we can readily extend our definitions of substitutes and complements as follows:

$\left(\hat{\mathbf{S}}\right)$ *Substitutes:* $\partial_j D_i\left(\hat{\mathbf{p}}\right) > 0$ for $j \neq i \in \mathcal{N}$;

$\left(\hat{\mathbf{C}}\right)$ *Complements:* $\partial_j D_i\left(\hat{\mathbf{p}}\right) < 0$ for $j \neq i \in \mathcal{N}$.

Finally, we still assume that:

- there exists a unique Nash equilibrium in the unconstrained pricing game, which we denote by $\hat{\mathbf{p}}^N = \left(\hat{p}_i^N\right)_{i\in\mathcal{N}}$ (together with $\mathbf{s}^N = \left(s_i^M\right)_{i\in\mathcal{N}}$);
- the best-responses satisfy, for $j \in \mathcal{N}$:

$$\sum_{i\in\mathcal{N}\setminus\{j\}} \partial_j \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) < 1; \tag{10}$$

- the industry profit

$$\Pi\left(\mathbf{p},\mathbf{s}\right) \equiv \sum_{i\in\mathcal{N}} \pi_i\left(\mathbf{p},\mathbf{s}\right) = \sum_{i\in\mathcal{N}} \hat{\pi}_i\left(\hat{\mathbf{p}}, s_i\right) \equiv \hat{\Pi}\left(\hat{\mathbf{p}},\mathbf{s}\right)$$

  is strictly quasi-concave in $\mathbf{p}$ and achieves its maximum at $\hat{\mathbf{p}}^M = \left(p_i^M\right)_{i\in\mathcal{N}}$ (together with $(\mathbf{s}^N = \left(s_i^M\right)_{i\in\mathcal{N}})$.

As the monopoly outcome and the individual best-responses both involve $s_i = s^M$ for every $i \in \mathcal{N}$, the proofs of first two Lemmas readily carry over; that is, letting

$$\hat{c}_i^M \equiv c_i\left(s_i^M\right) - s_i^M$$

denote the net cost corresponding to the optimal quality s_i^M, we have:

Lemma 7 (unconstrained net prices) *For any firm $i \in \mathcal{N}$ and any $\hat{\mathbf{p}}_{-i} \in \mathbb{R}_+^{n-1}$, $\hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) > \hat{c}_i^M$.*

(i) $\left(\hat{\mathbf{S}}\right) \Longrightarrow \forall i \in \mathcal{N}, \hat{p}_i^M > \hat{c}_i^M$ *and* $\hat{p}_i^M > \hat{R}_i\left(\hat{\mathbf{p}}_{-i}^M\right)$.

(ii) $\left(\hat{\mathbf{C}}\right) \Longrightarrow \exists\left(i,j\right) \in \mathcal{N}^2$ *such that* $\hat{p}_i^M > \hat{c}_i^M$ *and* $\hat{p}_j^M < \hat{R}_j\left(\hat{\mathbf{p}}_{-j}^M\right)$*; furthermore, if* $n = 2$*, then* $\hat{p}_i^M > \hat{c}_i^M \Longrightarrow \hat{p}_j^M < \hat{R}_j\left(\hat{p}_i^M\right)$ *for* $j \neq i$.

Proof. We first show that (net) best-responses exceed (net) marginal costs. Starting from $s_i = s_i^M$ and $\hat{\mathbf{p}} = (\hat{p}_i, \hat{\mathbf{p}}_{-i})|_{\hat{p}_i = \hat{R}_i(\hat{\mathbf{p}}_{-i})}$, the impact of a slight increase in $\hat{p}_i$ on firm i's profit $\hat{\pi}_i$ is given by $\left(\hat{p}_i - \hat{c}_i^M\right) \partial_i D_i (\hat{\mathbf{p}}) + D_i (\hat{\mathbf{p}})$. If firm i's margin were non-positive, this impact would be positive (as $\partial_i D_i (\cdot) < 0 < D_i (\cdot)$), a contradiction. Hence, $\hat{R}_i (\hat{\mathbf{p}}_{-i}) > \hat{c}_i^M$.

Next, we show that monopoly prices exceed marginal costs for at least one firm. Suppose instead that $\hat{p}_i^M \leq \hat{c}_i^M$ for all $i \in \mathcal{N}$, and consider a small and uniform increase in net prices: $d\hat{p}_i = d\hat{p} > 0$ for $i \in \mathcal{N}$. We then have $dq_j = \sum_{i \in \mathcal{N}} \partial_i D_j \left(\hat{\mathbf{p}}^M\right) d\hat{p} \leq 0$ for all $j \in \mathcal{N}$, and thus:

$$d\hat{\Pi} = \sum_{j \in \mathcal{N}} \left(\hat{p}_j^M - \hat{c}_j^M\right) dq_j + \sum_{j \in \mathcal{N}} q_j^M d\hat{p} > 0,$$

a contradiction. Therefore, $\hat{p}_i^M > \hat{c}_i^M$ for some $i \in \mathcal{N}$.

We now show that, under $\left(\hat{\mathbf{S}}\right)$, $\hat{p}_i^M > \hat{c}_i^M$ for every $i \in \mathcal{N}$. To see this, suppose that there exists a non-empty subset of $\mathcal{N}$, $\mathcal{N}^-$, such that $\hat{p}_j^M \leq \hat{c}_j^M$ for every $j \in \mathcal{N}^-$, and consider a small and uniform increase in these net prices: $d\hat{p}_j = d\hat{p} > 0$ for $j \in \mathcal{N}^-$. Under $\left(\hat{\mathbf{S}}\right)$, we then have:

- for $i \in \mathcal{N} \setminus \mathcal{N}^-$, $dq_i = \sum_{j \in \mathcal{N}^-} \partial_j D_i \left(\hat{\mathbf{p}}^M\right) d\hat{p} > 0$, as $\partial_j D_i \left(\hat{\mathbf{p}}^M\right) > 0$ for $j \neq i$.
- for $i \in \mathcal{N}^-$, $dq_i = \sum_{j \in \mathcal{N}^-} \partial_j D_i \left(\hat{\mathbf{p}}^M\right) d\hat{p} \leq \sum_{j \in \mathcal{N}} \partial_j D_i \left(\hat{\mathbf{p}}^M\right) d\hat{p} \leq 0$.

Therefore:

$$d\hat{\Pi} = \sum_{j \in \mathcal{N} \setminus \mathcal{N}^-} \underbrace{\left(\hat{p}_j^M - \hat{c}_j^M\right)}_{>0} \underbrace{dq_j}_{>0} + \sum_{j \in \mathcal{N}^-} \underbrace{q_j^M}_{>0} \underbrace{d\hat{p}}_{>0} + \sum_{j \in \mathcal{N}^-} \underbrace{\left(\hat{p}_j^M - \hat{c}_j^M\right)}_{\leq 0} \underbrace{dq_j}_{\leq 0} > 0,$$

a contradiction. Therefore, under $\left(\hat{\mathbf{S}}\right)$, $p_i^M > \hat{c}_i^M$ for every $i \in \mathcal{N}$.

We now compare monopoly prices to firms' best-responses. The monopoly prices satisfy, for $i \in \mathcal{N}$:

$$0 = \partial_i \hat{\Pi} \left(\hat{\mathbf{p}}^M\right) = \partial_i \hat{\pi}_i \left(\hat{\mathbf{p}}^M, s_i^M\right) + \sum_{j \in \mathcal{N} \setminus \{i\}} \partial_i \hat{\pi}_j \left(\mathbf{p}^M, s_j^M\right),$$

and thus:

$$\partial_i \hat{\pi}_i \left(\hat{\mathbf{p}}^M, s_i^M\right) = - \sum_{j \in \mathcal{N} \setminus \{i\}} \partial_i \hat{\pi}_j \left(\hat{\mathbf{p}}^M, s_j^M\right) = - \sum_{j \in \mathcal{N} \setminus \{i\}} \left\{ \left(\hat{p}_j^M - \hat{c}_j^M\right) \partial_i D_j \left(\hat{\mathbf{p}}^M\right) \right\}. \quad (11)$$

Therefore:

(i) Under $\left(\hat{\mathbf{S}}\right)$, the right-hand side of (11) is negative, as $\hat{p}_j^M > \hat{c}_j^M$, from the first part of the lemma, and $\partial_j D_i \left(\hat{\mathbf{p}}^M\right) > 0$ for $j \neq i \in \mathcal{N}$; hence, for $i \in \mathcal{N}$, we have $\partial_i \hat{\pi}_i \left(\hat{\mathbf{p}}^M, s_i^M\right) < 0$, which, together with the quasi-concavity of $\hat{\pi}_i$ with respect to $\hat{p}_i$, implies $\hat{p}_i^M > \hat{R}_i \left(\hat{\mathbf{p}}_{-i}^M\right)$.

(ii) Suppose that for all $j \in \mathcal{N}$, $\hat{p}_j^M \geq \hat{R}_j\left(\hat{\mathbf{p}}_{-j}^M\right)$, implying $\partial_j \hat{\pi}_j\left(\hat{\mathbf{p}}^M, s_j^M\right) \leq 0$. We then have, for $j \in \mathcal{N}$:

$$0 \geq \partial_j \hat{\pi}_j\left(\hat{\mathbf{p}}^M, s_j^M\right) = D_j\left(\hat{\mathbf{p}}^M\right) + \left(\hat{p}_j^M - \hat{c}_j^M\right) \partial_j D_j\left(\hat{\mathbf{p}}^M\right),$$

and thus, under $\left(\hat{\mathbf{C}}\right)$, $\hat{p}_j^M > \hat{c}_j^M$ for every $j \in \mathcal{N}$. But then, as $\partial_j D_i\left(\hat{\mathbf{p}}^M\right) < 0$ for $j \neq i \in \mathcal{N}$ under $\left(\hat{\mathbf{C}}\right)$, (11) implies $\partial_i \hat{\pi}_i\left(\hat{\mathbf{p}}^M, s_i^M\right) > 0$, a contradiction. Hence, the monopoly outcome satisfies $\hat{p}_j^M < \hat{R}_j\left(\hat{\mathbf{p}}_{-j}^M\right)$ for some firm j.

Finally, when $n = 2$, (11) implies, for $j \neq i \in \{1, 2\}$:

$$\partial_j \hat{\pi}_j\left(\hat{\mathbf{p}}^M, s_j^M\right) = -\left(p_i^M - \hat{c}_i^M\right) \partial_j D_i\left(\hat{\mathbf{p}}^M\right).$$

Under $\left(\hat{\mathbf{C}}\right)$, $\partial_j D_i\left(\hat{\mathbf{p}}^M\right) < 0$ and thus $\hat{p}_i^M > \hat{c}_i^M$ implies $\hat{p}_j^M < \hat{R}_j\left(\hat{p}_j^M\right)$. ■

G.2 Constrained net best-responses

Suppose now that caps $\bar{\mathbf{p}} = (\bar{p}_i)_{i \in \mathcal{N}}$ are imposed on the prices of the goods. Each firm $i \in \mathcal{N}$ then chooses p_i and s_i so as to maximize:

$$\max_{p_i, s_i} \quad \begin{array}{c} \pi_i(\mathbf{p}, \mathbf{s}) \\ \text{s.t. } p_i \leq \bar{p}_i \end{array} \quad .$$

Let $p_i = \rho_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)$ and $s_i = \sigma_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)$ denote the constrained best-responses in terms of price and quality, and $\hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)$ the resulting best-response in terms of net price. These best-responses can equivalently be expressed as:

$$\left(\hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right), \sigma_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)\right) = \arg\max_{\hat{p}_i, s_i} \quad \begin{array}{c} \hat{\pi}_i(\hat{\mathbf{p}}, s_i) \equiv \left[\hat{p}_i + s_i - c_i(s_i)\right] D_i(\hat{\mathbf{p}}) \\ \text{s.t. } \hat{p}_i + s_i \leq \bar{p}_i \end{array} \quad . \qquad (\hat{P})$$

For the sake of comparative statics, we will assume:

Assumption D: For any $i \in \mathcal{N}$, any $\hat{\mathbf{p}}_{-i}$, any $(\hat{p}_i, s_i)$ satisfying $s_i \leq s_i^M$ and $\hat{p}_i + s_i \leq \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) + s_i^M$:

$$\frac{c_i''(s_i)}{\left|c_i'(s_i) - 1\right|} > \frac{\left|\partial_i D_i\left(\hat{p}_i, \hat{\mathbf{p}}_{-i}\right)\right|}{D_i\left(\hat{p}_i, \hat{\mathbf{p}}_{-i}\right)}.$$

This condition asserts that the curvature of the "net cost" function $c(s) - s$ exceeds the semi-elasticity of demand. It is trivially satisfied when the price cap is close to the best-response price (i.e., $\bar{p}_i$ close to $\hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) + s_i^M$), as then s_i is close to s_i^M and thus $c_i'(s_i)$ is close to 1.

We have:

Lemma 8 (constrained net best-responses) *For any $i \in \mathcal{N}$ and any $\hat{\mathbf{p}}_{-i}$, as long as $\bar{p}_i \geq \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$, the price cap is not binding:*

$$\hat{\rho}_i(\hat{\mathbf{p}}_{-i};\bar{p}_i) = \hat{R}_i(\hat{\mathbf{p}}_{-i}), \sigma_i(\hat{\mathbf{p}}_{-i};\bar{p}_i) = s_i^M \text{ and thus } \rho_i(\hat{\mathbf{p}}_{-i};\bar{p}_i) = \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M.$$

When instead $\bar{p}_i < \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$:

(*i*) *the price cap is binding (i.e., $\rho_i(\hat{\mathbf{p}}_{-i};\bar{p}_i) = \bar{p}_i$) and the constrained optimal quality is strictly lower than the unconstrained optimum: $\sigma_i(\hat{\mathbf{p}}_{-i};\bar{p}_i) < s_i^M$;*

(*ii*) *in addition, under Assumption D the constrained optimal net price $\hat{\rho}_i(\hat{\mathbf{p}}_{-i};\bar{p}_i)$ strictly increases with $\bar{p}_i$, from 0 for $\bar{p}_i = 0$ to $\hat{R}_i(\hat{\mathbf{p}}_{-i})$ for $\bar{p}_i = \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$.*

Proof. It follows from Lemma 6 that the optimization program $\left(\hat{\mathcal{P}}\right)$ has a unique solution. As long as $\bar{p}_i \geq \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$, the price cap is not binding and the firm sticks to $\hat{p}_i = \hat{R}_i(\hat{\mathbf{p}}_{-i})$ and $s_i = s_i^M$, and thus to $p_i = \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$.

Part (*i*). It also follows from Lemma 6 that, when $\bar{p}_i < \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$, the price cap is binding: $\hat{p}_i + s_i = \bar{p}_i$. Letting $\lambda_i > 0$ denote the Lagrangian multiplier associated with the constraint, the constrained best-response $s_i = \sigma_i(\hat{\mathbf{p}}_{-i};\bar{p}_i)$ is then characterized by the first-order condition:

$$[1 - c_i'(s_i)]\, D_i(\hat{\mathbf{p}}) = \lambda_i,$$

which implies $c_i'(s_i) < 1$. It follows from the strict quasi-concavity of the profit function $\hat{\pi}$ with respect to s_i that $s_i = \sigma_i(\hat{\mathbf{p}}_{-i};\bar{p}_i) < s_i^M$.

Part (*ii*). When $\bar{p}_i < \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$, the price cap is binding and thus $p_i = \bar{p}_i$. It follows that strict quasi-concavity of the profit function $\pi_i(\mathbf{p}, \mathbf{s})$ with respect to s_i that the constrained optimal quality $s_i = \sigma_i(\hat{\mathbf{p}}_{-i};\bar{p}_i)$ is the unique solution to (with a slight abuse of notation, noting that, for $j \neq i$, π_i depends on p_j and s_j only through $\hat{p}_j = p_j - s_j$):

$$\max_{s_i} \pi_i(\bar{p}_i, s_i; \hat{\mathbf{p}}_{-i}) = [\bar{p}_i - c_i(s_i)]\, D_i(\bar{p}_i - s_i; \hat{\mathbf{p}}_{-i}),$$

and is characterized by the first-order condition:

$$c_i'(s_i)\, D_i(\bar{p}_i - s_i; \hat{\mathbf{p}}_{-i}) + [\bar{p}_i - c_i(s_i)]\, \partial_i D_i(\bar{p}_i - s_i; \hat{\mathbf{p}}_{-i}) = 0. \tag{12}$$

Differentiating this condition with respect to s_i and $\bar{p}_i$ shows that the net price $\hat{\rho}_i(\hat{\mathbf{p}}_{-i};\bar{p}_i) = \bar{p}_i - \sigma_i(\hat{\mathbf{p}}_{-i};\bar{p}_i)$ increases with the price cap $\bar{p}_i$ (i.e., $\partial\sigma_i/\partial\bar{p}_i < 1$) if and only Assumption D is satisfied.

From the above, $\hat{\rho}_i(\hat{\mathbf{p}}_{-i};\bar{p}_i)$ is equal to $\hat{R}_i(\hat{\mathbf{p}}_{-i})$ for $\bar{p}_i \geq \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M$, and strictly increases with $\bar{p}_i$ in the range $\bar{p}_i \in \left[0, \hat{R}_i(\hat{\mathbf{p}}_{-i}) + s_i^M\right]$. To conclude the argument, it suffices to note that $s_i = 0$ constitutes the constrained optimal quality when $\bar{p}_i = 0$, as firm i's profit is then given by $-c_i(s_i)\, D_i(\hat{\mathbf{p}})$; hence, $\hat{\rho}_i(\hat{\mathbf{p}}_{-i};0) = 0 - \sigma_i(\hat{\mathbf{p}}_{-i};0) = 0$. ■

G.3 Price-cap implementable net prices

The following proposition shows that, as in our baseline setting, price caps can sustain any net prices lying below firms' best responses, and only these prices:

Proposition 17 (price-cap implementable net prices) *Under Assumption D:*

(*i*) *The set $\hat{\mathcal{P}}$ of net prices that are sustainable through price caps is:*

$$\hat{\mathcal{P}} = \left\{ \hat{\mathbf{p}} \in \mathbb{R}^n_+ \mid 0 \leq \hat{p}_i \leq \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) \text{ for } i \in \mathcal{N} \right\}.$$

(*ii*) *In particular, the Nash price vector $\hat{\mathbf{p}}^N$ belongs to $\hat{\mathcal{P}}$ and, for any other price vector $\hat{p}$ in $\hat{\mathcal{P}}$, $\hat{p}_i < \hat{p}_i^N$ for some $i \in \mathcal{N}$.*

Proof. (*i*) We first show that price caps can sustain only prices in $\hat{\mathcal{P}}$. Consider net prices and qualities $(\hat{p}_i, s_i)_{i\in\mathcal{N}}$ that are sustainable through price caps $(\bar{p}_i)_{i\in\mathcal{N}}$; that is, they satisfy $\hat{p}_i = \hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)$ and $s_i = \sigma_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)$ for every $i \in \mathcal{N}$). It follows from Lemma 8 that $\hat{\mathbf{p}} \geq 0$. Suppose now that $\hat{p}_i > \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right)$ for some $i \in \mathcal{N}$. This, in turn, implies $\bar{p}_i < \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) + s_i^M$ – otherwise, the price cap $\bar{p}_i$ would not be binding, and firm i could thus profitably deviate to its unconstrained best-responses $\hat{p}_i' = \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right)$ and $s_i' = s_i^M$. But from Lemma 8, in the range $\bar{p}_i' \in \left[0, \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) + s_i^M\right]$ the constrained best-response $\hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i'\right)$ increases with $\bar{p}_i'$ from 0 up to $\hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right)$; it follows that $\hat{p}_i = \hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i'\right) < \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right)$, a contradiction.

Conversely, for any net price vector $\hat{\mathbf{p}} \in \hat{\mathcal{P}}$ and every $i \in \mathcal{N}$, there exists $\bar{p}_i \in \left[0, \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) + s_i^M\right]$ such that $\hat{p}_i = \hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)$. But then, the net price vector $\hat{\mathbf{p}}$ is sustainable through the price caps $\bar{\mathbf{p}} = (\bar{p}_i)_{i\in\mathcal{N}}$.

(*ii*) By construction, $\hat{\mathbf{p}}^N$ lies on firms' unconstrained best-responses, and thus belongs to $\hat{\mathcal{P}}$. Consider now a price vector $\hat{\mathbf{p}}$ in $\hat{\mathcal{P}} \setminus \left\{\hat{\mathbf{p}}^N\right\}$, and suppose that $\hat{p}_i \geq p_i^N$ for all $i \in \mathcal{N}$. For every $i \in \mathcal{N}$, we then have:

$$\begin{aligned}
\hat{p}_i - p_i^N &\leq \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) - p_i^N, \\
&= \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) - R_i\left(\mathbf{p}_{-i}^N\right) \\
&= \int_0^1 \frac{d}{d\lambda} \left\{ \hat{R}_i\left(\lambda \hat{\mathbf{p}}_{-i} + (1-\lambda)\, \mathbf{p}_{-i}^N\right) \right\} d\lambda \\
&= \int_0^1 \{ \sum_{j\in\mathcal{N}\setminus\{i\}} \partial_j \hat{R}_i\left(\lambda \hat{\mathbf{p}}_{-i} + (1-\lambda)\, \mathbf{p}_{-i}^N\right) \left(\hat{p}_j - p_j^N\right) \} d\lambda.
\end{aligned}$$

Summing up these inequalities for $i \in \mathcal{N}$ yields:

$$\begin{aligned}
\sum_{i\in\mathcal{N}} \left(\hat{p}_i - p_i^N\right) &\leq \sum_{i\in\mathcal{N}} \int_0^1 \{ \sum_{j\in\mathcal{N}\setminus\{i\}} \partial_j \hat{R}_i \left(\lambda \hat{\mathbf{p}}_{-i} + (1-\lambda)\, \mathbf{p}_{-i}^N\right) \left(\hat{p}_j - p_j^N\right) \} d\lambda \\
&= \sum_{j\in\mathcal{N}} \left(\hat{p}_j - p_j^N\right) \int_0^1 \{ [\sum_{i\in\mathcal{N}\setminus\{j\}} \partial_j \hat{R}_i \left(\lambda \hat{\mathbf{p}}_{-i} + (1-\lambda)\, \mathbf{p}_{-i}^N\right)] \} d\lambda \\
&< \sum_{j\in\mathcal{N}} \left(\hat{p}_j - p_j^N\right),
\end{aligned}$$

where the last inequality follows from (10). We thus obtain a contradiction, implying that $\hat{p}_i < \hat{p}_i^N$ for some $i \in \mathcal{N}$. ■

G.4 Duopoly

We now focus on the case of a duopoly: $\mathcal{N} = \{1, 2\}$. We first consider the impact of price caps and consumers. We start with the observation that, like the demand, consumer surplus only depends on net prices, $\hat{p}_1$ and $\hat{p}_2$; indeed, letting $S(p_1, p_2, s_1, s_2)$ denote consumer surplus, we have:

$$\frac{\partial S}{\partial p_i}(\cdot) = -D_i\left(p_i - s_i, p_j - s_j\right) = -\frac{\partial S}{\partial s_i}(\cdot).$$

Hence, consumer surplus can be expressed as $\hat{S}\left(\hat{p}_i, \hat{p}_j\right)$. We therefore adapt the previous baseline Assumption A, introduced to ensure that price caps can only benefit consumers, as follows:

Assumption Â: For any $i \neq j \in \{1, 2\}$ and any net price $\hat{p}_i \in [0, \hat{p}_i^N)$, if $\hat{R}_j(\hat{p}_i) > \hat{p}_i^N$, then:

$$\hat{R}_j'(p_i) > -\frac{D_i\left(\hat{p}_i, \hat{R}_j(\hat{p}_i)\right)}{D_j\left(\hat{R}_j(\hat{p}_i), \hat{p}_i\right)}.$$

Assumption Â holds again trivially when net prices are strategic complements (i.e., $\hat{R}_j'(\cdot) > 0$). We have:

Proposition 18 (non-verifiable quality: price caps - duopoly) *Under Assumptions D and $\hat{A}$, any vector of net prices $\hat{\mathbf{p}} \neq \hat{\mathbf{p}}^N$ that is sustainable through price caps yields a higher consumer surplus than $\hat{\mathbf{p}}^N$. Therefore: (i) for any vector of price caps $\bar{\mathbf{p}}$, consumers are weakly better off under $G_{\bar{\mathbf{p}}}$ than under G_∞; and (ii) in $\mathcal{G}$, it is optimal for the competition authority to allow price caps.*

Proof. Consider a price vector $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2)$ in $\hat{\mathcal{P}} \setminus \left\{\hat{\mathbf{p}}^N\right\}$. From Proposition 17, $\hat{p}_i < \hat{p}_i^N$ for some $i \in \{1, 2\}$. If the price of the other firm, j, satisfies $\hat{p}_j \leq \hat{p}_j^N$, then consumers clearly prefer $\hat{\mathbf{p}}$ to $\hat{\mathbf{p}}^N$. Suppose now that $\hat{p}_j > \hat{p}_j^N$; from Proposition 17 we then have $\hat{R}_j(\hat{p}_i) \geq \hat{p}_j > \hat{p}_j^N$, let:[24]

$$\hat{p}_i' \equiv \inf\left\{\hat{p}_i'' \geq \hat{p}_i \mid \hat{R}_j(\hat{p}_i'') \leq \hat{p}_j^N\right\}.$$

By construction, $\hat{p}_i' \in (\hat{p}_i, \hat{p}_i^N]$ and $\hat{R}_j(\hat{p}_i') = \hat{p}_j^N$. Letting $\hat{S}(\hat{p}_i, \hat{p}_j)$ denote total consumer surplus, we then have:

$$\hat{S}(\hat{p}_i, \hat{p}_j) \geq \hat{S}\left(\hat{p}_i, \hat{R}_j(\hat{p}_i)\right) > \hat{S}\left(\hat{p}_i', \hat{R}_j(\hat{p}_i')\right) \geq \hat{S}\left(\hat{p}_i^N, \hat{p}_j^N\right),$$

where the first inequality follows from Proposition 17, the last follows from $\hat{p}_i' \leq \hat{p}_i^N$ and $\hat{R}_j(\hat{p}_i') = \hat{p}_j^N$, and the strict one follows from $\hat{p}_i' > \hat{p}_i$ and Assumption Â, which together imply:

$$\hat{S}\left(\hat{p}_i, \hat{R}_j(\hat{p}_i)\right) - \hat{S}\left(\hat{p}_i', \hat{R}_j(\hat{p}_i')\right) = \int_{\hat{p}_i}^{\hat{p}_i'} \left[D_i\left(\hat{p}_i, \hat{R}_j(\hat{p}_i)\right) + D_j\left(\hat{R}_j(\hat{p}_i), \hat{p}_i\right)\hat{R}_j'(\hat{p}_i)\right] d\hat{p}_i > 0.$$

■

Hence, under Assumption Â, firms' use of price caps can only benefit consumers. Consider now firms' incentives to introduce price caps. The following Proposition extends our previous insights:

Proposition 19 (non-verifiable quality: firms' incentives - duopoly)

(i) Under $\left(\hat{\mathbf{S}}\right)$ and Assumption D, firms cannot use price caps to increase both of their profits; if in addition net prices are strategic complements (i.e., if $\hat{R}_i'(\cdot) > 0$ for $i = 1, 2$), then firms cannot use price caps to increase any of their profits (and thus, a fortiori, their joint profit).

(ii) Under $\left(\hat{\mathbf{C}}\right)$, firms can use price caps to increase both profits (and thus, a fortiori, their joint profit); any such price caps benefit consumers as well.

Proof. (i) Consider a price vector $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2)$ in $\hat{\mathcal{P}} \backslash \left\{\hat{\mathbf{p}}^N\right\}$ and let $\mathbf{s} = (s_1, s_2)$ denote the associated qualities. From Proposition 17, $\hat{p}_j < p_j^N$ for some $j \in \{1, 2\}$. Under $\left(\hat{\mathbf{S}}\right)$, we have, for $i \neq j \in \{1, 2\}$:

$$\begin{aligned}\hat{\pi}_i(\hat{p}_i, \hat{p}_j, s_i) &\leq \max_{\hat{p}_i, s_i} \hat{\pi}_i(\hat{p}_i, \hat{p}_j, s_i) \\ &< \max_{\hat{p}_i, s_i} \hat{\pi}_i\left(\hat{p}_i, \hat{p}_j^N, s_i\right) \\ &= \hat{\pi}_i\left(\hat{p}_i^N, \hat{p}_j^N, s_i^M\right),\end{aligned}$$

[24] The reasoning that follows relies on the range $[\hat{p}_i, \hat{p}_i']$, because Assumption A is required to hold only for the prices $\hat{p}_i < \hat{p}_i^N$ that satisfy $\hat{R}_j(\hat{p}_i) > \hat{p}_i^N$.

where the strict inequality follows from $\hat{p}_j < p_j^N$ and

$$\frac{d}{d\hat{p}_j}\left\{\max_{\hat{p}_i,s_i}\hat{\pi}_i\left(\hat{p}_i,\hat{p}_j,s_i\right)\right\} = \partial_j\hat{\pi}_i\left(\hat{R}_i\left(\hat{p}_j\right),\hat{p}_j,s_i^M\right)$$
$$= \left[\hat{R}_i\left(\hat{p}_j\right)-\hat{c}_i^M\right]\partial_j D_i\left(\hat{R}_i\left(\hat{p}_j\right),\hat{p}_j\right), \quad (13)$$

where, from Lemma 7, the last expression is positive under $\left(\hat{\mathbf{S}}\right)$. Therefore, firms cannot use price caps (with or without transfers) to increase both of their profits.

Furthermore, if prices are strategic complements, then any price vector $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2)$ in $\hat{\mathcal{P}}\setminus\left\{\hat{\mathbf{p}}^N\right\}$ is such that $\hat{p}_i < \hat{p}_i^N$ for $i = 1, 2$.[25] The above argument then implies that both firms obtain strictly less profit than in the Nash equilibrium. Hence, in that case firms cannot use price caps to increase any of their profits.

(ii) By contrast, under $\left(\hat{\mathbf{C}}\right)$, there exist prices in $\hat{\mathcal{P}}$ that increase both firms' profits. To see this, note first that, from Lemma 7, both firms' margins are positive at the Nash equilibrium. It follows that, starting from the Nash equilibrium prices $\left(\hat{p}_1^N, \hat{p}_2^N\right)$, a small and uniform reduction in both prices increases both firms' profits, as reducing one firm's price (and adjusting the quality accordingly) has only a second-order effect on the profit of that firm, and a first-order, positive effect on the other firm's profit (as it increases that firm's demand). To conclude the argument, it suffices to note that any $(\hat{p}_1, \hat{p}_2) = \left(\hat{p}_2^N - \varepsilon, \hat{p}_2^N - \varepsilon\right)$, with $\varepsilon > 0$, belongs to $\hat{\mathcal{P}}$, as:

- using condition (10), we have:

$$\hat{R}_j\left(\hat{p}_i\right) - \hat{p}_j = \int_0^{\varepsilon}\left[1 - \hat{R}_j'\left(\hat{p}_i^N - x\right)\right]dx > 0.$$

- Assumption D is trivially satisfied in the neighborhood of $\hat{\mathbf{p}}^N$, as $c_i'\left(s_i\right)$ is then close to 1; hence, in this neighborhood, the constrained net best-responses are increasing with price caps as long as these are binding, and are equal to unconstrained best-responses otherwise. It follows that, for ε small enough, there exists appropriately chosen price caps that sustain $(\hat{p}_1, \hat{p}_2) = \left(\hat{p}_2^N - \varepsilon, \hat{p}_2^N - \varepsilon\right)$.

Therefore, there are prices in $\hat{\mathcal{P}}$ that give both firms more profit than the Nash equilibrium prices, as is required for a price cap vector to be agreed upon in the absence of transfer.

To conclude the proof, it suffices to note that increasing both firms' profits requires lowering prices below the Nash level. To see this, consider a price vector $\hat{\mathbf{p}} \in \hat{\mathcal{P}}$ that increases both firms' profits above their Nash levels, and let $\hat{\mathbf{s}} = (\hat{s}_1, \hat{s}_2)$ denote the

[25] From Proposition 1, this has to be the case for at least one firm i; Proposition 1 and strategic complementarity then together imply that, for the other firm, j: $\hat{p}_j \leq \hat{R}_j\left(\hat{p}_i\right) < \hat{R}_j\left(\hat{p}_i^N\right) = \hat{p}_j^N$.

associated qualities; we then have, for $i \neq j \in \{1,2\}$:

$$\max_{\hat{p}_i, s_i} \hat{\pi}_i \left(\hat{p}_i, \hat{p}_j, s_i\right) \geq \hat{\pi}_i \left(\hat{\mathbf{p}}, \hat{s}_i\right) \geq \hat{\pi}_i^N = \max_{\hat{p}_i, s_i} \hat{\pi}_i \left(\hat{p}_i, \hat{p}_j^N, s_i\right),$$

which, using (13) and Lemma 7, implies $\hat{p}_j \leq \hat{p}_j^N$ under $\left(\hat{\mathbf{C}}\right)$. Hence, $\hat{\mathbf{p}} \leq \hat{\mathbf{p}}^N$. ∎

Proposition 19 (which does not hinge on Assumption Â) confirms that: (i) firms will not select price caps when they offer substitutable goods (case $\left(\hat{\mathbf{S}}\right)$); and (ii) price caps enable the firms to cooperate when they offer complements (case $\left(\hat{\mathbf{C}}\right)$), in which case firms' interests are aligned with those of consumers – both long for lower net prices. Furthermore, price caps then benefit consumers whenever they enhance both firms' profits. It is worth noting that the profitability of price caps for complements does not depend on any specific assumption on (the curvature of the net) cost function or (on the semi-elasticity of) demand. This is because the argument relies on net prices that are in the neighborhood of the unconstrained Nash outcome, where Assumption D is automatically satisfied (as s_i is close to s_i^M, $c_i'\left(s_i\right)$ is close to 1).

Remark: Mergers versus price caps. It would be straightforward to extend condition $(\mathbf{M_S})$ and show that, under this condition, a merger harms consumers whereas price caps can only benefit them. However, under the alternative condition $(\mathbf{M_C})$, a merger and joint-profit maximizing price caps would produce different results: the merger would lead to monopoly prices and qualities, whereas price caps would lead to lower qualities and, thus, to different prices.

G.5 Symmetric oligopoly

We now extend the analysis to an arbitrary number of firms, $n \geq 2$, and first focus on symmetric firms and outcomes; specifically, firms:

- face the same constant unit cost: $c_i\left(s_i\right) = c\left(s_i\right)$ for all $i \in \mathcal{N}$;
- face symmetric demands, in the sense that other firms' prices are interchangeable: $D_i\left(\hat{\mathbf{p}}\right) = D\left(\hat{p}_i; \hat{\mathbf{p}}_{-i}\right)$, where $\hat{p}_j = p_j - s_j$ denotes firm j's net price, adjusted for quality, and $D\left(\hat{p}_i; \hat{\mathbf{p}}_{-i}\right) = D\left(\hat{p}_i; \sigma\left(\hat{\mathbf{p}}_{-i}\right)\right)$ for any permutation $\sigma\left(\cdot\right)$ of the net prices $\hat{\mathbf{p}}_{-i}$, for all $i \in \mathcal{N}$ and $\left(\hat{p}_i, \hat{\mathbf{p}}_{-i}\right) \in \mathbb{R}_+^n$.

We maintain the strict quasi-concavity assumption for the individual profit functions. Our symmetry assumption implies that all firms have the same best-responses:

$$\hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) = \hat{R}\left(\hat{\mathbf{p}}_{-i}\right),$$

which is moreover invariant under any permutation of the other firms' prices. We further assume that

$$\partial_1 \hat{R}(\cdot) > -1, \tag{14}$$

where $\partial_1 \hat{R}(\cdot)$ denotes the partial derivative of $\hat{R}(\cdot)$ with respect to its first argument (by symmetry, the same condition applies to the other derivatives).

We further assume that the unique Nash equilibrium and monopoly outcome are symmetric: $\hat{p}_i^N = \hat{p}^N$ and $\hat{p}_i^M = \hat{p}^M$. We have:

From Proposition 17, firms cannot use price caps to raise their prices uniformly above the Nash level. Therefore:

Proposition 20 (non-verifiable quality: price caps - symmetric oligopoly) *Under Assumption D, any symmetric vector of net prices $\hat{\mathbf{p}} \neq \mathbf{p}^N$ that is sustainable through price caps yields a higher consumer surplus than $\mathbf{p}^N$.*

Proof. This follows directly from Proposition 17, which ensures that any such symmetric vector of price caps is such that $\hat{p}_i = \hat{p} < p^N$. ■

This Proposition extends our previous insights in that price caps can only result in lower symmetric net prices that benefit consumers. It also implies that firms have no incentives to introduce a price cap under (**S**), and can instead use them to increase their profits under (**C**):

Proposition 21 (non-verifiable quality: incentives - symmetric oligopoly)

(*i*) *Under $\left(\hat{\mathbf{S}}\right)$ and Assumption D, firms cannot use price caps to sustain a more profitable symmetric outcome than that of the Nash equilibrium.*

(*ii*) *Under $\left(\hat{\mathbf{C}}\right)$, firms can use price caps to generate a symmetric outcome that increases their profits, compared with the Nash equilibrium outcome; any such use of price caps benefits consumers as well.*

Proof. (*i*) From the above analysis, any symmetric vector of net prices that is sustainable through price caps lies below the Nash level: $\hat{p}_i = \hat{p} < \hat{p}^N$. It follows that, under $\left(\hat{\mathbf{S}}\right)$, any such vector of net prices is less profitable than the Nash equilibrium outcome; letting s denote the associated (symmetric) level of quality,[26] we have:

$$\begin{aligned}
\hat{\pi}_i(\hat{\mathbf{p}}, s) &= \max_{\hat{p}_i, s_i} \ [\hat{p}_i + s_i - c_i(s_i)] \, D(\hat{p}_i, \hat{\mathbf{p}}_{-i}) \\
&\qquad\qquad \text{s.t. } \hat{p}_i + s_i \leq \bar{p} \\
&\leq \max_{\hat{p}_i, s_i} [\hat{p}_i + s_i - c_i(s_i)] \, D(\hat{p}_i, \hat{\mathbf{p}}_{-i}) \\
&\leq \max_{\hat{p}_i, s_i} [\hat{p}_i + s_i - c_i(s_i)] \, D\left(\hat{p}_i, \hat{\mathbf{p}}_{-i}^N\right) \\
&= \hat{\pi}_i\left(\hat{\mathbf{p}}^N, s^M\right).
\end{aligned}$$

[26] Firm i's associated quality level is given by $s_i = \arg\max_s \hat{\pi}_i(\hat{\mathbf{p}}, s)$ and is thus symmetric: $s_i = s$ for all $i \in \mathcal{N}$.

(ii) As for Proposition 19, it suffices to note that, under $\left(\hat{\mathbf{C}}\right)$, a small and uniform reduction in all prices:

- increases all firms' profits, as reducing one firm's price (and adjusting the quality accordingly) has only a second-order effect on the profit of that firm, and a first-order, positive effect on all other firms' profits (as it increases their demands); and
- generates a symmetric vector of net prices that belongs to $\hat{\mathcal{P}}$ as (Assumption D is trivially satisfied in the neighborhood of $\hat{\mathbf{p}}^N$, so that constrained net best-responses strictly increase with relevant price caps, and):

$$\hat{R}(\hat{p},...,\hat{p}) - \hat{p} = \int_0^{\varepsilon} \left[1 - \sum_{j=1}^{n-1} \partial_j \hat{R}\left(\hat{p}^N - x\right)\right] dx > 0,$$

where the inequality follows from condition (10).Using symmetry, we have:

$$\sum_{j=1}^{n-1} \partial_j \hat{R}(\hat{p}) = \frac{1}{n} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{i\}} \partial_j \hat{R}_i(\hat{\mathbf{p}}) = \frac{1}{n} \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{N} \setminus \{j\}} \partial_j \hat{R}_i(\hat{\mathbf{p}}) < 1,$$

where the inequality stems from (10).

Hence, there are symmetric net prices in $\hat{\mathcal{P}}$ that give all firms more profit than the Nash equilibrium outcome, as is required for price caps to be agreed upon in the absence of transfer. To conclude the proof, it suffices to note that increasing all firms' profits requires lowering the symmetric net price below the Nash level. To see this, consider a symmetric price vector $\hat{\mathbf{p}}$ that increases all firms' profits above their Nash levels; letting s denote the associated (symmetric) quality, we then have, for $i \neq j \in \mathcal{N}$:

$$\max_{\hat{p}_i, s_i} \hat{\pi}_i(\hat{p}_i, \hat{\mathbf{p}}_{-i}, s_i) \geq \hat{\pi}_i(\hat{\mathbf{p}}, s) \geq \hat{\pi}_i^N = \max_{\hat{p}_i, s_i} \hat{\pi}_i\left(\hat{p}_i, \hat{\mathbf{p}}_{-i}^N, s_i\right),$$

which, using (13) and Lemma 7, implies $\hat{p} \leq \hat{p}^N$ under $\left(\hat{\mathbf{C}}\right)$. Hence, $\hat{\mathbf{p}} \leq \hat{\mathbf{p}}^N$. ■

G.6 Oligopoly under strategic complementarity

We extend here the analysis to oligopolies where net prices are strategic complements:

$\left(\widehat{\mathbf{SC}}\right)$ *Strategic complementarity:* for every $i \in \mathcal{N}$, $\hat{R}_i(\hat{\mathbf{p}}_{-i})$ increases in $\hat{p}_j$ for any $j \in \mathcal{N} \setminus \{i\}$.

We show that, under this assumption (together with our previous assumptions on the quasi-concavity of profit functions, the regularity of best-responses and Nash equilibrium uniqueness), price caps cannot generate higher equilibrium net prices (regardless of whether goods are complements or substitutes). It follows that price caps can only benefit consumers, and are useful for suppliers of complements, but not for competitors offering substitutes.

Suppose that each firm $i \in \mathcal{N}$ faces a price cap $\bar{p}_i$. Any resulting equilibrium price vector $\hat{\mathbf{p}} = (\hat{p}_i)_{i\in\mathcal{N}}$ satisfies $\hat{p}_i = \hat{\rho}_i (\hat{\mathbf{p}}_{-i}; \bar{p}_i)$ for $i \in \mathcal{N}$, where:

$$\hat{\boldsymbol{\rho}}_i (\mathbf{p}_{-i}; \bar{\mathbf{p}}_i) = (\hat{\rho}_i (\mathbf{p}_{-i}; \bar{\mathbf{p}}_i))_{j\in\mathcal{M}_i} \equiv \arg\max_{\mathbf{p}_i \le \bar{\mathbf{p}}_i} \pi_i (\mathbf{p}_i, \mathbf{p}_{-i})$$

denotes firm i's best-response, constrained by the price caps it faces.

We now show that price caps cannot be used to raise equilibrium prices:

Proposition 22 (non-verifiable quality: price caps benefit consumers under $\left(\widehat{\mathbf{SC}}\right)$) *Under $\left(\widehat{\mathbf{SC}}\right)$ and Assumption D, any vector of net prices $\hat{\mathbf{p}} = (\hat{p}_i)_{i\in\mathcal{N}}$ that is sustainable through price caps satisfies $\hat{\mathbf{p}} \le \hat{\mathbf{p}}^N$. Therefore, price caps can only benefit consumers.*

Proof. From Proposition 17, the set of net prices $\hat{\mathbf{p}} = (\hat{p}_i)_{i\in\mathcal{N}}$ that are sustainable through price caps is

$$\hat{\mathcal{P}} \equiv \left\{ \hat{\mathbf{p}} \in \mathbb{R}^n_+ \mid 0 \le \hat{p}_i \le \hat{R}_i (\hat{\mathbf{p}}_{-i}) \text{ for } i \in \mathcal{N} \right\},$$

where by assumption the best-response function $\hat{R}_i (\hat{\mathbf{p}}_{-i})$ is bounded above by some $\hat{B}_i$. Hence, without loss of generality, we can restrict the net price $\hat{p}_i$ to belong to $\hat{S}_i \equiv \left[0, \hat{B}_i\right]$. Note that, by construction, the Nash equilibrium net price vector $\hat{\mathbf{p}}^N = \left(\hat{p}_i^N\right)_{i\in\mathcal{N}}$ is such that $\hat{p}_i^N \in \hat{S}_i$.

Next, define $\hat{\boldsymbol{\phi}} (\hat{\mathbf{p}}) \equiv \left(\hat{\phi}_i (\hat{\mathbf{p}})\right)_{i\in\mathcal{N}}$, where $\hat{\phi}_i (\hat{\mathbf{p}}) = \hat{R}_i (\hat{\mathbf{p}}_{-i})$. The Nash equilibrium vector of net prices, $\hat{\mathbf{p}}^N$, obviously constitutes a fixed point of $\hat{\boldsymbol{\phi}} (\cdot)$. Furthermore, using the same reasoning as in the proof of Lemma 4, it can be checked that $\hat{\boldsymbol{\phi}}$ is a contraction mapping from $\hat{S} \equiv \hat{S}_1 \times ... \times \hat{S}_n$ to $\hat{S}$, endowed with the ℓ_1 norm: for any $\hat{\mathbf{p}} \in \hat{S}$, $\hat{\boldsymbol{\phi}} (\hat{\mathbf{p}}) \in \hat{S}$ and, in addition, for any $\hat{\mathbf{p}}' \in \hat{S}$:

$$\begin{aligned}
\left\|\hat{\boldsymbol{\phi}} (\hat{\mathbf{p}}') - \hat{\boldsymbol{\phi}} (\hat{\mathbf{p}})\right\| &= \sum_{i\in\mathcal{N}} \left|\hat{\phi}_i (\hat{\mathbf{p}}') - \hat{\phi}_i (\hat{\mathbf{p}})\right| \\
&= \sum_{i\in\mathcal{N}} \left|\hat{R}_i \left(\hat{\mathbf{p}}'_{-i}\right) - \hat{R}_i (\hat{\mathbf{p}}_{-i})\right| \\
&= \sum_{i\in\mathcal{N}} \left|\int_0^1 \frac{d}{d\lambda} \left\{\hat{R}_i \left(\lambda\hat{\mathbf{p}}'_{-i} + (1-\lambda)\, \hat{\mathbf{p}}_{-i}\right)\right\} d\lambda\right| \\
&\le \sum_{i\in\mathcal{N}} \int_0^1 \{ \sum_{j\in\mathcal{N}\setminus\{i\}} \left|\partial_j \hat{R}_i \left(\lambda\hat{\mathbf{p}}'_{-i} + (1-\lambda)\, \hat{\mathbf{p}}_{-i}\right)\right| \, |\hat{p}'_j - \hat{p}_j| \} d\lambda \\
&= \int_0^1 \{ \sum_{j\in\mathcal{N}} [\sum_{i\in\mathcal{N}\setminus\{j\}} \partial_j \hat{R}_i \left(\lambda\hat{\mathbf{p}}'_{-i} + (1-\lambda)\, \hat{\mathbf{p}}_{-i}\right)] \, |\hat{p}'_j - \hat{p}_j| \} d\lambda \\
&\le \sum_{j\in\mathcal{N}} \hat{k} \, |\hat{p}'_j - \hat{p}_j| \\
&= \hat{k} \, \|\hat{p}' - \hat{p}\|,
\end{aligned}$$

where the fourth equality uses $\left(\widehat{\mathbf{SC}}\right)$ and:

$$\hat{k} \equiv \max_{\hat{\mathbf{p}} \in \hat{S}, j \in \mathcal{N}} \sum_{i \in \mathcal{N} \setminus \{j\}} \partial_j \hat{R}_i \left(\hat{\mathbf{p}}_{-i}\right) < 1,$$

where the inequality stems from (10). It follows from the Banach fixed point theorem that $\hat{\mathbf{p}}^N$ is the unique fixed point in $\hat{S}$, and that any sequence $\{\hat{\mathbf{p}}_k\}_{k\in\mathbb{N}}$ satisfying $\hat{\mathbf{p}}_{k+1} = \hat{\boldsymbol{\phi}}\left(\hat{\mathbf{p}}_k\right)$ converges to $\hat{\mathbf{p}}^N$.

Next, we show that any $\hat{\mathbf{p}} \in \hat{\mathcal{P}}$ is such that $\hat{\boldsymbol{\phi}}\left(\hat{\mathbf{p}}\right) \in \hat{\mathcal{P}}$. Thus, fix $\hat{\mathbf{p}} \in \hat{\mathcal{P}}$ and $i \in \mathcal{N}$. Using the constrained net price best-response $\hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i\right)$ defined by $\left(\hat{P}\right)$, it follows from Lemma 8 that

$$\hat{\phi}_i\left(\hat{\mathbf{p}}\right) = \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) \geq \hat{\rho}_i\left(\hat{\mathbf{p}}_{-i}; \bar{p}_i = 0\right) = 0.$$

It remains to show that $\hat{\phi}_i\left(\hat{\mathbf{p}}\right) \leq \hat{R}_i\left(\hat{\boldsymbol{\phi}}_{-i}\left(\hat{\mathbf{p}}_{-i}\right)\right)$; using $\hat{p}_j\left(\alpha\right) = \alpha \hat{R}_j\left(\hat{\boldsymbol{\phi}}_{-j}\left(\hat{\mathbf{p}}\right)\right) + \left(1-\alpha\right)\hat{\phi}_j\left(\hat{\mathbf{p}}\right)$, we have:

$$\begin{aligned}
& \hat{R}_i\left(\hat{\boldsymbol{\phi}}_{-i}\left(\hat{\mathbf{p}}\right)\right) - \hat{\phi}_i\left(\hat{\mathbf{p}}\right) \\
= \;& \hat{R}_i\left(\hat{\boldsymbol{\phi}}_{-i}\left(\hat{\mathbf{p}}\right)\right) - \hat{R}_i\left(\hat{\mathbf{p}}_{-i}\right) \\
= \;& \hat{R}_i\left(\alpha\hat{\boldsymbol{\phi}}_{-i}\left(\hat{\mathbf{p}}\right) + \left(1-\alpha\right)\hat{\mathbf{p}}_{-i}\right)\Big|_{\alpha=1} - \hat{R}_i\left(\alpha\hat{\boldsymbol{\phi}}_{-i}\left(\hat{\mathbf{p}}\right) + \left(1-\alpha\right)\hat{\mathbf{p}}_{-i}\right)\Big|_{\alpha=1} \\
= \;& \int_0^1 \frac{d}{d\alpha}\left\{\hat{R}_i\left(\alpha\hat{\boldsymbol{\phi}}_{-i}\left(\hat{\mathbf{p}}\right) + \left(1-\alpha\right)\hat{\mathbf{p}}_{-i}\right)\right\} d\alpha \\
= \;& \int_0^1 \frac{d}{d\alpha}\left\{\hat{R}_i\left(\alpha\left(\hat{R}_j\left(\hat{\mathbf{p}}_{-j}\right)\right)_{j\in\mathcal{N}\setminus\{i\}} + \left(1-\alpha\right)\left(\hat{p}_j\right)_{j\in\mathcal{N}\setminus\{i\}}\right)\right\} d\alpha \\
= \;& \int_0^1 \left\{\sum_{j\in\mathcal{N}\setminus\{i\}} \partial_j \hat{R}_i\left(\alpha\left(\hat{R}_j\left(\hat{\mathbf{p}}_{-j}\right)\right)_{j\in\mathcal{N}\setminus\{i\}} + \left(1-\alpha\right)\left(\hat{p}_j\right)_{j\in\mathcal{N}\setminus\{i\}}\right)\left(\hat{R}_j\left(\hat{\mathbf{p}}_{-j}\right) - \hat{p}_j\right)\right\} d\alpha \\
\geq \;& 0,
\end{aligned}$$

where the inequality stems from $\left(\widehat{\mathbf{SC}}\right)$ (which implies $\partial_j \hat{R}_i \geq 0$ for all $i \neq j \in \mathcal{N}$) and $\hat{\mathbf{p}} \in \hat{\mathcal{P}}$ (which implies $\hat{R}_j\left(\hat{\mathbf{p}}_{-j}\right) \geq \hat{p}_j$ for all $j \in \mathcal{N}$).

It follows that the sequence $\{\hat{\mathbf{p}}_k\}_{k\in\mathbb{N}}$ defined by $\hat{\mathbf{p}}_0 = \hat{\mathbf{p}}$ and $\hat{\mathbf{p}}_{k+1} = \hat{\boldsymbol{\phi}}\left(\hat{\mathbf{p}}_k\right)$ not only converges to $\hat{\mathbf{p}}^N$, but remains within $\hat{\mathcal{P}}$ and is therefore such that $\hat{\mathbf{p}}_{k+1} \geq \hat{\mathbf{p}}_k$. Hence, $\hat{\mathbf{p}}^N = \hat{\mathbf{p}}_\infty \geq \hat{\mathbf{p}}_0 = \hat{\mathbf{p}}$. ■

It follows from the above that, again, firms have an incentive to agree on price caps when they offer complements, but not when they offer substitutes:

Corollary 2 (non-verifiable quality: firms' incentives under $\left(\widehat{\mathbf{SC}}\right)$)

(i) Under $\left(\hat{\mathbf{S}}\right)$ and Assumption D, price caps cannot increase the profit of any firm.

(ii) Under $\left(\hat{\mathbf{C}}\right)$, price caps can be used to increase all firms' profits.

Proof. If goods are all substitutes, then for any $\hat{\mathbf{p}} \in \hat{\mathcal{P}}$ and associated qualities $\mathbf{s}$, and any $i \in \mathcal{N}$, we have:

$$\hat{\pi}_i(\hat{\mathbf{p}}, s_i) \leq \max_{\hat{\mathbf{p}}_i', s_i'} \hat{\pi}_i(\hat{p}_i', \hat{\mathbf{p}}_{-i}, s_i') \leq \max_{\hat{\mathbf{p}}_i', s_i'} \hat{\pi}_i\left(\hat{p}_i', \hat{\mathbf{p}}_{-i}^N, s_i'\right) = \hat{\pi}_i^N,$$

where the first inequality reflects the fact that firm i may be constrained by its price cap $\bar{p}_i$, and the second inequality stems from the fact that price caps can only sustain net prices that are lower than $\hat{\mathbf{p}}^N$.

If goods are all complements, the reasoning used in the proof of Proposition 19 extends to the case of an asymmetric oligopoly: starting from the Nash equilibrium prices $\hat{\mathbf{p}}^N$, reducing all net prices by a small amount ε (and adjusting the qualities accordingly) increases all firms' profits, as firms' margins are positive from Lemma 7, and reducing one firm's price has only a second-order effect on the profit of that firm, and a first-order positive effect on the other firms' demands. To conclude the argument, it suffices to show that the new price vector, $\hat{\mathbf{p}}(\varepsilon) = \left(\hat{p}_i^N - \varepsilon\right)_{i \in \mathcal{N}}$, belongs to $\hat{\mathcal{P}}$; indeed, for $i \in \mathcal{N}$ we have $\hat{p}_i(\varepsilon) = \hat{p}_i^N - \varepsilon > 0$ for ε small enough and:

$$\hat{R}_i(\hat{\mathbf{p}}_{-i}(\varepsilon)) - \hat{p}_i(\varepsilon) = \int_0^\varepsilon [1 - \sum_{j \in \mathcal{N} \setminus \{i\}} \partial_j \hat{R}_i(\hat{\mathbf{p}}_{-i}(x))] dx > 0,$$

where the inequality stems from (10). As Assumption D is trivially satisfied in the neighborhood of $\hat{\mathbf{p}}^N$, it follows from the proof of Proposition 17 that $\hat{\mathbf{p}}(\varepsilon) \in \hat{\mathcal{P}}$ for ε small enough. ■

H Post-investment price caps

H.1 Substitutes

Consider the multi-product firm oligopoly setting developed in online Appendix C, in which each firm $i \in \mathcal{N}$ can offer a set $\mathcal{M}_i \equiv \{1, ..., m_i\}$ of products, and now suppose that in addition firms must make investment decisions. These decisions may correspond to entering or staying in the market, developing new products, improving the quality or lowering the production cost of existing ones; different firms may moreover be facing different choices.

Let $\mathcal{I}_i$ denote the set of feasible investment decisions for firm i, and $\mathbf{I} = (I_1, ..., I_n) \in \mathcal{I} = \mathcal{I}_1 \times ... \times \mathcal{I}_n$ denote the vector of these decisions. Firm i's total cost is now given by $C_i(\mathbf{q}_i; I_i)$, and the demand for firm i's goods is $\mathbf{D}_i(\mathbf{p}; \mathbf{I}) = (D_i^1(\mathbf{p}; \mathbf{I}), ..., D_i^{m_i}(\mathbf{p}; \mathbf{I}))$. As before, we will assume that, for all $\mathbf{I} \in \mathcal{I}$ and $i \in \mathcal{N}$, $\mathbf{D}_i(\cdot)$ and $C_i(\cdot)$ are both C^2 and that, for every $i \in \mathcal{N}$:

- the profit function $\pi_i(\mathbf{p}; \mathbf{I}) \equiv \sum_{j \in \mathcal{M}_i} p_i^j D_i^j(\mathbf{p}; \mathbf{I}) - C_i(\mathbf{D}_i(\mathbf{p}; \mathbf{I}); I_i)$ is strictly quasi-concave in $\mathbf{p}_i$;

- for every $j \in \mathcal{M}_j$, the "product-by-product" best-response function $r_i^j\left(\mathbf{p}_i^{\mathcal{M}_i\backslash\{j\}}, \mathbf{p}_{-i}; \mathbf{I}\right) \equiv \arg\max_{p_i^j} \pi_i\left(p_i^j, \mathbf{p}_i^{\mathcal{M}_i\backslash\{j\}}, \mathbf{p}_{-i}; \mathbf{I}\right)$ is well-defined and bounded above.

We further focus on substitutes and strategic complementarity, and assume price equilibrium uniqueness:

- For every $i \in \mathcal{N}$ and any $\mathbf{I} \in \mathcal{I}$:
  - (**S**) products are *substitutes*: $\partial_{p_j} D_i(\cdot) < 0$ for $j \neq i \in \mathcal{N}$.
  - (**SC**) prices are *strategic complements:* $\partial_{p_j} R_i(\cdot) > 0$ for $j \neq i \in \mathcal{N}$.
- For any investment decisions $\mathbf{I} \in \mathcal{I}$, in the absence of price caps there exists a unique Nash equilibrium in prices, which we denote by $\mathbf{p}^{\mathbf{I}} = \left(p_i^{\mathbf{I}}\right)_{i \in \mathcal{N}}$.

Suppose that investment decisions are publicly made in stage $2a$, and firms can then agree on price caps in stage $2b$, before setting prices in stage 3. From Proposition 8, for any vector of investment decisions $\mathbf{I} \in \mathcal{I}$ made in stage $2a$, firms have no incentive to adopt price-caps agreements in stage $2b$; therefore, in stage 3 the continuation price equilibrium is $\mathbf{p}^{\mathbf{I}}$, as in when price caps are not allowed. It follows that thus allowing price caps in stage 1 has no impact on the set of investment and price equilibria in stages 2 and 3.

H.2 Complements

H.2.1 On Assumption C

Suppliers of complements can always sign a mutually profitable agreement that benefit all of them: as shown in the proof of Corollary 1, starting from the Nash equilibrium prices $\mathbf{p}^N$, reducing all prices by a small amount ε is sustainable through price caps, and it increases all firms' profits, as firms' margins are positive from Lemma 2, and reducing one firm's price has only a second-order effect on the profit of that firm, and a first-order positive effect on the other firms' profits.

Other agreements may not share this feature: The price-caps agreement signed by a coalition of firms may benefit them, but hurt others. For example, suppose that there are three firms $i = 1, 2, 3$ producing at no cost products $1, 2, 3$ respectively, and facing demand $D_i(\mathbf{p}) = d_i(p_{i-1}) - p_i$ (with the convention that $0 = 3$), where $d_i(\cdot) > 0 > d_i'(\cdot)$ (that is, product i is a complement for product $i+1$). It is easy to show that this leads to best-responses $R_i(\mathbf{p}_{-i}) = d_i(p_{i-1})/2$ and to a unique Nash equilibrium $\mathbf{p}^N$.[27] Introducing

[27] To see this, it suffices to note that $\rho(p_1) \equiv R_1(R_3(R_2(p_1))) - p_1 = d_1(d_3(d_2(p_1)/2)/2)/2 - p_1$ is such that $\rho(0) > 0$ and $\rho'(p_1) = d_1'(\cdot)\, d_3'(\cdot)\, d_2'(p_1)/8 - 1 < -1$. Therefore, there exists a unique p_1^N satisfying $\rho(p_1) = 0$; the price vector $\left(p_1^N, p_2^N \equiv d_2\left(p_1^N\right)/2, p_3^N \equiv d_3\left(p_2^N\right)/2\right)$ then constitutes the unique Nash equilibrium.

a price cap $\bar{p}_1$ slightly below p_1^N then induces firm 1 to set $p_1 = \bar{p}_1$, which in turn leads firm 2 to raise p_2 slightly above p_2^N and firm 3 to reduce p_3 slightly below p_3^N. As a result, firms 1 and 2 benefit from the introduction of the cap (as their demands are boosted by the reductions in p_1 and p_3, respectively, and because they either best-respond or are close to their best-response), whereas firm 3 is hurt (as its demand is harmed by the increase in p_2).

The fact that price caps may benefit a coalition at the expense of outsiders need not imply that they will be adopted by the coalition, however; other agreements may be more profitable and benefit others as well. For example, in the above example, introducing the price cap $\bar{p}_1$ led to small reductions $\Delta_1 \left(= p_1^N - \bar{p}_1\right)$ and Δ_3 in the prices of firms 1 and 3, and to a small increase in the price of firm 2; a uniform slight reduction in all three prices, by $\Delta \equiv \max\{\Delta_1, \Delta_3\}$, would instead benefit firm 3 as well, while giving at least the same benefits to firms 1 and 2. We will not develop here a full-fledged model of negotiations over price caps, and simply assume that any price-caps agreement benefits non-signatories:

Assumption C: Any active unconstrained firm is at least as well off when other active firms are constrained by price caps than when all active firms are unconstrained.

Intuitively, this Assumption is likely to hold when firms are in a rather symmetric position. For example, under (**SS**) it holds for symmetric demands with an "aggregative" nature, that is, when there exist an aggregator $A(p_1, ..., p_{n-1})$, which is symmetric and increasing in all prices, and a function $D(p, A)$, which decreases with both p and A, such that

$$D_i(\mathbf{p}) = D(p_i, A(\mathbf{p}_{-i})).$$

A classic example is the linear demand $D_i(\mathbf{p}) = d - ap_i - b\sum_{j\in\mathcal{N}\setminus\{i\}} p_j$ (with $d > 0$ and $a > b > 0$).[28]

Suppose for simplicity that all firms are active.[29] In the absence of price caps, the resulting equilibrium is symmetric ($p_i = p^N$ for all $i \in \mathcal{N}$) and satisfies $p^N = R\left(\mathbf{p}_{-i}^N\right)$; we will assume that the symmetric best-response satisfies $\partial_1 R(\cdot) \in (-1, 0)$;[30] that is, prices are strategic substitutes (**SS**), but they do not respond excessively to each other – the latter condition is implied by the usual stability condition which requires $\sum_{j\in\mathcal{N}\setminus\{i\}} \partial_j R(\mathbf{p}_{-i}) > -1$.

Suppose now that firms are constrained by price caps $\{\bar{p}_i\}_{i\in\mathcal{N}}$, where $\bar{p}_i = +\infty$ for at

[28] For substitutes, such demand systems include multinomial logit ($D_i(\mathbf{p}) = d\exp(a - bp_i) / \sum_{j\in\mathcal{N}} \exp(a - bp_j)$, with $a, b, d > 0$) and CES ($D_i(\mathbf{p}) = dp_i^{-\sigma} / \sum_{j\in\mathcal{N}} ap_j^{1-\sigma}$, with $a, d > 0$ and $\sigma > 1$). See Nocke and Schutz (2017) for a recent analysis of such aggregative demand systems.

[29] The reasoning applies to any smaller set of active firms, with the convention that $p_i = +\infty$ for any inactive firms.

[30] $\partial_1 R$ denotes here the partial derivative of $R(\cdot)$ with respect to its first argument; by symmetry, it applies to the other arguments as well.

least one firm, and let $\hat{\mathbf{p}}$ and $\hat{\boldsymbol{\pi}}$ denote the resulting equilibrium prices and profits. We first note that all unconstrained firms charge the same price $\hat{p}$. To see this, suppose that two unconstrained firms i and j charge different prices, e.g., $\hat{p}_i > \hat{p}_j$. Using symmetry, we then have:

$$\begin{aligned}
\hat{p}_i - \hat{p}_j &= R(\hat{\mathbf{p}}_{-i}) - R(\hat{\mathbf{p}}_{-j}) \\
&= R\left(\hat{p}_j, \hat{\mathbf{p}}_{-\{i,j\}}\right) - R\left(\hat{p}_i, \hat{\mathbf{p}}_{-\{i,j\}}\right) \\
&= \int_{\hat{p}_i}^{\hat{p}_j} \partial_1 R\left(p, \hat{\mathbf{p}}_{-\{i,j\}}\right) dp \\
&< \hat{p}_i - \hat{p}_j,
\end{aligned}$$

a contradiction.

Next, we show that the symmetric unconstrained price $\hat{p}$ lies above the highest binding price cap. To see this, let $\mathcal{C}$ denote the set of firms for which the price cap is binding, $\bar{\imath}$ denote the firm with the highest binding price cap (that is, $\bar{p}_{\bar{\imath}} = \max_{i\in\mathcal{C}}\{\bar{p}_i\}$), and suppose that $\hat{p} < \bar{p}_{\bar{\imath}}$. We then have $\bar{p}_{\bar{\imath}} \leq R(\hat{\mathbf{p}}_{-\bar{\imath}})$ and $\hat{p} = R(\hat{\mathbf{p}}_{-i})$ for any $i \in \mathcal{N}\setminus\mathcal{C}$; therefore:

$$\begin{aligned}
\bar{p}_{\bar{\imath}} - \hat{p} &\leq R(\hat{\mathbf{p}}_{-\bar{\imath}}) - R(\hat{\mathbf{p}}_{-i}) \\
&= R\left(\hat{p}, \hat{\mathbf{p}}_{-\{i,\bar{\imath}\}}\right) - R\left(\bar{p}_{\bar{\imath}}, \hat{\mathbf{p}}_{-\{i,\bar{\imath}\}}\right) \\
&= \int_{\bar{p}_{\bar{\imath}}}^{\hat{p}} \partial_1 R\left(p, \hat{\mathbf{p}}_{-\{i,\bar{\imath}\}}\right) dp \\
&< \bar{p}_{\bar{\imath}} - \hat{p},
\end{aligned}$$

a contradiction.

We thus have $p_i = \bar{p}_i \leq \hat{p}$ for all $i \in \mathcal{C}$, and $p_i = \hat{p}$ for all $i \in \mathcal{N}\setminus\mathcal{C}$. From (**SS**), $\hat{p} < p^N$ would then imply $\hat{p}_i < p_i^N$ for all $i \in \mathcal{N}$ and thus, for any $j \in \mathcal{N}\setminus\mathcal{C}$: $\hat{p} = R(\hat{\mathbf{p}}_{-i}) > R\left(\mathbf{p}_{-i}^N\right) = p^N$, a contradiction. Therefore, $\hat{p} \geq p^N$. Finally, for aggregative games the best-response $R(\cdot)$ is of the form $R(\mathbf{p}_{-i}) = \hat{R}(A(\mathbf{p}_{-i}))$, where $\hat{R}' < 0$ from (**SS**). Therefore, for any unconstrained firm $i \in \mathcal{N}\setminus\mathcal{C}$: $A(\mathbf{p}_{-i}) = \hat{R}^{-1}(\hat{p}) \leq \hat{R}^{-1}\left(p^N\right) = A\left(\mathbf{p}_{-i}^N\right)$, implying that firm i obtains at least as much profit as in the unconstrained Nash equilibrium.[31]

H.2.2 Entry/exit game

Consider a setting in which each firm $i \in \mathcal{N}$ must decide whether to enter (or stay in) the market and suppose further that Assumption C holds. We first note that this assumption implies that, as intuition suggests, the development of complementary products boosts demand and enhances profits. To see this, consider two situations which only differ in that one firm (firm i, say), is either active or not, and let $\hat{p}_i$ denote firm i's (unconstrained)

[31] To see this, note that the price at which firm i can sell any quantity q_i is decreasing with $A(\mathbf{p}_{-i})$.

equilibrium price when it is active. From the standpoint of the other firms, the entry of firm i has the same impact as imposing a price cap $\bar{p}_i = \hat{p}_i$ on a firm producing the same good as firm i but with a very large marginal cost: that firm would thus charge $+\infty$ in the absence of a cap, and $\hat{p}_i$ when facing the cap.

Let $\mathcal{A}$ denote the set of active firms in an equilibrium that arises in the absence of price caps; each firm in $\mathcal{A}$ is thus better off being active (given the presence of the others), and it would benefit from the presence of any additional firms. Hence, if price caps are now allowed, each firm in $\mathcal{A}$ finds it profitable to be active if the others do, regardless of the decisions of firms outside $\mathcal{A}$, and regardless of any price caps that the other firms may agree to. The possibility of price caps can moreover be used to increase all active firms' profits (as noted at the beginning of Section H.2.1), and from Assumption C this may induce some of the outsiders to enter.

Summarizing this discussion yields Proposition 10.

I Pre-investment price caps

A potential concern is the use of (artificially low) price caps as a way of softening competition, by inducing exit, deterring entry or stifling investment.

A first issue is the possible use of price caps as a commitment to maintain low prices, so as to deter entry or discourage investment. This is indeed a serious concern if firms can sign long-term contracts with their customers: firms could then credibly commit themselves to maintain low prices, for example, by adopting most favored nation clauses promising a compensation for any price increase: this would *de facto* allow customers to buy at the initially agreed price caps, even if these caps are then renegotiated away. In such a case, incumbent firms could adopt low price caps so as to deter entry, as in the limit pricing model of Sylos Labini (1957) and Modigliani (1958). Ruling out this possibility leads to:

Policy recommendation 1: Customers are not part of the price-caps agreements.

Second, a low price cap (possibly against compensation) may act as a commitment to exit the market. Indeed, if firm i accepts a price cap $\bar{p}_i$ that is too low for operating profitably (e.g., lower than its minimum average cost), it will then choose to leave the market. Firms could therefore use such price-caps agreements so as to bribe some rivals out of the market; likewise, incumbent firms could induce potential entrants to stay out of the market.[32] These commitments are not credible, however, as the firms would have no incentive to enforce the price caps. Taking advantage of these incentives leads to:

[32] Consider for example a symmetric duopoly in which each firm faces a constant marginal cost c and a fixed cost $f > 0$, and obtains a gross profit $\pi^D > f$. If either firm were alone, it would instead obtain the monopoly profit π^M, where, due to competition $\pi^M > 2\pi^D$. Firm 1, say, would then have an incentive to "bribe" firm 2 into a price-caps agreement of the form $\left(\bar{p}_1 \geq p^M, \bar{p}_2 \leq c\right)$: this would induce firm 2 to exit, and enable firm 1 to increase its profit by $\Delta\pi_1 = \pi^M - \pi^D$; as $\Delta\pi_1 > \pi_2 = \pi^D - f$, these price caps,

Policy recommendation 2: The agreement becomes void if none of the parties wishes to enforce it.

This requirement implies that, in order to remain in place, a price-caps agreement must be "confirmed" by at least one party to the agreement; this contributes to undermine the credibility of the "threats" discussed above.

We now show that these policy recommendations indeed alleviate the above concerns about the use of price caps as a way to deter entry or stifle investment.

Consider the same setting as in Section H.1 (with quasi-concavity, strategic complementarity and unique continuation price equilibria), but modify the overall game $\mathcal{G}$ as follows:

2. (a) Firms choose price caps if such agreements are allowed.

   (b) Firms make observable investment or entry decisions.

3. (a) If an agreement has been signed, firms choose whether to confirm it; the agreement is enforced if and only if at least one firm confirms it.

   (b) Firms set their prices.

This timing allows the firms to sign price-caps agreements in order to influence investment decisions, but rules out non-credible threats by asking firms to confirm their willingness to enforce the agreement, once investment decisions have been made. To avoid coordination issues, we rule out weakly dominated strategies.

From Proposition 8, for any vector of investment decisions $\mathbf{I} \in \mathcal{I}$ made in stage $2b$, in stage $3a$ firms have no incentive to enforce any price-caps agreement, regardless of what they may have agreed to in stage $2a$; therefore, in stage $3b$ the continuation price equilibrium is $\mathbf{p}^{\mathbf{I}}$, as when price caps are not allowed. It follows that allowing price caps in stage 1 has no impact on the set of investment and price equilibria in stages 2 and 3.

Summarizing the above analysis yields:

Proposition 23 (commitment effects of price caps) *Consider the following two prerequisites to setting price caps: a) customers are not part of the price-caps agreements; b) the agreement becomes void if none of the parties wishes to enforce it. Then, when* (**S**) *and* (**SC**) *hold, price caps have no impact on investment/entry/exit and therefore no impact on consumers.*

together with any transfer $T \in (\pi_2, \Delta\pi_1)$, would make both firms better off. Furthermore, transfers may no longer be needed when several markets are involved, as price caps could then be used to divide these markets in a mutually profitable way.

J Foreclosure

Let us first recall the jest of the Choi-Stefanadis foreclosure model.

• *Integration and foreclosure.* There are two firms: an incumbent and a potential entrant. An integrated incumbent costlessly produces two perfect complements, A and B. We first assume an inelastic demand for the system: A and B together bring value v to consumers. An entrant can invest I to develop with probability $\rho \in (0,1)$ product A', which is an alternative to A and brings extra surplus $\Delta \in (I/\rho, \min\{I/\rho^2, v\})$; and similarly with product B' (A' and B' combined thus deliver consumer value $v + 2\Delta$). The two R&D processes are independent.

Prior to the entrant deciding whether to undertake R&D on A', B' or both, the incumbent makes a technological choice: it can choose an open standard, in which case consumers can mix and match developed products as they like (e.g., combine A and B', for value $v + \Delta$); alternatively, it can choose a closed standard, in which case A and B can only be consumed together (combining A and B', say, thus brings no value). The technological choice is costless. It is easy to check that the entrant always invests in both markets or none.

The entrant obtains no profit when both R&D projects fail, and 2Δ when both succeed. When a single R&D project succeeds, under a closed standard the entrant obtains again no profit. Under an open standard, the incumbent and the entrant are in a Nash demand game. As Nash observed, introducing a small noise on consumer valuation would deliver equal sharing $(v + \Delta)/2$, except that the incumbent can secure v (regardless of the entrant's price) by charging v for the monopolized component and offering the other one at cost. As $\Delta < v$, the resulting outcome is that the incumbent obtains the base value v and the entrant obtains its added value Δ.

It follows that, under an open standard, the entrant invests and obtains a profit equal to

$$\rho^2 (2\Delta) + 2\rho(1-\rho)(\Delta) + (1-\rho)^2 (0) - 2I = 2(\rho\Delta - I) > 0,$$

and the incumbent thus obtains:

$$\rho^2 (0) + 2\rho(1-\rho) v + (1-\rho)^2 (v) = \left(1 - \rho^2\right) v.$$

Under a closed standard, entry becomes riskier and the entrant does not invest, as

$$\rho^2 (2\Delta) - 2I = 2\left(\rho^2\Delta - I\right) < 0.$$

The incumbent's profit is then $v > (1-\rho^2) v$. So the incumbent is better off preventing investment by choosing the closed standard, as entry may lead to full system competition.

• *Absence of merger.* Suppose now that A and B are produced by two distinct incumbent firms, a and b, and each can choose an open or closed standard. It is then a dominant strategy for the incumbent firms to choose the open standard. For example, if A' is developed but not B', then with an open standard b can appropriate the entire surplus v: the entrant charges Δ, a charges 0, and b can charge v as consumers are willing to pay $v+\Delta$ for the pair $\{A', B\}$; with a closed standard, b can only appropriate $v/2$.[33] If A' is not developed, or both A' and B' are developed, then b's choice of technology is irrelevant. We thus conclude that no foreclosure occurs under separate ownership. Furthermore, allowing price-caps agreements does not enable the incumbents to deter entry, as the adoption of price caps can only boost the demand for the alternative components.

• *Downward sloping demand.* An elastic demand creates a private and social benefit from either a merger or price caps: the elimination of the double marginalization. However, price caps are a socially superior way of avoiding double marginalization, as they do not enable foreclosure.

To illustrate this, suppose that:

- There are two types of consumers: a fraction f have value v_H, whereas the others have value v_L, where $0 < v_L < v_H$ and there is double marginalization by independent producers:
$$v_L > f v_H > \frac{1+f}{2} v_L. \tag{15}$$
- The R&D cost can take two values, 0 (with probability γ) and I (with probability $1-\gamma$).

We consider three scenarios: (i) in the *benchmark* case, the two components are initially produced by independent firms; (ii) in case of a *merger*, these incumbents are integrated; (iii) in the *price caps* scenario, the independent incumbents can enter into price-caps agreements. The timing is as follows.

- Stage 0: incumbent firms choose between open and closed standards.
- Stage 1: the entrant decides whether to invest.
- Stage 2: R&D outcomes are observed by all firms; in the last scenario, they can moreover agree on price caps
- Stage 3: firms set their prices.

From the above analysis, when both alternative components are developed, price competition drives the incumbent firms' prices down to 0 (note that incumbents would never agree on negative price caps). When a single alternative component is developed and the monopolized component opted for an open standard (which is the case in the absence of a merger), price competition drives again the price of the incumbent component

[33] The two incumbents are again in a Nash demand game, and introducing a small noise on consumer valuation then delivers equal sharing.

down to 0. The producer of the monopolized component (or the integrated firm, in case of a merger) charges v_L as, from first inequality in (15), the corresponding profit, v_L, exceeds the profit derived from targeting the high-end segment, fv_H. The entrant obtains Δ under an open standard, and 0 otherwise.

In the absence of any alternative component, the equilibrium prices vary across scenarios. In the benchmark case, each incumbent charges $v_H/2$ and thus obtains a profit equal to $fv_H/2$. This indeed constitutes an equilibrium, as serving the low-end segment would require charging $v_L - v_H/2$ and thus generate a profit $v_L - v_H/2$, which, from the second inequality in (15), is lower than $fv_H/2$. To see that each incumbent charging $v_L/2$ and obtaining a profit $v_L/2$ is not an equilibrium,[34] it suffices to note that deviating and targeting the high-end segment would generate a profit equal to $f(v_H - v_L/2)$, which under (15) is higher than $v_L/2$.

In case of a merger, the integrated incumbent charges v_L as, from first inequality in (15), the corresponding profit, v_L, exceeds the profit derived from targeting the high-end segment, fv_H. For the same reason, in the price caps scenario, the incumbents agree on price caps equal to $v_L/2$.

Building on these insights:

- In the benchmark scenario, the entrant invests and consumers obtain an expected surplus equal to:

$$S^* = 2\rho(1-\rho) f(v_H - v_L) + \rho^2 [fv_H + (1-f) v_L].$$

- In the merger scenario, the integrated incumbent opts for a closed standard and the entrant does not invest; the merger however eliminates double marginalization and consumers thus obtain an expected surplus equal to:

$$\begin{aligned} S^m &= f(v_H - v_L) \\ &= S^* + (1-\rho)^2 f(v_H - v_L) - \rho^2 v_L. \end{aligned}$$

- In the price caps scenario, the entrant again invests *and* price caps eliminate double marginalization when R&D projects fail; as a result, consumers obtain an expected surplus equal:

$$\begin{aligned} S^p &= (1-\rho)^2 f(v_H - v_L) + 2\rho(1-\rho) f(v_H - v_L) + \rho^2 [fv_H + (1-f) v_L] \\ &= S^* + (1-\rho)^2 f(v_H - v_L) \\ &= S^m + \rho^2 v_L. \end{aligned}$$

Whether the merger benefits consumers depend on the balance between the ex-

[34] The same argument as before allows us to focus on symmetric equilibria.

pected gain from eliminating double marginalization in case of failed R&D projects, $(1-\rho)^2 f(v_H - v_L)$, and the harm resulting from the loss of competition in case of successful projects, $\rho^2 v_L$. By contrast, allowing price caps enables the firms to eliminate double marginalization without giving them incentives to opt for closed standards and deter entry in the other component. Hence, price caps do benefit consumers, and constitute a better alternative to mergers.

• *Bundling and price squeezes.*

Finally, let us ignore technological choices (that is, the only standard is an open one) but assume that the incumbent can commit to specific pricing policies.[35] A first option it to engage in *pure bundling* (i.e., to sell the two products only as a bundle). This is irrelevant when the entrant develops 0 or 2 products, but forces consumers to buy the bundle $\{A, B\}$ when the entrant develops only one product; contrary to the case of a closed standard, the entrant can still sell its component (which consumers can use as a replacement of the bundled component), but the profitability of doing so however depends on the level of production costs. To see this, suppose now that all goods are produced at the same constant unit cost c, and interpret the above values as "net" of this cost. Absent entry, the incumbent sells the bundle at price $2c+v$; when instead the entrant develops both products, it sells them at total price $2c+\Delta$. Consider now the case when A', say, is developed but not B'. Absent bundling, the incumbent sells B at price $c+v$ and the entrant sells A' at price $c+\Delta$. In case of bundling, the incumbent sells A and B at bundled price $2c+v$; the entrant can then sell A' at price Δ, but earns a profit of $\Delta - c$ (instead of Δ, absent bundling). Thus, when c is high (e.g., $c > \Delta$), bundling plays the same role of a closed standard: it deprives the entrant of a profit when it develops a single product, and therefore deters entry.

Another option for the incumbent is to commit to (entry-contingent) prices. Even if it is constrained by a system-wide no-loss condition (e.g., if regulators can demonstrate the existence of a financial loss, although not on a given product line, cross-subsidies being hard to monitor), it can use this instrument to extract (all of part of) the added value brought by the entrant when a single R&D project succeeds. For example, when A' is developed but not B', offering A at below-cost price $c-s$, where $s \in [0, \Delta]$, forces the entrant to sell A' at price $\Delta - s$. Opting for s close to Δ thus acts like bundling or the choice an incompatible technology: the price squeeze deprives the entrant of any profit when it develops a single product and not the entire system itself, which deters entry. However, as entry is welfare-enhancing, a better option consists in setting s so as to induce the entrant to invest, and appropriate all or most of the expected profit.[36]

• *Sequential entry.* Carlton and Waldman (2002) consider a related setting, and show that an integrated incumbent may again deter entry when it is sequential rather than

[35] One may have in mind an incumbent facing repeated entry in new segments and developing a reputation for these practices.

[36] That is, s should be set to that $\rho(1-\rho)s = \rho\Delta - I$.

uncertain. To see this, consider a two-period ($t = 1, 2$) variant of the above setting in which: (i) R&D is always successful (i.e., $\rho = 1$); and (ii) developing B' is possible in both periods, at cost I_B, whereas developing A' can only take place in period 2, at cost I_A. In this case, if the development costs satisfy $I_A/\Delta < 1$ and $I_B/\Delta < 1 + \delta$, then with an open standard E develops B' in period 1 and A' in period 2, but if in addition $(I_A + I_B)/\Delta > 2$, then, with a closed standard, E does not develop any product. By contrast, in case of independent incumbents, b's choice of standard is irrelevant, and a opts for an open standard, inducing entry, in order to appropriate the full value v in the first period: in this way, a obtains a profit equal to v, which exceeds its total discounted profit under foreclosure, which is equal to $(1 + \delta)\, v/2$.

Proposition 24 (price caps versus mergers: foreclosure concerns) *In the Choi-Stefanadis and Carlton-Waldman frameworks, a merger of complements allows foreclosure while price caps do not. Accordingly, price caps are a socially superior way of handling double marginalization.*

K Repeated interaction in the technology adoption model

Suppose that the firms play the technology adoption game repeatedly, with discount factor $\delta \in (0, 1)$. Let

$$v \equiv (1 - \delta)\, \Sigma_{t \geq 0} \delta^t \frac{\pi_1^t + \pi_2^t}{2}$$

denote the average of firms' discounted profits over a pure-strategy equilibrium path, $\mathcal{V}^+$ denote the set of these equilibrium payoffs that are weakly more profitable than Nash (i.e., such that $v \geq \pi^N$), and v^* denote the maximal equilibrium payoff.[37] Tacit coordination raises profits only if $v^* > \pi^N$.

The location of e affects not only the nature of tacit coordination, but also the minmax profit:

Lemma 9 (minmax) *Let $\underline{\pi}$ denote the minmax profit.*

(i) If $e \leq \hat{p}$, the static Nash equilibrium (e, e) gives each firm the minmax profit: $\underline{\pi} = \pi^N = \pi(e)$.

(ii) If $e > \hat{p}$, the minmax profit is the incomplete-technology per-period monopoly profit: $\underline{\pi} = \tilde{\pi}^M(e) < \pi^N = \pi(\hat{p})$.

[37] This maximum is well defined, as the set $\mathcal{V}^+$ of Nash-dominating subgame perfect equilibrium payoffs is non-empty (it includes π^N) and compact (see Mailath and Samuelson (2006), chapter 2). Also, although we restrict attention to pure-strategy subgame perfect equilibria here, the analysis could be extended to public mixed strategies (where players condition their strategies on public signals) or, in the case of private mixed strategies, to perfect public equilibria (relying on strategies that do not condition future actions on private past history); see Mailath and Samuelson (2006), chapter 7.

Proof. To establish part (i), note that firm i can secure its presence in the users' basket by charging e, thus obtaining $eD(e+p_j)$ if $p_j \leq e$ and $eD(2e)$ if $p_j > e$. Either way it can secure at least $\pi(e) = eD(2e)$. Because for $e \leq \hat{p}$ this lower bound is equal to the static Nash profit, we have $\underline{\pi} = \pi^N = \pi(e)$.

We now turn to part (ii). If firm j sets a price $p_j \geq e$, firm i can obtain at most $\max_{p \leq p_j} pD(e+p) = \tilde{\pi}^M(e)$ (as $\tilde{p}^M(e) = r(e) < \hat{p} < e \leq p_j$). Setting instead a price $p_j < e$ allows firm i to obtain at least $\max_{p \leq e} pD(p_j + p) > \max_{p \leq e} pD(e+p) = \tilde{\pi}^M(e)$. Therefore, setting any price above e minmaxes firm i, which then obtains $\tilde{\pi}^M(e)$. ■

Hence, when $e \leq \hat{p}$, the static Nash equilibrium (e, e) yields the minmax profit; it thus constitutes the toughest punishment for both firms. When instead $e > \hat{p}$, each firm can guarantee itself the incomplete-technology monopoly profit $\tilde{\pi}^M(e)$, which is then lower than the profit of the static Nash equilibrium $(\hat{p}, \hat{p})$; Abreu (1988)'s optimal penal codes can then be used to sustain the toughest punishment.

We now characterize the scope for tacit coordination in the case of rivalry and of complementors.

a) Rivalry: $p^N < p^M$ This case arises when $e < p^M$, implying $p^N = e$ and $\underline{\pi} = \pi^N = \pi(e)$; collusion then implies selling the incomplete technology, and the loss in demand due to partial consumption grows with essentiality. In particular, if e is close to p^M, the Nash equilibrium payoff $\pi(e)$ approaches the highest possible profit π^M, whereas pricing above e substantially reduces the demand for the patents; as each firm can guarantee itself $\pi(e)$, there is no collusion. Specifically, this occurs when patents are weak substitutes, namely, when $e \geq \underline{e}$, where $\underline{e}$ is the unique solution to

$$\tilde{\pi}^M(\underline{e}) = 2\pi(\underline{e}).$$

By contrast, for e close to 0, this loss in demand is small and the Nash profit is negligible; and so collusion, if feasible, is attractive for the firms. Because users then buy only one license, each firm can attract all users by slightly undercutting the collusive price. Like in standard Bertrand oligopolies, *maximal* collusion (on $\tilde{p}^M(e)$) is sustainable whenever *some* collusion is sustainable. As symmetric collusion is easier to sustain, and deviations are optimally punished by reverting to static Nash behavior, such collusion is indeed sustainable if:

$$\frac{\tilde{\pi}^M(e)}{2} \geq (1-\delta)\,\tilde{\pi}^M(e) + \delta\pi(e) \iff \delta \geq \delta^R(e) \equiv \frac{1}{2}\frac{1}{1 - \frac{\pi(e)}{\tilde{\pi}^M(e)}}, \tag{16}$$

where $\delta^R(e)$ is increasing in e and exceeds 1 for $e \geq \underline{e}$. Building on these insights, we have:

Proposition 25 (rivalry) *When $e < \underline{e}$ and $\delta \geq \delta^R(e)$, $\mathcal{V}^+ = \left[\pi^N, v^*\right]$, and $v^* = \tilde{\pi}^M(e)/2$: tacit collusion is feasible and the most profitable collusion occurs at price $\tilde{p}^M(e)$; otherwise, the unique equilibrium is the repetition of the static Nash one.*

Proof. Let $\pi_i(p_i, p_j)$ denote firm i's profit. Prices such that $\min\{p_1, p_2\} \leq e$ cannot yield greater profits than the static Nash:

- If $p_1, p_2 \leq e$, total price P is below $2e$; as the aggregate profit $PD(P)$ is concave in P and maximal for $P^M = 2p^M > 2e$, total profit is smaller than the Nash level.
- If instead $p_i \leq e < p_j$, then

$$\pi_1(p_1, p_2) + \pi_2(p_2, p_1) = p_i D(e + p_i) \leq eD(2e) \leq 2eD(2e) = 2\pi^N,$$

  where the first inequality stems from the fact that the profit $\tilde{\pi}(p) = pD(e+p)$ is concave in p and maximal for $\tilde{p}^M(e) = r(e)$, which exceeds e in the rivalry case (as then $e < p^M < \hat{p} = r(\hat{p})$).

Therefore, to generate more profits than the static Nash profit in a given period, both firms must charge more than e; this, in turn, implies that users buy at most one license, and thus aggregate profits cannot exceed $\tilde{\pi}^M(e)$. It follows that collusion cannot enhance profits if $\tilde{\pi}^M(e) \leq 2\pi^N = 2\pi(e)$. Keeping V and thus p^M constant, increasing e from 0 to p^M decreases $\tilde{\pi}^M(e) = \max_p pD(p+e)$ but increases $\pi(e)$; as $\tilde{\pi}^M(0) = 2\pi\left(p^M\right) = 2\pi^M$, there exists a unique $\underline{e} < p^M$ such that, in the range $e \in \left[0, p^M\right]$, $\tilde{\pi}^M(e) < 2\pi^N$ if and only if $e > \underline{e}$.

Thus, when $e > \underline{e}$, the static Nash payoff π^N constitutes an upper bound on average discounted equilibrium payoffs. But the static Nash equilibrium here yields minmax profits, and thus also constitutes a lower bound on equilibrium payoffs. Hence, π^N is the unique average discounted equilibrium payoff, which in turn implies that the static Nash outcome must be played along any equilibrium path.

Consider now the case $e < \underline{e}$, and suppose that collusion raises profits: $v^* > \pi^N$, where, recall, v^* is the maximal average discounted equilibrium payoff. As v^* is a weighted average of per-period profits, along the associated equilibrium path there must exist some period $\tau \geq 0$ in which the aggregate profit, $\pi_1^\tau + \pi_2^\tau$, is at least equal to $2v^*$. This, in turn, implies that users must buy an incomplete version of the technology; thus, there exists p^* such that:

$$\tilde{\pi}(p^*) = \pi_1^\tau + \pi_2^\tau \geq 2v^*.$$

By undercutting its rival, each firm i can obtain the whole profit $\tilde{\pi}(p^*)$ in that period; as this deviation could at most be punished by reverting forever to the static Nash behavior, a necessary equilibrium condition is, for $i = 1, 2$:

$$(1-\delta)\pi_i^\tau + \delta v_i^{\tau+1} \geq (1-\delta)\tilde{\pi}(p^*) + \delta\underline{\pi},$$

where $v_i^{\tau+1}$ denotes firm i's continuation equilibrium payoff from period $\tau+1$ onwards. Combining these conditions for the two firms yields:

$$(1-\delta)\,\tilde{\pi}\left(p^*\right)+\delta\underline{\pi} \leq (1-\delta)\,\frac{\pi_1^\tau+\pi_2^\tau}{2}+\delta\frac{v_1^{\tau+1}+v_2^{\tau+1}}{2} \leq (1-\delta)\,\frac{\tilde{\pi}\left(p^*\right)}{2}+\delta\frac{\tilde{\pi}\left(p^*\right)}{2},$$

where the second inequality stems from $v_1^{\tau+1}+v_2^{\tau+1} \leq 2v^* \leq \pi_1^\tau+\pi_2^\tau = \tilde{\pi}\left(p^*\right)$. This condition amounts to

$$\left(\delta-\frac{1}{2}\right)\tilde{\pi}\left(p^*\right) \geq \delta\underline{\pi} = \delta\pi\left(e\right), \tag{17}$$

which requires $\delta \geq 1/2$ (with a strict inequality if $e>0$). This, in turn, implies that (17) must hold for $\tilde{\pi}^M\left(e\right)=\max_p \tilde{\pi}\left(p\right)$:

$$\left(\delta-\frac{1}{2}\right)\tilde{\pi}^M\left(e\right) \geq \delta\pi\left(e\right). \tag{18}$$

Conversely, if (18) is satisfied, then the stationary path $\left(\tilde{p}^M(e),\tilde{p}^M(e)\right)$ (with equal market shares) is an equilibrium path, as the threat of reverting to the static Nash behavior ensures that no firm has an incentive to deviate:

$$\frac{\tilde{\pi}^M\left(e\right)}{2} \geq (1-\delta)\,\tilde{\pi}^M\left(e\right)+\delta\pi\left(e\right),$$

or

$$\delta \geq \delta^R\left(e\right) \equiv \frac{1}{2}\frac{1}{1-\frac{\pi(e)}{\tilde{\pi}^M(e)}}.$$

Finally, $\delta^R\left(e\right)$ increases with e, as $\pi\left(e\right)$ increases with e in that range, whereas $\tilde{\pi}^M(e)=\max_p\{pD\left(p+e\right)\}$ decreases as e increases. ■

Hence, greater essentiality hinders collusion, which is not feasible if $e \geq \underline{e}$; furthermore, as the threshold $\delta^R\left(e\right)$ increases with e, for any given $\delta \in (1/2,1)$, in the entire rivalry range $e \in \left[0,p^M\right]$ there exits a unique $\hat{e}\left(\delta\right) \in (0,\underline{e})$ such that collusion is feasible if and only if $e<\hat{e}\left(\delta\right)$. This is because the toughest punishment, given by the static Nash profit, becomes less effective as essentiality increases; although the gains from deviation also decrease, which facilitates collusion, this effect is always dominated.

b) Complementors: $p^M<p^N$ This case arises when $e>p^M$. Like when $e \in [\underline{e},p^M]$, selling the incomplete technology cannot be more profitable than the static Nash outcome.[38] Firms can however increase their profit by *lowering* their price below the Nash level. Furthermore, when demand is convex, it can be checked that cooperation on

[38]This follows from Lemma 9 for $e>\hat{p}$. For $p^M<e\leq\hat{p}$, we have $\tilde{p}^M\left(e\right)=r\left(e\right)\geq r\left(\hat{p}\right)=\hat{p}\geq e$ and thus $e+\tilde{p}^M\left(e\right)\geq 2e>2p^M=P^M$; hence, $\tilde{\pi}^M\left(e\right)=\tilde{p}^M\left(e\right)D\left(e+\tilde{p}^M\left(e\right)\right)<\left(e+\tilde{p}^M\left(e\right)\right)D\left(e+\tilde{p}^M\left(e\right)\right)\leq 2eD\left(2e\right)$, where the first inequality stems from $e>0$ and the second one from the fact that the aggregate profit $PD\left(P\right)$ is concave in P and maximal for $P^M<e+\tilde{p}^M\left(e\right)$.

some total price $P < 2e$ is easiest when it is symmetric (i.e., when $p_i = P/2$). As $p^N = \min\{e, \hat{p}\}$, we can distinguish two cases:

- *Weak complementors:* $e < \hat{p}$, in which case $p^M < p^N \equiv e$. The static Nash equilibrium $p^N = e$ still yields minmax profits and thus remains the toughest punishment in case of deviation. As $p_j \leq e < \hat{p} = r(\hat{p}) < r(p_j)$, firm i's best deviation then consists in charging e. In particular, *perfect* cooperation on p^M is sustainable if and only if:

$$\pi^M \geq (1-\delta)\, eD\left(p^M + e\right) + \delta\pi(e), \tag{19}$$

which is satisfied for δ close enough to 1.

The following proposition characterizes the scope for tacit coordination in this case:

Proposition 26 (weak complementors) *When $p^M < e \leq \hat{p}$:*

(*i*) *Perfect cooperation on price p^M is feasible (i.e., $v^* = \pi^M$) if and only if*

$$\delta \geq \overline{\delta}^C(e) \equiv \frac{eD\left(p^M + e\right) - \pi^M}{eD\left(p^M + e\right) - \pi(e)},$$

where $\overline{\delta}^C(e)$ lies strictly below 1 for $e > p^M$, and is decreasing for e close to p^M.

(*ii*) *Furthermore, if $D'' \geq 0$, then profitable cooperation is sustainable (i.e., $v^* > \pi^N$) if and only if*

$$\delta \geq \underline{\delta}^C(e),$$

where $\underline{\delta}^C(e)$ lies below $\overline{\delta}^C(e)$, is decreasing in e, and is equal to 0 for $e = \hat{p}$. The set of sustainable Nash-dominating per-firm payoffs is then $\mathcal{V}^+ = [\pi(e), v^*(e,\delta)]$, where $v^*(e,\delta) \in (\pi(e), \pi^M]$ is (weakly) increasing in δ.

Proof. (*i*) That perfect cooperation (on $p_i^t = p^M$ for $i = 1,2$ and $t = 0,1,...$) is sustainable if and only if

$$\delta \geq \overline{\delta}^C(e) = \frac{eD\left(p^M + e\right) - \pi^M}{eD\left(p^M + e\right) - \pi(e)} = \frac{1}{1 + \frac{\pi^M - \pi(e)}{eD(p^M+e) - \pi^M}}$$

derives directly from (19).

For $e \in (p^M, \hat{p}]$, $\pi^M > \pi(e)$ and $eD\left(p^M + e\right) > \pi^M$ (as $r\left(p^M\right) > r(e) \geq \hat{p} \geq e$); therefore, $\overline{\delta}^C(e) < 1$. Also, for ε positive but small, we have:

$$\overline{\delta}^C\left(p^M + \varepsilon\right) \simeq \frac{1}{1 - \frac{\pi''(p^M)}{D(2p^M) + p^M D'(2p^M)} \frac{\varepsilon}{2}},$$

which decreases with ε, as $\pi''\left(p^M\right) < 0$ and

$$D\left(2p^M\right) + p^M D'\left(2p^M\right) = -p^M D'\left(2p^M\right) > 0.$$

(ii) Suppose that collusion enhances profits: $v^* > \pi^N = \pi(e)$. In the most profitable collusive equilibrium, there exists again some period τ in which the average profit is at least v^*. And as $v^* > \pi(e) > \tilde{\pi}^M(e)/2$,[39] users must buy the complete technology in that period; thus, each firm i must charge a price p_i^τ not exceeding e, and the average price $p^\tau = \frac{p_1^\tau + p_2^\tau}{2}$ must moreover satisfy

$$\pi(p^\tau) = \frac{\pi_1^\tau + \pi_2^\tau}{2} \geq v^*.$$

As $p_j^\tau \leq e \leq \hat{p} = r(\hat{p}) \leq r(p_j^\tau)$, firm i's best deviation consists in charging e. Hence, to ensure that firm i has no incentive to deviate, we must have:

$$(1-\delta)\pi_i^\tau + \delta v_i^{\tau+1} \geq (1-\delta) eD\left(p_j^\tau + e\right) + \delta \underline{\pi}.$$

Combining these conditions for the two firms yields, using $\pi(p^\tau) = \frac{\pi_1^\tau + \pi_2^\tau}{2}$ and $\underline{\pi} = \pi(e)$:

$$(1-\delta) e \frac{D(p_1^\tau + e) + D(p_2^\tau + e)}{2} + \delta\pi(e) \leq (1-\delta)\pi(p^\tau) + \delta \frac{v_2^{\tau+1} + v_2^{\tau+1}}{2} \leq \pi(p^\tau),$$

where the second inequality stems from $\frac{v_1^{\tau+1} + v_2^{\tau+1}}{2} \leq v^* \leq \pi(p^\tau)$. If the demand function is (weakly) convex (i.e., $D'' \geq 0$ whenever $D > 0$), then this condition implies $H(p^\tau; e, \delta) \geq 0$, where

$$H(p; e, \delta) \equiv \pi(p) - (1-\delta) eD(p+e) - \delta\pi(e). \tag{20}$$

Conversely, if $H(p^*; e, \delta) \geq 0$, then the stationary path (p^*, p^*) is an equilibrium path.

Summing-up, when $D'' \geq 0$, $v^* > \pi^N$ if and only if there exists $p^* < e$ satisfying $\pi(p^*) > \pi^N$ and $H(p^*; e, \delta) \geq 0$. By construction, $H(e; e, \delta) = 0$. In addition,

$$\frac{\partial H}{\partial p}(p; e, \delta) = D(2p) + 2pD'(2p) - (1-\delta) eD'(p+e).$$

Hence, $D'' \geq 0$ and Assumption A (which implies that $PD'(P)$ decreases with P) ensure that

$$\frac{\partial^2 H}{\partial p^2}(p; e, \delta) < 0.$$

Therefore, if $J(e, \delta) \geq 0$, where:

$$J(e, \delta) \equiv \frac{\partial H}{\partial p}(e; e, \delta) = D(2e) + (1+\delta) eD'(2e),$$

then no cooperation is feasible, as then $H(p; e, \delta) < 0$ for $p < e$. Conversely, if $J(e, \delta) < 0$, then tacit cooperation on p^* is feasible for $p^* \in \left[\underline{p}(e, \delta), e\right]$, where $p = \underline{p}(e, \delta)$ is the unique

[39] See footnote 38.

solution (other than $p = e$) to $H(p; e, \delta) = 0$. Note that

$$\frac{\partial J}{\partial \delta}(e, \delta) = eD'(2e) < 0,$$

and

$$J(e, 0) = D(2e) + eD'(2e) \geq 0,$$

as $e \leq \hat{p} \leq r(e)$, whereas

$$J(e, 1) = D(2e) + 2eD'(2e) < 0,$$

as $e > p^M$. Therefore, there exists a unique $\underline{\delta}^C(e)$ such that tacit cooperation can be profitable for $\delta > \underline{\delta}^C(e)$. Furthermore, Assumption A implies that $eD'(2e)$ is decreasing and so

$$\frac{\partial J}{\partial e}(e, \delta) = 2D'(2e) + (1 + \delta)\frac{d}{de}(eD'(2e)) < 0.$$

Hence the threshold $\underline{\delta}^C(e)$ decreases with e; furthermore, $\underline{\delta}^C(\hat{p}) = 0$, as $J(\hat{p}, 0) = D(2\hat{p}) + \hat{p}D'(2\hat{p}) = 0$ (as $\hat{p} = r(\hat{p})$).

Finally, when $\delta > \underline{\delta}^C(e)$, the set of sustainable Nash-dominating per-firm payoffs is $[\pi(e), v^*(e, \delta)]$, where $v^*(e, \delta) \equiv \pi\left(\max\left\{p^M, \underline{p}(e, \delta)\right\}\right)$, and $\underline{p}(e, \delta)$ is the lower solution to $H(p; e, \delta) = 0$; as H increases in δ,[40] $\underline{p}(e, \delta)$ decreases with δ and thus $v^*(e, \delta)$ weakly increases with δ. ■

- *Strong complementors:* $e > \hat{p}$, in which case $p^M < p^N = \hat{p}$. Starting from a symmetric price $p \in \left[p^M, p^N\right]$, the best deviation profit is then given by $\max_{\tilde{p} \leq e} \tilde{p}D(\tilde{p} + p)$. The static Nash equilibrium $(\hat{p}, \hat{p})$ however no longer yields the minmax payoff, equal here to the incomplete-technology monopoly profit: $\underline{\pi} = \tilde{\pi}^M(e)$; Abreu (1988)'s optimal penal codes then provide more severe punishments than the static Nash outcome. If firms are sufficiently patient, these punishments can be as severe as the minmax profits,[41] in which case perfect cooperation on p^M is sustainable if in addition:

$$\pi^M \geq (1 - \delta) \max_{\tilde{p} \leq e} \tilde{p}D(\tilde{p} + p) + \delta\tilde{\pi}^M(e).$$

In order to characterize the scope for tacit coordination in this case, we first show that Abreu's penal codes (even when restricting attention to symmetric on- and off-equilibrium paths) can sustain minmax profits when firms are sufficiently patient:

[40] For any $p < e$:

$$\frac{\partial H}{\partial \delta}(p; e, \delta) = e[D(p + e) - D(2e)] > 0.$$

[41] See Lemma 10 below.

Lemma 10 (minmax with strong complementors) *The minmax payoff is sustainable whenever*

$$\delta \geq \underline{\delta}(e) \equiv \frac{\tilde{\pi}^M(e) - \pi(e)}{\pi(\hat{p}) - \pi(e)},$$

where $\underline{\delta}(e) \in (0,1)$ *for* $e \in (\hat{p}, V)$, *and* $\underline{\delta}(V) = \lim_{e \longrightarrow \hat{p}} \underline{\delta}(e) = 0$.

Proof. In order to sustain the minmax profit $\underline{\pi} = \tilde{\pi}^M(e)$, consider the following two-phase, symmetric penal code. In the first phase (periods $t = 1, ..., T$ for some $T \geq 1$), both firms charge e, so that the profit is equal to $\pi(e)$. In the first period of the second phase (i.e., period $T+1$), with probability $1-x$ both firms charge e, and with probability x they switch to the best collusive price that can be sustained with minmax punishments, which is defined as:

$$p^C(e, \delta) \equiv \arg\max_p pD(2p),$$

subject to the constraint

$$(1-\delta) \max_{\tilde{p} \leq e} \tilde{p} D(p + \tilde{p}) + \delta \underline{\pi} \leq pD(2p). \tag{21}$$

Then, in all following periods, both firms charge p^C. Letting $\Delta = (1-\delta) x \delta^T + \delta^{T+1} \in (0, \delta)$ denote the fraction of (discounted) time in the second phase, the average discounted per-period punishment profit is equal to

$$\pi^p = (1-\Delta)\pi(e) + \Delta \pi\left(p^C\right),$$

which ranges from $\pi(e) < \underline{\pi} = \tilde{\pi}^M(e)$ (for $T = +\infty$) to $(1-\delta)\pi(e) + \delta\pi\left(p^C\right)$ (for $T = 1$ and $x = 1$). Thus, as long as this upper bound exceeds $\tilde{\pi}^M(e)$, there exists $T \geq 1$ and $x \in [0,1]$ such that the penal code yields the minmax: $\pi^p = \tilde{\pi}^M(e) = \underline{\pi}$.

As p^C satisfies (21), the final phase of this penal code (for $t > T+1$, and for $t = T+1$ with probability x) is sustainable. Furthermore, in the first $T+1$ periods the expected payoff increases over time (as the switch to p^C comes closer), whereas the maximal profit from a deviation remains constant and equal to $\max_{p \leq e} pD(e+p) = \tilde{\pi}^M(e)$ (as $\tilde{p}^M(e) = r(e) < e$ for $e > \hat{p}$). Hence, to show that the penal code is sustainable it suffices to check that firms have no incentive to deviate in the first period, which is indeed the case if deviations are punished with the penal code:

$$\tilde{\pi}^M(e) = (1-\Delta)\pi(e) + \Delta\pi\left(p^C\right) \geq (1-\delta)\tilde{\pi}^M(e) + \delta\tilde{\pi}^M(e) = \tilde{\pi}^M(e).$$

There thus exists a penal code sustaining the minmax whenever the upper bound $(1-\delta)\pi(e) + \delta\pi\left(p^C\right)$ exceeds $\tilde{\pi}^M(e)$; as by construction $\pi\left(p^C\right) \geq \pi^N = \pi(\hat{p})$, this is in particular the case whenever

$$(1-\delta)\pi(e) + \delta\pi(\hat{p}) \geq \tilde{\pi}^M(e),$$

which amounts to $\delta \geq \underline{\delta}(e)$. Finally:

- $\underline{\delta}(e) \in (0,1)$ for any $e \in (\hat{p}, V)$, as then:

$$\pi(\hat{p}) = \max_p pD(\hat{p}+p) > \tilde{\pi}^M(e) = \max_p pD(e+p) > \pi(e) = eD(2e);$$

- $\underline{\delta}(V) = 0$, as $\tilde{\pi}^M(V) = \pi(V) = 0$, and

$$\lim_{e \longrightarrow \hat{p}} \frac{\tilde{\pi}^M(e) - \pi(e)}{\pi(\hat{p}) - \pi(e)} = \left. \frac{\frac{d\tilde{\pi}^M(e)}{de} - \frac{d\pi(e)}{de}}{-\frac{d\pi(e)}{de}} \right|_{e=\hat{p}} = \frac{D(2\hat{p}) + \hat{p}D'(2\hat{p})}{D(2\hat{p}) + 2\hat{p}D'(2\hat{p})} = 0,$$

  where the last equality stems from $\hat{p} = r(\hat{p}) = \arg\max_p pD(\hat{p}+p)$.

■

The following proposition now characterizes the scope for tacit coordination in case of strong complementors:

Proposition 27 (strong complementors) *When $e > \hat{p}$:*

(*i*) *$v^* > \pi^N$: some profitable cooperation is always sustainable. Perfect cooperation on price p^M is feasible (i.e., $v^* = \pi^M$) if $\delta \geq \overline{\delta}^C(e)$, where $\overline{\delta}^C(e)$ continuously prolongs the function defined in Proposition 26, lies strictly below 1, and is decreasing for e close to V.*

(*ii*) *Furthermore, if $D'' \geq 0$, then there exists $v^*(e,\delta) \in (\pi^N, \pi^M]$, which continuously prolongs the function defined in Proposition 26 and is (weakly) increasing in δ, such that the set of Nash-dominating sustainable payoffs is $\mathcal{V}^+ = [\pi(\hat{p}), v^*(e,\delta)]$.*

Proof. (*i*) We first show that, using reversal to Nash as punishment, firms can always sustain a stationary, symmetric equilibrium path in which they both charge constant price $p < \hat{p}$, for p close enough to $\hat{p}$. This amounts to $\hat{K}(p; e, \delta) \geq 0$, where

$$\hat{K}(p; e, \delta) \equiv \pi(p) - (1-\delta)\pi^D(p; e) - \delta\pi(\hat{p}),$$

where

$$\pi^D(p; e) \equiv \max_{\tilde{p} \leq e} \tilde{p}D(p + \tilde{p}) = \begin{cases} r(p)D(p + r(p)) & \text{if } r(p) \leq e, \\ eD(p+e) & \text{if } r(p) > e. \end{cases}$$

Because $\pi^D(\hat{p}; e) = \pi(\hat{p})$, $\hat{K}(\hat{p}; e, \delta) = 0$ for any e, δ. Furthermore:

$$\frac{\partial \hat{K}}{\partial p}(\hat{p}; e, \delta) = \pi'(\hat{p}) - (1-\delta)\hat{p}D'(2\hat{p}),$$

which using $\pi'(\hat{p}) = \hat{p}D'(2\hat{p})$, reduces to:

$$\frac{\partial \hat{K}}{\partial p}(\hat{p}; e, \delta) = \delta \hat{p} D'(2\hat{p}) < 0.$$

Hence, for p close to $\hat{p}$, $\hat{K}(p; e, \delta) > 0$ for any $\delta \in (0, 1]$. If follows that cooperation on such price p is always sustainable, and thus $v^* > \pi^N$.

We now turn to perfect cooperation. Note first that it can be sustained by the minmax punishment $\underline{\pi} = \tilde{\pi}^M(e)$ whenever

$$\pi^M \geq (1 - \delta)\pi^D\left(p^M; e\right) + \delta \tilde{\pi}^M(e),$$

or:

$$\delta \geq \overline{\delta}_1^C(e) \equiv \frac{\pi^D\left(p^M; e\right) - \pi^M}{\pi^D\left(p^M; e\right) - \tilde{\pi}^M(e)}.$$

Conversely, minmax punishments can be sustained using Abreu's optimal symmetric penal code whenever

$$(1 - \delta)\underline{\pi}(e) + \delta \pi^M \geq \tilde{\pi}^M(e), \tag{22}$$

or:

$$\delta \geq \overline{\delta}_2^C(e) \equiv \frac{\tilde{\pi}^M(e) - \underline{\pi}(e)}{\pi^M - \underline{\pi}(e)}.$$

Therefore, we can take $\overline{\delta}^C(e) \equiv \max\left\{\overline{\delta}_1^C(e), \overline{\delta}_2^C(e)\right\}$.

As $\overline{\delta}_1^C(\hat{p}) > \overline{\delta}_2^C(\hat{p}) = 0$ and $\overline{\delta}_1^C(V) > \overline{\delta}_2^C(V) = 0$, $\overline{\delta}^C(e) = \overline{\delta}_1^C(e) \geq \overline{\delta}_2^C(e)$ for e close to $\hat{p}$ and for e close to V. Furthermore, as $\tilde{\pi}^M(e)$ is continuous and coincides with $\pi(e)$ for $e = \hat{p}$, and $\pi^D\left(p^M; e\right) = eD\left(p^M + e\right)$ as long as $e < r\left(p^M\right)$ (where $r\left(p^M\right) > \hat{p}$), $\overline{\delta}_1^C(e)$ continuously prolongs the function $\overline{\delta}^C(e)$ defined in Proposition 26. Finally, both $\overline{\delta}_1^C(e)$ and $\overline{\delta}_2^C(e)$ lie below 1 (as $\tilde{\pi}^M(e) \leq \tilde{\pi}^M(\hat{p}) = \pi(\hat{p}) < \pi^M = \pi\left(p^M\right)$).

Finally, we note that

$$\overline{\delta}_1^C(e) = \frac{1}{1 + \dfrac{\pi^M - \tilde{\pi}^M(e)}{\pi^D\left(p^M; e\right) - \pi^M}}$$

decreases with e for $e \geq r\left(p^M\right)$: $\pi^D\left(p^M; e\right) = r\left(p^M\right) D\left(p^M + r\left(p^M\right)\right)$ does not vary with e whereas $\tilde{\pi}^M(e) = \max_p pD(e + p)$ decreases with e; and so $\overline{\delta}_1^C(e)$ decreases with e.

(ii) As in the case of weak complementors, selling the incomplete technology cannot be more profitable than the static Nash:

$$\tilde{\pi}^M(e) = \max_p pD(e + p) < 2\pi^N = 2\pi(\hat{p}) = 2 \max_p pD(\hat{p} + p).$$

Therefore, if collusion enhances profits ($v^* > \pi^N$), there must exist some period $\tau \geq 0$ in which each firm i charges a price p_i^* not exceeding e, and the average price $p^* = \frac{p_1^* + p_2^*}{2}$

moreover satisfies

$$\pi\left(p^*\right)=\frac{\pi_1^\tau+\pi_2^\tau}{2}\geq v^*.$$

To ensure that firm i has no incentive to deviate, and for a given punishment payoff $\underline{v}$, we must have:

$$(1-\delta)\,\pi_i^\tau+\delta v_i^{\tau+1}\geq(1-\delta)\,\pi^D\left(p_j^*;e\right)+\delta\underline{v}.$$

Combining these conditions for the two firms yields:

$$(1-\delta)\,\frac{\pi^D\left(p_i^*;e\right)+\pi^D\left(p_j^*;e\right)}{2}+\delta\underline{v}\leq(1-\delta)\,\pi\left(p^*\right)+\delta\frac{v_2^{\tau+1}+v_2^{\tau+1}}{2}\leq\pi\left(p^*\right), \tag{23}$$

where the last inequality stems from $\frac{v_1^{\tau+1}+v_2^{\tau+1}}{2}\leq v^*\leq\pi\left(p^*\right)$. But the deviation profit $\pi^D\left(p;e\right)$ is convex in p when $D''\geq 0$,[42] and thus condition (23) implies $\underline{K}\left(p^*;e,\delta,\underline{v}\right)\geq 0$, where

$$\underline{K}\left(p;e,\delta,\underline{v}\right)\equiv\pi\left(p\right)-(1-\delta)\max_{\tilde{p}\leq e}\tilde{p}D\left(p+\tilde{p}\right)-\delta\underline{v}. \tag{24}$$

Conversely, if $\underline{K}\left(p^*;e,\delta,\underline{v}\right)\geq 0$, then the stationary path $\left(p^*,p^*\right)$ is an equilibrium path.

For any δ, from Lemma 10 the minmax $\tilde{\pi}^M\left(e\right)$ can be used as punishment payoff for e close to $\hat{p}$; the sustainability condition then amounts to $K\left(p;e,\delta\right)\geq 0$, where

$$K\left(p;e,\delta\right)\equiv\pi\left(p\right)-(1-\delta)\max_{\tilde{p}\leq e}\tilde{p}D\left(p+\tilde{p}\right)-\delta\tilde{\pi}^M\left(e\right).$$

Using $\tilde{\pi}^M\left(e\right)=\max_p pD\left(e+p\right)$ and noting that $\hat{p}=r\left(\hat{p}\right)<e$ implies $\pi\left(\hat{p}\right)=\max_p pD\left(\hat{p}+p\right)=\max_{p\leq e} pD\left(\hat{p}+p\right)$ for $\delta>0$, we have:

$$K\left(\hat{p};e,\delta\right)=\delta\left[\max_p pD\left(\hat{p}+p\right)-\max_p pD\left(e+p\right)\right]>0.$$

Furthermore, K is concave in p if $\pi^D\left(p;e\right)$ is convex in p, which is the case when $D''\geq 0$. Thus, there exists $\underline{p}(e,\delta)\in[p^M,\hat{p})$ such that cooperation at price p is feasible if and only if $\underline{p}(e,\delta)\leq p<\hat{p}$, and the set of sustainable Nash-dominating per-firm payoffs is then $\left[\pi\left(e\right),v^*\left(e,\delta\right)\right]$, where $v^*\left(e,\delta\right)\equiv\pi\left(\max\left\{p^M,\underline{p}\left(e,\delta\right)\right\}\right)$. Furthermore, using $\tilde{p}^M\left(e\right)=$

[42] In the range where $r\left(p\right)<e$, $\frac{\partial\pi^D}{\partial p}\left(p;e\right)=r\left(p\right)D'\left(p+r\left(p\right)\right)$ and thus (using $-1<r'<0$):

$$\frac{\partial^2\pi^D}{\partial p^2}\left(p;e\right)=r'D'+rD''(1+r')>0.$$

In the range where $r\left(p\right)>e$, $\frac{\partial\pi^D}{\partial p}\left(p;e\right)=eD'\left(p+e\right)$ and thus π^D is convex if $D''\geq 0$. Furthermore, the derivative of π^D is continuous at $p=p_e\equiv r^{-1}\left(e\right)$:

$$\lim_{\substack{p\to p_e\\ p<p_e}}\frac{\partial\pi^D}{\partial p}\left(p;e\right)=\lim_{p\to p_e}eD'\left(p+e\right)=eD'\left(p_e+e\right)=\lim_{p\to p_e}r\left(p\right)D'\left(p+r\left(p\right)\right)=\lim_{\substack{p\to p_e\\ p>p_e}}\frac{\partial\pi^D}{\partial p}\left(p;e\right).$$

$r(e) < \hat{p} < e$; we have, for $p < \hat{p} < e$:

$$\frac{\partial K}{\partial \delta}(p; e, \delta) = \pi^D(p; e) - \tilde{\pi}^M(e) = \max_{\tilde{p} \leq e} \tilde{p} D(p + \tilde{p}) - \max_{\tilde{p}} \tilde{p} D(e + \tilde{p}) > 0.$$

Therefore, $\underline{p}(e, \delta)$ decreases with δ, and thus $v^*(e, \delta)$ weakly increases with δ. Finally, note that $K(p; \hat{p}, \delta) = H(p; \hat{p}, \delta)$, where H is defined by (20); hence the function $v^*(e, \delta)$ defined here prolongs that of Proposition 26.

The function $v^*(e, \delta) = \pi\left(\max\left\{p^M, \underline{p}(e, \delta)\right\}\right)$ remains relevant as long as the minmax $\tilde{\pi}^M(e)$ is sustainable. When this is not the case, then $\underline{v}$ can be replaced with the lowest symmetric equilibrium payoff, which, using Abreu's optimal symmetric penal code, is of the form $(1 - \delta)\pi(p^p) + \delta\pi(p^*)$, where p^p is the highest price in $[\hat{p}, e]$ satisfying $\pi^D(p^p; e) - \pi(p^p) \leq \delta[\pi(p^*) - \pi(p^p)]$, and p^* is the lowest price in $\left[p^M, \hat{p}\right]$ satisfying $\pi^D(p^*; e) - \pi(p^*) \leq \delta[\pi(p^*) - \pi(p^p)]$; we then have $v^*(e, \delta) = \pi(p^*)$ and the monotonicity stems from p^* and p^p being respectively (weakly) decreasing and increasing with δ. ■

Together, Propositions 26 and 27 lead to:

Proposition 28 (complementors) *When $p^M < p^N$:*

(*i*) *There exists $\overline{\delta}^C(e) < 1$ and $\underline{\delta}^C(e) < \overline{\delta}^C(e)$, where $\overline{\delta}^C(e)$ is decreasing for e close to p^M, close to $\hat{p}$ and close to V, and $\underline{\delta}^C(e)$ is decreasing in e, and equal to 0 for $e = \hat{p}$, such that*

- *perfect cooperation on price p^M is feasible (i.e., $v^* = \pi^M$) whenever $\delta \geq \overline{\delta}^C(e)$;*
- *profitable cooperation is sustainable (i.e., $v^* > \pi^N$) whenever $\delta \geq \underline{\delta}^C(e)$.*

(*ii*) *Furthermore, if $D'' \geq 0$, then there exists $v^*(e, \delta) \in (\pi^N, \pi^M]$, which is (weakly) increasing in δ, such that the set of Nash-dominating sustainable payoffs is $\mathcal{V}^+ = \left[\pi^N, v^*(e, \delta)\right]$.*

By contrast with the case of rivalry, where collusion inefficiently induces users to adopt the incomplete technology, avoiding double marginalization unambiguously raises profits here. It follows that some cooperation (and even perfect cooperation) is always sustainable, for any degree of essentiality, when firms are sufficiently patient; furthermore, in the case of strong complementors (i.e., $e > \hat{p}$), firms can always sustain some cooperation on a price $p < p^N = \hat{p}$, regardless of their discount factor: this is because starting from the static Nash price $\hat{p}$, a small reduction in the price then generates a first-order increase in profits, but only a second-order incentive to deviate.

L Proof of Proposition 12

When users acquire both licenses at total price P, welfare has the familiar expression:

$$W(P) = S(P) + PD(P),$$

where $S(P) \equiv \int_P^V D(\tilde{P})d\tilde{P}$. When instead users acquire a single license at price p, welfare is

$$\tilde{W}(p) = S(p+e) + pD(p+e).$$

Thus under rivalry ($e < p^M$), welfare is $W(2e)$ in the absence of collusion and $\tilde{W}(p)$ in the collusive outcome, for some $p > e$. Note that

$$\tilde{W}(p) = W(p+e) - eD(p+e).$$

This expression identifies the two facets of the collusive cost. First, the total price, $p+e$, exceeds the competitive price $2e$ as $p > e$. Second, there is a foregone surplus e on actual consumption $D(p+e)$ due to incomplete consumption. Collusion harms consumers and reduces total welfare under rivalry.

In the case of complementors, tacit coordination is profitable when firms cooperate in offering the complete technology at a price lower than the static Nash price; it then benefits users and increases total welfare.

M Proof of Propositions 13 and 14

We prove Proposition 14 in the extended setting described in Section 4.2.3, in which firms may have asymmetric offerings; this, in turn, establishes Proposition 13 for the case of symmetric offerings.

The case of complementors (part (ii), where $e_1 + e_2 \geq P^M = 2p^M$) is straightforward, as any vector of price caps $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2)$ satisfying $\bar{p}_1 + \bar{p}_2 = P^M$ and $\bar{p}_i \leq e_i$ induces $\mathbf{p} = \bar{\mathbf{p}}$ as unique continuation equilibrium: starting from any price vector $\mathbf{p} \leq \bar{\mathbf{p}}$, any firm offering $p_i < \bar{p}_i$ would have an incentive to increase its price towards $\bar{p}_i$, as (using $-1 < r'(p_j) < 0$) $\bar{p}_i + \bar{p}_j = P^M = 0 + r(0) < \bar{p}_j + r(\bar{p}_j)$ implies $\bar{p}_i < r(\bar{p}_j) \leq r(p_j)$, for $i \neq j \in \{1,2\}$.

We now turn to the case of rivalry (part (i), where $e_1 + e_2 < P^M$). We first show that, as noted in the text, this implies that both firms are constrained in the static Nash equilibrium. Indeed, if both firms were unconstrained, then we would have $p_1^N = p_2^N = \hat{p} = r(\hat{p}) \leq e_2 \leq e_1$ and thus $e_1 + e_2 \geq 2\hat{p} > P^M$, a contradiction. If instead firm i is unconstrained whereas firm j is constrained, for some $i \neq j \in \{1,2\}$, then $p_j^N = e_j$ and $p_i^N = r(e_j) \leq e_i$; hence, $e_i + e_j \geq r(e_j) + e_j > 0 + r(0) = P^M$, again a contradiction. Therefore, it must be the case that both firms are constrained: $p_i^N = e_i \leq r(e_j)$ for

$i \neq j \in \{1, 2\}$.

This, in turn, implies that reducing prices below their Nash levels would reduce both firms' profits: for any $\mathbf{p} \leq \mathbf{p}^N = \mathbf{e} = (e_1, e_2)$, we have $p_i \leq e_i \leq r(e_j)$ for $i \neq j \in \{1, 2\}$, and thus: $\pi_i(\mathbf{p}) \leq \pi_i(p_i, e_j) \leq \pi_i(e_i, e_j)$, where the first inequality stems from $p_j \leq e_j$, and the second one from $p_i \leq e_i \leq r(e_j)$ and quasi-concavity. Furthermore, offering a price $\bar{p}_i > V$ would be irrelevant. Thus, without loss of generality, suppose now that a price cap $\bar{p}_i \in [e_i, V]$ is introduced for each patent $i = 1, 2$.

Next, we show that the minmax profits: (a) are the same as without price caps, and (b) can be sustained by the repetition of the (unconstrained) static Nash outcome, $\mathbf{p}^N = \mathbf{e}$. To establish (a), it suffices to note that the minmaxing strategy $p_j = e_j\,(\leq \bar{p}_j)$ remains available to firm i's rival, and firm i's best response, $p_i = e_i\,(\leq \bar{p}_i)$, also remains available. To establish (b), it suffices to note that the static Nash outcome $\mathbf{p}^N = \mathbf{e}$ remains feasible, and that deviations are only more limited than in the absence of price caps.

We now show that any profitable collusion that is sustainable through price caps is also sustainable without them. Recall that the set of pure-strategy equilibrium payoffs can be characterized as the largest self-generating set of payoffs, where, as minmax profits are sustainable, a self-generating set of payoffs $\tilde{W}$ (where $\tilde{W} = W$ in the absence of price caps, and $\tilde{W} = W^c$ with price caps) is such that, for any payoff (π_1, π_2) in $\tilde{W}$, there exists a continuation payoff (π_1^*, π_2^*) in $\tilde{W}$ and a price profile $(p_1^*, p_2^*) \in \tilde{\mathcal{R}}_1 \times \tilde{\mathcal{R}}_2$, where $\tilde{\mathcal{R}}_i$ is the set of *relevant* prices for firm i (more on this below), that satisfy, for $i \neq j \in \{1, 2\}$:

$$\pi_i = (1 - \delta)\, \pi_i\left(p_i^*, p_j^*\right) + \delta \pi_i^* \geq \max_{p_i \in \tilde{\mathcal{R}}_i} \pi_i\left(p_i, p_j^*\right) + \delta \underline{\pi}_i. \tag{25}$$

To establish that the equilibrium payoffs that are weakly more profitable than Nash under price caps are also equilibrium payoffs without price caps, it suffices to show that any self-generating set with price caps $(\bar{p}_1, \bar{p}_2)$ satisfying $\bar{p}_i \in [e_i, V]$ for $i = 1, 2$, is also a self-generating set in the absence of price caps.

In the absence of price caps, without loss of generality the set of relevant prices for firm i is $\mathcal{R}_i \equiv [0, V]$; when a price cap $\bar{p}_i$ is introduced, then the set of relevant prices becomes $\mathcal{R}_i^c \equiv [0, \bar{p}_i]$. Consider now a self-generating set W^c for given price caps $(\bar{p}_1, \bar{p}_2)$ satisfying $\bar{p}_i \in [e_i, V]$ for $i = 1, 2$, and given payoffs $(\pi_1, \pi_2) \in W^c$, with associated payoffs $(\pi_1^*, \pi_2^*) \in W^c$ and prices $\left(\underline{p}_1^*, \underline{p}_2^*\right) \in \mathcal{R}_1^c \times \mathcal{R}_2^c$ satisfying, for $i \neq j \in \{1, 2\}$, $\underline{p}_i^* \leq \bar{p}_i$ and

$$\pi_i = (1 - \delta)\, \pi_i\left(\underline{p}_i^*, \underline{p}_j^*\right) + \delta \pi_i^* \geq \max_{\underline{p}_i \in \mathcal{R}_i^c} \pi_i\left(\underline{p}_i, \underline{p}_j^*\right) + \delta \underline{\pi}_i. \tag{26}$$

By construction, the associated price profile $\left(\underline{p}_1^*, \underline{p}_2^*\right)$ also belongs to $\mathcal{R}_1 \times \mathcal{R}_2$. However, the gain from a deviation may be lower than in the absence of price caps, as the set of relevant deviating prices is smaller. To conclude the proof, we now show that, for any

$\left(\underline{p}_1^*, \underline{p}_2^*\right) \in \mathcal{R}_1^c \times \mathcal{R}_2^c$ satisfying (26), there exists $(p_1^*, p_2^*) \in \mathcal{R}_1 \times \mathcal{R}_2$ satisfying

$$\pi_i = (1-\delta)\,\pi_i\left(p_i^*, p_j^*\right) + \delta \pi_i^* \geq \max_{p_i \in \mathcal{R}_i} \pi_i\left(p_i, p_j^*\right) + \delta \underline{\pi}_i. \tag{27}$$

For this, it suffices to exhibit a profile $(p_1^*, p_2^*) \in \mathcal{R}_1 \times \mathcal{R}_2$ yielding the same profits (i.e., $\pi_i\left(p_i^*, p_j^*\right) = \pi_i\left(\underline{p}_i^*, \underline{p}_j^*\right)$ for $i = 1, 2$) without increasing the scope for deviations (i.e., $\max_{p_i \in \mathcal{R}_i} \pi_i\left(p_i, p_j^*\right) \leq \max_{\underline{p}_i \in \mathcal{R}_i^c} \pi_i\left(\underline{p}_i, \underline{p}_j^*\right)$ for $i = 1, 2$). We can distinguish four cases for the associated price profile $\left(\underline{p}_1^*, \underline{p}_2^*\right)$:

Case a: $\underline{p}_1^* \leq e_1, \underline{p}_2^* \leq e_2$. In that case, we can pick $(p_1^*, p_2^*) = \left(\underline{p}_1^*, \underline{p}_2^*\right)$; as firm i's profit from deviating to p_i is then given by

$$\pi_i\left(p_i, \underline{p}_j^*\right) = \begin{cases} p_i D\left(\underline{p}_j^* + p_i\right) \text{ if } p_i \leq e_i \\ \quad 0 \text{ otherwise} \end{cases},$$

the best deviation is

$$\arg\max_{p_i \leq e_i} p_i D\left(\underline{p}_j^* + p_i\right) = e_i,$$

which belongs to both $\mathcal{R}_i$ and $\mathcal{R}_i^c$. Hence, $\max_{\underline{p}_i \in \mathcal{R}_i^c} \pi_i\left(\underline{p}_i, \underline{p}_j^*\right) = \max_{p_i \in \mathcal{R}_i} \pi_i\left(p_i, \underline{p}_j^*\right)$.

Case b: $\underline{p}_i^* - e_i \leq 0 < \underline{p}_j^* - e_j$, for $i \neq j \in \{1, 2\}$. In that case, the profile $\left(\underline{p}_1^*, \underline{p}_2^*\right)$ yields profits $\pi_j\left(\underline{p}_j^*, \underline{p}_i^*\right) = 0$ and $\pi_i\left(\underline{p}_i^*, \underline{p}_j^*\right) = \underline{p}_i^* D\left(e_j + \underline{p}_i^*\right)$, and best deviations are respectively given by:

$$\begin{aligned} \arg\max_{p_j} \pi_j\left(p_j, \underline{p}_i^*\right) &= \arg\max_{p_j \leq e_j} p_j D\left(\underline{p}_i^* + p_j\right) = e_j, \\ \arg\max_{p_i} \pi_i\left(p_i, \underline{p}_j^*\right) &= \arg\max_{p_i \leq \underline{p}_j^* + e_i - e_j} p_i D\left(e_j + p_i\right) = \min\left\{\underline{p}_j^* + e_i - e_j, p_i^M\right\}. \end{aligned}$$

As $e_j \in \mathcal{R}_j \cap \mathcal{R}_j^c$, $\max_{\underline{p}_j \in \mathcal{R}_j^c} \pi_j\left(\underline{p}_j, \underline{p}_i^*\right) = \max_{p_j \in \mathcal{R}_j} \pi_j\left(p_j, \underline{p}_i^*\right)$. Therefore, if $\min\left\{\underline{p}_j^* + e_i - e_j, p_i^M\right\} \leq \bar{p}_i$ (and thus $\min\left\{\underline{p}_j^* + e_i - e_j, p_i^M\right\} \in \mathcal{R}_i \cap \mathcal{R}_i^c$), we can pick $(p_1^*, p_2^*) = \left(\underline{p}_1^*, \underline{p}_2^*\right)$, as then we also have $\max_{\underline{p}_i \in \mathcal{R}_i^c} \pi_i\left(\underline{p}_i, \underline{p}_j^*\right) = \max_{p_i \in \mathcal{R}_i} \pi_i\left(p_i, \underline{p}_j^*\right)$. If instead $\min\left\{\underline{p}_j^* + e_i - e_j, p_i^M\right\} > \bar{p}_i$, then we can pick $p_i^* = \underline{p}_i^*$ and $p_j^* \in (e_j, e_j + \bar{p}_i - e_i)$:[43] the profile (p_1^*, p_2^*) yields the same profits as $\left(\underline{p}_1^*, \underline{p}_2^*\right)$, and, as the best deviations are the same, with or without price caps:

$$\begin{aligned} \arg\max_{p_j} \pi_j\left(p_j, p_i^*\right) &= \arg\max_{p_j} \pi_j\left(p_j, \underline{p}_i^*\right) = e_j \in \mathcal{R}_j \cap \mathcal{R}_j^c, \\ \arg\max_{p_i} \pi_i\left(p_i, p_j^*\right) &= \arg\max_{p_i \leq p_j^* + e_i - e_j} p_i D\left(e_j + p_i\right) = \min\left\{p_j^* + e_i - e_j, p_i^M\right\} \in \mathcal{R}_i \cap \mathcal{R}_i^c, \end{aligned}$$

as $\min\left\{p_j^* + e_i - e_j, p_i^M\right\} \leq p_j^* + e_i - e_j < \bar{p}_i$.

[43] This interval is not empty, as $\bar{p}_i \geq e_i$ by assumption.

Case c: $0 < \underline{p}_i^* - e_i = \underline{p}_j^* - e_j$. In that case, we can pick $(p_1^*, p_2^*) = \left(\underline{p}_1^*, \underline{p}_2^*\right)$, as best deviations consist in undercutting the other firm, and this is feasible with or without price caps.

Case d: $0 < \underline{p}_i^* - e_i < \underline{p}_j^* - e_j$, for $i \neq j \in \{1, 2\}$. In that case, the same payoff could be sustained through $p_i^* = \underline{p}_i^*$ and $p_j^* = \underline{p}_i^* + e_j - e_i \left(< \underline{p}_j^*\right)$, with the convention that technology adopters, being indifferent between buying a single license from i or from j, all favor i: the profile (p_1^*, p_2^*) yields the same profits as $\left(\underline{p}_1^*, \underline{p}_2^*\right)$, $\pi_j = 0$ and $\pi_i = \underline{p}_i^* D\left(e_j + \underline{p}_i^*\right)$, but reduces the scope for deviations, which now boil down to undercutting the rival:

$$\begin{aligned}
\max_{p_j \in \mathcal{R}_j} \pi_j\left(p_j, p_i^*\right) &= \max_{\underline{p}_j \in \mathcal{R}_j^c} \pi_j\left(\underline{p}_j, \underline{p}_i^*\right) = \max_{p_j \leq \underline{p}_i^* + e_j - e_i} p_j D\left(e_i + p_j\right), \\
\max_{p_i \in \mathcal{R}_i} \pi_i\left(p_i, p_j^*\right) &= \max_{p_i \leq p_j^* + e_i - e_j} p_i D\left(e_j + p_i\right) \leq \max_{\underline{p}_i \in \mathcal{R}_i^c} \pi_i\left(\underline{p}_i, \underline{p}_j^*\right) = \max_{p_i \leq \underline{p}_j^* + e_i - e_j} p_i D\left(e_j + p_i\right).
\end{aligned}$$

This moreover implies that, as in case c above, these best deviations were already feasible with price caps. Indeed, as $p_k^* = p_h^* + e_k - e_h$, for $h \neq k \in \{1, 2\}$, we have:

$$\begin{aligned}
\arg\max_{p_j} \pi_j\left(p_j, p_i^*\right) &= \arg\max_{\underline{p}_j} \pi_j\left(\underline{p}_j, \underline{p}_i^*\right) = \arg \max_{p_j \leq \underline{p}_i^* + e_j - e_i} p_j D\left(e_i + p_j\right) = \min\left\{p_j^*, p_j^M\right\}, \\
\arg\max_{p_i} \pi_i\left(p_i, p_j^*\right) &= \arg \max_{p_i \leq p_j^* + e_i - e_j} p_i D\left(e_j + p_i\right) = \min\left\{p_i^*, p_i^M\right\},
\end{aligned}$$

where $\min\left\{p_j^*, p_j^M\right\} \in \mathcal{R}_j \cap \mathcal{R}_j^c$, as $\min\left\{p_j^*, p_j^M\right\} \leq p_j^* < \underline{p}_j^* \in \mathcal{R}_j^c \left(\subset \mathcal{R}_j\right)$, and likewise $\min\left\{p_i^*, p_i^M\right\} \in \mathcal{R}_i \cap \mathcal{R}_i^c$, as $\min\left\{p_i^*, p_i^M\right\} \leq p_i^* = \underline{p}_i^* \in \mathcal{R}_i^c \left(\subset \mathcal{R}_i\right)$.

N Screening through independent licensing

Let us introduce a pool subject to independent licensing in the repeated game considered in Section 4.2.1. The pool sets the price of the bundle[44] and specifies a sharing rule for its dividends: some fraction $\alpha_i \geq 0$ (with $\alpha_1 + \alpha_2 = 1$) goes to firm i. In addition, each pool member can offer licenses on a stand-alone basis if it chooses to. The game thus operates as follows:

1. At date 0, the firms form a pool and fix a pool price P for the bundle, as well as the dividend sharing rule.

2. Then at dates $t = 1, 2, ...$, the firms non-cooperatively set prices p_i^t for their individual licenses; the profits of the pool are then shared according to the agreed rule.

[44] It can be checked that the firms cannot gain from asking the pool to offer unbundled prices as well.

We characterize below the set of equilibria that are sustainable through a pool subject to independent licensing; comparing it to the equilibria without a pool, or sustainable through a pool not subject to independent licensing, leads to the following proposition:

Proposition 29 (screening through independent licensing) *Independent licensing provides a useful but imperfect screen:*

(i) *Appending independent licensing to a pool is always welfare-enhancing.*

Relative to the absence of a pool:

(ii) *In case of complementors, a pool with independent licensing enables the firms to achieve perfect cooperation, which is welfare-enhancing.*

(iii) *In case of rivalry, if some collusion is already sustainable without a pool, then a pool with independent licensing enables the firms to collude more efficiently, which results in lower prices and is thus welfare-enhancing; however, there exists* $\underline{\delta}^R(e)$*, which increases from* $\delta^R(0) = 1/2$ *to* 1 *as* e *increases from* 0 *to* p^M*, and lies strictly below* $\delta^R(e)$ *for* $e \in (0, p^M)$*, such that, for* $\delta \in [\underline{\delta}^R(e), \delta^R(e))$*, the pool raises prices by enabling the firms to collude.*

To establish this Proposition, we first characterize the scope for tacit coordination for rival and complementary patents, before drawing the implications for the impact of a pool subject to independent licensing.

N.1 Rivalry: $e < p^M$

The firms can of course collude as before, by not forming a pool or, equivalently, by setting the pool price P at a prohibitive level ($P \geq V$, say); firms can then collude on selling the incomplete technology if $\delta \geq \delta^R(e)$. Alternatively, they can use the pool to sell the bundle at a higher price:

Lemma 11 *In order to raise firms' profits, the pool must charge a price* $P^P > 2p^N = 2e$.

Proof. Suppose that the pool charges a price $P^P \leq 2e$, and consider a period t, with individual licenses offered at prices p_1^t and p_2^t. Let $p^t = \min\{p_1^t, p_2^t\}$ denote the lower one.

• Users buy the complete technology from the pool only if $P^P \leq p^t + e$; the industry profit is then $P^P D(P^P) \leq 2\pi^N = 2\pi(e)$, as the aggregate profit function $PD(P)$ is concave and maximal for $2p^M > 2e \geq P^P$.

• Users buy the complete technology by combining individual licenses only if $p_i \leq e$ for $i = 1, 2$, in which case $p_1 + p_2 \leq 2e$ and the industry profit is $(p_1 + p_2) D(p_1 + p_2) \leq 2\pi^N$.

• Finally, users buy an incomplete version of the technology only if $p^t + e \leq P^P$, which in turn implies $p^t \leq e$ (as then $p^t \leq P^P - e$, and by assumption $P^P \leq 2e$); the industry profit is then $p^t D(p^t + e) \leq (p^t + e) D(p^t + e) \leq 2\pi^N$, as $p^t + e \leq 2e$.

Therefore, the industry profit can never exceed the static Nash level. ■

Thus, to be profitable, the pool must adopt a price $P^P > 2e$. This, in turn, implies that the repetition of static Nash outcome through independent licensing remains an equilibrium: If the other firm offers $p_j^t = e$ for all $t \geq 0$, buying an individual license from firm j (corresponding to quality-adjusted total price $2e$) strictly dominates buying from the pool, and so the pool is irrelevant (firm i will never receive any dividend from the pool); it is thus optimal for firm i to set $p_i^t = e$ for all $t \geq 0$. Furthermore, this individual licensing equilibrium, which yields $\pi(e)$, still minmaxes all firms, as in every period each firm can secure $eD\left(e + \min\left\{e, p_j^t\right\}\right) \geq \pi(e)$ by undercutting the pool and offering an individual license at price $p_i^t = e$.

Suppose that tacit coordination enhances profits: $v^* > \pi^N = \pi(e)$, where v^* denotes the maximal average discounted equilibrium per firm payoff. In the associated equilibrium, there exists some period $\tau \geq 0$ in which the aggregate profit, $\pi_1^\tau + \pi_2^\tau$, is at least equal to $2v^*$. If users buy an incomplete version of the technology in that period, then each firm can attract all users by undercutting the equilibrium price; the same reasoning as before then implies that collusion on $p_i^t = \tilde{p}^M(e)$ is sustainable, and requires $\delta \geq \delta^R(e)$.

If instead users buy the complete technology in period τ, then they must buy it from the pool,[45] and the per-patent price $p^P \equiv P^P/2$ must satisfy:

$$2\pi\left(p^P\right) = \pi_1^\tau + \pi_2^\tau \geq 2v^* > 2\pi(e),$$

implying $p^P > e$. In order to undercut the pool, a deviating firm cannot charge more for its individual license than p^D, the price that leaves users indifferent between buying the incomplete technology from the firm and buying the complete technology from the pool; that is, the price p^D is such that:

$$(V - e) - p^D = V - 2p^P,$$

or $p^D = 2p^P - e\,(> e)$; by offering its individual license at this price, the deviating firm obtains a profit equal to:

$$\pi^D = \left(2p^P - e\right) D\left(2p^P\right) = \pi\left(p^P\right) + \left(p^P - e\right) D\left(2p^P\right) > \pi\left(p^P\right). \tag{28}$$

[45] Users would combine individual licenses only if the latter were offered at prices not exceeding e; hence, the total price P would not exceed $2e$. But $PD(P) = \pi_1^\tau + \pi_2^\tau \geq 2v^* > 2\pi(e)$ implies $P > 2e$.

Thus, for the price p^P to be sustainable, there must exist continuation payoffs $\left(v_1^{\tau+1}, v_2^{\tau+1}\right)$ such that, for $i = 1, 2$:

$$(1-\delta)\,\pi_i^\tau + \delta v_i^{\tau+1} \geq (1-\delta)\left[\pi\left(p^P\right) + \left(p^P - e\right) D\left(2p^P\right)\right] + \delta\pi\left(e\right).$$

Combining these two conditions and using $\frac{v_1^{\tau+1}+v_2^{\tau+1}}{2} \leq v^* \leq \frac{\pi_1^\tau+\pi_2^\tau}{2} = \pi\left(p^P\right)$ yields:

$$\pi\left(p^P\right) \geq (1-\delta)[\pi\left(p^P\right) + \left(p^P - e\right) D\left(2p^P\right)] + \delta\pi\left(e\right). \tag{29}$$

Conversely, a pool price $p^P \in (e, p^M]$ satisfying this condition is *stable*: a bundle price $P^P = 2p^P$, together with an equal profit-sharing rule and firms charging high enough individual prices (e.g., $p_i^t \geq V$ for all $t \geq 0$), ensures that no firm has an incentive to undercut the pool, and each firm obtains $\pi\left(p^P\right)$. To see this, it suffices to note that the expression of π^D given by (28) represents the highest deviation profit when $p^P \leq p^M$, as the deviating profit $pD\left(p+e\right)$ is concave and maximal for $\tilde{p}^M\left(e\right) = r\left(e\right)$, and $e + \tilde{p}^M\left(e\right) = e + r\left(e\right) \geq 0 + r\left(0\right) = 2p^M$ implies $\tilde{p}^M\left(e\right) > 2p^M - e \geq 2p^P - e$. Building on this insight yields:

Proposition 30 (pool in the rivalry region) *Suppose $e \leq p^M$. As before, if $\delta \geq \delta^R\left(e\right)$ the firms can sell the incomplete technology at the monopoly price $\tilde{p}^M$ and share the associated profit, $\tilde{\pi}^M$. In addition, a per-license pool price p^P, yielding profit $\pi\left(p^P\right)$, is stable if* (29) *holds. As a result:*

(*i*) *Perfect collusion (i.e., on a pool price $p^P = p^M$) is feasible if*

$$\delta \geq \bar{\delta}^P\left(e\right) \equiv \frac{1}{2 - \frac{e}{p^M - e}\frac{D(2e) - D(2p^M)}{D(2p^M)}},$$

where the threshold $\bar{\delta}^P\left(e\right)$ is increasing in e.

(*ii*) *If the firms can already collude without a pool (i.e., if $\delta \geq \delta^R\left(e\right)$), then the pool enables them to sustain a more profitable collusion, which benefits consumers as well.*

(*iii*) *There exists $\underline{\delta}^R\left(e\right)$, which coincides with $\delta^R\left(e\right)$ for $e = 0$, and lies strictly below $\delta^R\left(e\right)$ for $e > 0$, such that some collusion (i.e., on a stable pool price $p^P \in (e, p^M]$) is feasible when $\delta \geq \underline{\delta}^R\left(e\right)$.*

Proof. (*i*) We have established that a pool price p^P is stable if and only if $L\left(p^P; e, \delta\right) \geq 0$, where

$$\begin{aligned} L\left(p; e, \delta\right) &\equiv \pi(p) - (1-\delta)\left[\pi\left(p\right) + \left(p - e\right) D\left(2p\right)\right] - \delta\pi\left(e\right) \\ &= \delta p D\left(2p\right) - (1-\delta)\left(p - e\right) D\left(2p\right) - \delta e D\left(2e\right). \end{aligned}$$

In the particular case of perfect substitutes (i.e., $e = 0$), this expression reduces to $(2\delta - 1)\,\pi(p) \geq 0$. Therefore, any pool price $p^P \geq 0$ is stable – including the monopoly price p^M – if and only if $\delta \geq 1/2$. For $e > 0$, sustaining a price $p^P \in (e, p^M]$ requires $\delta > 1/2$:

$$L(p; e, \delta) = (2\delta - 1)\left[pD(2p) - eD(2e)\right] + (1 - \delta)\, e\left[D(2p) - D(2e)\right],$$

where the second term is negative and, in the first term, $\pi(p) > \pi(e)$.

In particular, collusion on p^M is feasible if $L\left(p^M; e, \delta\right) \geq 0$, or:

$$\delta \geq \bar{\delta}^P(e) = \frac{\left(p^M - e\right) D\left(2p^M\right)}{\left(p^M - e\right) D\left(2p^M\right) + \pi^M - \pi(e)} = \frac{1}{2 - \frac{e}{p^M - e}\frac{D(2e) - D(2p^M)}{D(2p^M)}},$$

where

$$\frac{d\bar{\delta}^P}{de}\left(e, \bar{\delta}^P(e)\right) = -\frac{\frac{\partial L}{\partial e}\left(p^M; e, \bar{\delta}^P(e)\right)}{\frac{\partial L}{\partial \delta}\left(p^M; e, \bar{\delta}^P(e)\right)}.$$

Clearly $\partial L/\partial\delta > 0$. Furthermore

$$\frac{\partial L}{\partial e}\left(p^M; e, \bar{\delta}^P(e)\right) = [1 - \bar{\delta}^P(e)]D(2p^M) - \bar{\delta}^P(e)\pi'(e).$$

Using the fact that $L\left(p^M; e, \bar{\delta}^P(e)\right) = 0$,

$$\frac{\partial L}{\partial e}\left(p^M; e, \bar{\delta}^P(e)\right) \propto [\pi^M - \pi(e) - (p^M - e)\pi'(e)] < 0,$$

from the concavity of π. And so

$$\frac{d\bar{\delta}^P}{de} > 0.$$

(*ii*) In the absence of a pool, collusion is inefficient (users buy only one license) and is therefore unprofitable (and thus unsustainable) when $\tilde{\pi}^M(e) \leq 2\pi^N = 2\pi(e)$ (i.e., $e \geq \underline{e}$). When instead

$$\tilde{\pi}^M(e) > 2\pi^N = 2\pi(e), \tag{30}$$

then (i) inefficient collusion on $p \in (e, \tilde{p}^M(e)]$ is profitable for p close enough to $\tilde{p}^M(e)$; in this case, maximal collusion (on $\tilde{p}^M(e)$) is sustainable whenever some collusion is sustainable, and it is indeed sustainable if $\delta \geq \delta^R(e)$. We now show that the pool then enables the firms to sustain a more efficient and more profitable collusion, which benefits consumers as well as the firms. To be as profitable, the pool must charge a price P^P satisfying:

$$P^P D\left(P^P\right) \geq \tilde{\pi}^M(e).$$

Let $\tilde{P}(e)$ denote the lowest of these prices, which satisfies $\tilde{P}D\left(\tilde{P}\right) = \tilde{\pi}^M(e)$.[46] The pool price $\tilde{p}(e) = \tilde{P}(e)/2$ is stable if and only if $L(\tilde{p}(e), e, \delta) \geq 0$, which amounts to:

$$\begin{aligned} 0 \leq G(e,\delta) &\equiv \delta\tilde{p}(e) D\left(\tilde{P}(e)\right) - (1-\delta)(\tilde{p}(e) - e) D\left(\tilde{P}(e)\right) - \delta e D(2e) \\ &= (2\delta - 1)\left[\frac{\tilde{\pi}^M(e)}{2} - \pi(e)\right] + (1-\delta) e \left[D\left(\tilde{P}(e)\right) - D(2e)\right]. \end{aligned}$$

We have:

$$\begin{aligned} \frac{\partial G}{\partial \delta}(e,\delta) &= \tilde{p}(e) D\left(\tilde{P}(e)\right) + (\tilde{p}(e) - e) D\left(\tilde{P}(e)\right) - eD(2e) \\ &= \left[\frac{\tilde{\pi}^M(e)}{2} - \pi(e)\right] + \left(1 - \frac{e}{\tilde{p}(e)}\right)\frac{\tilde{\pi}^M(e)}{2} \\ &> 0, \end{aligned}$$

where the inequality follows from $\tilde{\pi}^M > 2\pi(e)$ (using (30)), which in turn implies $e < \tilde{p}(e)$ (as $2\tilde{p}D(2\tilde{p}) = \tilde{\pi}^M > 2\pi(e) = 2eD(2e)$, and the profit function $PD(P)$ is concave); as

$$G(e, 1/2) = \frac{e}{2}\left[D\left(\tilde{P}(e)\right) - D(2e)\right] < 0 < G(e,1) = \frac{\tilde{\pi}^M(e)}{2} - \pi(e),$$

where the inequalities follow again from $\tilde{\pi}^M(e) > 2\pi(e)$ and $e < \tilde{p}(e)$, then some collusion is feasible if δ is large enough, namely, if $\delta \geq \underline{\delta}_1^R(e)$, where:

$$\underline{\delta}_1^R(e) \equiv \frac{[\tilde{p}(e) - e] D\left(\tilde{P}(e)\right)}{\tilde{\pi}^M(e) - \pi(e) - eD\left(\tilde{P}(e)\right)}.$$

From the proof of Proposition 25, the inefficient collusion on $\tilde{p}^M(e)$ is instead sustainable (i.e., $\delta \geq \delta^R(e)$) when:

$$0 \leq \tilde{G}(e,\delta) \equiv (2\delta - 1)\frac{\tilde{\pi}^M(e)}{2} - \delta\pi(e).$$

In the case of perfect substitutes, this condition boils down again to $\delta \geq 1/2$. Therefore, when collusion is sustainable without the pool, the pool enables the firms to sustain perfect efficient collusion. Furthermore, for $e > 0$, $G(e,\delta) - \tilde{G}(e,\delta) = (1-\delta) eD\left(\tilde{P}(e)\right) > 0$ and thus, if some collusion is sustainable without a pool, then the pool enables again the firms to sustain a more efficient and more profitable collusion: as $G(e,\delta) > 0$ in this case, it follows that a pool price p^P slightly higher (and thus more profitable) than $\tilde{p}$ is also

[46] In the rivalry region, we have that $e < p^M < \hat{p} < r(e) = \tilde{p}^M(e)$; hence, the left-hand side increases from

$$2\pi^N = 2eD(2e) < 2\tilde{\pi}^M(e) = 2r(e) D(e + r(e))$$

to $\Pi^M = 2\pi^M > \tilde{\pi}^M(e)$ as P increases from $2e$ to $2p^M$; there thus exists a unique $P \in (2e, 2p^M)$ satisfying $PD(P) = \tilde{p}^M(e) D(\tilde{p}^M(e) + e)$.

stable. Finally, note that the (quality-adjusted) price is lower when collusion is efficient: the most profitable sustainable price lies below P^M,[47] and

$$P^M = 0 + r(0) < e + r(e) = \tilde{p}^M(e) + e.$$

(iii) Note that $L(e; e, \delta) = 0$ for all e. Therefore, some collusion is sustainable (i.e., there exists a stable pool price $p^P \in (e, p^M)$) whenever $I(e) > 0$, where

$$I(e, \delta) \equiv \frac{\partial L}{\partial p}(e; e, \delta) = (2\delta - 1)D(2e) + 2\delta e D'(2e).$$

We have:

$$\frac{\partial I}{\partial \delta}(e, \delta) = 2\left[D(2e) + eD'(2e)\right] > 0,$$

where the inequality follows from $e < r(e)$ (as here $e < p^M$ $(< \hat{p})$); as

$$I(e, 1/2) = eD'(2e) < 0 < I(e, 1) = D(2e) + 2eD'(2e),$$

where the last inequality stems from $e < p^M$, then some collusion is feasible if δ is large enough, namely, if $\delta \geq \underline{\delta}_2^R(e)$, where:

$$\underline{\delta}_2^R(e) \equiv \frac{1}{2}\frac{1}{1 + \frac{eD'(2e)}{D(2e)}}. \tag{31}$$

Furthermore:

$$\frac{\partial I}{\partial e}(e, \delta) = 2(3\delta - 1)\left[D'(2e) + \frac{\delta}{3\delta - 1}2eD''(2e)\right].$$

But $D'(2e) + 2eD''(2e) < 0$ from Assumption B and $\delta/(3\delta - 1) < 1$ from $\delta > 1/2$; and so

$$\frac{\partial I}{\partial e}(e, \delta) < 0,$$

implying that the threshold $\underline{\delta}_2^R(e)$ increases with e; it moreover coincides with $\delta^R(0) = 1/2$ for $e = 0$, and is equal to 1 for $e = p^M$ (in which case $D(2e) + 2eD'(2e) = 0$, and thus $I(p^M, \delta) = -(1 - \delta)D(2p^M)$).

To conclude the argument, it suffices to note that the statement of part (iii) holds for $\underline{\delta}^R(e) = \min\left\{\underline{\delta}_1^R(e), \underline{\delta}_2^R(e)\right\}$:

- For $e = 0$, perfect collusion is sustainable for $\delta \geq 1/2$, which coincides with the range where inefficient collusion at $\tilde{p}^M(e)$ would be sustainable without a pool.

[47] A price $P^P > P^M$ cannot be the most profitable stable price:

$$L(p^M; e, \delta) - L(p^P; e, \delta) = (2\delta - 1)\left[\pi^M - \pi(p^P)\right] + (1 - \delta)e\left[D(P^M) - D(P^P)\right],$$

which is positive for $P^P > P^M$, as $\pi^M \geq \pi(P^P)$ and $D(P^M) > D(P^P)$. Hence, whenever a pool price $P^P > P^M$ is stable, then $P = P^M$ is also stable.

- For $e \in (e, \underline{e})$ (in which case, without a pool, inefficient collusion at $\tilde{p}^M(e)$ is sustainable if and only if $\delta \geq \delta^R(e)$), $\underline{\delta}^R(e) \leq \underline{\delta}_1^R(e) < \delta^R(e)$.
- Finally, for $e \in [\underline{e}, p^M]$, no collusion is sustainable in the absence of a pool, whereas a pool enables the firms to collude on some price $p^P \in (e, p^M]$ whenever $\delta \geq \underline{\delta}^R(e)$, where $\underline{\delta}^R(e) \leq \underline{\delta}_2^R(e) < 1$.

■

Remark: If $D'' \leq 0$, then L is concave in p.[48] Hence, in that case, some collusion is feasible *if and only if* $\delta \geq \underline{\delta}_2^R(e)$, where $\underline{\delta}_2^R(e)$ lies strictly below $\delta^R(e)$ for $e \in \left(0, p^M\right)$ and increases from $\delta^R(0) = 1/2$ to 1 as e increases from 0 to p^M.

N.2 Weak or strong complementors: $p^M \leq e$

In case of complementary patents, a pool enables the firms to cooperate perfectly:

Proposition 31 (pool with complements) *With weak or strong complementors, a pool allows for perfect cooperation (even if independent licensing remains allowed) and gives each firm a profit equal to π^M.*

Proof. Suppose that the pool charges $P^M = 2p^M$ for the whole technology and shares the profit equally. No deviation is then profitable: as noted above, the best price for an individual license is then $\tilde{p} = 2p^M - e$ (that is, the pool price minus a discount reflecting the essentiality of the foregone license), which is here lower than p^M (since $p^M \leq e$) and thus yields a profit satisfying:

$$\left(2p^M - e\right) D\left(2p^M\right) < p^M D\left(2p^M\right) = \pi^M.$$

■

N.3 Impact of a pool subject to independent licensing

Comparing the most profitable equilibrium outcomes with and without a pool (subject to independent licensing) yields the following observations:

• In the rivalry region, a pool can only benefit users whenever some collusion would already be sustained in the absence of a pool (i.e., when $\delta \geq \delta^R(e)$). In this case, a pool enables the firms to sustain a more efficient collusion, which is more profitable but also benefits users: they can then buy a license for the complete technology at a price

[48] As $pD(2p)$ is concave from Assumption B and $\delta > 1/2$, we have:

$$\frac{\partial^2 L}{\partial p^2}(p, e, \delta) = (2\delta - 1)(pD(2p))'' + 4(1-\delta)eD''(2p) < 0.$$

$P \leq P^M = 2p^M$, which is preferable to buying a license for the incomplete technology at price $\tilde{p}^M(e)$: as $r'(\cdot) > -1$,

$$e + \tilde{p}^M(e) = e + r(e) > 0 + r(0) = P^M = 2p^M.$$

- By contrast, when collusion could not be sustained in the absence of a pool (i.e., when $\delta < \delta^R(e)$), then a pool harms users whenever it enables the firms to sustain some collusion, as users then face an increase in the price from $p^N(e) = e$ to some $p > e$. This happens in particular when $\delta \in [\underline{\delta}^R(e), \delta^R(e))$ (if $D''(\cdot) \leq 0$, it happens only in this case), where $\underline{\delta}^R(e)$ increases from $\delta^R(0) = 1/2$ to 1 as e increases from 0 to p^M, and lies strictly below $\delta^R(e)$ for $e \in (0, p^M)$.
- With weak or strong complementors, a pool enables perfect cooperation and benefits users as well as the firms: in the absence of the pool, the firms would either not cooperate and thus set $p = p^N(e) = \min\{\hat{p}, e\} > p^M$, or cooperate and charge per-license price $p \in [p^M, p^N)$, as opposed to the (weakly) lower price, p^M, under a pool.

Finally, note that, in the absence of the independent licensing requirement, a pool would always enable the firms to achieve the monopoly outcome. Appending independent licensing is therefore always welfare-enhancing, as it can only lead to lower prices in the case of rivalry, and does not prevent the firms from achieving perfect cooperation in the case of complementors.

O Proof of Proposition 15

We start by noting that, if all patents are priced below $\tilde{p}$, then technology adopters acquire all licenses:

Lemma 12 *Offering each license i at a price $p_i \leq \tilde{p}$ induces users to acquire all of them.*

Proof. Without loss of generality, suppose that the patents are ranked in such a way that $p_1 \leq ... \leq p_n$. If users strictly prefer acquiring only $m < n$ licenses, we must have:

$$V(m) - \sum_{k=1}^{m} p_k > V(n) - \sum_{k=1}^{n} p_k \iff \sum_{k=m+1}^{n} p_k > V(n) - V(m).$$

From from the definition of $\tilde{p}$, we also have:

$$V(n) - n\tilde{p} \geq V(m) - m\tilde{p} \iff V(n) - V(m) \geq (n-m)\tilde{p}.$$

Combining these conditions yields:

$$\sum_{k=m+1}^{n} p_k > (n-m)\tilde{p},$$

implying that some licenses are priced strictly above p. Conversely, if all licenses are priced below $\tilde{p}$, users are willing to acquire all of them. ■

To establish part (i) of Proposition 15, suppose that $p^N < p^M$ (which implies $p^N = \tilde{p}$ and $\pi^N = \tilde{p}D(n\tilde{p})$), that each firm faces a given price cap $\bar{p}_i$, and consider a stationary symmetric path in which all firms repeatedly charge the same price p^* (which thus must satisfies $p^* \leq \bar{p}_i$ for $i \in \mathcal{N}$), and obtain the same profit $\pi^* > \pi^N = \tilde{p}D(n\tilde{p})$. We first note that this last condition requires selling an incomplete bundle:

Lemma 13 *When $p^N < p^M$, generating more profit than the static Nash level requires selling less than n licenses.*

Proof. Suppose that a price profile $(p_1, ..., p_n)$ induces users to acquire all n licenses. The aggregate profit is then $\Pi(P) = PD(P)$, where $P = \sum_{k=1}^n p_k$ denotes the total price. But this profit function is concave in P under Assumption B, and thus increases with P in the range $P \leq P^M = np^M$. From Lemma 12, selling all n licenses require $P \leq n\tilde{p}$, where by assumption $n\tilde{p} < P^M$; therefore, the aggregate profit $PD(P)$ cannot exceed that of the (unconstrained) static Nash, $n\tilde{p}D(n\tilde{p})$. ■

From Lemma 13, $\pi^* > \pi^N$ implies that users must buy $m^* < n$ patents; Lemma 12 then implies $p^* > \tilde{p}$; the per-firm equilibrium profit is then:

$$\pi^* = \frac{m^*}{n}p^*D(m^*p^* + V(n) - V(m^*)).$$

Furthermore, as $\bar{p}_i \geq p^* > \tilde{p}$ for all $i \in \mathcal{N}$, the price caps do not affect the static Nash equilibrium, in which all firms still charge $p^N = \tilde{p}$. The price p^* can therefore be sustained by reversal to Nash if and only if:

$$\pi^* \geq (1 - \delta)\pi^D(p^*) + \delta\pi^N,$$

where $\pi^N = \tilde{p}D(n\tilde{p})$ and $\pi^D(p^*)$ denotes the most profitable deviation from p^*, subject to charging a price $p^D \leq \bar{p}_i$. But as the deviating price must lie below p^* (otherwise, the member's patent would be excluded from users' basket), it is not constrained by the price cap $\bar{p}_i \geq p^*$; therefore, the deviation cannot be less profitable than in an alternative candidate equilibrium in which, in the absence of price caps, all members would charge p^*. Hence, price caps cannot sustain higher symmetric prices than what the firms could already sustain in a symmetric equilibrium in the absence of price caps.

To establish part (ii) of the Proposition, suppose that all firms face the same price cap $\bar{p} = p^M < p^N = \min\{\tilde{p}, \hat{p}\}$. As no firm can charge more than $\bar{p} < \tilde{p}$, Lemma 12 implies that, by charging $p_i = p^M$, each firm i can ensure that technology adopters acquire its license, and thus secure a profit at least equal to:

$$\pi_i = p^M D(p^M + \sum_{j\in\mathcal{N}\setminus\{i\}} p_j) \geq p^M D(p^M + (n-1)\,\bar{p}) = \pi^M = p^M D\left(P^M\right).$$

As each firm can secure π^M, and the industry profit is maximal for P^M, it follows that the unique candidate equilibrium is such that each firm charges $\bar{p} = p^M$. Conversely, all firms charging p^M indeed constitutes an equilibrium: a deviating firm can only charge a price $p < \bar{p} = p^M$, and the deviating profit is thus given by:

$$pD\left(p + (n-1)\,p^M\right).$$

The conclusion then follows from the fact that this profit is concave in p, and maximal for (using $r'(\cdot) < 0$ and $p^M < \hat{p}$):

$$r\left((n-1)\,p^M\right) > r\left((n-1)\,\hat{p}\right) = \hat{p} > p^M = \bar{p}.$$