Limit value for optimal control with general means*

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Abstract

We consider optimal control problem with an integral cost which is a mean of a given function. As a particular case, the cost concerned is the Cesàro average. The limit of the value with Cesàro mean when the horizon tends to infinity is widely studied in the literature. We address the more general question of the existence of a limit when the averaging parameter converges, for values defined with means of general types.

We consider a given function and a family of costs defined as the mean of the function with respect to a family of probability measures–the evaluations–on $\mathbb{R}^+$. We give several conditions on the evaluations in order to obtain the uniform convergence of the associated value function (when the parameter of the family converges).

Our main result gives a necessary and sufficient condition in term of the total variation of the family of probability measures on $\mathbb{R}^+$. As a byproduct, we obtain the existence of a limit value (for general means) for control systems having a compact invariant set and satisfying suitable nonexpansive property.

Key words limit value, general means, long time average value

1 Introduction

We consider a control system defined on $\mathbb{R}^d$ whose dynamic is given by

$$y'(t) = f\big(y(t), u(t)\big) \tag{1.1}$$

where $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ and $u(\cdot)$ is a measurable function – called the control – from $\mathbb{R}^+$ to U a fixed metric space. We will make later on assumptions on (1.1) ensuring that for any initial condition $y(0) = y_0$, and any measurable control $u(\cdot)$, the equation (1.1) has a unique solution $t \mapsto y(t, u, y_0)$ defined on $\mathbb{R}^+$.

To any pair $\big(y_0, u(\cdot)\big)$, we associate to a cost

$$\int_0^{+\infty} g\big(y(t, u, y_0), u(t)\big)\mathrm{d}\theta(t),$$

where $g : \mathbb{R}^d \times U \times \mathbb{R}$ is Borel measurable bounded and θ is a Borel probability measure on $\mathbb{R}^+$ (called an *evaluation* throughout the article). Denote by $\Delta(\mathbb{R}^+)$ the set of such measures .

We will refer the previously described optimal control problem by the short notation $\mathcal{J} = \langle U, g, f\rangle$. Let $\theta \in \Delta(\mathbb{R}^+)$, we define for $\mathcal{J} = \langle U, g, f\rangle$ the following value function:

$$V_\theta(y_0) = \inf_{u\in\mathcal{U}} \int_0^{+\infty} g\big(y(t, u, y_0), u(t)\big)\mathrm{d}\theta(t), \tag{1.2}$$

where $\mathcal{U}$ denotes the set of measurable controls $u : [0, +\infty) \to U$.

Typical means in the definition (1.2) of the value function are well studied in the literature for

Cesàro mean: $\forall t > 0$, $\overline{\theta}_t$ with density $s \mapsto f_{\overline{\theta}_t}(s) = \frac{1}{t}\mathbb{1}_{[0,t]}(s)$, and the t-horizon value is

$$V_{\overline{\theta}_t}(y_0) = \inf_{u\in\mathcal{U}} \frac{1}{t}\int_{s=0}^{t} g\big(y(s, u, y_0), u(s)\big)\mathrm{d}s$$

Abel mean: $\forall \lambda \in (0,1]$, θ_λ with density $s \mapsto f_{\theta_\lambda}(s) = \lambda e^{-\lambda s}$, and the λ-discounted value is

$$V_{\theta_\lambda}(y_0) = \inf_{u \in \mathcal{U}} \int_{s=0}^{+\infty} \lambda e^{-\lambda s} g\big(y(s,u,y_0), u(s)\big) \mathrm{d}s$$

The limit of the above value functions as t tends to infinity or as λ tends to zero are well investigated in the control literature, (cf. [1], [2], [3], [4], [6], [7] and the references therein), which are often called ergodic control.

When $\theta \in \Delta(\mathbb{R}^+)$ is given, the contribution of the interval $[T, +\infty)$ in the mean (1.2) is less and less significant as T becomes large. Thus the control problem is essentially interesting only on $[0, T_0]$ for certain T_0, roughly named the "duration" for the problem. In this article, we are interested in the long-run property of $\mathcal{J}$, *i.e.*, the asymptotic behavior of the function $\theta \mapsto V_\theta$ when the "duration" of θ tends to infinity. In the particular examples of Cesàro mean and Abel mean, the uniform convergence of $V_{\overline{\theta}_t}$ as t tends to infinity and of V_{θ_λ} as λ tends to 0 are studied. It is a priori unclear how to define the "duration" of a general evaluation θ over $\mathbb{R}_+$. If one just assumes the expectation of θ to be very high, we can obtain very different value functions, as shown by the following

Example 1.1 *Consider the uncontrolled dynamic* $y(t) = t$*, the running cost* $t \mapsto g(t) = \mathbb{1}_{\cup_{m=1}^{\infty}[2m-1,2m]}(t)$*, and two sequences of evaluations* $(\mu^k)_{k\geq 1}$ *and* $(\nu^k)_{k\geq 1}$ *with densities:* $f_{\mu^k} = \frac{1}{k}\mathbb{1}_{\cup_{m=1}^{k}[2m-1,2m]}$ *and* $f_{\nu^k} = \frac{1}{k}\mathbb{1}_{\cup_{m=1}^{k}[2m-2,2m-1]}$*. Clearly,* $V_{\mu^k} = 1$ *and* $V_{\nu^k} = 0$*,* $\forall k \geq 1$*.*

For this reason, we introduce an asymptotic regularity condition for evaluations, called the *long-term condition* (LTC for short), to express the "extremely long duration" and the "asymptotic uniformity of distributions over $\mathbb{R}_+$", and we will study the convergence of the value functions along a sequence of evaluations satisfying the LTC.

More precisely, for any $s \geq 0$, we define the *s-total variation* of an evaluation θ to be the total variation between the measure θ and its s-shift along $\mathbb{R}_+$:

$$TV_s(\theta) = \max_{Q \in \mathcal{B}(\mathbb{R}_+)} |\theta(Q) - \theta(Q+s)|.$$

We say that a sequence of evaluations $(\theta^k)_{k\geq 1}$ satisfies the LTC if:

$$\forall S > 0 \ \sup_{0 \leq s \leq S} TV_s(\theta^k) \xrightarrow[k\to\infty]{} 0.$$

The optimal control problem $\mathcal{J} = \langle U, g, f \rangle$ is said to have a *general limit value* given by some function V^* defined on $\mathbb{R}^d$ if for any sequence $(\theta^k)_k$ satisfying the LTC, $(V_{\theta^k}(y_0))_k$ converges uniformly to V^* as k tends to infinity.

Our main result (Theorem 4.1) states that for any $(\theta^k)_k$ satisfying the LTC, $(V_{\theta^k})_k$ converges uniformly if and only if the family $\{V_{\theta^k}\}$ is totally bounded with respect to the uniform norm. Moreover, in this case, the limit is characterized by the following:

$$V^*(y_0) =_{def} \sup_{\theta \in \Delta(\mathbb{R}_+)} \inf_{s \in \mathbb{R}_+} \inf_{u \in \mathcal{U}} \int_{t=0}^{\infty} g\left(y(t+s,u,y_0), u(t+s)\right) \mathrm{d}\theta(t), \quad \forall y_0 \in \mathbb{R}^d. \tag{1.3}$$

The above function V^* naturally appears to be the unique possible long-term value function of the control problem.

As a byproduct of our main result, we obtain the existence of general limit value for any control problem $\mathcal{J} = \langle U, g, f \rangle$ with a running cost g that does not depend on u and with a control dynamic (1.1) which is non-expansive and has a compact invariant set. This can be viewed as a generalization of already obtained results in [8] for optimal control with Cesàro mean.

Existing results in the erdogic control literature are concerned mainly with the convergence of the t-horizon Cesàro mean values or the convergence of the λ-discounted Abel mean values. To the best of the authors' knowledge, this paper is the first to consider general long-term evaluations for optimal control problems.

Also it is worth pointing out that while many works (including [1], [2], [3], [4], [6], [7]) suppose controllability or ergodicity conditions, the present approach does not reply on such conditions. This could be understood by the fact that the limit value V^* may depend on the initial state y_0 (which does not occur under ergodic or controlability assumptions).

We also make here a link with the discrete time framework, where an evaluation $\theta = (\theta_m)_{m \geq 1}$ is a probability measure over positive integers $\mathbb{N}^* = \mathbb{N} \backslash \{0\}$, and θ_t is the weight for the stage-t payoff. The analogue notion of *total variation* is defined for any $\theta \in \Delta(\mathbb{N}^*)$: $TV(\theta) = \sum_{m=1}^{\infty} |\theta_{m+1} - \theta_m|$ (cf. [12] and [10]). Recently, the existence of general limit value of dynamic optimization problems in several discrete time frameworks has been obtained in [10], [11] and [13]. Our work is partially inspired by [10]. Similar idea and tool for the proof appeared in [9].

The article is organized as follows. Section 2 contains some preliminary notations and basic examples. The long-term condition is introduced and studied in Section 3. Section 4 contains our main result and its consequences. We discuss in the end of this section two (counter)examples. Section 5 is devoted to the proof of the main result. A weaker notation of LTC is discussed in Section 6.

2 Preliminaries

Consider now the optimal control problem $\mathcal{J} = \langle U, g, f \rangle$ described by (1.1)-(1.2). We make the floowing assumptions on g and f:

$$\begin{cases} \text{the function } g : \mathbb{R}^d \times U \to \mathbb{R} \text{ is Borel measurable and bounded;} \\ \text{the function } f : \mathbb{R}^d \times U \to \mathbb{R}^d \text{ is Borel measurable, and satisfies:} \\ (*).\ \exists L \geq 0, \forall (y, \overline{y}) \in \mathbb{R}^{2d}, \forall u \in U, ||f(y,u) - f(\overline{y}, u)|| \leq L||y - \overline{y}||, \\ (**).\ \exists a > 0, \forall (y,u) \in \mathbb{R}^d \times U, ||f(y,u)|| \leq a(1 + ||y||). \end{cases} \tag{2.1}$$

Under these hypotheses, given any control u in $\mathcal{U}$, (1.1) has a unique absolutely continuous solution $t \mapsto y(t, u, y_0)$ defined on defined on $[0, +\infty)$. As the running cost function $g : \mathbb{R}^d \times U \to \mathbb{R}$ is bounded, we can always assume that $g : \mathbb{R}^d \times U \to [0, 1]$ after some affine transformation.

Below we introduce several notations.

θ-evaluated cost $\gamma_\theta(y_0,u)$ Given $\theta \in \Delta(\mathbb{R}_+)$ and $y_0 \in \mathbb{R}^d$, the θ-evaluated cost induced by a control u is denoted by:

$$\gamma_\theta(y_0,u) = \int_0^\infty g\left(y(s,u,y_0), u(s)\right) \mathrm{d}\theta(s),$$

With this notation, the θ-value function in (1.2) writes as $V_\theta(y_0) = \inf_{u\in\mathcal{U}} \gamma_\theta(y_0,u)$.

Reachable map R_t For any $y_0 \in \mathbb{R}^d$, the reachable map in $\mathbb{R}_+$, $t \mapsto R_t(y_0)$, is defined as:

$$R_t(y_0) = \{\overline{y} \in \mathbb{R}^d | \exists\ u \in \mathcal{U} : y(t,u,y_0) = \overline{y}\}. \tag{2.2}$$

$R_t(y_0)$ represents the set of states that via certain control the dynamic can reach in at time t, starting from the initial state y_0 at time 0. We write $R^t(y_0) = \cup_{s=0}^t R_s(y_0)$ and $R(y_0) = \cup_{s=0}^\infty R_s(y_0)$. $R(y_0)$ is the set of states that can be reached in any finite time starting from y_0.

Image measure $\mathcal{T}_t\sharp\theta$ and the auxiliary value function $V_{\mathcal{T}_t\sharp\theta}$ Given $t \in \mathbb{R}$ and θ in $\Delta(\mathbb{R}_+)$, we use $\mathcal{T}_t\sharp\theta$ to denote the image (push-forward) measure of θ by the function $s \mapsto s+t$, *i.e.*,

$$\mathcal{T}_t\sharp\theta(Q) = \theta\left(\mathcal{T}_t^{-1}(Q)\right) = \theta(Q-t),\ \forall Q \in \mathcal{B}(\mathbb{R}_+),$$

where $\mathcal{B}(\mathbb{R}_+)$ denotes the set of all Borel subsets in $\mathbb{R}_+$. This leads us to write the t-shift θ-evaluated cost induced by a control u as following:

$$\gamma_{\mathcal{T}_t\sharp\theta}(y_0,u) = \int_0^\infty g\left(y(s+t,u,y_0), u(s+t)\right) \mathrm{d}\theta(s),\ \forall t \geq 0. \tag{2.3}$$

Taking on both sides of (2.3) the infimum over $u \in \mathcal{U}$ and using the notation of reachable map R_t, we obtain the t-shift θ-value function

$$V_{\mathcal{T}_t\sharp\theta}(y_0) = \inf_{u\in\mathcal{U}} \int_0^{+\infty} g(y(s+t,u,y_0), u(s+t))\mathrm{d}\theta(s) = \inf_{\overline{y}\in R_t(y_0)} V_\theta(\overline{y}). \tag{2.4}$$

The interpretation of the function $V_{\mathcal{T}_t\sharp\theta}(y_0)$ is the following: consider the control problem where the controller is allowed to choose (with no cost) a "good" initial state in the reachable set $R_t(y_0)$ and the evaluation θ begins from the time t on, and $V_{\mathcal{T}_t\sharp\theta}(y_0)$ is the corresponding value.

s-total variation Given an evaluation θ, define its s-total variation for each $s \geq 0$:

$$TV_s(\theta) = \sup_{Q\in\mathcal{B}(\mathbb{R}_+)} |\theta(Q) - \theta(Q+s)|. \tag{2.5}$$

Long-term condition (LTC) A sequence of evaluations $(\theta^k)_{k\geq1}$ satisfies the LTC if:

$$\forall S > 0,\ \overline{TV}_S(\theta^k) =_{def} \sup_{0\leq s\leq S} TV_s(\theta^k) \xrightarrow[k\to\infty]{} 0. \tag{2.6}$$

Definition 2.1 *Let V be a function defined on $\mathbb{R}^d$. The optimal control problem $\mathcal{J}$ has V as a **general limit value** if and only if: for any sequence of evaluations $(\theta^k)_{k\geq1}$ satisfying the LTC, for all y_0 in $\mathbb{R}^d$, $(V_{\theta^k}(y_0))$ converges to $V(y_0)$ as k tends to infinity, and moreover the convergence is uniform in y_0.*

Below are several some basic examples.

Example 2.2 *Here y lies in $\mathbb{R}^2$ seen as the complex plane, there is no control, and the dynamic is given by $f(y,u) = i\ y$, where $i^2 = -1$. We clearly have*

$$V_{\theta^k}(y_0) \xrightarrow[k\to\infty]{} \frac{1}{2\pi}\int_0^{2\pi} g(|y_0|e^{rit})\mathrm{d}t,$$

for any sequence of evaluations $(\theta^k)_k$ satisfying the LTC.

Example 2.3 *Here y lies in the complex plane again, with $f(y,u) = i\ y\ u$, where $u \in U$ is a given bounded subset of $\mathbb{R}$, and g is any continuous function in y (which thus does not depend on u).*

Example 2.4 *$f(y,u) = -y + u$, where $u \in U$ a given bounded subset of $\mathbb{R}^d$, and g is any continuous function in y (which thus does not depend on u).*

We show later (using Corollary 4.6) that the general limit value exists in Examples 2.3 and 2.4.

3 On the long-term condition (LTC)

In this section, we discuss the LTC. First, we give the following remarks.

Remark 3.1 *(a). From the definition, one obtains*

$$\forall s \geq 0, \forall t \geq 0, \forall \theta \in \Delta(\mathbb{R}_+),\ TV_{s+t}(\theta) \leq TV_s(\theta) + TV_t(\theta).$$

This implies that $(\theta^k)_{k\geq 1}$ satisfies the LTC if and only if $\exists S_0 > 0$, s.t. $\overline{TV}_{S_0}(\theta^k) \xrightarrow[k\to\infty]{} 0$.
(b). If one takes $Q = \mathbb{R}_+$ in definition of $TV_s(\theta^k)$ for each $s \geq 0$ and each $k \geq 1$, we deduce that if $(\theta^k)_{k\geq 1}$ satisfies the LTC, then $\theta^k([0,s]) \xrightarrow[k\to\infty]{} 0$ for any $s \geq 0$.

Remark 3.2 *Let θ be an evaluation absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+$, and f_θ its density. Scheffé Theorem (cf. [5], Theorem 1 in p.12) implies that:*

$$\forall s \geq 0,\ 2TV_s(\theta) = I_s(\theta) =_{def} \int_{t=0}^{\infty} |f_\theta(t+s) - f_\theta(t)|\mathrm{d}t.$$

Thus, if $(\theta^k)_{k\geq 1}$ is a sequence of evaluations with densities $(f_{\theta^k})_{k\geq 1}$:
(a). $(\theta^k)_{k\geq 1}$ satisfies the LTC if and only if $\sup_{0\leq s\leq 1} I_s(\theta^k) \xrightarrow[k\to\infty]{} 0$. If moreover, for each $k \geq 1$, $t \mapsto f_{\theta^k}(t)$ is non increasing on $\mathbb{R}_+$, then $(\theta^k)_{k\geq 1}$ satisfies the LTC if and only if $\forall s \geq 0$, $\theta^k([0,s]) = \int_{t=0}^{\infty} f_\theta(t)\mathrm{d}t - \int_{t=0}^{\infty} f_\theta(t+s)\mathrm{d}t \xrightarrow[k\to\infty]{} 0$.
(b). if $(\theta^k)_{k\geq 1}$ satisfies the LTC, then $\int_{t=0}^{\infty} t f_{\theta^k}(t)\mathrm{d}t \xrightarrow[k\to\infty]{} \infty$. Indeed, Chebychev's inequality gives that $\int_{t=0}^{\infty} t f_{\theta^k}(t)\mathrm{d}t \geq M\left(1 - \theta^k([0,M])\right)$ for all $M > 0$.

Here we discuss several cases where the LTC condition is satisfied.

Example 3.3 *(Uniform distributions) Assume that for each k, θ^k is the uniform law over the interval $[a_k, b_k]$, with $0 \leq a_k \leq b_k$. For each k,*

- $\underline{s \geq b_k - a_k}$: $I_s(\theta^k) = \begin{cases} \frac{2}{b_k - a_k} & \text{if} \quad 0 < s < a_k \\ \frac{1+(b_k - s)}{b_k - a_k} & \text{if} \quad a_k < s < b_k \\ \frac{1}{b_k - a_k} & \text{if} \quad b_k < s \end{cases}$,
- $\underline{s < b_k - a_k}$: $I_s(\theta^k) = \begin{cases} \frac{2s}{b_k - a_k} & \text{if} \quad 0 < s < a_k \\ \frac{2s}{b_k - a_k} & \text{if} \quad a_k < s < b_k \end{cases}$.

One can check easily that $(\theta^k)_k$ satisfies the LTC if and only if $b_k - a_k \xrightarrow[k\to\infty]{} \infty$. Indeed, by Remark 3.2 (a), it is sufficient to look at $I_s(\theta^k)$ for $s \in [0, 1]$.

Example 3.4 *(Abel average) Assume that for each k, θ^k has density $s \mapsto f_{\theta^k}(s) = \lambda_k e^{-\lambda_k s} \mathbb{1}_{\mathbb{R}_+}(s)$, with $\lambda_k > 0$. Since $\forall k \geq 1$, $s \mapsto f_{\theta^k}(s)$ is non increasing, so Remark 3.2 (a) implies that $(\theta^k)_k$ satisfies the LTC if and only if: $\forall T > 0$, $\theta^k([0, T]) = \int_{s=0}^{T} \lambda_k e^{-\lambda_k s} \mathrm{d}s = 1 - e^{-T/\lambda_k} \xrightarrow[k\to\infty]{} 0$, which is again equivalent to $\lambda_k \xrightarrow[k\to\infty]{} 0$.*

Example 3.5 *(Folded normal distributions) Assume that for each k, θ^k is the distribution of a random variable $|X^k|$, where X^k follows a normal law $\mathcal{N}(m_k, \sigma_k^2)$. The density of θ^k is given by:*

$$\forall t \geq 0, \ f_{\theta^k}(t) = \frac{1}{\sigma_k \sqrt{2\pi}} \left[\exp\left(-\frac{1}{2} \Big(\frac{t - m_k}{\sigma_k} \Big)^2 \right) + \exp\left(-\frac{1}{2} \Big(\frac{t + m_k}{\sigma_k} \Big)^2 \right) \right].$$

Claim 3.1 *$(\theta^k)_k$ satisfies the LTC if and only if $\sigma_k \xrightarrow[k\to\infty]{} \infty$.*

Our argument relies on the following lemma, whose proof is put in the **Appendix**. Without loss of generality, we may assume that m_k is non-negative for each k.

Lemma 1 *Let θ be the distribution of X where $|X|$ follows the normal law $\mathcal{N}(m, \sigma)$ with $m, \sigma > 0$. There exists some $t^* \in [0, m)$ such that $f'_\theta(t) > 0$ for any $t \in (0, t^*)$ and $f'_\theta(t) < 0$ for any $t \in (t^*, \infty)$.*

Proof for Claim 3.1 We apply Lemma 1 to each evaluation θ_k to obtain some $t_k^* \in [0, m_k)$ such that: $f_{\theta^k}(\cdot)$ is increasing on $[0, t_k^*)$ and decreasing on $[t_k^*, \infty)$. This enables us to write:

$$\forall s \leq t_k^*, \quad I_s(\theta^k) = \int_{t_k^* - s}^{t_k^*} f_{\theta^k}(t) \mathrm{d}t + \int_{t_k^* - s}^{t_k^*} |f_{\theta^k}(t + s) - f_{\theta^k}(t)| \mathrm{d}t + \int_{t_k^*}^{t_k^* + s} f_{\theta^k}(t) \mathrm{d}t.$$

We deduce then $sf_{\theta^k}(t_k^*-s) \leq I_{\theta^k}(s) \leq 4sf_{\theta^k}(t_*^k)$ for $s \leq t_k^*$. Assume below $\hat{t}^* =_{def} \liminf_{k\to\infty} t_k^* > 0$, and the analysis is analogue for $\hat{t}^*=0$, which we omit here.
(*). Suppose that $\sigma_k \to \infty$, then

$$f_{\theta^k}(t_k^*) = \frac{1}{\sigma_k\sqrt{2\pi}}\left[\exp\left(-\frac{1}{2}\left(\frac{t_k^*-m_k}{\sigma_k}\right)^2\right)+\exp(-\frac{1}{2}\left(\frac{t_k^*+m_k}{\sigma_k}\right)^2)\right] \leq \frac{2}{\sigma_k\sqrt{2\pi}} \xrightarrow[k\to\infty]{} 0.$$

This implies that: $\forall S \in [0,\hat{t}^*]$, $\sup_{0\leq s\leq S} I_s(\theta^k) \xrightarrow[k\to\infty]{} 0$.
(**). Conversely, suppose that $(\theta^k)_k$ satisfies the LTC. Then for any $s<\hat{t}^*$, $I_s(\theta^k)$ thus $f_{\theta^k}(t_k^*-s)$ vanishes as k tends to infinity. This implies that either $\sigma_k \to \infty$ or $(\sigma_k)_k$ is bounded and $\big(m_k-(t_k^*-s)\big)_k \to \infty$. Lemma 1 shows that the specified point t_k^* for the evaluation θ_k satisfies $(t_k^*)^2 \geq m_k^2-\sigma_k^2$, thus $m_k-(t_k^*+s) \leq m_k-t_k^* \leq \frac{\sigma_k^2}{m_k+t_k^*} \leq \frac{\sigma_k^2}{m_k}$. If $(\sigma_k)_k$ is bounded, $(m_k-t_k^*)_k$ thus $(m_k)_k$ should tend to infinity, but this leads to a contradiction with $m_k-t_k^* \leq \frac{\sigma_k^2}{m_k}$. □

Below we link the LTC condition to the discrete time framework. In a discrete time dynamic optimization problem, a general evaluation on the payoff stream is a probability distribution over $\mathbb{N}^* = \mathbb{N}/\{0\}$ the set of postive integers. For any $\theta=(\theta_1,...,\theta_t,...)$ in $\Delta(\mathbb{N}^*)$, its "total variation" $TV(\theta)=\sum_{m=1}^{\infty}|\theta_{m+1}-\theta_m|$ is the stage by stage absolute difference between the measure θ and its one-stage "shift" measure $\theta'=(\theta_2,...,\theta_{t+1},...)$. (cf. Sorin [12] or Renault [11]).

Compare with any ξ in $\Delta(\mathbb{R}_+)$: $TV_s(\xi)=\sup_{Q\in\mathcal{B}(\mathbb{R}_+)}|\xi(Q)-\xi(Q+s)|$ is the total variation of the measure ξ and its s-shift image measure $\mathcal{T}_{-s}\sharp\xi$. For ξ being absolutely continuous, $2TV_s(\xi)=I_s(\xi)=\int_{t=0}^{\infty}|f_\xi(t+s)-f_\xi(t)|\mathrm{d}t$. When the sequence of evaluations admits step functions as densities, this link is much clearer as seen by the following

Proposition 3.6 *Let $(\theta^k)_k$ be a sequence of absolutely continuous evaluations in $\Delta(\mathbb{R}_+)$, and their densities are given as: $\forall k\geq 1$, $f_{\theta^k}=\sum_{m=1}^{\infty}\overline{\theta}_m^k \mathbb{1}_{[m-1,m)}$, where $(\overline{\theta}_m^k)_{m\geq 1}$ is a non negative sequence summing to 1. Then $(\theta^k)_k$ satisfies the LTC if and only if $\sum_{m=1}^{\infty}|\overline{\theta}_{m+1}^k-\overline{\theta}_m^k| \xrightarrow[k\to\infty]{} 0$.*

Proof: Fix $s\in[0,1]$. We shall write for each k,

$$I_s(\theta^k)=\sum_{m=1}^{\infty}\int_{[m-1,m)}\Big|f_{\theta^k}(t+s)-f_{\theta^k}(t)\Big|\mathrm{d}t.$$

For each $m=1,2,...$, we have

$$\begin{aligned}\int_{[m-1,m)}\Big|f_{\theta^k}(t+s)-f_{\theta^k}(t)\Big|\mathrm{d}t=&\int_{[m-1,m-s)}\Big|f_{\theta^k}(t+s)-f_{\theta^k}(t)\Big|\mathrm{d}t+\int_{[m-s,m)}\Big(f_{\theta^k}(t+s)-f_{\theta^k}(t)\Big)\mathrm{d}t\\ =&s|\overline{\theta}_{m+1}^k-\overline{\theta}_m^k|.\end{aligned}$$

As a consequence,$I_s(\theta^k)=s\sum_{m=1}^{\infty}|\overline{\theta}_{m+1}^k-\overline{\theta}_m^k| \leq \sum_{m=1}^{\infty}|\overline{\theta}_{m+1}^k-\overline{\theta}_m^k|$, $\forall s\in[0,1]$. In view of Remark 3.2, $(\theta^k)_k$ satisfies the LTC if and only if $\sum_{m=1}^{\infty}|\overline{\theta}_{m+1}^k-\overline{\theta}_m^k| \xrightarrow[k\to\infty]{} 0$. □

We end this section by a technical lemma, which will be useful in later results.

Lemma 3.7 *Fix any $\theta \in \Delta(\mathbb{R}_+)$ and any $t \in \mathbb{R}_+$, we have*

$$\left|\int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[0,+\infty)} h(s-t)\mathrm{d}\theta(s)\right| \leq TV_t(\theta)$$

and

$$\left|\int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[0,+\infty)} h(s+t)\mathrm{d}\theta(s)\right| \leq 2TV_t(\theta),$$

for any $h \in \mathcal{M}(\mathbb{R}_+, [0,1])$, where $\mathcal{M}(\mathbb{R}_+, [0,1]) = \Big\{h \Big| h : \mathbb{R}_+ \to [0,1], \text{Borel measurable}\Big\}$.

Proof: By definition of $\mathcal{T}_s\sharp\theta$ for any θ in $\Delta(\mathbb{R}_+)$ and $s \in \mathbb{R}$, we have

$$\int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[t,+\infty)} h(s-t)\mathrm{d}\theta(s) = \int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[0,+\infty)} h(s)\mathrm{d}\mathcal{T}_{-t}\sharp\theta(s) \quad (3.1)$$

and

$$\int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[0,+\infty)} h(s+t)\mathrm{d}\theta(s) = \int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[0,+\infty)} h(s)\mathrm{d}\mathcal{T}_t\sharp\theta(s). \quad (3.2)$$

Since $\mathcal{T}_{-t}\sharp\theta$ and $\mathcal{T}_t\sharp\theta$ are both Borel measures on $\mathbb{R}_+$, "$\theta - \mathcal{T}_{-t}\sharp\theta$" and "$\theta - \mathcal{T}_t\sharp\theta$" are both signed measures. Hahn's decomposition theorem[1] implies that:

$$\sup_{h\in\mathcal{M}(\mathbb{R}_+,[0,1])}\left|\int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[0,+\infty)} h(s)\mathrm{d}\mathcal{T}_{-t}\sharp\theta(s)\right| = \sup_{Q\in\mathcal{B}(\mathbb{R}_+)}\Big|\theta(Q) - \mathcal{T}_{-t}\sharp\theta(Q)\Big|.$$

and

$$\sup_{h\in\mathcal{M}(\mathbb{R}_+,[0,1])}\left|\int_{[0,+\infty)} h(s)\mathrm{d}\theta(s) - \int_{[0,+\infty)} h(s)\mathrm{d}\mathcal{T}_t\sharp\theta(s)\right| = \sup_{Q\in\mathcal{B}(\mathbb{R}_+)}\Big|\theta(Q) - \mathcal{T}_t\sharp\theta(Q)\Big|.$$

Combining with (3.1)-(3.2), we obtain:

$$\left|\int_{s=0}^{\infty} h(s)\mathrm{d}\theta(s) - \int_{s=t}^{\infty} h(s-t)\mathrm{d}\theta(s)\right| = \sup_{Q\in\mathcal{B}(\mathbb{R}_+)}\Big|\theta(Q) - \theta(Q+t)\Big| = TV_t(\theta)$$

and

$$\left|\int_{s=0}^{\infty} h(s)\mathrm{d}\theta(s) - \int_{s=0}^{\infty} h(s+t)\mathrm{d}\theta(s)\right| \leq \sup_{Q\in\mathcal{B}(\mathbb{R}_+)}\Big|\theta(Q) - \theta(Q-t)\Big| \leq \theta\big([0,t)\big) + TV_t(\theta) \leq 2TV_t(\theta).$$

The proof for the lemma is complete. □

[1] The first author acknowledges Eilon Solan for the discussion on using Hahn's decomposition theorem.

4 Main Result

First rewrite the function $V^*(y_0)$ (defined in (1.3)) as:

$$V^*(y_0) = \sup_{\theta \in \Delta(\mathbb{R}_+)} \inf_{t \in \mathbb{R}_+} V_{\mathcal{T}_t \sharp \theta}(y_0) = \sup_{\theta \in \Delta(\mathbb{R}_+)} \inf_{\overline{y} \in R(y_0)} V_\theta(\overline{y}).$$

We shall give the following interpretation: consider the auxiliary optimal control problem (game) where an adversary of the controller chooses a worst evaluation θ, and then knowing the θ as given, the controller chooses a "good" initial state in the reachable set of any finite time t. The running cost from the time t is evaluated by θ. $V^*(y_0)$ is then the value of this auxiliary problem starting from y_0.

Recall that a metric space X is *totally bounded* if for each $\varepsilon > 0$, X can be covered by finitely many balls of radius ε.

Theorem 4.1 *Let $(\theta^k)_{k\geq 1}$ be a sequence of evaluations satisfying the LTC. Assume (2.1) for the optimal control problem $\mathcal{J} = \langle U, g, f \rangle$. Then,*
(i). $V^* = \sup_{k\in\mathbb{N}} \inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t \sharp \theta^k}$.
(ii). Any accumulation point (for the uniform convergence) of the sequence $(V_{\theta^k})_k$ is equal to V^*.
(iii). The sequence $(V_{\theta^k})_k$ uniformly converges if and only if the space $(\{V_{\theta^k}\}, ||\cdot||_\infty)$ is totally bounded.

Remark 4.2 *Let $(\theta^k)_k$ be a sequence of evaluations which contains a subsequence $(\theta^{\varphi_k})_k$ satisfying the LTC. Then Part (i) of Theorem 4.1 still holds true for $(\theta^k)_k$.*

A more precise convergence result is obtained if we suppose that there exists a compact set $Y \subseteq \mathbb{R}^d$ which is *invariant* for (1.1), *i.e.*, $y(t, u, y_0) \in Y$ for all $u \in \mathcal{U}$, $t \geq 0$ and y_0 in Y.

Corollary 4.3 *Assume (1.1) and (2.1) for the optimal control problem $\mathcal{J} = \langle U, g, f \rangle$. Suppose that there is a compact set $Y \subseteq \mathbb{R}^d$ which is invariant for (1.1), and that the family $\{V_\theta : \theta \in \Delta(\mathbb{R}_+)\}$ is uniformly equicontinuous on Y. Then there is general uniform convergence of the value functions $\{V_\theta\}$ to V^*, namely,*

$$\forall \varepsilon > 0, \exists S > 0, \exists \eta > 0 \text{ s.t. } \forall \theta \in \Delta(\mathbb{R}_+), \text{with } \overline{TV}_S(\theta) \leq \eta,\ ||V_\theta - V^*||_\infty \leq \varepsilon.$$

Proof: By assumption, the family of value functions $\{V_\theta : \theta \in \Delta(\mathbb{R}_+)\}$ is both uniformly bounded and uniformly equicontinuous on the compact invariant set Y, so we can use Ascoli's theorem to deduce the totally boundedness of the space $(\{V_\theta\}, ||\cdot||_\infty)$. Theorem 4.1 implies that: for any $(\theta^k)_k$ satisfying the LTC, the corresponding sequence of value functions (V_{θ^k}) converges uniformly to V^* as k tends to infinity. Thus $\mathcal{J}$ has a general limit value given as V^*.

Next we show that the existence of general limit value given as V^* is sufficient to deduce the general uniform convergence of $\{V_\theta\}$ to V^*. Suppose by contradiction that there is no general uniform convergence of $\{V_\theta\}$ to V^*, *i.e.*,

$$\exists \varepsilon_0 > 0,\ \forall S > 0,\ \forall \eta^k > 0,\ \exists \theta^k \in \Delta(\mathbb{R}_+) \text{ with } \overline{TV}_S(\theta^k) \leq \eta^k, \text{ and } ||V_{\theta^k} - V^*||_\infty > \varepsilon_0,\ \forall k \geq 1.$$

Let $\varepsilon_0 > 0$ be fixed as above. We take a vanishing positive sequence $(\eta^k)_k$ and some $S_0 > 0$, then there is a sequence of evaluations (θ^k) with $\overline{TV}_{S_0}(\theta^k) \leq \eta^k \xrightarrow[k\to\infty]{} 0$, and $\liminf_k ||V_{\theta^k} - V^*||_\infty \geq \varepsilon_0$. According to Remark 3.1 (a), such $(\theta^k)_k$ satisfies the LTC, while (V_{θ^k}) does not converges uniformly to V^*. This is a contradiction. The proof is complete. □

Remark 4.4 *From the proof of Corollary 4.3, we see that the general uniform convergence of $\{V_\theta\}$ to V^* and the existence of general limit value V^* are equivalent.*

We shall give the existence result of general limit value under sufficient conditions expressed directly in terms of properties of the control dynamic (1.1) and of the running cost g.

Let us introduce the following *non expansive* condition (cf. [8]). The control dynamic (1.1) is non expansive if

$$\forall y_1, y_2 \in \mathbb{R}^d, \ \sup_{a\in U} \inf_{b\in U} \langle y_1 - y_2, f(y_1, a) - f(y_2, b)\rangle \leq 0.$$

Definition 4.5 *The optimal control problem $\mathcal{J} = \langle U, g, f\rangle$ is called* ***compact non expansive*** *if it satisfies the following three conditions:*
$(A.1)$ there is a compact set $Y \subseteq \mathbb{R}^d$ is the invariant for (1.1);
$(A.2)$ the running cost function $g(\cdot)$ does not depend on u, and is continuous in y;
$(A.3)$ the control dynamic (1.1) is non expansive on Y.

Corollary 4.6 *Assume (2.1) for the optimal control problem $\mathcal{J} = \langle U, g, f\rangle$. Suppose that that $\mathcal{J}$ is compact non expansive, then the general limit value exists in $\mathcal{J}$ and is given as V^*.*

Proof: Under $(A.1)$ and $(A.3)$, Proposition 3.7 in [8] implies that:

$$\forall (y_1, y_2) \in Y^2, \forall u \in \mathcal{U}, \exists v \in \mathcal{U}, s.t.\ \forall t \geq 0,\ ||y(t, u, y_1) - y(t, v, y_2)|| \leq ||y_1 - y_2||. \tag{4.1}$$

We claim that the family $(V_\theta)_{\theta\in\Delta(\mathbb{R}_+)}$ is uniformly equicontinuous on Y, thus Corollary 4.3 and Remark 4.4 apply. Fix any $(y_1, y_2) \in Y^2$, $\theta \in \Delta(\mathbb{R}_+)$, and $\varepsilon > 0$. Let u be ε-optimal for $V_\theta(y_1)$:

$$V_\theta(y_1) \geq \int_{s=0}^{+\infty} g\big(y(s, u, y_1)\big) \mathrm{d}\theta(s) - \varepsilon.$$

By the non expansive property, there exists v in $\mathcal{U}$ as in (4.1) such that

$$||y(s, u, y_1) - y(s, v, y_2)|| \leq ||y_1 - y_2||,\ \forall s \geq 0. \tag{4.2}$$

By definition, $V_\theta(y_2) \leq \int_{s=0}^{+\infty} g\big(y(s, y_2, v)\big)\mathrm{d}\theta(s)$, hence

$$V_\theta(y_2) - V_\theta(y_1) \leq \int_{s=0}^{+\infty} \Big[g\big(y(s, v, y_2)\big) - g\big(y(s, u, y_1)\big)\Big]\mathrm{d}\theta(s) + \varepsilon.$$

Denoting ω_g the modulus of continuity of g, we obtain in view of (4.2):

$$V_\theta(y_2) - V_\theta(y_1) \ \leq \ \int_{s=0}^{+\infty} \Big[g\big(y(s, v, y_2)\big) - g\big(y(s, u, y_1)\big)\Big]\mathrm{d}\theta(s) + \varepsilon \leq \omega_g(||y_1 - y_2||) + \varepsilon.$$

Interchanging y_1 and y_2 and taking into account of $\varepsilon > 0$ being arbitrary, we deduce that $(V_\theta)_{\theta\in\Delta(\mathbb{R}_+)}$ is uniformly equicontinuous on the invariant set Y. This finishes the proof. □

Remark 4.7 *Both Example 2.3 and Example 2.4 satisfy conditions in Corollary 4.6, so there is general uniform convergence of the value functions $\{V_\theta\}$ (the existence of general limit value).*

Remark 4.8 *Our result generalizes Proposition 3.3 in [8] which proved the uniform convergence of the t-horizon values in compact non expansive optimal control problems.*

We end this section by presenting two (counter)examples, showing that the results in Theorem 4.1 are not valid if some of their conditions is not satisfied.

The first example is an uncontrolled dynamic. We show that if $(\theta^k)_k$ contains no subsequence satisfying the LTC, then the result in Part (i) of Theorem 4.1 does not hold, *i.e.*, $\sup_{k\geq 1}\inf_{t\geq 0} V_{\mathcal{T}_t\sharp\theta^k}(y_0) < \sup_{\theta\in\Delta(\mathbb{R}_+)}\inf_{t\geq 0} V_{\mathcal{T}_t\sharp\theta}(y_0)$ for some y_0 (cf. Remark 4.2).

Counter-example 4.9 *Consider the uncontrolled dynamic on $\mathbb{R}$: $y(0) = y_0$ and $y'(t) = -(y(t)-1), \forall t \geq 0$. The trajectory is then $y(t) = 1 + (y_0 - 1)e^{-t}$. The running cost function $g : \mathbb{R} \to [0,1]$ is given by:*

$$g(y) = \begin{cases} 0 & \textit{if} \quad y < 0 \\ y & \textit{if} \quad 0 \leq y \leq 1 \\ 1 & \textit{if} \quad y > 1 \end{cases}$$

We have that $V^*(y_0) = \sup_{\theta\in\Delta(\mathbb{R}_+)}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta}(y_0) = 1$, $\forall y_0 \in \mathbb{R}$. Indeed, let y_0 be given and fix any $\varepsilon > 0$, there is some $T_\varepsilon > 0$ such that $|y(T) - 1| \leq \varepsilon$ for all $T \geq T_\varepsilon$. Take an evaluation θ in $\Delta(\mathbb{R}_+)$ with $\theta([0, T_\varepsilon]) = 0$. This enables us to deduce that: for all $t \geq 0$,

$$V_{\mathcal{T}_t\sharp\theta}(y_0) = \int_{t=T_\varepsilon}^{\infty} g\big(y(s+t)\big)\mathrm{d}\theta(s) \geq \int_{s=T_\varepsilon}^{\infty} g\big(y(T_\varepsilon)\big)\mathrm{d}\theta(s) \geq \big(1 - y(T_\varepsilon)\big)\theta([T_\varepsilon, \infty]) \geq 1 - \varepsilon.$$

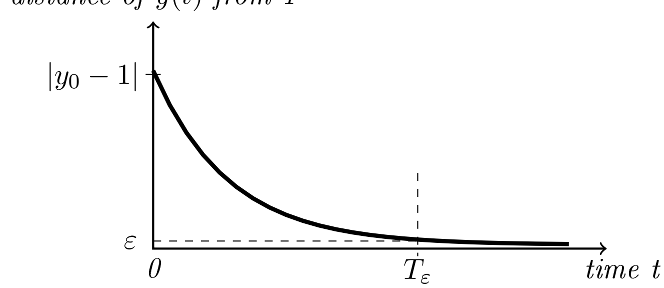

Figure 5.1: *The solution $y(t) = 1 + (y_0 - 1)e^{-t}$ to the dynamic is represented. Here, the thick curve represents the distance of $y(t)$ from the point 1, which is $|y_0 - 1|e^{-t}$ the exponential function e^{-t} times the initial distance $|y_0 - 1|$. For $\varepsilon > 0$, $T_\varepsilon > 0$ is chosen such that $|y(T_\varepsilon) - 1| = \varepsilon$.*

Consider now any sequence of evaluations $(\theta^k)_k$ which does not contain any subsequence satisfying the LTC. Under the assumption that the density f_{θ^k} for each evaluation θ^k is non increasing, we show that Part (i) of Theorem 4.1 is not valid: $V^* \neq \sup_{k\in\mathbb{N}}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^k}$.

Indeed, let us take any $y_0 < 1$ and suppose that $\sup_{k\in\mathbb{N}}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^k}(y_0) = V^*(y_0)$, which is equal to 1 as was proved. Let $\varphi(k)$ be a subsequence such that $\lim_{k\to\infty}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^{\varphi(k)}}(y_0) = 1$.

$(\theta^{\varphi(k)})_k$ does not satisfy the LTC by assumption, so Remark 3.2 (a) implies that there exists some $T > 0$ with $\theta^{\varphi(k)}([0,T]) \nrightarrow 0$. Let φ_m be the subsequence of φ and $\eta > 0$ such that $\theta^{\varphi_m(k)}([0,T]) \xrightarrow[k\to\infty]{} \eta$. We obtain for any $k \geq 1$,

$$\begin{aligned}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^{\varphi_m(k)}}(y_0) \leq V_{\theta^{\varphi_m(k)}}(y_0) &= \int_{t=0}^{T} g\left(y(t)\right) \mathrm{d}\theta^{\varphi_m(k)}(t) + \int_{t=T}^{\infty} g\left(y(t)\right) \mathrm{d}\theta^{\varphi_m(k)}(t) \\ &\leq y(T)\theta^{\varphi_k(m)}([0,T]) + \theta^{\varphi_k(m)}([T,\infty]).\end{aligned}$$

This implies that for such fixed $y_0 < 1$, $\lim_k \inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^{\varphi_m(k)}}(y_0) \leq y(T)\eta + (1-\eta) < 1$. This contradicts the assumption that $\sup_{k\in\mathbb{N}} \inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^k}(y_0) = 1$, and our claim is proved.

In the second example, we study the convergence of the value functions of a control problem along two different sequences of evaluations satisfying the LTC. Along the first sequence, the value functions converge uniformly to V^*; while along the second, the value functions pointwisely converge, but not uniformly (thus the family of value functions is not totally bounded for the uniform norm), to a limit function which is different from V^*.

Counter-example 4.10 *Consider the control problem on the state space $\mathbb{R} = (-\infty, +\infty)$, where the control set is $U = \{+1, -1\}$; the dynamic is[2] :$f(y,u) = u$ for all $(y,u) \in \mathbb{R}_+ \times U$ and $f(y,u) = -1$ for all $(y,u) \in \mathbb{R}_-^* \times U$, where $\mathbb{R}_-^* = \mathbb{R}_-/\{0\}$; and the running cost function is:*

$$g(y,u) = \begin{cases} +1 & \text{if} \quad u = +1,\ y \geq 0 \\ 0 & \text{if} \quad u = -1,\ y \geq 0 \\ +K & \text{if} \qquad y < 0 \end{cases}$$

Suppose that $K > 1$ big enough, so the cost on $\mathbb{R}_-$ is positive and high. Whenever the state reaches $y = 0$, it is optimal to choose control $u = +1$ and this drives the state back to $\mathbb{R}_+$; on $\mathbb{R}_-^*$, the dynamic is $f = -1$, independent of control and state. $V_\theta(y_0) = K$ for all y_0 in $\mathbb{R}_-^*$ and θ in $\Delta(\mathbb{R}_+)$, so the reduced state space is $\mathbb{R}_+$, and we consider value functions defined on it.

$V^*(y_0) = \sup_\theta \inf_{t\geq 0} V_{\mathcal{T}_t\sharp\theta}(y_0) = 0$ for any $y_0 \geq 0$. Fix any $y_0 \geq 0$. For any $\theta \in \Delta(\mathbb{R}_+)$ and $\varepsilon > 0$, let $t^\varepsilon \geq 0$ such that $\theta([0,t^\varepsilon]) \geq 1 - \varepsilon$. Define now the control $u^\varepsilon(\cdot)$ to be: $u^\varepsilon(t) = +1$, if $t \in [0,t^\varepsilon]$ and $u^\varepsilon(t) = -1$ if $t \in (t^\varepsilon, \infty)$, which gives: $\gamma_{\mathcal{T}_{t^\varepsilon}\sharp\theta}(y_0, u^\varepsilon) \leq \varepsilon K$.

Consider $(\theta^k)_k$ the sequence of evaluations with density $f_{\theta^k}(s) = \frac{1}{k}\mathbb{1}_{[k,2k]}(s)$ for each k, and $(\bar\theta^k)_k$ the sequence of k-horizon evaluations with density $f_{\bar\theta^k}(s) = \frac{1}{k}\mathbb{1}_{[0,k]}(s)$ for each k. We show that:

$(\{V_{\theta^k}\}, ||\cdot||_\infty)$ is totally bounded and (V_{θ^k}) converges uniformly to V^*; while $(\{V_{\bar\theta^k}\}, ||\cdot||_\infty)$ is not totally bounded and $(V_{\bar\theta^k})$ does not converge to V^*.

Let $y_0 \geq 0$, we have that:

1. *$V_{\theta^k}(y_0) = 0$, for all $k \geq 1$. Indeed, one optimal control for $V_{\theta^k}(y_0)$ can be taken as: $u^*(t) = +1,\ t \in [0,k]$ and $u^*(t) = -1,\ t \in (k, 2k]$;*

[2]Notice that the dynamic is discontinuous at $y = 0$ when $u = +1$. To get the desired asymptotic result under the Liptchitz regularity, one can slightly modify dynamic to set $f(y,+1) = y$ for $y \in [0,1]$ and others unchanged.

2. $V_{\overline{\theta}^k}(y_0) = 0$ *if* $k \leq y_0$ *and* $V_{\overline{\theta}^k}(y_0) = \frac{1}{2} - \frac{y_0}{2k}$ *if* $k > y_0$. *Indeed, for* $k \leq y_0$, *one optimal control for* $V_{\overline{\theta}^k}(y_0)$ *can be taken as:* $u^*(t) = -1,\ t \in [0,k]$; *for* $k > y_0$, *one optimal control for* $V_{\overline{\theta}^k}(y_0)$ *can be taken as:* $u^*(t) = +1,\ t \in [0, \frac{k-y_0}{2}]$ *and* $u^*(t) = -1,\ t \in (\frac{k-y_0}{2}, k]$, *so* $\gamma_{\overline{\theta}^k}(y_0, u^*) = \frac{(k-y_0)/2}{k} = \frac{1}{2} - \frac{y_0}{2k}$.

See the following two pictures for illustration.

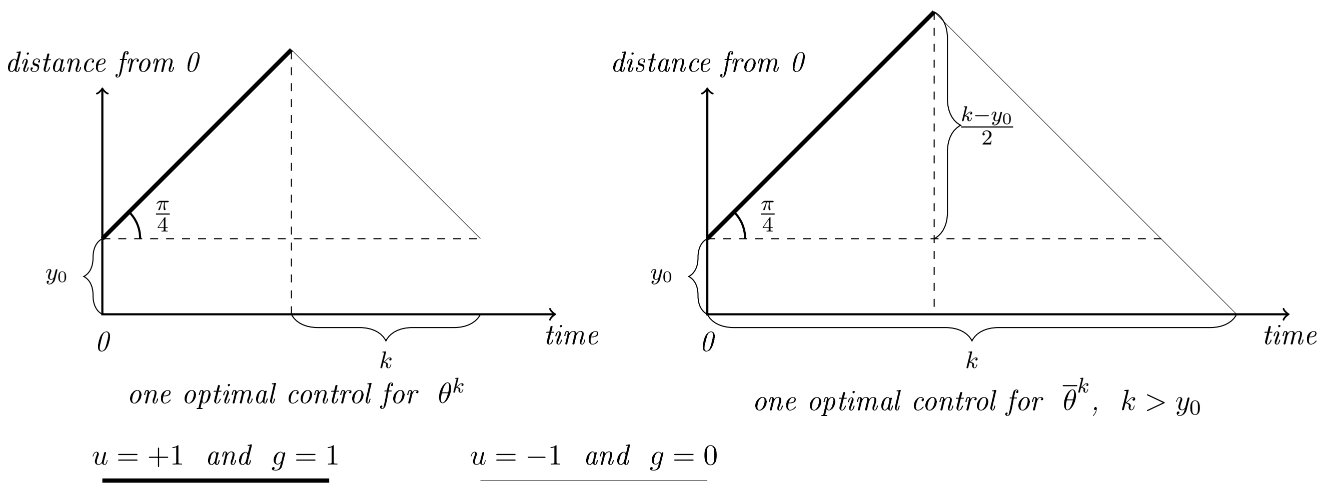

Figure 5.2: *The left figure describes the dynamic of one optimal control for the evaluation* θ^k, *which is* $u^* = +1$ *on* $[0,k]$ *and* $u^* = -1$ *on* $(k, 2k]$; *the right figure describes the dynamic of one optimal control for the evaluation* $\bar{\theta}^k$ *with* $k > y_0$, *which is* $u^* = +1$ *on* $[0, \frac{k-y_0}{2}]$ *and* $u^* = -1$ *on* $(\frac{k-y_0}{2}, k]$. *Here, the vertical axis represents the distance of* $y(t)$ *from zero and the thick trajectory (resp. thin trajectory) corresponds to state on which* $u = +1$ *and* $g = 1$ *(resp.* $u = -1$ *and* $g = 0$*).*

We deduce that $(V_{\theta^k}(y_0))_k$ *converges uniformly to* $V^*(y_0) = 0$ *on* $\mathbb{R}_+$; *and that* $V_{\overline{\theta}^k}(y_0) \xrightarrow[k\to\infty]{} \frac{1}{2}$, *while the convergence is not uniformly in* $y_0 \in \mathbb{R}_+$: *indeed, for all* $k \geq 1$, $V_{\overline{\theta}^k}(k) = 0$.

5 Proof of main result: Theorem 4.1

Let's fix through out this section a sequence of evaluations $(\theta^k)_k$ that satisfies the LTC. As the proof is rather long, we divide it into two main parts:

- in Subsection 5.1, we present the first preliminary result, Proposition 5.1. It is used in two ways: first, we obtain an immediate consequence of it for later use, which bounds $\liminf_k V_{\theta^k}$ from below in terms of the auxiliary value functions $\{V_{\mathcal{T}_t \sharp \theta^k} : k \in \mathbb{N}^*, t \in \mathbb{R}_+\}$; second, we deduce from it in Corollary 5.2 the proof for Part (i) of Theorem 4.1.

- In Subsection 5.2, we prove Parts (ii)-(iii) of Theorem 4.1. Lemma 5.4 gives an upper bound of $\limsup_k V_{\theta^k}$ in terms of the auxiliary value functions $\{V_{\mathcal{T}_t \sharp \theta^k} : k \in \mathbb{N}^*, t \in \mathbb{R}_+\}$, which is, together with the result from Proposition 5.1, used to end the proof.

5.1 A first preliminary result and proof for Part (i)

Proposition 5.1 *For any θ in $\Delta(\mathbb{R}_+)$, and any initial state y_0 in $\mathbb{R}^d$,*

$$\inf_{\overline{y}\in R(y_0)} V_\theta(\overline{y}) = \inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta}(y_0) \le \liminf_k V_{\theta^k}(y_0).$$

In particular, we have for all y_0 in $\mathbb{R}^d$,

$$\sup_{k\in\mathbb{N}^*}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^k}(y_0) \le \liminf_k V_{\theta^k}(y_0).$$

Proof: Fix y_0 and θ, we set $\beta =_{def} \inf_{\overline{y}\in R(y_0)} V_\theta(\overline{y})$. For any $\varepsilon > 0$ fixed, there exists some $T_0 > 0$ such that $\theta([T_0,\infty)) < \varepsilon$. Take any control u in $\mathcal{U}$. By definition of β, we have that

$$\forall T \ge 0,\ \int_{t=0}^{\infty} g\left(y(t+T,u,y_0),u(t+T)\right)\mathrm{d}\theta(t) \ge \beta,$$

thus

$$\forall T \ge 0,\ \int_{t=0}^{T_0} g\left(y(t+T,u,y_0),u(t+T)\right)\mathrm{d}\theta(t) \ge \beta-\varepsilon. \tag{5.1}$$

For each $k \ge 1$, integrating both sides of (5.1) over $T\in[0,\infty)$ w.r.t. the evaluation θ^k, we obtain

$$\int_{T=0}^{\infty}\int_{t=0}^{T_0} g\left(y(t+T,u,y_0),u(t+T)\right)\mathrm{d}\theta(t)\mathrm{d}\theta^k(T) \ge \beta-\varepsilon. \tag{5.2}$$

Applying Fubini's Theorem to (5.2) yields

$$\beta-\varepsilon \le \int_{t=0}^{T_0}\left[\int_{T=0}^{\infty} g\left(y(t+T,u,y_0),u(t+T)\right)\mathrm{d}\theta^k(T)\right]\mathrm{d}\theta(t) = \int_{t=0}^{T_0}\left[\gamma_{\mathcal{T}_t\sharp\theta^k}(y_0,u)\right]\mathrm{d}\theta(t), \tag{5.3}$$

where $\gamma_{\mathcal{T}_t\sharp\theta^k}(y_0,u) = \int_{T=0}^{\infty} g\left(y(t+T,u,y_0),u(t+T)\right)\mathrm{d}\theta^k(T)$. According to Lemma 3.7, we have $|\gamma_{\theta^k}(y_0,u)-\gamma_{\mathcal{T}_t\sharp\theta^k}(y_0,u)| \le 2TV_t(\theta^k)$. This enables us to rewrite (5.3) as:

$$\begin{aligned}\beta-\varepsilon &\le \int_{t=0}^{T_0}\left(\gamma_{\theta^k}(y_0,u)+2TV_t(\theta^k)\right)\mathrm{d}\theta(t)\\ &\le \left(\gamma_{\theta^k}(y_0,u)+2\overline{TV}_{T_0}(\theta^k)\right)\theta([0,T_0])\\ &\le \gamma_{\theta^k}(y_0,u)+2\overline{TV}_{T_0}(\theta^k).\end{aligned}$$

The control $u\in\mathcal{U}$ being taken arbitrarily, we deduce that

$$\beta-\varepsilon \le V_{\theta^k}(y_0)+2\overline{TV}_{T_0}(\theta^k).$$

Since (θ^k) satisfies the LTC, $\overline{TV}_{T_0}(\theta^k)$ vanishes as k tends to infinity. The proof is achieved. □

We end the proof for Part (i) of Theorem 4.1 by the following corollary of Proposition 5.1.

Corollary 5.2 *[Proof for Part (i) of Theorem 4.1]*

$$\sup_{\theta\in\Delta(\mathbb{R}_+)}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta}(y_0) = \sup_{k\ge1}\inf_{t\in\mathbb{R}_+} V_{\mathcal{T}_t\sharp\theta^k}(y_0),\ \forall y_0\in\mathbb{R}^d.$$

Proof: Fix $y_0 \in \mathbb{R}^d$, and denote $\varrho = \sup_{k\geq 1} \inf_{t\geq 0} V_{\mathcal{T}_t \sharp \theta^k}(y_0)$. It is clear that $\varrho \leq \sup_{\theta\in\Delta(\mathbb{R}_+)} \inf_{t\geq 0} V_{\mathcal{T}_t\sharp\theta^k}(y_0)$. Now for each $k \geq 1$ there exists $m(k)$ in $\mathbb{R}_+$ such that $V_{\mathcal{T}_{m(k)}\sharp\theta^k}(y_0) \leq \varrho + 1/k$. Since $\mathcal{T}_{m(k)}\sharp\theta^k$ – the image measure of θ^k by the function $s \mapsto s + m(k)$ – is also an evaluation on $\mathbb{R}_+$ we have:

$$\forall s \geq 0,\ TV_s(\mathcal{T}_{m(k)}\sharp\theta^k) = \sup_{Q\in\mathcal{B}(\mathbb{R}_+)} |\theta(Q - m(k)) - \theta(Q - m(k) + s)| \leq TV_s(\theta^k) + \theta^k([0, s]).$$

We deduce that $(\mathcal{T}_{m(k)}\sharp\theta^k)_k$ satisfies the LTC whenever $(\theta^k)_k$ does so. According to Proposition 5.1, $\forall\theta \in \Delta(\mathbb{R}_+)$, $\inf_{\overline{y}\in R(y_0)} V_\theta(\overline{y}) \leq \liminf_k V_{\mathcal{T}_{m(k)}\sharp\theta^k}(y_0) \leq \varrho$, thus $\sup_{\theta\in\Delta(\mathbb{R}_+)} \inf_{\overline{y}\in R(y_0)} V_\theta(\overline{y}) \leq \varrho$. The proof is complete. □

5.2 Proof for Parts (ii)-(iii)

In this subsection, we give the proof for Parts (ii)-(iii) of Theorem 4.1. We begin with the following result, which compares the values under evaluation θ and its t-"shifted" evaluation $\mathcal{T}_t\sharp\theta$ for any $t > 0$.

Lemma 5.3 *Let θ in $\Delta(\mathbb{R}_+)$ be any evaluation. Then: for all $t \geq 0$ and $y_0 \in \mathbb{R}^d$,*

$$V_\theta(y_0) \leq \inf_{\overline{y}\in R_t(y_0)} V_\theta(\overline{y}) + 2TV_t(\theta).$$

Proof: Fix $\theta \in \Delta(\mathbb{R}_+)$, $t \geq 0$, $y_0 \in \mathbb{R}^d$. By Lemma 3.7, we have

$$\gamma_\theta(y_0, u) \leq \gamma_{\mathcal{T}_t\sharp\theta}(y_0, u) + 2TV_t(\theta), \forall u \in \mathcal{U}.$$

For all $\varepsilon > 0$, take $u^\varepsilon \in \mathcal{U}$ be an ε-optimal control for $V_{\mathcal{T}_t\sharp\theta}(y_0)$, *i.e.*, $\gamma_{\mathcal{T}_t\sharp\theta}(y_0, u^\varepsilon) \leq V_{\mathcal{T}_t\sharp\theta}(y_0) + \varepsilon$. We obtain that

$$\gamma_\theta(y_0, u^\varepsilon) \leq V_{\mathcal{T}_t\sharp\theta}(y_0) + \varepsilon + 2TV_t(\theta).$$

Since $V_\theta(y_0) = \inf_{u\in\mathcal{U}} \gamma_\theta(y_0, u)$ and $\varepsilon > 0$ being arbitrary, we deduce that

$$V_\theta(y_0) \leq V_{\mathcal{T}_t\sharp\theta}(y_0) + 2TV_t(\theta).$$

Finally notice that $\inf_{\overline{y}\in R_t(y_0)} V_\theta(\overline{y}) = V_{\mathcal{T}_t\sharp\theta}(y_0)$. The proof is complete. □

The following result gives an upper bound on $\limsup_k V_{\theta^k}$ in terms of the auxiliary value functions $\{V_{\mathcal{T}_t\sharp\theta^k} : k \in \mathbb{N}^*, t \in \mathbb{R}_+\}$.

Lemma 5.4 *For all $T_0 \geq 0$ and any y_0 in $\mathbb{R}^d$,*

$$\limsup_k V_{\theta^k}(y_0) = \limsup_k \inf_{t\leq T_0} V_{\mathcal{T}_t\sharp\theta^k}(y_0).$$

In particular, for all $T_0 \geq 0$ and any y_0 in $\mathbb{R}^d$,

$$\limsup_k V_{\theta^k}(y_0) \leq \sup_{k\in\mathbb{N}^*} \inf_{t\leq T_0} V_{\mathcal{T}_t\sharp\theta^k}(y_0).$$

Proof: Fix $T_0 \geq 0$ and $y_0 \in \mathbb{R}^d$. The inequality $'' \limsup_k \inf_{t \leq T_0} V_{\mathcal{T}_t \sharp \theta^k} \leq \limsup_k V''_{\theta^k}$ is clear by taking $t = 0$ for each k. Now for the converse inequality $'' \limsup_k \inf_{t \leq T_0} V_{\mathcal{T}_t \sharp \theta^k} \geq \limsup_k V''_{\theta^k}$: according to Proposition 5.3, we have that for all k and $t \leq T_0$,

$$V_{\theta^k}(y_0) \leq V_{\mathcal{T}_t \sharp \theta^k}(y_0) + 2TV_t(\theta^k).$$

For each $k \geq 1$, take $t^k \leq T_0$ with $V_{\mathcal{T}_{t^k} \sharp \theta^k}(y_0) \leq \inf_{0 \leq t \leq T_0} V_{\mathcal{T}_{t^k} \sharp \theta^k} + \frac{1}{k}$, which gives us:

$$\begin{aligned} V_{\theta^k}(y_0) &\leq \inf_{0 \leq t \leq T_0} V_{\mathcal{T}_t \sharp \theta^k}(y_0) + \frac{1}{k} + 2TV_{t^k}(\theta^k) \\ &\leq \inf_{0 \leq t \leq T_0} V_{\mathcal{T}_t \sharp \theta^k}(y_0) + \frac{1}{k} + 2\overline{TV}_{T_0}(\theta^k). \end{aligned}$$

Since $(\theta^k)_k$ satisfies the LTC, $\overline{TV}_{T_0}(\theta^k)$ vanishes as k tends to infinity. By taking "$\limsup_k$" on both sides of above inequality, the proof for the lemma is complete. □

Now we end the proof for Theorem 4.1. To do this, we first summarize results in Proposition 5.1 and Lemma 5.4 in the following chain form, which is then used for the study of the convergence of $(V_{\theta^k})_k$.

Corollary 5.5 *For all $T_0 \geq 0$ and y_0 in $\mathbb{R}^d$,*

$$\sup_{k \geq 1} \inf_{t \leq T_0} V_{\mathcal{T}_t \sharp \theta^k}(y_0) \geq \limsup_k V_{\theta^k}(y_0) \geq \liminf_k V_{\theta^k}(y_0) \geq \sup_{k \geq 1} \inf_{t \geq 0} V_{\mathcal{T}_t \sharp \theta^k}(y_0)$$

Remark 5.6 *Corollary 5.5 states that the uniform convergence of $\sup_{k \geq 1} \inf_{t \leq T_0} V_{\mathcal{T}_t \sharp \theta^k}$ to $\sup_{k \geq 1} \inf_{t \geq 0} V_{\mathcal{T}_t \sharp \theta^k}$ as T_0 tends to infinity implies the uniform convergence of $(V_{\theta^k})_k$ as k tends to infinity. Moreover, according to Corollary 5.2, in case of uniform convergence, the limit function is V^*.*

For any states y and $\overline{y}$ in $\mathbb{R}^d$, let us define $\tilde{d}(y, \overline{y}) = \sup_{k \geq 1} |V_{\theta^k}(y) - V_{\theta^k}(\overline{y})|$. The space $(\mathbb{R}^d, \tilde{d})$ is now a *pseudometric* space (may not be Hausdorff).

This following is similar to the proof of Theorem 2.5 in [10], and is also similar to the proof of Theorem 3.10 in [9]. We rewrite it here for sake of completeness. Roughly speaking, we shall use the total boundedness of the space $\{V_{\theta^k}\}$ for the uniform metric so as to make the state space $(\mathbb{R}^d, \tilde{d})$ totally bounded for the pseudometric metric $\tilde{d}$. This allows us to prove the convergence of the reachable set R^T in finite time to R at infinity. We are then able to prove the convergence of $\sup_{k \geq 1} \inf_{t \leq T_0} V_{\mathcal{T}_t \sharp \theta}$ to $\sup_{k \geq 1} \inf_{t \geq 0} V_{\mathcal{T}_t \sharp \theta}$.

Proof for Theorem 4.1, Parts (ii)-(iii).

We first prove Part (iii). One direction is easy: the uniform convergence of (V_{θ^k}) implies the totally boundedness of the space $(\{V_{\theta^k}\}, ||\cdot||_\infty)$.

Let us prove the converse. Suppose that $(\{V_{\theta^k}\}, ||\cdot||_\infty)$ is totally bounded, so there exists a finite set of indices I such that for all $k \geq 1$, there exists $i \in I$ satisfying $||V_{\theta^k} - V_{\theta^i})||_\infty \leq \varepsilon$. The set $\{\big(V_{\theta^i}(y)\big), y \in \mathbb{R}^d\}$, it is a subset of the compact metric space $[0,1]^I$ with the uniform norm, thus it is itself totally bounded and so there exists a finite subset C of states in $\mathbb{R}^d$ such that

$$\forall y \in \mathbb{R}^d, \exists c \in C, \forall i \in I, |V_{\theta^i}(y) - V_{\theta^i}(c)| \leq \varepsilon.$$

We have obtained that for each $\varepsilon > 0$, there exists a finite subset C of $\mathbb{R}^d$ such that for every $y \in \mathbb{R}^d$, there is $c \in C$ with $\tilde{d}(y, c) \leq \varepsilon$. The pseudometric space $(\mathbb{R}^d, \tilde{d})$ is itself totally bounded. Equivalently, any sequence in $\mathbb{R}^d$ admits a Cauchy subsequence for $\tilde{d}$. Notice that all value functions V_{θ^k} are clearly 1-Lipschitz for $\tilde{d}$.

Fix y in $\mathbb{R}^d$, we observe that:

$$\forall T, S \in \mathbb{R}_+, \quad R^T(y) \subset R^{T+S}(y).$$

From the precompacity of $(\mathbb{R}^d, \tilde{d})$ it is not difficult to show (cf. Step 2 in the proof of Theorem 3.7 in [9]) that R^T converges to R in the following sense

$$\forall \varepsilon > 0, \exists T \geq 0, \forall \overline{y} \in R(y), \exists \tilde{y} \in R^T(y), \tilde{d}(\overline{y}, \tilde{y}) \leq \varepsilon. \tag{5.4}$$

By Corollary 5.5, for all $T \geq 0$:

$$\sup_{k\geq 1} \inf_{\overline{y} \in R^T(y_0)} V_{\theta^k}(\overline{y}) \geq \limsup_k V_{\theta^k}(y) \geq \liminf_k V_{\theta^k}(y) \geq \sup_{k\geq 1} \inf_{\overline{y} \in R(y)} V_{\theta^k}(\overline{y}).$$

Fix finally $\varepsilon > 0$, and consider $k \geq 1$ and $T \geq 0$ given by assertion (5.4). Let $\overline{y} \in R(y)$ be such that $V_{\theta^k}(\overline{y}) \leq \inf_{\overline{y} \in R(y)} V_{\theta^k}(\overline{y}) + \varepsilon$. Let $\tilde{y}$ in $R^T(y)$ be such that $d(\overline{y}, \tilde{y}) \leq \varepsilon$. Since V_{θ^k} is 1-Lipschitz for $\tilde{d}$, we obtain $V_{\theta^k}(\tilde{y}) \leq \inf_{\overline{y} \in R(y)} V_{\theta^k}(\overline{y}) + 2\varepsilon$. Consequently, $\inf_{\overline{y} \in R^T(y)} V_{\theta^k}(\overline{y}) \leq \inf_{\overline{y} \in R(y)} V_{\theta^k}(\overline{y}) + 2\varepsilon$ for all k, so

$$\sup_{k\geq 1} \inf_{\overline{y} \in R^T(z)(y)} V_{\theta^k}(\overline{y}) \leq \sup_{k\geq 1} \inf_{\overline{y} \in R(y)} V_{\theta^k}(\overline{y}) + 2\varepsilon.$$

One obtains that $\limsup_{k\geq 1} V_{\theta^k}(y) \leq \liminf_{k\geq 1} V_{\theta^k}(y) + 2\varepsilon$, and so $\big(V_{\theta^k}(y)\big)_k$ converges. Since $(\mathbb{R}^d, \tilde{d})$ is precompact and all V_{θ^k} is 1-Liptschitz, the convergence is uniform.

Next, Part (ii) can be deduced from the proof of Part (iii). Let $(\theta^{\varphi(k)})$ be any subsequence of (θ^k) that converges uniformly to some function V. This implies that $(\{V_{\theta^{\varphi(k)}}\}, ||\cdot||_\infty)$ is totally bounded. As we have shown in the proof of Part (iii) that if $(\{V_{\theta^{\varphi(k)}}\}, ||\cdot||_\infty)$ is totally bounded, $(V_{\theta^{\varphi(k)}})$ converges uniformly to $V = V^*$, which implies Part (ii) that V^* is the unique accumulation point (for the uniform convergence) of the sequence $(V_{\theta^k})_k$. □

6 Discussion on a weaker long-term condition

One might state the long-term condition (LTC) in the following weaker form:

Long-term condition' (LTC') A sequence of evaluations $(\theta^k)_{k\geq 1}$ satisfies the LTC' if:

$$\forall s > 0, \ TV_s(\theta^k) \xrightarrow[k\to\infty]{} 0. \tag{6.1}$$

It is not clear whether the LTC' is strictly weaker than the LTC. One might want to construct an example of (θ^k) such that $TV_s(\theta^k) \xrightarrow[k\to\infty]{} 0$ for all $s > 0$ while $\overline{TV}_{s_0}(\theta^k) \xrightarrow[k\to\infty]{} \alpha > 0$ for some $s_0 > 0$ and $\alpha > 0$. The following example shows that this is possible if we consider only s being rational numbers. In general, the question is still open.

Example 6.1 *Given a positive integer k, consider the density θ^k with support included in $[0,k]$ by dividing $[0,k]$ in k^2 consecutive small intervals of length $1/k$, and θ^k is uniform over the union of all small odd intervals... and puts no weight on even small intervals. Define the support*

$$S_k = \bigcup_{l\in\mathbb{N}, l\leq \frac{k^2-1}{2},} \left[\frac{2l}{k}, \frac{2l+1}{k}\right).$$

θ^k has density:

$$f_k(x) = \frac{2}{k}\,\mathbb{1}_{x\in S_k} = \frac{2}{k}\,\mathbb{1}_{x\in[0,k], E(kx)\in 2\mathbb{N}}$$

(where $2\mathbb{N}$ is the set of even numbers in $\mathbb{N}$, $E(x)$ is the integer part of x).

For each k, we have (consider $s = 1/k$):

$$\sup_{0\leq s\leq 1} \int_{x\geq 0} \left|f_k(x+s) - f_k(x)\right| \mathrm{d}x \geq 2 - 1/k$$

Consider now only k of the form $n!$, and we define the density $g_n = f_{n!}$ for each n in $\mathbb{N}$. For all $x \geq 0$,

$$g_n(x+s) - g_n(x) = \frac{2}{n!}\left(\mathbb{1}_{E(n!(x+s))\in 2\mathbb{N}, x+s\leq n!} - \mathbb{1}_{E(n!x)\in 2\mathbb{N}, x\leq n!}\right).$$

Assume s is a rational number. Then for n large enough, $n!s$ is an even integer, so for all x such that $0 \leq x \leq n! - s$, we have $g_n(x+s) - g_n(x) = 0$. Consequently

$$\int_{x\geq 0} \left|g_n(x+s) - g_n(x)\right| \mathrm{d}x \xrightarrow[n\to\infty]{} 0.$$

7 Appendix

Proof for Lemma 1: The following calculation of $f_t'(\theta)$ is straight:

$$\forall t > 0,\quad f_\theta'(t) = \frac{1}{\sigma\sqrt{2\pi}}\left[\exp\left(-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2\right)\frac{m-t}{\sigma^2} - \exp\left(-\frac{1}{2}\left(\frac{t+m}{\sigma}\right)^2\right)\frac{m+t}{\sigma^2}\right],$$

thus

$$f_\theta'(t) > 0 \ (resp. < 0) \iff (m-t)\exp\left(-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2\right) - (m+t)\exp\left(-\frac{1}{2}\left(\frac{t+m}{\sigma}\right)^2\right) > 0 \ (resp. < 0).$$

As a consequence, one obtains that

$$f_\theta'(t) < 0,\ \forall t \geq m.$$

Now we look at $t \in (0,m)$. Let us denote $H(t) =_{def} \exp\left(\frac{2mt}{\sigma^2}\right) - \frac{m+t}{m-t}$, which yields:

$$f_\theta'(t) > 0 \ (resp. < 0) \iff H(t) > 0 \ (resp. < 0),\quad \forall t \in (0,m).$$

From above we deduce that, to prove the lemma, it is essentially reduced to the following

Claim *There is some $t^* \in [0,m)$ such that $H(t) < 0$ for $t \in (0,t^*)$ and $H(t) > 0$ for $t \in (t^*,m)$.*

Below we prove this claim through the study of the variation of $H(\cdot)$ on $[0,m)$ (on which it is obviously C^∞). For this aim, one calculates:

- the values at the end point, $H(0) = 0$ and $\lim_{t \to m^-} H(t) = -\infty$;
- the first-order derivative at any $t \in [0, m)$,

$$H'(t) = \exp\left(\frac{2mt}{\sigma^2}\right)\frac{2m}{\sigma^2} - \frac{2m}{(m-t)^2} \tag{7.1}$$

- at any rest point $t^e \in [0, m)$ (*i.e.*, $H(t^e) = 0$),

$$\exp\left(\frac{2mt^e}{\sigma^2}\right) = \frac{m + t^e}{m - t^e}, \tag{7.2}$$

  which is thus substituted back into (7.1), to yield

$$H'(t^e) > 0 \ \big(\ \textit{resp. } H'(t^e) < 0\ \big) \iff (t^e)^2 < m^2 - \sigma^2 \ \big(\ \textit{resp. } (t^e)^2 > m^2 - \sigma^2\ \big). \tag{7.3}$$

First, it is easy to prove the following result:

Let $t_1^e \in [0, m)$ be a rest point for $H(\cdot)$, and suppose that $t_2^e \in (t_1^e, m)$ is the smallest rest point after t_1^e. Then $H'(t_1^e)H'(t_2^e) \leq 0$ and if $H'(t_1^e) \leq 0$, such t_2^e does not exist.

Indeed, $H'(t_1^e)H'(t_2^e) \leq 0$ can be derived from the continuity of $H(\cdot)$; suppose that $H'(t_1^e) \leq 0$, we have $(t_1^e)^2 \geq m^2 - \sigma^2$ and $H'(t_2^e) \geq 0$, which leads to a controdiction (with $t_2^e > t_1^e$) as the later implies that $(t_2^e)^2 \leq m^2 - \sigma^2$.

Finally, remark that $H(0) = 0$, thus $t = 0$ is a rest point. We discuss the following two cases:

<u>Case 1. $m^2 - \sigma^2 \leq 0$, thus $H'(0) \leq 0$.</u>

This implies that no rest point exists after 0. Since $\lim_{t \to m^-} H(t) = -\infty$, we deduce that $H(t) < 0$, $\forall t \in (0, m)$. The claim is proved for $t^* = 0$.

<u>Case 2. $m^2 - \sigma^2 > 0$, thus $H'(0) > 0$.</u>

$\lim_{t \to m^-} H(t) = -\infty$ implies that some rest point exists in $(0, m)$. Take t^e the closet to 0, implying that $H(t) > 0$, $\forall t \in (0, t^e)$. Further, we obtain that $H'(t^e) \leq 0$ by the continuity of $H(\cdot)$. Again, there exists no other rest point after t^e. Since $\lim_{t \to m^-} H(t) = -\infty$, we deduce that $H(t) < 0$, $\forall t \in (t^e, m)$. The claim is proved for $t^* = t^e$.

To conclude, we see that in both cases, the claim is proved. This finishes the proof for the lemma. □

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