On a Class of Smooth Preferences*

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February 19, 2018

Abstract

We construct a complete space of smooth strictly convex preferences defined over commodities and monetary transfers. Our model extends the classical one in that preferences are strictly monotone in monetary transfers, but need not be monotone in all commodities. We thereby provide a natural framework for performing genericity analyses in situations involving inventory costs or decisions under risk. The constructed space of preferences is contractible, which allows for a natural aggregation procedure in collective decision situations.

Keywords: Smooth Preferences, Nonmonotonicity, Collective Choice.
JEL Classification: C60, D11.

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*We thank the editor and two anonymous referees for very thoughtful and detailed comments. We also thank Michel Le Breton and Jérôme Renault for extremely valuable feedback.

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1 Introduction

Applications of classical demand theory tend to treat each commodity either as a good or as a bad. This implies that we can, if necessary, redefine the commodities so as to treat all of them as goods.[1] Thus leisure time is the flip side of hours worked, and clean air or water is the flip side of pollution. Technically, this allows us to focus on preferences that are strictly monotone in each commodity.[2]

However, even in this sense, the monotonicity of preferences cannot always be taken for granted. If a consumer or a firm cannot freely dispose of a stock of unwanted commodities, they will typically have to bear inventory costs for holding this stock; for instance, they may need to acquire a new storage facility. In the presence of such costs, or, more generally, when an individual's objective function is an indirect utility function derived under technological constraints, there may be situations where, beyond a certain point, more of a commodity actually reduces his utility or profit. We can then no longer unambiguously classify such a commodity as a good or as a bad.

Example 1 Nonmonotone preferences naturally arise in the modeling of portfolio decisions under risk. From an investor's viewpoint, larger holdings of a risk-free asset always increase his utility, but this is not true of risky assets if he is risk-averse: for him, moderate holdings of such assets are typically beneficial for hedging or speculative purposes, but larger holdings involve excessive risk.

Example 2 Nonmonotone preferences also arise in the modeling of choices in groups such as households, families, firms, unions, or clubs. In such situations, an individual is expected, and to some extent willing, to contribute to a collective good at some privately borne cost: for him, moderate contributions are directly or indirectly beneficial, but larger contributions involve excessive costs.

In such situations, the classical construction of a complete space of smooth strictly convex preferences (Mas-Colell (1985, Chapter 2)) needs to be amended. We provide a canonical model of preferences in which, while there is at least one commodity that is always desirable, this need not be the case for all the other commodities. Individual monetary transfers are

[1] See, for instance, Varian (1992, Chapter 7) or Mas-Colell, Whinston, and Green (1995, Chapter 3).

[2] To be fair, much of classical demand theory can be developed by relying on the weaker local-nonsatiation assumption; yet strict monotonicity appears to be the rule in applications. The implications of nonmonotone preferences for the existence and efficiency of competitive equilibria have been examined by Polemarchakis and Siconolfi (1993), among others. However, they take individual preferences as given and do not provide a framework for genericity analysis.

a straightforward example of the first commodity, and we will stick to this interpretation in most of the paper; this, in particular, is relevant for principal-agent models where transfers are allowed. But other examples of such a commodity are easy to find: think of leisure time in a household or the practice of a shared activity in a club.

Our construction gives rise to a topologically complete space of smooth strictly convex nonmonotone preferences. This space in turn provides a natural framework for genericity analyses: we may, for instance, use it when checking the robustness of results obtained in portfolio-choice theory under the usual CARA-normal specification, with no need to assume that investors' primitive preferences over state-contingent consumption have an expected-utility representation.[3]

An important topological property of our space of preferences is that it is contractible; that is, it can be continuously deformed into a single preference relation. This reflects a compensation principle requiring that, as long as his subsistence is not at stake, an individual can, through appropriate transfers of some uniformly desirable commodity, be compensated for holding any amounts of the other commodities. In line with Chichilnisky and Heal (1983), contractibility can be interpreted as a topological unanimity condition; in particular, profiles of such preferences can be continuously deformed into unanimous profiles. We show that this implies the existence of a collective choice rule over individual preference profiles that is anonymous, continuous, and respects unanimity. Although this finding is in line with the literature on topological social choice initiated by Chichilnisky (1980), it is not a direct consequence of known results; instead, it follows in a natural way from the specific contraction we construct on the space of preferences.

The paper is organized as follows. Section 2 describes a space of basic preferences. Section 3 introduces our compensation principle and a space of normalized preferences. Section 4 achieves the construction of our complete space of preferences. Section 5 draws the implications of our analysis for collective choice.

2 Basic Preferences

In this section, we introduce a space $\mathbf{P}$ of basic preferences. There are $\ell + 1$ commodities, the last of which is interpreted as transfers to the individual. We denote by q a vector of the first ℓ commodities, by t a scalar amount of transfers, and by 0_ℓ the null vector in $\mathbb{R}^\ell$. We

[3]Notice in that respect that, when the state space is infinite, conducting a genericity analysis in the finite-dimensional space of portfolio choices is mathematically much simpler than doing so in the infinite-dimensional space of state-contingent consumption choices.

will consider regular preference relations over an open subset V of $\mathbb{R}^{\ell+1}$. We require that V contain the no-trade point $(0_\ell, 0)$, which corresponds to the individual's endowment point, that it be convex with a nonempty interior, and that it be *comprehensive with respect to transfers* in the sense that, if $(q,t) \in V$ and $t' > t$, then $(q,t') \in V$; thus V is unbounded from above in the direction of transfers. We let $Q \equiv \mathrm{proj}_{\mathbb{R}^\ell} V$ and, for each $q \in Q$, we let $\underline{t}(q) \equiv \inf\{t \in \mathbb{R} : (q,t) \in V\}$. Notice that, because V is convex, if $\underline{t}(q) = -\infty$ for some $q \in Q$, then $\underline{t}(q') = -\infty$ for all $q' \in Q$. Figure 1 below illustrates these assumptions.

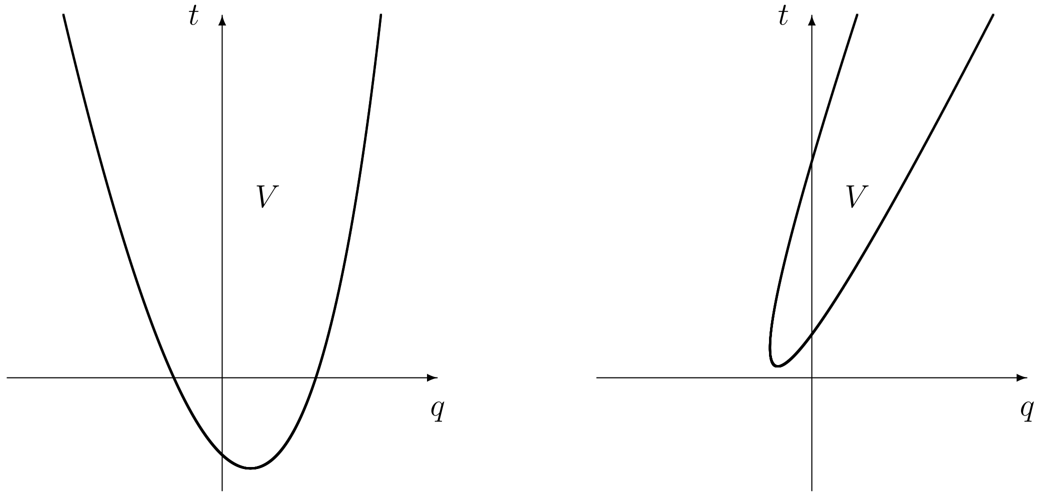

Figure 1.a An admissible domain V. **Figure 1.b** A nonadmissible domain V.

We focus on complete, reflexive, and transitive preference relations over V, generically denoted by $\succeq$; we denote by $\succ$ the corresponding strict preference relations. The following axioms are maintained throughout the paper.

A1 *$\succeq$ is closed relative to $V \times V$.*

A2 *$\succeq$ is strictly monotone in transfers: if $(q,t) \in V$ and $t' > t$, then $(q,t') \succ (q,t)$.*

A3 *$\succeq$ is convex: if $(q,t) \succeq (q',t')$ and $\lambda \in [0,1]$, then $\lambda(q,t) + (1-\lambda)(q',t') \succeq (q',t')$.*

A4 *$\succeq$ has closed upper contour sets relative to $\mathbb{R}^{\ell+1}$.*

A5 *$\succeq$ has a boundary in $V \times V$ that is a C^2 manifold.*

A1 and A3 are standard. A2 requires that preferences be strictly monotone in transfers. This generalizes the standard assumption that preferences be strictly monotone in each commodity; if we were to impose this stronger requirement, we would obtain the class of preferences studied in Mas-Colell (1985, Chapter 2). A4 describes the boundary behavior of preferences.[4] Finally, A5 requires that preferences be sufficiently regular.

Example 3 Assume that a risk-averse investor with constant absolute risk-aversion α can invest in ℓ risky assets with payoffs that are jointly normally distributed with mean vector π and covariance matrix Γ, as well as in a risk-free asset with payoff 1. Then his preferences over portfolios of risky and risk-free assets $(q, t) \in V \equiv \mathbb{R}^{\ell+1}$ are represented by

$$u(q, t) \equiv q^\top \pi - \frac{\alpha}{2} q^\top \Gamma \, q + t$$

and thus satisfy A1–A5. The nonmonotonity of $u(q, t)$ in q reflects that the investor does not want to hold an excessively risky asset position.

Although its role is mainly technical, A4 also has an economic interpretation, as the following example suggests.

Example 4 Let $V \equiv \prod_{l=1}^{\ell} (q_l^-, q_l^+) \times \mathbb{R}$ for some positive and negative numbers $(q_l^+)_{l=1}^{\ell}$ and $(q_l^-)_{l=1}^{\ell}$, respectively. Figure 2 below illustrates a possible shape for the individual's indifference sets for $\succeq$ under A4.

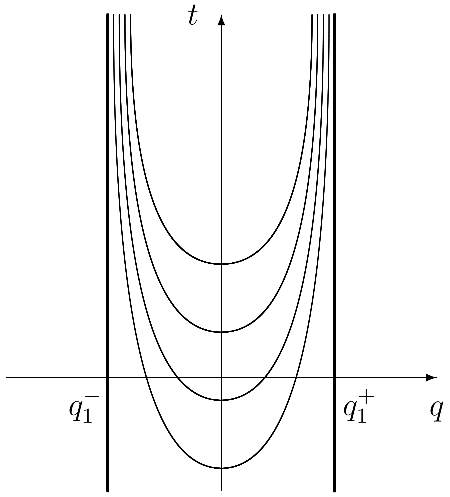

Figure 2 $\succeq$ satisfies A4 on a rectangular domain V.

In this example, A4 expresses that, if the consumption or sale of commodity $l = 1, \ldots, \ell$ attain the thresholds q_l^+ or $|q_l^-|$, respectively, subsistence becomes impossible, in the sense

[4] It should be noted that A4 does not follow from A1 if V is a proper subset of $\mathbb{R}^{\ell+1}$; indeed, in this case, $\succeq$ can be closed relative to $V \times V$, though its upper contour sets are adherent to the boundary of V in $\mathbb{R}^{\ell+1}$ and are thus not closed relative to $\mathbb{R}^{\ell+1}$.

that the individual can no longer be compensated by transfers. As a result, the individual's indifference sets for $\succeq$ do not cross the hyperplanes $q_l = q_l^+$ and $q_l = q_l^-$; in Figure 2, they admit vertical asymptotes at the boundary of V. We may analogously introduce a negative lower bound t^- on transfers below which subsistence is impossible.

Our first task is to characterize the space $\mathbf{P}$ of preferences $\succeq$ that satisfy A1–A5. The following notation will be useful. Let $U_{(q,t)}$ and $L_{(q,t)}$ be the upper and lower contour sets of (q,t) for $\succeq$, respectively, and let $I_{(q,t)} \equiv U_{(q,t)} \cap L_{(q,t)}$ be the indifference set of (q,t) for $\succeq$. Observe by A2 that $U_{(q,t)}$ is comprehensive with respect to transfers, just as V. Also denote by cl and ∂ the closure and boundary operators relative to V or $V \times V$, depending on the context. We start with two technical lemmas.

Lemma 1 *If $\succeq$ satisfies A1–A2, then, for each $(q,t) \in V$, $U_{(q,t)}$ has a nonempty interior relative to $\mathbb{R}^{\ell+1}$ and $I_{(q,t)} = \partial U_{(q,t)}$.*

Proof. To prove the first claim, observe that, as $\succeq$ is closed relative to $V \times V$ by A1, $V \setminus L_{(q,t)}$ is open relative to V and thus also relative to $\mathbb{R}^{\ell+1}$ because V is an open subset of $\mathbb{R}^{\ell+1}$. Hence, as $V \setminus L_{(q,t)}$ is nonempty by A2, $U_{(q,t)} \supset V \setminus L_{(q,t)}$ has a nonempty interior relative to $\mathbb{R}^{\ell+1}$. To prove the second claim, observe that, by A1 again, $U_{(q,t)}$ and $L_{(q,t)}$ are closed relative to V. Therefore,

$$\partial U_{(q,t)} \equiv \mathrm{cl}(U_{(q,t)}) \cap \mathrm{cl}(V \setminus U_{(q,t)}) = U_{(q,t)} \cap \mathrm{cl}(V \setminus U_{(q,t)}) \subset U_{(q,t)} \cap L_{(q,t)} = I_{(q,t)}.$$

The reverse inclusion is satisfied if $I_{(q,t)} \subset \mathrm{cl}(V \setminus U_{(q,t)})$, which is obviously true because, for each $(q',t') \in I_{(q,t)}$, $t' - \varepsilon > \underline{t}(q')$ for any small enough $\varepsilon > 0$ as V is open, and $(q',t') \succ (q', t' - \varepsilon)$ for any such ε by A2. The result follows. ■

Lemma 2 *If $\succeq$ satisfies A1–A4, then, for each $(q,t) \in V$, $I_{(q,t)}$ is connected.*

Proof. By Lemma 1, $I_{(q,t)} = \partial U_{(q,t)}$, so that we can focus on the topological properties of $U_{(q,t)}$. By A3–A4, $U_{(q,t)}$ is a closed convex subset of $\mathbb{R}^{\ell+1}$; moreover, $U_{(q,t)}$ has a nonempty interior by Lemma 1. Hence two cases may arise (Klee (1953, III.1.6)).

Case 1 Either the asymptotic cone

$$\mathbf{A}U_{(q,t)} \equiv \{x \in \mathbb{R}^{\ell+1} : (q',t') + \lambda x \in U_{(q,t)} \text{ for all } (q',t',\lambda) \in U_{(q,t)} \times \mathbb{R}_+\}$$

of $U_{(q,t)}$ is not a linear subspace. Then $U_{(q,t)}$ is homeomorphic with $\mathbb{R}^\ell \times [0,1)$ and $\partial U_{(q,t)}$ with $\mathbb{R}^\ell$. In particular, $\partial U_{(q,t)}$ is connected.

Case 2 Or the asymptotic cone $\mathbf{A}U_{(q,t)}$ of $U_{(q,t)}$ is an $\ell+1-k$-dimensional linear subspace for some integer $k \leq \ell+1$. Because $U_{(q,t)}$ is comprehensive with respect to transfers, we must have $k \leq \ell$ and $(0_\ell, 1) \in \mathbf{A}U_{(q,t)}$. As $\mathbf{A}U_{(q,t)}$ is a linear subspace, it follows that $(0_\ell, -1) \in \mathbf{A}U_{(q,t)}$. This implies that $(q, t') \succeq (q,t)$ for all $t' \in (\underline{t}(q'), t)$, which is ruled out by A2. This case is thus impossible. The result follows. ■

Let $\mathbf{U}$ be the space of quasiconcave C^2 functions $u : V \to \mathbb{R}$ such that $\partial u/\partial t > 0$ over V and $u^{-1}([v, \infty))$ is closed relative to $\mathbb{R}^{\ell+1}$ for all $v \in \mathbb{R}$. Lemmas 1–2 then imply the following representation result.

Proposition 1 *$\succeq$ satisfies A1–A5 if and only if it admits a utility function $u \in \mathbf{U}$.*

Proof. (Direct part) Suppose that $\succeq$ admits a utility function $u \in \mathbf{U}$. Then $\succeq$ trivially satisfies A1–A3. Moreover, as $u^{-1}([v,\infty))$ is closed relative to $\mathbb{R}^{\ell+1}$ for all $v \in \mathbb{R}$, $\succeq$ satisfies A4. Finally, because u clearly has no critical point, that is, $\partial u \neq 0$ over V, it follows as in Mas-Colell (1985, Proposition 2.3.5) that $\succeq$ satisfies A5.

(Indirect part) By A2, $\succeq$ is locally nonsatiated, and by A5, $\partial \succeq$ is a C^2 manifold in $V \times V$. Thus $\succeq$ is of class C^2 (Mas-Colell (1985, Definition 2.3.4)). Moreover, by Lemmas 1–2, $\succeq$ has connected indifference sets $I_{(q,t)}$. Hence $\succeq$ admits a C^2 utility function u over V with no critical point (Mas-Colell (1985, Proposition 2.3.9)). That u is quasiconcave follows from A3. To show that $\partial u/\partial t > 0$ over V, observe first that $\partial u/\partial t \geq 0$ over V by A2. Now, suppose, by way of contradiction, that $(\partial u/\partial t)(q,t) = 0$ for some $(q,t) \in V$. Then $(\partial u/\partial q)(q,t) \neq 0_\ell$ as u has no critical point. Thus the hyperplane through (q,t) orthogonal to $\partial u(q,t)$ that supports the convex set $U_{(q,t)}$ is vertical. It follows that the strict upper contour set $U_{(q,t)} \setminus L_{(q,t)}$ of (q,t) for $\succeq$ strictly lies on one side or the other of this hyperplane. But then it does not include the half line $\{(q,t') : t' > t\}$, which contradicts A2. Hence $\partial u/\partial t > 0$ over V, as claimed. Finally, that $u^{-1}([v,\infty))$ is closed relative to $\mathbb{R}^{\ell+1}$ for all $v \in \mathbb{R}$ is a direct consequence of A4. Hence the result. ■

Proposition 1 states that any preference relation in $\mathbf{P}$ can be represented by some function in $\mathbf{U}$ and, conversely, that any function in $\mathbf{U}$ represents a preference relation in $\mathbf{P}$. For each $u \in \mathbf{U}$, let $P(u) \subset V \times V$ be the preference relation represented by u. In line with Mas-Colell (1985, Chapter 2, Section 4), a topology over $\mathbf{P}$ can be constructed as follows. Note that $\mathbf{U}$ is a subspace of $C^2(V)$, the Polish space of real-valued C^2 functions over V endowed with the topology of uniform convergence over compact subsets of V of functions and of their derivatives up to the order 2 (Mas-Colell (1985, Chapter 1, K.1.2)). Then we can endow $\mathbf{P}$

with the identification topology from P; that is, we let O be open in $\mathbf{P}$ if $P^{-1}(O)$ is open in $\mathbf{U}$. Notice that P is not one-to-one; however, we can show as in Mas-Colell (1985, Chapter 2, Proposition 2.4.2) that P is open, which implies that a sequence $(\succeq_n)_{n\in\mathbb{N}}$ converges to $\succeq$ in $\mathbf{P}$ if and only if there exists a sequence of representations $(u_n)_{n\in\mathbb{N}}$ for the preferences $(\succeq_n)_{n\in\mathbb{N}}$ that converges in $\mathbf{U}$ to a representation u of $\succeq$.

3 Normalized Preferences

In this section, we introduce a subspace $\mathbf{P}_v$ of $\mathbf{P}$, the elements of which admit convenient normalized representations. To this end, we add a further restriction on preferences in the form of the following axiom.

A6 *For all $(q,t) \in V$ and $q' \in Q$, there exists $t' > \underline{t}(q')$ such that $(q',t') \succeq (q,t)$.*

A6 expresses a compensation principle: through appropriate transfers, the individual can be compensated for holding any amounts of the first ℓ commodities, as long as they are consistent with subsistence. The following lemma shows that exact compensation is then possible up to any utility level.

Lemma 3 *If $\succeq$ satisfies A1–A4 and A6, then, for all $(q,t) \in V$ and $q' \in Q$, there exists $t' > \underline{t}(q')$ such that $(q',t') \sim (q,t)$.*

Proof. Suppose, by way of contradiction, that the result does not hold for some $(q,t) \in V$ and $q' \in Q$. Then, by A1–A2 and A6, $(q',t') \succeq (q,t)$ for all $t' > \underline{t}(q')$. Two cases may arise. First, if $\underline{t}(q') \in \mathbb{R}$, we have $(q',t') \succeq (q,t)$ for all $t' > \underline{t}(q')$. However, $(q',\underline{t}(q')) \not\succeq (q,t)$ as $(q',\underline{t}(q')) \notin V$. But then $U_{(q,t)}$ is not closed relative to $\mathbb{R}^{\ell+1}$, which contradicts A4. Second, if $\underline{t}(q') = -\infty$, we have $(q',t') \succeq (q,t)$ for all $t' \in \mathbb{R}$; that is, $U_{(q,t)}$ contains a vertical line. But then it must be that

$$\left(1+\frac{1}{t'}\right)(q,t) - \frac{1}{t'}(q',t') \succeq (q,t)$$

for all $t' < -1$ by A3. Letting t' go to $-\infty$, we obtain by A1 that $(q,t-1) \succeq (q,t)$, which contradicts A2. The result follows. ■

The geometrical interpretation of Lemma 3 is that any vertical line that intersects V must intersect all the indifference sets for $\succeq$. That is, the indifference sets for $\succeq$ do not admit vertical asymptotes, except perhaps at the boundary of V, as illustrated in Figure 2. This property plays a role analogous, in our model of possibly nonmonotone preferences, to the standard property that the indifference sets for strictly monotone preferences defined over the interior of the positive orthant must intersect any ray in the latter that emanates from the origin. Figure 3 below illustrates possible shapes for the individual's indifference sets for $\succeq$ when it satisfies or violates A6.

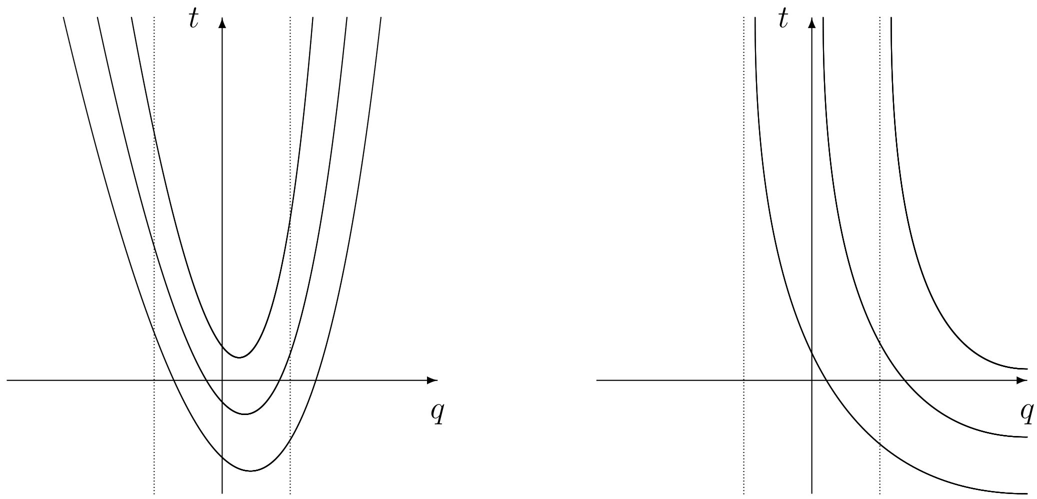

Figure 3.a $\succeq$ satisfies A6. **Figure 3.b** $\succeq$ violates A6.

Our task in this section is to characterize the space $\mathbf{P}_v$ of preferences $\succeq$ that satisfy A1–A6. Given Lemma 3, it will be convenient to work with a space of normalized utility functions, defined as

$$\mathbf{U}_v \equiv \{u \in \mathbf{U} : \operatorname{range} u(q, \cdot) = \operatorname{range} u(0_\ell, \cdot) \text{ for all } q \in Q \text{ and } u(0_\ell, t) = t \text{ if } t > \underline{t}(0_\ell)\}.$$

The normalization along the vertical axis satisfied by the utility functions in $\mathbf{U}_v$ differs from the standard radial one (Wold and Juréen (1953), Kannai (1970)), reflecting that preferences are strictly monotone in transfers, but not necessarily in the other commodities. We have the following characterization result.

Proposition 2 $\mathbf{U}_v$ *is homeomorphic with* $\mathbf{P}_v$ *under the natural map* P.

Proof. As a preliminary remark, let us observe that, for each $u \in \mathbf{U}_v$, $P(u)$ satisfies A1–A5 by Proposition 1; moreover, the assumption that $\operatorname{range} u(q, \cdot) = \operatorname{range} u(0_\ell, \cdot)$ for all $q \in Q$ implies that $P(u)$ satisfies the property stated in Lemma 3 and thus, a fortiori, A6. Hence $P(u) \in \mathbf{P}_v$ for all $u \in \mathbf{U}_v$. We must prove that the mapping $P_{|\mathbf{U}_v} : \mathbf{U}_v \to \mathbf{P}_v : u \mapsto P(u)$ is one-to-one, onto, continuous, and open.

(One-to-one) Let $(u, u') \in \mathbf{U}_v \times \mathbf{U}_v$ be such that $P(u) = P(u')$. Then $u = \zeta \circ u'$, where $\zeta : u'(V) \to \mathbb{R}$ is C^2, strictly increasing, and regular; that is, $\partial\zeta > 0$ over $u'(V)$ (Mas-Colell (1985, Proposition 2.3.11)). This implies that $\zeta(v) = \zeta(u'(0_\ell, v)) = u(0_\ell, v) = v$ for all $v \in u'(V)$, so that $u = u'$.

(Onto) Let $\succeq \in \mathbf{P}_v$. By Proposition 1, there exists some $u \in \mathbf{U}$ such that $\succeq = P(u)$. As $\operatorname{range} u(q, \cdot) = \operatorname{range} u(0_\ell, \cdot)$ for all $q \in Q$, we can implicitly define a function $u' : V \to \mathbb{R}$ by $u(q, t) = u(0_\ell, u'(q, t))$. We clearly have $P(u') = \succeq$; there remains to check that $u' \in \mathbf{U}_v$. That u' is quasiconcave follows from the fact that $\{(q, t) \in V : u'(q, t) \geq v\} = \{(q, t) \in V : u(q, t) \geq u(0_\ell, v)\}$ for all $v > \underline{t}(0_\ell)$; observe, moreover, that $(u')^{-1}([v, \infty))$ is closed relative to $\mathbb{R}^{\ell+1}$ for any such v. That u' is C^2 follows from the implicit function theorem, taking advantage of the fact that $\partial u/\partial t > 0$ over V. We then have $(\partial u/\partial t)(q, t) = (\partial u/\partial t)(0_\ell, u'(q, t))(\partial u'/\partial t)(q, t)$, which in turn implies that $\partial u'/\partial t > 0$ over V. We also obtain that $\operatorname{range} u'(q, \cdot) = u^{-1}(0_\ell, \operatorname{range} u(q, \cdot)) = u^{-1}(0_\ell, \operatorname{range} u(0_\ell, \cdot)) = \operatorname{range} u'(0_\ell, \cdot)$ for all $q \in Q$. Last, by construction, $u(0_\ell, t) = u(0, u'(0_\ell, t))$ for all $t > \underline{t}(0_\ell)$, so that, by A2, $u'(0_\ell, t) = t$ for any such t. Thus $u' \in \mathbf{U}_v$, as claimed.

(Continuous) This follows from the definition of the topology of $\mathbf{P}$.

(Open) Mimic the proof of Mas-Colell (1985, Proposition 2.4.2)). Hence the result. ■

4 Differentiably Strictly Convex Preferences

We are now ready to complete the construction of the complete and contractible space of preferences announced in the introduction. Preferences in $\mathbf{P}_v$ are not necessarily strictly convex. We impose this as an additional restriction.

A7 $\succeq$ *is strictly convex: if* $(q, t) \succeq (q', t')$, $(q, t) \neq (q', t')$, *and* $\lambda \in (0, 1)$, *then* $\lambda(q, t) + (1 - \lambda)(q', t') \succ (q', t')$.

Finally, to obtain a topologically complete space of preferences, we require preferences to be nonlinear, even in a local sense. To this end, observe that, because a utility function $u \in \mathbf{U}_v$ representing a preference relation $\succeq \in \mathbf{P}_v$ has no critical point, the Gaussian curvature of the indifference set $I_{(q,t)}$ of (q, t) for $\succeq$ is well defined and given by

$$c(q, t) \equiv \frac{1}{\|\partial u(q, t)\|^3} \begin{vmatrix} -\partial^2 u(q, t) & \partial u(q, t) \\ -\partial u^\top(q, t) & 0 \end{vmatrix},$$

see, for instance, Debreu (1972). The last restriction we impose on preferences is that this curvature nowhere vanish.

A8 $\succeq$ *is regular: for each* $(q,t) \in V$, $c(q,t) \neq 0$.

Preferences that satisfy A7–A8 are said to be *differentiably strictly convex* (Mas-Colell (1985, Definition 2.6.1)).

We can now define our fundamental space of preferences as the space $\mathbf{P}_{v,dsc}$ of preferences over V that satisfy A1–A8. Our central theorem states two key topological properties of $\mathbf{P}_{v,dsc}$, namely, completeness and contractibility.

Theorem 1 $\mathbf{P}_{v,dsc}$ *is a contractible Polish space.*

Proof. As a preliminary remark, let us observe that, by Proposition 2, $\mathbf{U}_{v,dsc} \equiv P^{-1}(\mathbf{P}_{v,dsc})$ and $\mathbf{P}_{v,dsc}$ are homeomorphic under the natural map P, so that we can indifferently work with preferences in $\mathbf{P}_{v,dsc}$ or their normalized representations in $\mathbf{U}_{v,dsc}$.

(Polish) To prove that $\mathbf{U}_{v,dsc}$ is a Polish space, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ decreasing to $\underline{t}(0_\ell)$, and let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact convex subsets of V such that $\bigcup_{n\in\mathbb{N}} K_n = V$. Then $\mathbf{U}_{v,dsc}$ is the intersection of the following countable families of open sets:

$$\left\{u \in C^2(V) : \frac{\partial u}{\partial t}(q,t) > 0 \text{ for all } (q,t) \in K_n\right\},$$

$$\Big\{u \in C^2(V) : \text{there exists } \varepsilon > 0 \text{ such that } u(q,t) < u(0_\ell, t_n) \text{ if } (q,t) \in K_n$$
$$\text{and } \inf_{(q',t')\in\mathbb{R}^{\ell+1}\setminus V} \|(q',t') - (q,t)\| \leq \varepsilon\Big\},$$

$$\left\{u \in C^2(V) : \left|\min_{(0_\ell,t)\in K_n} u(0_\ell,t) - \min_{(q,t)\in K_n} u(q,t)\right| \vee \left|\max_{(0_\ell,t)\in K_n} u(0_\ell,t) - \max_{(q,t)\in K_n} u(q,t)\right| < \frac{1}{n}\right\},$$

$$\left\{u \in C^2(V) : \max_{(0_\ell,t)\in K_n} |u(0_\ell,t) - t| < \frac{1}{n}\right\},$$

$$\Big\{u \in C^2(V) : \text{there exists } \xi : u(V) \to \mathbb{R} \text{ such that } \partial\xi > 0 \text{ over } u(V)$$
$$\text{and } \partial^2(\xi \circ u) \text{ is negative definite over } K_n\Big\}.$$

The first family deals with the strict monotonicity in transfers (A2), the second family with the boundary behavior of preferences (A4), the third and fourth families with the normalization (A6), and the fifth family with the differential strict convexity of preferences (A7–A8), bearing in mind that differentiably strictly convex preferences can be represented over any compact convex subset K of V by a C^2 utility function u with no critical point such that $\partial^2 u$ is negative definite over K (Mas-Colell (1985, Proposition 2.6.4)). Hence $\mathbf{U}_{v,dsc}$ is a G_δ in the Polish space $C^2(V)$ and thus, by Alexandrov's lemma (Mas-Colell (1985, Chapter 1, A.3.4)), a Polish space itself in the relative topology.

(Contractible) To prove that $\mathbf{U}_{v,dsc}$ is contractible, we show that the identity function over $\mathbf{U}_{v,dsc}$ is homotopic to a constant function; that is, there exists some $\overline{u} \in \mathbf{U}_{v,dsc}$ and a continuous function $h_{\overline{u}} : \mathbf{U}_{v,dsc} \times [0,1] \to \mathbf{U}_{v,dsc}$, called a *contraction*, such that $h_{\overline{u}}(u,0) = u$ and $h_{\overline{u}}(u,1) = \overline{u}$ for all $u \in \mathbf{U}_{v,dsc}$. Thus pick an arbitrary $\overline{u} \in \mathbf{U}_{v,dsc}$ and, to each $(u,\xi) \in \mathbf{U}_{v,dsc} \times [0,1]$, associate a utility function u_ξ as follows. First, let $u_0 \equiv u$ and $u_1 \equiv \overline{u}$. Next, for all $\xi \in (0,1)$ and $(q,t) \in V$, consider the following equation in μ:

$$\overline{u}(q,\mu) = u\left(q, \frac{t-\xi\mu}{1-\xi}\right). \tag{1}$$

We claim that (1) has a unique solution in the admissible range for μ,

$$\underline{t}(q) < \mu < \underline{t}(q) + \frac{1}{\xi}\,[t - \underline{t}(q)],$$

with $-\infty + \infty/\xi = \infty$ by convention. Indeed, the left-hand side of (1) is strictly increasing in μ, whereas the right-hand side of (1) is strictly decreasing in μ. Moreover, as both u and $\overline{u}$ belong to $\mathbf{U}_{v,dsc}$, we have $\inf \operatorname{range} u(q,\cdot) = \inf \operatorname{range} \overline{u}(q,\cdot) = \underline{t}(0_\ell)$. Therefore,

$$\lim_{\mu \downarrow \underline{t}(q)} \overline{u}(q,\mu) = \underline{t}(0_\ell) < \lim_{\mu \downarrow \underline{t}(q)} u\left(q, \frac{t-\xi\mu}{1-\xi}\right),$$

whereas

$$\lim_{\mu \uparrow \underline{t}(q) + \frac{1}{\xi}[t-\underline{t}(q)]} \overline{u}(q,\mu) > \underline{t}(0_\ell) = \lim_{\mu \uparrow \underline{t}(q) + \frac{1}{\xi}[t-\underline{t}(q)]} u\left(q, \frac{t-\xi\mu}{1-\xi}\right),$$

so that there exists a unique solution $\mu_\xi(q,t)$ to (1), as claimed. We can then define the utility function u_ξ by

$$u_\xi(q,t) \equiv \overline{u}(q, \mu_\xi(q,t)) = u\left(q, \frac{t - \xi\mu_\xi(q,t)}{1-\xi}\right) \tag{2}$$

for all $(q,t) \in V$. Geometrically, to each $t' > \underline{t}(0_\ell)$, this transformation assigns the t'-indifference set $u_\xi^{-1}(\{t'\})$ for u_ξ, which is obtained by taking the vertical convex combination with weights ξ and $1-\xi$ of the t'-indifference sets $\overline{u}^{-1}(\{t'\})$ and $u^{-1}(\{t'\})$ for $\overline{u}$ and u, respectively, bearing in mind that, by normalization, $\overline{u}(0_\ell, t') = u(0_\ell, t') = t'$. Figure 4 below illustrates this construction.

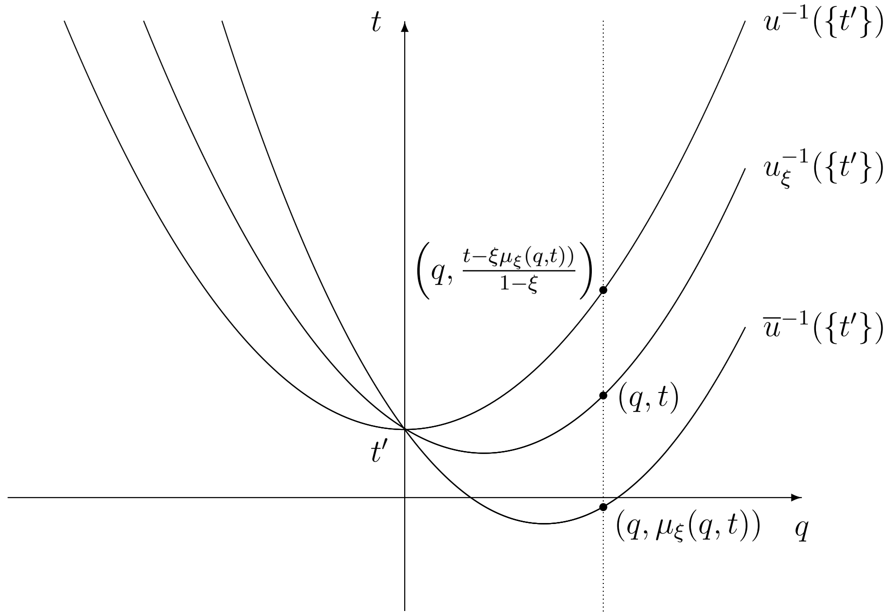

Figure 4 The contraction $h_{\overline{u}}$.

To complete the proof, we show that the mapping $(u, \xi) \mapsto u_\xi$ is an homotopy. The proof consists of two steps.

Step 1 We first check that $u_\xi \in \mathbf{U}_{v,dsc}$ for all $\xi \in [0, 1]$. This is obvious for $\xi = 0, 1$. Now, fix some $\xi \in (0, 1)$. We must prove that $\text{range}\, u_\xi(q, \cdot) = \text{range}\, u_\xi(0_\ell, \cdot)$ for all $q \in Q$ and that $u_\xi(0_\ell, t) = t$ for all $t > \underline{t}(0_\ell)$, that u_ξ is C^2, with $\partial u_\xi/\partial t > 0$ over V, and is strictly quasiconcave, that $u_\xi^{-1}([v, \infty))$ is closed relative to V for all $v \in \mathbb{R}$, and that the curvature of the indifference sets for u_ξ nowhere vanishes.

(Normalization) By construction, we have $\text{range}\, u_\xi(q, \cdot) = \text{range}\, u(q, \cdot) = \text{range}\, \overline{u}(q, \cdot) = (\underline{t}(0_\ell), \infty)$ for all $q \in Q$, and $u_\xi(0_\ell, t) = \overline{u}(0_\ell, t) = u(0_\ell, t) = t$ for all $t > \underline{t}(0_\ell)$. Hence u_ξ is normalized.

(Regularity) By (1), for each $(q, t) \in V$, $\mu_\xi(q, t)$ is the unique solution to $f(q, t, \mu) = 0$, where $f(q, t, \mu) \equiv \overline{u}(q, \mu) - u(q, (t - \xi\mu)/(1 - \xi))$. That μ_ξ is C^2 follows from the implicit function theorem along with the fact that $\partial f/\partial\mu > 0$ as $\partial\overline{u}/\partial t > 0$ and $\partial u/\partial t > 0$ over V. That u_ξ is C^2 then follows from (2).

(Strict monotonicity in transfers) Differentiating (2) with respect to t and using the fact that $\partial \overline{u}/\partial t > 0$ and $\partial u/\partial t > 0$ over V yields, for each $(q,t) \in V$,

$$\frac{\partial \mu_\xi}{\partial t}(q,t) = \frac{\frac{\partial u}{\partial t}\left(q, \frac{t-\xi\mu_\xi(q,t)}{1-\xi}\right)}{\xi \frac{\partial u}{\partial t}\left(q, \frac{t-\xi\mu_\xi(q,t)}{1-\xi}\right) + (1-\xi)\frac{\partial \overline{u}}{\partial t}(q, \mu_\xi(q,t))} > 0.$$

That $\partial u_\xi/\partial t > 0$ over V then follows from A2 and (2).

(Strict quasiconcavity) For each $t' > \underline{t}(0_\ell)$, the t'-indifference sets $\overline{u}^{-1}(\{t'\})$, $u^{-1}(\{t'\})$, and $u_\xi^{-1}(\{t'\})$ for $\overline{u}$, u, and u_ξ can be parameterized as $t = \overline{\tau}(q,t')$, $t = \tau(q,t')$, and $t = \tau_\xi(q,t') = \xi\overline{\tau}(q,t') + (1-\xi)\tau(q,t')$, respectively. Because $\overline{u}$ and u are strictly quasiconcave, the mappings $q \mapsto \overline{\tau}(q,t')$ and $q \mapsto \tau(q,t')$ are strictly convex, and so is the mapping $q \mapsto \tau_\xi(q,t')$ by convex combination. That u_ξ is strictly quasiconcave then follows from this observation along with the strict monotonicity of u_ξ in transfers.

(Boundary behavior) Fix some $\upsilon \in \operatorname{range} u_\xi = \operatorname{range} u = \operatorname{range} \overline{u}$ and let $((q_n,t_n))_{n\in\mathbb{N}}$ be a sequence in $u_\xi^{-1}([\upsilon,\infty))$ that converges to $(q,t) \in \mathbb{R}^{\ell+1}$. We must prove that $(q,t) \in u_\xi^{-1}([\upsilon,\infty))$. Because u_ξ is continuous over V, we only need to check that (q,t) does not belong to the boundary of V in $\mathbb{R}^{\ell+1}$. Recall first that, by construction, (q_n,t_n) is for each $n \in \mathbb{N}$ a convex combination with weights ξ and $1-\xi$ of $(q_n, \mu_\xi(q_n,t_n)) \in \overline{u}^{-1}([\upsilon,\infty))$ and $(q_n, [t_n - \xi\mu_\xi(q_n,t_n)]/(1-\xi)) \in u^{-1}([\upsilon,\infty))$. We claim that an implication of this is that the sequences $((q_n, \mu_\xi(q_n,t_n)))_{n\in\mathbb{N}}$ and $((q_n, [t_n - \xi\mu_\xi(q_n,t_n)]/(1-\xi)))_{n\in\mathbb{N}}$ are bounded. Indeed, if they are not, then, as the sequence $((q_n,t_n))_{n\in\mathbb{N}}$ is bounded, we can extract two divergent subsequences $(\mu_\xi(q_{n_k},t_{n_k}))_{k\in\mathbb{N}}$ and $([t_{n_k} - \xi\mu_\xi(q_{n_k},t_{n_k})]/(1-\xi))_{k\in\mathbb{N}}$ of transfers with opposite signs; suppose with no loss of generality that $\lim_{k\to\infty} \mu_\xi(q_{n_k},t_{n_k}) = -\infty$. Now, fix some (q',t') such that $\overline{u}(q',t') = \upsilon$. Because $\overline{u}$ is quasiconcave and $\overline{u}(q_{n_k}, \mu_\xi(q_{n_k},t_{n_k})) \geq \upsilon$,

$$\overline{u}\left(\left[1 + \frac{1}{\mu_\xi(q_{n_k},t_{n_k})}\right](q',t') - \frac{1}{\mu_\xi(q_{n_k},t_{n_k})}(q_{n_k}, \mu_\xi(q_{n_k},t_{n_k}))\right) \geq \upsilon$$

for all k such that $\mu_\xi(q_{n_k},t_{n_k}) < -1$. Letting k go to ∞, we obtain by continuity of $\overline{u}$ that $\overline{u}(q',t'-1) \geq \upsilon = \overline{u}(q',t')$, which contradicts the fact that $\overline{u}(q',\cdot)$ is strictly increasing. Thus the sequences $((q_n, \mu_\xi(q_n,t_n)))_{n\in\mathbb{N}}$ and $((q_n, [t_n - \xi\mu_\xi(q_n,t_n)]/(1-\xi)))_{n\in\mathbb{N}}$ are bounded, as claimed. Extracting subsequences if necessary, let us denote by $(q, t_{\overline{u}})$ and (q, t_u) their respective limits; suppose with no loss of generality that $t_{\overline{u}} \leq t \leq t_u$. As $\overline{u}^{-1}([\upsilon,\infty))$ is closed and comprehensive with respect to transfers, and as $(q_n, \mu_\xi(q_n,t_n)) \in \overline{u}^{-1}([\upsilon,\infty))$ for all $n \in \mathbb{N}$, we have $(q, t_{\overline{u}}) \in \overline{u}^{-1}([\upsilon,\infty))$ and thus $(q,t) \in \overline{u}^{-1}([\upsilon,\infty))$. That (q,t) cannot belong to the common boundary of $\mathbb{R}^{\ell+1} \setminus V$ and V in $\mathbb{R}^{\ell+1}$ then follows from the fact that the disjoint closed sets $\overline{u}^{-1}([\upsilon,\infty))$ and $\mathbb{R}^{\ell+1} \setminus V$ can be separated by open sets in $\mathbb{R}^{\ell+1}$.

(Curvature) Given the parametrization $t = \tau_\xi(q,t') = \xi\overline{\tau}(q,t') + (1-\xi)\tau(q,t')$ of the t'-indifference set $u_\xi^{-1}(\{t'\})$ for u_ξ, the Hessian $\partial^2\tau_\xi(q,t') = \xi\partial^2\overline{\tau}(q,t') + (1-\xi)\partial^2\tau(q,t')$ is positive definite, because so must be the Hessians $\partial^2\overline{\tau}(q,t')$ and $\partial^2\tau(q,t')$ for the curvatures of the t'-indifference sets $\overline{u}^{-1}(\{t'\})$ and $u^{-1}(\{t'\})$ for $\overline{u}$ and u, respectively, to nowhere vanish (Mas-Colell (1985, Chapter 1, H.3)). This proves that the curvature of the t'-indifference set $u_\xi^{-1}(\{t'\})$ for u_ξ nowhere vanishes.

Step 2 There remains to check that the mapping $h_{\overline{u}} : \mathbf{U}_{v,dsc} \times [0,1] \to \mathbf{U}_{v,dsc} : (u,\xi) \mapsto u_\xi$ is continuous. Let $((u_n,\xi_n))_{n\in\mathbb{N}}$ be a sequence in $\mathbf{U}_{v,dsc} \times [0,1]$ converging to (u,ξ). To avoid trivial cases, let us assume that, for each $n \in \mathbb{N}$, there exists $m \geq n$ such that $\xi_m \neq 0, 1$. We must prove that the sequences $(h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$, $(\partial h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$, and $(\partial^2 h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$ converge uniformly to $h_{\overline{u}}(u,\xi)$, $\partial h_{\overline{u}}(u,\xi)$, and $\partial^2 h_{\overline{u}}(u,\xi)$, respectively, over any compact subset K of V. The proof consists of three substeps.

Step 2.1 For each $n \in \mathbb{N}$ such that $\xi_n \in (0,1)$ and for each $(q,t) \in K$, define $\mu_{n,\xi_n}(q,t)$ as the unique solution to

$$\overline{u}(q,\mu) = u_n\left(q, \frac{t - \xi_n\mu}{1-\xi_n}\right). \tag{3}$$

We first claim that the sequences $(\mu_{n,\xi_n}(q,t))_{n\in\mathbb{N}}$ and $([t - \xi_n\mu_{n,\xi_n}(q,t)]/(1-\xi_n))_{n\in\mathbb{N}}$ are bounded, uniformly in $(q,t) \in K$. Suppose for instance, by way of contradiction, that there exists a divergent sequence $(\mu_{n_k,\xi_{n_k}}(q_k,t_k))_{k\in\mathbb{N}}$ such that $(q_k,t_k) \in K$ for all $k \in \mathbb{N}$; suppose also with no loss of generality that the sequence $((q_k,t_k))_{k\in\mathbb{N}}$ converges to some $(q,t) \in K$ and that $\lim_{k\to\infty} \mu_{n_k,\xi_{n_k}}(q_k,t_k) = -\infty$. (The other cases can be handled in a similar way.) Then, as the sequence $(q_k)_{k\in\mathbb{N}}$ converges to q, we have $\underline{t}(q) = -\infty$ and thus $\underline{t}(0_\ell) = -\infty$, which in turn implies $\operatorname{range}\overline{u}(q,\cdot) = \operatorname{range}\overline{u}(0_\ell,\cdot) = \mathbb{R}$ by normalization. Now, for each $(t',\varepsilon) \in \mathbb{R} \times \mathbb{R}_{++}$, we have $\mu_{n_k,\xi_{n_k}}(q_k,t_k) \leq t'$ and $\overline{u}(q_k, \mu_{n_k,\xi_{n_k}}(q_k,t_k)) \leq \overline{u}(q,t') + \varepsilon$ for k large enough. Because t' can take any value in $\mathbb{R}$ and $\operatorname{range}\overline{u}(q,\cdot) = \mathbb{R}$, we obtain

$$\lim_{k\to\infty} \overline{u}(q_k, \mu_{n_k,\xi_{n_k}}(q_k,t_k)) = -\infty. \tag{4}$$

On the other hand, as the sequence $(t_k)_{k\in\mathbb{N}}$ converges, $\lim_{k\to\infty} \mu_{n_k,\xi_{n_k}}(q_k,t_k) = -\infty$ implies that $\mu_{n_k,\xi_{n_k}}(q_k,t_k) \leq t_k$ and thus $[t_k - \xi_{n_k}\mu_{n_k,\xi_{n_k}}(q_k,t_k)]/(1-\xi_{n_k}) \geq t_k$ for k large enough. Therefore,

$$\liminf_{k\to\infty} u_{n_k}\left(q_k, \frac{t_k - \xi_{n_k}\mu_{n_k,\xi_{n_k}}(q_k,t_k)}{1-\xi_{n_k}}\right) \geq \lim_{k\to\infty} u_{n_k}(q_k,t_k) = u(q,t), \tag{5}$$

where the equality follows from the fact that the sequence $(u_n)_{n\in\mathbb{N}}$ converges uniformly to u over K and that the sequence $((q_k,t_k))_{k\in\mathbb{N}}$ converges to $(q,t) \in K$. But then, in light of (4)–(5), (3) cannot hold for $(n, q, t, \mu) = (n_k, q_k, t_k, \mu_{n_k,\xi_{n_k}}(q_k, t_k))$ for k large enough, a contradiction. The claim follows.

Step 2.2 We next claim that the sequence $(\mu_{n,\xi_n})_{n\in\mathbb{N}}$ converges uniformly to μ_ξ over K when the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to $\xi \in (0,1)$, so that $\xi_n \neq 0, 1$ for $n \in \mathbb{N}$ large enough. For any such n and for each $(q,t) \in K$, we have, by (3),

$$\overline{u}(q, \mu_{n,\xi_n}(q,t)) - u\left(q, \frac{t - \xi\mu_{n,\xi_n}(q,t)}{1-\xi}\right) = \Delta_{1,n}(q,t) + \Delta_{2,n}(q,t), \tag{6}$$

where

$$\Delta_{1,n}(q,t) \equiv u_n\left(q, \frac{t - \xi_n\mu_{n,\xi_n}(q,t)}{1-\xi_n}\right) - u\left(q, \frac{t - \xi_n\mu_{n,\xi_n}(q,t)}{1-\xi_n}\right)$$

and

$$\Delta_{2,n}(q,t) \equiv u\left(q, \frac{t - \xi_n\mu_{n,\xi_n}(q,t)}{1-\xi_n}\right) - u\left(q, \frac{t - \xi\mu_{n,\xi_n}(q,t)}{1-\xi}\right).$$

By Step 2.1, there exists some compact subset K' of V such that

$$\max_{(q,t)\in K} |\Delta_{1,n}(q,t)| \leq \|u_n - u\|_{K'}$$

and

$$\max_{(q,t)\in K} |\Delta_{2,n}(q,t)| \leq \left\|\frac{\partial u}{\partial t}\right\|_{K'} \sup_{(q,t)\in K} \frac{|(\xi_n - \xi)[t - \mu_{n,\xi_n}(q,t)]|}{(1-\xi)(1-\xi_n)}.$$

Taking limits as n goes to ∞ yields, by (6),

$$\begin{aligned}\lim_{n\to\infty} \overline{u}(q, \mu_{n,\xi_n}(q,t)) - u\left(q, \frac{t - \xi\mu_{n,\xi_n}(q,t)}{1-\xi}\right) &= 0 \\ &= \overline{u}(q, \mu_\xi(q,t)t) - u\left(q, \frac{t - \xi\mu_\xi(q,t)}{1-\xi}\right),\end{aligned}$$

uniformly in $(q,t) \in K$. By Step 2.1, this implies that

$$\lim_{n\to\infty} \left[\min_{(q',t')\in K'} \frac{\partial \overline{u}}{\partial t}(q',t') + \frac{\xi}{1-\xi} \min_{(q',t')\in K'} \frac{\partial u}{\partial t}(q',t')\right] |\mu_{n,\xi_n}(q,t) - \mu_\xi(q,t)| = 0,$$

uniformly in $(q,t) \in K$, from which the claim follows as both $\partial\overline{u}/\partial t$ and $\partial u/\partial t$ are positive and bounded away from 0 over any compact subset of V.

Step 2.3 We are now ready to prove the required convergence results, first for the functions $(h_{\overline{u}}(u_n, \xi_n))_{n\in\mathbb{N}}$, and then for their first- and second-order derivatives $(\partial h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$ and $(\partial^2 h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$.

(Functions) Suppose first that the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to $\xi \in (0,1)$. For all $(q,t) \in K$ and $n \in \mathbb{N}$, we have $h_{\overline{u}}(u_n,\xi_n)(q,t) = \overline{u}(q,\mu_{n,\xi_n}(q,t))$ and $h_{\overline{u}}(u,\xi)(q,t) = \overline{u}(q,\mu_\xi(q,t))$. Hence, by Step 2.1,

$$\|h_{\overline{u}}(u_n,\xi_n) - h_{\overline{u}}(u,\xi)\|_K \leq \left\|\frac{\partial \overline{u}}{\partial t}\right\|_{K'} \|\mu_{n,\xi_n} - \mu_\xi\|_K,$$

which converges to 0 by Step 2.2. Suppose next that the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to 0. We can focus on the terms of the sequence $(h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$ such that $\xi_n \neq 0$, for the other terms are equal to $h_{\overline{u}}(u_n,0) = u_n$ and the sequence $(u_n)_{n\in\mathbb{N}}$ converges uniformly to $u = h_{\overline{u}}(u,0)$ over K. Then (6) holds for $\xi = 0$, and reasoning as in Step 2.2 yields that

$$\|h_{\overline{u}}(u_n,\xi_n) - u\|_K = \|h_{\overline{u}}(u_n,\xi_n) - h_{\overline{u}}(u,0)\|_K$$

converges to 0. Suppose finally that the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to 1. We can focus on the terms of the sequence $(h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$ such that $\xi_n \neq 1$, for the other terms are equal to $h_{\overline{u}}(u_n,1) = \overline{u}$. Then, letting $\nu_{n,\xi_n}(q,t) \equiv [t - \xi_n\mu_{n,\xi_n}(q,t)]/(1-\xi_n)$ for all $(q,t) \in K$, we have $h_{\overline{u}}(u_n,\xi_n) = \overline{u}(q,[t-(1-\xi_n)\nu_{n,\xi_n}(q,t)]/\xi_n)$. Bearing in mind that, as shown in Step 2.1, the sequence $(\nu_{n,\xi_n}(q,t))_{n\in\mathbb{N}}$ is bounded, uniformly in $(q,t) \in K$, we obtain as above that

$$\|h_{\overline{u}}(u_n,\xi_n) - \overline{u}\|_K = \|h_{\overline{u}}(u_n,\xi_n) - h_{\overline{u}}(u,1)\|_K$$

converges to 0. Therefore, in any case, the sequence $(h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$ converges uniformly to $h_{\overline{u}}(u,\xi)$ over K.

(Derivatives) We focus on the first-order derivatives $(\partial h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$. (The proof for the second-order derivatives $(\partial^2 h_{\overline{u}}(u_n,\xi_n))_{n\in\mathbb{N}}$ is similar and is, therefore, omitted.) Suppose first that the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to $\xi \in (0,1)$, so that $\xi_n \neq 0, 1$ for $n \in \mathbb{N}$ large enough. For any such n and for each $(q,t) \in K$, we have, by the implicit function theorem,

$$\begin{pmatrix} \vdots \\ \dfrac{\partial \mu_{n,\xi_n}}{\partial q_l}(q,t) \\ \vdots \\ \dfrac{\partial \mu_{n,\xi_n}}{\partial t}(q,t) \end{pmatrix} = \begin{pmatrix} \vdots \\ \dfrac{(1-\xi_n)\left[\frac{\partial u_n}{\partial q_l}\left(q, \frac{t-\xi_n\mu_{n,\xi_n}(q,t)}{1-\xi_n}\right) - \frac{\partial \overline{u}}{\partial q_l}(q,\mu_{n,\xi_n}(q,t))\right]}{\xi_n \frac{\partial u_n}{\partial t}\left(q, \frac{t-\xi_n\mu_{n,\xi_n}(q,t)}{1-\xi_n}\right) + (1-\xi_n)\frac{\partial \overline{u}}{\partial t}(q,\mu_{n,\xi_n}(q,t))} \\ \vdots \\ \dfrac{\frac{\partial u_n}{\partial t}\left(q, \frac{t-\xi_n\mu_{n,\xi_n}(q,t)}{1-\xi_n}\right)}{\xi_n \frac{\partial u_n}{\partial t}\left(q, \frac{t-\xi_n\mu_{n,\xi_n}(q,t)}{1-\xi_n}\right) + (1-\xi_n)\frac{\partial \overline{u}}{\partial t}(q,\mu_{n,\xi_n}(q,t))} \end{pmatrix}.$$

Because, as shown in Step 2.2, the sequence $(\mu_{n,\xi_n})_{n\in\mathbb{N}}$ converges uniformly to μ_ξ over K, and the sequences $(\partial u_n/\partial q_l)_{n\in\mathbb{N}}$ and $(\partial u_n/\partial t)_{n\in\mathbb{N}}$ converge uniformly to $\partial u/\partial q_l$ and $\partial u/\partial t$

over compact subsets of V, this converges to

$$\begin{pmatrix} \vdots \\ \dfrac{(1-\xi)\left[\frac{\partial u}{\partial q_l}\left(q, \frac{t-\xi\mu_\xi(q,t)}{1-\xi}\right) - \frac{\partial \overline{u}}{\partial q_l}(q, \mu_\xi(q,t))\right]}{\xi\frac{\partial u}{\partial t}\left(q, \frac{t-\xi\mu_\xi(q,t)}{1-\xi}\right) + (1-\xi)\frac{\partial \overline{u}}{\partial t}(q, \mu_\xi(q,t))} \\ \vdots \\ \dfrac{\frac{\partial u}{\partial t}\left(q, \frac{t-\xi\mu_\xi(q,t)}{1-\xi}\right)}{\xi\frac{\partial u}{\partial t}\left(q, \frac{t-\xi\mu_\xi(q,t)}{1-\xi}\right) + (1-\xi)\frac{\partial \overline{u}}{\partial t}(q, \mu_\xi(q,t))} \end{pmatrix} = \begin{pmatrix} \vdots \\ \dfrac{\partial \mu_\xi}{\partial q_l}(q,t) \\ \vdots \\ \dfrac{\partial \mu_\xi}{\partial t}(q,t) \end{pmatrix},$$

uniformly in $(q,t) \in K$. As a result,

$$\partial h_{\overline{u}}(u_n, \xi_n)(q,t) = \begin{pmatrix} \vdots \\ \dfrac{\partial \overline{u}}{\partial q_l}(q, \mu_{n,\xi_n}(q,t)) + \dfrac{\partial \overline{u}}{\partial t}(q, \mu_{n,\xi_n}(q,t))\,\dfrac{\partial \mu_{n,\xi_n}}{\partial q_l}(q,t) \\ \vdots \\ \dfrac{\partial \overline{u}}{\partial t}(q, \mu_{n,\xi_n}(q,t))\,\dfrac{\partial \mu_{n,\xi_n}}{\partial t}(q,t) \end{pmatrix}$$

converges to

$$\begin{pmatrix} \vdots \\ \dfrac{\partial \overline{u}}{\partial q_l}(q, \mu_\xi(q,t)) + \dfrac{\partial \overline{u}}{\partial t}(q, \mu_\xi(q,t))\,\dfrac{\partial \mu_\xi}{\partial q_l}(q,t) \\ \vdots \\ \dfrac{\partial \overline{u}}{\partial t}(q, \mu_\xi(q,t))\,\dfrac{\partial \mu_\xi}{\partial t}(q,t) \end{pmatrix} = \partial h_{\overline{u}}(u, \xi)(q,t),$$

uniformly in $(q,t) \in K$. Suppose finally that the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to 0. (The proof for the case where the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to 1 is similar and is, therefore, omitted.) We can focus on the terms of the sequence $(\partial h_{\overline{u}}(u_n, \xi_n))_{n\in\mathbb{N}}$ such that $\xi_n \neq 0$, for the other terms are equal to $\partial h_{\overline{u}}(u_n, 0) = \partial u_n$ and the sequence $(\partial u_n)_{n\in\mathbb{N}}$ converges uniformly to $\partial u = \partial h_{\overline{u}}(u, 0)$ over K. For any such n and for each $(q,t) \in K$, we have

$$\partial h_{\overline{u}}(u_n, \xi_n)(q,t) = \partial \overline{u}(q, \mu_{n,\xi_n}(q,t)) = \partial u_n\left(q, \frac{t - \xi_n \mu_{n,\xi_n}(q,t)}{1-\xi_n}\right),$$

which converges to $\partial u(q,t)$, uniformly in $(q,t) \in K$, because the sequence $(\mu_{n,\xi_n}(q,t))_{n\in\mathbb{N}}$ is bounded, uniformly in $(q,t) \in K$, and the sequence $(\partial u_n)_{n\in\mathbb{N}}$ converges uniformly to ∂u over compact subsets of V. Hence the result. ■

Observe that the compensation principle expressed by A6 plays a key role in the proof that $\mathbf{P}_{v,dsc}$ is contractible. Indeed, A6 ensures that the vertical convex combination of any two indifference sets for u and $\overline{u}$ going through the same point of the vertical axis is well defined, which in turn allows us to construct the contraction $h_{\overline{u}}$ in a straightforward way. The first part of the proof of Theorem 1 shows that the space $\mathbf{P}_{dsc}$ of preferences that satisfy A1–A5 and A7–A8, but not necessarily A6, is topologically complete. However, it is an open question whether it is contractible.

5 Collective Choice

We now draw the implications of our analysis for the aggregation of individual preference relations in $\mathbf{P}_{v,dsc}$. The collective choice framework we have in mind is as follows. Consider a group of I individuals $i = 1, \dots, I$, which may be thought of as a household, a family, a firm, a union, or a club. Each individual in the group has preferences about the consumption of private and collective goods within the group. In particular, an individual may care about the consumption of private goods by other members of the group, as in Becker's (1981) model of altruism in the family or Chiappori's (1988, 1992) collective model of household labor supply. Individuals also privately contribute to the supply of collective goods, such as household chores or meeting participations.

We allow individual preferences to be nonmonotone in such contributions. This captures the idea that a small contribution comes at a negligible personal marginal cost but generate a nonnegligible personal marginal benefit, both directly and perhaps also indirectly through altruistic concerns. By contrast, larger contributions generate substantial marginal costs that outweigh marginal benefits.[5] Finally, although individuals may value in different ways the consumption of private and collective goods, as well as their contributions to the supply of collective goods, there is at least one collective good that is always desirable from each individual's viewpoint. An example in the case of a household or a family may be an index of the amount and quality of time spent together on vacation; in the case of a club, the practice of common-interest activities.

Formally, we will assume that, as in the general model of Section 2, individuals have preferences defined over ℓ commodities representing the amounts of private and collective goods consumed within the group as well as the private contributions to the latter, and a uniformly desirable collective good denoted by $\ell + 1$. Each individual i is endowed with a preference relation $\succeq_i \in \mathbf{P}_{v,dsc}$ over V. The problem is how to aggregate a profile of

[5] Conversely, negative contributions to collective goods can, in the presence of altruistic concerns, come at a personal cost in the form of guilt or shame. For instance, if we interpret q in Figure 2 as the individual's contribution to a collective good, then, for any fixed level of t, he equally loses from contributing too little, $q = q_1^- + \varepsilon$, as from contributing too much, $q = q_1^+ - \varepsilon$. A4 implies that both situations involve prohibitive costs as $\varepsilon > 0$ converges to zero.

individual preferences relations $(\succeq_1, \ldots, \succeq_I) \in \mathbf{P}^I_{v,dsc}$ into a collective preference relation in $\mathbf{P}_{v,dsc}$ through a collective choice rule

$$\Phi_I : \mathbf{P}^I_{v,dsc} \to \mathbf{P}_{v,dsc} : (\succeq_1, \ldots, \succeq_I) \mapsto \Phi_I(\succeq_1, \ldots, \succeq_I). \tag{7}$$

Following Chichilnisky's (1980) classical formulation of the topological social choice problem, we restrict ourselves to collective choice rules that satisfy the following axioms.

Anonymity *For each* $(\succeq_1, \ldots, \succeq_I) \in \mathbf{P}^I_{v,dsc}$ *and for any permutation* σ *of* $\{1, \ldots, I\}$, $\Phi_I(\succeq_{\sigma(1)}, \ldots, \succeq_{\sigma(I)}) = \Phi_I(\succeq_1, \ldots, \succeq_I)$.

Anonymity requires that, if individuals exchange their preferences, then the collective preferences remain the same; this axiom is stronger than Arrow's (1951) nondictatorship axiom.

Unanimity *For each* $\succeq \in \mathbf{P}_{v,dsc}$, $\Phi_I(\succeq, \ldots, \succeq) = \succeq$.

Unanimity requires that, if all individuals have the same preferences, then the collective preferences coincide with the individual preferences; this axiom is weaker than Arrow's (1951) Pareto axiom.

Continuity Φ_I *is continuous.*

Continuity requires that, if two individual preference profiles are close to each other, then so are the collective preferences; this axiom replaces Arrow's (1951) independence axiom as the interprofile consistency condition.[6]

The question is then whether there exists a collective choice rule Φ_I over $\mathbf{P}^I_{v,dsc}$ that is anonymous, continuous, and respects unanimity. Because $\mathbf{P}_{v,dsc}$ is contractible by Theorem 1, it is tempting to invoke Chichilnisky and Heal (1983, Theorem 1) to infer that such a collective choice rule indeed exists. However, there are two important obstacles to such a hasty conclusion.

The first obstacle is that, following Debreu (1972), each individual i's preferences in Chichilnisky and Heal (1983) are represented by a locally integrable C^1 normalized vector field g_i over V, which locally describes the gradient of individual i's utility function. When the space of admissible preferences is convex, a natural collective choice rule simply consists in averaging the vector fields g_i, $i = 1, \ldots, I$; clearly, this rule is anonymous, continuous, and respects unanimity. Moreover, it can be extended to the case where the space of admissible preferences is a retract of its convex hull. However, these operations typically do not preserve

[6] The surveys by Lauwers (2000, 2009) and Baigent (2011) offer useful discussions of these axioms.

the convexity of preferences, as required by (7); besides, it is unclear whether $\mathbf{P}_{v,dsc}$ satisfies the required retractability property.

The second obstacle is that the space of admissible preferences in Chichilnisky and Heal (1983) is assumed to have a very particular topological structure; specifically, it must be a path-connected parafinite CW complex.[7] Roughly speaking, this means that this space is built in a countable number of stages, each stage being obtained from the previous one by adding cells of a given finite dimension; a typical example is the space of linear preferences over V. Horwath (2001) allows for a much broader class of spaces of admissible preferences, but it is unclear whether the space $\mathbf{P}_{v,dsc}$ belongs to it.

These difficulties prevent us from directly applying Chichilnisky and Heal's (1983) result. Fortunately, the explicit construction of the contraction $h_{\overline{u}}$ for an arbitrary $\overline{u} \in \mathbf{U}_{v,dsc}$ suggests an easy alternative. As in the proof of Theorem 1, we identify preferences in $\mathbf{P}_{v,dsc}$ with their representations in $\mathbf{U}_{v,dsc}$. The following aggregation result then holds.

Theorem 2 *Define inductively a sequence of collective choice rules as follows: for each* $u_1 \in \mathbf{U}_{v,dsc}$, *let*

$$\Phi_1(u_1) \equiv u_1,$$

and for all $I \geq 2$ *and* $(u_1, \ldots, u_I) \in \mathbf{U}^I_{v,dsc}$, *let*

$$\Phi_I(u_1, \ldots, u_I) \equiv h_{\Phi_{I-1}(u_1,\ldots,u_{I-1})}\left(u_I, \frac{I-1}{I}\right).$$

Then, for each $I \geq 1$, $\Phi_I : \mathbf{U}^I_{v,dsc} \to \mathbf{U}_{v,dsc}$ *satisfies Anonymity, Unanimity, and Continuity.*

Proof. We use a straightforward induction on I. Proceeding along the lines of the proof of Theorem 1 first shows that $\Phi_I(u_1, \ldots, u_I) \in \mathbf{U}_{v,dsc}$ for all $I \geq 1$ and $(u_1, \ldots, u_I) \in \mathbf{U}^I_{v,dsc}$. We next check each axiom in turn.

(Anonymity) We only need to prove that, for all I and $t' > \underline{t}(0_\ell)$, the t'-indifference set for $\Phi_I(u_1, \ldots, u_I)$ is the vertical convex combination with identical weights $1/I$ of the indifference sets $u_1^{-1}(\{t'\}), \ldots, u_I^{-1}(\{t'\})$ for $u_1, \ldots, u_I$, respectively; that Φ_I is anonymous then follows from the symmetry of this construction. The result trivially holds for Φ_1. Let then $I \geq 2$, and suppose that the result holds for Φ_{I-1}. By definition of the contraction $h_{\Phi_{I-1}(u_1,\ldots,u_{I-1})}$, the t'-indifference set for $\Phi_I(u_1, \ldots, u_I)$ is the vertical convex combination with weights $(I-1)/I$ and $1/I$ of the t'-indifference sets $(\Phi_{I-1}(u_1, \ldots, u_{I-1}))^{-1}(\{t'\})$ and

[7] See Spanier (1966) for precise definitions of these terms.

$u_I^{-1}(\{t'\})$ for $\Phi_{I-1}(u_1, \ldots, u_{I-1})$ and u_I, respectively. Using the induction hypothesis then completes the induction step.

(Unanimity) The result trivially holds for Φ_1. Let then $I \geq 2$, and suppose that the result holds for Φ_{I-1}. Then, for each $u \in \mathbf{U}_{v,dsc}$, we have $\Phi_I(u, \ldots, u) = h_{\Phi_{I-1}(u,\ldots,u)}(u, (I-1)/I) = h_u(u, (I-1)/I) = u$. This completes the induction step.

(Continuity) The result trivially holds for Φ_1. Let then $I \geq 2$, and suppose that the result holds for Φ_{I-1}. Using similar arguments as for the contractibility part of Theorem 1, we can show that, for each $\xi \in (0,1)$, the mapping $(\overline{u}, u) \mapsto h_{\overline{u}}(u, \xi)$ is continuous from $\mathbf{U}_{v,dsc} \times \mathbf{U}_{v,dsc}$ to $\mathbf{U}_{v,dsc}$. Letting $\xi \equiv (I-1)/I$, $\overline{u} \equiv \Phi_{I-1}(u_1, \ldots, u_{I-1})$, and $u \equiv u_I$, and using the induction hypothesis then completes the induction step. Hence the result. ■

Observe again that, through the use of the contraction $h_{\overline{u}}$, the compensation principle expressed by A6 plays a key role in the proof of Theorem 2. It is interesting, by contrast, to consider what may happen when A6 is violated by each individual preference relation in a profile. Figure 5 below illustrates this situation.

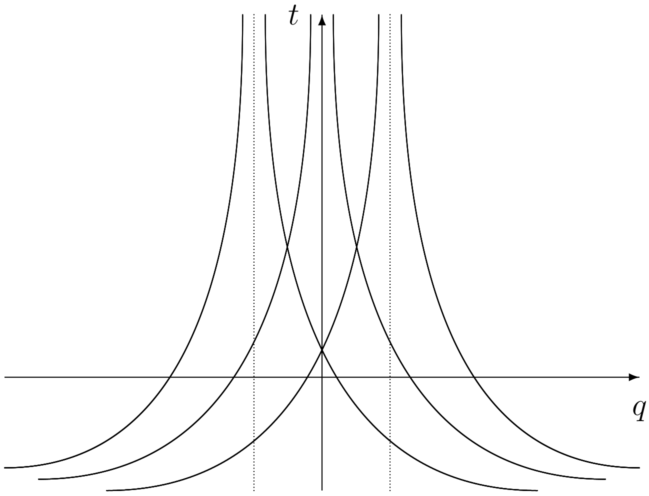

Figure 5 $\succeq_1$ and $\succeq_2$ violate A6.

The two individuals whose preferences are depicted in Figure 5 have opposite views regarding the desirability of the first commodity, and transfers are not effective enough to compensate either of them for holding large (in absolute value) undesired amounts of that commodity. It is then unclear how to aggregate their preferences into a collective preference relation that satisfies, in particular, strict convexity. Notice in that respect that the additive rule for gradient fields will not do. Indeed, suppose for instance that the preferences $\succeq_1$ and $\succeq_2$ admit utility functions $u_1(q,t) = -q - 1/(t+1)$ and $u_2(q,t) = q - 1/(t+1)$ for $(q,t) \in V \equiv \mathbb{R} \times (-1,\infty)$, so that indifference sets are convex hyperbolas. Then the addition of the normalized gradients of u_1 and u_2 is everywhere equal to $(0,1)$, which does not correspond to strictly convex preferences: the candidate collective indifference sets are flat, reflecting that, at each point of V, individuals 1 and 2 only agree on the fact that higher transfers would be desirable. It is thus an open question whether the space $\mathbf{P}_{dsc}$ of preferences that satisfy A1–A5 and A7–A8, but not necessarily A6, admits a collective choice rule that is anonymous, continuous, and respects unanimity.

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