Beta Risk in the Cross-Section of Equities*

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We develop a conditional capital asset pricing model in continuous-time that allows for stochastic beta exposure. When beta co-moves with market variance and the stochastic discount factor (SDF), beta risk is priced, and the expected return on a stock deviates from the security market line. The model predicts that low-beta stocks earn high returns because their beta co-moves positively with market variance and the SDF. The opposite is true for high-beta stocks. Estimating the model on equity and option data, we find that beta risk explains expected returns on low- and high-beta stocks, resolving the "betting against beta" anomaly. (*JEL* G10, G12, G13)

The exposure of a stock to market fluctuations is arguably its most important risk characteristic. But after more than 50 years of research, the pricing of this market risk exposure remains unclear. Black, Jensen, and Scholes (1972) show that the cross-sectional relationship between estimates of beta and average stock return is too flat compared with the prediction of the classic capital asset pricing model (CAPM; Sharpe, 1964; Lintner, 1965; Mossin, 1966). An extensive subsequent literature documents pronounced time-variation in the market exposure of stock returns.[1] However, the empirical performance of the conditional CAPM is still controversial.[2]

On top of the overall lack of consensus, various seemingly conflicting results exist in the literature. Frazzini and Pedersen (2014) estimate betas using daily data and find that low-beta stocks offer relatively high returns on average and vice versa generating a beta anomaly. Gilbert et al. (2014) document that this conclusion depends on the frequency of data used in beta estimation. Cederburg and O'Doherty (2016) model regression-based conditional betas using instrumental variables. They find that time-variation in beta explains unconditional alphas of betting against beta strategies. Lewellen and Nagel (2006) forcefully argue that the pricing errors of conditional CAPM are too large to be explained by fluctuation in regression-betas. Buss and Vilkov (2012) document that option-implied betas improve the performance of conditional CAPM by generating a steeper security market line (SML). In view of the state of the literature, further evidence on the impact of fluctuation in betas on expected stock returns is therefore of paramount importance.

In this paper, we formalize the notion of beta (instability) risk put forward by Jagannathan

[1] De Bondt and Thaler (1987) provide regression-based evidence of time-variation in market beta. Shanken (1990) studies a model where market beta is a function of state variables. Jagannathan and Wang (1996) allow innovations in market beta to depend on a market risk premium and a residual component. Bollerslev, Li and Todorov (2016) study continuous and jump market betas. Another strand of the literature analyzes the dynamics of consumption betas. For instance, see, Lettau and Ludvigson (2001), Santos and Veronesi (2006), and Lustig and van Nieuwerburgh (2005).

[2] Among others, see, Ghysels (1998), Lewellen and Nagel (2006), Nagel and Singleton (2011), and the recent surveys by Goyal (2012) and Nagel (2013).

and Wang (1996). We develop a new asset pricing model in which individual equity and market returns covary dynamically and where beta itself is stochastic. In the model, the expected return on a stock deviates from the conditional SML when beta risk is priced, that is, when beta covaries with the market variance and, more generally, with the stochastic discount factor (SDF). Our model extends traditional frameworks by providing explicit dynamics for stochastic beta. This, in turn, enables us to study the pricing implications of beta risk in the cross-section. Our main contribution is to show that, when beta is stochastic, the covariance of beta with the SDF and market variance results in economically large deviations of expected stock returns from the conditional SML which helps resolve the beta anomaly.

The key research questions we pose are the following: Is beta risk priced in the cross-section of stock returns? If so, can it help explain qualitatively and quantitatively the relatively flat relation between expected equity returns and beta observed in practice? Is there any leftover alpha to be explained? Finally, is the steeper SML generated by option-implied betas related to the pricing of beta risk in the cross-section? The answer turns out to be "yes" to all of these questions.

To address our research questions, we proceed by first specifying physical factor dynamics for the equity market. Second, we assume a SDF allowing for market return and variance risks to be priced. Third, we derive equity return premiums and derivative prices from the physical dynamics and SDF. The pricing kernel we assume allows for a market variance risk premium and can be interpreted as a reduced-form approximation of the equilibrium SDF implied by an intertemporal CAPM with stochastic volatility in consumption. Our work therefore builds on the literature on the market variance risk premium including the seminal contributions by Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), and Driessen, Maenhout, and Vilkov (2009).

To capture equity market dynamics, we develop a bivariate extension of Heston's (1993) stochastic volatility model in which the variance-covariance matrix of index and equity returns follows a Wishart process. Our model allows individual equity and market returns to covary stochastically and to price equity and index options in closed-form.[3]

Several studies document that the relation between expected equity returns and beta in the US is flatter than predicted by the SML (Black, Jensen, and Scholes, 1972; Fama and French, 1992). In Frazzini and Pedersen (2014)'s model, leverage-constrained investors tilt their portfolios toward high-beta assets bidding up their prices relative to the low-beta stocks that require financial leverage. Our model provides an alternative channel based on systematic beta risk to explain the weak empirical relationship between estimated average stock returns and beta.

We argue that the equity return premium consists partly of a premium for beta risk. Our model predicts that when beta is low it co-moves more strongly with the SDF and market variance than when it is high. Thus, low-betas tend to increase in bad times which results in a form of "wrong-way" beta risk. To compensate low-beta firms for this added risk, they earn an additional premium in our model. The expected return of high-beta firms is correspondingly less than what the standard market model predicts because their betas covary negatively with the SDF and the market variance. Empirically, we show that beta risk generates large upward and downward deviations of expected stock returns from the conditional SML that explain the abnormal performance of low- and high-beta stocks observed in practice.

We estimate the model by maximizing the joint return and option likelihood for a cross-section of 344 stocks observed over 19 years. Our sample includes the tech bubble and the recent

[3]For one-factor model extensions of Heston (1993) with constant loadings on market innovations, see, Serban, Lehoczky, and Seppi (2008), Christoffersen, Fournier and Jacobs (2018) and Bégin, Dorion, and Gauthier (2018), among others.

financial and sovereign debt crises. This allows us to study fluctuations in betas over pronounced economic cycles. Overall, we find that the model fits the return and option data well.

So far, the conditional CAPM literature has primarily focused on analyzing the implications of time-variation in regression-based betas on unconditional alphas (among others, see, Jagannathan and Wang, 1996; Cederburg and O'Doherty, 2016). Our work extends prior studies in one important way. By fully specifying the dynamics of beta and the SDF, our model delivers a closed-form solution for conditional expected returns at any horizon (i.e., term structure of expected equity return). This allows us to analyze return deviations from the conditional SML over time.

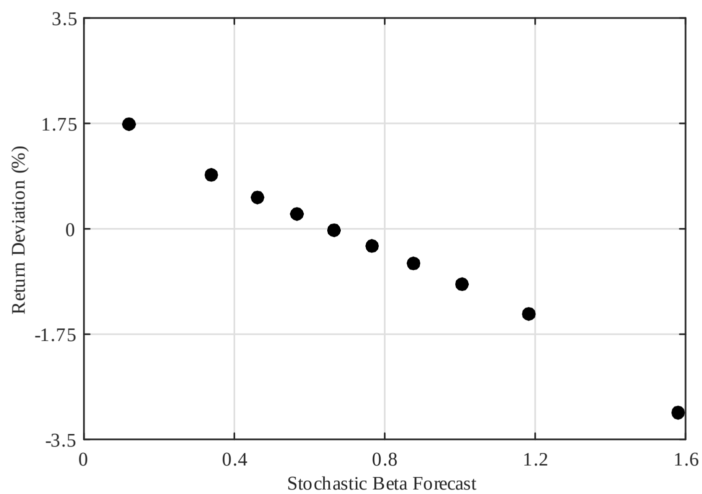

Figure 1

Annualized return deviation induced by beta risk

The figure presents model-implied deviations of expected stock returns from the conditional SML induced by beta risk. On each day, we sort stocks into decile portfolios based on the model one-month stochastic beta forecast. We then calculate for each portfolio the daily value-weighted average of annualized return deviation implied by our model and scatter plot the sample average of these deviations against beta. The sample period is from January 8^{th}, 1996 to December 30^{th}, 2016.

We document substantial cross-sectional and temporal variation in return deviations from the conditional SML. On average, the return deviations implied by our model for the high-minus-low beta strategy is -4.79% during our sample period. We further validate the model predictions using ordinary least squares (OLS) betas for portfolios of NYSE stocks. Variation in OLS betas causes an average deviation of the high-minus-low beta strategy from the SML of -5.16%. Our model thus explains about 90% of observed SML mispricing. To validate further the model predictions, we compare ex-ante and ex-post OLS betas. We show that the ex-post beta of the high-beta portfolio subsequently co-moves negatively with the SDF and market variance the year following the sorting. We further show that the opposite is true for the low-beta portfolio.

To capture variation in betas, the common practice is to adopt a rolling-window beta estimation method using OLS.[4] A key challenge faced by this approach is to find the right balance between bias and efficiency. On one hand, the longer the estimation window, the more potential bias there is if beta is truly dynamic. On the other hand, the shorter the estimation window, the greater the loss of efficiency is in estimation. Our dynamic model offers two advantages over the standard approach. First, we can use the convenient particle filter to extract latent conditional stochastic betas from a single daily return observation once the model parameters are estimated.

[4] This is done in Petkova and Zhang (2005), Lewellen and Nagel (2006), and Ang, Chen, and Xing (2006), among others.

Second, we can forecast beta across horizons using the stock's most recent filtered beta and the model-implied beta dynamics.

There is comprehensive evidence on variance and correlation risk premiums.[5] In contrast, the study of the beta risk premium and its implications for equity expected return is unexplored. When market variance risk is priced, beta co-moves with the SDF in the model which generates a wedge between physical and risk-neutral betas (i.e., a beta risk premium). Because the beta risk premium of low-beta stocks is negative while it is positive for high-beta stocks, risk-neutral betas shrink toward one resulting in a steeper SML. This prediction provides theoretical support to Buss and Vilkov (2012) and Christoffersen, Jacobs, and Vainberg (2012), who empirically show that option-implied betas generate a more pronounced SML than OLS betas. To validate our model predictions, we first show that model-implied beta risk premiums quantitatively match model-free premium measures for beta-sorted decile portfolios. This result proves that using option prices in the estimation helps us pin down the dynamics of the beta risk premium and extract relevant information about beta's co-movement with the SDF.[6] Using ex-post excess stock returns, we then study the average slopes of the conditional SML implied by model's physical and risk-neutral beta forecasts. Consistent with our model, we find that model risk-neutral betas result in a steeper SML than model physical betas. Impressively, both measures of beta obtain positive loadings that are highly significant when predicting the cross-section of ex-post equity returns up to three months ahead.

Gouriéroux and Sufana (2006) apply Wishart processes for the pricing of credit risk and

[5] For example, see, Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), Driessen, Maenhout, and Vilkov (2009), and Todorov (2010).

[6] This is related to various existing studies that show that option prices are highly informative about underlying risk and premium dynamics. Among others, see, Pan (2002), Carr and Wu (2009), and Bollerslev, Todorov, and Xu (2015).

the modeling of interest rates. Buraschi, Porchia, and Trojani (2010) solve an intertemporal portfolio allocation where the dependence across countries and asset classes is captured by a Wishart dynamic. Gruber, Tebaldi, and Trojani (2015) develop an index model with time-varying Wishart jump intensity. To our knowledge, our study is the first to use Wishart processes to capture the joint dynamic of market index and individual equity returns.[7]

Our work is also related to the empirical literature that studies correlation dynamics. Engle and Kelly (2012) study a time-varying equicorrelation model in which the correlation of various pairs of stocks is equal in the cross-section. Engle (2016), Bali, Engle, and Tang (2017), and Bali and Zhou (2016) develop GARCH-style beta models.[8] Campbell, Giglio, Polk, and Turley (2018) study the pricing of volatility risk in stock returns using an intertemporal CAPM with stochastic volatility but constant correlation. Patton and Verardo (2012) investigate whether stock betas vary with earning announcements using daily betas estimated from intraday prices. We complement these papers by developing a new model that allows for stochastic beta exposure and, by showing that, co-movements of beta with aggregate risks, have important pricing implications.

1 Building a Market Model with Stochastic Beta

First, we define the modeling framework allowing for stochastic beta in a market index model. We then provide details on the stochastic discount factor that allows us to pin down risk-premiums and option prices.

[7] See Bru (1991), Gouriéroux (2006), Da Fonseca, Grasselli, and Tebaldi (2007), Da Fonseca and Grasselli (2011), and Mayerhofer (2012) for other studies on Wishart processes.

[8] See also Engle (2002) and Bali (2008) for other studies on the modeling of assets' dynamic dependence within GARCH-style models.

1.1 The Modeling Framework

Consider a market index, I_t, and stock price, S_t, with physical dynamics of the form

$$\begin{bmatrix} \frac{dI_t}{I_t} \\ \\ \frac{dS_t}{S_t} \end{bmatrix} = \begin{bmatrix} r + \mu_{I,t} \\ \\ r + \beta_t \mu_{I,t} \end{bmatrix} dt + \sqrt{\Sigma_t} \begin{bmatrix} dZ_{I,t} \\ \\ dZ_{S,t} \end{bmatrix}, \tag{1}$$

where we specify the stochastic instantaneous (spot) beta as $\beta_t \equiv \sigma_{SI,t}/\sigma^2_{I,t}$, and where $\sigma_{SI,t}$ is the spot covariance between the stock and the index, and $\sigma^2_{I,t}$ is the market index spot variance. In equation (1), $dZ_{I,t}$ denotes market return risk and $dZ_{S,t}$ is the idiosyncratic equity shock. We thus assume that the continuous-time conditional CAPM holds and that the equity premium, $\mu_{I,t}$, is the slope of the instantaneous SML.

The matrix square root of the conditional variance of market and equity returns, $\sqrt{\Sigma_t}$, is specified as

$$\sqrt{\Sigma_t} = \begin{bmatrix} \sigma_{I,t} & 0 \\ \\ \sigma_{SI,t}/\sigma_{I,t} & \sqrt{\sigma^2_{S,t} - \sigma^2_{SI,t}/\sigma^2_{I,t}} \end{bmatrix} \Longrightarrow \Sigma_t \equiv \sqrt{\Sigma_t}\sqrt{\Sigma_t}' = \begin{bmatrix} \sigma^2_{I,t} & \sigma_{SI,t} \\ \\ \sigma_{SI,t} & \sigma^2_{S,t} \end{bmatrix}, \tag{2}$$

where $\sigma^2_{S,t}$ is the spot variance of the stock.

We model the dynamics of Σ_t as a bivariate Wishart process

$$d\Sigma_t = \left(K\left(\Theta - \Sigma_t\right) + \left(\Theta - \Sigma_t\right)K'\right)dt + \sqrt{\Sigma_t}dW_tQ + \left(\sqrt{\Sigma_t}dW_tQ\right)', \tag{3}$$

where all components are 2×2 matrices.[9] K captures the mean-reversion speed of Σ_t toward

[9] Σ_t and Θ are two symmetric positive definite matrices. Q is a square matrix. W_t is square matrix of Brownian motions. K is a positive semi-definite matrix with its upper off-diagonal element set to 0. We discuss the importance of this restriction in Section 2.2 below.

the long-run values Θ. Q is the matrix of volatilities and co-volatilities of Σ_t. Whenever $Q^{'}Q$ is invertible, the existence and uniqueness of the long-run variance matrix, Θ, is guaranteed and Θ solves the system of equations given by $\gamma Q^{'}Q - K\Theta - \Theta K' = 0$, where γ is a scalar to be estimated.[10]

Four independent Brownian motions in W_t drive the dynamics of the variance matrix, Σ_t in (3). We label them as follows

$$W_t \equiv \begin{bmatrix} W^1_{I,t} & W^2_{I,t} \\ W^1_{S,t} & W^2_{S,t} \end{bmatrix}, \tag{4}$$

where $W^1_{I,t}$ and $W^2_{I,t}$ capture market variance risks and $W^1_{S,t}$ and $W^2_{S,t}$ denote firm specific variance risks, respectively.

We account for the leverage effect (Black, 1976; Christie, 1982) by linking the Brownian motions in returns and variances according to

$$dZ_{I,t} = \sqrt{1-\rho^2}dB_{I,t} + \rho dW^1_{I,t} \text{ and } dZ_{S,t} = \sqrt{1-\rho^2}dB_{S,t} + \rho dW^1_{S,t}, \tag{5}$$

where ρ is the leverage correlation parameter, and $B_{I,t}$ and $B_{S,t}$ are two independent Brownian motions. Our bivariate model with variance and covariance dynamics thus has a total number of 6 independent shocks.

In summary, we are following a conditional CAPM approach but with a fully specified dynamic structure on the covariance matrix of the shocks.[11]

[10] The parameter restriction $\gamma \geqslant N+1$ in a N-dimensional set-up ensures that the Wishart process admits a unique strong solution in the set of positive-definite matrices. In our bivariate setting it implies $\gamma \geqslant 3$. For a given mean-reversion matrix K and long-run matrix Θ, γ defines the wedge between the level of volatilities and co-volatilities and the level of long-run variances and covariance as $\gamma = \left(Q^{'}Q\right)^{-1}(K\Theta + \Theta K')$.

[11] In our view, the Wishart covariance process provides a good balance between flexibility and parameter parsimony. It enables us to build in mean-reversion in variances and covariance, a leverage effect, and non-trivial

1.2 Variance Drifts

The drift terms for the variances and covariances can be written

$$\frac{E_t\left[d\sigma^2_{I,t}\right]}{dt} = 2K_I(\theta_I - \sigma^2_{I,t}), \tag{6}$$

$$\frac{E_t\left[d\sigma_{SI,t}\right]}{dt} = (K_I + K_S)(\theta_{SI} - \sigma_{SI,t}) + K_{SI}(\theta_I - \sigma^2_{I,t}), \tag{7}$$

$$\frac{E_t\left[d\sigma^2_{S,t}\right]}{dt} = 2K_S(\theta_S - \sigma^2_{S,t}) + 2K_{SI}(\theta_{SI} - \sigma_{SI,t}), \tag{8}$$

where $E_t[\cdot]$ denotes the physical conditional expectations operator, and where we have defined

$$K \equiv \begin{bmatrix} K_I & 0 \\ K_{SI} & K_S \end{bmatrix}, \text{ and } \Theta \equiv \begin{bmatrix} \theta_I & \theta_{SI} \\ \theta_{SI} & \theta_S \end{bmatrix}.$$

Note that we have set an off-diagonal element in K to zero to ensure that the stochastic market index variance is independent of the individual stock covariance term.

From equations (6), (7), and (8), we see that $2K_I$, $2K_S$, and $K_I + K_S$ capture the mean reversion speed of market variance, covariance and stock variance, respectively. The second term in equations (7) and (8) reveal an interesting property of our model. Whenever $K_{SI} \neq 0$, fluctuations in market variance influence the covariance dynamic which in turn impacts the stock variance drift. This allows the model to generate important co-movements in equity variances and covariances.

1.3 Variance Diffusions and Leverage Effects

The model's implied market variance dynamics is closely related to the square-root model in Heston (1993). This is apparent from (6) and from the diffusion coefficient of the market index

covariance dynamics which, in turn, generates non-trivial dynamics for β_t.

variance

$$d\sigma_{I,t}^2 - E_t\left[d\sigma_{I,t}^2\right] = \sigma_{I,t} \cdot 2\left(Q_I^1 dW_{I,t}^1 \ + Q_I^2 dW_{I,t}^2\right), \tag{9}$$

where we have used the square root of the variance matrix in (2) as well as the definition

$$Q = \begin{bmatrix} Q_I^1 & Q_S^1 \\ Q_I^2 & Q_S^2 \end{bmatrix}. \tag{10}$$

The key difference with Heston (1993) is that $d\sigma_{I,t}^2$ in our model in (9) is driven by two shocks instead of one which provides additional flexibility that is important empirically.

Despite its parsimony, the model produces important contemporaneous co-movements. The diffusion of the total stock variance is

$$\begin{aligned} d\sigma_{S,t}^2 - E_t\left[d\sigma_{S,t}^2\right] &= \beta_t \sigma_{I,t} \cdot 2\left(Q_S^1 dW_{I,t}^1 + Q_S^2 dW_{I,t}^2\right) \\ &+ \sqrt{\sigma_{S,t}^2 - \beta_t^2 \sigma_{I,t}^2} \cdot 2\left(Q_S^1 dW_{S,t}^1 + Q_S^2 dW_{S,t}^2\right), \end{aligned} \tag{11}$$

where we have used the definition of spot beta, $\beta_t \equiv \sigma_{SI,t}/\sigma_{I,t}^2$.

The diffusion term in the stock's total variance in (11) follows a factor structure. On one hand, the stock's systematic volatility, $\sigma_{SI,t} = \beta_t \sigma_{I,t}$, defines the loading of $d\sigma_{S,t}^2$ on market level risks. On the other hand, the stock's idiosyncratic volatility, $\sqrt{\sigma_{S,t}^2 - \beta_t^2 \sigma_{I,t}^2}$, defines the way $d\sigma_{S,t}^2$ loads on firm-specific innovations. Similar conclusions can be drawn from the dependence of the covariance diffusion on aggregate and firm-specific shocks (see Appendix A).

From equations (3), (5), (10), and the dynamics of $\sigma_{I,t}^2$ and $\sigma_{S,t}^2$, we can show that the market and equity leverage effects are given by ρ and the Q matrix as follows

$$\begin{aligned}\rho_I &\equiv Corr_t(\tfrac{dI_t}{I_t}, d\sigma^2_{I,t}) = \rho\frac{Q^1_I}{\sqrt{\left(Q^1_I\right)^2+\left(Q^2_I\right)^2}},\\ \rho_S &\equiv Corr_t(\tfrac{dS_t}{S_t}, d\sigma^2_{S,t}) = \rho\frac{Q^1_S}{\sqrt{\left(Q^1_S\right)^2+\left(Q^2_S\right)^2}}.\end{aligned} \tag{12}$$

So, while the specification in (5) relies on a single correlation parameter, ρ, the model generates different leverage effects for the market index, ρ_I, and individual equity, ρ_S, via the parameters in the Q matrix. Finally, note that including ρ, our model has a total of 9 parameters under the physical measure. We next discuss the dynamics of the SDF, which leads to the introduction of three additional price-of-risk parameters.

1.4 A Stochastic Discount Factor

The stochastic discount factor (SDF) in the model depends linearly on market index return and variance risks. More precisely, the SDF follows the dynamics

$$\frac{d\zeta_t}{\zeta_t} = -rdt - \sigma_{I,t}\left(\lambda^{R_I} dB_{I,t} + \lambda^{\sigma_I}_1 dW^1_{I,t} + \lambda^{\sigma_I}_2 dW^2_{I,t}\right), \tag{13}$$

where $\sigma_{I,t}\lambda^{R_I}$ is the price of market return-specific risk, $B_{I,t}$, and $\sigma_{I,t}\lambda^{\sigma_I}_1$ and $\sigma_{I,t}\lambda^{\sigma_I}_2$ are the prices of the market variance risks $W^1_{I,t}$ and $W^2_{I,t}$, respectively. The stock-specific innovations $Z_{S,t}$, $W^1_{S,t}$, and $W^2_{S,t}$ are deliberately assumed not to be priced in the model. Unlike standard factor models (Jagannathan and Wang, 1996; Lewellen and Nagel, 2006), our SDF allows for market and equity systematic variance risk premiums. As shown in the Online Appendix, the specification (13) is a reduced-form approximation of the equilibrium SDF implied by an intertemporal CAPM with long-run risk and stochastic volatility in consumption. Among other things, we show that $W^1_{I,t}$ and $W^2_{I,t}$ in (13) can be interpreted as the shocks driving consumption variance while $B_{I,t}$ captures both consumption-specific innovation and the shock to consumption growth. The dynamics of the SDF in the model is thus consistent with the long-run risk and variance risk premium literatures which provide theoretical and empirical support for priced variance risks (for example, see, Bansal and Yaron, 2004; Bollerslev, Tauchen, and Zhou, 2009; Koijen, Lustig, Van Nieuwerburgh, and Verdelhan, 2010).[12]

The SDF will enable us to pin down the equity risk premium, $\mu_{I,t}$, and to derive the risk-neutral dynamics which in turn enables us to price index and equity options. The linear form of the SDF in equation (13) ensures that the risk-neutral dynamics of the model will be similar to the physical dynamics in equation (1) above.

2 Model Properties

We now explore some key properties of the model. First, we present the model's risk-neutral dynamics and derive instantaneous return premiums. Then, we investigate the model's implications for variation in beta. Finally, we derive expressions for the term structure of return risk premiums and present model-implied expected future betas.

[12]For empirical evidence on the variance risk premium, see, Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), and Driessen, Maenhout, and Vilkov (2009).

2.1 The Risk-Neutral Dynamics

The physical return dynamics in (1) and the stochastic discount factor in (13) imply that the risk-neutral dynamics (see Appendix B) used in option valuation is given by

$$\begin{bmatrix} \frac{dI_t}{I_t} \\ \\ \frac{dS_t}{S_t} \end{bmatrix} = \begin{bmatrix} r \\ \\ r \end{bmatrix} dt + \sqrt{\Sigma_t} \begin{bmatrix} d\tilde{Z}_{I,t} \\ \\ d\tilde{Z}_{S,t} \end{bmatrix}. \tag{14}$$

In our setup, the spot variance-covariance matrix Σ_t is the same under the two measures, which implies that the instantaneous spot beta, $\beta_t \equiv \sigma_{SI,t}/\sigma^2_{I,t}$, is identical under the two measures as well.[13] However, the dynamics of Σ_t will differ under the two measures which, in turn, has interesting implication for the term-structure of risk premiums as investigated below. The risk-neutral returns shocks are defined by $d\tilde{Z}_{I,t} = dZ_{I,t} + \sigma_{I,t}(\sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda_1^{\sigma_I})$ and $d\tilde{Z}_{S,t} = dZ_{S,t}$.

The dynamics for Σ_t under the risk-neutral measure is as in equation (3) only with the matrix of mean-reversion parameters, K, replaced by

$$\tilde{K} = K + \begin{bmatrix} \lambda_1^{\sigma_I} Q_I^1 + \lambda_2^{\sigma_I} Q_I^2 & 0 \\ \\ \lambda_1^{\sigma_I} Q_S^1 + \lambda_2^{\sigma_I} Q_S^2 & 0 \end{bmatrix}, \tag{15}$$

and the matrix of long-term mean Θ replaced by $\tilde{\Theta}$ which solves $\tilde{K}\tilde{\Theta} + \tilde{\Theta}\tilde{K}' = \gamma Q'Q$. The risk-neutral variance shocks, $d\tilde{W}_t$, are provided in Appendix B.

[13]This, again, resembles the Heston (1993) model where the instantaneous variance is the same under the two measures.

2.2 Instantaneous Return Premiums

Comparing (1) and (14) we see that the instantaneous return risk premium for the market index is

$$E_t\left[\frac{dI_t}{I_t}\right] - E_t^Q\left[\frac{dI_t}{I_t}\right] \equiv \mu_{I,t}dt = \Lambda^I\sigma_{I,t}^2 dt, \tag{16}$$

where $E_t^Q[\cdot]$ is the risk-neutral expectation operator and where $\Lambda^I \equiv \left(\sqrt{1-\rho^2}\right)\lambda^{R_I} + \rho\lambda_1^{\sigma_I}$, as shown in Appendix B. For the empirically relevant case $\rho < 0$ (i.e., negative leverage effect), $\lambda^{R_I} > 0$ (i.e., positive price of return risk), and $\lambda_1^{\sigma_I} < 0$ (i.e., negative price of variance risk), we have $\Lambda^I > 0$.

For the stock we have the instantaneous risk premium

$$E_t\left[\frac{dS_t}{S_t}\right] - E_t^Q\left[\frac{dS_t}{S_t}\right] \equiv \beta_t\mu_{I,t}dt = \beta_t\left(\Lambda^I\sigma_{I,t}^2\right)dt = \Lambda^I\sigma_{SI,t}dt. \tag{17}$$

As long as the leverage correlation, ρ, is non-zero, the market variance price of risk, $\lambda_1^{\sigma_I}$, impacts the instantaneous market equity price of risk, Λ^I, and thus the instantaneous risk-premium on the stock.

2.3 Beta Risk

We next present some implications for the dynamics of beta that are intended to provide further intuition for the model.

Proposition 1 *Given (3), the physical (P) dynamics of market beta, $\beta_t \equiv \sigma_{SI,t}/\sigma_{I,t}^2$, is such that*

$$d\beta_t = \beta_t\left(\frac{(d\sigma_{SI,t} - \Phi_{SI}dt)}{\sigma_{SI,t}} - \frac{(d\sigma_{I,t}^2 - \Phi_I dt)}{\sigma_{I,t}^2}\right), \tag{18}$$

where $\Phi_{SI} \equiv 2\left(Q_I^1 Q_S^1 + Q_I^2 Q_S^2\right)$ *and* $\Phi_I \equiv 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$. *The risk-neutral (Q) dynamics of market beta satisfies*

$$d\beta_t = \beta_t \left(\frac{\left(d\sigma_{SI,t} - \Phi_{SI} dt\right)}{\sigma_{SI,t}} - \frac{\left(d\sigma_{I,t}^2 - \Phi_I dt\right)}{\sigma_{I,t}^2} \right) + cov_t \left(d\beta_t, \frac{d\zeta_t}{\zeta_t} \right), \tag{19}$$

where $cov_t\left(d\beta_t, \frac{d\zeta_t}{\zeta_t}\right)$ *is the instantaneous conditional covariance (i.e., quadratic covariation) of changes in market beta with the returns on the SDF.*

Proof. See Appendix C. ■

We see from Proposition 1 that β_t follows a two-factor dynamics under P. By definition, the market beta is proportional to equity covariance with the market index and is inversely proportional to the market index variance. Accordingly, the first factor in equation (18), $\left(d\sigma_{SI,t} - \Phi_{IS} dt\right)/\sigma_{SI,t}$, corresponds to the relative change in covariance which positively impacts the change in beta. The second factor, $\left(d\sigma_{I,t}^2 - \Phi_I dt\right)/\sigma_{I,t}^2$, captures the relative change in the market index variance and is negatively related to $d\beta_t$.

Proposition 1 has important implications for the dynamics of equity risk under the risk-neutral measure. Comparing equation (18) with (19), we see that the covariance between beta and the SDF generates the wedge between physical and risk-neutral beta dynamics. A positive covariance implies that the level of beta under the risk-neutral measure is higher than the level of beta under the objective measure. Reciprocally, the more negative the covariance, the lower the level of risk neutral beta is compared to physical beta. As shown in Appendix C, beta covaries with the SDF in our model whenever market variance risks are priced (i.e., $\lambda_1^{\sigma_I}, \lambda_2^{\sigma_I} \neq 0$). We now discuss the implications of beta's co-movement with the SDF for the integrated beta risk premium. We discuss the impact of the integrated beta risk premium on the term structure of expected stock returns in the next section.

In the spirit of return and variance risk premia, the h-day expected integrated beta risk premium, $BRP_{t,h}$, captures the difference between expected integrated physical and risk-neutral stochastic betas. It satisfies

$$BRP_{t,h} \equiv E_t\left[\int_t^{t+h/252} \beta_u du\right] - E_t^Q\left[\int_t^{t+h/252} \beta_u du\right] = \beta_{t,h} - \beta_{t,h}^Q, \tag{20}$$

where we have assumed 252 trading days in a year, and where $\beta_{t,h}$ and $\beta_{t,h}^Q$ are the h-day expected integrated physical and risk-neutral betas, respectively. When beta co-moves positively with the SDF, we have $cov_t\left(d\beta_t, d\zeta_t/\zeta_t\right) > 0$ in (19) and the level of risk-neutral beta will be relatively higher than the level of physical beta. All things being equal, this will result in a negative integrated beta risk premium (i.e., $\beta_{t,h} < \beta_{t,h}^Q \Leftrightarrow BRP_{t,h} < 0$). In contrast, the beta risk premium will be positive on average when beta covaries negatively with the SDF.

Appendix C shows the way the sign of the instantaneous covariance of beta with the SDF (i.e., $cov_t\left(d\beta_t, d\zeta_t/\zeta_t\right)$) is related to the level of β_t. A byproduct of this result is that the sign of the instantaneous beta risk premium (i.e., $E_t\left[d\beta_t\right] - E_t^Q\left[d\beta_t\right]$) is also impacted by the level of beta in our framework as $E_t\left[d\beta_t\right] - E_t^Q\left[d\beta_t\right] \equiv -cov_t\left(d\beta_t, d\zeta_t/\zeta_t\right)$. Unfortunately, while $\beta_{t,h} - \beta_{t,h}^Q$ factors in the integrated instantaneous covariance of beta with the SDF, it is also influenced by additional terms that cannot be solved analytically (See Appendix C).[14] Empirically, we find that $BRP_{t,h} > 0$ on average for stocks with average conditional beta above 0.83 which is broadly consistent with the idea that the beta of these stocks co-moves negatively with the SDF. In contrast, stocks with an average conditional beta below 0.83 have a negative integrated beta risk premium because their

[14] As shown in Appendix C, $BRP_{t,h}$ is also impacted by the difference of the physical and risk-neutral expectations of the integrals of $\beta_t\left(d\sigma_{I,t}^2 - \Phi_I dt\right)/\sigma_{I,t}^2$ and $\beta_t\left(d\sigma_{I,t}^2 - \Phi_I dt\right)/\sigma_{I,t}^2$.

beta co-moves positively with the SDF.

Equation (18) shows the way changes in beta depend on $d\sigma_{I,t}^2$ under the P-measure. Because changes in beta depend on market variance innovations, beta co-moves with market variance in the model. Empirically, we find that the betas of low-beta stocks co-move positively with the market variance while the betas of high-beta stocks tend to co-move negatively with $\sigma_{I,t}^2$. As we discuss next, this has important implications for the term structure of expected stock returns.

2.4 The Term Structure of Return Premiums

While our theoretical model is written in continuous time, when evaluating it, we need to decide on a return frequency of interest, say monthly, and we therefore now explore the model's implications for the return premium at different horizons.

To this end, the following proposition provides the expressions of the conditional risk premium for market index and individual equity returns for horizon h. For ease of notation, we define by $X_{t,h} \equiv \int_t^{t+h/252} E_t\left[X_s\right] ds$ and $X_{t,h}^Q \equiv \int_t^{t+h/252} E_t^Q\left[X_s\right] ds$ the h-day physical and risk-neutral integrated expectations of variable X at time t, respectively. Armed with this notation, we now present the main theoretical result.

Proposition 2 *Given (1), (3), and (13), the h-day integrated market return premium at time t, $RP_{t,h}^I$, is given by*

$$RP_{t,h}^I \equiv E_t\left[\int_t^{t+h/252} \frac{dI_u}{I_u}\right] - E_t^Q\left[\int_t^{t+h/252} \frac{dI_u}{I_u}\right] = \Lambda^I \sigma_{I,t,h}^2, \tag{21}$$

where $\Lambda^I = \left(\sqrt{1-\rho^2}\right)\lambda^{R_I} + \rho\lambda_1^{\sigma_I}$ and $\sigma_{I,t,h}^2$ is the h-day expected integrated market variance under *the physical measure. The stock's h-day expected integrated return premium, $RP^S_{t,h}$, is given by*

$$RP^S_{t,h} \equiv E_t\left[\int_t^{t+h/252}\frac{dS_u}{S_u}\right] - E^Q_t\left[\int_t^{t+h/252}\frac{dS_u}{S_u}\right] = \Lambda^I\sigma^2_{SI,t,h} \tag{22}$$

$$= RP^{SML}_{t,h} + RP^{BRP}_{t,h}, \tag{23}$$

where $RP^{SML}_{t,h}$ is the return premium predicted by the conditional security market line

$$RP^{SML}_{t,h} \equiv \int_t^{t+h/252}\left(E_t\left[\beta_u\right]E_t\left[\frac{dI_u}{I_u}\right] - E^Q_t\left[\beta_u\right]E^Q_t\left[\frac{dI_u}{I_u}\right]\right), \tag{24}$$

and where $RP^{BRP}_{t,h}$ is the beta return premium

$$RP^{BRP}_{t,h} \equiv \int_t^{t+h/252} cov_t\left(\beta_u, \mu_{I,u}\right)du - r\left(\beta_{t,h} - \beta^Q_{t,h}\right)$$

$$= \Lambda^I\int_t^{t+h/252} cov_t\left(\beta_u, \sigma^2_{I,u}\right)du - r\left(BRP_{t,h}\right), \tag{25}$$

where $\sigma_{SI,t,h}$, $\beta_{t,h}$, and $\beta^Q_{t,h}$ are the h-day expected integrated physical covariance, and physical and risk-neutral betas, respectively.

Proof. See Appendix D. ■

The market premium is thus the product of the integrated market variance and Λ^I which reflects the representative investor's aversion to return and variance risks. Similarly, we see from (22) that the individual equity premium is Λ^I times the integrated covariance of the stock and the market.

Proposition 2 provides a decomposition of the equity return premium. From (23), we see that the equity's total return premium is composed of two parts. The first component ($RP_{t,h}^{SML}$) corresponds to the difference between expected beta times expected market return under the physical and risk-neutral measures. It is the return premium predicted for the stock by the conditional security market line. The second component ($RP_{t,h}^{BRP}$) is the covariance of beta with the market risk premium along with a term capturing the beta risk premium multiplied by the risk-free rate. We refer to $RP_{t,h}^{BRP}$ as the beta return premium as it captures the component of the return premium induced by beta's co-movements with market variance and the SDF.

Because the beta of high-beta firms co-moves negatively with market variance and vice versa, $\Lambda^I \left(\int_t^{t+h/252} cov_t \left(\beta_u, \sigma_{I,u}^2 \right) du \right)$ is negative for high-beta firms and positive for low-beta firms given $\Lambda^I > 0$. The second term in $RP_{t,h}^{BRP}$ captures the difference between physical and risk-neutral expected integrated betas. As noted in Section 2.3, the sign of $BRP_{t,h}$ is driven by the way beta co-moves with the SDF. It is positive on average for high-beta firms because their beta co-moves negatively with the SDF while it is negative for low-beta firms. As a result, $-r \left(\beta_{t,h} - \beta_{t,h}^Q \right)$ and $\Lambda^I \left(\int_t^{t+h/252} cov_t \left(\beta_u, \sigma_{I,u}^2 \right) du \right)$ do not cancel out each other and take on negative values on average for high-beta stocks and positive values for low-beta stocks. Overall, $RP_{t,h}^S > RP_{t,h}^{SML}$ on average for low-beta firms while $RP_{t,h}^S < RP_{t,h}^{SML}$ on average for high-beta firms.

In the limit when $h \longrightarrow 0$, $RP_{t,0}^{SML} = E_t \left[\beta_t \right] E_t \left[\frac{dI_t}{I_t} \right] - E_t^Q \left[\beta_t \right] E_t^Q \left[\frac{dI_t}{I_t} \right]$, which further simplifies to

$$\beta_t \left(E_t \left[\frac{dI_t}{I_t} \right] - E_t^Q \left[\frac{dI_t}{I_t} \right] \right) = \beta_t \left(\Lambda^I \sigma_{I,t}^2 \right) dt = \Lambda^I \sigma_{SI,t} dt,$$

where we have used the definition of instantaneous market return premium in equation (16) combined with the fact that $E_t \left[\beta_t \right] = E_t^Q \left[\beta_t \right] = \beta_t$ by absolute continuity of the two probability measures. Moreover, when $h \longrightarrow 0$ then we have

$$
\begin{aligned}
RP_{t,0}^{BRP} &= \Lambda^I \int_t^{t+0} cov_t\left(\beta_u, \sigma_{I,u}^2\right) du - r\left(E_t\left[\int_t^{t+0} \beta_u du\right] - E_t^Q\left[\int_t^{t+0} \beta_u du\right]\right) \\
&= \Lambda^I cov_t\left(\beta_t, \sigma_{I,t}^2\right) dt - r\left(E_t\left[\beta_t\right] - E_t^Q\left[\beta_t\right]\right) dt = 0,
\end{aligned}
$$

where we have used that, conditional on time t, β_t and $\sigma_{I,t}^2$ are known and do not covary and that $E_t\left[\beta_t\right] = E_t^Q\left[\beta_t\right] = \beta_t$. In other words, even if beta co-moves with market variance and the SDF, it has no impact on instantaneous expected returns and $RP_{t,0}^{BRP} = 0$ always holds. This is because the instantaneous co-movements of beta with market variance and the SDF only impact $RP_{t,h}^{BRP}$ when the forecast horizon is greater than an instant (i.e., $h > 0$). This result is related to the solution of optimal portfolio allocation problems. When the investment horizon is equal to the discretization step, intertemporal hedging demands are zero and the representative investor holds her mean-variance allocation. As the horizon increases, intertemporal hedging demands of the non-myopic agent kick in and the agent's holdings deviate from her mean-variance allocation. A similar mechanism is at play in our set-up. Instantaneously, the SML holds perfectly (i.e., $RP_{t,0}^{BRP} = 0$). As the horizon increases, expected co-movements of betas with the market variance and the SDF generate deviations from the conditional SML (i.e., $RP_{t,h}^{BRP} \neq 0$ for $h > 0$).

The results in Proposition 2 complement an extensive literature that studies the way variation in conditional betas impacts unconditional CAPM alphas. Lewellen and Nagel (2006) show that unconditional alphas are function of the way beta co-moves with market return premium and market volatility. Frazzini and Pedersen (2014) show that high-beta stocks earn negative unconditional alphas. In their model, constrained investors tilt their portfolios toward high-beta assets bidding up their prices. Consequently, high-beta stocks require relatively low risk-adjusted returns compared with low-beta stocks, which require leverage. Our model provides an alternative explanation to this stylized fact. High-beta stocks have lower expected returns than what the SML predicts because of the way their betas co-move with market variance and the SDF. More recently, Cederburg and O'Doherty (2016) provide evidence that the beta of high-minus-low beta trading strategies tends to co-move negatively with the market premium which impacts unconditional CAPM alphas.[15] This result is broadly consistent with our model which predicts that $\int_t^{t+h/252} cov_t\left(\beta_u, \mu_{I,u}\right) du = \Lambda^I \left(\int_t^{t+h/252} cov_t\left(\beta_u, \sigma_{I,u}^2\right) du\right)$ is negative for high-beta stocks and positive for low-beta stocks on average.

Other evidence in the literature suggests that beta's co-movement with market returns helps explain the cross-section of stock conditional expected returns. Petkova and Zhang (2005) show that variation in betas helps explain the value-premium puzzle. Ang, Chen, and Xing (2006) document that stocks which co-move positively with the market index when market index returns are low (i.e., high downside betas) have a positive risk-adjusted alpha while stocks with low downside betas have a negative risk-adjusted alpha. Our model complements this result by identifying the type of firms prone to high or low downside betas. Because the beta of low-beta stocks co-moves positively with the SDF and negatively with market returns (i.e., market returns are low when the SDF is high), low-beta stocks are thus inclined to high downside betas in the model. In contrast, high-beta stocks have low downside betas because their betas co-move positively with market returns (i.e., negatively with the SDF). This prediction combined with the fact that $RP_{t,h}^{BRP}$ is positive for low-beta stocks and negative for high-beta stocks is consistent with Ang, Chen, and Xing (2006)'s finding.

[15] See, also, Liu, Stambaugh and Yuan (2018) for discussions on the impact of idiosyncratic volatility on the unconditional abnormal return of beta strategies.

2.5 Risk-Neutral Betas and the Slope of the Conditional SML

The pricing implications of option-implied betas is a subject of recent interest. For example, Buss and Vilkov (2012) and Chang, Christoffersen, Jacobs, and Vainberg (2012) estimate betas from risk-neutral market index and individual stock moments and show that option-implied betas generate a more pronounced SML than OLS betas.

The pricing of beta risk has implications for the slope of the conditional SML estimated using risk-neutral betas. To see why, let us consider two hypothetical stocks. The first stock has a high (h-day expected integrated) physical beta which we denote $\beta_{t,h}^{H}$. The second has a low (physical) beta, $\beta_{t,h}^{L}$, so that $\beta_{t,h}^{H} > \beta_{t,h}^{L}$. Furthermore, let us assume that the h-day expected excess return of the high and low-beta stocks satisfy $RP_{t,h}^{H} > RP_{t,h}^{L}$.[16]

Recall that the beta risk premium of low-beta firms is negative on average which implies that $\beta_{t,h}^{L} - \beta_{t,h}^{Q,L} < 0 \Leftrightarrow \beta_{t,h}^{L} < \beta_{t,h}^{Q,L}$. Reciprocally, BRP is positive for high-beta stocks on average and thus $\beta_{t,h}^{H} > \beta_{t,h}^{Q,H}$. Now, suppose that we want to estimate the slope of the conditional SML at time t for horizon h. Using physical betas, the slope of the conditional SML is approximated by $\left(RP_{t,h}^{H} - RP_{t,h}^{L}\right) / \left(\beta_{t,h}^{H} - \beta_{t,h}^{L}\right)$. We can now compare this estimate to the one that would be obtained when using risk-neutral betas. Given that $\beta_{t,h}^{L} < \beta_{t,h}^{Q,L} \Leftrightarrow -\beta_{t,h}^{L} > -\beta_{t,h}^{Q,L}$ and $\beta_{t,h}^{H} > \beta_{t,h}^{Q,H}$, we have $\beta_{t,h}^{H} - \beta_{t,h}^{L} > \beta_{t,h}^{Q,H} - \beta_{t,h}^{Q,L}$ which, in turn, implies that

$$\frac{RP_{t,h}^{H} - RP_{t,h}^{L}}{\beta_{t,h}^{H} - \beta_{t,h}^{L}} < \frac{RP_{t,h}^{H} - RP_{t,h}^{L}}{\beta_{t,h}^{Q,H} - \beta_{t,h}^{Q,L}}.$$

When beta risk is priced, the slope of the conditional SML estimated using risk-neutral betas

[16] The joint assumption that $\beta_{t,h}^{H} > \beta_{t,h}^{L}$ and $RP_{t,h}^{H} > RP_{t,h}^{L}$ hold rules out the case of inverted conditional SML (i.e., negative relationship between expected excess return and beta).

will be steeper than the one obtained using physical betas. This prediction is consistent with the empirical findings in Buss and Vilkov (2012) and Chang, Christoffersen, Jacobs, and Vainberg (2012).[17]

2.6 The Term Structure of Beta

Below, we want to empirically validate the model partly based on its ability to forecast ex-post realized beta. To this end we need to derive model-based expected future betas.

Proposition 3 *Conditional on time t, the h-day ahead expected integrated variance-covariance matrix under the physical measure is*

$$\Sigma_{t,h} = E_t\left[\int_t^{t+h/252} \Sigma_u du\right] = \int_t^{t+h/252} \left(e^{-K(u-t)}\Sigma_t e^{-K'(u-t)} + \Gamma_{t,u}\right) du, \tag{26}$$

where the expression for $\Gamma_{t,u}$, which is a function of γ, Q, and K, is given in Appendix E. The h-day ahead expected integrated beta under the physical measure is at first-order equal to

$$\beta_{t,h} = E_t\left[\int_t^{t+h/252} \beta_u du\right] \approx \frac{\sigma_{SI,t,h}}{\theta_I} - \frac{\left(\sigma^2_{I,t,h} - \theta_I \frac{h}{252}\right)\theta_{SI}}{(\theta_I)^2}, \tag{27}$$

where $\sigma^2_{I,t,h}$ and $\sigma_{SI,t,h}$ denote the h-day ahead expected integrated market variance and covariance under the physical measure, respectively. They are given by $\sigma^2_{I,t,h} = \Sigma^{(1,1)}_{t,h}$ and $\sigma_{SI,t,h} = \Sigma^{(2,1)}_{t,h}$ where $\Sigma^{(i,j)}_{t,h}$ corresponds to the element on the i^{th} row, j^{th} column of $\Sigma_{t,h}$.

Proof. See Appendix E. ■

[17] These studies show that option-implied betas are practically unbiased relative to the future realized betas which also contributes to generate a more pronounced SML.

Based on these results, the annualized h-day ahead integrated stochastic variance-covariance matrix and beta can be defined by

$$\Sigma_{t,h}^{SB} \equiv \frac{252}{h}\Sigma_{t,h} \quad \text{and} \quad \beta_{t,h}^{SB} \equiv \frac{252}{h}\beta_{t,h}, \tag{28}$$

respectively. Using (27) and (28), we can obtain the model's forecast of future realized beta. By construction, beta is a non-linear function of the state variables, and an exact analytical expression for its conditional expectation does not exist. Equation (27) approximates expected integrated future beta based on a first-order Taylor expansion around the long-term variance-covariance means. While this expression is not exact, we have verified using Monte Carlo simulations that it closely approximates the true expected integrated beta.

In summary, our contribution is threefold. First, the results in Sections 2.3 and 2.4 provide new insights on the impact of beta's co-movements with market variance and the SDF on conditional expected returns on low- and high-beta stocks. Second, we provide in Section 2.5 new theoretical support for explaining the better fit of the SML by risk-neutral betas through a beta risk premium channel. Combined, these results suggest that the betting-against-beta return anomaly and the better SML fit generated by option-implied betas documented in the literature are two tales of the same story, in the sense that, both are related to the pricing of beta risk. Third, by fully-specifying the dynamics of the variance and covariance matrix, our model offers an innovative way to estimate latent betas jointly from return and option data. We discuss model estimation and provide an extensive empirical investigation of model's predictions in the next section.

3 Empirical Results

We first present the data, the model estimation strategy, and parameter estimates. We then analyze the ability of stochastic betas to predict future realized OLS betas. Subsequently, we analyze conditional beta risk premiums and study the slope of the SML implied by model betas. Finally, we analyze the impact of beta risk on the cross-section of expected stock returns and compare it to the security market line.

3.1 Data and Model Estimation

Our empirical analysis relies on two main datasets. We obtain daily return data from CRSP and end-of-day implied-volatility surfaces from OptionMetrics. We consider two sample periods. The first sample starts on January 8, 1996 and ends on December 30, 2014 and is used to estimate the structural parameters of our model. The second starts on January 8, 1996 and ends on December 30, 2016. We use this extended sample, which includes two additional years relative to the estimation sample, to analyze the empirical performance of our model. To assess model performance in the cross-section, it is important to have a sufficiently large number of stocks. To this end, we obtain all the constituents of the S&P 500 index at the end of the estimation sample. We retain all stocks with complete return data and with quoted options for the 1996-2014 sample period. In total, 344 stocks meet these criteria. We use the S&P 500 index to proxy for the market factor.

Table 1 presents summary statistics of excess returns for various value-weighted stock portfolios for the 1996-2016 sample period. We consider the value-weighted portfolio composed of all stocks and decile portfolios of stocks sorted unconditionally on their sample OLS betas with respect to S&P 500 returns. We use sample OLS betas to measure unconditional OLS betas. The first two columns report the sample mean and volatility of excess returns annualized. Columns three to five report the OLS R-squared, correlation, and beta of each portfolio estimated by regressing daily excess portfolio returns on daily excess S&P 500 returns over the entire sample. In column six, we report an alternative measure of beta for each portfolio estimated by OLS using the value-weighted portfolio as a factor. This allows us to assess the representativeness of our sample with respect to the S&P 500 index. Column 7 presents the t-statistics of the difference in betas. We report the average market capitalization of the constituents of each portfolio in billion dollars in the last column. Comparing the betas in column 5 with those reported in column 6, we see that the estimates are close and not statistically significantly different one from another. We conclude that our cross-section of 344 stocks is fairly representative of the S&P 500 index for beta analysis.

When estimating the model it is important to combine returns with option data in order to obtain the best possible estimates of risk premiums and physical and risk-neutral beta dynamics. Precisely estimating physical and risk-neutral dynamics is critically important when analyzing the properties of physical and risk-neutral betas as we do. Appendix F contains the closed-form option pricing formula implied by our model.[18]

Recall that we need to estimate the paths of the unobserved market variance $\{\sigma_{I,t}^2\}$, equity variance $\{\sigma_{S,t}^2\}$, and covariance $\{\sigma_{SI,t}\}$ and two sets of structural parameters $\{\Psi_I,\ \Psi_S\}$, where $\Psi_I \equiv \{\gamma, K_I, Q_I^1, Q_I^2, \rho, \lambda_1^{\sigma_I}, \lambda_2^{\sigma_I}\}$ and $\Psi_S \equiv \{K_S, K_{SI}, Q_S^1, Q_S^2\}$. To reduce the dimensionality of the market estimation, we impose $\lambda_1^{\sigma_I} = \lambda_2^{\sigma_I} = \lambda^{\sigma_I}$. With this restriction, the set of market parameters Ψ_I becomes $\{\gamma, K_I, Q_I^1, Q_I^2, \rho, \lambda^{\sigma_I}\}$.

[18]Firm-level statistics for return and option data are available from the authors upon request.

Our methodology to estimate the model is based on a two-step procedure and a daily discretization of the model dynamics. In both steps, we filter the paths of the unobserved latent variables from returns only. In the first step, we estimate the market index dynamics $\{\Psi_I, \{\sigma^2_{I,t}\}\}$ by maximizing the sum of S&P 500 returns and options log-likelihoods

$$\hat{\Psi}_I, \{\hat{\sigma}^2_{I,t}\} = argmax(\mathcal{L}^R_I(\Psi_I) + \mathcal{L}^O_I(\Psi_I)), \tag{29}$$

where $\mathcal{L}^R_I(\cdot)$ and $\mathcal{L}^O_I(\cdot)$ denote the index returns and options log-likelihoods, respectively. In the second step, we use S&P 500 returns, and equity returns and options, to estimate stock-specific dynamics $\{\Psi_S, \{\sigma^2_{S,t}, \sigma_{SI,t}\}\}$ taking market parameters from the first step as given. Similarly to the market estimation, we estimate stock-specific dynamics by maximizing the sum of returns and options log-likelihoods such that

$$\hat{\Psi}_S, \{\hat{\sigma}^2_{S,t}, \hat{\sigma}_{SI,t}\} = argmax(\mathcal{L}^R_{S,I}(\hat{\Psi}_I, \Psi_S) + \mathcal{L}^O_S(\hat{\Psi}_I, \Psi_S)), \tag{30}$$

where $\mathcal{L}^R_{S,I}(\cdot)$ denotes the index-equity returns joint log-likelihood, and $\mathcal{L}^O_S(\cdot)$ is the equity options log-likelihood.[19]

This estimation procedure enables us to ensure that the same dynamics is imposed for the market index for each firm. Our estimation strategy uses particle filter because it provides a convenient method for obtaining real-time estimates of the daily latent variables and betas, $\beta_t \equiv \sigma_{SI,t}/\sigma^2_{I,t}$, for each stock. Using this estimation procedure we estimate our model for the 344 equities over 19 years between 1996 and 2014. Using the structural parameters obtained, we then

[19] For alternative estimation approaches, see, Renault and Touzi (1996), Gouriéroux, Monfort, and Renault (1993), Pan (2002), Gagliardini, Gouriéroux, and Renault (2011), Eraker (2004), and Christoffersen, Jacobs, and Mimouni (2010).

filter daily latent variables and betas for the 1996-2016 extended sample period. Specific details on the estimation strategy including the construction of the likelihoods, the particle filter, and on the way options enter into the estimation can be found in the Online Appendix.[20]

3.2 Unconditional Model Fit and Risk Premiums

We first discuss parameter estimates. We then assess whether the unconditional betas and variance risk premiums implied by the model are reasonable. Finally, we discuss model unconditional fit.

In Panel A of Table 2 we report the estimated model parameters. The top of the panel presents the market parameters and below we report the value-weighted average of stock-specific parameters for decile portfolios of stocks sorted unconditionally on sample OLS betas (i.e., as in Table 1). We relegate stock level results to Table A.1 in the Online Appendix.

For the market index, the estimated θ_I corresponds to a 20.21% average volatility which is close to the 19.34% volatility of S&P 500 daily returns during the 1996-2016 sample. For equities, the long term volatility $\sqrt{\theta_S}$ ranges from 26.26% for the low-beta portfolio to 40.08% for the high-beta portfolio.[21] The equity leverage effect, ρ_S, is -0.40 on average across portfolios and much lower (in absolute value) than the market index leverage effect (ρ_I) of -0.61. Because the leverage effects drives the skewness of the return distribution, this result is consistent with Bakshi, Kapadia, and Madan (2003) who document empirically that the market index skewness is more negative than individual equity skewness on average. The estimated price of variance risk λ^{σ_I} is large and negative which is important to allow the model to generate a negative market variance risk premium. Comparing the estimated unconditional stochastic betas, $\hat{\beta} = \hat{\theta}_{SI}/\hat{\theta}_I$, in Panel A

[20] A simulation-based assessment of the performance of the particle filter in the context of our model is also discussed in the Online Appendix.

[21] Unlike the portfolio return risk measures reported in Table 1, the variance measures reported in Table 2 are value-weighted averages of stock-level variances which do not account for diversification benefits.

of Table 2 with the unconditional OLS betas in column 5 of Table 1, we see that model and OLS betas are close to each other. The divergence of model and OLS betas is, however, slightly larger for stocks with high-betas, which are also the most volatile as suggested by Tables 1 and 2.

To further investigate this, we scatter plot unconditional OLS beta against unconditional stochastic betas. Figure 2 shows the result. The solid line corresponds to the regression fit obtained from regressing OLS betas against our stochastic model betas. The figure uncovers the close relationship between unconditional OLS and unconditional stochastic betas. The coefficient obtained when regressing OLS betas on stochastic betas is 1.07 and the regression R-squared is 65%. No particular outliers are apparent.

Panel A of Table 2 also reports unconditional risk-premiums. For a given risk measure (i.e., variance, covariance, or beta), the risk premium is defined as the difference between physical and risk-neutral expectations of the integrated risk measure. For each firm and the market index, we apply the results (26), (27), and (28) in Proposition 3 and compute annual expected integrated risk measures given the filtered stochastic variance-covariance matrices on each day. The physical expectations are calculated based on parameter K and Θ, while risk-neutral model forecasts are obtained using the risk-neutral parameters, $\tilde{K}$ and $\tilde{\Theta}$. Armed with the daily expectations of each measure, we then take the difference for each stock and the market index. For equities, the values reported in the table correspond to the sample average of the daily value-weighted stock-level premiums.

Unconditionally, the market variance risk premium, VRP_I, is -1.8% which compares well to what has been documented in the literature (see Carr and Wu, 2009). For equities, the unconditional variance risk premiums are all negative but their magnitudes vary widely. The equity variance risk premiums, VRP_S, are particularly small (in absolute value) for the first five decile portfolios compared to the top deciles. As beta increases, the dependence of equity variance on market variance risks increases. Thus, the variance risk premium of stocks with high-beta is larger in absolute value as these stocks load more on the market variance risk premium.[22] Little is known about unconditional covariance and beta risk premiums. In that regard, Table 2 is informative about the size of these premiums and their distribution across beta-sorted portfolios. For low to medium beta portfolios, covariance risk premiums, CRP, are larger (in absolute value) than equity variance risk premiums. Unconditionally, the beta risk premium, BRP, can be positive or negative, but its magnitude is small.

Panel B of Table 2 reports various measures of fit for the return and option data. For each firm, the model R-squared corresponds to $\hat{\beta}^2\hat{\theta}_I/\hat{\theta}_S$. Not surprisingly, the fit obtained for the market index is better than for equities. The sample log-likelihood for the index return is $17,004$ and the model IVRMSE for index options is 3.50%. For equities, the model fit is generally good with conditional log-likelihoods of equity returns (i.e., difference between index-equity joint log-likelihood and index log-likelihood) ranging from $14,893$ to $16,817$. The equity options IVRMSE ranges from 5.25% to 9.22% which is noteworthy given that our sample includes the financial crisis. The model fits the return and option data of low-beta stocks better than that of high-beta stocks. This is to be expected as the returns of high-beta stocks are more volatile than the returns of low-beta stocks.

[22] Accounting for a separate price of idiosyncratic variance risk would allow the model to generate both positive and negative equity total variance risk premiums. For recent empirical evidence on equity variance risk premium, we refer to Buss, Schoenleber, and Vilkov (2016).

3.3 A Comparison of Stochastic, Option-Implied, and OLS Betas

We now assess the information content of model stochastic beta forecasts relative to alternative measures of beta. We assume 21 trading days per month in all calculations. We first construct annualized one-month model forecasts of future stochastic betas, $\beta_{t,21}^{SB} = \beta_{t,21} \times \frac{252}{21}$, for each firm. We also construct daily estimates of OLS beta. To obtain OLS beta predictions for the h days horizon of future betas on day t, we regress daily excess equity returns against excess S&P 500 returns over the last h days

$$R_u^S = \alpha_{t,h}^{OLS} + \beta_{t,h}^{OLS} \times R_u^I + \varepsilon_u, \text{ for } u \in \{t-h+1, ..., t\}. \tag{31}$$

Accordingly, we define the OLS beta forecast for the h-day future realized beta on day t by the loading $\beta_{t,h}^{OLS}$ of the above regression. We take the h-day ex-post realized beta on day t to be the OLS beta for the period starting on day $t+1$ and ending on day $t+h+1$, and we denote it $\beta_{t+h+1,h}^{OLS}$. Setting $h = 21$, we run the regression above on every day and for each firm to obtain the time-series of one-month OLS betas over the sample period.

In Figure 3, we plot the time-series of daily value-weighted average of one-month OLS betas (grey) and stochastic betas (black) for decile portfolios of stocks sorted on $\beta_{t,21}^{SB}$ each day (i.e., conditional sorting). Note that the results we document are robust to the use of one-month OLS betas for the sorting. Overall, the patterns in the two beta time-series are similar across portfolios. For instance, note the way OLS and stochastic betas substantially increases during the Tech bubble and the financial crisis for the top decile portfolio. This is encouraging because it demonstrates the ability of our model to adequately capture large variation in equity risks during periods of high uncertainty.

Buss and Vilkov (2012) and Chang, Christoffersen, Jacobs, and Vainberg (2012) among others find that option-implied betas have good predictive properties for future stock betas. Following Chang, Christoffersen, Jacobs, and Vainberg (2012), we construct measures of option-implied beta on each day for each firm. We use options with one-month to maturity to construct 21-day option-implied betas. We denote them $\beta_{t,21}^{OI}$.

To investigate the information content of our stochastic betas, we regress one-month ex-post realized betas against one-month expected integrated stochastic betas controlling for one-month option-implied and OLS betas. We take the 21-day ex-post realized beta on day t to be the OLS beta for the period starting on day $t+1$ and ending on day $t+22$, and we denote it $\beta_{t+22,21}^{OLS}$. Table 3 presents the regression coefficient estimates, t-statistics, and adjusted R-squared by $\beta_{t,21}^{SB}$-sorted portfolios. The stochastic beta forecasts are statistically significant for predicting future OLS betas for all portfolios. Note the way the magnitude of the loadings on the stochastic beta increases for more extreme portfolios. It is the largest for the top and bottom decile portfolios, respectively. Overall, this evidence suggests that model stochastic beta forecasts are highly informative about future realized betas, and especially for the high- and low-beta portfolios.

3.4 Conditional Beta Risk Premiums

An extensive literature document pronounced variation in betas but the question of whether a premium compensates unexpected innovations in betas is mainly open. Unconditionally, the magnitude of the beta risk premium is small as discussed in Section 3.2. We now investigate whether this is also the case when stocks are dynamically sorted on conditional betas and the forecast horizon is shorter than a year.

We construct daily estimates of annualized model-implied 21-day beta risk premiums for each stock. The model conditional premium is calculated as the difference between the annualized 21-day integrated physical and risk-neutral expectations of stochastic beta. More precisely, it corresponds to $\beta_{t,21}^{SB} - \beta_{t,21}^{SB,Q}$ where $\beta_{t,21}^{SB,Q} = \beta_{t,21}^{Q} \times \frac{252}{21}$ and $\beta_{t,21}^{Q}$ is obtained by applying the results in Proposition 3 using the risk-neutral parameters, $\tilde{K}$ and $\tilde{\Theta}$. For comparison, we also construct daily "model-free" measures of beta risk premium defined as the difference between OLS (physical) and option-implied (risk-neutral) betas. We argue that option-implied betas are valid measures of risk-neutral beta because they are constructed from market and equity risk-neutral moments. We use options with maturity of one-month to construct 21-day option-implied betas.

Table 4 presents the results for portfolios of stock sorted each day on $\beta_{t,21}^{SB}$. The model-implied measures of one-month risk premium compare well to the model-free ones both from a sign and a magnitude perspective. For both model-based and model-free measures, it is negative for the low-beta portfolio and increases as beta increases to become positive for the top decile portfolio. The difference between the top and bottom decile portfolios of model-free conditional beta risk premiums is 0.65. This is close to the model-implied difference of 0.43. This provides further support that our model adequately captures equity risk physical and risk-neutral dynamics. Recall that the sign and magnitude of the beta risk premium reflect the way beta co-moves with the SDF. We see that low-beta firms have a negative conditional beta risk premium on average while it is positive for high-beta firms. For low-beta firms, the fact that $\beta_{t,21}^{SB} - \beta_{t,21}^{SB,Q} < 0$ implies that the betas of these firms co-move positively with the SDF. In contrast, $\beta_{t,21}^{SB} - \beta_{t,21}^{SB,Q} > 0$ for high-beta firms is consistent with the idea that the beta of these stocks co-moves negatively with the SDF. It is worth noting that the beta risk premium (i.e., $\beta_{t,21} - \beta_{t,21}^{Q}$) is negative for portfolio 6 but positive for portfolio 7. Because portfolio 6 has an average stochastic beta of 0.77 while portfolio 7 has an average stochastic beta of 0.88, it implies a beta threshold of about 0.83 above or below which the beta risk premium changes sign. Overall, we conclude that the conditional beta risk premiums implied by our model match qualitatively and quantitatively model-free measures of premium.

3.5 Beta Risk and the Slope of the Conditional SML

Recall that our model predicts that the slope of the conditional SML estimated using risk-neutral betas will be steeper than the one obtained using physical betas when beta risk is priced. To investigate whether this is the case, we construct daily measures of one-week, and one- and three-month compounded ex-post realized returns. We denote by $R^S_{t,h}$ the h-day ahead compounded excess return of a given equity on day t. Using these measures, we run cross-sectional Fama-MacBeth regressions. On each day, we regress the cross-section of future realized excess equity returns on betas. The first specification we consider uses expected physical stochastic betas. The second uses expected risk-neutral stochastic betas. Each day, we estimate

$$R^S_{t+1,h} = b^0_{t,h} + b^{SB}_{t,h} \times \beta^{Model}_{t,h} + b^{OLS}_{t,h} \times \beta^{OLS}_{t,h} + b^{OI}_{t,h} \times \beta^{OI}_{t,h} + \varepsilon_{t+1,h} \tag{32}$$

for all equities where $\beta^{Model}_{t,h}$ is either set to $\beta^{SB}_{t,h}$ or to $\beta^{SB,Q}_{t,h}$. We do not consider a specification with model physical and risk-neutral betas together because of collinearity issues. We present robustness results for alternative regression specifications in Table A.3 in the Online Appendix.

Table 5 presents the average of the coefficients, their t-statistics computed using the Newey-West approach, and the average of the daily regressions R-squared. We set the Newey-West autocorrelation lags to the number of trading days considered for each horizon (i.e., 5, 21, and 63 lags, respectively). Because of data limitations, we use one-month option implied betas to predict one-week ahead excess stock returns. The R-squared obtained across horizons are high. Comparing the coefficients obtained for stochastic physical and risk-neutral betas with the ones of OLS and option-implied betas reveal an interesting pattern. Relative to the coefficients estimated for stochastic betas, the coefficients obtained for OLS and option-implied betas are small in magnitude, and are less significant. These results are robust across forecast horizons and sample periods. The weak relation between OLS betas and stock expected returns we document is consistent with previous studies (e.g., Fama and French, 1992).

Comparing the coefficients obtained for the model physical and risk-neutral betas uncovers an important insight. We see that the average loading obtained for risk-neutral betas is larger than the average loading estimated for physical beta forecasts for all forecast horizons. For any given stock, model physical and risk-neutral beta forecasts are constructed based on the same filtered latent variables and the same dynamics (i.e., the dynamics of the variance-covariance matrix used to forecast beta over time is affine under P and Q). This implies that model beta forecasts are identically impacted by any model misspecification and estimation errors. In other words, the only channel that could explain that the difference in the average loadings obtained for $\beta_{t,h}^{SB,Q}$ and $\beta_{t,h}^{SB}$ is the fact that $\beta_{t,h}^{SB,Q}$ incorporates a beta risk premium while $\beta_{t,h}^{SB}$ does not. We conclude that the higher average loading estimated for model risk-neutral betas is consistent with the idea that beta risk is priced in the cross-section.

Finally, we see that the intercept estimates are all statistically significant except for the three-month horizon in Panel B. This result is consistent with Proposition 2 which shows that, when beta risk is priced, it generates deviation of expected stock returns from the conditional SML. Quantifying the impact of beta risk on expected stock returns is the subject of the next sections.

3.6 Beta Risk and Deviation from the Conditional SML

While it is commonly acknowledged that beta varies over time, little is known about the impact of beta risk on expected stock returns. Equation (23) in Proposition 2 shows the way beta risk influences expected stock returns in our model.

To test Proposition 2, we construct model-based estimates of one-month beta return premiums, $RP_{t,21}^{BRP}$, for each firm as follows

$$RP_{t,21}^{BRP} = \Lambda^I \left(\int_t^{t+\frac{21}{252}} \left(E_t \left[\sigma_{SI,u} \right] - E_t \left[\beta_u \right] E_t \left[\sigma_{I,u}^2 \right] \right) du \right) - r \left(\beta_{t,21} - \beta_{t,21}^Q \right).$$

We set the risk-free rate, r, to its 1996-2016 sample average of 2.34% and Λ^I to 1.77 to match the sample average of S&P 500 excess returns. For a given firm, we construct daily measures of $\beta_{t,21}$ and $\beta_{t,21}^Q$. For the first term in RP^{BRP}, we use the results in Proposition 3 to obtain estimates of the physical $E_t \left[\sigma_{SI,u} \right]$, $E_t \left[\beta_u \right]$, and $E_t \left[\sigma_{I,u}^2 \right]$ which we integrate over u. We then calculate the daily value-weighted average of the stock-level one-month beta return premiums. We then take the value-weighted average for each decile portfolio of stocks sorted on one-month stochastic beta each day.

Figure 4 presents the time-series of the conditional beta return premium for the low- and the high-decile portfolios, and a high-minus-low beta strategy that buys the high-beta portfolio and shorts the low-beta portfolio. We report the sample average of the daily one-month beta return premium and its components for each portfolio in Table 6. Table 6 provides further insights into the decomposition of beta return premium by portfolio. First, the higher the conditional beta, the lower the co-movement of beta with the market variance is as indicated by the results in column 1 of Table 6 given $\Lambda^I > 0$. Second, the higher the conditional beta the lower the co-movement of beta with the SDF and the higher the beta risk premium is. As a result, $-r\left(\beta_{t,21} - \beta^Q_{t,21}\right)$ decreases with the level of conditional betas. Interestingly, we see that both patterns in columns 1 and 3 are linear in beta and decrease as beta increases. As a result, stocks with relatively high conditional beta have a negative beta return premium while low-beta portfolios have a positive beta return premium. During the 1996-2016 sample period, the average beta return premium of the high-minus-low beta strategy is -0.40% monthly (i.e., -4.79% annually). This deviation is highly statistically significant with a Newey-West t-statistic of -6.54 adjusted for 21 autocorrelation lags.

3.7 Beta Risk in the Cross-Section of NYSE Stocks

We now investigate whether the model implications hold across the entire cross-section of NYSE stocks for which we of course do not have options, and so cannot directly use the dynamic stochastic beta model developed. Effectively, this constitutes a tough out-of-sample assessment of the predictions of the model. We obtain daily stock excess return data from CRSP for all common shares traded on the NYSE, and use CRSP market returns to proxy for the return on the market portfolio. In each month t, we sort stocks into decile portfolios based on ex-ante betas obtained from regressing daily stock excess returns against daily market excess returns over the last 252 trading days, that is $\beta^{OLS}_{t,252}$. We compute five measures of ex-post betas for each stock. The year following sorting, we regress daily excess stock returns against daily market excess returns to obtain firms' ex-post beta, $\beta^{OLS}_{t+253,252}$. Following Ang, Chen, and Xing (2006), we compute ex-post measures of high and low market return betas, $\beta^{H,Ret}_{t+253,252}$ and $\beta^{L,Ret}_{t+253,252}$, calculated by regressing excess stock returns against excess market returns on the subset of days with market returns above and below its yearly average, respectively.[23] Arguably, the sign of $\beta^{H,Ret}_{t+253,252} - \beta^{L,Ret}_{t+253,252}$ is informative about the ex-post co-movement of stock beta with market returns. Note that $\beta^{H,Ret}_{t+253,252} - \beta^{L,Ret}_{t+253,252} > 0$ indicates that the beta of the stock co-moves positively with market returns (i.e., negatively with the SDF) in the subsequent year and vice versa. We also compute ex-post measures of high and low market variance betas, $\beta^{H,Var}_{t+253,252}$ and $\beta^{L,Var}_{t+253,252}$, calculated by regressing excess stock returns against excess market returns on the subset of days with market squared-returns above and below its yearly median, respectively. For robustness purposes, we compute similar measures when using the VIX index to identify high and low market variance days. When $\beta^{H,Var}_{t+253,252} - \beta^{L,Var}_{t+253,252} > 0$, it indicates that the beta of the stock co-moves positively with market variance in the year following the sorting and vice versa.

Table 7 presents the value-weighted results of sorting stocks into decile portfolios based on ex-ante OLS betas for the 1996-2016 sample period. Comparing the first with the third column confirms that the beta of high-beta stocks co-moves positively with market returns as $\beta^{H,Ret}_{t+253,252} - \beta^{L,Ret}_{t+253,252} > 0$. In contrast, the beta of firms with relatively low beta co-moves negatively with market returns ex-post as $\beta^{H,Ret}_{t+253,252} - \beta^{L,Ret}_{t+253,252} < 0$ for the low beta portfolio. Comparing the first with the fourth and fifth columns provides evidence that the beta of high-beta stocks co-moves negatively with market variance as $\beta^{H,Var}_{t+253,252} - \beta^{L,Var}_{t+253,252} < 0$. The opposite is true for firms with relatively low beta. In the sixth column, we report the one-year ex-post abnormal return which we use to measure the deviation from the SML. We see that the high-minus-low beta strategy earns a -5.16% abnormal return with a t-statistics of -1.71. The average conditional alpha of the high-minus-low beta trading strategy reported in Table 7 is comparable to -4.79%, which is

[23] Ang, Chen, and Xing (2006) sort stocks on ex-post betas whereas we sort on ex-ante betas.

the average conditional SML deviation of such a strategy implied by the model. The model thus explains about 92% (i.e., $-4.79/-5.16$) of the abnormal performance of such a strategy.

We conclude that our model's predictions are qualitatively and quantitatively supported in the cross-section of NYSE equity returns and that co-movements of beta with market variance and the SDF explain the abnormal return of "betting against beta" trading strategies.

4 Summary and Conclusions

We study the implications of beta dynamics and beta risk for the cross-section of stock returns. To this end we develop a new dynamic factor model with stochastic beta. In the model, individual equity and market returns covary dynamically and their variance-covariance matrix follows a bivariate Wishart process.

Our model can be used to filter conditional betas from daily returns and it allows for closed-form option pricing formulas. The model implies a term-structure of beta that can be used to forecast future realized betas. We develop an estimation methodology that maximizes the joint likelihood of returns and options for a large cross-section of stocks observed over a period of twenty one years.

The model makes a series of predictions. First, the model shows that part of the equity premium corresponds to compensation for risky betas. Second, it predicts that deviations from the SML are related to the co-movements of beta with the SDF and market variance. When beta is relatively low, it co-moves more positively with the stochastic discount factor (SDF) and negatively with market returns. To compensate low-beta firms for this risk, they earn an additional premium beyond the SML. Empirically we find that the model predictions hold in the cross-section of firms we study.

Several issues are left for future research. First, it may be useful to extend the model, for instance by allowing for jumps in the market price (see, e.g., Bates, 2008; Bollerslev and Todorov, 2011; Kelly, Lustig, and van Nieuwerburgh, 2016). Second, combining option information with high-frequency returns when estimating the parameters in our model may lead to even better inference on beta (see, e.g., Andersen, Fusari, and Todorov, 2015; Patton and Verardo, 2012; Bollerslev, Li, and Todorov, 2016). Finally, we have focused on analyzing the implications of beta dynamics and risk for stock returns, but additionally analyzing option returns through the lens of our model would be of great interest (see, e.g., An, Ang, Bali, and Caciki, 2014).

Appendix A. The Physical Dynamics of $\sigma^2_{I,t}$, $\sigma_{SI,t}$, and $\sigma^2_{S,t}$

We use the dynamic of Σ_t in equation (3) and the form of $\sqrt{\Sigma_t}$ in (2) to express the dynamics of $\sigma^2_{I,t}$, $\sigma_{SI,t}$, and $\sigma^2_{S,t}$ setting the upper off-diagonal element in K to 0. The dynamic of $\sigma^2_{I,t}$ is

$$d\sigma^2_{I,t} = 2K_I(\theta_I - \sigma^2_{I,t})dt + 2\sigma_{I,t}\left(Q^1_I dW^1_{I,t} + Q^2_I dW^2_{I,t}\right), \tag{A.1}$$

while the total individual equity variance follows

$$\begin{aligned} d\sigma^2_{S,t} &= \left(2K_S(\theta_S - \sigma^2_{S,t}) + 2K_{SI}(\theta_{SI} - \sigma_{SI,t})\right)dt \\ &\quad + 2\beta_t\sigma_{I,t}\left(Q^1_S dW^1_{I,t} + Q^2_S dW^2_{I,t}\right) \\ &\quad + 2\sqrt{\sigma^2_{S,t} - \beta^2_t\sigma^2_{I,t}}\left(Q^2_S dW^2_{S,t} + Q^1_S dW^1_{S,t}\right), \end{aligned} \tag{A.2}$$

and the covariance dynamics follows

$$\begin{aligned} d\sigma_{SI,t} &= \left(K_{SI}(\theta_I - \sigma^2_{I,t}) + (K_S + K_I)(\theta_{SI} - \sigma_{SI,t})\right)dt \\ &\quad + \left(\sigma_{I,t}Q^1_S + \beta_t\sigma_{I,t}Q^1_I\right)dW^1_{I,t} + \left(\sigma_{I,t}Q^2_S + \beta_t\sigma_{I,t}Q^2_I\right)dW^2_{I,t} \\ &\quad + \sqrt{\sigma^2_{S,t} - \beta^2_t\sigma^2_{I,t}}\left(Q^1_I dW^1_{S,t} + Q^2_I dW^2_{S,t}\right). \end{aligned} \tag{A.3}$$

Appendix B. Return and Variance-Covariance Risk-Neutral Dynamics

We now derive the return and variance-covariance dynamics under the risk-neutral measure. We make use of these results in Appendix D and Appendix F.

We proceed in two steps. First, we derive the model implication for the risk-neutralization of the Brownian motions driving the dynamics of the economy. Based on the risk-neutral shocks obtained, we subsequently risk-neutralize dI_t, dS_t, and $d\Sigma_t$.

We now derive the risk-neutralization of the Brownian motions $Z_{I,t}$, $Z_{S,t}$, and W_t consistent with the SDF ζ_t. To this end, let us define $L_R \equiv \begin{bmatrix} \lambda^{R_I} \\ 0 \end{bmatrix}$ and $L_V \equiv \begin{bmatrix} \lambda_1^{\sigma_I} & \lambda_2^{\sigma_I} \\ 0 & 0 \end{bmatrix}$. We can re-write the SDF using the following matrix notation

$$\frac{d\zeta_t}{\zeta_t} = -rdt - Tr\left[L_V' \sqrt{\Sigma_t} dW_t\right] - L_R' \sqrt{\Sigma_t} dB_t. \tag{A.4}$$

where $B_t \equiv [B_{I,t}\ B_{S,t}]'$ and $Tr[\cdot]$ is the trace operator. By application of the multivariate Girsanov theorem, we have

$$d\begin{bmatrix} \tilde{B}_{I,t} \\ \tilde{B}_{S,t} \end{bmatrix} = d\begin{bmatrix} B_{I,t} \\ B_{S,t} \end{bmatrix} + \sqrt{\Sigma_t}' L_R dt \tag{A.5}$$

$$d\begin{bmatrix} \tilde{W}_{I,t}^1 & \tilde{W}_{I,t}^2 \\ \tilde{W}_{S,t}^1 & \tilde{W}_{S,t}^2 \end{bmatrix} = d\begin{bmatrix} W_{I,t}^1 & W_{I,t}^2 \\ W_{S,t}^1 & W_{S,t}^2 \end{bmatrix} + \sqrt{\Sigma_t}' L_V dt, \tag{A.6}$$

where the tildes denote risk-neutral Brownian motions. Note that the previous system is equivalent to

$$d\begin{bmatrix} \tilde{B}_{I,t} \\ \tilde{B}_{S,t} \end{bmatrix} = d\begin{bmatrix} B_{I,t} \\ B_{S,t} \end{bmatrix} + \sigma_{I,t} \begin{bmatrix} \lambda^{R_I} \\ 0 \end{bmatrix} dt$$

$$d\begin{bmatrix} \tilde{W}_{I,t}^1 & \tilde{W}_{I,t}^2 \\ \tilde{W}_{S,t}^1 & \tilde{W}_{S,t}^2 \end{bmatrix} = d\begin{bmatrix} W_{I,t}^1 & W_{I,t}^2 \\ W_{S,t}^1 & W_{S,t}^2 \end{bmatrix} + \sigma_{I,t} \begin{bmatrix} \lambda_1^{\sigma_I} & \lambda_2^{\sigma_I} \\ 0 & 0 \end{bmatrix} dt. \tag{A.7}$$

Combining these results with the leverage effect decomposition of $dZ_{I,t}$, and $dZ_{S,t}$, we can infer the risk-neutral expression for the return shock dynamics

$$\begin{aligned}
d\begin{bmatrix} \tilde{Z}_{I,t} \\ \tilde{Z}_{S,t} \end{bmatrix} &= \left(\sqrt{1-\rho^2}\right) d\begin{bmatrix} \tilde{B}_{I,t} \\ \tilde{B}_{S,t} \end{bmatrix} + \rho d\begin{bmatrix} \tilde{W}^1_{I,t} & \tilde{W}^2_{I,t} \\ \tilde{W}^1_{S,t} & \tilde{W}^2_{S,t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \left(\sqrt{1-\rho^2}\right) d\begin{bmatrix} B_{I,t} \\ B_{j,t} \end{bmatrix} + \left(\sqrt{1-\rho^2}\right)\sqrt{\Sigma_t}' L_R dt \\
&\quad +\rho\left(d\begin{bmatrix} W^1_{I,t} & W^2_{I,t} \\ W^1_{S,t} & W^2_{S,t} \end{bmatrix} + \sqrt{\Sigma_t}' L_V dt\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= d\begin{bmatrix} Z_{I,t} \\ Z_{S,t} \end{bmatrix} + \sqrt{\Sigma_t}'\left(\left(\sqrt{1-\rho^2}\right) L_R + \rho L_V \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) dt. \qquad \text{(A.8)}
\end{aligned}$$

Combining (A.7) and (A.8) implies that

$$\begin{aligned}
d\tilde{Z}_{I,t} &= dZ_{I,t} + \sigma_{I,t}(\sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda_1^{\sigma_I}) \\
d\tilde{Z}_{S,t} &= dZ_{S,t} \\
d\tilde{W}^1_{I,t} &= dW^1_{I,t} + \sigma_{I,t}\lambda_1^{\sigma_I} dt \\
d\tilde{W}^2_{I,t} &= dW^2_{I,t} + \sigma_{I,t}\lambda_2^{\sigma_I} dt \\
d\tilde{W}^1_{S,t} &= dW^1_{S,t} \\
d\tilde{W}^2_{S,t} &= dW^2_{S,t}.
\end{aligned} \qquad \text{(A.9)}$$

The absence of arbitrage opportunities implies that $I_t = E_t\left[\frac{\zeta_T}{\zeta_t} I_T\right]$ and $S_t = E_t\left[\frac{\zeta_T}{\zeta_t} S_T\right]$ which, in turn, implies that the instantaneous return premium on the market index is

$$\mu_{I,t} = (\sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda_1^{\sigma_I})\sigma^2_{I,t} = \Lambda^I \sigma^2_{I,t}, \qquad \text{(A.10)}$$

where $\Lambda^I \equiv \sqrt{1-\rho^2}\lambda^{R_I} + \rho\lambda_1^{\sigma_I}$ and that the individual equity instantaneous return premium is

$$\beta_t \mu_{I,t} = \beta_t \left(\Lambda^I \sigma_{I,t}^2\right) = \Lambda^I \sigma_{SI,t}. \tag{A.11}$$

Thus, the risk-neutral dynamics for market index and individual equity returns is given by

$$\begin{bmatrix} \frac{dI_t}{I_t} \\ \\ \frac{dS_t}{S_t} \end{bmatrix} = \begin{bmatrix} r \\ \\ r \end{bmatrix} dt + \sqrt{\Sigma_t} \begin{bmatrix} d\tilde{Z}_{I,t} \\ \\ d\tilde{Z}_{S,t} \end{bmatrix}. \tag{A.12}$$

The Wishart dynamics are also impacted by the change of measure. Using $\gamma Q^{'}Q = K\Theta + \Theta K'$, we can rewrite the physical dynamics of the Wishart process

$$\begin{aligned} d\Sigma_t &= \left(K\left(\Theta - \Sigma_t\right) + \left(\Theta - \Sigma_t\right)K'\right)dt + \sqrt{\Sigma_t}dW_t Q + \left(\sqrt{\Sigma_t}dW_t Q\right)^{'} \\ &= \left(\gamma Q^{'}Q - K\Sigma_t - \Sigma_t K^{'}\right)dt + \sqrt{\Sigma_t}dW_t Q + Q' dW_t' \sqrt{\Sigma_t}^{'}. \end{aligned} \tag{A.13}$$

Given the physical dynamics above and the risk-neutralization (A.6), that is $d\tilde{W}_t = dW_t + \sqrt{\Sigma_t}^{'} L_V$ with $L_V \equiv \begin{bmatrix} \lambda_1^{\sigma_I} & \lambda_2^{\sigma_I} \\ \\ 0 & 0 \end{bmatrix}$, we have under the risk-neutral measure

$$\begin{aligned} d\Sigma_t &= \left(\gamma Q^{'}Q - K\Sigma_t - \Sigma_t K^{'}\right)dt + \sqrt{\Sigma_t}\left(d\tilde{W}_t - \sqrt{\Sigma_t}^{'} L_V dt\right)Q \\ &\quad + Q'\left(d\tilde{W}_t - \sqrt{\Sigma_t}^{'} L_V dt\right)^{'} \sqrt{\Sigma_t}^{'} \end{aligned}$$

$$\Leftrightarrow \quad d\Sigma_t = \left(\gamma Q^{'}Q - K\Sigma_t - \Sigma_t K^{'}\right)dt + \sqrt{\Sigma_t}d\tilde{W}_t Q - \Sigma_t L_V Q dt$$
$$+Q' d\tilde{W}_t^{'}\sqrt{\Sigma_t}^{'} - Q' L_V^{'}\Sigma_t dt$$

$$\Leftrightarrow d\Sigma_t = \left(\gamma Q^{'}Q - \tilde{K}\Sigma_t - \Sigma_t\tilde{K}'\right)dt + \sqrt{\Sigma_t}d\tilde{W}_t Q + Q' d\tilde{W}_t^{'}\sqrt{\Sigma_t}^{'} \qquad \text{(A.14)}$$

$$\Leftrightarrow d\Sigma_t = \left(\tilde{K}\left(\tilde{\Theta} - \Sigma_t\right) + \left(\tilde{\Theta} - \Sigma_t\right)\tilde{K}'\right)dt + \sqrt{\Sigma_t}d\tilde{W}_t Q + \left(\sqrt{\Sigma_t}d\tilde{W}_t Q\right)^{'}, \qquad \text{(A.15)}$$

where

$$\tilde{K} = K + Q' L_V^{'} = K + \begin{bmatrix} \lambda_1^{\sigma_I} Q_I^1 + \lambda_2^{\sigma_I} Q_I^2 & 0 \\ \lambda_1^{\sigma_I} Q_S^1 + \lambda_2^{\sigma_I} Q_S^2 & 0 \end{bmatrix}, \quad \text{and} \quad \gamma Q^{'}Q = \tilde{K}\tilde{\Theta} + \tilde{\Theta}\tilde{K}'. \qquad \text{(A.16)}$$

Together, equations (A.12), (A.15), and (A.16) define the joint risk-neutral dynamics of the market index and equity returns, and of the variance-covariance matrix.

Appendix C. The Physical and Risk-Neutral Dynamics of Beta

We first derive the dynamics of equity risk under the physical measure. By definition, the stock beta satisfies $\beta_t = \sigma_{SI,t}/\sigma_{I,t}^2$. A straightforward application of Itô's lemma implies that

$$d\beta_t = \frac{1}{\sigma_{I,t}^2}d\sigma_{SI,t} - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2}d\sigma_{I,t}^2 + \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^3}cov_t\left(d\sigma_{I,t}^2, d\sigma_{I,t}^2\right) - \frac{1}{\left(\sigma_{I,t}^2\right)^2}cov_t\left(d\sigma_{I,t}^2, d\sigma_{SI,t}\right), \qquad \text{(A.17)}$$

where $cov_t(\cdot,\cdot)$ denotes the instantaneous covariance operator. From (A.1) and (A.3), we see that the quadratic variations $cov_t\left(d\sigma_{I,t}^2, d\sigma_{I,t}^2\right)$ and $cov_t\left(d\sigma_{I,t}^2, d\sigma_{SI,t}\right)$ satisfy

$$\frac{cov_t\left(d\sigma_{I,t}^2, d\sigma_{I,t}^2\right)}{dt} = \sigma_{I,t}^2 \cdot \left(4\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)\right) = \sigma_{I,t}^2 \cdot A, \qquad \text{(A.18)}$$

where $A \equiv 4\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$, and

$$\begin{aligned}
\frac{cov_t\left(d\sigma_{I,t}^2, d\sigma_{SI,t}\right)}{dt} &= \sigma_{I,t}^2 \cdot 2\left(Q_I^1 Q_S^1 + Q_I^2 Q_S^2\right) + \sigma_{SI,t} \cdot 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right) \\
&= \sigma_{I,t}^2 \cdot B + \sigma_{SI,t} \cdot C, \qquad \text{(A.19)}
\end{aligned}$$

where $B \equiv 2\left(Q_I^1 Q_S^1 + Q_I^2 Q_S^2\right)$, and $C \equiv 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$, respectively. We can use these results to obtain

$$\begin{aligned}
d\beta_t &= \frac{1}{\sigma_{I,t}^2} d\sigma_{SI,t} - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} d\sigma_{I,t}^2 + \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^3} \sigma_{I,t}^2 A dt - \frac{1}{\left(\sigma_{I,t}^2\right)^2}\left(\sigma_{I,t}^2 \cdot B + \sigma_{SI,t} \cdot C\right) dt \\
&= \frac{1}{\sigma_{I,t}^2} d\sigma_{SI,t} - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} d\sigma_{I,t}^2 + \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} A dt - \frac{1}{\sigma_{I,t}^2} B dt - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2} C dt \\
&= \frac{1}{\sigma_{I,t}^2}\left(d\sigma_{SI,t} - B dt\right) - \frac{\sigma_{SI,t}}{\left(\sigma_{I,t}^2\right)^2}\left(d\sigma_{I,t}^2 - (A - C)\, dt\right) \\
&= \beta_t \left(\frac{\left(d\sigma_{SI,t} - \Phi_{IS} dt\right)}{\sigma_{SI,t}} - \frac{\left(d\sigma_{I,t}^2 - \Phi_I dt\right)}{\sigma_{I,t}^2}\right), \qquad \text{(A.20)}
\end{aligned}$$

where $\Phi_{IS} \equiv 2\left(Q_I^1 Q_S^1 + Q_I^2 Q_S^2\right)$ and $\Phi_I \equiv 2\left(\left(Q_I^1\right)^2 + \left(Q_I^2\right)^2\right)$.

The SDF (13) combined with the dynamics (A.20) implies the following risk-neutral dynamics of equity risk

$$d\beta_t = \beta_t \left(\frac{\left(d\sigma_{SI,t} - \Phi_{SI} dt\right)}{\sigma_{SI,t}} - \frac{\left(d\sigma_{I,t}^2 - \Phi_I dt\right)}{\sigma_{I,t}^2}\right) + cov_t\left(d\beta_t, \frac{d\zeta_t}{\zeta_t}\right), \qquad \text{(A.21)}$$

where $cov_t(\cdot, \cdot)$ denotes the conditional covariance. Given equation (A.21) and the SDF dynamic (13), beta's instantaneous co-movement with the SDF takes the form

$$cov_t\left(d\beta_t, \frac{d\zeta_t}{\zeta_t}\right) = \left(\lambda_I^{VRP} \beta_t - \lambda_S^{\beta}\right) dt,$$

where $\lambda_I^{VRP} \equiv \lambda_1^{\sigma_I} Q_I^1 + \lambda_2^{\sigma_I} Q_I^2$, and $\lambda_S^{\beta} \equiv \lambda_1^{\sigma_I} Q_S^1 + \lambda_2^{\sigma_I} Q_S^2$.

We now discuss the implications of priced variance risks for the sign of $cov_t\left(d\beta_t, \frac{d\zeta_t}{\zeta_t}\right)$. The instantaneous covariance of beta with the SDF in our model is implied directly by the prices of the market variance risks. When market variance risks are not priced (i.e., $\lambda_1^{\sigma_I} = \lambda_2^{\sigma_I} = 0$), we have $\lambda_I^{VRP} = 0$ and $\lambda_S^{\beta} = 0$. In this case, $cov_t\left(d\beta_t, \frac{d\zeta_t}{\zeta_t}\right)$ is zero regardless of β_t. The instantaneous covariance of beta with the SDF is linear in β_t and its sign and magnitude depend on the level of $\lambda_I^{VRP}\beta_t$ relative to λ_S^{β}. The instantaneous covariance is positive whenever $\lambda_I^{VRP}\beta_t > \lambda_S^{\beta} \Leftrightarrow \beta_t < \frac{\lambda_S^{\beta}}{\lambda_I^{VRP}}$ for the empirical relevant case of negative market variance risk premium, $\lambda_I^{VRP} < 0$.[24] Empirically, we find that $\lambda_S^{\beta} < 0$ on average and $\frac{\lambda_S^{\beta}}{\lambda_I^{VRP}} > 0$. Thus, the instantaneous covariance is positive when β_t is relatively low. In contrast, the betas of high-beta stocks instead tend to co-move negatively with the SDF.

By definition, the instantaneous beta risk premium satisfies

$$E_t\left[d\beta_t\right] - E_t^Q\left[d\beta_t\right] \equiv -cov_t\left(d\beta_t, d\zeta_t/\zeta_t\right) = \lambda_I^{VRP}\left(\frac{\lambda_S^{\beta}}{\lambda_I^{VRP}} - \beta_t\right)dt.$$

From the previous equation, we see that the instantaneous beta risk premium is positive when β_t is relatively high (i.e., $\beta_t > \frac{\lambda_S^{\beta}}{\lambda_I^{VRP}}$) given $\lambda_I^{VRP} < 0$. In contrast, the instantaneous beta risk premium is negative when β_t is relatively low. Together, the definition of the integrated beta risk premium (20) and the dynamics (A.20) and (A.21) imply that $BRP_{t,h}$ is given by

[24] See, among others, Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), and Driessen, Maenhout, and Vilkov (2009).

$$
\begin{aligned}
BRP_{t,h} &= \beta_{t,h} - \beta_{t,h}^{Q} \\
&= E_t\left[\int_t^{t+h/252}\left\{\int_t^u \beta_v\left(\frac{(d\sigma_{SI,v} - \Phi_{SI}dv)}{\sigma_{SI,v}} - \frac{\left(d\sigma_{I,v}^2 - \Phi_I dv\right)}{\sigma_{I,v}^2}\right)\right\}du\right] \\
&\quad - E_t^Q\left[\int_t^{t+h/252}\left\{\int_t^u \beta_v\left(\frac{(d\sigma_{SI,v} - \Phi_{SI}dv)}{\sigma_{SI,v}} - \frac{\left(d\sigma_{I,v}^2 - \Phi_I dv\right)}{\sigma_{I,v}^2}\right)\right\}du\right] \\
&\quad - E_t^Q\left[\int_t^{t+h/252}\left\{\int_t^u \left(E_v\left[d\beta_v\right] - E_v^Q\left[d\beta_v\right]\right)\right\}du\right],
\end{aligned}
$$

which cannot be solved explicitly.

Appendix D. Market Index and Individual Equity Return Premiums

We now derive integrated return premiums from t to $t+\tau$ where we define $\tau \equiv \frac{h}{252}$ for ease of notation. From equations (1) and (14), we have

$$
\begin{aligned}
E_t\left[\int_t^{t+\tau}\frac{dI_u}{I_u}\right] - E_t^Q\left[\int_t^{t+\tau}\frac{dI_u}{I_u}\right] &= E_t\left[\int_t^{t+\tau}\left(r + \mu_{I,u}\right)du\right] - E_t^Q\left[\int_t^{t+\tau} r du\right] \\
= E_t\left[\int_t^{t+\tau}\mu_{I,u}du\right] &= \Lambda^I\int_t^{t+\tau}E_t\left[\sigma_{I,u}^2\right]du = \Lambda^I\sigma_{I,t,h}^2, \qquad \text{(A.22)}
\end{aligned}
$$

where we have used the form of the market return premium $\mu_{I,t} = \Lambda^I\sigma_{I,t}^2$ with $\Lambda^I = \left(\sqrt{1-\rho^2}\right)\lambda^{R_I} + \rho\lambda_1^{\sigma_I}$ (see Appendix B) and $\sigma_{I,t,h}^2 \equiv \int_t^{t+\tau}E_t\left[\sigma_{I,u}^2\right]du$ is the h-day expected integrated market variance under the P-measure.

Comparing (1) and (14), we see that the instantaneous individual equity return premium is equal to $\beta_t \cdot \mu_{I,t}$. As a result, we have

$$
\begin{aligned}
E_t\left[\int_t^{t+\tau}\frac{dS_u}{S_u}\right]-E_t^Q\left[\int_t^{t+\tau}\frac{dS_u}{S_u}\right] &= E_t\left[\int_t^{t+\tau}\left(r+\beta_u\mu_{I,u}\right)du\right]-E_t^Q\left[\int_t^{t+\tau} rdu\right] \\
&= E_t\left[\int_t^{t+\tau}\beta_u\left(\Lambda^I\sigma_{I,u}^2\right)du\right]=\Lambda^I\int_t^{t+\tau}E_t\left[\sigma_{SI,u}\right]du=\Lambda^I\sigma_{SI,t,h}, \qquad \text{(A.23)}
\end{aligned}
$$

where $\sigma_{SI,t,h} \equiv \int_t^{t+\tau} E_t\left[\sigma_{SI,u}\right] du$ is the h-day expected integrated covariance under the P-measure.

The return dynamic for the individual equity in (1) given the form of $\sqrt{\Sigma_t}$ in (2) and the risk-neutralization (A.9) satisfies

$$
\frac{dS_t}{S_t}=rdt+\beta_t\left(\frac{dI_t}{I_t}-rdt\right)+\left(\sqrt{\sigma_{S,t}^2-\beta_t^2\sigma_{I,t}^2}\right)dZ_{S,t}, \qquad \text{(A.24)}
$$

under the P-measure where $E_t\left[\frac{dI_t}{I_t}\right]=\left(r+\mu_{I,t}\right)dt$ and

$$
\frac{dS_t}{S_t}=rdt+\beta_t\left(\frac{dI_t}{I_t}-rdt\right)+\left(\sqrt{\sigma_{S,t}^2-\beta_t^2\sigma_{I,t}^2}\right)d\tilde{Z}_{S,t}, \qquad \text{(A.25)}
$$

under the Q-measure where $E_t^Q\left[\frac{dI_t}{I_t}\right]=rdt$. The factor structure of individual equity return (A.24) and (A.25) implies that

$$
\begin{aligned}
& E_t\left[\int_t^{t+\tau}\frac{dS_u}{S_u}\right]-E_t^Q\left[\int_t^{t+\tau}\frac{dS_u}{S_u}\right] \\
&= E_t\left[\int_t^{t+\tau} rdu+\beta_u\left(\frac{dI_u}{I_u}-rdu\right)\right]-E_t^Q\left[\int_t^{t+\tau} rdu+\beta_u\left(\frac{dI_u}{I_u}-rdu\right)\right] \\
&= E_t\left[\int_t^{t+\tau}\beta_u\frac{dI_u}{I_u}\right]-E_t^Q\left[\int_t^{t+\tau}\beta_u\frac{dI_u}{I_u}\right]-r\int_t^{t+\tau}\left(E_t\left[\beta_u\right]-E_t^Q\left[\beta_u\right]\right)du \\
&= E_t\left[\int_t^{t+\tau}\beta_u\frac{dI_u}{I_u}\right]-E_t^Q\left[\int_t^{t+\tau}\beta_u\frac{dI_u}{I_u}\right]-r\left(\beta_{t,h}-\beta_{t,h}^Q\right), \qquad \text{(A.26)}
\end{aligned}
$$

where $\beta_{t,h} \equiv \int_t^{t+\tau} E_t[\beta_u]\, du$ and $\beta_{t,h}^Q \equiv \int_t^{t+\tau} E_t^Q[\beta_u]\, du$ are the h-day expected integrated physical and risk-neutral betas, respectively. Note that we have used the fact that idiosyncratic risk is not priced in the model and thus $E_t\left[\int_t^{t+\tau} dZ_{S,t}\right] = E_t^Q\left[\int_t^{t+\tau} d\tilde{Z}_{S,t}\right] = 0$.

We can now further develop the first term in (A.26). By Fubini's theorem, we have

$$E_t\left[\int_t^{t+\tau} \beta_u \frac{dI_u}{I_u}\right] - E_t^Q\left[\int_t^{t+\tau} \beta_u \frac{dI_u}{I_u}\right] = \int_t^{t+\tau} E_t\left[\beta_u \frac{dI_u}{I_u}\right] - \int_t^{t+\tau} E_t^Q\left[\beta_u \frac{dI_u}{I_u}\right].$$

Noting that

$$E_t\left[\beta_u \frac{dI_u}{I_u}\right] = E_t[\beta_u]\, E_t\left[\frac{dI_u}{I_u}\right] + cov_t\left(\beta_u, \frac{dI_u}{I_u}\right),$$

independently of the measure considered, we have

$$\begin{aligned} & E_t\left[\int_t^{t+\tau} \beta_u \frac{dI_u}{I_u}\right] - E_t^Q\left[\int_t^{t+\tau} \beta_u \frac{dI_u}{I_u}\right] \\ = \; & \int_t^{t+\tau} \left(E_t[\beta_u]\, E_t\left[\frac{dI_u}{I_u}\right] - E_t^Q[\beta_u]\, E_t^Q\left[\frac{dI_u}{I_u}\right]\right) + \int_t^{t+\tau} cov_t\left(\beta_u, \frac{dI_u}{I_u}\right) - cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right). \end{aligned} \tag{A.27}$$

We now show that

$$\int_t^{t+\tau} cov_t\left(\beta_u, \frac{dI_u}{I_u}\right) - cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right) = \int_t^{t+\tau} cov_t\left(\beta_u, \mu_{I,u}\right) du \tag{A.28}$$

$$= \Lambda^I \int_t^{t+\tau} cov_t\left(\beta_u, \sigma_{I,u}^2\right) du. \tag{A.29}$$

First, consider $cov_t\left(\beta_u, \frac{dI_u}{I_u}\right)$ and $cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right)$ which satisfy

$$\begin{aligned} cov_t\left(\beta_u, \frac{dI_u}{I_u}\right) &= E_t\left[(\beta_u - E_t[\beta_u])\left(\frac{dI_u}{I_u} - E_t\left[\frac{dI_u}{I_u}\right]\right)\right] \\ cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right) &= E_t^Q\left[\left(\beta_u - E_t^Q[\beta_u]\right)\left(\frac{dI_u}{I_u} - E_t^Q\left[\frac{dI_u}{I_u}\right]\right)\right]. \end{aligned}$$

Under the risk-neutral measure, we have $dI_u/I_u - E_t^Q[dI_u/I_u] = \sigma_{I,u} d\tilde{Z}_{I,u}$ which implies that $cov_t^Q\left(\beta_u, \frac{dI_u}{I_u}\right)$ is equal to

$$\begin{aligned}
& E_t^Q\left[\left(\beta_u - E_t^Q[\beta_u]\right)\left(\sigma_{I,u} d\tilde{Z}_{I,u}\right)\right] \\
= & E_t^Q\left[\left(\beta_u - E_t^Q[\beta_u]\right) E_u^Q\left[\left(\sigma_{I,u} d\tilde{Z}_{I,u}\right)\right]\right] \\
= & 0,
\end{aligned} \tag{A.30}$$

while under the physical measure, we have $dI_u/I_u - E_t[dI_u/I_u] = \mu_{I,u} - E_t\left[\mu_{I,u}\right] + \sigma_{I,u} dZ_{I,u}$ and $cov_t\left(\beta_u, \frac{dI_u}{I_u}\right)$ satisfies

$$\begin{aligned}
& E_t\left[(\beta_u - E_t[\beta_u])\left(\left(\mu_{I,u} - E_t\left[\mu_{I,u}\right]\right) + \sigma_{I,u} dZ_{I,u}\right)\right] \\
= & E_t\left[(\beta_u - E_t[\beta_u])\left(\mu_{I,u} - E_t\left[\mu_{I,u}\right]\right)\right] + 0 \\
= & cov_t\left(\beta_u, \mu_{I,u}\right).
\end{aligned} \tag{A.31}$$

Combining the results in equations (A.26), (A.27), and (A.28), we get

$$\begin{aligned}
& E_t\left[\int_t^{t+\tau} \frac{dS_u}{S_u}\right] - E_t^Q\left[\int_t^{t+\tau} \frac{dS_u}{S_u}\right] \\
= & \int_t^{t+\tau}\left(E_t[\beta_u] E_t\left[\frac{dI_u}{I_u}\right] - E_t^Q[\beta_u] E_t^Q\left[\frac{dI_u}{I_u}\right]\right) + \int_t^{t+\tau} cov_t\left(\beta_u, \mu_{I,u}\right) du - r\left(\beta_{t,h} - \beta_{t,h}^Q\right) \\
= & RP_{t,h}^{SML} + RP_{t,h}^{BRP},
\end{aligned} \tag{A.32}$$

where

$$
\begin{aligned}
RP_{t,h}^{SML} &\equiv \int_t^{t+\tau} \left(E_t\left[\beta_u\right] E_t\left[\frac{dI_u}{I_u}\right] - E_t^Q\left[\beta_u\right] E_t^Q\left[\frac{dI_u}{I_u}\right] \right) && \text{(A.33)} \\
RP_{t,h}^{BRP} &\equiv \int_t^{t+\tau} cov_t\left(\beta_u, \mu_{I,u}\right) du - r\left(\beta_{t,h} - \beta_{t,h}^Q\right) && \text{(A.34)} \\
&= \Lambda^I \left(\int_t^{t+\tau} cov_t\left(\beta_u, \sigma_{I,u}^2\right) du \right) - r\left(\beta_{t,h} - \beta_{t,h}^Q\right), && \text{(A.35)}
\end{aligned}
$$

where we have used the definition $\mu_{I,u} = \Lambda^I \sigma_{I,u}^2$ to obtain the last equality, which completes the proof.

Appendix E. Term Structure of Risks

We start by deriving the model's prediction for the expected integrated variance-covariance matrix. We then apply this result to find the expression for expected integrated beta.

An application of Itô's Lemma to $e^{Kt}\Sigma_t e^{tK'}$ where Σ_t follows

$$
d\Sigma_t = \left(\gamma Q^{'} Q - K\Sigma_t - \Sigma_t K^{'}\right) dt + \sqrt{\Sigma_t} dW_t Q + Q' dW_t' \sqrt{\Sigma_t}^{'},
$$

with $\gamma Q^{'} Q = K\Theta + \Theta K'$ implies

$$
\begin{aligned}
d\left(e^{Kt}\Sigma_t e^{tK'}\right) &= \left(e^{Kt} K \Sigma_t e^{tK'} + e^{Kt}\Sigma_t K' e^{tK'}\right) dt + e^{Kt} d\Sigma_t e^{tK'} \\
&= \gamma e^{Kt} Q^{'} Q e^{tK'} dt + e^{Kt}\left(\sqrt{\Sigma_t} dW_t Q + Q' dW_t' \sqrt{\Sigma_t}^{'}\right) e^{tK'}. && \text{(A.36)}
\end{aligned}
$$

Integrating both sides of the previous equation from t to u with $u > t$ gives

$$e^{Ku}\Sigma_u e^{uK'} - e^{Kt}\Sigma_t e^{tK'} = \gamma \int_t^u e^{Kv} Q' Q e^{vK'} dv + \int_t^u e^{Kv}\left(\sqrt{\Sigma_v} dW_v Q + Q' dW_v' \sqrt{\Sigma_v}\right) e^{vK'}, \quad \text{(A.37)}$$

which implies that

$$E_t\left[e^{Ku}\Sigma_u e^{uK'}\right] = e^{Kt}\Sigma_t e^{tK'} + \gamma \int_t^u e^{Kv} Q' Q e^{vK'} dv. \quad \text{(A.38)}$$

Multiplying by e^{-Ku} from the left and $e^{-uK'}$ from the right, we get

$$E_t\left[\Sigma_u\right] = e^{-Ku}\left(e^{Kt}\Sigma_t e^{tK'} + \gamma \int_t^u e^{Kv} Q' Q e^{vK'} dv\right) e^{-uK'} = e^{-K(u-t)}\Sigma_t e^{-K'(u-t)} + \Gamma_{t,u}, \quad \text{(A.39)}$$

where $\Gamma_{t,u} \equiv \gamma \int_t^u e^{-K(u-v)} Q' Q e^{-K'(u-v)} dv$. To obtain the model's h-day expected integrated variance-covariance matrix, we need to integrate the previous expression over h days. This gives

$$\Sigma_{t,h} = \int_t^{t+\tau} E_t\left[\Sigma_u\right] du = \int_t^{t+\tau} \left(e^{-K(u-t)}\Sigma_t e^{-K'(u-t)} + \Gamma_{t,u}\right) du, \quad \text{(A.40)}$$

where $\tau = \frac{h}{252}$. We now derive an approximation formula for the h-day ahead model forecast of future realized beta: $\beta_{t,h} = \left(\int_t^{t+\tau} E_t\left[\beta_u\right] du\right)$. A first-order Taylor-expansion of conditional beta around θ_{SI}/θ_I leads to

$$\beta_u = \frac{\sigma_{SI,u}}{\sigma_{I,u}^2} = \frac{\theta_{SI}}{\theta_I} + \frac{\left(\sigma_{SI,u} - \theta_{SI}\right)}{\theta_I} - \frac{\left(\sigma_{I,u}^2 - \theta_I\right)\theta_{SI}}{\left(\theta_I\right)^2} + O, \quad \text{(A.41)}$$

where O is the error terms. Taking the expectation and ignoring the errors of order greater than two, we get

$$E_t[\beta_u] \approx \frac{\theta_{SI}}{\theta_I} + \frac{(E_t[\sigma_{SI,u}] - \theta_{SI})}{\theta_I} - \frac{\left(E_t\left[\sigma^2_{I,u}\right] - \theta_I\right)\theta_{SI}}{(\theta_I)^2} \tag{A.42}$$

$$\approx \frac{E_t[\sigma_{SI,u}]}{\theta_I} - \frac{\left(E_t\left[\sigma^2_{I,u}\right] - \theta_I\right)\theta_{SI}}{(\theta_I)^2}. \tag{A.43}$$

Integrating the previous expression from t to $t+\tau$, we obtain

$$\beta_{t,h} \approx \frac{\sigma_{SI,t,h}}{\theta_I} - \frac{\left(\sigma^2_{I,t,h} - \theta_I\tau\right)\theta_{SI}}{(\theta_I)^2}, \tag{A.44}$$

where $\sigma^2_{I,t,h} = \int_t^{t+\tau} E_t\left[\sigma^2_{I,u}\right] du = (\Sigma_{t,h})^{(1,1)}$, $\sigma_{SI,t,h} = \int_t^{t+\tau} E_t[\sigma_{SI,u}]\, du = (\Sigma_{t,h})^{(2,1)}$, and $\tau = \frac{h}{252}$.

Appendix F. Index and Individual Equity Option Prices

For ease of notation, we define the integrated Brownian $\tilde{Z}_{\Sigma,t,\tau} \equiv \int\limits_t^{t+\tau} \sqrt{\Sigma_u} d \begin{bmatrix} \tilde{Z}_{I,u} \\ \tilde{Z}_{S,u} \end{bmatrix}$ and the integrated variance-covariance matrix $\Sigma^{Int}_{t,\tau} \equiv \int\limits_t^{t+\tau} \Sigma_u du$. Given the Q-dynamics in Appendix B for dI_t and dS_t, we can apply Itô's lemma to $\ln(P_t)$ where $P_t \equiv [I_t\ S_t]'$ and obtain after integration the following expression for log-returns

$$\ln(P_T) - \ln(P_t) = r\mathbf{1}\tau - \frac{1}{2} diag\left(\Sigma^{Int}_{t,\tau}\right) + \tilde{Z}_{\Sigma,t,\tau}, \tag{A.45}$$

where $\mathbf{1}$ is a 2×1 vector of ones and $T = t + \tau$. Therefore, the conditional characteristic function of the risk-neutral log-returns takes the form

$$\begin{aligned}
\tilde{\phi}_t^{LR}(\tau, u_I, u_S) &= E_t^Q\left[\exp\left(iu'\left(\ln\left(P_T\right) - \ln\left(P_t\right)\right)\right)\right] \\
&= E_t^Q\left[\exp\left(iu'\left(r\mathbf{1}\tau - \frac{1}{2}diag\left(\Sigma_{t,\tau}^{Int}\right) + \tilde{Z}_{\Sigma,t,\tau}\right)\right)\right], \qquad \text{(A.46)}
\end{aligned}$$

where $u \equiv [u_I \; u_S]'$ is a 2×1 vector. Let us introduce the stochastic exponential $\xi(\cdot)$ defined by

$$\xi\left(\eta'\tilde{Z}_{\Sigma,t,\tau}\right) = \exp\left(\eta'\tilde{Z}_{\Sigma,t,\tau} - \frac{1}{2}\eta' var_t\left(\tilde{Z}_{\Sigma,t,\tau}\right)\eta\right) = \exp\left(\eta'\tilde{Z}_{\Sigma,t,\tau} - \frac{1}{2}\eta'\Sigma_{t,\tau}^{Int}\eta\right). \qquad \text{(A.47)}$$

Then, we can write (A.46) as

$$\begin{aligned}
\tilde{\phi}_t^{LR}(\tau, u_I, u_S) &= \exp(iu'r\mathbf{1}\tau) \cdot E_t^Q\left[\xi\left(iu'\tilde{Z}_{\Sigma,t,\tau}\right)\exp\left(\frac{iu'\Sigma_{t,\tau}^{Int}iu}{2}\right)\exp\left(-\frac{iu'diag\left(\Sigma_{t,\tau}^{Int}\right)}{2}\right)\right] \\
&= \exp(iu'r\mathbf{1}\tau) \cdot E_t^Q\left[\xi\left(iu'\tilde{Z}_{\Sigma,t,\tau}\right)\exp\left(\frac{iu'\left(\Sigma_{t,\tau}^{Int}iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2}\right)\right]. \qquad \text{(A.48)}
\end{aligned}$$

We can define the following change-of-measure

$$\frac{dC}{dQ}(t) \equiv \xi\left(iu'\tilde{Z}_{\Sigma,0,t}\right). \qquad \text{(A.49)}$$

Combining (A.48) with the change of measure (A.49), we can write

$$\tilde{\phi}_t^{LR}(\tau, u_I, u_S) = \exp(iu'r\mathbf{1}\tau)E_t^Q\left[\frac{\frac{dC}{dQ}(T)}{\frac{dC}{dQ}(t)}\exp\left(\frac{iu'\left(\Sigma_{t,\tau}^{Int}iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2}\right)\right]$$

$$\Rightarrow \tilde{\phi}_t^{LR}(\tau, u_I, u_S) = \exp(iu^{'} r \mathbf{1}\tau) E_t^C \left[\exp\left(\frac{iu^{'}\left(\Sigma_{t,\tau}^{Int} iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2} \right) \right].$$

Because

$$\frac{iu^{'}\left(\Sigma_{t,\tau}^{Int} iu - diag\left(\Sigma_{t,\tau}^{Int}\right)\right)}{2} = Tr\left[\Gamma(u_I, u_S)\Sigma_{t,\tau}^{Int}\right]$$

where

$$\Gamma(u_I, u_S) \equiv \frac{1}{2} \times \begin{bmatrix} -(u_I)^2 - iu_I & -u_I u_S \\ -u_I u_S & -(u_S)^2 - iu_S \end{bmatrix},$$

we have

$$\tilde{\phi}_t^{LR}(\tau, u_I, u_S) = \exp(iu^{'} r \mathbf{1}\tau) E_t^C \left[\exp\left(Tr\left[\Gamma(u_I, u_S) \cdot \Sigma_{t,\tau}^{Int}\right]\right)\right]. \tag{A.50}$$

While $\Gamma(\cdot, \cdot)$ is function of u_I and u_S, we drop the two input arguments in the rest of the proof for ease of notation. Thus, we now refer to it simply as Γ. An extension of the multivariate Girsanov theorem to the complex plane implies that under the C-measure, we have

$$dZ_t^C = d\tilde{Z}_t - i\sqrt{\Sigma_t}^{'} u dt,$$

where $\tilde{Z}_t \equiv \left[\tilde{Z}_{I,t} \ \tilde{Z}_{S,t}\right]^{'}$ and

$$dW_t^C = d\tilde{W}_t - i\rho\sqrt{\Sigma_t}^{'}\left[u\ \mathbf{0}\right]dt,$$

where $\mathbf{0}$ is a 2×1 vector of zeros. We can now infer the Wishart dynamic under the new measure given the risk-neutral dynamic (A.14)

$$d\Sigma_t = \left(\gamma Q^{'} Q - \tilde{K}\Sigma_t - \Sigma_t \tilde{K}^{'}\right) dt + \sqrt{\Sigma_t} d\tilde{W}_t Q + Q' d\tilde{W}_t' \sqrt{\Sigma_t}^{'}$$

which satisfies

$$\Leftrightarrow \;\; d\Sigma_t = \left(\gamma Q^{'} Q - \tilde{K}\Sigma_t - \Sigma_t \tilde{K}^{'}\right) dt + \sqrt{\Sigma_t}\left(d\tilde{W}_t^C + i\rho\sqrt{\Sigma_t}^{'}\left[u\ \mathbf{0}\right] dt\right) Q$$
$$+Q'\left(d\tilde{W}_t^C + i\rho\sqrt{\Sigma_t}^{'}\left[u\ \mathbf{0}\right] dt\right)^{'}\sqrt{\Sigma_t}^{'}$$

$$\Leftrightarrow \;\; d\Sigma_t = \left(\gamma Q^{'} Q - \tilde{K}\Sigma_t - \Sigma_t \tilde{K}^{'}\right) dt + \sqrt{\Sigma_t} d\tilde{W}_t^C Q + i\rho\Sigma_t\left[u\ \mathbf{0}\right] Q dt$$
$$+Q' d\tilde{W}_t^{C'}\sqrt{\Sigma_t}^{'} + i\rho Q'\left[u\ \mathbf{0}\right]^{'}\Sigma_t dt$$

$$\Leftrightarrow d\Sigma_t = \left(\gamma Q^{'} Q - K^C\Sigma_t - \Sigma_t K^{C\prime}\right) dt + \sqrt{\Sigma_t} d\tilde{W}_t^C Q + Q' d\tilde{W}_t^{C'}\sqrt{\Sigma_t}, \tag{A.51}$$

where

$$K^C = \tilde{K} - i\rho Q'\left[u\ \mathbf{0}\right]^{'}.$$

We can now make use of the closed-form solution for the moment generating function $E_t^C[\exp\left(Tr\left[\Gamma \cdot \Sigma_{t,\tau}^{Int}\right]\right)]$ to obtain the following expression for $\tilde{\phi}_t^{LR}(\cdot)$,

$$\tilde{\phi}_t^{LR}\left(\tau, u_I, u_S\right) = \exp\left(Tr\left[A(\tau)\cdot\Sigma_t\right] + B(\tau)\right), \tag{A.52}$$

with

$$A(\tau) = \left(a^{22}\left(\tau\right)\right)^{-1}\cdot\left(a^{21}\left(\tau\right)\right), \tag{A.53}$$

where

$$\begin{pmatrix} a^{11}\left(\tau\right) & a^{12}\left(\tau\right) \\ a^{21}\left(\tau\right) & a^{22}\left(\tau\right) \end{pmatrix} = \exp\left(\tau\begin{bmatrix} -K^C & -2Q^{'}Q \\ \Gamma & K^{C'} \end{bmatrix}\right),$$

and

$$B(\tau) = -\frac{\gamma}{2} Tr\left[\log\left(a^{22}(\tau)\right) - \tau K^{C'} + i\tau r u'\mathbf{1}\right], \tag{A.54}$$

where

$$\begin{aligned} u &= [u_I \; u_S]' \\ K^C &= \tilde{K} - i\rho Q' [u \; \mathbf{0}]', \end{aligned}$$

and

$$\Gamma = \frac{1}{2} \times \begin{bmatrix} -(u_I)^2 - iu_I & -u_I u_S \\ -u_I u_S & -(u_S)^2 - iu_S \end{bmatrix}.$$

Given the characteristic function above the price of a call written on the market index with strike price X is

$$C_t^I(I_t, X, \tau) = I_t\left(\frac{1}{2} - \Pi_{t,\tau}^I\right), \tag{A.55}$$

and the price of a call written on the individual equity is

$$C_t^S(S_t, X, \tau) = S_t\left(\frac{1}{2} - \Pi_{t,\tau}^S\right), \tag{A.56}$$

where the risk-neutral probabilities $\Pi_{t,\tau}^I$ and $\Pi_{t,\tau}^S$ are defined by

$$\begin{aligned} \Pi_{t,\tau}^I &= \frac{e^{-r\tau}}{2\pi} \int_0^\infty \text{Re}\left[\frac{e^{-iu_I \ln X/I_t}\tilde{\phi}_t^{LR}(\tau, u_I - i, 0)}{(u_I)^2 - iu_I} + \frac{e^{iu_I \ln X/I_t}\tilde{\phi}_t^{LR}(\tau, -u_I - i, 0)}{(u_I)^2 + iu_I}\right] du_I, \\ \Pi_{t,\tau}^S &= \frac{e^{-r\tau}}{2\pi} \int_0^\infty \text{Re}\left[\frac{e^{-iu_S \ln X/S_t}\tilde{\phi}_t^{LR}(\tau, 0, u_S - i)}{(u_S)^2 - iu_S} + \frac{e^{iu_S \ln X/S_t}\tilde{\phi}_t^{LR}(\tau, 0, -u_S - i)}{(u_S)^2 + iu_S}\right] du_S. \end{aligned}$$

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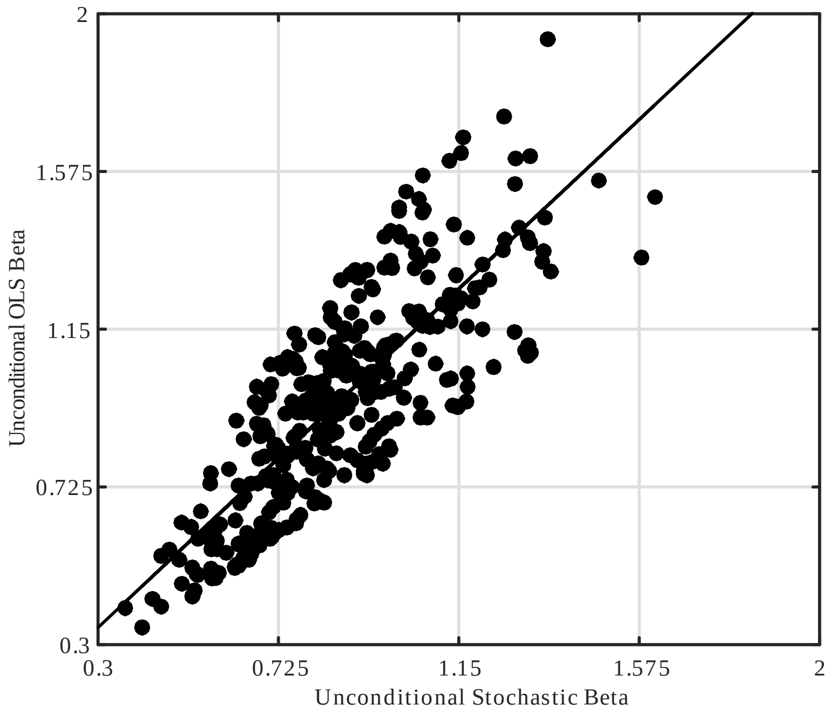

Figure 2

Unconditional OLS beta versus unconditional stochastic beta

We scatter plot the unconditional OLS betas against the unconditional stochastic betas, θ_{SI}/θ_I, for the 344 firms. We compute the unconditional OLS beta for each stock by regressing daily excess stock returns on daily excess S&P 500 returns over the entire sample.

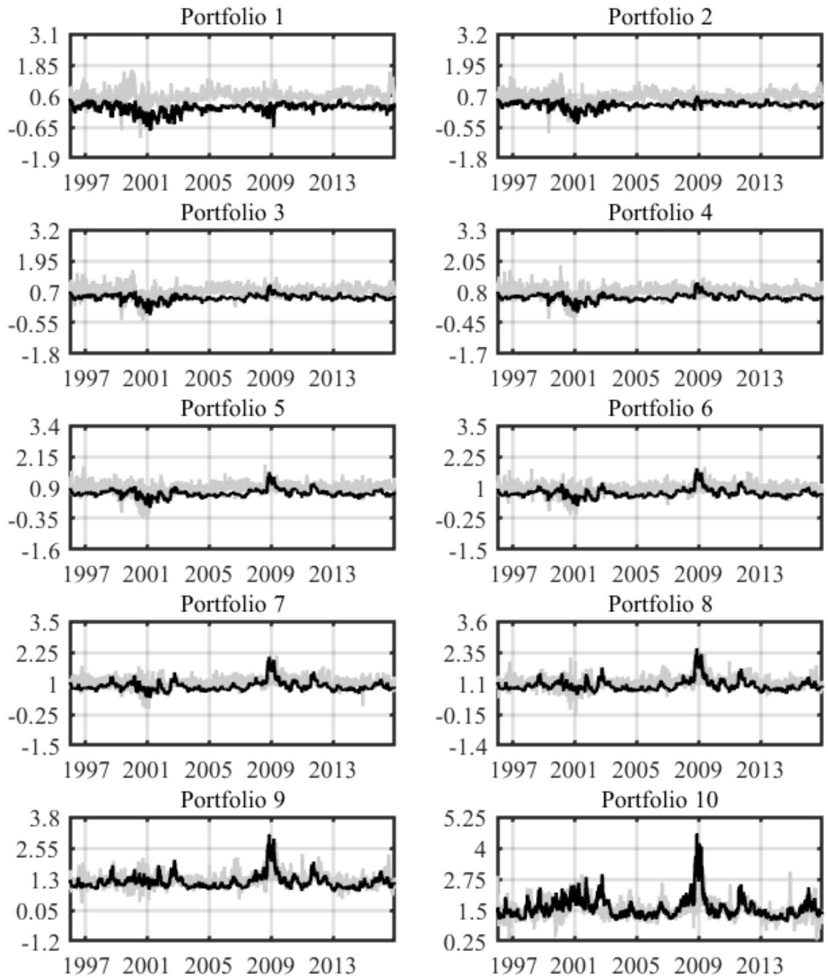

Figure 3

One-month OLS beta (grey) and one-month expected integrated stochastic beta (black)

We plot the value-weighted average of the one-month OLS and stochastic expected integrated betas over time for decile portfolios of stocks. On each day, we sort stocks into decile portfolios based on the model's one-month beta forecast (i.e., 21-day expected integrated physical beta). We then calculate the daily value-weighted average of the OLS and model betas for each portfolio. Portfolio 1 corresponds to the low-beta portfolio while portfolio 10 corresponds to the high-beta portfolio.

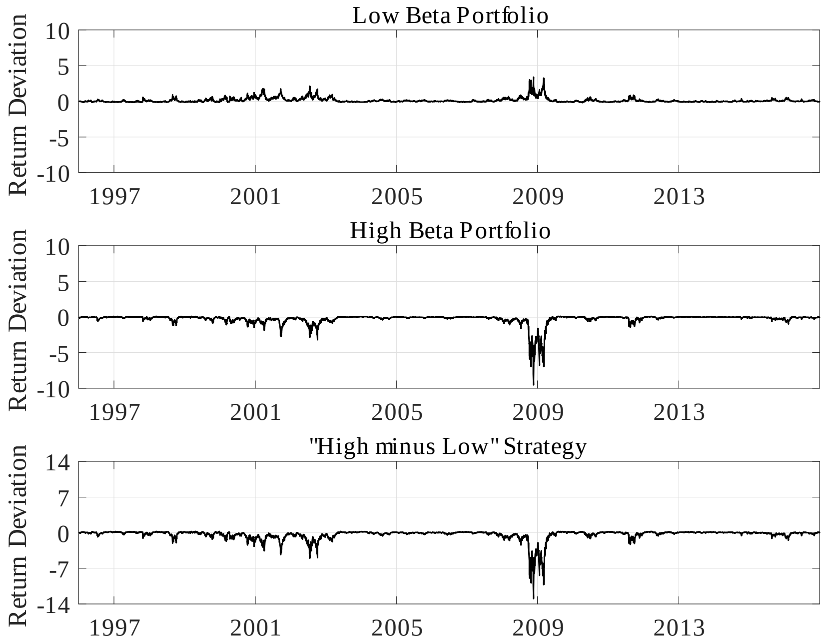

Figure 4

Monthly conditional beta return premia

We plot the time-series of the daily value-weighted average of monthly beta return premiums for various beta-sorted portfolios. On each day, we sort stocks into decile portfolios based on the model one-month beta forecast (i.e., 21-day expected integrated physical beta). We consider three portfolios, the low, and the high beta-sorted portfolios, and a high-minus-low beta portfolio, respectively. For each stock in a given portfolio, we compute $RP_{t,21}^{BRP} \times 100$ on each day t setting r to its sample mean of 2.34% and Λ^I to 1.77. We then calculate the daily value-weighted average of the stock-level one-month beta return premiums for each portfolio, and plot the results obtained.

Table 1
Summary statistics for daily excess portfolio returns

Portfolio	Mean (%)	Standard Deviation (%)	S&P 500			Portfolio of All Stocks	t-Stat. of Difference in Betas	Market Cap. ($ Billions)
			R-Squared (%)	Correlation	Beta	Beta		
Portfolio of All Stocks	9.59	19.15	98.90	0.99	0.98	1.00	-0.51	24.83
1. Low Beta	8.71	15.04	46.81	0.68	0.53	0.55	-0.99	21.49
2.	8.36	15.64	58.45	0.76	0.62	0.64	-1.14	23.86
3.	8.60	17.54	70.01	0.84	0.76	0.78	-1.22	32.05
4.	10.53	19.75	69.58	0.83	0.85	0.87	-0.88	32.16
5	10.90	20.26	81.30	0.90	0.94	0.96	-1.57	17.44
6	10.21	22.28	79.77	0.89	1.03	1.04	-0.68	13.56
7	11.47	23.35	80.76	0.90	1.08	1.10	-0.92	18.24
8	11.26	23.68	83.98	0.92	1.12	1.14	-1.41	36.77
9	11.73	27.97	80.08	0.89	1.29	1.31	-0.78	26.22
10. High Beta	11.36	32.30	80.14	0.90	1.49	1.50	-0.30	26.20

The table reports various summary statistics for excess portfolio returns. We consider the value-weighted portfolio composed of all stocks and decile portfolios of stocks sorted unconditionally on their sample OLS betas with respect to S&P 500 returns. The first two columns report the sample mean and volatility of excess returns annualized. Columns three to five report the OLS R-squared, correlation, and beta of each portfolio estimated by regressing daily excess portfolio returns on daily excess S&P 500 returns over the entire sample. In column six, we report an alternative measure of beta for each portfolio estimated by OLS using the value-weighted portfolio of all stocks as a factor. Column 7 presents the t-statistics of the difference in betas calculated using Newey-West methodology with 5 autocorrelation lags. In the last column, we report the average market capitalization of the stocks constituting each portfolio in $ billions. For each stock, the market capitalization is held constant over the month and updated on the last trading day of each month. Each portfolio is value-weighted. The sample period is from January 8th, 1996 to December 30th, 2016.

Table 2

Parameters, unconditional risk premiums, and model fit

Panel A: Model Parameters, Unconditional Second Moments, and Unconditional Risk Premiums

	γ	K_I	Q_I^1	Q_I^2	ρ	λ^{σ_I}	θ_I	ρ_I	VRP_I
Index	6.392	1.802	0.145	0.045	-0.640	-3.894	0.041	-0.611	-0.018

Portfolio	K_S	K_{SI}	Q_S^1	Q_S^2	θ_S	θ_{SI}	β	ρ_S	VRP_S	CRP	BRP
1. Low Beta	1.585	-0.786	0.081	-0.085	0.069	0.025	0.600	-0.330	-0.001	-0.007	0.039
2.	1.248	-0.413	0.099	-0.098	0.079	0.027	0.652	-0.409	-0.001	-0.007	0.062
3.	1.236	-0.138	0.086	0.012	0.084	0.029	0.720	-0.333	-0.009	-0.012	-0.007
4.	1.226	0.103	0.102	0.014	0.080	0.031	0.764	-0.404	-0.010	-0.013	0.001
5.	0.941	0.121	0.117	-0.057	0.106	0.032	0.780	-0.441	-0.006	-0.010	0.051
6.	0.931	-0.087	0.115	-0.039	0.114	0.036	0.892	-0.438	-0.010	-0.013	0.029
7.	1.000	-0.157	0.115	-0.044	0.130	0.036	0.879	-0.398	-0.010	-0.013	0.038
8.	1.005	0.179	0.125	0.026	0.140	0.041	1.016	-0.389	-0.021	-0.018	-0.020
9.	0.939	-0.052	0.122	-0.009	0.157	0.041	1.016	-0.392	-0.017	-0.017	0.005
10. High Beta	0.808	0.129	0.153	-0.050	0.161	0.047	1.149	-0.491	-0.016	-0.017	0.045

Panel B: Goodness of Fit

	Return		Option	
	Log-likelihood	R-Squared (%)	Log-likelihood	IVRMSE (%)
Index	17 004	100.00	51 047	3.50
Portfolio				
1. Low Beta	16 778	23.38	40 754	5.42
2.	16 817	23.92	41 864	5.31
3.	16 319	26.48	39 429	5.53
4.	16 418	33.20	41 569	5.25
5.	16 016	25.03	37 290	6.42
6.	15 588	30.60	36 231	6.35
7.	15 125	26.18	36 682	6.59
8.	15 442	34.17	33 164	7.36
9.	14 893	28.85	30 765	8.20
10. High Beta	15 120	36.02	28 610	9.22

The table reports parameter estimates, unconditional risk premiums, and goodness of fit measures for the market index and portfolios of stocks sorted on sample OLS betas. Based on this sorting, we calculate the value-weighted average of the stock-specific parameters, risk premiums, and measures of fit by portfolio. Panel A presents the parameters for return and variance dynamics and risk premiums. We adopt a two-step procedure to estimate the model. In the first step, we estimate the market parameters. In the second step, we estimate the equity parameters for each stock paired with the S&P500 index setting the market parameters to the values obtained in the first step. For each step, the parameters are estimated by maximizing the composite log-likelihoods of returns and options over the 1996-2014 sample period. We construct measures of unconditional risk premium as follow. For a given risk measure (i.e. variance, covariance, or beta), the premium is defined as the difference between physical and risk-neutral expectations. For each firm and the market index, we compute the 252-day expected integrated risk measures under P and Q for the 1996-2016 sample period given the filtered latent variables, and take the difference. For equities, we then calculate the value-weighted average of the daily stock-level premiums. The table reports the sample average of these measures. Panel B reports various goodness of fit measures calculated over the 1996-2016 sample including log-likelihood values of returns and options, R-squared, and IVRMSE. Note that the return log-likelihoods for equities correspond to the conditional log-likelihood (i.e., joint log-likelihood of a given pair of equity and market index minus market index return log-likelihood).

Table 3
Forecasting realized beta

Model: $\beta^{OLS}_{t+22,21} = a_{cte} + a_{SB} \times \beta^{SB}_{t,21} + a_{OI} \times \beta^{OI}_{t,21} + a_{OLS} \times \beta^{OLS}_{t,21} + \varepsilon_{t+22,21}$

Portfolio	a_{cte}	*t-Stat.*	a_{SB}	*t-Stat.*	a_{OI}	*t-Stat.*	a_{OLS}	*t-Stat.*	Adjusted R-Squared (%)
1. Low Beta	0.178	*3.78*	0.353	*5.95*	0.172	*4.15*	0.505	*7.66*	36.33
2.	0.158	*4.06*	0.305	*4.49*	0.090	*3.69*	0.559	*15.77*	39.42
3.	0.232	*5.13*	0.236	*3.29*	0.024	*1.33*	0.559	*17.03*	39.46
4.	0.277	*6.33*	0.163	*2.76*	-0.012	*-0.72*	0.596	*16.25*	41.26
5.	0.337	*6.79*	0.104	*2.56*	-0.024	*-1.48*	0.582	*14.42*	38.65
6.	0.396	*8.71*	0.128	*4.80*	-0.002	*-0.10*	0.489	*11.09*	29.71
7.	0.309	*7.24*	0.161	*6.68*	0.036	*2.27*	0.515	*12.65*	36.97
8.	0.273	*6.52*	0.203	*6.99*	0.084	*5.43*	0.471	*13.77*	38.70
9.	0.231	*4.84*	0.294	*7.99*	0.124	*7.43*	0.381	*10.10*	43.77
10. High Beta	0.107	*1.66*	0.356	*8.09*	0.171	*5.40*	0.329	*7.48*	47.99

The table presents the loadings and t-statistics from daily portfolio-level regressions of one-month future realized beta on our model expected stochastic beta controlling for option-implied and lagged OLS betas. On each day, we obtain future realized beta for each stock by estimating the CAPM regression using the 21-day-ahead index and stock excess returns. The 21-day forecast from our stochastic model on a given day is computed using the filtered conditional latent variables from the previous day. For option-implied betas, we follow Chang, Christoffersen, Jacobs, and Vainberg (2012) and construct daily measures of beta for each stock. On each day, we sort stocks into decile portfolios based on the model one-month expected stochastic beta. For a given beta measure, we then calculate the value-weighted average of the stock-level measures to obtain one single estimate for each portfolio on each day. The t-statistics (in italics) are calculated using Newey-West methodology with 21 lags. The sample period is from January 8th, 1996 to December 30th, 2016.

Table 4
One-month conditional beta risk premiums

Portfolio	Stochastic Beta Forecast $\beta_{t,21}^{SB}$	*t-Stat.*	Model-Free $\beta_{t,21}^{OLS}$ - $\beta_{t,21}^{OI}$	*t-Stat.*	Model-Implied $\beta_{t,21}^{SB}$ - $\beta_{t,21}^{SB,Q}$	*t-Stat.*
1. Low Beta	0.120	*11.33*	-0.514	*-26.89*	-0.231	*-40.31*
2.	0.338	*43.05*	-0.405	*-23.02*	-0.139	*-36.96*
3.	0.461	*60.16*	-0.336	*-19.68*	-0.104	*-33.03*
4.	0.566	*69.46*	-0.270	*-16.71*	-0.076	*-23.41*
5.	0.665	*72.69*	-0.225	*-13.58*	-0.049	*-14.46*
6.	0.766	*73.61*	-0.184	*-11.16*	-0.023	*-5.73*
7.	0.876	*72.81*	-0.122	*-7.69*	0.004	*0.96*
8.	1.006	*71.77*	-0.065	*-3.87*	0.039	*6.92*
9.	1.183	*69.31*	0.002	*0.10*	0.087	*12.51*
10. High Beta	1.580	*61.44*	0.138	*6.46*	0.200	*17.86*

The table presents the sample average of daily one-month integrated stochastic betas, and daily model-free and model-implied beta risk premiums by portfolio. Beta risk premium is defined as the difference between physical and risk-neutral expected beta. On each day, we obtain model-free measures of physical beta for a horizon of 21 days by estimating the CAPM regression using the most recent 21-day index and stock excess returns. To obtain data-based measures of risk-neutral beta, we follow Chang, Christoffersen, Jacobs, and Vainberg (2012) and compute option-implied betas from index and equity risk-neutral moments. We use options of maturity of one-month to construct 21-day measures of option-implied beta. The model conditional beta risk premium corresponds to the difference between the physical and risk-neutral expectations of integrated stochastic beta. The model-free beta risk premium is defined as the difference between OLS and option-implied betas. On each day, we sort stocks into decile portfolios based on the model one-month expected integrated physical beta. We then construct one single daily measure of conditional beta risk premium (model-free or model-implied) for each portfolio by value-weighting stock-level measures. The t-statistics (in italics) are calculated using Newey-West methodology with 21 lags. The sample period is from January 8th, 1996 to December 30th, 2016.

Table 5

Predictive cross-sectional regressions. Various horizons, and specifications

Panel A: Multivariate Regressions Based on Model Physical Beta

Dependent Variable: $R^S_{t+1,h}$

	Weekly h = 5 days		Monthly h = 21 days		Quarterly h = 63 days	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.0018	*3.72*	0.0058	*3.05*	0.0137	*2.11*
time-t h- day Expected Integrated Physical Beta	0.0018	*3.91*	0.0042	*2.32*	0.0073	*1.31*
time-t h- day OLS Beta	0.0000	*0.06*	0.0016	*1.19*	0.0050	*0.78*
time-t h- day Option-Implied Beta	-0.0006	*-2.51*	-0.0003	*-0.43*	0.0053	*1.22*
R-Squared (%)	7.94		8.20		9.76	

Panel B: Multivariate Regressions Based on Model Risk-Neutral Beta

Dependent Variable: $R^S_{t+1,h}$

	Weekly h = 5 days		Monthly h = 21 days		Quarterly h = 63 days	
	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*	Coeff.	*t-Stat.*
Intercept	0.0012	*2.26*	0.0043	*2.14*	0.0102	*1.40*
time-t h- day Expected Integrated Risk-Neutral Beta	0.0024	*3.70*	0.0059	*2.35*	0.0138	*2.04*
time-t h- day OLS Beta	0.0001	*0.50*	0.0017	*1.35*	0.0038	*0.62*
time-t h- day Option-Implied Beta	-0.0006	*-2.57*	-0.0003	*-0.41*	0.0050	*1.17*
R-Squared (%)	7.94		8.23		9.56	

The table shows the results of cross-sectional Fama-MacBeth predictive regressions. On each day, we regress future realized excess stock returns on model expected integrated physical betas in Panel A, and on model expected integrated risk-neutral betas in Panel B. In both panels, we further control for OLS and option-implied betas. The table reports the sample average of the daily coefficients, their t-statistics, and the average of the regression R-squared. We consider three horizons. For a given horizon of h days, we compute future realized excess stock returns on each day by compounding the h-day-ahead daily excess returns. The daily stochastic beta forecasts (i.e., the h-day expected integrated physical and risk-neutral betas) are calculated based on the latent variables filtered on day t. The daily OLS beta measures for a given horizon are obtained by regressing past excess stock returns on S&P 500 excess returns using an estimation window of length equal to the horizon considered. To construct option-implied betas, we follow Chang, Christoffersen, Jacobs, and Vainberg (2012) and construct daily measures of risk-neutral beta for each stock. Because of data limitations, we use one-month option implied betas to predict one-week ahead stock excess returns. We use one-month and three-month option-implied betas for the 21 and 63-day forecast horizons, respectively. The t-statistics (in italic) are calculated using the Newey-West methodology allowing for h autocorrelation lags. The sample period is from January 8th, 1996 to December 30th, 2016.

Table 6

One-month conditional beta return premiums

Portfolio	$\Lambda^I \int_t^{t+\frac{21}{252}} cov_t(\beta_u; \sigma_{I,u}^2) du \times 100$	*t-Stat.*	$-r \times (\beta_{t,21} - \beta_{t,21}^Q) \times 100$	*t-Stat.*	$RP_{t,21}^{BRP} \times 100$	*t-Stat.*	$RP_{t,21}^{BRP} \times 100 \times 12$
1. Low Beta	0.0997	*5.26*	0.0451	*40.31*	0.1447	*7.52*	1.7366
2.	0.0474	*4.67*	0.0271	*36.96*	0.0745	*7.15*	0.8942
3.	0.0229	*3.42*	0.0204	*33.03*	0.0433	*6.17*	0.5197
4.	0.0057	*0.97*	0.0148	*23.41*	0.0205	*3.27*	0.2461
5.	-0.0117	*-1.60*	0.0097	*14.46*	-0.0020	*-0.26*	-0.0239
6.	-0.0284	*-2.93*	0.0044	*5.73*	-0.0240	*-2.35*	-0.2880
7.	-0.0471	*-3.59*	-0.0009	*-0.96*	-0.0480	*-3.49*	-0.5763
8.	-0.0692	*-3.97*	-0.0076	*-6.92*	-0.0767	*-4.21*	-0.9209
9.	-0.1009	*-4.30*	-0.0170	*-12.51*	-0.1179	*-4.80*	-1.4148
10. High Beta	-0.2155	*-5.18*	-0.0391	*-17.86*	-0.2546	*-5.87*	-3.0552
H-L	-0.3152	*-5.33*	-0.0841	*-31.43*	-0.3993	*-6.54*	-4.7918

The table presents the sample average of the daily model-implied one-month beta return premium and of its components by portfolio. For each firm on each day, we calculate the model one-month expected integrated covariance between beta and market variance, the negative of the risk-free rate times the one-month beta risk premium, and the sum of the two (i.e., the one-month beta return premium). Each day, we sort stocks into decile portfolios based on the model one-month expected integrated physical beta. We then construct daily portfolio measure of these variables by taking the value-weighted average of the stock-level measures. The t-statistics (in italics) are calculated using Newey-West methodology with 21 lags. The sample period is from January 8th, 1996 to December 30th, 2016.

Table 7
Value-weighted decile portfolio sorting results for all NYSE stocks

Portfolio	Ex-ante Beta	Ex-ante minus Ex-Post Betas	$\beta^{H,Ret}_{t+253,252}$-$\beta^{L,Ret}_{t+253,252}$ (High and Low based on Average Market Return)	$\beta^{H,Var}_{t+253,252}$-$\beta^{L,Var}_{t+253,252}$ (High and Low based on Median Squared-Market Return)	$\beta^{H,Var}_{t+253,252}$-$\beta^{L,Var}_{t+253,252}$ (High and Low based on Average VIX)	Ex-Post Annual Abnormal Return	CAPM Ex-post Alpha	FFC Ex-Post Alpha
1. Low	0.35	-0.15	-2.66%	5.18%	4.15%	1.96%	1.92%	1.95%
2.	0.55	-0.09	-1.24%	3.27%	2.72%	2.61%	2.54%	2.24%
3.	0.68	-0.07	-2.75%	2.83%	2.54%	1.36%	1.80%	0.94%
4.	0.79	-0.04	-2.41%	1.95%	2.14%	0.41%	1.32%	0.28%
5.	0.89	-0.03	0.24%	3.11%	3.01%	0.17%	1.32%	0.21%
6.	0.99	-0.01	1.19%	0.54%	2.00%	-0.01%	1.28%	0.53%
7.	1.10	0.03	1.39%	-0.50%	-0.03%	-1.16%	0.35%	-0.69%
8.	1.24	0.06	3.72%	0.68%	-0.10%	-1.88%	-0.28%	-1.51%
9.	1.42	0.10	6.93%	-0.80%	-0.90%	-3.23%	-1.33%	-2.91%
10. High	1.81	0.23	6.79%	-10.60%	-4.31%	-3.20%	-2.62%	-4.58%
H-L	1.46	0.37	9.45%	-15.77%	-8.46%	-5.16%	-4.55%	-6.54%
t-Stat.	37.48	6.84	2.01	-2.73	-2.48	-1.71	-1.57	-1.77

Each month, we sort stocks into decile portfolios based on ex-ante betas obtained by regressing daily stock excess returns on daily market excess returns from the last 252 trading days. In the first column, we report the value-weighted average ex-ante betas for each portfolio. In the second column, we report the value-weighted average of the difference between ex-ante and ex-post betas, where the ex-post betas are obtained by regressing daily excess stock returns against daily excess market returns during the 252 days following the sorting. In the third column, we report the difference between a high and low market return ex-post beta (times 100), which we calculate by regressing excess stock returns against excess market returns during the next year for above- and below-average market return days, separately. In the fourth column, we report the difference between high and low squared market return ex-post betas (times 100), which we calculate by regressing excess stock returns against excess market returns during the next year for above- and below-median market squared return days separately. In the fifth column, we report the difference between high and low ex-post betas (times 100), where we use average VIX to identify high and low market variance days. In the sixth column, the ex-post annual abnormal returns are obtained for each stock by taking the difference between the compounded daily excess equity return over the next year and the product of ex-post beta and the compounded daily excess market return. The value-weighted abnormal returns are subsequently calculated for each portfolio. Finally, we report CAPM and Fama-French-Carhart (FFC) alphas for each portfolio as well as for high minus low where the alphas are estimated over the full sample. The t-statistics are from Newey-West using 12 lags. The sample period is from January 8th, 1996 to December 30th, 2016.