Decreasing aversion under ambiguity

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The wealth effect in the maxmax portfolio problem

(Appendix not for publication)

In this Appendix, we present a short analysis of the portfolio choice problem when the investor has a maxmax preference functional:

$$\max_{\alpha} \max_{\theta} Eu(z + \alpha\widetilde{x}_{\theta}). \tag{27}$$

In the family of $\alpha-$MEU preferences, this model is the polar one to the maxmin criterion. Of course, u DC is necessary to guarantee that α is increasing in z in intervals of wealth levels where the argument of the maximum of $Eu(z+\alpha\widetilde{x}_{\theta})$ with respect to θ does not change. Because the objective function is not concave with respect to the decision variable α, we also need to take care of the possible bifurcations. In Figure 4, we describe a situation in which the demand for the risky asset goes discountinuously down from $\alpha(z)$ to $\alpha(z') < \alpha(z)$ when wealth goes up from z to z'.

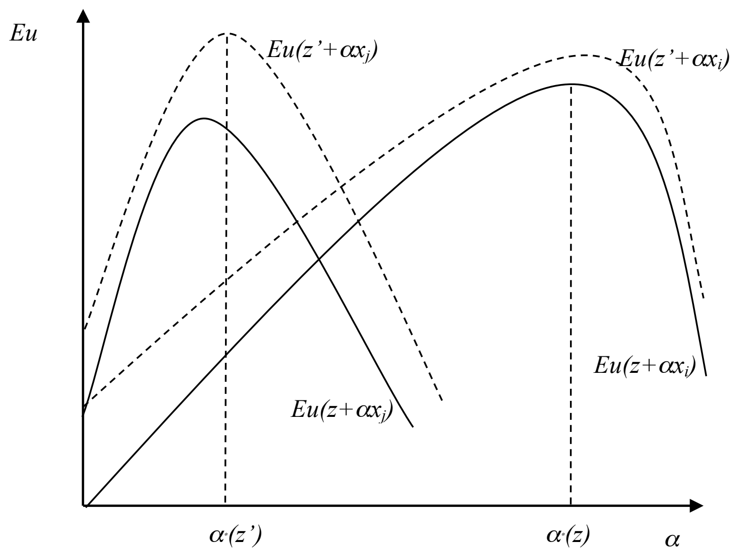

Figure 4: A bifurcation yielding a decreasing demand for the risky asset in the maxmax portfolio model.

We first show that bifurcations never occur in the case of HARA utility functions defined by (21). Let us define

$$V_\theta(z) = \max_\alpha Eu(z + \alpha\widetilde{x}_\theta)$$

The absence of bifurcation comes from the property that, for all $(i,j) \in \{1,...,n\}^2$,

$$V_i(z) \geq V_j(z) \implies \forall z' > -\eta\gamma : \quad V_i(z') \geq V_j(z').$$

In other words, the θ that maximizes expected utility with the optimal portfolio never switches in the case of HARA preferences (21). To show this, let us observe that conditional to θ, the expected-utility-maximizing investment in the risky asset is linear in the wealth level z. The first-order condition to the maximization of $Eu(z + \alpha\widetilde{x}_\theta)$ can be written as follows:

$$E\widetilde{x}_\theta \left(\eta\gamma + z + \alpha_\theta^*(z)\widetilde{x}_\theta\right)^{-\gamma} = 0.$$

Let us define α_θ^* as the unique root of the follwoing equation:

$$E\widetilde{x}_\theta \left(1 + \alpha_\theta^*\widetilde{x}_\theta\right)^{-\gamma} = 0.$$

By comparing the last two equations, it is immediate that the optimal solution conditional to θ is $\alpha_\theta^*(z) = \alpha_\theta^*(\eta\gamma + z)$ for all $z > -\eta\gamma$. We can now compute $V_\theta(z)$. We obtain:

$$V_\theta(z) = \xi E\left(\frac{(\eta\gamma + z)(1 + \alpha_\theta^*\widetilde{x}_\theta)}{\gamma}\right)^{1-\gamma} = (\eta\gamma + z)^{1-\gamma} v_\theta^*,$$

with

$$v_\theta^* = \xi E\left(\frac{1 + \alpha_\theta^*\widetilde{x}_\theta}{\gamma}\right)^{1-\gamma}.$$

It implies that the $(V_1(z), ..., V_n(z))$ can be ordered in the same way as $(v_1^*, ..., v_n^*)$, which is independent of z. This implies in particular that the largest element in $(V_1(z), ..., V_n(z))$ is independent of z. Hence, there is no bifurcation. This concludes the proof of the following result.

Proposition 8 *Consider the maxmax portfolio problem (27) with a HARA utility function u. In this framework, there is no bifurcation when wealth* *increases in the sense that the θ that maximizes the expected utility along the optimal portfolio strategy is independent of z. This implies that u DC is sufficient to guarantee that the demand for the risky asset is increasing with wealth in the maxmax-HARA portfolio model.*

One can easily find a counterexample where the bifurcation yields a downward jump in the demand for the risky asset when wealth increases in spite of DC. Consider the DC utility function $u(c) = c - k_1 e^{-k_2 c}$ with $k_1 = 10$ and $k_2 = 1$. Consider an ambiguous situation with two possible priors: $\widetilde{x}_1 \sim (-1, 1/2; 2, 1/2)$ and $\widetilde{x}_2 \sim (-1, 1/3; 1, 2/3)$. The demand for the risky asset as a function of wealth is represented in Figure 5. This illustrates the fact that DARA is not sufficient for a monotone relationship between wealth and the optimal exposure to risk in the maxmax model.

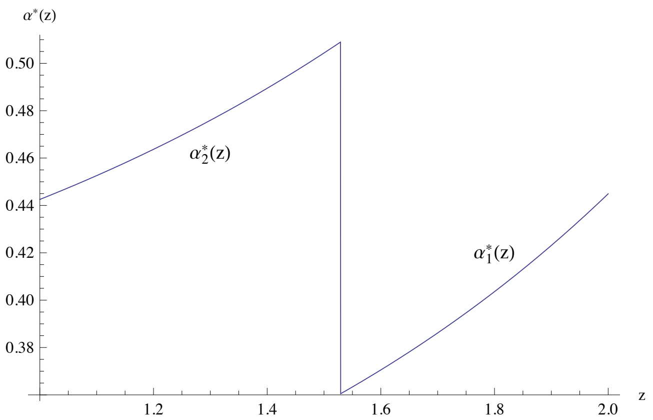

Figure 5: The demand for the risky asset as a function of wealth in the maxmax model.

Let us now characterize bifurcations. A bifurcation occurs at some wealth level z if two global maxima to problem (27) prevails for that z. Let $\widetilde{y}_\theta$ denote $\alpha^*_\theta \widetilde{x}_\theta$. The following set of conditions is necessary for (α^*_i, α^*_j) to be two global maxima:

$$E\widetilde{y}_i u'(z + \widetilde{y}_i) = 0, \tag{28}$$

$$E\widetilde{y}_j u'(z + \widetilde{y}_j) = 0, \tag{29}$$

and

$$Eu(z + \widetilde{y}_i) = Eu(z + \widetilde{y}_j). \tag{30}$$

A marginal increase in wealth yields a bifurcation from α_j^* to α_i^*iff

$$Eu'(z+\widetilde{y}_i) > Eu'(z+\widetilde{y}_j). \tag{31}$$

Whether this bifurcation is demand-increasing or demand-decreasing depends upon whether $\alpha_i^* \geq \alpha_j^*$ or $\alpha_i^* \leq \alpha_j^*$, respectively. If it is a demand-increasing bifurcation, we can build another ambiguous context with the same utility function in which there is a demand-decreasing bifurcation.

Proposition 9 *Consider the maxmax model with utility function u. Consider an ambiguous context $(\widetilde{x}_1, ..., \widetilde{x}_n)$ with $E\widetilde{x}_\theta > 0$ for all $\theta = 1, ..., n$ such that there is a demand-increasing bifurcation at wealth level z. Then, there exists another ambiguous context $(\widetilde{x}'_1, ..., \widetilde{x}'_n)$ such that there is a demand-decreasing bifurcation at wealth z.*

Proof: Observe first that condition $E\widetilde{x}_\theta > 0$ implies that the local maximum α_θ^* is positive. Suppose that the two local maxima under $(\widetilde{x}_1, ..., \widetilde{x}_n)$ are α_i^* and α_j^*, with $\alpha_i^* > \alpha_j^*$. It implies that $\widetilde{y}_i = \alpha_i^*\widetilde{x}_i$ and $\widetilde{y}_j = \alpha_j^*\widetilde{x}_j$ satisfy conditions (28)-(31), yielding a demand-increasing bifurcation at z. Consider now the alternative ambiguous context $(\widetilde{x}'_1, ..., \widetilde{x}'_n)$ such that $\widetilde{x}'_\theta = \widetilde{x}_\theta$ for all $\theta \neq i$ and $\widetilde{x}'_i = k\widetilde{x}_i$ where k is a positive scalar larger than $\alpha_i^*/\alpha_j^* > 0$. It is then immediate that the pair $(\alpha_i^{*\prime} = \alpha_i^*/k, \alpha_j^*)$ describes two global maxima under the new ambiguous context. Indeed, we have that $\alpha_i^{*\prime}\widetilde{x}'_i = \alpha_i^*\widetilde{x}_i = \widetilde{y}_i$, so that the pair of conditions (28)-(31) is preserved by the joint change of ambiguity context and in (α_i^*, α_j^*). But in the new context, we have that

$$\alpha_i^{*\prime} = \frac{\alpha_i^*}{k} < \frac{\alpha_i^*\alpha_j^*}{\alpha_i^*} = \alpha_j^*,$$

so that the bifurcation is now demand-decreasing. ■

This proposition tells us that as soon as an ambiguity context yields a bifurcation in the demand for the risky asset as some wealth level, we can make it demand-decreasing, thereby violating the desired comparative statics property. We have seen earlier that there is never any bifurcation if u is HARA. In the remainder of this section, we examine the case of small risks.

Lemma 3 *Suppose that there are two local maxima α_i^* and α_j^* to program (27) at wealth level z, and that they are small. This requires that*

$$m_{1\theta} - \alpha_\theta^* m_{2\theta} A + 0.5\alpha_\theta^{*2} m_{3\theta} P = 0 \tag{32}$$

for $\theta = i$ and j, together with

$$\alpha_i^{*2} m_{2i} - \alpha_j^{*2} m_{2j} = \frac{2}{3} \left[\alpha_i^{*3} m_{3i} - \alpha_j^{*3} m_{3j} \right] P, \tag{33}$$

where $m_{k\theta}$ is the kth moment of $\widetilde{x}_\theta$ and A and P are respectively the absolute risk aversion and the absolute prudence evaluated at wealth level z. A marginal increase in z implies a bifurcation from α_j^* to α_i^* if and only if

$$\left[\alpha_i^{*2} m_{2i} - \alpha_j^{*2} m_{2j} \right] [A + T - 2P] \leq 0, \tag{34}$$

where $T = -u''''(z)/u'''(z)$ is the index of absolute temperance.

Proof: Condition (32) is obtained from (28) and (29) via second-order Taylor expansion $u'(z+\alpha x)$ around z. Using third-degree Taylor expansions, condition (30) can be rewritten as follows:

$$\alpha_i^* m_{1i} - \frac{1}{2}\alpha_i^{*2} m_{2i} A + \frac{1}{6}\alpha_i^{*3} m_{3i} AP = \alpha_j^* m_{1j} - \frac{1}{2}\alpha_j^{*2} m_{2j} A + \frac{1}{6}\alpha_j^{*3} m_{3j} AP.$$

Replacing m_{1i} and m_{1j} in this equation by their expression derived from (32) yields condition (33). Using the same method for inequality (31) yields

$$\left[\alpha_i^{*2} m_{2i} - \alpha_j^{*2} m_{2j} \right] \left[A - \frac{1}{2} P \right] \leq \left[\alpha_i^{*3} m_{3i} - \alpha_j^{*3} m_{3j} \right] \left[\frac{A}{2} - \frac{T}{6} \right] P.$$

Using condition (33) to eliminate $\alpha_i^{*3} m_{3i} - \alpha_j^{*3} m_{3j}$ in this inequality yields condition (34). ■

Observe that in the HARA case, $A + T - 2P$ is uniformly zero, so that if there are two global maxima to program (27) for some z, this is the case for all z. With a non-HARA utility function, whether bifurcations are compatible with an increasing demand for the risky asset depends upon a complex condition linking the signs of $A + T - 2P$ and of $\alpha_i^{*2} m_{2i} - \alpha_j^{*2} m_{2j}$, where α_i^* and α_j^* are defined by (32) under constraint (33). Notice that skewness is important. On the contrary, if $\widetilde{x}_i$ and $\widetilde{x}_j$ are such that $m_{3i} = m_{3j} = 0$ implies that $m_{2i} = m_{2j}$ by condition (33) to guarantee that the two local maxima are also global maxima. But this latter condition implies that condition (34) is satisfied as an equality, which means that there is no bifurcation.