Competitive Screening under Heterogeneous Information

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First Version: November, 2013.
This Version: October, 2018.

Abstract

We study the interplay between informational frictions and second-degree price discrimination. Our theory recognizes that consumers differ in their tastes for quality as well as in the information they possess about available offers, which leads to dispersion over price-quality menus in equilibrium. While firms are ex-ante identical, we show that their menus are ordered so that more generous menus leave more surplus to consumers of all valuations. We explore the cross-section of equilibrium menus and variations in market conditions to generate empirical predictions on prices, qualities and markups across firms, and within a firm's product line. For instance, more competition may raise prices for low-quality goods; yet, consumers are better off, as the qualities they receive also increase. The predictions of our model illuminate empirical findings in many markets, such as those for cell phone plans, yellow-pages advertising, cable TV and air travel.

JEL Classification: D82

Keywords: competition, screening, heterogeneous information, price discrimination, adverse selection

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1 Introduction

Consumer information is a key determinant of market outcomes. As long recognized by economists, informational frictions, preventing consumers from properly comparing all available options, are a major source of firms' market power.[1] Recent empirical work documents that consumers are imperfectly and heterogeneously informed about firms' offers in many markets,[2] and points to the importance of accounting for informational frictions when estimating consumer demand and performing welfare analysis.[3] Recent developments in the way consumers shop also call for a better understanding of the impact of informational frictions on firms' behavior. Arguably, with the surge in online shopping and the advent of comparison websites, consumers have access to (and knowledge of) a greater number of firms.[4]

An important feature of many product markets is the practice of price discrimination, whereby firms screen consumers' (unobservable) tastes by offering goods with different combinations of quality and price.[5] The goal of this paper is to study the interplay between informational frictions and price discrimination. Specifically, how do improvements in market transparency affect pricing and quality provision across firms and within a firm's product line? How are different consumers affected by changes in market transparency?

To answer these questions, we embed a canonical price discrimination problem (à la Mussa and Rosen (1978)) into an imperfect information framework. Our theory recognizes that consumers differ in their tastes for quality as well as in the information they possess about available offers. Firms then have to design their menu of products accounting for consumers' choices of which firm to patronize (among those in one's information set), *and* which product to buy (within the firm menu). Crucially, a firm's choice of how much utility to leave to one type of consumer (which determines the level of sales to that type) affects its ability to provide quality to other types (as incentive constraints have to be satisfied). This interdependency makes the firms' offers differ on *both* price and quality provision, which work as substitute instruments for increasing sales. As described below, our theory generates a number of testable predictions, and unveils different patterns of firm behavior regarding pricing and quality provision under competition.

[1] Early theoretical work making this point includes Butters (1977), Varian (1980) and Burdett and Judd (1983).

[2] Draganska and Klapper (2011) provide evidence in the market for ground coffee, Honka et al (2017) for banking services, Moraga-González et al (2017) for cars, Sengupta and Wiggins (2014) for airline tickets, and Clark et al (2009) for 25 broad categories of goods.

[3] In this regard, see, for instance, Sovinsky Goeree (2008), Chandra and Tappata (2011), Honka (2014) and Moraga-González et al (2017).

[4] See, for instance, Ellison and Ellison (2009).

[5] Common examples include flight tickets with different terms and conditions, cell phone plans with increasing usage allowances, and television subscriptions in basic and premium versions. The French engineer Jules Dupuit provided perhaps one of the first explanations of how menus of vertically differentiated products serve to screen consumer tastes - see Dupuit (1849).

Our framework departs from existing theoretical work on second-degree price discrimination, which assumes that consumers enjoy *perfect* (and therefore homogeneous) information about the offers available in the market. This literature postulates that imperfect competition is due to product differentiation; see Champsaur and Rochet (1989), Rochet and Stole (1997, 2002) and Armstrong and Vickers (2001).[6] Our focus on *heterogeneously* informed consumers also stands in contrast to most of the large empirical literature on product differentiation which follows the tradition of Berry, Levinsohn and Pakes (1995). Although a growing empirical literature now investigates the role of informational frictions on price and quality provision, theoretical work on competitive price discrimination with heterogeneously informed consumers has to date been missing.

Model and Results

To isolate the effects of information heterogeneity on competition, we assume that consumer tastes only differ with respect to their valuation for quality (their "type"). That is, consumers have no "brand" preferences, and so evaluate the offers of different firms symmetrically. An offer is a menu of products, consisting of different combinations of price and quality.

We model information heterogeneity in the baseline model following the sample-size search framework of Burdett and Judd (1983); accordingly, each consumer observes a random sample of menus offered by the competing firms, and purchases from the menu with the most attractive product among those observed. As described later, our findings extend to a much broader class of models of informational frictions, including the urn-ball model of Butters (1977) and the on-the-job search model of Burdett and Mortensen (1998). Our analysis is amenable to comparative statics over the degree of informational frictions, spanning the entire spectrum of competitive intensity, from perfect competition to monopoly.

Similarly to the aforementioned works, we consider ex-ante identical firms and anticipate that they will make different offers in equilibrium. An equilibrium is then a distribution over menus such that every menu in its support is a profit-maximizing response to that distribution. As consumer preferences are private information, the menus offered by firms have to satisfy the self-selection (or incentive) constraints inherent to price discrimination.

It is useful to note that, were consumer preferences public information, firms would provide quality efficiently to consumers of all valuations. The reason is that, holding constant consumer payoffs, firms' profits would increase by setting qualities at their efficient levels. Under asymmetric information, however, a firm's choice of how much utility to leave to one type of consumer affects its ability to provide quality to other types, as incentive constraints must be satisfied. In the case of a monopoly, the relevant incentive constraint is the one that prevents high-valuation consumers from buying the low-quality product (which results in under-provision of quality to low-valuation consumers). As we shall see, under competition, the relevant incentive constraint remains the same;

[6] See also Stole (2007) for a comprehensive survey.

however, firms in equilibrium differ in the indirect utilities offered to consumers of both low and high valuations (as, in contrast to the monopoly case, both individual rationality constraints are slack).

A key step in our analysis is then to express incentive constraints and firms' profits in terms of the indirect utilities offered to consumers. This allows us to establish a crucial property of a firm's profit function: It satisfies *increasing differences* in the indirect utility left to consumers of low and high valuations. To understand why, note that leaving more indirect utility to consumers with high valuations relaxes (downward) incentive constraints, which enables firms to raise the quality of the low-quality product. This, in turn, increases the firms' marginal profit associated with increasing the indirect utility left to low-valuation consumers, as marginal sales generate greater surplus.[7]

Building on this monotonicity property, we characterize an equilibrium of this economy, which, under mild qualifications, is the unique one. This equilibrium, which we call the *ordered equilibrium*, has the property that, for any two menus, one of them leaves more indirect utility to consumers of all valuations. Accordingly, firms sort themselves on how *generous* they are (i.e., how much indirect utility they leave) to all consumers. In equilibrium, all firms expect the same profits, as the less generous ones make fewer sales to consumers of all valuations.

We leverage this characterization to understand how generosity affects quality provision and prices across the product line. A crucial property of equilibrium is that more generous firms leave relatively more indirect utility to consumers of high valuations, for whom sales are more profitable. As a result, as generosity increases, incentive constraints are relaxed, and firms provide quality more efficiently to all consumer types. The effect on prices is more subtle. As one should expect, high-quality goods are cheaper at more generous firms. By contrast, the prices for low-quality goods are non-monotone in generosity: They first increase, reflecting steep quality improvements, but then decrease, as quality approaches its efficient level. This non-monotonicity underscores how firms of varying generosities employ price and quality instruments differently to compete for consumers.[8]

Empirical Implications

To connect our theory to empirical work on price discrimination, we generate testable predictions by analyzing the cross-section of firm behavior and exploring variations in market conditions.

In terms of the cross-section, we find that firms that offer low-quality goods of higher quality charge lower prices for high-quality goods. Moreover, such firms exhibit a lower price differential between high and low-quality goods as well as a smaller difference in markups across their goods. These predictions, which can be tested with cross-sectional data about firms' menus, set our analysis

[7]With more than two types, the profit function satisfies increasing differences in the indirect utility left to consumers of adjacent types. The relaxation of downward-adjacent incentive constraints trickles down from the highest to the lowest type, enabling higher quality provision and increasing the marginal profit from extra sales for each type.

[8]These ideas extend to models with an arbitrary number of types. The new insight from studying such models is that the sets of valuations that play the role of "low" and "high" types, as well as the quality levels deemed "low" or "high," are endogenously determined in equilibrium.

apart from the previous theoretical literature on oligopolistic price discrimination, which restricts attention to equilibria that are ex-post symmetric.[9]

Alternatively, consider an increase in competition, as modeled by consumers observing (probabilistically) larger samples of offers. One interpretation is that competition increases as the market becomes more transparent; for example, due to the emergence of online shopping. Another interpretation is that consumers observe a larger sample of firms because there are more firms in the market (holding constant the level of information frictions).

One effect of a competition increase is that, if competition is not too intense, prices for the low-quality product rise (stochastically). The reason is that low-valuation consumers benefit less from competition than those with high valuations. As a result, more competition relaxes (downward) incentive constraints, which implies that low-valuation qualities sharply increase, and so do the prices charged for the low-quality good.

Another prediction is that, as competition intensifies, markups decrease faster for consumers with high valuations than for those with low valuations. The reason is that, unlike for high-valuation consumers, much (in some cases, all) of the higher payoffs appropriated by low-valuation consumers take the form of higher quality provision, rather than lower prices. These implications can be tested by comparing how the cross-sectional distributions of firms' menus varies across markets that are similar except that competition is more intense in one of them (see, for example, Busse and Rysman (2005)). Another possibility is to use longitudinal data containing the distribution of firms' offers before and after the change in the degree of competition (see, for example, Chu (2010)).

Relatedly, our theory predicts that firms enjoy higher profits per sale of high rather than low-quality goods. This prediction is shared by models of monopolistic price discrimination (e.g., Mussa and Rosen (1978), Maskin and Riley (1984)), but stands in sharp contrast to other models of competition. Armstrong and Vickers (2001) and Rochet and Stole (2002), who study oligopolistic price discrimination in a setting where market power stems from brand preferences, find that markups are constant across the product line when brand preferences are narrowly dispersed. Empirical evidence on oligopolistic price discrimination supports the prediction of our model: Verboven (2002) and Thomassen (2017) document higher markups on premium variants of car models in Europe, while Song (2015) finds higher markups for premium versions of personal computers.

We also assess the impact of changes in consumer tastes on quality provision and price differentials within firms' menus. We show that the quality of firms' low-quality options increases with the proportion of low-valuation consumers, and that the price differentials between firms' high and low quality goods decline.[10] Such effects are also observed when competition in the market increases. Related, we find that the impact of competition on price differentials is stronger in markets with

[9] One notable exception is Champsaur and Rochet (1989), who obtain an equilibrium exhibiting market segmentation by assuming that firms commit to a range of qualities before choosing prices. The literature is reviewed in Section 6.

[10] These effects are also found in the monopoly models of Mussa and Rosen (1978) and Maskin and Riley (1984).

fewer low-valuation consumers. Intuitively, the degree of competition and the fraction of consumers with low valuations work as substitutes for reducing price differentials. These predictions can be tested by comparing markets with different compositions of demand (see, in the context of airline travel, Gerardi and Shapiro (2009), who identify "big-city" routes as having the highest proportion of business, or "high-valuation," passengers).

Although our model is not conceived to describe in detail any specific industry, the predictions above find empirical support in a variety of markets, such as those for cell phone plans, yellow-pages advertising, cable TV, and airline tickets. This is reviewed in Section 4, where we also discuss in detail which markets seems to better fit the assumptions of our model, and comment on what data is needed to further test its implications.

Paper Outline

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes equilibrium, focusing for analytical ease on the space of utilities that consumers derive from equilibrium menus. Section 4 then translates the findings in Section 3 into key testable implications, relates them to existing empirical findings, and discuss some alternatives to further test the model predictions. Section 5 describes important extensions to our baseline model. Section 6 describes related literature, and Section 7 concludes. All proofs are in Appendix A at the end of the document.

2 Model and Preliminaries

The economy is populated by a unit-mass continuum of consumers with single-unit demands for a vertically differentiated good. If a consumer with valuation per quality θ purchases a unit of the good with quality q at a price x, his utility is

$$u(q, x, \theta) \equiv \theta q - x.$$

Consumers are heterogeneous in their valuations per unit of quality: the valuation of each consumer is an iid draw from a discrete distribution with support $\{\theta_l, \theta_h\}$, where $\Delta\theta \equiv \theta_h - \theta_l > 0$, and associated probabilities p_l and p_h, with $p_l, p_h > 0$ (and $p_l + p_h = 1$).[11] Consumers privately observe their valuations per unit of quality. The utility from not buying the good is normalized to zero.

A unit-mass continuum of firms compete by posting *menus* of contracts with different combinations of quality and price. Firms have no capacity constraints and share a technology that exhibits constant returns to scale. The per-unit profit of a firm which sells a good with quality q at a price x is

$$x - \varphi(q),$$

[11] We discuss the cases of multiple discrete types, and a continuum of them, in Section 5, and show in the Supplementary Material that the results are robust to these possibilities.

where $\varphi(q)$ is the per-unit cost to the firm of providing quality q. We assume that $\varphi(\cdot)$ is twice continuously differentiable, strictly increasing and strictly convex, with $\varphi(0) = \varphi'(0) = 0$. Furthermore, we require that $\lim_{q\to\infty} \varphi'(q) = \infty$, which guarantees that surplus-maximizing qualities are interior.

We assume that firms' offers stipulate simply that consumers choose a combination of quality and price from a menu of options. Given the absence of capacity constraints, a consumer is assured to receive his choice. We thus rule out stochastic mechanisms as well as mechanisms which condition on the choices of other buyers or on the offers of other firms.[12] Given our restriction to menus of price-quality pairs, it is without loss of generality to suppose firms' menus include only two pairs: $\mathcal{M} \equiv ((q_l, x_l), (q_h, x_h)) \subset (\mathbb{R}_+ \times \mathbb{R})^2$, where (q_k, x_k) is the contract designed for consumers of valuation θ_k, where $k \in \{l, h\}$.[13] Furthermore, every menu has to satisfy the following incentive-compatibility constraints: For each $k \in \{l, h\}$,

$$IC_k : \qquad u(q_k, x_k, \theta_k) = \max_{\hat{k}\in\{l,h\}} \theta_k q_{\hat{k}} - x_{\hat{k}}.$$

This constraint requires that consumers of valuation θ_k are better off choosing the contract (q_k, x_k) rather than the contract designed for consumers of valuation $\theta_{\hat{k}}$, where $\hat{k} \neq k$.

Contracts which offer negative utility to consumers would never be selected. It is therefore without loss of generality to assume that menus are individually rational (IR), i.e. $u(q_k, x_k, \theta_k) \geq 0$ for each k. According to our convention, a firm that does not want to serve consumers of valuation θ_k offers an incentive compatible menu with contract $(q_k, x_k) = (0, 0)$. A menu $\mathcal{M}$ that satisfies the IC and IR constraints is said to be *implementable*. The set of implementable menus is denoted by $\mathbb{I}$.

One key feature of our analysis is that there is heterogeneity in the information possessed by consumers about the menus offered by firms. To simplify the exposition, we model this heterogeneity in the baseline model according to the simultaneous-search framework of Burdett and Judd (1983). Each consumer observes the menus of a sample of firms independently and uniformly drawn from the set of all firms. For each consumer, the size of the observed sample is $j \in \{1, 2, \ldots\}$ with probability $\omega_j(v)$. Consumers select the best contract among all menus in their samples.

The distribution over sample sizes $\Omega(v) \equiv \{\omega_j(v) : j = 1, 2, \ldots\}$ is indexed by the *degree of competition* $v > 0$.[14] To rule out the possibility that all firms are local monopolists, while ensuring that all firms have some market power, we assume there is some $k > 1$ such that $\omega_j(v) > 0$ for all

[12] There is no loss of generality in considering deterministic mechanisms, provided that one assumes that each consumer can contract with at most one firm. The difficulties associated with stochastic mechanisms in environments where consumers can try firms sequentially (e.g., a consumer might look for a second firm if the lottery offered by the first firm resulted in a bad outcome) are discussed in Rochet and Stole (2002).

[13] Assume a seller offers a menu with more than two price-quality pairs and that at least one type chooses two or more options with positive probability. It is easy to show that there is a menu, with a single option intended for each consumer type, which yields the same payoff to each type but strictly increases the seller's profit (see Lemma 1 below).

[14] For every v, we assume that $\Omega(v)$ has a finite mean.

$j \leq k$. This guarantees that the offer of any given firm is the only one received by *some* consumers, but that *other* consumers observe that as well as other offers.

Improvements in market competitiveness are captured by increasing the degree of competition v. Namely, as v increases, the distribution $\Omega(v)$ increases in the likelihood-ratio order,[15] capturing the idea that consumers are more likely to observe a larger sample of firms. One interpretation is that v represents the level of informational frictions in the market; more transparent (for example, online) markets are described by higher v's. Another interpretation is that v represents the ratio of firms to consumers (holding constant the level of information frictions). This possibility is explored in Section 5, where we describe other matching technologies, and derive conditions under which a greater number of firms in the market implies that consumers observe more offers.

We assume that $\lim_{v\to 0}\omega_1(v) = 1$ (i.e., firms are local monopolies as the degree of competition vanishes), and $\lim_{v\to\infty}\omega_1(v) = 0$ (i.e., consumers are sure to know at least two firms as the degree of competition grows unbounded). For a simple example satisfying these requirements, let $\Omega(v)$ be a shifted Poisson distribution with support on $\{1, 2, \ldots\}$ and mean $v + 1$.[16] As v increases from zero to infinity, our model spans the entire spectrum of competitive intensity.

It is convenient to denote by $\tilde{F}$ the probability measure over menus prevailing in the economy. In intuitive terms, this measure describes the cross-section distribution of menus offered by all firms. This measure, which has support $\mathbb{S}$ contained in the set of implementable menus $\mathbb{I}$, induces, for each k, the marginal distribution over indirect utilities offered to consumers of valuation θ_k:

$$F_k(\tilde{u}_k) \equiv \text{Prob}_{\tilde{F}}\left[\mathcal{M} : u(q_k, x_k, \theta_k) \leq \tilde{u}_k\right].$$

We denote by $\Upsilon_k \subseteq \mathbb{R}_+$ the support of indirect utilities offered to consumers of valuation θ_k, and by f_k the density of F_k, whenever it exists.

The sales a firm expects to make to consumers of valuation θ_k depend on the ranking of the indirect utility that the firm's offer yields to those consumers (relative to other offers in the market).[17] If the type-k contract is (q_k, x_k), then this ranking is $F_k(u_k)$, where $u_k \equiv u(q_k, x_k, \theta_k)$. We define the *sales function* $\Lambda\left(\cdot|v\right)$ such that, if the distribution F_k is continuous at u_k, then any firm offering the contract (q_k, x_k) expects sales $p_k\Lambda\left(F_k(u_k)|v\right)$ to consumers of valuation θ_k. In our baseline model (following Burdett and Judd (1983)), we have, for any $y \in [0, 1]$,

$$\Lambda\left(y|v\right) = \sum_{j=1}^{\infty} j\omega_j(v)y^{j-1} \tag{1}$$

[15]The likelihood-ratio order is commonly used in economics (see Jewitt (1991) for a detailed account). For discrete random variables, we say that X dominates $\hat{X}$ in the sense of the likelihood-ratio order if $\frac{\rho(x)}{\hat{\rho}(x)}$ is increasing in x over the union of the supports of X and $\hat{X}$, where ρ and $\hat{\rho}$ are the probability mass functions of X and $\hat{X}$.

[16]Another simple example is the one where $\omega_1(v) = \exp\{-v\}$ and $\omega_2(v) = 1 - \omega_1(v)$.

[17]A firm's sales to consumers of valuation θ_k depends neither on the offers it makes to consumers of valuation θ_l, with $l \neq k$, nor on the distribution over contracts offered to other types by competing firms. This is an immediate consequence of the fact that all menus are incentive compatible.

(where we adopt the convention that $0^0 \equiv 1$).[18] Section 5 describes the sales functions induced by other models of information heterogeneity, and establishes the robustness of our results in a large class of such models.

In turn, if the distribution F_k has an atom at u_k, expected sales are determined according to a uniform rationing rule (which amounts to assuming that consumers evenly randomize across offers generating the same payoffs). This implies that slightly undercutting competitors whose indirect utility to consumers of valuation θ_k is u_k generates a discrete increase in expected sales.[19] For this reason, mass points in the marginal distributions F_k will not occur in equilibrium.[20] Finally, expected sales equal $\Lambda(1|v)$ if $u_k > \tilde{u}_k$ for all $\tilde{u}_k \in \Upsilon_k$, and equal $\Lambda(0|v)$ if $0 \leq u_k < \tilde{u}_k$ for all $\tilde{u}_k \in \Upsilon_k$.

A firm that faces a measure over menus $\tilde{F}$ (inducing, for each k, the continuous marginal cdf F_k over type-k indirect utilities) chooses a menu $((q_l, x_l); (q_h, x_h)) \in \mathbb{I}$ to maximize profits

$$\sum_{k=l,h} p_k \Lambda\left(F_k(u(q_k, x_k, \theta_k))|v\right)\left(x_k - \varphi(q_k)\right). \tag{2}$$

The next definition formalizes our notion of equilibrium in terms of the probability measure over menus $\tilde{F}$.

Definition 1 ***[Equilibrium]*** *An equilibrium is a probability measure over menus $\tilde{F}$ such that, if $\mathcal{M}$ is in the support of $\tilde{F} \subset \mathbb{I}$, then $\mathcal{M}$ maximizes firm profits.*

This equilibrium definition renders itself to different interpretations. For instance, one interpretation is that firms follow symmetric mixed strategies by randomizing over menus according to the probability $\tilde{F}$. Another interpretation is that each firm follows a pure strategy that consists in posting some menu in the support of $\tilde{F}$.

2.1 Incentive Compatibility and Indirect Utilities

A key step in our analysis is to formulate the firms' maximization problem in terms of the of indirect utilities offered to consumers. To this end, denote by

$$q_k^* \equiv \arg\max_q \ \theta_k q - \varphi(q),$$

the efficient quality for consumers of valuation θ_k, and let $S_k^* \equiv \theta_k q_k^* - \varphi(q_k^*)$ be the social surplus associated with the efficient quality provision. The next lemma uses the incentive constraints and the optimality of equilibrium contracts to map indirect utilities into quality levels.

[18] In the case where the sample-size distribution $\Omega(v)$ is a shifted-Poisson distribution, the sales function (1) can be written as $\Lambda(y|v) = (1+v)\exp\{-v(1-y)\}$.

[19] See Equation (12) in Appendix A for a formal statement.

[20] This is established in Lemma 3 below.

Lemma 1 *[Incentive Compatibility] Consider a menu $\mathcal{M} = \{(q_l, x_l), (q_h, x_h)\}$ in the support of the equilibrium probability over menus, $\tilde{F}$, and let $u_k \equiv u(q_k, x_k, \theta_k)$. Then the menu's qualities are given by*

$$q_l(u_l, u_h) = \begin{cases} \frac{u_h - u_l}{\triangle\theta} & \text{if} \quad u_h - u_l < q_l^* \triangle\theta \\ q_l^* & \text{if} \quad u_h - u_l \geq q_l^* \triangle\theta \end{cases} \quad \text{and} \quad q_h(u_l, u_h) = \begin{cases} \frac{u_h - u_l}{\triangle\theta} & \text{if} \quad u_h - u_l > q_h^* \triangle\theta \\ q_h^* & \text{if} \quad u_h - u_l \leq q_h^* \triangle\theta. \end{cases}$$

The result above is standard in adverse selection models. Consider some menu $\mathcal{M} \in \text{supp}(\tilde{F})$ offered in equilibrium, and let (u_l, u_h) be its profile of indirect utilities. Suppose for illustration that $u_h - u_l < q_h^* \triangle\theta$. Then we have efficiency at the top, $q_h = q_h^*$, reflecting the fact that the low-valuation incentive constraint (IC_l) is slack. The low-valuation quality is then efficient if $u_h - u_l \geq \Delta\theta q_l^*$, and downward distorted otherwise. This is because the high-valuation incentive constraint (IC_h) is slack in the former case, and binds otherwise.

We can now describe each menu in the support of $\tilde{F}$ in terms of the indirect utilities (u_l, u_h) induced by $\mathcal{M}$. Given (u_l, u_h), one can determine via Lemma 1 the equilibrium quality levels consumed by each consumer type (q_l, q_h), and hence also the prices paid (x_l, x_h).

Two natural benchmarks play an important role in the analysis that follows. The first one is the static monopolistic (or Mussa-Rosen) solution. Under this benchmark, the quality provided to low-valuation consumers, denote it q_l^m, is implicitly defined by:

$$\varphi'(q_l^m) = \max\left\{\theta_l - \frac{p_h}{p_l}\Delta\theta, 0\right\}. \tag{3}$$

We interpret $q_l^m = 0$ as meaning that low-valuation consumers are not served under the monopolistic solution. In turn, quality provision for high-valuation consumers is efficient: $q_h^m = q_h^*$. Finally, recall that, in the monopolistic solution, the indirect utility left to low-valuation consumers is zero, $u_l^m = 0$ (as the IR constraint is binding), and the indirect utility left to high-valuation consumers is $u_h^m = q_l^m \triangle\theta$, as the constraint IC_h is binding. Written in terms of indirect utilities, the menu $\mathcal{M}^m \equiv (0, q_l^m \triangle\theta)$ is the *monopolist* (or *Mussa-Rosen*) *menu.*

The second benchmark is the competitive (or Bertrand) solution. Under this benchmark, quality provision is efficient to consumers of all valuations, and firms derive zero profits from each contract in the menu. Written in terms of indirect utilities, the menu $\mathcal{M}^* \equiv (S_l^*, S_h^*)$ is the *competitive* (or *Bertrand*) *menu.*

3 Equilibrium Characterization

We start by studying the firms' profit-maximization problem, and then characterize equilibrium. For analytical convenience, the analysis of this section is developed in the space of indirect utilities. The next section then translates the equilibrium characterization into properties of observable variables.

3.1 Firm Problem

For each menu $\mathcal{M} = (u_l, u_h)$ offered in equilibrium, let

$$S_k(u_l, u_h) \equiv \theta_k q_k(u_l, u_h) - \varphi(q_k(u_l, u_h)) \tag{4}$$

be the social surplus induced by $\mathcal{M}$ for each consumer of valuation θ_k, where the quality levels $q_k(u_l, u_h)$ are computed according to Lemma 1. We can then write the profit per sale to consumers of valuation θ_k produced by the menu $\mathcal{M} = (u_l, u_h)$ as $S_k(u_l, u_h) - u_k$.

The firm's profit-maximization problem (assuming the cdf's over indirect utilities F_l and F_h are continuous, as will be true in equilibrium — see Lemma 3) is then to choose a menu (u_l, u_h) to maximize

$$\pi(u_l, u_h) \equiv \sum_{k=l,h} p_k \Lambda\left(F_k(u_k)|v\right)\left(S_k(u_l, u_h) - u_k\right), \tag{5}$$

subject to the constraint $u_h \geq u_l \geq 0$. This constraint guarantees that menus are individually rational. The requirement that $u_h \geq u_l$ captures incentive compatibility, as it guarantees that the indirect utility profile (u_l, u_h) can be generated by a pair of contracts satisfying the incentive constraints.

To better understand the firms' trade-offs, we now analyze the first-order conditions for (5). We will follow the common practice in mechanism design of assuming that IC_l is slack in equilibrium, in which case IC_h is the only constraint that may bind in equilibrium. We also assume that each F_k is differentiable; in Appendix A, we verify that these properties hold in all equilibria. First-order conditions for each firm's problem are then

$$\underbrace{p_h \Lambda_1\left(F_h(u_h)|v\right) f_h(u_h)\left(S_h^* - u_h\right)}_{\text{sales gains}} - \underbrace{p_h \Lambda\left(F_h(u_h)|v\right)}_{\text{profit losses}} + \underbrace{p_l \Lambda\left(F_l(u_l)|v\right) \frac{\partial S_l}{\partial u_h}(u_l, u_h)}_{\text{efficiency gains}} = 0 \tag{6}$$

for u_h, and

$$\underbrace{p_l \Lambda_1\left(F_l(u_l)|v\right) f_l(u_l)\left(S_l(u_l, u_h) - u_l\right)}_{\text{sales gains}} - \underbrace{p_l \Lambda\left(F_l(u_l)|v\right)}_{\text{profit losses}} + \underbrace{p_l \Lambda\left(F_l(u_l)|v\right) \frac{\partial S_l}{\partial u_l}(u_l, u_h)}_{\text{efficiency losses}} = 0 \tag{7}$$

for u_l. Intuitively, the firms' choice of menus balances sales, profit, and efficiency considerations.

Let us start with the first-order condition for high-valuation payoffs, given by Equation (6). The first two terms in (6) are familiar from models without asymmetric information on consumer valuations. By increasing the indirect utility u_h, the firm makes its menu more attractive to high-valuation consumers, increasing sales (the first term). However, the higher indirect utility reduces profits per sale (the second term). The third term captures the effect of an increase in u_h on the quality offered to low-valuation consumers. When IC_h binds (in which case $u_h = u_l + \Delta\theta q_l$), increasing u_h increases the quality q_l that can be supplied to low-valuation consumers, which, in turn, increases the surplus per sale to these consumers. Holding u_l fixed, the seller appropriates all of this additional surplus, and this is reflected in profits. On the other hand, when IC$_h$ is slack (i.e., $u_h > u_l + \Delta\theta q_l^*$), increasing u_h does not affect the quality supplied to low-valuation consumers (which is equal to the efficient level), and so the efficiency effect on profits is absent.

The first-order condition for low-valuation utilities is given by Equation (7). The first two terms are familiar from (6). In contrast to (6), however, increasing u_l has the effect of tightening IC$_h$, so the quality distortion present in the low-valuations' contract increases. The efficiency loss is the third term in Equation (7). It has the same magnitude as the third term in (6), but the opposite sign. Our equilibrium analysis of the next subsections will clarify how firms simultaneously resolve the efficiency-rent-extraction and the rent-extraction-sales-volume trade-offs in equilibrium.

Equations (6) and (7) reveal that, when consumer valuations are private information, the problems of choosing u_l and u_h are interdependent whenever incentive constraints bind. As the next lemma shows, this interdependency is materialized in the following crucial property of the profit function π.

Lemma 2 *[**Increasing differences**] Consider the profit function π defined over the set of implementable menus $\{(u_l, u_h) : u_h \geq u_l \geq 0\}$. This function satisfies strict increasing differences in (u_l, u_h) if some incentive constraint binds, but constant differences otherwise.*

Intuitively, the result above means that firms generating a high u_h have a comparative advantage in generating a high u_l. To understand why this is true, let us consider the first-order conditions (6) and (7) and suppose that IC$_h$ is binding.[21] In this case, $q_l = \frac{u_h - u_l}{\Delta\theta}$, and increasing u_h raises the quality supplied to low-valuation consumers. This increases the marginal profit of raising u_l for two reasons. First, the sales gains from raising u_l (which is the first term in (7)) go up as u_h increases. Second, the efficiency losses from raising u_l (which is the third term in (7)) go down (in absolute value) as u_h increases. This is so because the cost of quality φ is convex, which means that a marginal reduction in low-valuation quality has less effect on surplus when this quality is closer to its first-best level. These effects are summarized by the cross derivative of the profit function π at any menu for which IC$_h$ is binding:

$$\frac{\partial^2 \pi(u_l, u_h)}{\partial u_h \partial u_l} = p_l f_l(u_l) \Lambda_1 \left(F_l(u_l)|v\right) \left(\frac{\theta_l - \varphi'(q_l)}{\Delta\theta}\right) + \frac{p_l \Lambda \left(F_l(u_l)|v\right) \varphi''(q_l)}{(\Delta\theta)^2} > 0, \tag{8}$$

as can be directly computed from either (6) or (7).[22] The first term captures the effect of u_h on the sales gain from raising u_l, while the second term captures the effect of u_h on the efficiency loss from

[21] The intuition for the case where the low types' incentive constraint binds is similar. However, we will show that this constraint does not bind in equilibrium.

[22] Differentiability of F_l holds in equilibrium, but is not assumed in the proof of Lemma 2.

raising u_l. Both terms are positive.[23]

By contrast, if no incentive constraints bind at some menu (u_l, u_h), the profit function π exhibits constant differences; i.e., $\frac{\partial^2 \pi(u_l,u_h)}{\partial u_h \partial u_l} = 0$. In this case, as established by Lemma 1, optimality requires that qualities are fixed at their efficient levels for consumers of both low and high valuations, and the effects of u_l and u_h on profits are locally separable.

Before characterizing the equilibrium, we will use Lemma 2 to establish that, in any equilibrium, the distributions over indirect utilities, F_l and F_h, are absolutely continuous, and have support on an interval that starts at the indirect utility associated with the monopolistic (Mussa-Rosen) menu.

Lemma 3 *[**Support**] In any equilibrium of this economy, the marginal cdf F_k over indirect utilities offered to consumers of valuation θ_k is absolutely continuous, for $k \in \{l, h\}$. Its support is $\Upsilon_k = [u_k^m, \bar{u}_k]$, where $\bar{u}_k < S_k^*$.*

The lemma above has a number of important implications. First, that the distributions F_k are absolutely continuous confirms the result anticipated above: no equilibria exist in which, for any $k \in \{l, h\}$, a positive mass of firms post menus with the same indirect utilities u_k. Second, there are no gaps in the supports of the distributions F_k. Third, the minimum indirect utilities offered in equilibrium are those induced by the monopoly menu.

3.2 Ordered Equilibrium

We construct an equilibrium in which firms that cede high indirect utilities to high types also cede high indirect utilities to low types. We say that equilibria that satisfy this property are *ordered.*

Definition 2 *[**Ordered Equilibrium**] An equilibrium is said to be ordered if, for any two menus $\mathcal{M} = (u_l, u_h)$ and $\mathcal{M}' = (u_l', u_h')$ offered in equilibrium, $u_l < u_l'$ if and only if $u_h < u_h'$. In this case, the menu (u_l', u_h') is said to be more generous than the menu (u_l, u_h).*

Note that menus offered in an ordered equilibrium can be described via a *support function* relating the payoffs of high-valuation consumers to those appropriated by low-valuation consumers in any menu. This function, denoted by $\hat{u}_l : \Upsilon_h \to \Upsilon_l$, is strictly increasing and bijective. Accordingly, for every menu $\mathcal{M} = (u_l, u_h)$ in $\Upsilon_l \times \Upsilon_h$, we have that $u_l = \hat{u}_l(u_h)$.

Theorem 1 characterizes the unique ordered equilibrium of the economy. We find it notationally convenient to denote the identity function by $\hat{u}_h(u_h) = u_h$ for all $u_h \in \Upsilon_h$.

[23]As described in the Supplementary Material, Lemma 2 can be generalized to a model with an arbitrary number of types provided downward adjacent incentive constraints bind. Namely, the firm's profit satisfies strict increasing differences in the indirect utilities left to consumers of any two adjacent valuations.

Theorem 1 *[**Equilibrium Characterization**] There exists a unique ordered equilibrium. In this equilibrium, the support of indirect utilities offered by firms is described by the support function $\hat{u}_l : [u_h^m, \bar{u}_h] \to [0, \bar{u}_l]$ that is the unique solution to the differential equation*

$$\hat{u}_l'(u_h) = \frac{S_l(\hat{u}_l(u_h), u_h) - \hat{u}_l(u_h)}{S_h^* - u_h} \frac{1 - \frac{p_l}{p_h} \frac{\partial S_l}{\partial u_h}(\hat{u}_l(u_h), u_h)}{1 - \frac{\partial S_l}{\partial u_l}(\hat{u}_l(u_h), u_h)} \tag{9}$$

with boundary condition $\hat{u}_l(u_h^m) = 0$.

The equilibrium distribution over menus solves

$$\frac{\Lambda\left(F_h(u_h)|v\right)}{\Lambda\left(0|v\right)} = \frac{\sum_{k=l,h} p_k \left(S_k(0, u_h^m) - u_k^m\right)}{\sum_{k=l,h} p_k \left(S_k(\hat{u}_l(u_h), u_h) - \hat{u}_k(u_h)\right)}, \tag{10}$$

and the supremum point $\bar{u}_h$ is determined by $F_h(\bar{u}_h) = 1$.

The existence of an ordered equilibrium is intimately related to the increasing differences property of firms' profit functions established in Lemma 2. To understand how, consider a firm that raises the indirect utility of high-valuation consumers u_h (or, equivalently, reduces their price x_h). As noted above, this relaxes IC_h (i.e., it relaxes the constraint $u_h \geq u_l + \Delta\theta q_l$), permitting an increase in the quality supplied to low-valuation consumers q_l without a violation of incentive compatibility. If u_l were to remain fixed, then the firm would extract all of the additional surplus associated with the increase in q_l. But since the firm now makes more profits per sale to low-valuation consumers, it is worthwhile sharing some of the surplus with those consumers in order to realize more sales. This explains why u_h and u_l increase together in equilibrium.

The support function $\hat{u}_l(\cdot)$, together with the supremum point $\bar{u}_h$, determines the set of menus offered in equilibrium; i.e., the support of the equilibrium distribution $\tilde{F}$. Completing the characterization of $\tilde{F}$, Equation (10) determines the marginal distribution over high-valuation payoffs F_h. Note that this equation is simply an indifference condition which requires that all menus in the support of $\tilde{F}$ generate the same expected profits (more generous menus, i.e., those with a higher u_h, make more sales in expectation, but smaller expected profits per sale).

A striking feature of equilibrium is that the support function $\hat{u}_l\left(\cdot\right)$ does not depend on the sales function Λ. This means that $\hat{u}_l\left(\cdot\right)$ is invariant to the sample-size distribution $\Omega(v)$, and hence to any variation in the level of competition, v. By contrast, the support of equilibrium menus *does* depend on $\Omega(v)$, but *only* through the supremum indirect utility $\bar{u}_h$, determined by the indifference condition (10). $\Omega(v)$ also plays an important role in determining the equilibrium distribution over menus, as seen from (10).

The slope of the support function, implicitly determined by the differential equation (9), describes how payoffs change across consumers of different valuations as menus become more generous. A slope $\hat{u}_l'(u_h)$ close to zero means that, as a menu offers one extra unit of indirect utility to high-valuation consumers, it cedes very little indirect utility to low-valuation consumers. By contrast, as the slope

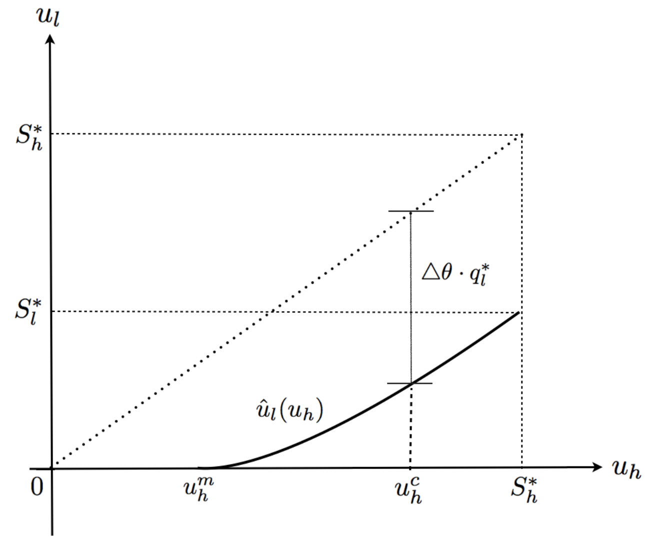

Figure 1: The equilibrium support function $\hat{u}_l(\cdot)$. The dotted line is the 45-degree line.

$\hat{u}_l'(u_h)$ increases, menus that are more generous increase the payoffs to both types of consumers in a "more balanced" manner. The next corollary to Theorem 1 describes how the slope $\hat{u}_l'(\cdot)$ varies across equilibrium menus.

Corollary 1 ***[Relative Generosity]*** *The equilibrium support function is such that* $0 < \hat{u}_l'(u_h) < 1$ *for all* $u_h \in (u_h^m, \bar{u}_h)$. *Moreover, the slope* $\hat{u}_l'(u_h)$ *is increasing in* u_h *(i.e.,* $\hat{u}_l(\cdot)$ *is convex) and satisfies* $\hat{u}_l'(u_h^m) = 0$.

Corollary 1 states that the slope $\hat{u}_l'(u_h)$ is always less than unity; accordingly, more generous menus increase payoffs relatively more for high- as compared to low-valuation consumers. There are two reasons for this. First, purchases by high-valuation consumers generate more profits per sale than those by consumers with low valuations. Second, as noted above, increasing the high-valuation payoff relaxes the incentive constraint IC$_h$; when this constraint is binding, it permits higher quality to be supplied to low-valuation consumers in an incentive-compatible manner. These two forces imply that sales to high-valuation consumers effectively contribute more to *total expected profits* than those to consumers with low valuations, resulting in equilibrium menus that cede relatively more indirect utility to high-valuation consumers.

The slope of the support function determines how low-valuation quality varies across menus. In particular, recall that

$$q_l\left(\hat{u}_l\left(u_h\right), u_h\right) = \frac{u_h - \hat{u}_l\left(u_h\right)}{\Delta\theta}$$

whenever $u_h - \hat{u}_l\left(u_h\right) \leq \triangle\theta q_l^*$ (i.e., when IC$_h$ binds). Because $\hat{u}_l'(u_h) < 1$, the quality provided to low types is strictly increasing in u_h (and so is the low-valuation surplus $S_l(\hat{u}_l(u_h), u_h)$). This quality is at its efficient level in menus which generosity u_h is above a threshold, call it u_h^c, that uniquely solves $u_h - \hat{u}_l\left(u_h\right) = \triangle\theta q_l^*$.

The corollary above also establishes that the slope $\hat{u}_l'(u_h)$ is increasing in u_h. The reason is that low-valuation quality increases with u_h, which implies that sales to low-valuation consumers generate greater profits per sale the more generous is the menu. As a result, in equilibrium, more generous menus leave increasingly more indirect utility to those consumers.[24]

The last property established by Corollary 1 is that the slope of the support function is zero in a neighborhood of the Mussa-Rosen menu. To gain intuition, suppose that $u_h^m > 0$. For menus in a neighborhood of the monopoly menu $(0, u_h^m)$, increasing u_h has only a second-order effect on expected profits per sale, since u_h^m is an interior maximizer of these profits. Increasing u_l, however, leads to a first-order loss in expected profits per sale. As such, for menus close to the monopolistic one, ceding indirect utility to high-valuation consumers is arbitrarily less costly than doing so for low-valuation consumers. This implies $\hat{u}_l'(u_h^m) = 0$ in equilibrium.

Figure 1 illustrates these observations. This figure represents the entire graph of the support function, $\{(\hat{u}_l(u_h), u_h) : u_h \in [u_h^m, S_h^*)\}$. Which of these offers are made in equilibrium depends on the supremum point $\bar{u}_h$.

Before developing some testable implications of our model, it is useful to understand how the degree of competition in the market affects the equilibrium distribution over menus. To this end, consider the indifference condition (10) from Proposition 1. Its right-hand side is independent of the degree of competition v (similarly to the support function), while its left-hand side increases with v.[25] It then follows that, as the degree of competition increases, (i) firms become more likely to offer more generous menus, and (ii) the support of equilibrium menus expands. To see the latter, note that expected profits fall as competition increases. Hence, the firm which makes the largest number of expected sales, by offering the most generous menu $(\hat{u}_l(\bar{u}_h), \bar{u}_h)$, must be yielding more rent to consumers in equilibrium (i.e., $\bar{u}_h$ must increase with competition).

To formalize these results, and to state the key empirical predictions of our theory in the next section, we shall say that a random variable *increases probabilistically* if its distribution increases in the sense of first-order stochastic dominance.

Corollary 2 *[**Comparative Statics**] The indirect utilities obtained by consumers of either low or high valuations increase probabilistically with competition (i.e., as v increases). Moreover, the top of the support of these indirect utilities $\bar{u}_k$ increases with v, for $k \in \{l, h\}$.*

The next remark clarifies when the ordered equilibrium is unique in the class of all possible equilibria.

[24] There is also a second reason why menus are more generous towards low-valuation consumers as u_h increases. Because the marginal surplus created by additional low-valuation quality is smaller the more efficient is quality provision, the gains from raising u_h (through the relaxation of IC_h) are smaller the higher is u_h.

[25] This follows from the fact that $\frac{\Lambda(y|v)}{\Lambda(0|v)}$ increases with v for all $y \in (0, 1]$, as implied by Equation (1) and that $\Omega(v)$ increases in the likelihood-ratio order with v.

Remark 1 ***[Equilibrium Uniqueness]*** *In deriving testable predictions below, we focus on the unique ordered equilibrium characterized in Theorem 1. As we show in Appendix B, little (if anything) is lost by restricting attention to this equilibrium. Namely, when the degree of competition is not too large, the ordered equilibrium is the unique equilibrium. By contrast, if the degree of competition is sufficiently large, there are equilibria which are not ordered. However, all equilibria (including the non-ordered ones) lead to the same marginal distributions over indirect utilities* $F_k(\cdot)$*, and the same ex-ante profits for firms.*

4 Empirical Implications

For analytical convenience, the equilibrium characterization of the previous section was developed in the space of indirect utilities. In this section, we employ this characterization to derive implications in terms of prices, qualities and markups. We then relate these implications to the available empirical evidence, and discuss how empirical researchers can test the predictions of our model.

4.1 Competing on price and quality

Purchasing decisions depend on consumer preferences as well as on price and quality differentials across products. Absent asymmetric information about consumer preferences, firms would offer menus containing all the efficient qualities associated with consumer valuations, although at different prices. Under asymmetric information, a firm's choice of how much utility to leave to one type of consumer affects its ability to provide quality to other types (as incentive constraints have to be satisfied). As a consequence, firms' offers differ on *both* price and quality provision, which work as substitute instruments for increasing sales.

Our theory helps to explain how firms combine these instruments as a function of the degree of competition in the market. To explore this important question, it is convenient to write the prices of the low and high quality goods as functions of indirect utilities:

$$\begin{aligned} x_l\left(u_h\right) &\equiv \theta_l q_l\left(\hat{u}_l\left(u_h\right), u_h\right) - \hat{u}_l\left(u_h\right), \text{ and} \\ x_h\left(u_h\right) &\equiv \theta_h q_h^* - u_h \end{aligned}$$

for $u_h \in \Upsilon_h$. The next result is then a consequence of Theorem 1 and Corollary 1. To state it, recall that u_h^m is the high-valuation indirect utility at the monopolistic menu, and that the threshold u_h^c is such that all menus with $u_h \geq u_h^c$ serve low-valuation consumers efficiently.

Corollary 3 ***[Equilibrium Prices]*** *The price of the high-quality good* $x_h\left(u_h\right)$ *is strictly decreasing in* u_h *over* $[u_h^m, S_h^*)$*. There exists* $u_h^d \in (u_h^m, u_h^c]$ *such that the price of the low-quality good* $x_l\left(u_h\right)$ *is strictly increasing in* u_h *if* $u_h \leq u_h^d$*, and strictly decreasing over* $\left(u_h^d, S_h^*\right)$*.*

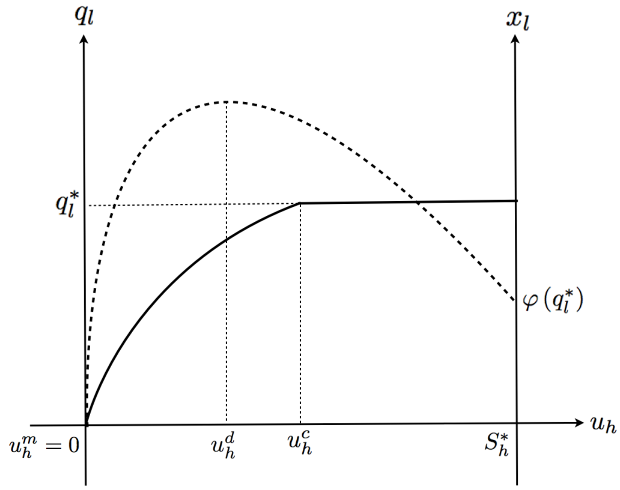

Figure 2: The low-valuation quality schedule (full curve and left-side Y-axis)) and the low-valuation price (dotted curve and right-side Y-axis) as a function of the generosity of the menu, u_h.

As established in Corollary 1, more generous menus increase payoffs relatively more for high- as compared to low-valuation consumers, and the difference is largest for menus close to the Mussa-Rosen menu, as $\hat{u}_l'(u_h^m) = 0$. This results in the relaxation of the high-valuation incentive constraint IC$_h$, leading to an increase in the quality of the low-valuation good relative to the Mussa-Rosen menu. Also because $\hat{u}_l'(u_h^m) = 0$, most of the additional surplus from higher low-valuation quality is appropriated by the firms. This implies that prices increase. Hence, firms offering the least generous menus (i.e., $u_h < u_h^d$) compete for additional consumers by increasing both the quality and the price of the low-quality option.

As menus become more generous, they cede utility more evenly to consumers of low and high valuations (reflecting the fact that $\hat{u}_l(\cdot)$ is convex). This implies that quality increases with generosity at a decreasing rate. When $u_h^d < u_h^c$, $x_l(u_h)$ is decreasing over the interval $\left(u_h^d, u_h^c\right)$, although the low-valuation quality $q_l(\hat{u}_l(u_h), u_h)$ increases in u_h. For still more generous menus (i.e., those with $u_h > u_h^c$), the low-valuation quality remains constant at its efficient level. As a result, the price of the low-quality good necessarily decreases in u_h. Taken together, these observations imply the price of the low-quality good is an inverse-U-shaped function of the menu generosity, as indexed by u_h.

Finally, because high-valuation quality is efficiently provided for all levels of u_h, the price of the high-quality good $x_h(u_h)$ strictly decreases in u_h. Figure 2 summarizes the findings of Corollary 3.

Combined with the comparative statics of Corollary 2, Corollary 3 leads to interesting predictions on how prices react to changes in the degree of competition in the market.

Proposition 1 ***[Price-Increasing Competition]*** *As the degree of competition increases, the prices for the high-quality good decrease and the quality levels of the low-quality good increase probabilistically. Moreover, there exists $v^d > 0$ such that, if the degree of competition is low (i.e., $v < v^d$), the prices for the low-quality good increase probabilistically as v increases to $\hat{v} < v^d$.*

Intuitively, more competition makes it more likely that firms offer high indirect utility to high-valuation consumers, which relaxes incentive constraints, and permits an increase in the quality offered to low-valuation consumers. Because, when competition is not too intense, firms are substantially more generous to high-valuation consumers, this increase in quality is accompanied by an increase in the price of the low-quality good. Figure 2 illustrates this result. Consider first the case where $v < v^d$. Because more competition shifts to the right the distribution of high-valuation indirect utilities u_h, and because the prices charged to low-value consumers increase in u_h, it follows that these low-valuation prices have to increase (probabilistically) following an increase in v.

When instead $v > v^d$, the effect of competition on the price of the low-quality good depends on features of the market, such as the level of competition and the proportion of high types. It is helpful to consider the limiting cases of monopoly and perfect competition (equivalently, to consider the limits $v \to 0$ and $v \to +\infty$ in our model). For instance, suppose that the proportion of high types, p_h, is small. Then, as we go from monopoly to perfect competition, the price of the low-quality good falls. This is because the low-valuation quality at the monopoly menu, q_l^m, is close to the efficient level q_l^*. Hence, while quality increases little in response to competition, the low type's rent goes from zero under monopoly to the entire surplus S_l^* at the competitive outcome. Conversely, if p_h is sufficiently large, then the change in the low type's quality is large, and the price paid by the low type increases as we move from monopoly to perfect competition.

Our finding that increased competition can raise prices is not without empirical counterparts. For instance, Chu (2010) documents that cable companies in the US reacted to new competition by satellite television by raising both price and quality (as determined by the available channels) in some markets, with consumers benefiting overall from the higher-priced offerings.[26] Miravete and Roller (2004) examine the market for cell-phone plans in the 1980s and find that, while prices per minute for high-consumption users are lower in markets with competition, there is little difference in prices for low-consumption users. A different conclusion is reached by Gerardi and Shapiro (2009), who study the US airline industry in the 1990s and early 2000s. Differences in ticket characteristics, such as refundability and Saturday night stay-over restrictions, seem to explain much of the variation in an airline's prices on any given route (see Sengupta and Wiggins 2014 for evidence). Gerardi and Shapiro (2009) find that both the 10th and 90th percentile of prices fall with increases in competition (yet, the former falls on average less than half as much as the latter). The muted relationship between competition and the prices charged to low-paying consumers found in the empirical literature seems consistent with the findings in Proposition 1.[27]

[26] See also Byrne (2014), who uses Canadian data to study how the entry of satellite television affected the pricing and quality provision of cable television in Canada. His results are comparable to those of Chu (2010).

[27] Although data on quality provision is scarce, some papers support the prediction that baseline quality increases with competition. Besides the work by Chu cited above, Mazzeo (2003) provide some direct evidence suggesting quality increases in response to competition.

Our theory also predicts that qualities are downward distorted for low-quality goods in all equilibrium menus provided competition is not too intense, but might be provided efficiently by some firms if there is enough competition. These results are consistent with the findings of McManus (2007), who empirically evaluates the magnitude of product design distortions in an oligopolistic market for specialty coffee. He documents that, for some coffee products, distortions are close to zero across the product line, whereas, for others, distortions decrease as one increases product size.[28]

The empirical literature also suggests a further prediction of our model; namely, that the difference between the menus' highest prices ($x_h(u_h)$) and lowest prices ($x_l(u_h)$) should decrease with competition. To see why, we can decompose the difference in prices into differences in consumption utility and differences in payoffs of the consumers purchasing high- and low-quality options. In particular, we can write the *price differential* for a menu with high-valuation utility u_h as

$$\begin{aligned} \Delta_p(u_h) &\equiv x_h(u_h) - x_l(u_h) \\ &= \underbrace{(\theta_h q_h^* - \theta_l q_l(\hat{u}_l(u_h), u_h))}_{\text{Difference in consumption utility}} - \underbrace{(u_h - \hat{u}_l(u_h))}_{\text{Difference in rents}}. \end{aligned} \tag{11}$$

Recall that both the low-valuation quality $q_l(\hat{u}_l(u_h), u_h)$ and the payoff differential $u_h - \hat{u}_l(u_h)$ are increasing in u_h. Together with Corollary 2 this implies the following result.[29]

Proposition 2 ***[Competition and Price Differentials]*** *As the degree of competition increases, the price differential $\Delta_p(u_h)$ decreases probabilistically.*

The logic behind this result is familiar from Proposition 1. High-quality prices decrease with more competition. On the other hand, the price of the low-quality good either (i) increases (when competition is originally low, following large quality improvements), or (ii) decreases less sharply (when competition is originally high, following small or null quality improvements). The former reflects that quality improvements are a superior way (in terms of profits) to compete for consumers than price reductions.

Much of the existing evidence on the role of competition in markets with price discrimination is in line with this prediction.[30] The studies of Miravete and Roller (2004) and Gerardi and Shapiro (2009) mentioned above are clear examples. Another is Busse and Rysman (2005), who find that, while prices for yellow-pages advertisements are lower on average in more competitive markets, this price reduction is most pronounced for the largest (and hence most expensive) ads.

[28]The latter finding cannot be rationalized by the cost-plus-fee pricing result of Armstrong and Vickers (2001) and Rochet and Stole (2002). This finding is however consistent with the monopolistic analysis of Mussa and Rosen (1978) and the circular-city model of Yang and Ye (2008).

[29]With slight abuse of notation, we interpret $\Delta_p(u_h)$ as a random variable, itself a function of the random variable u_h, distributed according to F_h.

[30]For a recent review of this literature, see Crawford (2012).

4.2 Distributional Effects of Competition

Our theory also generates predictions concerning how changes in the degree of competition affect the welfare across consumers of different valuations, and the profitability of sales across the product line. Competition in our model is beneficial for all consumers (even when the price of the low-valuation option increases with competition), as implied by Corollary 2. How consumers with different valuations fare in relative terms can then be understood by studying

$$\Delta_u \equiv \int_{\Upsilon_h} u_h dF_h - \int_{\Upsilon_l} u_l dF_l,$$

which is the difference (in expectation) of the indirect utility offered by firms to high and low types in equilibrium. Corollaries 1 and 2 imply that Δ_u increases with the degree of competition υ.[31] This implies that, on average, an increase in competition leads to larger gains to high-valuation consumers.[32]

Again, this prediction is consistent with many of the studies that examine the effects of competition on consumer welfare in markets with price discrimination. Miravete and Roller (2004) and Economides, Seim and Viard (2008) find that high-consumption users gain the most from increases in competition in cellular phone markets. Chu (2010) finds that those consumers with the highest value for quality in television packages gained most when satellite television began to compete with incumbent cable providers. Other studies, such as Busse and Rysman (2005) and Gerardi and Shapiro (2009) suggest that high-paying consumers have the most to gain from competition (due to lower prices), but do not estimate consumer preferences.

It is also interesting to compare how firms' profits from sales of low- and high-quality products depend on the intensity of competition. One natural way to do so is to consider the impact of competition on the *markup difference*, which is the difference in profits per sale for high and low types. For a menu with high-valuation utility u_h, this can be written

$$\begin{aligned} \Delta_m(u_h) &\equiv [x_h(u_h) - \varphi(q_h^*)] - [x_l(u_h) - \varphi(q_l(\hat{u}_l(u_h), u_h))] \\ &= \underbrace{(S_h^* - S_l(\hat{u}_l(u_h), u_h))}_{\text{Difference in surplus}} - \underbrace{(u_h - \hat{u}_l(u_h))}_{\text{Difference in rents}}. \end{aligned}$$

Recall that the low-valuation surplus $S_l((\hat{u}_l(u_h), u_h)$ is increasing in u_h (because low-valuation quality is increasing in u_h), while the difference in rents $u_h - \hat{u}_l(u_h)$ is also increasing in u_h. This, together with our finding that u_h increases probabilistically with competition (see Corollary 2), then implies the following.[33]

[31] Note that $\Delta_u = \int_{\Upsilon_h} (u_h - \hat{u}_l(u_h))\, dF_h$, after a change of variables. The result then follows, as $u_h - \hat{u}_l(u_h)$ is increasing in u_h and F_h increases according to first-order stochastic dominance as υ increases (by Corollaries 1 and 2).

[32] The result in the text refers to the expected difference in the utility *offered* to each consumer type. It is straightforward to show the expected difference in the utility *accepted* by different consumer types also increases in υ.

[33] As with the price differential $\Delta_p(u_h)$, we slightly abuse of notation and interpret $\Delta_m(u_h)$ as a random variable.

Proposition 3 ***[Competition and Markups]*** *As the degree of competition increases, the markup difference decreases probabilistically.*

In other words, markups decrease faster for high-quality goods than for low-quality goods as competition intensifies. Intuitively, this result reflects two facts. First, firms gain the most from sales to high-valuation consumers, for whom they compete by lowering prices. Second, much (in many cases, all) of the gains that low-valuation consumers obtain from more competition take the form of higher provision of quality, rather than lower prices. This contributes to moderate the effect of competition on low-valuation markups.

Although researchers often lack data on marginal costs, the empirical work examining the relationship between mark-ups and competition seems broadly in line with Proposition 3. Under assumptions on the structure of yellow-pages advertising, Busse and Rysman (2005) find that decreases in mark-ups associated with competition are greatest for the largest sizes of ads in yellow-pages phone books. Miravete and Roller (2004) estimate price-cost margins using a structural model, and find that marginal costs fall in response to competition (possibly due to improved efficiency in markets with competition). They argue that competition favors high-end users more than low-end users, as the markup difference across calling plans typically decreases as competition intensifies.

Finally, note that the mark-up difference is positive in all equilibrium menus.[34] As described in the Introduction, this prediction contrasts with the cost-plus-fixed-fee result (which implies a zero markup difference) found in Armstrong and Vickers (2001) and Rochet and Stole (2002), and finds support in a variety of markets.[35]

4.3 Competition and Preference Heterogeneity

Consumer preference heterogeneity is another important determinant of how competition impacts price and qualities. In this subsection, we describe how differences in consumer preferences help to explain variations in quality provision and pricing across markets.

Next, we address the following question: For markets with the same degree of competition, how does quality provision depend on the distribution of consumer preferences? While the answer to this question is already well understood when firms are monopolists – a higher proportion of low-value consumers implies higher quality for low types – we show that this answer extends to our setting for any degree of imperfect competition.

Proposition 4 ***[Quality Provision and Preference Heterogeneity]*** *The quality levels of the low-quality good probabilistically increase as the share of consumers with low willingness to pay (p_l) increases.*

[34] This is so because $\Delta_m(u_h)$ is a strictly decreasing function satisfying $\Delta_m(u_h^m) > 0$ and $\Delta_m(S_h^*) = 0$.

[35] See McManus (2007) for specialty coffee, Verboven (2002) and Thomassen (2017) for cars, and Song (2015) for personal computers.

This result combines the effects of changes in p_l on both the support function and the equilibrium distributions over indirect utilities. An increase in p_l shifts the support function downwards; equivalently, for each u_l, the associated u_h increases (and hence the quality supplied to the low-value consumer goes up). This reflects that, as the mass of low-valuation consumers increases, the marginal gains in expected profits from increasing u_h due to the relaxation of IC_h increase. This in turn is due to the increased profitability of providing a higher-quality good to low-value consumers.

In addition, the distribution over high-valuation indirect utilities shifts to the right as p_l increases. To understand why, recall that markup differences $\Delta_m(u_h)$ decrease in u_h. This implies that, as p_l goes up, the profits of the most generous firms are the ones to decrease the least. Therefore, to sustain indifference, firms have to be more likely in equilibrium to offer menus of high generosity. These two effects together guarantee that low-valuation qualities are higher (probabilistically) in markets with more low-valuation consumers. We are unaware of empirical work documenting such correlations.

Inspired by the empirical literature, the next proposition explores two other predictions of our model. First, for markets with the same degree of competition, how do price differentials depend on the distribution of consumer preferences? Second, in which markets should we expect a greater impact of competition on price differentials?

To answer these questions, it is convenient to denote by $\tilde{\Delta}_p(y)$ the *price differential of rank* y, that is, $\tilde{\Delta}_p(y) \equiv x_h(u_h) - x_l(u_h)$ such that $y = F_h(u_h)$. By tracking the price differential provided by firms in each quantile of the distribution of indirect utilities, we are able to describe in a transparent way the interplay between pricing and consumer heterogeneity.

Proposition 5 ***[Price Differentials and Preference Heterogeneity]*** *For any $y \in (0,1)$, the price differential of rank y, $\tilde{\Delta}_p(y)$, decreases in v and p_l. Moreover, if the degree of competition is sufficiently high (i.e., whenever $v > \bar{v}$ for some $\bar{v}$), the marginal impact of v on $\tilde{\Delta}_p(y)$ (in absolute terms) is decreasing in p_l.*

As established by the proposition above, the price differential of rank y is decreasing in both v and p_l, for any $y \in (0,1)$. That it decreases in v follows immediately from Proposition 2. That $\tilde{\Delta}_p(y)$ decreases in p_l follows from two facts. First, the low-valuation quality level of rank y (namely $\hat{q}_l(y) \equiv q_l(\hat{u}_l(u_h), u_h)$ such that $y = F_h(u_h)$) increases in p_l, as implied by Proposition 4. Second, the price differential of rank y is a decreasing function of the low-quality option $\hat{q}_l(y)$ offered in the menu (see Equation (11)).

More interesting, perhaps, is that the impact of competition on price differentials is stronger in markets with fewer low-valuation consumers. As such, the degree of competition and the fraction of low-valuation consumers work as substitutes for reducing price differentials. To get intuition, recall from Proposition 4 that, as p_l increases, the qualities of low-quality goods increase probabilistically, and the prices of high-quality goods go down. This means that firms are willing to cede more indirect utility to consumers of low valuations the higher is p_l. This works to moderate the impact of competition on price differentials.

In line with this prediction, Gerardi and Shapiro (2009) find that the compression of an airline's prices associated with increased competition is greatest on "big-city" routes, likely to have the greatest number of high-paying business travelers.

4.4 Discussion: From Theory to Data

As already suggested, our theory applies in a variety of markets that have been studied in existing empirical work. More generally, we expect our theory to be applicable in settings exhibiting the following key features. First, consumer information about available offers is imperfect and heterogeneous, which is firms' source of market power in our model. Second, consumers are heterogeneous in their preferences over product attributes, and firms engage in price discrimination by designing product lines exhibiting options which differ in both price and quality. These two features are shared by a number of markets, such as those for gasoline,[36] banking services, cars, mobile phone plans, cable TV, wine and liquor,[37] airline transportation, among others.[38]

These settings are amenable to various tests of our theory's predictions. A number of studies mentioned above examine the effects of different levels of competition in outcomes *across* markets, distinct in time or space.[39] One example is Chu (2010), who analyzes how the entry of satellite TV affected the price and quality provision of Cable TV providers. It does so by exploring longitudinal

[36] Evidence of imperfect (and heterogeneous) consumer information in gasoline markets abound (see Lach and Moraga-González (2017) and the references therein). The qualities of regular and premium gasoline vary by retailer, and are determined by the additives blended in the base fuel. Information on the latter is required by regulation to be easily available to consumers at the point of sale.

[37] For evidence of imperfect (and heterogeneous) consumer information about the firms' offers, see Honka et al (2017) for the case of banking services, Moraga-González et al (2017) for the case of cars, and Clark et al (2009) for the cases of mobile phone plans, cable TV and alcoholic beverages.

[38] In the markets for wine and airline transportation, specialized reviews, expert ratings and certification are likely to render product quality known to consumers before a purchase has been made (as opposed to after the purchase, as is the case with experience goods).

[39] The baseline model identifies more intense competition with a shift (in the sense likelihood-ratio dominance) of the distribution over the size of consumers' consideration sets. Data about consumers' consideration sets (as in Clark et al 2009, Honka 2014 or Honka et al 2017) enables testing whether this notion of an increase in the degree of competition is empirically satisfied. Namely, one has to compare the empirical distributions over the size of consumers' consideration sets as market conditions change. For instance, consider a market that moves from being off-line to being online. More competition is verified if the "online" empirical distribution over the size of consumers' consideration sets is a likelihood-ratio shift of its off-line counterpart. In the next section, we extend the baseline setup to other models of information heterogeneity, show how our notion of an increase in competition generalizes to these richer settings, and describe how to assess its empirical validity using sales data only (in case data about consumers' consideration sets is not available).

data containing the distribution of firms' offers before and after the entry of satellite TV. Another example is Busse and Rysman (2005), who uses cross-sectional data on yellow-page advertising in geographically distinct markets to assess the effects of competition (as proxied by the number of firms) on price and quality provision.

Future work in this direction might directly test Propositions 1, 2 and 3, regarding the effects of competition on the distributions of prices and markups. The typical case where markups are not observed (but have to be estimated) might require data on consumers' consideration sets, including, for instance, survey evidence on consumers' brand knowledge. Indeed, empirical researchers are increasingly making use of this kind of data (see Clark et al. (2009), Honka (2014), Moraga-González et al. (2017), and Honka et al. (2017)) to estimate demand and marginal costs in markets where information is imperfect.[40]

Our theory is also testable by examining predictions that relate to behavior *within* given markets, especially by considering the cross-sectional distribution of firm offers. For instance, one can directly examine whether the distribution of offers within a market exhibit the systematic relationships predicted by our theory. As an example, one can test whether firms that offer lower prices on their premium products also tend to offer higher-quality alternatives to these premium goods (i.e., higher-quality baseline options), and whether the price differentials and markup differences between premium and non-premium products are smaller for these firms (as implied by Theorem 1 and Corollary 1). Relative to the theoretical literature on oligopolistic price discrimination, these cross-section implications, which pertain to the *dispersion* of firms' offers, are unique to our model.

As discussed in Subsection 4.2, our theory also delivers implications regarding the relative impact on consumers' payoffs of changes in competition. The framework developed in this paper can form the basis of the structural modeling needed to compare consumers' surplus before and after the change. Finally, the implications of Subsection 4.3, regarding how the distribution of consumer preferences affects the firms' offers, can also be empirically evaluated (see, for instance, Gerardi and Shapiro (2009)).

5 Extensions and Robustness

5.1 Information Heterogeneity and Competition

For expositional simplicity, the baseline model considered a fixed number of firms, and identified the degree of competition with the level of transparency about market offers. In this interpretation, increases in competition arise from better consumer information; for instance, due to the advent of online marketplaces, or a reduction in the costs of advertising. Another possibility is to hold fixed the level of market transparency and increase the number of competing firms. The next example

[40] Another possibility is to use aggregate advertising data following, for instance, Sovinsky Goeree (2008).

illustrates this point with the "urn-ball" model of Butters (1977), which is a common alternative to the "simultaneous-search" framework of Burdett and Judd (1983). Our results are robust to either framework; in fact, to a much broader class of models of information heterogeneity, as described below.

Example 1 ***[Generalized Butters (1977)]*** *Let the menu offered by each firm be observed by exactly $n \geq 1$ consumers; we view n as the degree of market transparency. The size-n subset of consumers reached by each firm is uniformly (and independently) drawn from the set of all n-size subsets of consumers. When the number of firms and consumers in the market is large (with ratio γ), the analysis of Butters (1977) implies that the sales function faced by firms has the functional form*

$$\Lambda\left(F_k(u_k)|n,\gamma\right) = \exp\left\{-n\gamma(1-F_k(u_k))\right\}.$$

Under this model of information heterogeneity, equilibrium outcomes solely depend on the product $n\gamma$ between the degree of market transparency and the number of firms per consumer. In our baseline model, which follows Burdett and Judd, a similar role is played by the degree of competition v.

Other models of information heterogeneity exhibit a similar pattern of substitutability between the degree of market transparency and the number of firms in the market (see, for instance, the binomial model proposed by Lach and Moraga-Gonzalez (2017)). Another important example is the "on-the-job search" model of Burdett and Mortensen (1998).

Example 2 ***[Burdett and Mortensen (1998)]*** *Consider a dynamic economy in continuous time in which consumers receive ads (each ad describes the menu of a particular firm) according to independent Poisson processes with arrival rate λ. Consumers must make purchasing decisions as soon as an ad arrives, and there is no recall. Each matched consumer purchases continuously from the seller until the match is dissolved. This can occur exogenously due to an event which arrives at Poisson rate γ. Alternatively, consumers may switch firms if they receive (at rate λ) an ad describing a more attractive menu. The analysis of Burdett and Mortensen (1998) implies that, at the steady state of this economy, the expected sales to type-k consumers of a firm offering an indirect utility of rank y_k (relative to the other offers in the market) is given by $p_k\Lambda\left(y_k|\lambda\right)$, where the sales function $\Lambda\left(y_k|\lambda\right)$ has the form*

$$\Lambda\left(y_k|\lambda\right) = \gamma\left[\frac{1}{\gamma+\lambda(1-y_k)}\right]\left[\frac{1}{\gamma+r+\lambda(1-y_k)}\right].$$

Under this model of information heterogeneity, the parameter λ, describing the arrival rate of a new ad, plays an analogous role to the degree of competition v in the Burdett-Judd specification of the baseline model. As consumers get more exposed to ads (for instance, due to more intense internet use), the degree of market competition increases.

The examples above share a key feature: The volume of sales of a firm depends solely on its rank, in terms of indirect utility, relative to other firms in the market. This "ranking property" reflects the assumption that consumers are concerned only for the utility of consumption net of transfers (and thus pick the best offer available based on these features), and not with other characteristics of a firm's offer, such as transportation costs or the firm's identity. This feature, together with some other technical requirements detailed below, define a class of matching technologies under which all results from Sections 3 and 4 hold as stated.

Assumption 1 *Let $\tilde{F}$ be a measure over menus with supp $\tilde{F} \subset \mathbb{I}$, and marginal cdf over type-k indirect utilities F_k, with support Υ_k. At any continuity point $u_k \in \Upsilon_k$ of F_k, the expected sales to type-k consumers of a firm offering an indirect utility of rank $y_k = F_k(u_k)$ is given by $p_k \Lambda (y_k|v)$. The sales function $\Lambda (y|v)$ is*

(a) continuously differentiable, bounded, strictly increasing in y with derivative bounded away from zero, and

(b) such that $\frac{\Lambda(y|v)}{\Lambda(0|v)}$ is strictly increasing in v for any $y > 0$.

The matching technologies of Burdett and Judd (1983) and of the Examples 1 and 2 satisfy Assumption 1. We refer the reader to a previous version of this article (Garrett et al (2014)) for other examples, as well as for a detailed proof of Theorem 1 that covers any matching technology satisfying Assumption 1.

Note that the second property in Assumption 1 generalizes the notion of an increase in competition to a large class of models of heterogeneous information. Namely, competition increases when, relative to the least generous offer available in the market, the proportional gains in sales from offering a contract whose indirect utility lies in any quantile $y > 0$ increases. Note that this requirement is agnostic on whether competition increases due to more transparency or more competitors (or both). Sales and price data could be used to test this condition jointly with the predictions of our theory. In particular, our results suggest constructing a ranking of the generosity of firms' offerings and then considering how sales change (across the ranking) in response to any observed variation in market conditions.

5.2 Beyond the Binary-Type Model

The equilibrium characterization described above proceeded under the assumption of two consumer types, with no restrictions on the probability distribution. However, our approach can often be adapted to characterize equilibria with more than two types, and even a continuum. The Supplementary Material considers such models in detail. Our analysis there points to the robustness of the key economic predictions of our theory, adapted to the richer settings where firms screen consumers through menus with more than two distinct qualities of good.

We can show that, when types are discrete, under a standard regularity condition on the distribution of types, provided competition is not too intense (i.e., v is not too large), all equilibria are necessarily ordered. That is, firms' offers can be ranked in terms of generosity. In general, we find that the support of firms' offers, rather than being characterized by the solution to the differential equation (9) as given above, are characterized by a *system* of differential equations that embed similar trade-offs to those in the two-type case. Further, in an example with a continuum of types that are uniformly distributed, and with quadratic costs, we provide a full characterization of the ordered equilibrium (which we conjecture is the only equilibrium of this model). This example exhibits the (arguably realistic) feature that qualities are distorted for almost every consumer type in every equilibrium menu. This is distinct from discrete-type models, where quality is efficiently provided to consumers with the very highest valuation. Again, we show that our main insights remain true (pertaining to the existence of an ordered equilibrium, its implications for pricing and quality provision, and comparative statics) in the equilibrium constructed for this continuum-type setting.[41]

5.3 Information Acquisition by Consumers

In order to isolate the effects of competition on the firms' pricing and quality provision, the analysis above assumed that consumer information is exogenous. In reality, information might often be endogenous, as consumers may invest to learn the offers available in the market.

It is possible to incorporate information acquisition by consumers in our model of competitive nonlinear pricing. A natural alternative is to assume that, after learning their willingness to pay for quality, each consumer can make an investment that shifts her sample-size distribution according to first-order stochastic dominance. We explore this extension in a previous version of this article (Garrett et al (2014)), and show that high-valuation consumers invest more in information acquisition than those with low valuations. As a result, relative to the case where information is exogenous, consumer information acquisition makes high-valuation consumers "over-represented" in equilibrium, i.e., firms behave as if those consumers were more frequent than as implied by their actual masses.

6 Related Literature

This paper brings the theory of nonlinear pricing under asymmetric information (Mussa and Rosen (1978), Maskin and Riley (1984) and Goldman, Leland and Sibley (1984)) to a competitive set-

[41] The chief difficulty for analyzing other specifications stems from the fact that closed-form solutions are difficult to obtain.

ting where consumers are heterogeneously informed about the offers made by firms. Other related literature is described below.

Competition in Nonlinear Pricing. This article primarily contributes to the literature that studies imperfect competition in nonlinear pricing schedules when consumers make exclusive purchasing decisions (exclusive agency). In one strand of this literature, firms' market power stems from comparative advantages for serving consumer segments. In Stole (1995) such comparative advantages are exogenous, whereas in Champsaur and Rochet (1989) they are endogenous, as firms can commit to a range of qualities before choosing prices.[42]

Another strand of this literature generates market power by assuming that goods are horizontally differentiated, i.e. consumers have preferences over brands; see Spulber (1989) for a one-dimensional model where consumers are distributed on a Salop circle, and Rochet and Stole (1997, 2002), Armstrong and Vickers (2001), and Yang and Ye (2008) for multi-dimensional models where brand preferences enter utility additively. These papers study symmetric equilibria, and show that (i) the equilibrium outcome under duopoly often lies between the monopoly and the perfectly competitive outcome, and that (ii) when brand preferences are narrowly dispersed, quality provision may be fully efficient with cost-plus-fixed-fee pricing prevailing.[43]

Our model presents several advantages relative to the aforementioned papers. First, it parsimoniously explains different firm behavior on how to compete for consumers. Indeed, the ability to describe various responses to competition (in terms of price and quality decisions) highlights the *tractability* of our model. In the horizontal-differentiation literature mentioned above, results analogous to those of this paper (e.g., comparative statics on the degree of competition) are analytically elusive.

Second, the Hotelling-Salop approach of the aforementioned papers conflates changes in the degree of competition with changes in the outside option (as changes in "transportation costs" simultaneously affect the consumer choice between any two firms, and between any given firm and not consuming the good).[44] Our model isolates changes purely in the competitiveness of the market, contributing to its greater tractability.[45]

There is, of course, other work recognizing that consumers may not be perfectly informed about offers in competitive settings.[46] The works of Verboven (1999) and Ellison (2005) depart from the

[42] In work subsequent to ours, Fabra and Montero (2017) challenge the conclusions of Champsaur and Rochet by showing that an arbitrarily small amount of search frictions is enough to give rise to an equilibrium in which both firms offer a complete product line. Unlike in our work, the quality schedule describing product lines is assumed exogenous.

[43] See Borenstein (1985), Wilson (1993) and Borenstein *et al* (1994) for numerical results in closely related settings.

[44] See Bénabou and Tirole (2016) for further discussion of this point.

[45] Relatedly, understanding the impact of entry in a spatial model typically requires modeling the detailed spatial structure of the market. Our alternative approach simply views an increase in the number of firms as stochastically increasing the number of offers each consumer receives.

[46] There is also work where consumers have imperfect information about offers in the absence of competition. Most

benchmark of perfect consumer information by assuming that consumers observe the baseline prices offered by *all* firms, but have to pay a search cost to observe the price of upgrades (or add-on prices). The focus of these papers is on the strategic consequences of the hold-up problem faced by consumers *once* their store choices are made. By taking quality provision as exogenous, these papers ignore the mechanism design issues that are at the core of the present article. Katz (1984) studies a model of price discrimination where a measure of low-value consumers are uninformed about prices while other consumers are perfectly informed. Heterogeneity of information thus takes a very particular form in this model, and price dispersion does not arise (when quantity discounting is permitted, a unique price schedule is offered in equilibrium).

Assuming perfect consumer information, Stole (1991) and Ivaldi and Martimort (1994) study duopolistic competition in nonlinear price schedules when consumers can purchase from more than one firm (common agency).[47] In a related setting, Calzolari and Denicolo (2013) study the welfare effects of contracts for exclusivity and market-share discounts (i.e., discounts that depend on the seller's share of a consumer's total purchases). The analysis of these papers is relevant for markets where goods are divisible and/or exhibit some degree of complementarity, whereas our analysis is relevant for markets where exclusive purchases are prevalent (e.g., most markets for durable goods).

Price Dispersion. We borrow important insights from the seminal papers of Butters (1977), Salop and Stiglitz (1977), Varian (1980) and Burdett and Judd (1983), that study oligopolistic competition in settings where consumers are differently informed about the prices offered by firms. In these papers, there is complete information about consumer preferences, and firms compete only on prices.[48] Relative to this literature, we introduce asymmetric information about consumers' tastes, and allow firms to compete on price *and* quality.

Competing Auctioneers. McAfee (1993), Peters (1997), and Peters and Severinov (1997) study competition among principals who propose auction-like mechanisms. These papers assume that buyers perfectly observe the sellers' mechanisms, and that the meeting technology between buyers and sellers is perfectly non-rival. This last assumption is relaxed by Eeckhout and Kircher (2010), who show that posted prices prevail in equilibrium if the meeting technology is sufficiently rival. A key ingredient of these papers is that sellers face capacity constraints (each seller has *one* indivisible good to sell), and offer homogenous goods whose quality is exogenous. Our paper differs from this literature in three important respects. First, sellers in our model control both the price and the quality of the good to be sold. Second, we assume away capacity constraints. Third, buyers are heterogeneously informed about the offers made by sellers.

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closely related to our paper, Villas-Boas (2004) studies *monopoly* price discrimination where consumers randomly observe either some or all elements of the menu.

[47]See Stole (2007) for a comprehensive survey of the common agency literature.

[48]See, however, Grossman and Shapiro (1984) where consumers not only have heterogeneous information about offers, but also about brand preferences. Firms compete in prices and advertising intensities, but do not price discriminate.

Search and Matching. Inderst (2001) embeds the setup of Mussa and Rosen (1978) in a dynamic matching environment, where sellers and buyers meet pairwise and, in each match, each side may be chosen to make a take-it-or-leave offer. His main result shows that inefficiencies vanish when frictions (captured by discounting) are sufficiently small, thus providing a foundation for perfectly competitive outcomes.[49] Frictions in our model have a different nature (they are informational).

Faig and Jerez (2005) study the effect of buyers' private information in a general equilibrium model with directed search. They show that if sellers can use two-tier pricing, private information has no bite, and the equilibrium allocation is efficient. In turn, Guerrieri, Shimer and Wright (2010) show that private information leads to inefficiencies in a directed-search environment with common values. Our model is closer to Faig and Jerez (2005), as we study private values. In contrast to Faig and Jerez (2005), our model leads to menu dispersion and distortions.

In work subsequent to ours, Lester et al (2017) introduce information heterogeneity in a common-values setting where seller types are privately known. They show that an ordered equilibrium also exists in their environment, and study the welfare effects of market interventions common in insurance and financial markets. Finally, our paper is also related to Moen and Rosén (2011), who introduce private information on match quality and effort choice in a labor market with search frictions. We focus on private information about willingness to pay (which is the same for all firms), while workers have private information about the match-specific shock in their model.

7 Conclusion

This paper studies imperfect competition in price-quality schedules in a market with informational frictions. On the one hand, consumers have private information about their willingness to pay for quality. On the other, consumers are imperfectly informed about the offers in the market, which is the source of firms' market power. While firms are ex-ante identical, equilibrium menus are dispersed and can be ranked in terms of their generosity to consumer of all valuations. Our analysis illuminates how firms of different generosity employ price reductions and/or quality improvements to compete for consumers.

We build on the characterization of equilibrium to deliver a number of empirical implications that explore the cross-sectional behavior of firms and changes in market fundamentals. Namely, (i) we show how prices, qualities and markups over different goods co-move across firms in equilibrium, (ii) we study how prices and qualities react to changes in competition, (iii) we analyze the distributional effects of competition across consumers with different valuations, and (iv) we assess the impact of variations in consumer tastes and competition on the price differentials within firms' menus. These predictions find support in a number of markets (such as those for cell phone plans, yellow-pages

[49] In contrast, Inderst (2004) shows that if frictions affect agents' utilities through type-independent costs of search (or waiting), equilibrium contracts are always first-best.

advertising, airline tickets and cable television), and can be further tested in a variety of other settings.

Directions for Further Research. For the sake of tractability, we have assumed that firms' capacities are unconstrained, so they are able to fill all orders. An alternative to relax this assumption is to assume that the marginal benefit of an additional consumer is decreasing in the measure of consumers already served by each firm. If firms are heterogeneous in costs, we conjecture that market segmentation would occur in equilibrium, and more cost-effective firms would more often serve higher-valuation consumers.[50]

Finally, we assumed that consumers observe the entire menu of qualities offered by each firm. In practice, consumers may fail to consider all of the options that a firm offers; i.e., information imperfections may pertain also to a consumer's ability to observe the entire menu. This possibility has been explicitly recognized in empirical work (e.g., Sovinsky Goeree (2008)). In theoretical work, Villas-Boas (2004) studies a monopolist whose consumers may (randomly) observe only the low or the high-quality option; extending the analysis to a competitive setting raises additional challenges.

Acknowledgements

We are grateful to the editor, Philipp Kircher, and three anonymous referees for detailed comments and suggestions. We also thank, without implicating, Jean Tirole for extensive feedback at the early stages of this project, and Yeon-Koo Che, Andrew Clausen, Jacques Crémer, Wouter Dessein, Jan Eeckhout, Renaud Foucart, Ed Hopkins, Bruno Jullien, Tatiana Kornienko, Volker Nocke, Wojciech Olszewski, Michael Peters, Patrick Rey, Andrew Rhodes, Mike Riordan, Maher Said, Ron Siegel, Bruno Strulovici, André Veiga, and Yaron Yehezkel for very helpful conversations. For useful feedback, we thank seminar participants at Bilkent, Columbia, FGV-EPGE, the LMU of Munich, Melbourne, Monash, UNSW, Queensland, École Polytechnique, Northwestern, Oxford, Bonn, Edinburgh, Stonybrook, the 2013 Toulouse-Northwestern IO conference, the 2013 UBC-UHK Theory conference (Hong Kong), the 2014 CEPR Applied IO conference (Athens), INSPER, the 2014 EARIE conference (Milan), Mannheim, PSE, Pompeu Fabra, and Groningen. Daniel Garrett gratefully acknowledges financial support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 714147 - DYNMECH). Renato Gomes gratefully acknowledges financial support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 759733 - PLATFORM). Lucas Maestri gratefully acknowledges financial support from

[50] An examination of "congestion effects" when consumer tastes are private information and information is heterogeneous seems difficult but worthwhile. See Lester (2011) for a related model without private information on consumer tastes. The literature on competing auctioneers, reviewed in Section 6, assumes homogenous information across consumers.

the CNPq. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

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Appendix A: Proofs

Throughout this Appendix, we find it convenient to define $\Phi_k(u_k)$, for $k \in \{l, h\}$, to be the expected sales generated in equilibrium to consumers with valuations θ_k when the indirect utility offered by a menu is equal to u_k. If u_k is a continuity point of F_k (the equilibrium distribution over payoffs offered to consumers with valuation θ_k), then $\Phi_k(u_k) \equiv p_k \Lambda\left(F_k(u_k)|v\right)$. If instead u_k is a point of discontinuity, then $\Phi_k(u_k)$ is determined by a uniform rationing rule (as noted in the main text), and hence

$$\lim_{\tilde{u}_k \downarrow u_k} \Phi_k\left(\tilde{u}_k\right) > \Phi_k\left(u_k\right). \tag{12}$$

Proof of Lemma 1. If the low-valuation consumer is offered the quality q_l, then payoffs must satisfy IC_h, i.e.,

$$u_h \geq u_l + \Delta\theta q_l. \tag{13}$$

On the other hand, IC_l requires that

$$u_l \geq u_l - \Delta\theta q_h. \tag{14}$$

The firm would like to make its offer as efficient as possible subject to the payoffs it delivers to the consumer.

If $u_h - u_l < \bar{\Delta}_l \equiv \Delta\theta q_l^*$, then offering the efficient quality q_l^* for the low-valuation consumer is inconsistent with (13), and the firm does best to choose the highest possible value. That is, the firm chooses quality $q_l\left(u_l, u_h\right)$ which satisfies (13) with equality, or

$$q_l\left(u_l, u_h\right) \equiv \frac{u_h - u_l}{\Delta\theta}.$$

If $u_h - u_l \geq \bar{\Delta}_l$, then the constraint (13) does not bind, and the firm chooses low-valuation quality efficiently: $q_l\left(u_l, u_h\right) \equiv q_l^*$. Similarly, let $\bar{\Delta}_h \equiv \Delta\theta q_h^*$. If $u_h - u_l > \bar{\Delta}_h$, then asking the quality q_h^* for the high-valuation consumer violates (14), and so the best the firm can do is to choose $q_h\left(u_l, u_h\right)$ defined by

$$q_h\left(u_l, u_h\right) \equiv \frac{u_h - u_l}{\Delta\theta}.$$

If $u_h - u_l < \bar{\Delta}_h$, the firm offers high-valuation consumers the efficient quality: $q_h\left(u_l, u_h\right) \equiv q_h^*$. Q.E.D.

Proof of Lemma 2. To see this claim, note that $\pi\left(u_l, u_h^2\right) - \pi\left(u_l, u_h^1\right)$ equals

$$\begin{gathered}
\Phi_l\left(u_l\right)\left(S_l\left(u_l, u_h^2\right) - S_l\left(u_l, u_h^1\right)\right) \\
+\left(\Phi_h\left(u_h^2\right) - \Phi_h\left(u_h^1\right)\right)\left(S_h\left(u_l, u_h^2\right) - u_h^2\right) \\
+\Phi_h\left(u_h^1\right)\left(\begin{array}{c} S_h\left(u_l, u_h^2\right) - u_h^2 \\ -\left(S_h\left(u_l, u_h^1\right) - u_h^1\right) \end{array}\right).
\end{gathered} \tag{15}$$

The cross-partial $\frac{\partial^2}{\partial u_l \partial u_h} S_l(u_l, u_h)$ is positive if $u_h - u_l < \Delta\theta q_l^*$ and zero otherwise. Thus the first line of (15) is strictly increasing over u_l such that $u_h^1 - u_l \leq \Delta\theta q_l^*$ and constant otherwise. The function $S_h(\cdot, u_h^2)$ is strictly increasing if $q_h(u_l, u_h^2) > q_h^*$ and constant otherwise. Thus the second line in (15) is increasing in u_l. The cross-partial $\frac{\partial^2}{\partial u_l \partial u_h} S_h(u_l, u_h)$ is positive if $u_h - u_l > \Delta\theta q_h^*$ and is zero otherwise. Thus the third term is strictly increasing over u_l such that $u_h^2 - u_l \geq \Delta\theta q_h^*$ and constant otherwise. These arguments imply the result. Q.E.D.

Proof of Lemma 3.

Outline. We divide the proof into five steps. Steps 1 and 2 rule out mass points in the distributions F_h and F_l respectively, using the "undercutting" argument mentioned in the main text.

Step 3 then shows that the supports of F_k are intervals for each $k \in \{l, h\}$, which requires ruling out gaps in the support. Again, the argument is more involved than the one from models where consumers lack private information on preferences. In particular, suppose that $(\underline{u}_k, \bar{u}_k)$ represents a gap in the support of the indirect utilities obtained in equilibrium by consumers with valuations θ_k. Absent private information on preferences, a menu that yields consumers with valuations θ_k a utility $\bar{u}_k$ would not be optimal; prices charged to these consumers could be slightly increased without affecting their equilibrium expected sales, increasing profits. Since consumer preferences are private in our model, however, the indirect utility yielded to a consumer with valuation θ_k potentially affects the quality that can be supplied (in an incentive-compatible manner) to either type, requiring a more careful argument.

Step 4 then establishes that the minimum utility earned by consumers of any valuation θ_k is equal to the utility earned in the optimal menu when the seller is a monopolist. An important part of the argument is that, were the minimum utilities strictly above the monopoly ones, then expected profits per sale would be lower than for the monopoly menu, but the level of expected sales would be precisely the same.

Finally, Step 5 shows that F_k is absolutely continuous for each $k \in \{l, h\}$. This clearly goes further than the absence of mass points established in Steps 1 and 2.

Steps of the Proof.

Step 1 *No mass points in the distribution of high-type indirect utilities.*

To show that F_h has no mass points, assume towards a contradiction there is an atom at $\tilde{u}_h$ and let $(\tilde{u}_l, \tilde{u}_h)$ be an equilibrium offer.

We have $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h > 0$ for any equilibrium offer $(\tilde{u}_l, \tilde{u}_h)$. Suppose not: $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h \leq 0$. Then it must be that we also have $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l \leq 0$ (in case $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h \leq 0$ and $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l > 0$, offering only the option designed for the low type increases the seller's expected profit because positive sales are still made to high types). Hence $\pi(\tilde{u}_l, \tilde{u}_h) \leq 0$. This contradicts seller optimization. Indeed, the seller could offer a menu which yields the Mussa-and-Rosen utilities (u_l^m, u_h^m), which leads to profits at least as large as $(S_h^* - u_h^m)\, p_h \Lambda(0|v) > 0$ (recall that $\Lambda(0|v) > 0$ follows because some consumers observe only one offer).

We have $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l \geq 0$ **for any equilibrium offer** $(\tilde{u}_l, \tilde{u}_h)$. Suppose not. Then it must be that $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l < 0$. In this case, the seller could profitably deviate by offering the menu containing the contracts $(q_l, x_l) = (0, 0)$ and $(q_h, x_h) = (q_h^*, \theta_h q_h^* - \tilde{u}_h)$. Irrespective of whether the low type finds it incentive compatible to choose the option $(0, 0)$, the seller is guaranteed an expected profit at least as high as the one obtained by selling exclusively to the high type, which is greater than the profit from $(\tilde{u}_l, \tilde{u}_h)$ because $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l < 0$.

There exists $\varepsilon > 0$ **such that** $\pi(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon) > \pi(\tilde{u}_l, \tilde{u}_h)$. This assertion contradicts the optimality of $(\tilde{u}_l, \tilde{u}_h)$. Using the two assertions in bold above, we have

$$\begin{aligned}
& \lim_{\varepsilon \downarrow 0} \{\pi(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon) - \pi(\tilde{u}_l, \tilde{u}_h)\} \\
= \; & \lim_{\varepsilon \downarrow 0} \left\{ \sum_{k=l,h} \Phi_k(\tilde{u}_k + \varepsilon)(S_k(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon) - \tilde{u}_k - \varepsilon) - \sum_{k=l,h} \Phi_k(\tilde{u}_k)(S_k(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_k) \right\} \\
\geq \; & \lim_{\varepsilon \downarrow 0} \{(\Phi_h(\tilde{u}_h + \varepsilon) - \Phi_h(\tilde{u}_h))(S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h)\} > 0, \qquad (16)
\end{aligned}$$

where the positivity of the last term uses the assumption that F_h has an atom at $\tilde{u}_h$ and (12).

Step 2 *No mass points in the distribution of low-type indirect utilities.*

No mass points in F_l **at any** $u_l > 0$. Suppose towards a contradiction that F_l has a mass point at some $\tilde{u}_l > 0$. Consider a menu $(\tilde{u}_l, \tilde{u}_h)$. As shown in the proof of Step 1, we have $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l \geq 0$ and $S_h(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_h > 0$. We must consider two cases.

Case 1: $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l > 0$.

In this case, we show that there exists $\varepsilon > 0$ such that $\pi(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon) > \pi(\tilde{u}_l, \tilde{u}_h)$, which contradicts the optimality of $(\tilde{u}_l, \tilde{u}_h)$:

$$\begin{aligned}
& \lim_{\varepsilon \downarrow 0} \{\pi(\tilde{u}_l + \varepsilon, \tilde{u}_h + \varepsilon) - \pi(\tilde{u}_l, \tilde{u}_h)\} \\
\geq \; & \lim_{\varepsilon \downarrow 0} \{(\Phi_l(\tilde{u}_l + \varepsilon) - \Phi_l(\tilde{u}_l))(S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l)\} > 0, \qquad (17)
\end{aligned}$$

where the arguments are analogous to those for (16).

Case 2: $S_l(\tilde{u}_l, \tilde{u}_h) - \tilde{u}_l = 0$. We show that this case cannot arise. Let

$$\{(q_l, x_l), (q_h, x_h)\} = \{(q_l(\tilde{u}_l, \tilde{u}_h), x_l(\tilde{u}_l, \tilde{u}_h)), (q_h(\tilde{u}_l, \tilde{u}_h), x_h(\tilde{u}_l, \tilde{u}_h))\}$$

be the menu offered by the firm. Consider a deviation to the menu $\{(q_l, x_l + \varepsilon), (q_h, x_h)\}$, which is built by increasing the transfers only in the low-type contract by some $\varepsilon \in (0, \tilde{u}_l)$. This menu generates the same expected profits from high types, as a high type would still purchase the contract (q_h, x_h). Moreover, this menu is accepted with positive probability by low types (as a low type can guarantee a positive utility from the contract $(q_l, x_l + \varepsilon)$). Hence, profits obtained from $\{(q_l, x_l + \varepsilon), (q_h, x_h)\}$ are strictly higher than for $\{(q_l, x_l), (q_h, x_h)\}$, contradicting the optimality of the latter.

No mass points in F_l at $u_l = 0$. Assume towards a contradiction that $F_l(0) > 0$. From Step 1 above (i.e., since there are no mass points in the distribution of high-type offers), there is $\varepsilon > 0$ for which menus $(u_l, u_h) \in \{0\} \times [\varepsilon, \infty)$ are then offered with positive probability. Moreover, we can take ε small enough that $\theta_l(\frac{\varepsilon}{\triangle\theta}) - \varphi(\frac{\varepsilon}{\triangle\theta}) > 0$. For each menu $(u_l, u_h) \in \{0\} \times [\varepsilon, \infty)$, Lemma 1 implies the the quality in the low-type contract is given by $\min\{\frac{u_h - u_l}{\triangle\theta}, q_l^*\} = \min\{\frac{u_h}{\triangle\theta}, q_l^*\}$, and hence $S_l(0, u_h) \geq (\theta_l(\frac{\varepsilon}{\triangle\theta}) - \varphi(\frac{\varepsilon}{\triangle\theta})) > 0$ (because $S_l(0, u_h)$ is non-decreasing in u_h). Thus take an offer $(0, u_h) \in \{0\} \times [\varepsilon, \infty)$ in the support of the distribution of indirect utilities offered in equilibrium. We will show that the there exists a small $\eta > 0$ such that $\pi(\eta, u_h + \eta) > \pi(0, u_h)$, which contradicts the optimality of $(0, u_h)$:

$$\lim_{\eta \downarrow 0} \{\pi(\eta, u_h + \eta) - \pi(0, u_h)\} \geq \lim_{\eta \downarrow 0} \left\{ (\Phi_l(\eta) - \Phi_l(0)) \left(\theta_l \left(\frac{\varepsilon}{\triangle\theta} \right) - \varphi \left(\frac{\varepsilon}{\triangle\theta} \right) - \eta \right) \right\} > 0,$$

where the reason for this inequality is the same as the one in (17).

Step 3 *The supports Υ_k are bounded intervals, $\Upsilon_k = [\underline{u}_k, \bar{u}_k]$, where $\bar{u}_k < S_k^*$.*

The supports Υ_l and Υ_h are intervals. Assume first that Υ_l is not an interval. Then there exists $u_l' < u_l''$ and $F_l(u_l') = F_l(u_l'') = y \in (0, 1)$. Assume without loss that u_l' is the infimal utility level u_l such that $F_l(u_l) = y$ and u_l'' is the supremal utility level such that $F_l(u_l) = y$ (that $F_l(u_l') = F_l(u_l'') = y$ follows because F_l lacks mass points). Analogously, let u_h' be the infimal utility level such that $F_h(u_h) = y$ and u_h'' be the supremal utility level such that $F_h(u_h) = y$. (Clearly, for the case in which $F_h(u_h' + \varepsilon) > F_h(u_h')$ for every $\varepsilon > 0$, we have $u_h' = u_h''$.)

The increasing-differences property of π (see Lemma 2) and the absence of mass points in the F_l and F_h, as established above, imply that (u_l', u_h') and (u_h'', u_h'') are optimal menus.

If $u_h'' > u_h'$, then we can find an $\varepsilon > 0$ for which $F_l(u_l'' - \varepsilon) = F_l(u_l'')$ and $F_h(u_h'' - \varepsilon) = F_h(u_h'')$. But this implies that the menu $(u_l'' - \varepsilon, u_h'' - \varepsilon)$ generates the same sales to low and high types as (u_l'', u_h''), but strictly higher profits per sale (since the price of both low and high quality options are higher). This contradicts the optimality of (u_l'', u_h''), so we may assume that $u_h'' = u_h'$.

Optimality of (u_l'', u_h'') then requires that, for any $\varepsilon \in (0, u_l'' - u_l')$, $\pi(u_l'' - \varepsilon, u_h'') \leq \pi(u_l'', u_h'')$, which can only be true if IC_l binds at (u_l'', u_h''). In this case, using Lemma 1, $q_h(u_l, u_h'')$ is above the first-best level and strictly decreasing in u_l over (u_l', u_l''), which implies that $\frac{\partial^2 S_h(u_l, u_h'')}{\partial u_l^2} < 0$ for u_l in (u_l', u_l''). Using $\Phi_l(u_l') = \Phi_l(u_l'')$, it is then immediate that firm expected profits $\pi(\cdot, u_h'')$

are strictly concave over (u_l', u_l''). Hence, for any $\lambda \in (0,1)$, $\pi(\lambda u_l' + (1-\lambda) u_l'', u_h'') > \lambda \pi(u_l', u_h'') + (1-\lambda)\pi(u_l'', u_h'') = \pi(u_l'', u_h'')$, contradicting the optimality of (u_l'', u_h'').

The argument establishing that Υ_h is an interval is similar. Suppose it is not an interval. Then there exists $u_h' < u_h''$ and $F_h(u_h') = F_h(u_h'') = y \in (0,1)$, and we can take u_h' to be the infimal utility level u_h such that $F_h(u_h) = y$ and u_h'' to be the supremal utility level such that $F_h(u_h) = y$. Analogous to above, there must exist a unique value u_l'' for which $F_l(u_l'') = y$ and the menus (u_l'', u_h') and (u_l'', u_h'') are optimal. Optimality of (u_l'', u_h'') implies that IC_h must bind at (u_l'', u_h''), and we can conclude that $\pi(u_l'', \lambda u_h' + (1-\lambda) u_h'') > \lambda\pi(u_l'', u_h') + (1-\lambda)\pi(u_l'', u_h'') = \pi(u_l'', u_h'')$ for $\lambda \in (0,1)$, which contradicts the optimality of (u_l'', u_h'').

We have shown that Υ_k is an interval: $\Upsilon_k = [\underline{u}_k, \bar{u}_k]$. The fact that $\bar{u}_k < S_k^*$ for $k = l, h$ follows from our findings in Steps 1 and 2 that the seller makes positive profits from each type of consumer from each optimal menu (u_l, u_h) such that $u_l > 0$.

Step 4 *The minimum of the supports Υ_l and Υ_h are, respectively, $u_l^m = 0$ and u_h^m.*

Let $\underline{u}_l$ and $\underline{u}_h$ be the minimum of the supports of Υ_l and Υ_h respectively. Take $(\underline{u}_l, \tilde{u}_h)$ and $(\tilde{u}_l, \underline{u}_h)$ in the support of $\tilde{F}$, which are thus optimal menus. Since π has the increasing-difference property (see Lemma 2), it follows that $(\underline{u}_l, \underline{u}_h)$ is optimal. The profit from this menu is:

$$\Phi_l(\underline{u}_l)\left[S_l(\underline{u}_l, \underline{u}_h) - \underline{u}_l\right] + \Phi_h(\underline{u}_h)\left[S_h(\underline{u}_l, \underline{u}_h) - \underline{u}_h\right].$$

We have $\underline{u}_l = u_l^m = 0$. Individual rationality requires $\underline{u}_l \geq 0$. Suppose towards a contradiction that $\underline{u}_l > 0$. Note that $\underline{u}_h \geq \underline{u}_l$ and consider the deviating menu $(\underline{u}_l - \varepsilon, \underline{u}_h - \varepsilon)$ for some $\varepsilon \in (0, \underline{u}_l)$. Notice that since F_l and F_h are continuous (see Steps 1 and 2 above), the minimality of $(\underline{u}_l, \underline{u}_h)$ implies that $\Phi_k(\underline{u}_k) = \Phi_k(\underline{u}_k - \varepsilon)$ for $k = l, h$. Therefore

$$\begin{aligned}
& \pi(\underline{u}_l - \varepsilon, \underline{u}_h - \varepsilon) \\
= \; & \Phi_l(\underline{u}_l - \varepsilon)\left[S_l(\underline{u}_l - \varepsilon, \underline{u}_h - \varepsilon) - (\underline{u}_l - \varepsilon)\right] + \Phi_h(\underline{u}_h - \varepsilon)\left[S_h(\underline{u}_l - \varepsilon, \underline{u}_h - \varepsilon) - (\underline{u}_h - \varepsilon)\right] \\
= \; & \Phi_l(\underline{u}_l)\left[S_l(\underline{u}_l, \underline{u}_h) - (\underline{u}_l - \varepsilon)\right] + \Phi_h(\underline{u}_h)\left[S_h(\underline{u}_l, \underline{u}_h) - (\underline{u}_h - \varepsilon)\right] \\
= \; & \pi(\underline{u}_l, \underline{u}_h) + \varepsilon\left(\Phi_l(\underline{u}_l) + \Phi_h(\underline{u}_h)\right) > \pi(\underline{u}_l, \underline{u}_h),
\end{aligned}$$

which contradicts the optimality of $(\underline{u}_l, \underline{u}_h)$.

We have $\underline{u}_h = u_h^m$. Suppose towards a contradiction that $\underline{u}_h \neq u_h^m$. Notice that, since the Mussa-and-Rosen problem is strictly concave in u_h,

$$p_l S_l(0, u_h) + p_h(S_h(0, u_h) - u_h) < p_l S_l(0, u_h^m) + p_h(S_h(0, u_h^m) - u_h^m) \tag{18}$$

for every $u_h \geq 0$ with $u_h \neq u_h^m$. Therefore, we have:

$$\begin{aligned}
\pi\left(0, u_h^m\right) &= \Phi_l\left(0\right) S_l\left(0, u_h^m\right) + \Phi_h\left(u_h^m\right)\left(S_h\left(0, u_h^m\right) - u_h^m\right) \\
&\geq \Phi_l\left(0\right) S_l\left(0, u_h^m\right) + \Phi_h\left(\underline{u}_h\right)\left(S_h\left(0, u_h^m\right) - u_h^m\right) \\
&= \left(\frac{\Phi_l\left(0\right)}{p_l}\right)\left[p_l S_l\left(0, u_h^m\right) + p_h\left(S_h\left(0, u_h^m\right) - u_h^m\right)\right] \\
&> \left(\frac{\Phi_l\left(0\right)}{p_l}\right)\left[p_l S_l\left(0, \underline{u}_h\right) + p_h\left(S_h\left(0, \underline{u}_h\right) - \underline{u}_h\right)\right] = \pi\left(\underline{u}_l, \underline{u}_h\right),
\end{aligned}$$

where the first inequality uses $\Phi_h\left(u_h^m\right) \geq \Phi_h\left(\underline{u}_h\right)$, the next equality uses $\frac{\Phi_h(\underline{u}_h)}{\Phi_l(\underline{u}_l)} = \frac{\Phi_h(\underline{u}_h)}{\Phi_l(0)} = \frac{p_h}{p_l}$, and the last inequality uses (18). This contradicts the optimality of $\left(\underline{u}_l, \underline{u}_h\right)$. We have thus established that, for each $k \in \{l, h\}$, the support Υ_k is an interval $[u_k^m, \bar{u}_k]$, where $\bar{u}_k > u_k^m$.

Step 5 *F_l and F_h are absolutely continuous.*

Fact 1 *For every $\varepsilon > 0$, for every $(u_l, u_h) \in [u_l^m + \varepsilon, \bar{u}_l] \times [u_h^m + \varepsilon, \bar{u}_h]$ in the support of $\tilde{F}$, there exists $\psi_\varepsilon > 0$ such that* $\min\left\{S_l\left(u_l, u_h\right) - u_l, S_h\left(u_l, u_h\right) - u_h\right\} > \psi_\varepsilon$.

Assume towards a contradiction that the fact is false. Since the support is a closed set, its intersection with $[u_l^m + \varepsilon, \bar{u}_l] \times [u_h^m + \varepsilon, \bar{u}_h]$ is a compact set. Hence the contradiction assumption implies that we can find $(u_l, u_h) \in [u_l^m + \varepsilon, \bar{u}_l] \times [u_h^m + \varepsilon, \bar{u}_h]$ in the support of $\tilde{F}$ such that

$$\min\left\{S_l\left(u_l, u_h\right) - u_l, S_h\left(u_l, u_h\right) - u_h\right\} = 0.$$

Since $\tilde{F}$ is an equilibrium distribution, (u_l, u_h) is optimal. As claimed in Step 1, we have $S_h\left(u_l, u_h\right) - u_h > 0$. Therefore assume that $S_l\left(u_l, u_h\right) - u_l = 0$. But notice that the argument provided in Case 2 of Step 2 above shows that there exists a profitable deviation, a contradiction.

Fact 2 *Take $\varepsilon \in (0, \bar{u}_k - u_k^m)$ and pick K_ε such that the function Φ_k is K_ε-Lipschitz over $[u_k^m + \varepsilon, \bar{u}_k]$. Then F_k is $\frac{K_\varepsilon}{p_k 2\omega_2(v)}$-Lipschitz over the same domain.*

Take $u_k \in [u_k^m + \varepsilon, \bar{u}_k]$ and $\lambda > 0$, and pick K_ε such that the function Φ_k *is* K_ε-Lipschitz over $[u_k^m + \varepsilon, \bar{u}_k]$. Since $\omega_2(v) > 0$ by assumption, we have:

$$\begin{aligned}
& p_k 2\omega_2(v)\left(F_k\left(u_k + \lambda\right) - F_k\left(u_k\right)\right) \\
\leq\; & p_k \sum_{j=1}^{\infty} j\omega_j(v)\left(F_k\left(u_k + \lambda\right)^{j-1} - F_k\left(u_k\right)^{j-1}\right) \\
=\; & \Phi_k\left(u_k + \lambda\right) - \Phi_k\left(u_k\right) \\
\leq\; & K_\varepsilon \lambda,
\end{aligned}$$

and hence $\left(F_k\left(u_k + \lambda\right) - F_k\left(u_k\right)\right) \leq \frac{K_\varepsilon \lambda}{p_k 2\omega_2(v)}$, which proves this fact because F_k is nondecreasing.

Fact 3 *For every $\varepsilon > 0$, and any $\lambda > 0$, there exists $K_\varepsilon > 0$ such that, for every $k \in \{l, h\}$, we have* $\Phi_k\left(u_k + \lambda\right) \leq \Phi_k\left(u_k\right) + \lambda K_\varepsilon$ *for every* $u_k \in [u_k^m + \varepsilon, \bar{u}_k]$.

Take an optimal menu (u_l, u_h) such that $u_k \in [u_k^m + \varepsilon, \bar{u}_k]$. Since the seller cannot profitably deviate by offering the menu $(u_l + \lambda, u_h + \lambda)$, we have $\pi(u_l + \lambda, u_h + \lambda) \leq \pi(u_l, u_h)$, which is equivalent to

$$\sum_{k=l,h} \Phi_k(u_k + \lambda)(S_k(u_l + \lambda, u_h + \lambda) - u_k - \lambda) \leq \sum_{k=l,h} \Phi_k(u_k)(S_k(u_l, u_h) - u_k).$$

Using $S_k(u_l + \lambda, u_h + \lambda) = S_k(u_l, u_h)$ for $k = l, h$, we have:

$$\sum_{k=l,h} [\Phi_k(u_k + \lambda) - \Phi_k(u_k)](S_k(u_l, u_h) - u_k) \leq \lambda \sum_{k=l,h} \Phi_k(u_k + \lambda). \tag{19}$$

Since the R.H.S. of (19) is bounded by $\lambda \sum_{k=l,h} \Phi_k(\bar{u}_k)$ and $(S_k(u_l, u_h) - u_k) \geq \psi_\varepsilon$ by **Fact 1**, we have:

$$\Phi_k(u_k + \lambda) - \Phi_k(u_k) \leq \lambda \left(\frac{\sum_{k=l,h} \Phi_k(\bar{u}_k)}{\psi_\varepsilon} \right).$$

Therefore the fact follows if we set $K_\varepsilon \equiv \left(\frac{\sum_{k=l,h} \Phi_k(\bar{u}_k)}{\psi_\varepsilon} \right)$.

Conclusion of the proof. For each k, the function F_k is absolutely continuous on $[u_k^m, \bar{u}_k]$ if there exists a Lebesgue-integrable function f_k on $[u_k^m, \bar{u}_k]$ such that, for any $u_k \in [u_k^m, \bar{u}_k]$, $F_k(u_k) = F_k(\bar{u}_k) - \int_{u_k}^{\bar{u}_k} f_k(y)dy$. By Facts 2 and 3 above, for any $n \in \mathbb{N}$, F_k is absolutely continuous on $[u_k^m + \frac{1}{n}, \bar{u}_k]$, and hence there exists a function f_k^n such that, for any $u_k \in [u_k^m + \frac{1}{n}, \bar{u}_k]$, $F_k(u_k) = F_k(\bar{u}_k) - \int_{u_k}^{\bar{u}_k} f_k^n(y)dy$. We can pick, for instance, the sequence $(f_k^n)_{n=1}^\infty$ such that $f_k^n(y) = f_k^{n-1}(y)$ for y in $[u_k^m + \frac{1}{n-1}, \bar{u}_k]$ for all $n \geq 2$. Then define g_k^n by $g_k^n(y) = f_k^n(y)$ if $y \in [u_k^m + \frac{1}{n}, \bar{u}_k]$ and by $g_k^n(y) = 0$ if $y \in [u_k^m, u_k^m + \frac{1}{n})$. Then (g_k^n) is an increasing sequence of Lebesgue integrable functions on $[u_k^m, \bar{u}_k]$ and there exists a function f_k on this interval s.t. (g_k^n) converges to f_k pointwise on $(u_k^m, \bar{u}_k]$. It is easy to see that, for any $u_k \in (u_k^m, \bar{u}_k]$, $F_k(u_k) = F_k(\bar{u}_k) - \int_{u_k}^{\bar{u}_k} f_k(y)dy$. Further,

$$\begin{aligned}
& F_k(\bar{u}_k) - \int_{u_k^m}^{\bar{u}_k} f_k(y)dy \\
= & \lim_{n\to\infty} \left\{ F_k(\bar{u}_k) - \int_{u_k^m}^{\bar{u}_k} g_k^n(y)dy \right\} \\
= & \lim_{n\to\infty} \left\{ F_k(\bar{u}_k) - \int_{u_k^m + \frac{1}{n}}^{\bar{u}_k} f_k^n(y)dy \right\} \\
= & \lim_{n\to\infty} \left\{ F_k(u_k^m + \frac{1}{n}) \right\} \\
= & F_k(u_k^m).
\end{aligned}$$

The first equality follows by the Monotone Convergence Theorem, the second by definition of g_k^n, the third by choice of f_k^n, and the fourth follows because F_k is continuous at u_k^m, as established in Steps 1 and 2 above. Hence, $F_k(u_k) = F_k(\bar{u}_k) - \int_{u_k}^{\bar{u}_k} f_k(y)dy$ holds for any $u_k \in [u_k^m, \bar{u}_k]$, as required. Q.E.D.

Proof of Theorem 1. We proceed in three steps. First, we construct the support function $\hat{u}_l(\cdot)$. In the second step, we derive the equilibrium distribution over menus. In the last step, we show that firms cannot benefit from deviating to an out-of-equilibrium menu.

In order to better explain the Theorem, we relegate some technical steps to Section S3 of the Supplementary Material (Claims S1-S5).

Step 1 *Constructing the support function.*

Because of the ranking property of kernels, it follows that in any ordered equilibrium with support function $\hat{u}_l(\cdot)$,

$$\Lambda\left(F_h(u_h)|v\right) = \Lambda\left(F_l(\hat{u}_l(u_h))|v\right). \tag{20}$$

The equation above implies that sales to each type k are proportional to the probability of that type, p_k. Accordingly, the support function $\hat{u}_l(\cdot)$ describes the locus of indirect utility pairs $(\hat{u}_l(u_h), u_h)$ such that the proportion of sales to each type is constant.

Claim S1 shows that $\Phi_h(\cdot)$ and $\Phi_l(\cdot)$ (defined in (12)) are continuously differentiable, which justifies the first-order conditions (6) and (7). Claim S2 guarantees that if we differentiate (20) we obtain:

$$\hat{u}_l'(u_h) = \frac{\Lambda_1\left(F_h(u_h)|v\right) f_h(u_h)}{\Lambda\left(F_h(u_h)|v\right)} \left[\frac{\Lambda_1\left(F_l(\hat{u}_l(u_h))|v\right) f_l(\hat{u}_l(u_h))}{\Lambda\left(F_l(\hat{u}_l(u_h))|v\right)}\right]^{-1}. \tag{21}$$

Intuitively, the slope of the support function, $\hat{u}_l'(u_h)$, equals the ratio between the semi-elasticities of sales with respect to indirect utilities for each type of consumer.

The first-order conditions (6) and (7) provide an alternative expression for these semi-elasticities. Evaluated at the locus $(\hat{u}_l(u_h), u_h)$, with the help of (20), equations (6) and (7) can be rewritten as

$$p_k \frac{\Lambda_1\left(F_k(\hat{u}_k(u_h))|v\right) f_k(\hat{u}_k(u_h))}{\Lambda\left(F_k(\hat{u}_k(u_h))|v\right)} \left(S_k(\hat{u}_l(u_h), u_h) - u_k\right) = p_k - p_l \frac{\partial S_l}{\partial u_k}(\hat{u}_l(u_h), u_h), \tag{22}$$

for $k = h$ and $k = l$, respectively. In equilibrium, the optimality of firms' menus requires that the support function $\hat{u}_l(\cdot)$ simultaneously satisfies the first-order conditions (22) and equation (21). Combining these two equations leads to the differential equation (9) which describes how the utility of low-valuation consumers relates to the utility of high-valuation consumers in the equilibrium menus.

From Lemma 3, we know that the least generous menu in equilibrium is the Mussa and Rosen menu $(0, u_h^m)$. Hence, we require that the solution to (9) to satisfy the initial condition $\hat{u}_l(u_h^m) = 0$. Next we invoke Claim S3 to assert the existence and uniqueness of the differential equation (9) subject to $\hat{u}_l(u_h^m) = 0$. Moreover, Claim S3 shows that $\hat{u}_l'(u_h) > 0$ for all $u_h \in [u_h^m, \bar{u}_h]$. This will guarantee that the menus $(\hat{u}_l(u_h), u_h)$ are indeed ordered.

Finally, Claim S4 verifies that the constraint IC_l is never binding in any menu $(\hat{u}_l(u_h), u_h)$. Indeed, we are able to show that, for all $u_h \in [u_h^m, \bar{u}_h]$,

$$u_h - \hat{u}_l(u_h) \leq \bar{u}_h - \hat{u}_l(\bar{u}_h) < S_h^* - S_l^* < \triangle\theta q_h^*,$$

which, by Lemma 1, implies that IC_l is slack at any equilibrium menu.

Step 2 *Constructing the distribution over menus.*

In view of the support function $\hat{u}_l(\cdot)$, we can describe the equilibrium distribution over menus in terms of the distribution of indirect utilities to high-valuation consumers, $F_h(\cdot)$. For that we choose, for each u_h, the quantile $F_h(u_h)$ in such a way that all menus offered in equilibrium lead to the same expected profits as the Mussa-Rosen menu $\mathcal{M}^m$. This is reflected in the indifference condition (10). The argument in Claim S5 shows that the profits conditional on sale

$$p_l \left(S_l \left(\hat{u}_l \left(u_h\right), u_h\right) - \hat{u}_l \left(u_h\right)\right) + p_h \left(S_h \left(\hat{u}_l \left(u_h\right), u_h\right) - u_h\right)$$

are strictly decreasing in u_h. There then exists a unique increasing function $F_h(\cdot)$ that takes value zero at u_h^m and one at some $\bar{u}_h > u_h^m$ (and hence is a cdf); Claim S5 establishes that $\bar{u}_h < S_h^*$. This cdf, by Lemma 3, is the only candidate for an equilibrium distribution over high-type indirect utilities, given the support function $\hat{u}_l(\cdot)$ (by Lemma 3, if such a cdf describes the distribution over high-type payoffs in equilibrium, which we verify next, then it is absolutely continuous).

Step 3 *Verifying the optimality of equilibrium menus.*

It now remains to check that firms have no incentive to deviate from the putative equilibrium strategies. By construction, all menus (u_l, u_h) such that $u_h \in [u_h^m, \bar{u}_h]$ and $u_l = \hat{u}_l\left(u_h\right)$ yield the same profit. Moreover, it is easy to show that we may restrict attention to menus $(u_l', u_h') \in [u_h^m, \bar{u}_h] \times [u_l^m, \hat{u}_l\left(\bar{u}_h\right)]$. Hence, consider a menu $(u_l', u_h') \in [u_h^m, \bar{u}_h] \times [u_l^m, \hat{u}_l\left(\bar{u}_h\right)]$ such that $u_l' \neq \hat{u}_l\left(u_h'\right)$. We have that

$$\begin{aligned}
\pi(\hat{u}_l\left(u_h'\right), u_h') - \pi(u_l', u_h') &= \int_{u_l'}^{\hat{u}_l\left(u_h'\right)} \frac{\partial \pi\left(\tilde{u}_l, u_h'\right)}{\partial u_l} d\tilde{u}_l \\
&= \int_{u_l'}^{\hat{u}_l\left(u_h'\right)} \frac{\partial \pi\left(\tilde{u}_l, u_h'\right)}{\partial u_l} - \frac{\partial \pi\left(\tilde{u}_l, \hat{u}_l^{-1}\left(\tilde{u}_l\right)\right)}{\partial u_l} d\tilde{u}_l \\
&= \int_{u_l'}^{\hat{u}_l\left(u_h'\right)} \int_{\hat{u}_l^{-1}\left(\tilde{u}_l\right)}^{u_h'} \frac{\partial^2 \pi\left(\tilde{u}_l, \tilde{u}_h\right)}{\partial u_h \partial u_l} d\tilde{u}_h d\tilde{u}_l \\
&\geq 0.
\end{aligned}$$

The second equality follows because $\frac{\partial \pi(u_l,u_h)}{\partial u_l} = 0$ along the curve $\{(\hat{u}_l\left(u_h\right), u_h) : u_h \in [u_h^m, \bar{u}_h]\}$. The inequality follows because $\frac{\partial^2 \pi(\tilde{u}_l,\tilde{u}_h)}{\partial u_h \partial u_l} \geq 0$ for all $(\tilde{u}_l, \tilde{u}_h)$ by Lemma 2. Thus a deviation to menu (u_l', u_h') is unprofitable. This completes the proof of the Theorem 1. Q.E.D.

Proof of Corollary 1. The proof that $0 < \hat{u}_l'(u_h) < 1$ can be found in the proof of Claim S3 in the Supplementary Material. The proof that $\hat{u}_l(\cdot)$ is convex can be found in the proof of Corollary 3 below. Finally, the proof of $\hat{u}_l'\left(u_h^m\right) = 0$ follows from evaluating (9) at $(0, u_h^m)$. Q.E.D.

Proof of Corollary 2 Denote by F_k and $\hat{F}_k$ (with supports Υ_k and $\hat{\Upsilon}_k$) the distributions over indirect utilities in the ordered equilibrium associated with the degree of competition is v and $\hat{v}$, respectively. We will show that if $v > \hat{v}$ then F_k first-order stochastically dominates $\hat{F}_k$, and $\hat{\Upsilon}_k$ is a proper subset of Υ_k, for $k \in \{l, h\}$.

We first show that $F_h(u_h) \leq \hat{F}_h(u_h)$ for all u_h and hence F_h stochastically dominates $\hat{F}_h$. Since the support function does not depend on the degree of competition, the last claim implies that F_l stochastically dominates $\hat{F}_l$. Towards a contradiction, take $\tilde{u}_h$ such that $F_h(\tilde{u}_h) > \hat{F}_h(\tilde{u}_h)$. Without loss assume that $\tilde{u}_h \in \Upsilon_h$ (otherwise, replace $\tilde{u}_h$ with $\max \Upsilon_h$). Therefore, we have:

$$\begin{gathered}\Lambda(0 \mid v)\left[p_l S_l\left(0, u_h^m\right)+p_h\left(S_h\left(0, u_h^m\right)-u_h^m\right)\right] \\ =\Lambda\left(F_h(\tilde{u}_h) \mid v\right)\left[p_l\left(S_l\left(\tilde{u}_h, \hat{u}_l\left(\tilde{u}_h\right)\right)-\hat{u}_l\left(\tilde{u}_h\right)\right)+p_h\left(S_h\left(\tilde{u}_h, \hat{u}_l\left(\tilde{u}_h\right)\right)-\tilde{u}_h\right)\right]\end{gathered}$$

and

$$\begin{gathered}\Lambda(0 \mid \hat{v})\left[p_l S_l\left(0, u_h^m\right)+p_h\left(S_h\left(0, u_h^m\right)-u_h^m\right)\right] \\ =\Lambda\left(\hat{F}_h(\tilde{u}_h) \mid \hat{v}\right)\left[p_l\left(S_l\left(\tilde{u}_h, \hat{u}_l\left(\tilde{u}_h\right)\right)-\hat{u}_l\left(\tilde{u}_h\right)\right)+p_h\left(S_h\left(\tilde{u}_h, \hat{u}_l\left(\tilde{u}_h\right)\right)-\tilde{u}_h\right)\right],\end{gathered}$$

and hence

$$\frac{\Lambda\left(\hat{F}_h(\tilde{u}_h) \mid \hat{v}\right)}{\Lambda(0 \mid \hat{v})}=\frac{\Lambda\left(F_h(\tilde{u}_h) \mid v\right)}{\Lambda(0 \mid v)}. \tag{23}$$

On the other hand, $F_h(\tilde{u}_h) > \hat{F}_h(\tilde{u}_h)$ implies $\frac{\Lambda(F_h(\tilde{u}_h)|v)}{\Lambda(0|v)} > \frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)|v\right)}{\Lambda(0|v)}$.

Since $v > \hat{v}$ we have $\frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)|v\right)}{\Lambda(0|v)} > \frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)|\hat{v}\right)}{\Lambda(0|\hat{v})}$. To see this, notice that for a fixed $y \in (0,1)$ we have:

$$\begin{aligned}\frac{\Lambda(y|v)}{\Lambda(0|v)}-\frac{\Lambda(y|\hat{v})}{\Lambda(0|\hat{v})} &= \sum_{j=1}^{\infty} j\left(\frac{\omega_j(v)}{\omega_1(v)}\right) y^{j-1}-\sum_{j=1}^{\infty} j\left(\frac{\omega_j(\hat{v})}{\omega_1(\hat{v})}\right) y^{j-1} \\ &= \sum_{j=1}^{\infty} j y^{j-1}\left[\left(\frac{\omega_j(v)}{\omega_1(v)}\right)-\left(\frac{\omega_j(\hat{v})}{\omega_1(\hat{v})}\right)\right].\end{aligned}$$

By the assumption that $\Omega(v)$ dominates $\Omega(\hat{v})$ in the sense of likelihood-ratio order, it follows that if $\frac{\omega_j(v)}{\omega_j(\hat{v})}$ is increasing in j. Hence, for all j, we have $\frac{\omega_j(v)}{\omega_j(\hat{v})} \geq \frac{\omega_1(v)}{\omega_1(\hat{v})}$, which implies $\frac{\omega_j(v)}{\omega_1(v)} - \frac{\omega_j(\hat{v})}{\omega_1(\hat{v})} \geq 0$ for all j, with a strict inequality for at least one $j > 1$. Thus we have $\frac{\Lambda(F_h(\tilde{u}_h)|v)}{\Lambda(0|v)} > \frac{\Lambda\left(\hat{F}_h(\tilde{u}_h)|\hat{v}\right)}{\Lambda(0|\hat{v})}$, which contradicts (23).

Since F_h stochastically dominates $\hat{F}_h$ it follows that $\hat{\Upsilon}_k \subset \Upsilon_k$. Assume towards a contradiction that $\hat{\Upsilon}_k = \Upsilon_k$. In this case, let $\bar{u}_h$ the largest indirect utility that is provided to high-valuation consumers in both equilibria. By an argument similar to the one above, we obtain

$$\frac{\Lambda\left(\hat{F}_h(\bar{u}_h) \mid \hat{v}\right)}{\Lambda(0 \mid \hat{v})}=\frac{\Lambda(1 \mid \hat{v})}{\Lambda(0 \mid \hat{v})}=\frac{\Lambda(1 \mid v)}{\Lambda(0 \mid v)}=\frac{\Lambda\left(F_h(\bar{u}_h) \mid v\right)}{\Lambda(0 \mid v)},$$

which cannot happen when $v > \hat{v}$. Q.E.D.

Proof of Corollary 3. Because the high-valuation quality is constant at q_h^*, it is immediate that $x_h(\cdot)$ is decreasing in u_h. The same is true regarding the low-valuation price $x_l(\cdot)$ at any $u_h > u_h^c$ (since the low-valuation quality is constant at q_l^*). So take $u_h \in [u_h^m, u_h^c]$ and note that

$$x_l(u_h) = \theta_l \frac{u_h - \hat{u}_l(u_h)}{\Delta\theta} - \hat{u}_l(u_h).$$

Consider $\hat{u}_l'(u_h) = h(\hat{u}_l(u_h), u_h)$, where h is given by

$$h(u_l, u_h) = \left(\frac{S_l(u_l, u_h) - u_l}{S_h^* - u_h}\right) \frac{1 - \frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(u_l, u_h)}{1 - \frac{\partial S_l}{\partial u_l}(u_l, u_h)},$$

and note that $h(0, u_h^m) = 0$ (which implies that $\hat{u}_l'(u_h^m) = 0$). Therefore, since $\hat{u}_l(\cdot)$ and $h(\cdot,\cdot)$ are continuous,

$$x_l'(u_h) = \theta_l \frac{1 - \hat{u}_l'(u_h)}{\Delta\theta} - \hat{u}_l'(u_h) = \theta_l \frac{1 - h(\hat{u}_l(u_h), u_h)}{\Delta\theta} - h(\hat{u}_l(u_h), u_h) > 0$$

for all u_h sufficiently close to u_h^m.

Recall that there is a unique threshold u_h^c, above which low-valuation quality is efficient. The threshold u_h^c is the the unique value of u_h solving $u_h - \hat{u}_l(u_h) = \triangle\theta \cdot q_l^*$. Trivially we have $x_l'(u_h) < 0$ for all $u_h \geq u_h^c$. We will now show that $\hat{u}_l(\cdot)$ is convex for $u_h < u_h^c$. Note that the convexity of $\hat{u}_l(\cdot)$ and the facts that $x_l'(u_h^m) > 0$ and $x_l'(u_h^c) < 0$ imply that there exists a unique $u_h^d \in (u_h^m, u_h^c]$ such that $x_l'(u_h) > 0$ if and only if $u_h < u_h^d$. To see why $\hat{u}_l(\cdot)$ is convex, let us differentiate (9) to obtain that

$$\hat{u}_l''(u_h) = \frac{\frac{d}{du_h} S_l(\hat{u}_l(u_h), u_h) + \hat{u}_l'(u_h)\left(\frac{1}{\triangledown(u_h)} - 1\right)}{S_h^* - u_h} + \frac{S_l(\hat{u}_l(u_h), u_h) - \hat{u}_l(u_h)}{S_h^* - u_h} \triangledown'(u_h), \tag{24}$$

where

$$\triangledown(u_h) \equiv \frac{1 - \frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(\hat{u}_l(u_h), u_h)}{1 - \frac{\partial S_l}{\partial u_l}(\hat{u}_l(u_h), u_h)}.$$

It follows from the proof of Claim S3 in the Supplementary Material (in the paragraph before Equation (S44)) that $\triangledown(u_h) \in (0,1)$ for $u_h < u_h^c$ and that $\frac{d}{du_h} S_l(\hat{u}_l(u_h), u_h) > 0$. Moreover, straightforward differentiation shows that $\triangledown'(u_h) > 0$. Coupled together, these facts imply that $\hat{u}_l''(u_h) > 0$ for $u_h < u_h^c$, as claimed. Q.E.D.

Proof of Proposition 1. Follows directly from Corollary 3. Q.E.D.

Proof of Proposition 2. Recall that we have $x_l(u_h) \equiv \theta_l q_l(\hat{u}_l(u_h), u_h) - \hat{u}_l(u_h)$ and $x_h(u_h) \equiv \theta_h q_h^* - u_h$.

First assume that $u_h < u_h^c$. In this case, since $\Delta_p(u_h) \equiv x_h(u_h) - x_l(u_h)$, we have

$$\begin{aligned}
\Delta_p'(u_h) &= -1 - \left[\theta_l \frac{1 - h(\hat{u}_l(u_h), u_h)}{\Delta\theta} - h(\hat{u}_l(u_h), u_h)\right] \\
&= -\left(\frac{\theta_h}{\Delta\theta}\right)[1 - h(\hat{u}_l(u_h), u_h)] < 0,
\end{aligned}$$

because $h\left(\hat{u}_l\left(u_h\right), u_h\right) < 1$.

Next assume that $u_h \geq u_h^c$. In this case, we have $\Delta_p\left(u_h\right) = \hat{u}_l\left(u_h\right) - u_h$ and hence $\Delta_p'(u_h) = -\left[1 - h\left(\hat{u}_l\left(u_h\right), u_h\right)\right] < 0$. We conclude that $\Delta_p\left(u_h\right)$ is strictly decreasing in u_h, which immediately implies the proposition. Q.E.D.

Proof of Proposition 3. To show the result, it suffices to show that the markup difference is strictly decreasing in u_h. The markup difference can be written as: $\left[S_h^* - u_h\right] - \left[S_l\left(\hat{u}_l\left(u_h\right), u_h\right) - \hat{u}_l\left(u_h\right)\right]$. First notice that $\frac{d\left[S_h^* - u_h\right]}{du_h} = -1$. Moreover, we have

$$\begin{aligned}
& \frac{d}{du_h}\left[S_l\left(\hat{u}_l\left(u_h\right), u_h\right) - \hat{u}_l\left(u_h\right)\right] \\
= \quad & \frac{dS_l\left(q_l(\hat{u}_l\left(u_h\right), u_h)\right)}{dq_l}\left(\frac{\partial q_l(\hat{u}_l\left(u_h\right), u_h)}{\partial u_l}\hat{u}_l'\left(u_h\right) + \frac{\partial q_l(\hat{u}_l\left(u_h\right), u_h)}{\partial u_h}\right) - h\left(\hat{u}_l\left(u_h\right), u_h\right) \\
= \quad & \frac{dS_l\left(q_l(\hat{u}_l\left(u_h\right), u_h)\right)}{dq_l}\frac{\partial q_l(\hat{u}_l\left(u_h\right), u_h)}{\partial u_h}\left(1 - h\left(\hat{u}_l\left(u_h\right), u_h\right)\right) - h\left(\hat{u}_l\left(u_h\right), u_h\right).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
& \frac{d\left(\left[S_h^* - u_h\right] - \left[S_l\left(\hat{u}_l\left(u_h\right), u_h\right) - \hat{u}_l\left(u_h\right)\right]\right)}{du_h} \\
= \quad & -\left(1 - h\left(\hat{u}_l\left(u_h\right), u_h\right)\right) - \frac{dS_l\left(q_l(\hat{u}_l\left(u_h\right), u_h)\right)}{dq_l}\frac{\partial q_l(\hat{u}_l\left(u_h\right), u_h)}{\partial u_h}\left(1 - h\left(\hat{u}_l\left(u_h\right), u_h\right)\right) < 0,
\end{aligned}$$

which proves the Proposition. Q.E.D.

A New Indexation of Contracts

For $k \in \{l, h\}$, let $w_k\left(u_k\right) \equiv \frac{\Lambda(F_k(u_k)|v)}{\Lambda(0|v)}$ be the ratio between the sales to type k of an offer with payoff u_k and the sales of the monopoly offer (with payoff u_k^m). Since the monopolist's offer $(0, u_h^m)$ is always the less generous offer made in equilibrium (hence it yields the sales $\Lambda\left(0|v\right)$), every offer (u_l, u_h) that is optimal must satisfy:

$$w_l\left(u_l\right)p_l\left(S_l\left(u_l, u_h\right) - u_l\right) + w_h\left(u_h\right)p_h\left(S_h^* - u_h\right) = p_l S_l\left(u_l^m, u_h^m\right) + p_h\left[S_h^* - u_h^m\right]. \tag{25}$$

Recall that our sale function depends only on the ranking $y_k = F_k(u_k)$. Therefore, for every $w \in \left[1, \frac{\Lambda(1|v)}{\Lambda(0|v)}\right]$, there is a unique $y \in [0,1]$ such that $w = \frac{\Lambda(y|v)}{\Lambda(0|v)}$. Hence, for all $w \in \left[1, \frac{\Lambda(1|v)}{\Lambda(0|v)}\right]$, we may define $y(w)$ implicitly by: $w = \frac{\Lambda(y(w)|v)}{\Lambda(0|v)}$. Moreover, in any ordered equilibrium we have $w_l\left(u_l\right) = w_h\left(u_h\right)$ and the ranking of the menu ($y_k = F_k(u_k)$) is a strictly increasing function of u_k. Thus, since $w \to y(w)$ is strictly increasing, so is its inverse, which implies that the mapping $u_k \to w(u_k)$ is strictly increasing. It thus follows that we can index contracts by w and rewrite (25) as

$$wp_l\left(S_l\left(u_l(w), u_h(w)\right) - u_l(w)\right) + wp_h\left(S_h^* - u_h(w)\right) = p_l S_l\left(u_l^m, u_h^m\right) + p_h\left[S_h^* - u_h^m\right], \tag{26}$$

for every $w \in \left[1, \frac{\Lambda(1|v)}{\Lambda(0|v)}\right]$.

Proof of Proposition 4. From the discussion right above this proof, we know that there is a one-to-one mapping between the ranking of the offer $y = F_k(u_k)$ and the variable w. Hence, take $w \in \left[1, \frac{\Lambda(1|v)}{\Lambda(0|v)}\right)$, consider an increase in p_l from p_l^a to p_l^b and let, for $r \in \{a, b\}$ and $k \in \{l, h\}$, $u_k^r(w)$ be the indirect utility yielded by an offer with generosity w to type k in the unique ordered equilibrium in which the probability that consumer is a low type is p_l^r. Claim S7 (stated and proved in the Supplementary Material) shows that $u_h^b(w) > u_h^a(w)$.

First assume that $u_h^a(w) \leq u_h^b(0)$. In this case, we have

$$u_h^b(w) - u_l^b(w) > u_h^b(0) - u_l^b(0) = u_h^b(0) \geq u_h^a(w) \geq u_h^a(w) - u_l^a(w), \tag{27}$$

where the first inequality used the fact that $\hat{u}_l'(u_h) < 1$ for any equilibrium (see Corollary 1).

Next assume that $u_h^a(w) > u_h^b(0)$. In this case, there exists $\tilde{w} \in (1, w)$ such that $u_h^b(\tilde{w}) = u_h^a(w)$. It follows that

$$u_h^b(w) - u_l^b(w) > u_h^b(\tilde{w}) - u_l^b(\tilde{w}) = u_h^b(\tilde{w}) - \hat{u}_l^b\left(u_h^b(\tilde{w})\right), \tag{28}$$

where, for $r \in \{a, b\}$, we write $\hat{u}_l^r(\cdot)$ for the support function of the ordered equilibrium in which the probability that the consumer is a low type is p_l^r. Claim S6 (stated and proved in the Supplementary Material) shows that $\frac{\partial \hat{u}_l(u_h, p_l)}{\partial p_l} < 0$ and hence $\hat{u}_l^a\left(u_h^b(\tilde{w})\right) > \hat{u}_l^b\left(u_h^b(\tilde{w})\right)$, which implies $u_h^b(\tilde{w}) - \hat{u}_l^b\left(u_h^b(\tilde{w})\right) > u_h^b(\tilde{w}) - \hat{u}_l^a\left(u_h^b(\tilde{w})\right)$. Finally, since $u_h^b(\tilde{w}) = u_h^a(w)$ the last inequality implies $u_h^b(\tilde{w}) - \hat{u}_l^b\left(u_h^b(\tilde{w})\right) > u_h^a(w) - \hat{u}_l^a\left(u_h^a(w)\right) = u_h^a(w) - u_l^a(w)$. Combining this inequality with (28) we obtain

$$u_h^b(w) - u_l^b(w) > u_h^a(w) - u_l^a(w). \tag{29}$$

Therefore, we always have $u_h^b(w) - u_l^b(w) > u_h^a(w) - u_l^a(w)$. Since $q_l(u_l, u_h)$ is a nondecreasing function of the difference $(u_h - u_l)$, we conclude that for each w (and hence for each ranking $y(w)$), the low-valuation quality is weakly more efficient in equilibrium p_l^b. Q.E.D.

Proof of Proposition 5. The price differential satisfies

$$\Delta_p(u_h) = [\theta_h q_h^* - \theta_l q_l(\hat{u}_l(u_h), u_h)] - [u_h - \hat{u}_l(u_h)].$$

From Corollary 1 we have $\hat{u}_l'(u_h) < 1$ and hence $[u_h - \hat{u}_l(u_h)]$ strictly increases with u_h. Next notice that $q_l(\hat{u}_l(u_h), u_h)$ increases with the difference $[u_h - \hat{u}_l(u_h)]$ and hence $q_l(\hat{u}_l(u_h), u_h)$ increases with u_h. This implies that $\Delta_p(u_h)$ strictly decreases in u_h. Indexing the contracts for w instead of u_h, we immediately see that $\Delta_p(w)$ is strictly decreasing in w.

Now define $y(w, v)$ implicitly by $\left(\frac{\Lambda(y(w,v)|v)}{\Lambda(0|v)}\right) = w$. Take $v'' > v'$ and notice that $\left(\frac{\Lambda(y(w,v')|v'')}{\Lambda(0|v'')}\right) > \left(\frac{\Lambda(y(w,v')|v'')}{\Lambda(0|v'')}\right) = w$, which implies that $y(w, v'') < y(w, v')$. Hence $y(\tilde{w}, v'') = y(w, v')$ implies $\tilde{w} > w$. Therefore, since we have shown that $\Delta_p(w)$ is strictly decreasing in w, we conclude that $\tilde{\Delta}_p(y)$ is strictly decreasing in v.

Next we will show that $\tilde{\Delta}_p(y)$ is decreasing in p_l. Fix $y \in (0,1)$ and notice that the w that solves the equation $\left(\frac{\Lambda(y|v)}{\Lambda(0|v)}\right) = w$ does not depend on p_l. Hence, consider an increase in p_l from p_l^a to p_l^b. Using the notation introduced in the proof of Proposition 4, the argument that we used in that proof implies that $u_h^b(w) - u_l^b(w) > u_h^a(w) - u_l^a(w)$. Thus if, for $r \in \{a, b\}$, we let $\Delta_p^r(y)$ be the price differential when the probability that the consumer is a low type is p_l^r, we have

$$\Delta_p^b(y) - \Delta_p^a(y) = \theta_l\left[q_l\left(u_l^a(w), u_h^a(w)\right) - q_l\left(u_l^b(w), u_h^b(w)\right)\right] + \left[\left(u_h^a(w) - u_l^a(w)\right) - \left(u_h^b(w) - u_l^b(w)\right)\right],$$

which is less than zero, since

$$q_l\left(u_l^a(w), u_h^a(w)\right) \le q_l\left(u_l^b(w), u_h^b(w)\right)$$

and

$$u_h^a(w) - u_l^a(w) \le u_h^b(w) - u_l^b(w),$$

as shown in the proof of Proposition 4.

Finally, we show that if the degree of competition is sufficiently high ($v > \bar{v}$), the marginal impact of v on $\tilde{\Delta}_p(y)$ (in absolute terms) is decreasing in p_l. Let, for $k \in \{l, h\}$ and $y \in (0,1)$, u_k^y be defined by $F_k(u_k^y) = y$. Note first that there exists $\bar{v}$ such that for $v \ge \bar{v}$ we have $u_h^y - u_l^y > \Delta\theta q_l^*$. Then, for all such v, we have

$$\begin{aligned}
\frac{\partial \tilde{\Delta}_p(y)}{\partial v} = \frac{\partial\left(u_h^y - u_l^y\right)}{\partial v} &= \frac{\partial u_h^y}{\partial v}\left(1 - \frac{\frac{\partial u_l^y}{\partial v}}{\frac{\partial u_h^y}{\partial v}}\right) \\
&= \frac{\partial u_h^y}{\partial v}\left(1 - \hat{u}_l'\left(u_h^y\right)\right) \\
&= \frac{\partial u_h^y}{\partial v}\left(1 - \left(\frac{S_l(\hat{u}_l(u_h^y), u_h^y) - \hat{u}_l(u_h^y)}{S_h^* - u_h^y}\right)\frac{1 - \frac{p_l}{p_h}\cdot\frac{\partial S_l}{\partial u_h^y}(\hat{u}_l(u_h^y), u_h^y)}{1 - \frac{\partial S_l}{\partial u_l^y}(\hat{u}_l(u_h^y), u_h^y)}\right) \\
&= \frac{\partial u_h^y}{\partial v}\left(1 - \frac{S_l(\hat{u}_l(u_h^y), u_h^y) - \hat{u}_l(u_h^y)}{S_h^* - u_h^y}\right).
\end{aligned}$$

where the third equality uses (9) and the final equality uses the fact the incentive constraints are slack at the quantile y for large enough v.

Next, consider $p_l^{\#\#}, p_l^{\#}$ such that $p_l^{\#\#} < p_l^{\#}$. Therefore we have

$$\begin{aligned}
&\frac{d}{dp_l}\left(\frac{S_l^* - \hat{u}_l(u_h^y(p_l), p_l)}{S_h^* - u_h^y(p_l)}\right) \\
&= \left(\frac{-\frac{\partial \hat{u}_l(u_h^y(p_l), p_l)}{\partial u_h^y}\left[S_h^* - u_h^y(p_l)\right]\left(u_h^y\right)'(p_l) + \left(u_h^y\right)'(p_l)\left[S_l^* - \hat{u}_l(u_h^y(p_l), p_l)\right] - \frac{\partial \hat{u}_l(u_h^y(p_l), p_l)}{\partial p_l}\left[S_h^* - u_h^y(p_l)\right]}{\left[S_h^* - u_h^y(p_l)\right]^2}\right) \\
&= \left(\frac{-\frac{\partial \hat{u}_l(u_h^y(p_l), p_l)}{\partial p_l}\left[S_h^* - u_h^y(p_l)\right]}{\left[S_h^* - u_h^y(p_l)\right]^2}\right) > 0,
\end{aligned}$$

where the second line uses $-\frac{\partial \hat{u}_l(u_h^y(p_l),p_l)}{\partial u_h^y} = \frac{S_l^* - \hat{u}_l(u_h^y)}{S_h^* - u_h^y}$ and $\frac{\partial \hat{u}_l(u_h^y(p_l),p_l)}{\partial p_l} < 0$ follows from Claim S6 (stated and proved in the Supplementary Material). Thus we have

$$\frac{\partial}{\partial p_l}\left(1 - \frac{S_l(\hat{u}_l(u_h^y), u_h^y) - \hat{u}_l(u_h^y)}{S_h^* - u_h^y}\right) < 0. \tag{30}$$

Now, let $w_k = \frac{\Lambda(F_k(u_k)|v)}{\Lambda(0|v)}$ be the ratio of sales for an offer with payoff u_k to the sales for the monopoly offer to type k (with payoff u_k^m). Here we find convenient to index all contracts by the ratio of sales. Let $u_k(w_k)$ be the indirect utility of type k such that the sales ratio is w_k. It is easily verified that, for all large enough v, incentive constraints are slack when the sales ratio is $w_k = \frac{\Lambda(y|v)}{\Lambda(0|v)}$ for each k. It follows that we can write the expected profits by

$$w_l p_l \left(S_l^* - u_l^y(w_l)\right) + w_h p_h \left(S_h^* - u_h^y(w_h)\right)$$

for all (w_l, w_h) in a sufficiently small neighborhood of $\left(\frac{\Lambda(y|v)}{\Lambda(0|v)}, \frac{\Lambda(y|v)}{\Lambda(0|v)}\right)$. A first-order condition with respect to w_h yields

$$u_h'\left(\frac{\Lambda(y|v)}{\Lambda(0|v)}\right) = \frac{S_h^* - u_h^y(w_h)}{w_h} = \frac{S_h^* - u_h^y\left(\frac{\Lambda(y|v)}{\Lambda(0|v)}\right)}{\frac{\Lambda(y|v)}{\Lambda(0|v)}}.$$

Hence

$$\frac{\partial u_h^y}{\partial v} = u_h'\left(\frac{\Lambda(y|v)}{\Lambda(0|v)}\right)\frac{d}{dv}\left[\frac{\Lambda(y|v)}{\Lambda(0|v)}\right] = \left(S_h^* - u_h^y\left(\frac{\Lambda(y|v)}{\Lambda(0|v)}\right)\right)\frac{d}{dv}\left[\frac{\Lambda(y|v)}{\Lambda(0|v)}\right] > 0, \tag{31}$$

where $\frac{d}{dv}\left[\frac{\Lambda(y|v)}{\Lambda(0|v)}\right] > 0$ holds by our assumptions on the matching function. Again consider $p_l^{\#\#}, p_l^{\#}$ such that $p_l^{\#\#} < p_l^{\#}$, and index payoffs accordingly. Let $w = \frac{\Lambda(y|v)}{\Lambda(0|v)}$. Claim S7 (stated and proved in the Supplementary Material) shows that $u_h^{\#\#}(w) < u_h^{\#}(w)$, i.e. $S_h^* - u_h^y\left(\frac{\Lambda(y|v)}{\Lambda(0|v)}\right)$ is larger when p_l is smaller. Thus using (31) we conclude that

$$\frac{\partial^2 u_h^y}{\partial p_l \partial v} < 0. \tag{32}$$

Combining (30), (31) and (32) we obtain

$$\frac{\partial^2 \tilde{\Delta}_p(y)}{\partial p_l \partial v} = \frac{\partial^2\left(u_h^y - u_l^y\right)}{\partial p_l \partial v} = \frac{\partial^2 u_h^y}{\partial v \partial p_l}\left(1 - \frac{\frac{\partial u_l^y}{\partial v}}{\frac{\partial u_h^y}{\partial v}}\right) + \left(\frac{\partial u_h^y}{\partial v}\right)\left[\frac{\partial}{\partial p_l}\left(1 - \frac{\frac{\partial u_l^y}{\partial v}}{\frac{\partial u_h^y}{\partial v}}\right)\right] < 0.$$

Q.E.D.

Appendix B: Ordered and Unordered Equilibria

We now describe when the ordered equilibrium of Theorem 1 is the only equilibrium. Recall from Lemma 2 that a firm's expected profits π satisfy strict increasing differences when the incentive constraint IC_h binds (i.e., for menus with $u_h \leq u_h^c$, where u_h^c is the efficiency threshold). In this case, we argued that higher indirect utilities u_h must imply higher indirect utilities to low-valuation consumers u_l, i.e. the equilibrium menus must be ordered. Hence, if IC$_h$ binds for all equilibrium menus, ordering is enough to uniquely pin down equilibrium.

However, if IC$_h$ is slack, firms offering high indirect utilities u_h to high-valuation consumers no longer have a comparative advantage in offering high utilities u_l to low-valuation consumers, and so equilibrium menus need not be ordered. A multiplicity of equilibria then arises due to the possible non-ordering of menus for which the incentive constraint IC$_h$ is slack (i.e., menus for which $u_h > u_h^c$). We summarize these observations as follows.

Theorem 2 ***[Equilibrium Uniqueness].*** *There exists a threshold $v^c > 0$ on the degree of competition such that:*

(a) If $v \leq v^c$, the top of the support of F_h, $\bar{u}_h$, satisfies $\bar{u}_h \leq u_h^c$. The downward incentive constraint (IC$_h$) is then binding for all menus offered in the ordered equilibrium. In this case, the only equilibrium is the ordered equilibrium.

(b) If $v > v^c$, then $\bar{u}_h > u_h^c$, and the downward incentive constraint (IC$_h$) is slack for all menus in the ordered equilibrium with $u_h > u_h^c$, and binding for $u_h \leq u_h^c$. In this case, there exist multiple equilibria that differ only in the menus for which $u_h > u_h^c$ (i.e., the efficient menus). However, all equilibria (including the non-ordered ones) lead to the same marginal distributions over indirect utilities $F_k(\cdot)$, and the same ex-ante profits for firms.

The proof, presented below, shows that in *any* equilibrium, when the degree of competition is small (i.e., $v \leq v^c$), the support of utilities of type-k consumers, Υ_k, is contained in $[u_k^m, u_k^c]$. Using the increasing differences property (see Lemma 2), we show that this implies that all equilibria are equal to the ordered equilibrium.

By contrast, when the degree of competition is large (i.e., $v > v^c$), some menus offered in the ordered equilibrium exhibit non-binding incentive constraints. Consider such a menu $(\hat{u}_l(u_h), u_h)$, in which case $u_h \in (u_h^c, \bar{u}_h]$. For this menu, the profit function $\pi(u_l, u_h)$ is locally modular, i.e. its cross-partial derivative is zero. As a result, for some (small) $\varepsilon > 0$, both the menus $(\hat{u}_l(u_h - \varepsilon), u_h)$ and $(\hat{u}_l(u_h), u_h - \varepsilon)$ are profit-maximizing for the firm. Based on the ordered equilibrium, we can thus construct a non-ordered equilibrium by replacing the menus $(\hat{u}_l(u_h), u_h)$ and $(\hat{u}_l(u_h - \varepsilon), u_h - \varepsilon)$ by their non-ordered counterparts $(\hat{u}_l(u_h - \varepsilon), u_h)$ and $(\hat{u}_l(u_h), u_h - \varepsilon)$. Theorem 2 confirms that this is the unique source of multiplicity of equilibria in our economy.

Proof of Theorem 2. For the unique ordered distribution described in Theorem 1, there is a value v^c such that $v \leq v^c$ implies $\bar{u}_h \leq u_h^c$, while $v > v^c$ implies $\bar{u}_h > u_h^c$. What remains to show is that, for $v \leq v^c$, the only equilibrium is the ordered equilibrium (i.e., Part 1 of the Theorem) as well as the uniqueness claims in Part 2 (i.e., regarding menus with payoffs $u_h \leq u_h^c$ and regarding the marginal distributions F_k).

Let $\tilde{F}$ be any distribution over menus which describes a (not necessarily ordered) equilibrium. Let the marginal distributions over indirect utilities be given by F_k with supports Υ_k as given in Lemma 3. We begin with the following lemma.

Lemma 4 *Consider two equilibrium menus $(u_l, u_h), (u_l', u_h') \in \Upsilon_l \times \Upsilon_h$. If $u_h' > u_h$, then either $u_l' \geq u_l$ or both IC_h and IC_l are slack for both menus (i.e., $u_h - u_l, u_h' - u_l' \in [q_l^* \Delta\theta, q_h^* \Delta\theta]$).*

Proof of Lemma 4. Suppose $u_h' > u_h$ and $u_l' < u_l$, while either $u_h - u_l \notin [q_l^* \Delta\theta, q_h^* \Delta\theta]$ or $u_h' - u_l' \notin [q_l^* \Delta\theta, q_h^* \Delta\theta]$. By Lemma 2, we have

$$\pi\left(u_l, u_h'\right) + \pi\left(u_l', u_h\right) > \pi\left(u_l', u_h'\right) + \pi\left(u_l, u_h\right),$$

contradicting the optimality of (u_l, u_h) or (u_l', u_h'). Q.E.D.

An immediate implication of this lemma is that if (u_l, u_h) is a menu for which IC_l or IC_h binds (i.e., $u_h - u_l \notin [q_l^* \Delta\theta, q_h^* \Delta\theta]$), then there exists no other equilibrium menu (u_l', u_h') for which $u_l' < u_l$ and $u_h' > u_h$ or $u_l' > u_l$ and $u_h' < u_h$. Since F_l and F_h are absolutely continuous by Lemma 3, we can conclude hence that $F_l(u_l) = F_h(u_h)$.

Next, note that there exists $\varepsilon > 0$ such that IC_h binds for all $u_h \leq u_h^m + \varepsilon$. Thus, for every menu (u_l, u_h) with $u_h \leq u_h^m + \varepsilon$, we have $F_l(u_l) = F_h(u_h)$. Define a strictly increasing and continuous function κ by $\kappa(u_h) \equiv F_l^{-1}(F_h(u_h))$ (here we use Lemma 3, which guarantees both the continuity of F_l and F_h and that both are strictly increasing). Using Lemma 4, it is easy to see that there can be no menu (u_l, u_h) with $u_l < \kappa(u_h^m + \varepsilon)$ but $u_l \neq \kappa(u_h)$. Thus, we have established that, for any equilibrium menu (u_l, u_h), with $u_h < u_h^m + \varepsilon$ or $u_l < \kappa(u_h^m + \varepsilon)$, $u_l = \kappa(u_h)$. The arguments in Step 1 of the proof of Theorem 1 then imply that $\kappa(\cdot) = \hat{u}_l(\cdot)$ on $[u_h^m, u_h^m + \varepsilon)$.

We can extend the above argument to show that all menus (u_l, u_h) with $u_h < u_h^c$ or $u_l < \hat{u}_l(u_h^c)$ must also be given by $(\hat{u}_l(u_h), u_h)$ for some $u_h < u_h^c$. To see this, let

$$\breve{u}_h \equiv \sup\left\{u_h : \forall \text{ eqm menus } \left(u_l', u_h'\right), u_h' < u_h \text{ or } u_l' < \hat{u}_l(u_h) \text{ implies } u_l' = \hat{u}_l\left(u_h'\right)\right\}. \tag{33}$$

As argued above, $\breve{u}_h > u_h^m$. Suppose with a view to contradiction that $\breve{u}_h < u_h^c$. Since we must have $u_l \geq \hat{u}_l(\breve{u}_h)$ for any equilibrium menu with $u_h \geq \breve{u}_h$, there must exist $\eta > 0$ sufficiently small that IC_h binds for all $u_h \leq \breve{u}_h + \eta$ (indeed, this must follow because IC_h binds at $(\hat{u}_l(\breve{u}_h), \breve{u}_h)$). The same arguments as above then imply that, for any equilibrium menu (u_l, u_h) with $u_h < \breve{u}_h + \eta$ or $u_l < \hat{u}_l(\breve{u}_h + \eta)$, $u_l = \hat{u}_l(u_h)$. Hence, $\breve{u}_h$ cannot be the supremum in (33), a contradiction.

Thus, we have established that $\breve{u}_h \geq u_h^c$. This establishes Part 1 of the theorem: In case $v \leq v^c$, we have $u_h \leq u_h^c$ for all equilibrium menus, as implied by the requirement that all menus generate the same expected profits. This also establishes our claim in Part 2 that non-ordered equilibria differ only in menus for which $u_h > u_h^c$ (the existence of such non-ordered equilibria is straightforward).

To establish our remaining claims, we consider menus for which $u_h \geq u_h^c$ and $u_l \geq \hat{u}_l(u_h^c)$. We show that (recall that $\Phi_k(u_k) \equiv p_k \Lambda(F_k(u_k) \mid v)$)

$$\Phi_h'(u_h) = \frac{\Phi_h(u_h)}{S_h^* - u_h} \tag{34}$$

$$\Phi_l'(u_l) = \frac{\Phi_l(u_l)}{S_l^* - u_l} \tag{35}$$

for these values of u_h and u_l. This implies that Φ_l and Φ_h are precisely those functions determined in Theorem 1; hence, the marginal distributions F_k are identical in any equilibrium. As a result, as shown in the proof of Theorem 1, neither incentive constraint can bind for equilibrium menus with $u_h \geq u_h^c$ and $u_l \geq \hat{u}_l(u_h^c)$ (a binding incentive constraint at (u_l, u_h) would imply $F_l(u_l) = F_h(u_h)$, but then $u_l = \hat{u}_l(u_h)$ and neither incentive constraint binds at $(\hat{u}_l(u_h), u_h)$ as shown in the proof of Theorem 1).

It is easy to see that the equilibrium menu with high-valuation payoff u_h^c is unique and equal to $(\hat{u}_l(u_h^c), u_h^c)$.[51] Neither of the incentive constraints IC_l or IC_h binds at this menu. This allows us to establish that (34) and (35) hold at $(\hat{u}_l(u_h^c), u_h^c)$. We consider (34) as the case of (35) is analogous. We use a similar argument to that in Step 1 of the proof of Theorem 1. For any $\varepsilon \in \mathbb{R}$ such that $u_h^c + \varepsilon \in \Upsilon_h$, let $(u_{l,\varepsilon}, u_h^c + \varepsilon)$ be a corresponding equilibrium menu. The same arguments as in the proof of Theorem 1 imply

$$\begin{aligned}
&\frac{\left(\begin{array}{c} \Phi_l(u_{l,\varepsilon})\left[S_l(u_{l,\varepsilon}, u_h^c) - S_l(u_{l,\varepsilon}, u_h^c + \varepsilon)\right] \\ +\Phi_h(u_h^c)\left[S_h(u_{l,\varepsilon}, u_h^c) - S_h(u_{l,\varepsilon}, u_h^c + \varepsilon) + \varepsilon\right] \end{array}\right)}{\varepsilon\left[S_h(u_{l,\varepsilon}, u_h^c + \varepsilon) - u_h^c - \varepsilon\right]} \\
&\leq \frac{\Phi_h(u_h^c + \varepsilon) - \Phi_h(u_h^c)}{\varepsilon} \\
&\leq \frac{\left(\begin{array}{c} \Phi_l(\hat{u}_l(u_h^c))\left[S_l(\hat{u}_l(u_h^c), u_h^c) - S_l(\hat{u}_l(u_h^c), u_h^c + \varepsilon)\right] \\ +\Phi_h(u_h^c)\left[S_h(\hat{u}_l(u_h^c), u_h^c) - S_h(\hat{u}_l(u_h^c), u_h^c + \varepsilon) + \varepsilon\right] \end{array}\right)}{\varepsilon\left[S_h(\hat{u}_l\left(u_h^c\right), u_h^c + \varepsilon) - u_h^c - \varepsilon\right]}.
\end{aligned}$$

[51] By the previous argument, any equilibrium menu (u_l, u_h^c) must satisfy $u_l \geq \hat{u}_l(u_h^c)$. If $u_l > \hat{u}_l(u_h^c)$, then IC_h binds at (u_l, u_h^c) implying that $F_l(u_l) = F_h(u_h^c)$, a contradiction (since $F_h(u_h^c) = F_l(\hat{u}_l(u_h^c))$ by the previous argument and continuity of F_l and F_h).

We then use that[52]

$$\lim_{\varepsilon\to 0}\frac{S_l(u_{l,\varepsilon},u_h^c)-S_l(u_{l,\varepsilon},u_h^c+\varepsilon)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{S_l(\hat{u}_l\left(u_h^c\right),u_h^c)-S_l(\hat{u}_l\left(u_h^c\right),u_h^c+\varepsilon)}{\varepsilon}=0$$

and

$$S_h(u_{l,\varepsilon},u_h^c)-S_h(u_{l,\varepsilon},u_h^c+\varepsilon)=S_h(\hat{u}_l\left(u_h^c\right),u_h^c)-S_h(\hat{u}_l\left(u_h^c\right),u_h^c+\varepsilon)=0$$

to conclude that

$$\Phi_h'\left(u_h^c\right)=\frac{\Phi_h\left(u_h^c\right)}{S_h^*-u_h^c}.$$

Next, observe that there exists $\eta>0$ such that incentive constraints are slack for any equilibrium menu with $u_h\in[u_h^c,u_h^c+\eta]$. This is obtained from (i) the above observation that if (u_l,u_h) is a menu for which an incentive constraint IC_k binds, then $F_l\left(u_l\right)=F_h\left(u_h\right)$, and (ii) $u_h^c-\hat{u}_l\left(u_h^c\right)=q_l^*\Delta\theta$ together with $\Phi_h'\left(u_h^c\right)<\Phi_l'\left(\hat{u}_l\left(u_h^c\right)\right)$ (equivalently, $F_h'\left(u_h^c\right)<F_l'\left(\hat{u}_l\left(u_h^c\right)\right)$).

As with the derivatives $\Phi_h'\left(u_h^c\right)$ and $\Phi_l'\left(\hat{u}_l\left(u_h^c\right)\right)$, one obtains (34) and (35) on $[u_h^c,u_h^c+\eta]$. We then use again that $F_l\left(u_l\right)=F_h\left(u_h\right)$ for any menu (u_l,u_h) for which an incentive constraint binds to obtain that the constraints must be slack for any equilibrium menu with $u_h\geq u_h^c$. To see this, let

$$u_h^{\#}=\sup\left\{u_h: IC_l \text{ and } IC_h \text{ are slack for all eqm. menus } \left(u_l',u_h'\right) \text{ with } u_h'\in[u_h^c,u_h]\right\}.$$

The above property, together with continuity of F_l and F_h, implies that, if $u_h^{\#}<\bar{u}_h$, then $u_h^{\#}-u_l^{\#}\notin(\Delta\theta q_l^*,\Delta\theta q_h^*)$ for $u_l^{\#}$ satisfying $F_l(u_l^{\#})=F_h(u_h^{\#})$. However, F_l and F_h must agree with functions derived in Theorem 1 on, respectively, $[u_l^m,\hat{u}_l(u_h^{\#})]$ and $[u_h^m,u_h^{\#}]$. Hence $u_l^{\#}=\hat{u}_l(u_h^{\#})$. This contradicts our finding in the proof of Theorem 1 that $u_h^{\#}-\hat{u}_l(u_h^{\#})\in(\Delta\theta q_l^*,\Delta\theta q_h^*)$. Q.E.D.

[52] This follows after noticing that, for any $\nu>0$, there exists $\iota>0$ such that, for all $|\varepsilon|<\iota$, $u_h^c-u_{l,\varepsilon}\in(\Delta\theta q_l^*-\nu,\Delta\theta q_h^*]$. Indeed, either both incentive constraints are slack at $(u_{l,\varepsilon},u_h^c+\varepsilon)$, or one of IC_l or IC_h bind, in which case $u_{l,\varepsilon}=F_l^{-1}\left(F_h\left(u_h^c+\varepsilon\right)\right)$, which tends to $\hat{u}_l\left(u_h^c\right)$ as $\varepsilon\to 0$.