Chap. I: The Lucas Critique

Patrick Fève

Toulouse School of Economics

GREMAQ, IDEI and IUF

Adresse e-mail: `patrick.feve@tse-eu.fr`

Internet: See the Website of TSE

MF 508

L3–TSE: Macroeconomics [redacted]

[redacted]

No Dynamic Optimizing Problems in this chapter.

Just Examples that illustrates the Lucas critique.

But some technicalities about solving forward–looking stochastic models.

Next year in Macro-M1 course (doctoral and standard track): explicit solutions to Dynamic Optimizing Problems (stochastic permanent income model, dynamic labor demand /supply models, RBC models, all of them constitutes different illustrations of this critique)

These notes are self contained!!!

Please redo all the models and problems that are expounded here!

Warning!!! This notes can contain many (Mathematical and English) typos!!!

Figure 1: Robert Lucas

Figure 2: Thomas Sargent

Robert Lucas, Nobel Laureate 1995.

Professor of Economics, University of Chicago.

Wikipedia: http://fr.wikipedia.org/wiki/Robert_E._Lucas

Nobel Lecture:

http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/1995/ lucas-lecture.html

The paper related to this chapter

Lucas, Robert. 1976. "Econometric policy evaluation: A critique", Carnegie-Rochester Conference Series on Public Policy, vol. 1(1), pages 19-46, January.

Thomas Sargent, Nobel Laureate 2011.

Professor of Economics, New–York University.

Wikipedia: http://fr.wikipedia.org/wiki/Thomas_Sargent

Nobel Lecture:

http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2011/sargent-lecture_slides.pdf

The paper related to this chapter

Lucas, Robert and Thomas Sargent. 1979. "After Keynesian Macroceconomics", Federal Reserve Bank of Minneapolis Quarterly Review, 3:2, Spring, 1979..

Other (Simple, but Useful) References

Collard,F., Feve, P. and F. Langot "Structural Inference and the Lucas Critique", **Annals of Economics and Statistics**, 2002, issue 67-68.

Collard, F. and P. Feve, "Sur les causes et les effets en macroeconomie : les Contributions de Sargent et Sims, Prix Nobel d'Economie 2011", **Revue d'Economie Politique**, 2012, Dalloz, vol. 122(3), pages 335-364.

Feve, P. and S. Gregoir "The Econometrics of Economic Policy", **Annals of Economics and Statistics**, 2002, issue 67-68, pages 1-19.

Linde, J. "Testing for the Lucas Critique: A Quantitative Investigation", **American Economic Review**, 2001, vol. 91, no. 4, pp. 986-1005.

A Simple Intuition

Banque de France (in Paris) has never been robbed.

This does not mean the guards can safely be eliminated.

The incentive not to rob Banque de France depends on the presence of the guards.

Because of the heavy security that exists at the Banque de France today, criminals are unlikely to attempt a robbery because they know they are unlikely to succeed.

But a change in security policy, such as eliminating the guards, would lead criminals to reappraise the costs and benefits of robbing the fort.

So just because there are no robberies under the current policy does not mean this should be expected to continue under all possible policies.

In more technical (economic) terms:

"Given that the structure of an econometric model consist of optimal decision rules of economic agents, and that optimal decision rule vary systematically with change in the structure of series relevant of the decision maker, it follows that any change in policy will systematically alter the structure of econometric models".

LUCAS [1976, p. 40]

Introduction

- Why studying the Lucas critique? See Lucas, 1976.
- What are the implications of the Lucas critique? (macro modeling, policy evaluation), See Lucas and Sargent, 1979.
- Does the Lucas critique quantitatively matter? See the the references.

Why studying the Lucas critique?

This critique represents a huge break in the macro research agenda. (a revolution!)

The central role of expectations (rational expectations).

Dynamic economies (forward–looking behavior).

New tools for macroeconomics (dynamic optimization, dynamic programming, partial and general equilibrium, micro-foundations of macroeconomics, stochastic process, tools of time series econometrics, conduct of economic policy).

The Legacy of Lucas and Sargent: A Modern Macroeconomics must be able to manage all these aspects of macro-economics.

What are the implications of the Lucas critique?

The irrelevance of existing macro-Econometric Models for policy evaluation.

Changes in economic policy (fiscal policy, monetary policy) will affect the behavior of the private agents (households, firms)

The timing and the time profile of the economic policy.

Timing: new policy today or tomorrow.

Time profile: persistence of the policy (transitory versus permanent change in economic policy)

Really relevant for policy evaluation (see labor friendly fiscal reform in France)

Does the Lucas critique quantitatively matter?

Theoretically, the Lucas critique matters (except in very particular cases).

The key issue: the quantitative importance of this critique.

If not, we can still use "old–fashioned" macro-econometric model for policy evaluation.

If yes, we must abandon the previous setup and then develop models that are more immune to this critique. See the recent development of Dynamic Stochastic General Equilibrium (DSGE) modeling in central banks.

The question: how to test for the Lucas critique?

The setup: take advantage of the cross–equation restrictions to evaluate the empirical relevance of this critique.

Identify a shift in (fiscal and/or monetary) policy.

Then investigate if some behavioral equations (consumption, investment, labor demand/supply, money demand) display instability when policy changes.

Plan of this Chapter

I- Macroeconometric Modelling

- The Benchmark Macro-Econometric Modelling before the Critique
  - The Theoretical Foundations
  - Macro-Econometric Models
- The Limits of the Macro-Econometric Modelling.
  - Historical Events
  - Two Central Criticisms

II- The Lucas Crtique

- A Formal Representation of the Critique
  - Modeling Expectations
  - Examples
  - The Solution
- The Lucas Critique in Practise
  - The Formal Case
  - Illustrations

I- Macroeconometric Modelling

- The Benchmark Macro-Econometric Modelling before the Critique
  - The Theoretical Foundations
  - Macro-Econometric Models
- The Limits of the Macro-Econometric Modelling.
  - Historical Events
  - Two Central Criticisms

I- Macroeconometric Modelling

The Benchmark Macro-Econometric Modelling before the Critique

The Theoretical Foundations

The Keynesian model

The IS-LM Model

The AS–AD Model

The AS–AD Model with a Phillips curve

Property: Keynesian properties in the short–run (demand side), but neo-classical properties in the very long-run (supply side)

But: Adaptative (or Naive) expectations. Backward–looking behavior. No prospect of private agents (households, firms) + Systematic errors. Private agents cannot develop optimal strategic behaviors (in the Game Theory sense) following any change in the public (government, central bank) behavior.

Idea related to the Lucas Critique (and the Modern Macro): private agent can develop optimal responses to any change in their economic environment (exogenous shocks, economic policy).

An simple theoretical foundation of “old-fashioned” macro-models

The simple Keynesian model (demand side)

Consumption function

$$C_t = C_o + c(Y_t - T_t)$$

Aggregate ressource constraint

$$Y_t = C_t + I_t + G_t$$

Assumptions

$$I_t = \bar{I} \quad T_t = \bar{T}$$

Public spending is exogenous (and stochastic).

Suppose that we can summarize the government spending policy by the following (useful) stochastic process

$$G_t = \bar{G} + \tilde{G}_t$$

where $\bar{G}$ is a positive constant and $\tilde{G}_t$ follows an autoregressive process of order one (say AR(1) in the time series econometrics literature):

$$\tilde{G}_t = \rho\tilde{G}_{t-1} + \epsilon_t$$

Interpretation This representation has two component.

i) a rule for government spending: $\bar{G}$ and ρ.

$\bar{G}$: the government commits to maintain a fraction of G_t constant over time

ρ: the government announces that part of government spending display some degree of persistence, depending on the value of ρ.

Ex1: $\rho = 0$ no persistence. $\tilde{G}_t = \epsilon_t$ and $G_t = \bar{G} + \epsilon_t$

Ex2: $\rho = 0$ permanence (infinite persistence, unit root). $\tilde{G}_t = \tilde{G}_{t-1} + \epsilon_t$ and $G_t = \bar{G} + \tilde{G}_t$. After some computations for $\tilde{G}_t$ (**show the details on the blackboard**), this leads to

$$G_t = \bar{G} + \sum_{i=0}^{\infty} \epsilon_{t-i}$$

ii) a discretionary component for government spending: ϵ_t.

The variable ϵ_t simply represents the discretion in the government policy. The government can freely choose every period to change (randomly) its policy. **Say more on economic interpretations**

Important Questions: Does the type of government spending policy (say the value of ρ) matters for the consumption behavior and the aggregate government spending multiplier? Does the discretionary policy ϵ_t has different impact on the consumption function and the aggregate multiplier (compared to the permanent policy $\bar{G}$)?

Answer No!

Demonstration Solving this model.

Consumption

$$C_t = C_o + c(Y_t - \bar{T})$$

We immediately observe the the parameter c (Marginal Propensity to Consume) is invariant to economic policy (in the Keynesian setup, this parameter is considered as a “deep” parameter, unafected by changes in the economic environment of the consumer).

Aggregate Demand

$$Y_t^D = C_t + \bar{I} + G_t$$

So, because the equilibrium output is determined by the lel of demand in the

Keynesian model,

$$Y_t = Y_t^D$$

we deduce

$$Y_t = Y_t^D \equiv C_o + c(Y_t - \bar{T}) + \bar{I} + G_t$$

or equivalently

$$Y_t = \frac{1}{1-c}\left(C_o - c\bar{T} + \bar{I} + G_t\right)$$

We deduce the government spending multiplier (on impact)

$$\frac{\Delta Y_t}{\Delta G_t} = \frac{1}{1-c}$$

We obtain that the government spending multiplier does not depend on the form of the government spending policy ($\bar{G}$, ρ and ϵ_t)

In addition, we have on impact (we will show latter that this will be not true!)

$$\frac{\Delta G_t}{\Delta \bar{G}} = \frac{\Delta G_t}{\Delta \epsilon_t}$$

This is typically that we obtain from this theoretical setup and to its extension during the 60' and the beginning of 70'.

So, any policy experiment can be conducted whatever the form and the timing of the economic policy.

This type of model (and its extensions) has served as the theoretical foundations of modern macro-econometric modeling.

Macro-Econometric Models

On the basis of these theoretical foundations, Macro-Econometric Models have been developed since the second WWII (but especially during the 60' and 70').

These models have been widely used by the governments, central administrations and central banks to evaluate various policy options about different economic (fiscal and monetary) policies.

Question What are Macro-Econometric Models???

A collection (or a system) of equations.

- <u>Behavioral equations:</u> consumption, investment, factor demands, money demand, ...

- Technical equations: input-output matrix on intermediate goods consumption.

- Accounting equation: equilibrium on different markets (ex: $Y = C + I + G$ in a closed economy, or $Y = C + I + G + X - Im$ in an open economy)

- Additional equations (to close the model): the most well known example is the Phillips curve.

Formally, a Macro-Econometric Models can be written in the following general form

$$F(Y_t, X_t, \theta) = 0$$

where Y_t is a set of endogenous variables (GDP, consumption, investment, prices, wages, ...), X_t is a set of exogenous variables (fiscal variables, rest of the world,...) and θ a vector that includes all the parameters related to the behavior, technical and additional equations of the model.

The previous equation is the structural representation of the economy (so, with simultaneity between the elements of Y_t).

Example: in the simple previous Keynesian model, the structural form is given by the set of equations:

Consumption function

$$C_t = C_o + c(Y_t - T_t)$$

Aggregate ressource constraint

$$Y_t = C_t + I_t + G_t$$

It follows that Y_t cannot be considered as an exogenous variable in the consumption function.

Suppose that C increases because Y increases after a government spending expansion.

It follows that Y will thus increases.

So Y is endogenous in the consumption function.

The reduced form will account for that!

The reduced form expressed the set of endogenous variables as a function of exogenous (and pre-determined, if the model is dynamic) variables.

$$Y_t = f(X_t, \theta)$$

Example: in the simple previous keynesian model, the reduced form (given the assumptions on I and G) is given by:

$$Y_t = \frac{1}{1-c}\left(C_o - c\bar{T} + \bar{I} + G_t\right)$$

we can also deduce consumption

$$C_t = C_o + \frac{c}{1-c}\left(C_o - c\bar{T} + \bar{I} + G_t\right)$$

A difficulty: the vector is unknown, but we can estimate this vector using econometric techniques (see the L3 Econometric Course for in introduction to basic econometrics and next years M1 and M2 for extensions).

See the development of Econometrics in the 60' and 70', that provided useful tools to identify and thus consistently estimate the parameters of this type of model.

So, if we are able to do this, we can replace the unknown parameter θ by its consistent estimate $\hat{\theta}$

$$Y_t = f(X_t, \hat{\theta})$$

From this simple reduced form, the policy maker can conduct different types of useful quantitative experiments.

Two types of quantitative experiments:

- Forecasting
- Policy Evaluation

(others: optimal policy experiments)

We do not consider the first type of experiment, because the Lucas critique only applies for the second.

This is the object of this chapter!

The Limits of the Macro-Econometric Modelling

Historical Events

In the beginning, mid and end of the 70', these models faces serious difficulties to explain:

1) the instability of the Phillips curve

2) the huge decrease in aggregate activity after the two oil price shocks.

In addition, these model are very large scaled (some of them exceeds 1000 equation; ex: in France, the model METRIC includes more than 500 equations, the model DMS more than 1500 equations and the model PROPAGE around 6000 equations!!!!).

So very large maintenance costs (wage costs, informatics, database, ...) with relatively poor quantitative performances.

These types of models become less and less used (unless in academics–University, still used in large institutions).

Two Central Criticisms

- The Lucas Critique
- The Sims Critique

The first one will be now presented in details!

The second one is due to Chris. Sims, Nobel Laureate with Tom Sargent in 2001.

[Say more on Sims' contribution, if necessary!]

II- The Lucas Crtique

- A Formal Representation of the Critique
  - Modeling Expectations
  - Examples
  - The Solution
- The Lucas Critique in Practise
  - The Formal Case
  - Illustrations

II- The Lucas Crtique

A Formal Representation of the Lucas Critique

Let y_t a single endogenous variable and x_t a single exogenous (stochastic) variable.

Let us assume the following linear dynamic model

$$y_t = a\tilde{y}_{t+1} + bx_t$$

where $|a| < 1$ and $b \neq 0$. $\tilde{y}_{t+1}$ is the expectation of y_{t+1}

Difficulty: Future Expectation of an endogenous variable.

Key issue: How to represent (or specify) expectations.

Modelling Expectations

A Simple Representation: Naive expectations

$$\tilde{y}_{t+1} = y_t$$

The expectation of future realizations of y is equal to the current realization of y.

After replacement into the equation

$$y_t = a\tilde{y}_{t+1} + bx_t$$

one gets

$$y_t = ay_t + bx_t$$

or equivalently

$$y_t = \frac{b}{1-a} x_t$$

We obtain a time invariant (meaning that the parameter of the reduced form is time invariant) equation that expresses the endogenous variable as a linear function of x.

Notice that the properties of the stochastic variable x_t does not impact on the reduced form parameter $(a/(1-b))$. Very important in the sequel of the analysis.

A More Sophisticated Representation: Rational expectations

$$\tilde{y}_{t+1} = E_t y_{t+1}$$

The expectation of future realizations of y is equal to the rational expectations of the future value of y. E_t is the rational expectation.

After replacement into the equation

$$y_t = a\tilde{y}_{t+1} + bx_t$$

one gets

$$y_t = aE_t y_{t+1} + bx_t$$

Difficulty: How to solve the model?

Rmk What are rational expectations?

Expectations lie at the core of economic dynamics (individual behavior, but most importantly equilibrium properties).

Long tradition in economics, since Keynes, but put forward by the new classical economy (Lucas, Prescott, Sargent in the 70' and 80').

The term "rational expectations" is most closely associated to Robert Lucas (Nobel Laureate 1995, University of Chicago), but the rationality of expectations deeply examined before (see Muth, 1960)

We consider the following definition.

Def *Agents formulate expectations in such a way that their subjective probability distribution of economic variables (conditional on the available information) coincides with the objective probability distribution of the same variable (according to a measure of the state of nature) in an equilibrium.*

Expectations should be consistent with the model $\Rrightarrow$ Solving the model is finding an expectation function (see below when we will solve the model.)

With this definition, we assume that agents know the model and the probability distribution of exogenous variables (or shocks) that hit the economy.

$E_t \equiv E(./I_t)$ where I_t denotes the information set in period t, i.e. when agent must decide L_t.

Here, I_t includes all the histories of y and x, i.e. $\{y_t, y_{t-1}, ...; x_t, x_{t-1}, ...\}$.

$E_t y_{t+1}$ is thus the linear projection of

y_{t+1} on $\{y_t, y_{t-1}, ...; x_t, x_{t-1}, ...\}$,

Properties of rational expectations

Property 1: No systematic bias. Let the expectation error $\varepsilon^y_{t+1} = y_{t+1} - E_t y_{t+1}$.

This error term satisfies:

$$E_t \varepsilon^y_{t+1} = 0$$

Proof: Straightforward (using $E_t(A+B) = E_tA + E_tB$ and $E_tE_tA = E_tA$):

$$E_t\varepsilon^y_{t+1} = E_t(y_{t+1} - E_ty_{t+1}) = E_ty_{t+1} - E_tE_ty_{t+1}$$

$$= E_ty_{t+1} - E_ty_{t+1} = 0$$

Property 2: Expectation errors do not exhibit any serial correlation.

Proof: Straightforward using the conditional auto–covariance function (see the previous chapter for a formal definition of this function):

$$\begin{aligned} Cov_t(\varepsilon^y_{t+1}, \varepsilon^y_t) &= E_t(\varepsilon^y_{t+1}\varepsilon^y_t) - E_t(\varepsilon^y_{t+1})E_t(\varepsilon^y_t) \\ &= E_t(\varepsilon^y_{t+1})\varepsilon^y_t - E_t(\varepsilon^y_{t+1})\varepsilon^y_t \\ &= 0 \end{aligned} \tag{1}$$

An example: an AR(1) process (Autoregressive process of order one) for x_t

$$x_t = \rho x_{t-1} + \varepsilon_t^x$$

The information set in period t is given by all the realizations of the random variable x from period "0" to period t (all the history at period t of the variable x), i.e.

$$I_t = \{x_t, x_{t-1}, ...\}$$

From this definition, we get

$$E_t x_{t+1} = E(\rho x_t + \varepsilon_t^x) = \rho E_t x_t + E_t \varepsilon_{t+1}^x = \rho x_t + E_t \varepsilon_{t+1}^x$$

Since ε_{t+1}^x is an innovation, it is orthogonal to the information set and thus $E_t \varepsilon_{t+1}^x = 0$.

It follows that

$$E_t x_{t+1} = \rho x_t$$

The expectation errors

$$\varepsilon^x_{t+1} = y_{t+1} - E_t y_{t+1} = y_{t+1} - \rho y_t$$

thus satisfies

$$E_t \varepsilon^x_{t+1} = 0$$

This property of rational expectations coincides with the optimal forecast of an econometrician who use the observations and the AR(1) process to formulate an optimal forecast of x (one step–ahead) [**Show a Figure on the blackboard**]

Examples of this Linear Economy

Example 1: Asset Pricing

Suppose a risk–neutral agent who wants to invest into a risky asset and a safe asset, with a constant positive net return ($r > 0$).

Let P_t the price of a stock and D_t its dividend payment.

If an investor buys the stock at date t and sells it at date $t + 1$, the investor will earn a yield of

$$\frac{D_t}{P_t}$$

from the dividend and an expected yield

$$\frac{P_{t+1} - P_t}{P_t}$$

in capital gains.

So the total (expected) return from stock's holding is

$$E_t \frac{P_{t+1} - P_t + D_t}{P_t}$$

If this investor chooses to invest into the safe asset, the return is given by r.

The no–arbitrage condition implies

$$r = E_t \frac{P_{t+1} - P_t + D_t}{P_t}$$

or equivalently

$$P_t = \frac{1}{1+r} E_t P_{t+1} + \frac{1}{1+r} E_t D_t \equiv \frac{1}{1+r} E_t P_{t+1} + \frac{1}{1+r} D_t$$

because D_t is known in period t.

Since $r > 0$, then $1/(1+r) < 1$.

This equation is equivalent to the previous one

$$y_t = aE_t y_{t+1} + bx_t$$

where y_t is P_t, x_t is D_t and $a = b = 1/(1+r) < 1(\neq 0)$

Example 2: The Cagan Model

Already investigated in the previous Chapter (with Franck Portier)

A money demand function (in logs)

$$m_t^d - p_t = -\gamma E_t \pi_{t+1}$$

where $\gamma > 0$ and

$$\pi_{t+1} = p_{t+1} - p_t$$

Suppose that the supply of money $m_t^s = m_t$ is exogenous and stochastic (m_t is a stochastic variable).

Equilibrium on the money market

$$m_t^d = m_t^s (= m_t)$$

We deduce

$$m_t - p_t = -\gamma E_t(p_{t+1} - p_t)$$

or equivalently

$$p_t = \frac{\gamma}{1+\gamma} E_t p_{t+1} + \frac{1}{1+\gamma} m_t$$

This equation is equivalent to the previous one

$$y_t = a E_t y_{t+1} + b x_t$$

where y_t is p_t, x_t is m_t and $a = \gamma/(1+\gamma) < 1$, $b = 1/(1+\gamma) \neq 0$

Example 3: The Fisher equation and the Taylor rule

Fisher equation

$$r_t = i_t - E_t\pi_{t+1}$$

where r_t is the real interest rate, i_t the nominal interest rate and $E_t\pi_{t+1}$ is the expected inflation for the next period. This equation can be deduced from intertemporal decision problem (not done here, see M1 Macro course)

Taylor rule

$$i_t = \alpha\pi_t + s_t$$

where $\alpha > 1$ (aggressive monetary policy, i.e. the monetary policy increases the nominal interest rate more than the rate of inflation, the “Taylor” principle). s_t represents exogenous shocks to monetary policy.

Let us assume that the real interest rate is constant and zero (it is determined by other constant forces than the monetary policy).

So we deduce

$$i_t = E_t \pi_{t+1}$$

Combining this equation with the Taylor rule yields

$$\alpha \pi_t + s_t = E_t \pi_{t+1}$$

or equivalently

$$\pi_t = \frac{1}{\alpha} E_t \pi_{t+1} - \frac{1}{\alpha} s_t$$

This equation is equivalent to the previous one

$$y_t = a E_t y_{t+1} + b x_t$$

where y_t is i_t, x_t is s_t and $a = -b = 1/\alpha < 1(\neq 0)$.

Example 4: The Government Budget Constraint

The Government Budget Constraint at any point in time is given by

$$B_{t+1} = (1 + r_t)B_t + G_t - T_t$$

where r_t is the interest rate, B_t is the level of government debt, G_t is the government spending (excluding interest payments on public debt) and T_t represent taxes on the economy. For simplicity, we assume $r_t(= r > 0)$ constant. We can thus express the current value of public debt as a linear function of future (expected) public debt, taxes and government spending:

$$B_t = \frac{1}{1+r} E_t B_{t+1} + \frac{1}{1+r}(T_t - G_t)$$

So, the current value of public debt is equal to the discounted (expected) value of future debt, plus the discounted primary surplus $(T_t - G_t)$ (the primary surplus does not incorporate the interest payment on pubic debt).

This equation is equivalent to the previous one

$$y_t = aE_t y_{t+1} + bx_t$$

where y_t is B_t, x_t is $(T_t - G_t)$ and $a = b = 1/(1+r) < 1 (\neq 0)$.

Example 5: The Permanent Income Model of Consumption

The Permanent Income Model departs a lot from the Keynesian consumption function.

Here, the current consumption is not a linear function of the (net of taxation) current income, but it depends on the intertemporal wealth.

More formally, this consumption function writes:

$$C_t = k(A_t + H_t)$$

where k is a constant parameter. In a dynamic model of intertemporal choices on consumption, we have $k = r$ (see, Hall, 1978) and r is the (constant) real interest rate, A_t the level of the financial wealth in period t and H_t is the human (or non–financial wealth).

H_t represents all the (net of taxation) labor income that an agent can get from participating to the labor market.

Formally, the dynamic equation of this variable is given by

$$H_t = \frac{1}{1+r} H_{t+1} + \frac{1}{1+r} (Y_t - T_t)$$

where Y_t is the labor income (real wage rate times the number of hours worked for a given time period) and T_t are taxes on the labor income.

So, the current value of human wealth is equal to the discounted (expected) value of future human wealth, plus the discounted net labor income $(Y_t - T_t)$.

This equation is equivalent to the previous one

$$y_t = aE_t y_{t+1} + bx_t$$

where y_t is H_t, x_t is $(Y_t - T_t)$ and $a = b = 1/(1+r) < 1(\neq 0)$.

Solving the Model

Two (immediate) difficulties: the model is <u>dynamic</u> and the model is <u>stochastic</u>.

An additional difficulty: the model is <u>forward–looking</u>, i.e. the value of y_t today will depend on the expected value of y tomorrow ($E_t y_{t+1}$).

How to solve the model? For simplicity, assume deterministic version of the model (no expectations and $x_t = x$ is constant $\forall t$).

$$y_t = a y_{t+1} + bx$$

First, deterministic the steady state value of y.

A steady state value of y, denoted $\bar{y}$ (if it exists and unique, this is the case in our setup), satisfies

$$y_{t+1} = y_t = ... = \bar{y}$$

If we apply this definition to the dynamic equation on y, we get

$$\bar{y} = a\bar{y} + bx \Longleftrightarrow \bar{y} = \frac{b}{1-a}x$$

Now, substract the steady state value $\bar{y}$ from the dynamic equation

$$y_t - \bar{y} = a(y_{t+1} - \bar{y})$$

and let denote with a “hat”, the deviation of the endogenous variable y from its steady–state value $\bar{y}$

$$\hat{y}_t = a\hat{y}_{t+1}$$

This is a first difference equation in $\hat{y}_t$.

How to solve it? Backward, i.e. we determine the value of $\hat{y}_t$ as a function of its past realizations or Forward, we determine $\hat{y}_t$ as a function of its future realizations.

[Show the figures on the blackboard]

[Represent $\hat{y}_t$ as a function of $\hat{y}_{t+1}$ and $\hat{y}_{t+1}$ as a function of $\hat{y}_t$, when $|a| < 1$.]

[Represent $\hat{y}_t$ as a function of $\hat{y}_{t+1}$ and $\hat{y}_{t+1}$ as a function of $\hat{y}_t$, when $|a| > 1$.]

Solving the Forward Looking Model

Let us start from the initial (basic) representation of the economy

$$y_t = aE_t y_{t+1} + bx_t$$

We know that the model must be solved forwards when $|a| < 1$ (our benchmark case, see the extensions for the case $|a| > 1$).

So, we will use successive forward substitutions.

Let the previous equation in period $t+1$ (don't forget that this equation is satisfied every period),

$$y_{t+1} = aE_{t+1} y_{t+2} + bx_{t+1}$$

After replacement into the initial equation

$$y_t = aE_t y_{t+1} + bx_t$$

we obtain

$$y_t = a^2 E_t E_{t+1} y_{t+2} + abE_t x_{t+1} + bx_t$$

Now use the fact that

$$E_t E_{t+1} y_{t+2} = E_t y_{t+2}$$

Demonstration:

Let denote

$$E_{t+1} y_{t+2} = y_{t+2} + \varepsilon^y_{t+2}$$

where

$$E_{t+1}\varepsilon^y_{t+2} = 0$$

by the definition of rational expectations. ε^y_{t+2} is not in the information set in period $t+1$, but in the information set of period $t+2$.

Now, apply the condition expectation operator E_t on both sides of the previous equation

$$E_t E_{t+1} y_{t+2} = E_t y_{t+2} + E_t \varepsilon^y_{t+2}$$

ε^y_{t+2} is not in the information set in period $t+1$ and also it is not in the information set of period t. So, we have

$$E_t \varepsilon_{t+2}^y = 0$$

It follows that

$$E_t E_{t+1} y_{t+2} = E_t y_{t+2}$$

End of the demonstration

So the new equation reduces to

$$y_t = a^2 E_t y_{t+2} + ab E_t x_{t+1} + b x_t$$

Next, express the equation in period $t+2$

$$y_{t+2} = a E_{t+2} y_{t+3} + b x_{t+2}$$

and replace this equation into the previous one

$$y_t = a^3 E_t E_{t+2} y_{t+3} + a^2 b E_t x_{t+2} + ab E_t x_{t+1} + b x_t$$

and using the fact (again)

$$E_t E_{t+2} y_{t+3} = E_t y_{t+3}$$

we obtain

$$y_t = a^3 E_t y_{t+3} + a^2 b E_t x_{t+2} + ab E_t x_{t+1} + b x_t$$

and continue to substitute forward [**On the blackboard for more details**].

$$y_t = \lim_{T \to \infty} a^T E_t y_{t+T} + b \lim_{T \to \infty} E_t \sum_{i=0}^{T} a^i x_{t+i}$$

Transversality condition

$$\lim_{T\to\infty} a^T E_t y_{t+T} = 0$$

and taking the limit

$$y_t = bE_t \sum_{i=0}^{\infty} a^i x_{t+i}$$

So, the value of y_t today is an expected discounted sum of all the present and future values of x_t.

This represents a present value equation.

Application I: The intertemporal equation of public debt

Let us start with the period per period government budget constraint

$$B_{t+1} = (1 + r_t)B_t + G_t - T_t$$

or equivalently

$$B_t = \frac{1}{1+r} E_t B_{t+1} + \frac{1}{1+r}(T_t - G_t)$$

This equation holds every period, so in period $t + 1$

$$B_{t+1} = \frac{1}{1+r} E_t B_{t+2} + \frac{1}{1+r}(T_{t+1} - G_{t+1})$$

Now, substitute this equation into the previous one and take expectations

$$B_t = \left(\frac{1}{1+r}\right)^2 E_t B_{t+2} + \left(\frac{1}{1+r}\right)^2 E_t(T_{t+1} - G_{t+1}) + \frac{1}{1+r}(T_t - G_t)$$

Repeat the operation many times and take the limit

$$B_t = \lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T E_t B_{t+T} + \frac{1}{1+r}\lim_{T\to\infty} E_t \sum_{i=0}^{T}\left(\frac{1}{1+r}\right)^i (T_{t+i} - G_{t+i})$$

Now, impose the transversality condition (i.e. we exclude explosive paths)

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T E_t B_{t+T} = 0$$

This is a "no bubble" condition on public debt [**Discussion here on the meaning of this condition.**].

Now, the value of the public debt today is just equal to the expected sum of future tax revenues net of government spending

$$B_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i (T_{t+i} - G_{t+i})$$

This equation means that the value of the public debt today is given by the expected ability of the government to repay it by taxing more and/or spending less.

If there is no debt today, the intertemporal budget constraint of the government is given by

$$\frac{1}{1+r}\sum_{i=0}^{\infty} E_t \left(\frac{1}{1+r}\right)^i (T_{t+i} - G_{t+i}) = 0$$

or equivalently

$$E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i T_{t+i} = E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i G_{t+i}$$

This mean that the expected discounted tax revenues must equate the expected discounted government spending.

Application II: The Human Wealth

In the permanent income model, the consumption deeply depends on the human wealth, the expected discounted sum of (net) labor income.

$$H_t = \frac{1}{1+r} H_{t+1} + \frac{1}{1+r}(Y_t - T_t)$$

This equation holds every period. In period $t+1$ we have

$$H_{t+1} = \frac{1}{1+r} H_{t+2} + \frac{1}{1+r}(Y_{t+1} - T_{t+1})$$

Now, substitute this equation into the previous one and take expectations

$$H_t = \left(\frac{1}{1+r}\right)^2 E_t H_{t+2} + \left(\frac{1}{1+r}\right)^2 E_t(Y_{t+1} - T_{t+1}) + \frac{1}{1+r}(Y_t - T_t)$$

Repeat the operation many times and take the limit

$$H_t = \lim_{T \to \infty} \left(\frac{1}{1+r} \right)^T E_t H_{t+T} + \frac{1}{1+r} \lim_{T \to \infty} E_t \sum_{i=0}^{T} \left(\frac{1}{1+r} \right)^i (Y_{t+i} - T_{t+i})$$

Now, impose the transversality condition (i.e. we exclude explosive paths)

$$\lim_{T \to \infty} \left(\frac{1}{1+r} \right)^T E_t H_{t+T} = 0$$

Now, the value of the human wealth today is just equal to the expected sum of future labor incomes net of taxation

$$H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i (Y_{t+i} - T_{t+i})$$

An important consequence is that the current consumption does not depend on the current income only, but on the expected income net of taxation

$$C_t = k(A_t + H_t)$$

where $k = r$.

Suppose that the consumer is the only one agent who can save and then buy the public debt (closed economy).

Market clearing on the public bond market implies

$$A_t = B_t$$

where the intertemporal budget constraint of the government must satisfy

$$B_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i (T_{t+i} - G_{t+i})$$

So, we deduce

$$A_t + H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i (T_{t+i} - G_{t+i}) + \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i (Y_{t+i} - T_{t+i})$$

or equivalently

$$A_t + H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i (Y_{t+i} - G_{t+i})$$

So the consumption is given by

$$C_t = k \left(\frac{1}{1+r} E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i (Y_{t+i} - G_{t+i}) \right) \quad \text{where } \ k = r \ \text{ in the model.}$$

This equation simply reflects that the consumer fully internalizes the intertemporal budget constraint of the government. In other words, an increase in G mean an increase in taxation in order to satisfy the intertemporal budget constraint. So, the expected net income decreases and thus the consumption.

Computing the Final Solution

To do so, we need to specify how agents forecast the future values of x (or in our previous examples, dividends, money supply, discretionary monetary policy, primary surplus (government spending, tax revenues), net labor income)

To do so, we assume a stochastic process of the exogenous variable x_t.

The agents knows this stochastic process and the associated assumptions about the properties of the shocks (i.e. he/she knows the conditional pdf of the exogenous driving forces of the economy)

We need to specify a (parametric) stochastic process for the exogenous variable x_t.

For simplicity, we assume an AR(1) process for x_t:

$$x_t = \rho x_{t-1} + \varepsilon_t^x$$

where $\rho \in [0, 1]$. ε_t^x is an unpredictable random variable with zero mean and variance equals to σ^2.

$$E_t \varepsilon_{t+i}^x = 0 \quad \forall i > 0$$

Basically, with the above parametrizations and assumptions, the agents know the probability distribution of x_t.

A digression:

Discussion on ρ

Case 1. $\rho = 0$. In this case, the process of x reduces to

$$x_t = \varepsilon_t^x$$

It follows that

$$E_t x_t = x_t$$

because x_t is known in period t.

But, for the next period

$$E_t x_{t+1} = E_t \varepsilon_{t+1}^x = 0$$

The best prediction about the future value (one step ahead) of x is the average value of x (i.e. the average value of ε^x), that is zero.

We can repeat the computation for all the subsequent periods. For example, for period $t+2$, we have:

$$E_t x_{t+2} = E_t \varepsilon^x_{t+2} = 0$$

and so on. We have

$$E_t x_{t+i} = E_t \varepsilon^x_{t+i} = 0 \quad \forall i > 0$$

[Show a figure for an illustration]

Case 2. $\rho = 1$. In this case, the process of x rewites

$$x_t = x_{t-1} + \varepsilon^x_t$$

The variable x_t follows a random walk.

As before

$$E_t x_t = x_t$$

Next period expectation

$$E_t x_{t+1} = E_t(x_t + \varepsilon^x_{t+1}) = E_t x_t + E_t \varepsilon^x_{t+1} = x_t$$

So the best predictor of the variable x tomorrow is the observation of this variable today.

Redo the exercise in period $t + 2$

$$E_t x_{t+2} = E_t(x_{t+1} + \varepsilon^x_{t+2}) = E_t x_{t+1} + E_t \varepsilon^x_{t+2} = x_t$$

Again the best predictor of the variable x in 2 periods is the observation of this variable today.

We get

$$E_t x_{t+i} = x_t \quad \forall i \geq 0$$

[Show a figure for an illustration]

General case. $\rho \in [0, 1]$

As before

$$E_t x_t = x_t$$

Next period expectation

$$E_t x_{t+1} = E_t(\rho x_t + \varepsilon^x_{t+1}) = E_t \rho x_t + E_t \varepsilon^x_{t+1}) = \rho x_t$$

Redo the exercise in period $t + 2$

$$E_t x_{t+2} = E_t(\rho x_{t+1} + \varepsilon^x_{t+2}) = \rho E_t x_{t+1} + E_t \varepsilon^x_{t+2} = \rho^2 x_t$$

More generally, we get

$$E_t x_{t+i} = \rho^i x_t \quad \forall i \geq 0$$

[Show a figure for an illustration]

Now, we use the last formula to compute the successive forward looking expectations on x_t

We must compute

$$y_t = bE_t \sum_{i=0}^{\infty} a^i x_{t+i} \equiv bE_t \left(x_t + a x_{t+1} + a^2 x_{t+2} + ... \right)$$

We now that

$$E_t a^i x_{i+i} x_{t+i} = (a\rho)^i x_t$$

So, we obtain

$$y_t = bE_t \sum_{i=0}^{\infty} a^i x_{t+i} \equiv b \left(\sum_{i=0}^{\infty} (a\rho)^i \right) x_t$$

The sequence $(a\rho)^i$ will converge to zero as i becomes large (i.e. $\lim_{j \to \infty} (a\rho)^j = 0$),

since $|a| < 1$ and $\rho \in [0, 1]$.

So

$$\sum_{i=0}^{\infty} (a\rho)^i = \frac{1}{1 - a\rho}$$

After replacement into the forward looking equation, one gets

$$y_t = \frac{b}{1 - a\rho} x_t$$

This is the solution of the model, as we express the endogenous variable y_t as a (linear) function of the exogenous variable x_t.

If we combine this equation with the precess of x_t, we obtain

$$y_t = \rho y_{t-1} + \frac{b}{1 - a\rho} \varepsilon_t^x$$

An Application to the Permanent Income Model

The human wealth is given by

$$H_t = \frac{1}{1+r} E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i (Y_{t+i} - T_{t+i})$$

Suppose (for simplicity) that the labor income is constant

$$Y_t = \bar{Y}$$

but the labor income tax T_t is stochastic. More precisely, the labor income tax rule is the following:

$$T_t = \bar{T} + \tilde{T}_t$$

where $\bar{T}$ and $\tilde{T}_t$ follows an AR(1) process

$$\tilde{T}_t = \rho \tilde{T}_{t-1} + \epsilon_t$$

where $E_t \epsilon_{t+i} = 0, \forall i \geq 0$

Using the previous calculations [**Do the computations on the blackboard**],

we get

$$H_t = \frac{1}{r}(\bar{Y} - \bar{T}) - \frac{1}{1 + r - \rho}\tilde{T}_t$$

So, depending on the fiscal policy, the labor income taxes can affect differently the

human wealth.

If $\rho = 0$, smaller effect.

If $\rho = 1$, larger effect.

To see this more precisely, consider the Permanent Income Model of consumption.

We have:

$$C_t = k(A_t + H_t)$$

where $k = r$. After substitution of H_t into the consumption function, we obtain:

$$C_t = rA_t + (\bar{Y} - \bar{T}) - \frac{r}{1 + r - \rho}\tilde{T}_t$$

Before discussing the illustration of the Lucas Critique, compare this consumption to the Keynesian one. Use the following equation

$$C_t = rA_t + (\bar{Y} - \bar{T})$$

Now illustrate the Lucas critique using this consumption function

If $\rho = 0$, very small effect of taxes on consumption $r/(1+r)$ (transitory tax changes have almost no effect on consumption).

If $\rho = 1$, larger effect of taxes on consumption 1 (permanent tax changes have stronger effect on consumption).

Discussion

The Lucas Critique at Work

To illustrate the **Lucas critique**, let us consider the model with naive expectations, *i.e.* $E_t y_{t+1} = y_t$.

We know that in this case, we obtain a solution of the form:

$$y_t = \frac{b}{1-a} x_t$$

This is the solution of the model, as we express the endogenous variable y_t as a (linear) function of the exogenous variable x_t. If we combine this equation with the process of x_t, we obtain

$$y_t = \rho y_{t-1} + \frac{b}{1-a} \varepsilon_t^x$$

or equivalently

$$y_t = \rho y_{t-1} + \frac{b}{1-a}\varepsilon_t^x$$

Now consider the case of rational expectations. We have

$$y_t = \frac{b}{1-a\rho}x_t$$

Again, this is the solution of the model, as we express the endogenous variable y_t as a (linear) function of the exogenous variable x_t. If we combine this equation with the process of x_t, we obtain

$$y_t = \rho y_{t-1} + \frac{b}{1-a\rho}\varepsilon_t^x$$

The two solutions (with naive and rational expectations) do not coincide, except in the particular case where $\rho = 1$.

With naive expectations, the value of ρ as no impact on the value of the impact multiplier

$$\frac{\partial y_t}{\partial x_t} = \frac{b}{1-a}$$

The value of ρ only impacts the dynamic properties of the exogenous variable x_t. This is because, with naive expectations, agents do not care about expectations on the future (i.e. about the next periods realizations of the exogenous variable).

With rational expectations, the value of ρ does impact on the value of the impact multiplier

$$\frac{\partial y_t}{\partial x_t} = \frac{b}{1 - a\rho}$$

With rational expectations with prospective agents, agents really care about the future (i.e. about the next periods realizations of the exogenous variable).

Dynamic Responses or Impulse Response Function (IRF)

The Dynamic Responses or Impulse Response Function (IRF) at any horizon h ($h = 0, 1, 2, ...$) to a shock is given by

$$\frac{\partial y_{t+h}}{\partial \varepsilon_t^x}$$

<u>A simple first approach.</u>

Let ε_t^x is equal to 1 in period t, but 0 elsewhere (i.e. $..., t-2, t-1$ and $t+1, t+2, ...$).

Next compute the value of y_t in this case.

Application 1: The model with naive expectations.

With naive expectation, we obtain the following.

In period t, the value of y_t is

$$\frac{b}{1-a}$$

One period ahead

$$\rho\frac{b}{1-a}$$

Next

$$\rho^2\frac{b}{1-a}$$

More generally, we have

$$\frac{\partial y_{t+h}}{\partial \varepsilon_t^x} = \rho^h\frac{b}{1-a}$$

So, the value of ρ only matters for the persistence of the effect.

Application 1: The model with rational expectations.

In period t, the value of y_t is

$$\frac{b}{1-a\rho}$$

One period ahead

$$\rho\frac{b}{1-a\rho}$$

Next

$$\rho^2\frac{b}{1-a\rho}$$

More generally, we have

$$\frac{\partial y_{t+h}}{\partial \varepsilon_t^x} = \rho^h \frac{b}{1 - a\rho}$$

So, the value of ρ both matters for the size of the effect and the persistence of the effect.

This another important implication of the Lucas critique.

A More General Approach.

Let us introduce the lag operator

$$Ly_t = y_{t-1}$$

Using this lag operator the solution rewrites

$$y_t = \rho L y_t + \frac{b}{1-a\rho}\varepsilon_t^x$$

$$(1-\rho L)y_t = \frac{b}{1-a\rho}\varepsilon_t^x$$

$$y_t = \frac{1}{1-\rho L}\frac{b}{1-a\rho}\varepsilon_t^x$$

where

$$\frac{1}{1-\rho L} = \sum_{i=0}^{\infty}(\rho^i L^i)$$

From this inversion, we get

$$y_t = \frac{b}{1-a\rho}\sum_{i=0}^{\infty}(\rho^i L^i)\varepsilon_t^x$$

or equivalently

$$y_t = \frac{b}{1 - a\rho}(\varepsilon_t^x + \rho\varepsilon_{t-1}^x + \rho^2\varepsilon_{t-2}^x + ...)$$

We can then deduce the dynamic responses or IRFs to the innovation ε_t^x.

Indeed, we have

$$y_{t+1} = \frac{b}{1 - a\rho}(\varepsilon_{t+1}^x + \rho\varepsilon_t^x + \rho^2\varepsilon_{t-1}^x + ...)$$

$$y_{t+2} = \frac{b}{1 - a\rho}(\varepsilon_{t+2}^x + \rho\varepsilon_{t+1}^x + \rho^2\varepsilon_t^x + ...)$$

...

$$y_{t+h} = \frac{b}{1 - a\rho}(\varepsilon_{t+h}^x + \rho\varepsilon_{t+h-1}^x + \rho^2\varepsilon_{t+h-2}^x + ... + \rho^h\varepsilon_t^x + ...)$$

We can then directly deduce the dynamic responses from these representations using the formula

$$\frac{\partial y_{t+h}}{\partial \varepsilon_t^x}$$

An Application: Monetary Policy and Inflation

Let us consider the case of a Taylor rule with the Fisher equation (a simple model of inflation dynamics)

In the case of rational expectations and assuming an AR(1) monetary policy shock

[Redo the computations on the blackboard if necessary.]

$$\pi_t = -\frac{1}{\alpha - \rho} s_t$$

In the case of naive expectations and assuming an AR(1) monetary policy shock

$$\pi_t = -\frac{1}{\alpha - 1} s_t$$

In the first case, the response of inflation to a monetary policy shock will depend on the value of ρ.

If $\rho = 0$, the inflation responds very little, because the (restrictive) monetary policy is perceived as transitory.

Conversely, if $\rho = 1$, the inflation responds a lot, because the (restrictive) monetary policy is perceived as permanent.

This is not the case with naive expectations: the sensitivity of inflation to monetary policy shock is invariant to the persistence of the shock.

Testing the Lucas Critique: An Illustration

Change in monetary policy

In the 70', high inflation in US and probably a passive (or expansionary) monetary policy.

In the beginning 80', the Volker (chairman of the Fed) disinflation and more aggressive monetary policy.

The idea: the monetary policy parameter has changed the behavior of inflation.

To illustrate this, let us again consider the simple model with a Fisher equation and a Taylor rule.

The solution to this model (only with monetary policy shock) is given by

To simplify, assume that the monetary policy shock does not display persistence ($\rho = 0$).

The central parameter is α (in each case, i.e. each policy, we assume a different value for α but we still maintain the restriction that $\alpha > 1$).

$$\pi_t = -\frac{1}{\alpha} s_t$$

We can deduce that the variance of inflation is

$$V(\pi_t) = \left(\frac{1}{\alpha}\right)^2 V(s_t)$$

For $V(s_t)$ given (and constant), the variance of inflation is a decreasing function of α.

This is what we observed in the 80':

A decrease in $V(\pi_t)$ (inflation stabilisation) and an increase in α, more aggressive monetary policy.

Testing the Lucas Critique: A Quantitative Experiment

There exist a huge literature about the empirical relevance of the Lucas critique.

Notice that we question here *the empirical relevance* of the critique, not its *logical consistency*, because everybody agrees that this critique matters when agents are prospective.

Empirical works do not support so much this empirical relevance, but they mainly use a reduced form approach (linear regression on *ad-hoc* specifications of the consumption and money demand equations).

Here, we propose to quantitatively investigate the relevance of the Lucas critique in a Dynamic Stochastic General equilibrium model that fully account from the cross-equations restrictions created by this type of modeling.

To test for the Lucas critique, we need to develop dynamic models in which forward looking expectations matters.

> *"Ultimately, it [demanding that one's structural relations be derived from individual optimization] is the only way in which the 'observational equivalence' of a multitude of alternative possible structural interpretations of the co-movements of aggregate series can be resolved".*
>
> ROTEMBERG and WOODFORD [1998, p. 1]

Here, we propose a simple dynamic, stochastic, general equilibrium model with a monetary policy.

Collard,F., Feve, P. and F. Langot "Structural Inference and the Lucas Critique", **Annals of Economics and Statistics**, 2002, issue 67-68.

Plan of the paper

A brief description of the empirical strategy adopted to properly identify and illustrate the relevance of the Lucas critique.

A first look at actual US data.

The estimation of the model and some tests.

A illustration of the critique.

3 Testing the *Lucas critique*

Given this structural model, we now investigate an econometric evaluation of the *Lucas critique*. Our strategy can be summarized as follows. Consider the simple representation of the log-linearized version of our model

$$y_t = aE_t y_{t+1} + bx_t \text{ where } |a| < 1 \tag{12}$$

where y_t and x_t respectively denote the endogenous and the forcing variables. x_t is assumed to follow a rule of the form:

$$x_t = \rho x_{t-1} + \sigma \varepsilon_t \text{ with } \varepsilon_t iid(0,1) \tag{13}$$

The solution to this simple model is given by:

$$\begin{cases} y_t &= \dfrac{b}{1-a\rho} x_t \\ x_t &= \rho x_{t-1} + \sigma \varepsilon_t \end{cases}$$

This reduced form may be used to estimate the parameters of the model and evaluate the empirical relevance of the *Lucas critique*. Our strategy then proceeds as follows:

Step 1. *Testing for the potential instability of the US monetary Business Cycle:* We first estimate a set of moments that characterize the main features of observed fluctuations in the US economy. We then test for the instability of this set of moments and estimate the breakpoint. This leads us to select two sub– samples, whose characteristics differ significantly.

Step 2. *Estimating and testing the structural model:* We estimate the parameters of the policy rules holding the deep parameters constant ($a = \overline{a}$ and $b = \overline{b}$ in our example) over the whole sample. This amounts to compute moments generated by the following reduced form:

$$\begin{cases} y_t &= \dfrac{\overline{b}}{1-\overline{a}\rho} x_t \\ x_t &= \rho x_{t-1} + \sigma \varepsilon_t \end{cases}$$

and therefore estimate σ and ρ to match the moments computed on actual data. As, we impose that the number of moments is greater than the number of parameters in the rule, we can conduct an overidentifying restriction test for each sub-sample.

Step 3. *Testing for instability of the monetary policy rule:* If the model is not rejected in each sub-sample, it is possible to evaluate the empirical relevance of the *Lucas critique*, inspecting the stability of the policy rule parameters.

Step 4. *Illustrating the Lucas critique*: In order to make our quantitative illustration more explicit, we go back to the simple model we developed at the beginning of the previous section (equations (12) and (13)). The structural model in sub-sample i (M_i) is characterized by the following forward-looking equation:

$$y_t = aE_t y_{t+1} + bx_t \text{ where } |a| < 1$$

and the rule for the forcing variable x in sub-sample i (R_i) is given by:

$$x_t = \rho_i x_{t-1} + \sigma_i \varepsilon_t \text{ with } \varepsilon_t iid(0,1)$$

The proper evaluation of the implication of rule R_j in M_i is to solve the model M_i taken R_j into account. Misusing R_i then amounts to solve M_j given R_j and plugging R_i in the model without solving M_j again. This is summarized in table 1.

TABLE 1
The Lucas Critique at Work

	R_1	R_2
M_1	$\begin{cases} y_t = \dfrac{b}{1-a\rho_1}x_t \\ x_t = \rho_1 x_{t-1} + \sigma_1 \varepsilon_t \end{cases}$	$\begin{cases} y_t = \dfrac{b}{1-a\rho_1}x_t \\ x_t = \rho_2 x_{t-1} + \sigma_2 \varepsilon_t \end{cases}$
	$\Downarrow$	$\Uparrow$
M_2	$\begin{cases} y_t = \dfrac{b}{1-a\rho_2}x_t \\ x_t = \rho_1 x_{t-1} + \sigma_1 \varepsilon_t \end{cases}$	$\begin{cases} y_t = \dfrac{b}{1-a\rho_2}x_t \\ x_t = \rho_2 x_{t-1} + \sigma_2 \varepsilon_t \end{cases}$

We apply this methodology to our model and evaluate economic policy in the light of impulse response functions to a technological shock. In order to provide some metric to our evaluation, we compute the confidence interval for the well-specified model (M_i solved taken R_i into account, $i = 1,2$).

Figure 3: Structural Breaks Tests on Moments Summarizing US Business Cycles

FIGURE 1

Structural Break Test

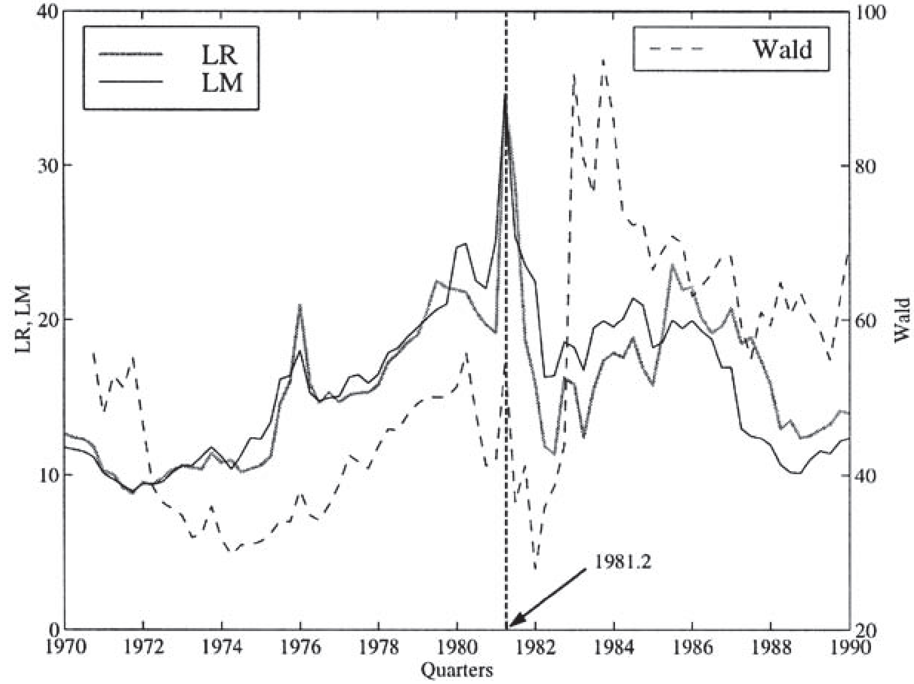


Figure 4: US Business Cycle

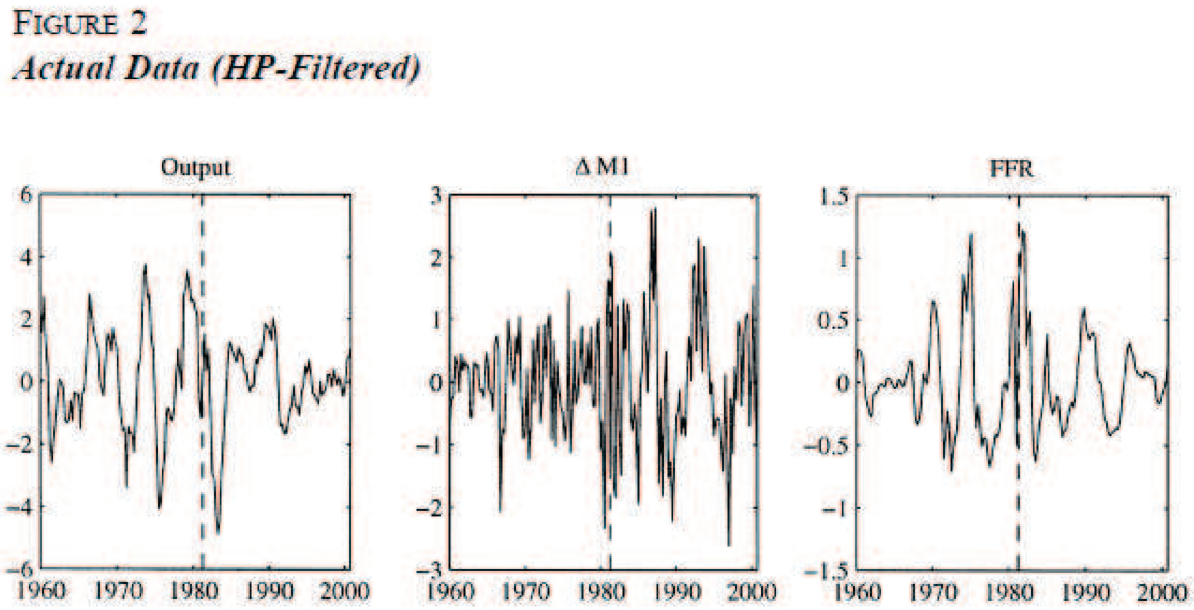


Figure 5: Moments on two Sub-Samples (US Data)

TABLE 4
Moments on US Data

	First sub-sample (1959.4-1981.2)	Second sub-sample (1981.3-2000.3)
σ_y	1.6252	1.3588
	(0.2313)	(0.4334)
σ_g	0.6656	1.2248
	(0.0561)	(0.0826)
σ_R	0.3720	0.3455
	(0.0736)	(0.0348)
$\rho(y,g(-1))$	0.2950	– 0.0733
	(0.1105)	(0.1253)
$\rho(y,g)$	0.1851	– 0.0995
	(0.1167)	(0.1393)
$\rho(y,g(+1))$	0.0419	– 0.2201
	(0.1101)	(0.1230)
$\rho(y,R(-1))$	0.0640	0.2858
	(0.2102)	(0.2234)
$\rho(y,R)$	0.3455	0.5132
	(0.1734)	(0.2113)
$\rho(y,R(+1))$	0.5618	0.6115
	(0.2315)	(0.2672)

Note: Estimates are robust to both heteroskedasticity and serial correlation. We used a VARHAC(1) estimator.

Figure 6: Estimated Parameters for the US Monetary Policy Rule

TABLE 6

Policy Rule Estimates

	First sub-sample (1959.4 – 1981.2)	Second sub-sample (1981.3 – 2000.2)
σ_g	0.0067	0.0108
	(0.0006)	(0.0010)
ρ_g	0.3310	0.3021
	(0.7909)	(0.0686)
π_π	– 0.0353	0.3157
	(0.6029)	(0.0651)
π_y	0.0724	0.0615
	(0.0318)	(0.0363)
J– stat	3.4690	1.5104
	[62.81]	[91.18]

Figure 7: Stability of Estimated Parameters for the US Monetary Policy Rule

TABLE 8

Stability Tests for Policy Rules Estimates

	LM	LR	Wald
σ_g	6.1429	12.0507	6.3994
	[1.32]	[0.05]	[1.14]
ρ_g	0.0002	0.0002	0.0037
	[98.87]	[98.88]	[95.15]
π_π	0.1520	3.1884	4.1577
	[69.66]	[7.42]	[4.14]
π_y	0.3828	0.0225	0.0229
	[53.61]	[88.08]	[87.97]
Global	33.7684	39.7108	28.0403
	[0.00]	[0.00]	[0.00]

Figure 8: Illustration of the Lucas Critique (1)

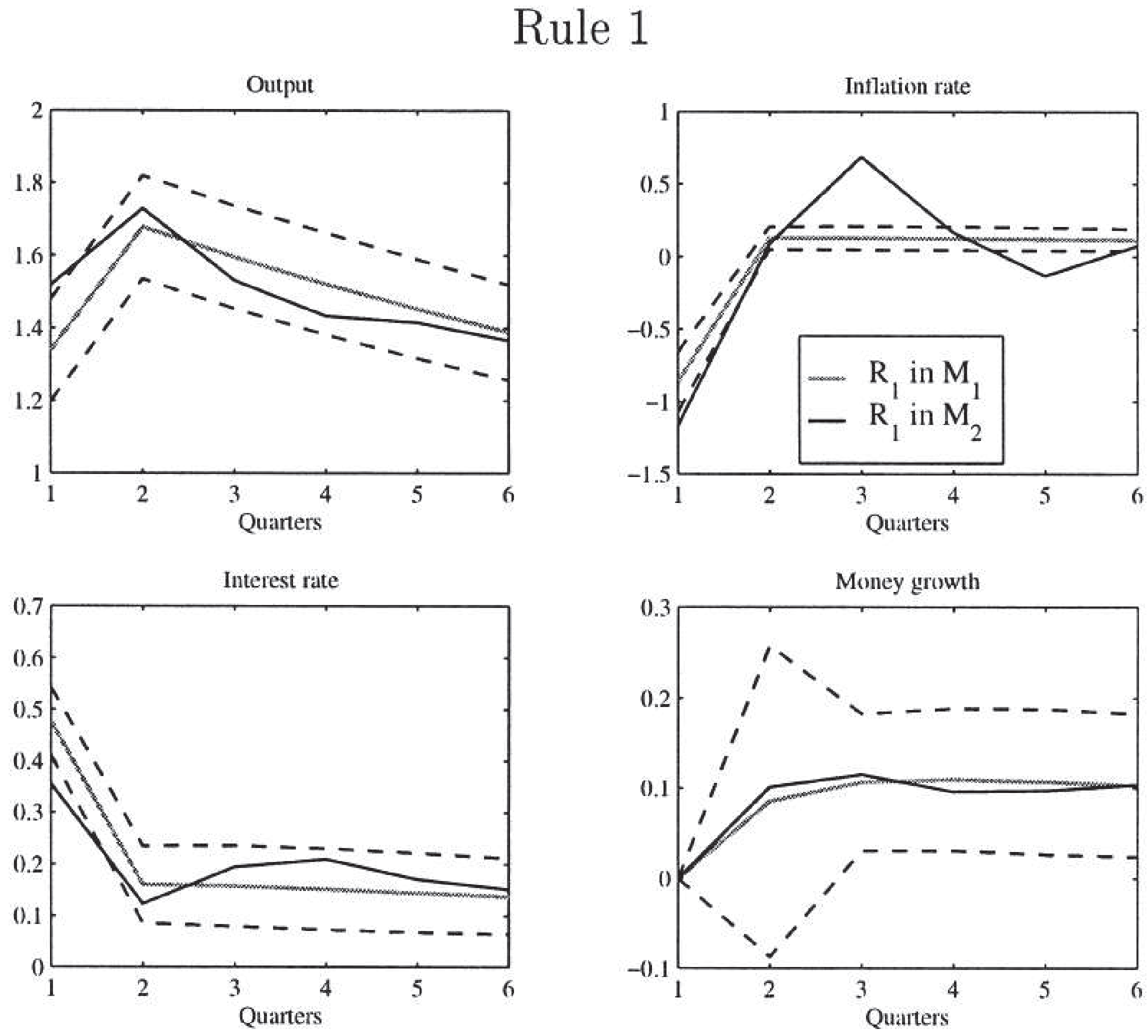
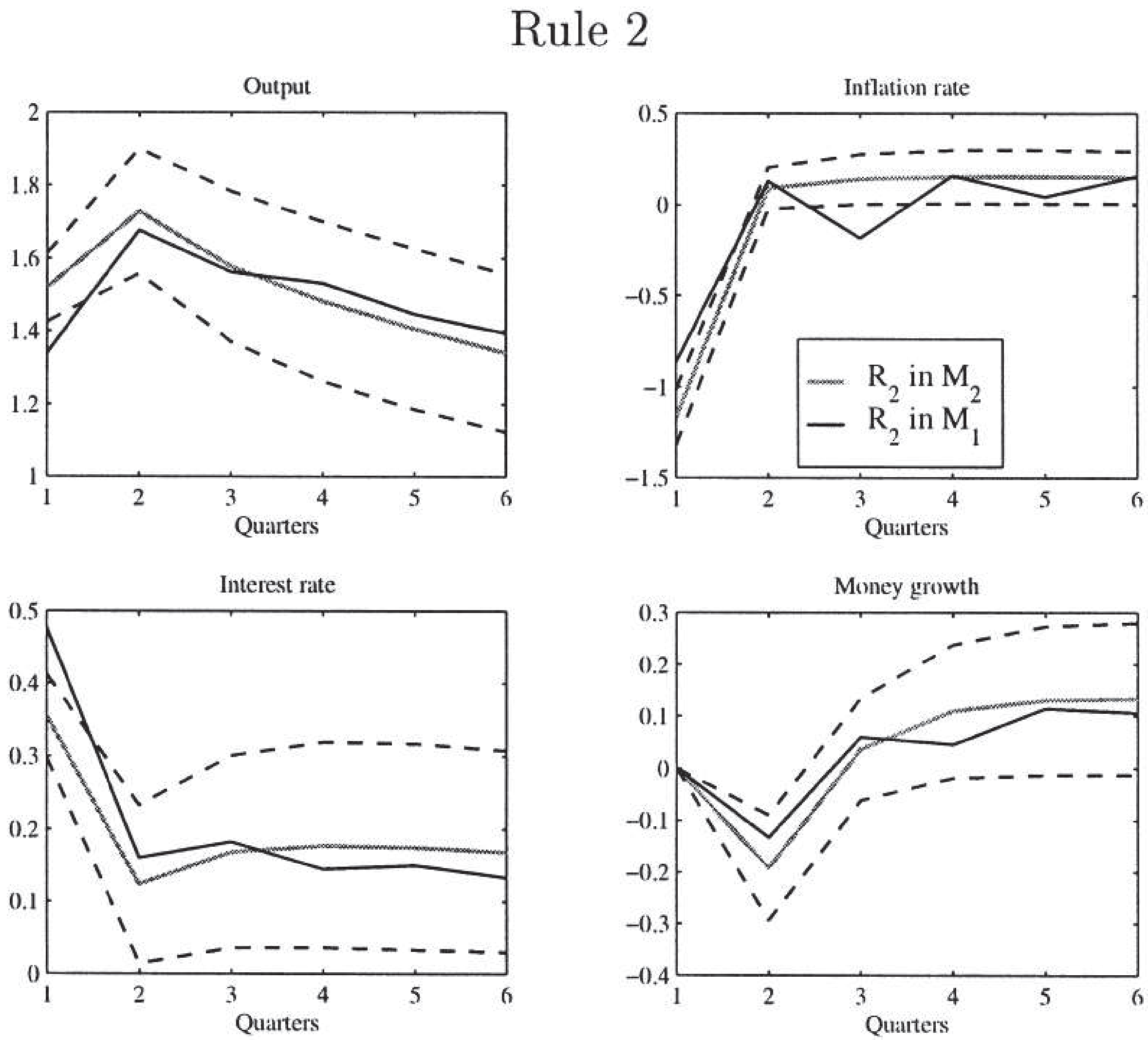


Figure 9: Illustration of the Lucas Critique (2)



Extensions

Three different extensions (and related discussions) are considered here.

1. The Method of undetermined coefficients (as an illustration of the Lucas critique).

2. Indeterminacy and the Lucas critique.

3. Optimal Policy and the Lucas critique.

1. The Method of undetermined coefficients

The simple model was solved by forward substitutions.

Another approach: The Method of undetermined coefficients

Let us assume that the solution is a time-invariant equation that expressed the endogenous variable as a linear function of the exogenous variable

$$y_t = \mu x_t$$

The question: how to identify the parameter μ.

Idea: this equation must be consistent with the model, i.e. the structural equation

$$y_t = aE_t y_{t+1} + bx_t$$

and the process of the exogenous variable

$$x_t = \rho x_{t-1} + \varepsilon_t^x$$

So we replace the assumed function for y_t into the dynamic forward-looking equation

$$\mu x_t = aE_t \mu x_{t+1} + bx_t$$

Using the process for x_t, we obtain

$$\mu x_t = a\rho\mu x_t + bx_t$$

Identification of μ:

$$\mu = \frac{b}{1 - a\rho}$$

So, we deduce

$$y_t = \frac{b}{1 - a\rho} x_t$$

2. Indeterminacy and the Lucas critique.

We assumed that $|a| < 1$ in our simple model

$$y_t = aE_t y_{t+1} + bx_t$$

Suppose that the theoretical model (preferences, technology, endogenous economic policy) implies $|a| > 1$.

So the theoretical model cannot be solved forward, but backward.

$$aE_t y_{t+1} = y_t - bx_t$$

or equivalently

$$E_t y_{t+1} = \frac{1}{a} y_t - \frac{b}{a} x_t$$

where $1/a < 1$.

Now use the expectation of y

$$y_{t+1} = E_t y_{t+1} + \nu_{t+1}$$

where

$$E_t \nu_{t+1} = 0$$

It follows

$$y_{t+1} = \frac{1}{a} y_t - \frac{b}{a} x_t + \nu_{t+1}$$

The random term ν_{t+1} is a sunspot shock (extrinsic beliefs) that can be arbitrary correlated with the fundamental shock ε^x_{t+1}

$$\nu_{t+1} = \pi\varepsilon^x_{t+1} + \eta_{t+1}$$

where π is an arbitrary parameter related to the fundamental shock and

$$E_t\eta_{t+1} = 0$$

Here the value of ρ does not affect the reduced form, so the Lucas critique does not appy (see, Farmer 2002).

3. Optimal Policy and the Lucas critique.

Consider again our simple model

$$y_t = aE_t y_{t+1} + bx_t$$

and the process of the exogenous variable

$$x_t = \rho x_{t-1} + \varepsilon_t^x$$

Now consider that the government (or Central Bank) aims at stabilizing the variable y_t (for example output).

The objective function is thus to find a value of ρ that minimizes the total variance of y.

Let us first consider that the model with naive expectations.

This variance of y is given by

$$V(y) = \left(\frac{b}{1-a}\right)^2 V(x)$$

where ([**See this on the blackboard**])

$$V(x) = \frac{\sigma_{\varepsilon^x}}{1-\rho^2}$$

So, the government must chose a value of ρ that minimizes this variance:

$$\bar{\rho} = arg \min_{\rho} V(y)$$

or equivalently

$$\bar{\rho} = arg \min_{\rho} \left(\frac{b}{1-a}\right)^2 V(x)$$

or

$$\bar{\rho} = arg\min_{\rho} \left(\frac{b}{1-a}\right)^2 \frac{\sigma_{\varepsilon^x}}{1-\rho^2}$$

The solution is trivial, since $V(x)$ is a decreasing function of ρ.

So the optimal decision is

$$\bar{\rho} = 0$$

Let us now consider the case of rational expectations.

The solution is more complicated because the short–run multiplier depends now on the value of ρ.

Given this, the variance of y is now given by:

$$V(y) = \left(\frac{b}{1 - a\rho}\right)^2 V(x)$$

where as previously the variance of x is

$$V(x) = \frac{\sigma_{\varepsilon^x}}{1 - \rho^2}$$

We see that now the optimal choice of ρ will depend on the deep parameter a.

So the variance of y is now given by

$$V(y) = \left(\frac{b}{1 - a\rho}\right)^2 \frac{\sigma_{\varepsilon^x}}{1 - \rho^2}$$

Now, the solution is not trivial because it depends on the value of a.

$$\rho^\star = arg\min_\rho \left(\frac{b}{1 - a\rho}\right)^2 \frac{\sigma_{\varepsilon^x}}{1 - \rho^2}$$

The optimal value $\rho^\star$ of ρ solves the following First Order Condition:

$$\left(\frac{a}{1 - a\rho^\star}\right) + \left(\frac{\rho^\star}{1 - \rho^{\star 2}}\right) = 0$$

The (two) roots of the (second order) polynomial are given by

$$\rho^\star = \frac{1 + \sqrt{1 + 8a^2}}{4a}$$

$$\rho^\star = \frac{1 - \sqrt{1 + 8a^2}}{4a}$$

One root is eliminated because it exceeds unity. So we select

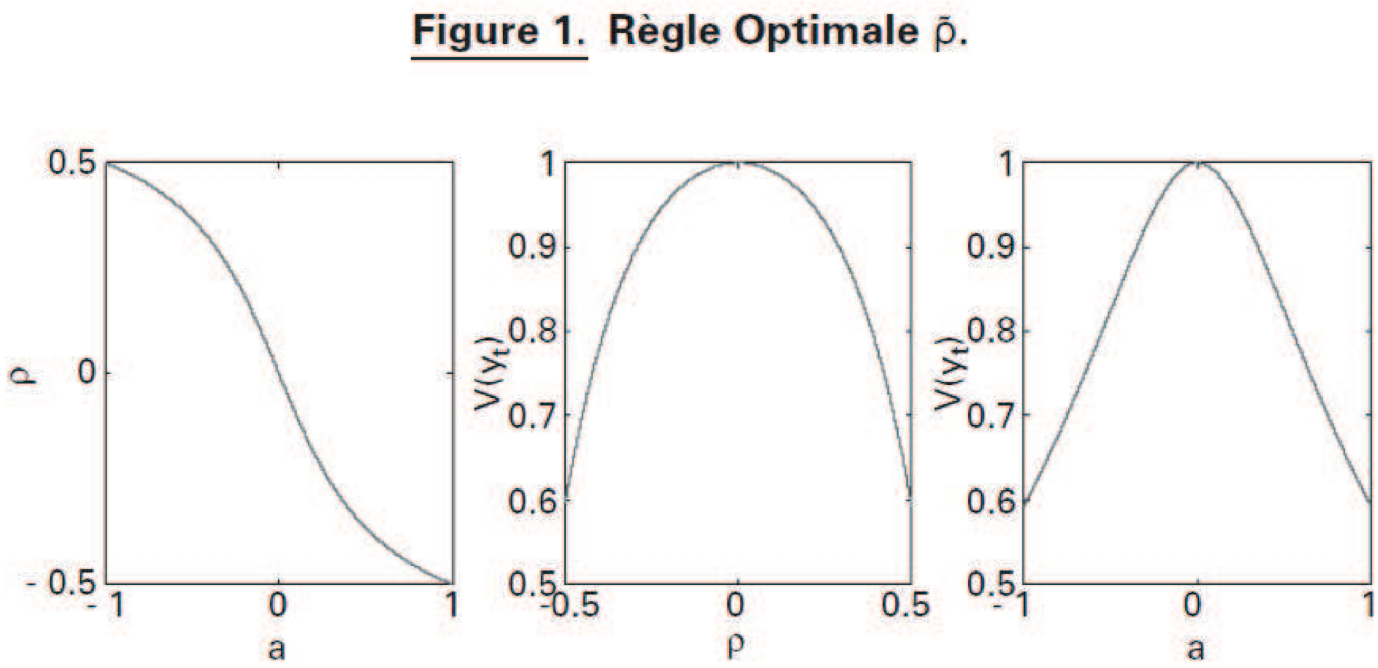
$$\rho^{\star} = \frac{1 - \sqrt{1 + 8a^2}}{4a}$$

Only in the case where $a = 0$, the two optimal policies are identical

$$\bar{\rho} = \rho^{\star} = 0$$

Conversely, for $a \neq 0$, the two policies differ, i.e. $\bar{\rho} \neq \rho^{\star}$; The figure below illustrates this finding.

Figure 10: Optimal value for ρ



The optimal policy (i.e. $\rho^\star$) is a decreasing function of a.

When agents positively value the future (i.e. $a > 0$) the optimal policy is negative.

This can lead to sizeable differences in economic performances, i.e. the stabilization (minimizing the variance of y).

For example, when a is close to one, the relative excess in terms of volatility implied by $\bar{\rho} = 0$ (relative to $\rho^\star$) is arround 67%.