

Persistent Government Spending and Fiscal Multipliers: the Investment-Channel

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Abstract

This paper inspects the mechanism shaping government spending multipliers in various small-scale DSGE setups with endogenous labor supply and capital accumulation. We analytically characterize the short-run investment multiplier, which in equilibrium can be either positive or negative. The investment multiplier increases with the persistence of the exogenous government spending process. The response of investment to government spending shocks strongly affects short-run multipliers on output and consumption.

Keywords: Government Spending Multipliers, DSGE models, Capital Accumulation, Labor Supply, Market Imperfections.

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Introduction

Policy debates related to stimulus packages and fiscal consolidation programs have renewed academic interest about the effects of government activity. The short-run government spending multiplier, *i.e.* the response of current GDP consecutive to a unit increase in government spending, has attracted considerable attention despite the large uncertainty surrounding its measurement (see Ramey (2011a) for a recent survey). Estimated values of the multiplier vary with many factors including the econometric approach, the identification strategy, the structural model, the nature and duration of the fiscal change, or the state of the economy (see among others, Cogan et al. (2010), Uhlig (2010), Christiano et al. (2011), Ramey (2011b), Auerbach and Gorodnichenko (2012), Coenen et al. (2012), Fève et al. (2013), or Erceg and Lindé (2014)).

Does the time profile of government spending affect fiscal multipliers? We revisit this classic question using a tractable business cycle model with physical capital accumulation, endogenous labor supply and stochastic government spending. Closed-form solutions for the equilibrium of that economy show that the persistence of government spending shapes short-run multipliers through the response of private investment.

The main contribution of this paper is to pin down the persistence of government spending for which a capital demand effect triggered by the increase in expected employment offsets the usual crowding-out effect on investment. This threshold persistence value also represents the equilibrium adjustment speed of consumption, which varies across economic environments. Should private investment increase, the output multiplier would be magnified compared to an economy where capital is held constant. Conversely, transitory fiscal stimuli do not provide any incentive to accumulate physical capital and give up a potentially important propagation mechanism.

Our analysis connects dynamic multipliers in an economy with capital accumulation to constant-capital multipliers. Constant-capital models are often used to deliver analytical results on fiscal multipliers, as in Hall (2009), Woodford (2011), Christiano et al. (2011) or Fève et al. (2013). In frictionless setups, constant-capital multipliers only result from the intra-temporal allocations (the marginal rate of substitution between consumption and leisure, the marginal productivity of labor and the aggregate resources constraint) but ignore expectations about the timing of government policy.¹

We also connect our results with long-run (non-stochastic steady-state) multipliers which take

¹This is not true a sticky price version in which expectations matters. See the discussion in Christiano et al. (2011).

into account full adjustment of physical capital. We show that long-run multipliers can be obtained as the limit case of dynamic multipliers after a permanent shock to government spending.

The model we use is sufficiently simple, given its functional forms on utility and production functions, to get analytical and insightful results. It nevertheless shares the key ingredients present in the DSGE literature: the utility is separable between consumption and leisure (consumption and leisure are deliberately maintained as normal goods), a constant return-to-scale technology combines labor and capital inputs, and the stochastic process of non-productive government spending is exogenous and persistent. These core assumptions are present in most current DSGE models (see Coenen et al. (2012) or Smets and Wouters (2003, 2007)). We show that our results still hold with capital adjustment costs, which makes the response of investment less sensitive to the persistence of government spending but does not modify the threshold value.

To extend our results, we extend our analysis to incorporate two key parameters of DSGE modeling: the intertemporal elasticity of substitution in consumption and the Frisch elasticity of labor supply. These two model versions nest our basic setup, making it simple to inspect the mechanism at work. The intertemporal elasticity of substitution in consumption only modifies the size of the constant capital multiplier, but does not alter the effects of the government spending driven by expectations. The elasticity of labor supply plays in two directions. First, when this elasticity is lower, the constant capital multiplier is smaller because the labor supply is less responsive after the negative income effect. Second, a smaller elasticity of labor supply reduces the adjustment speed of consumption (for a given level of physical capital). This implies that the threshold value of the autoregressive parameter on government spending must be higher to insure a positive response of saving.

Finally, we consider two types of market imperfections. First, we study external endogenous discounting, assuming that an increase in aggregate consumption makes agents more impatient (see Schmitt-Grohé and Uribe (2003), in a small open economy setup). Endogenous external discounting reinforces the investment channel and magnifies our previous results. As government spending crowds out private consumption, households become more patient and thus save more. In this economy, the threshold value on the persistence parameter is smaller, making the government spending policy more effective. Second, we allow for imperfect financial markets and imperfect competition on the labor market under a particular form of hand-to-mouth consumers (see Galí et al. (2007)). We make the special assumption that unions only consider the utility of savers when setting real wages. That

assumption amplifies the response of aggregate labor supply, which is the initial driving force in our setup. The existence of hand-to-mouth consumers magnifies output multipliers. When the fraction of these households is large enough, total consumption may increase after a government spending shock. However, a positive response of consumption is neither necessary nor sufficient to obtain an output multiplier above unity.

Our results build on the existing literature and make progress on several dimensions. As compared to Aiyagari et al. (1992), Baxter and King (1993) and Campbell (1994), we extend the analysis in three directions. First, we determine analytically under which conditions private investment increases after a positive shock to government spending (we show how our threshold value depends on preferences and technology). Second, we shed new light on different multiplier concepts. We decompose the short-run multipliers of all aggregate variables (output, consumption and investment) into a static component, the constant capital multiplier, and a term related to expectations about future government spending policy. We also analyze in a unified framework long-run and dynamic multipliers. Third, we consider economies with market imperfections. Leeper et al. (2011) show quantitatively, as we do analytically, that the persistence of the government spending shock is essential for obtaining a large output multipliers in calibrated DSGE models. Our results show under which conditions a larger multiplier can be obtained. Leeper et al. (2011) also find that the fraction of hand-to-mouth consumers matters a lot for multipliers. Again, we are able to disentangle the two key mechanisms at work (intra-temporal and inter-temporal) when considering that a fraction of households has no access to financial markets and no weight in the wage-bargaining process.

The paper is organized as follows. In the first two sections, we consider a prototypical model and derive closed-form solutions. We also consider adjustment costs on capital. In the third section, we extend the model in two directions: non-unit intertemporal elasticity of substitution in consumption and a finite elasticity of labor supply. In a fourth section, we consider two types of market imperfections and inspect how they modify multipliers. A last section concludes.

1 A Prototypical Model with Government Spending

We first consider a business cycle model with physical capital accumulation, endogenous labor supply and exogenous non-productive government spending. We also assume complete capital depreciation and utility function linear in leisure. Despite its simplicity, this model contains the key ingredients that we want to highlight. The last two restrictions will be relaxed in sections 3 and 4.

1.1 The Setup

The inter-temporal expected utility function of the representative household is given by

$$\mathbb{E}_t \sum_{i=0}^{fty} \beta^i \{ \log c_{t+i} + \eta(1 - n_{t+i}) \} + v(g_{t+i}) \quad (1)$$

where $\beta \in]0, 1[$ denotes the discount factor, \mathbb{E}_t is the expectation operator conditional on the information set available as of time t and $\eta > 0$. Time endowment is normalized to unity, c_t denotes period- t real consumption and n_t represents the household's labor supply. We follow Hansen (1985) and Rogerson (1988) and assume labor indivisibility, so that in a setup with perfect financial insurance, the utility is linear in leisure. Infinite labor supply elasticity magnifies fiscal multipliers and has technical consequences. First, this assumption simplifies computations a lot because the real wage and the real interest rate depend only on real consumption (see below). Second, this specification boosts the response of the economy to a government spending shock. We will investigate in section 3 the role of finite labor supply elasticity. The function $v(\cdot)$ is increasing and concave in g_t . Government spending delivers utility in an additively separable fashion and does not affect optimal choices on consumption and leisure. Without any normative perspective, this additive term in utility allows government spending to be useful.

The representative firm uses capital k_t and labor n_t to produce the homogeneous final good y_t . The technology is represented by the following constant returns-to-scale Cobb–Douglas production function²

$$y_t = Ak_t^\theta n_t^{1-\theta}, \quad (2)$$

where $A > 0$ is a scale parameter and $\theta \in]0, 1[$. Under full depreciation, the capital stock evolves according to

$$k_{t+1} = x_t \quad (3)$$

Finally, the final good can be either consumed, invested or devoted to unproductive government spending financed through lump-sum taxes:

$$y_t = c_t + x_t + g_t, \quad (4)$$

where g_t denotes exogenous stochastic government spending. For a given level of output, an increase in government spending reduces the resources available for consumption and investment. Agents

² All our main results are left unaffected if we consider a CES production function, allowing for a non unitary elasticity of substitution between inputs. Results are available from the authors upon request.

may respond to this shock through changes in their consumption, investment, and/or labor supply decisions.

The dynamic equilibrium of this economy is summarized by the following equations

$$k_{t+1} = y_t - c_t - g_t \quad (5)$$

$$y_t = Ak_t^\theta n_t^{1-\theta} \quad (6)$$

$$\eta c_t = (1 - \theta) \frac{y_t}{n_t} \quad (7)$$

$$\frac{1}{c_t} = \beta \theta \mathbf{E}_t \left[\left(\frac{y_{t+1}}{k_{t+1}} \right) \frac{1}{c_{t+1}} \right] \quad (8)$$

Equations (5) and (6) define the law of motion of physical capital (after substitution of the aggregate resource constraint (4) into (3)) and the production function (equation (2)). Equation (7) equates at equilibrium, the marginal rate of substitution between consumption and leisure with the marginal product of labor. Equation (8) represents the Euler equation on consumption. Equation (7) will simplify a lot further computations. To see this, let us combine equations (6) and (7). We deduce

$$y_t = A^{1/\theta} \left(\frac{1 - \theta}{\theta} \right)^{\frac{1-\theta}{\theta}} k_t c_t^{-\frac{1-\theta}{\theta}}$$

Using the definition of the real wage w_t and the real interest rate r_t (in a competitive decentralized equilibrium), we see that factor prices only depend on consumption levels: $w_t = \eta c_t$ and $r_t = A^{1/\theta} [(1 - \theta)/\theta]^{\frac{1-\theta}{\theta}} c_t^{-\frac{1-\theta}{\theta}} - 1$. After replacement into the Euler equation on consumption (8), one obtains

$$c_t^{-1} = \beta \theta \mathbf{E}_t \left[A^{1/\theta} \left(\frac{1 - \theta}{\theta} \right)^{\frac{1-\theta}{\theta}} c_{t+1}^{-\frac{1-\theta}{\theta}} \right],$$

so the Euler equation becomes an autonomous first order (non-linear) equation in consumption. It follows that this equation directly yields an expectation function about consumption when it comes at solving the permanent income problem. The lack of feedback effect of capital accumulation on prices in this equation allows to simply solve our problem. This property will remain true for other model's versions (incomplete depreciation, external endogenous discounting, hand-to-mouth consumers, CES technology, variable capital utilization) as soon as the infinite elasticity of labor supply is maintained. Conversely, when we consider finite elasticity of labor supply, both the real wage and the real interest rate will depend on the capital stock. This situation will be considered in Section 3.2.

Before analyzing fiscal multipliers in our dynamic stochastic economy, we present a restricted economy which provides an useful benchmark.

1.2 Constant Capital Government Spending Multipliers

We first consider a variant of our economy where the supply of physical capital is held fixed. Absent capital accumulation, intra-temporal allocations (between consumption and leisure) can be set statically, period by period, for successive values of government spending g_t . Fiscal multipliers are evaluated from the repeated static version of our model and depend on preferences, technology and steady-state ratios but they are independent from the process of government spending. To make sure the different economies we study have similar scales, we set the fixed levels of investment and capital, as well as the share of government spending in output, equal to their steady-state values in the variable-capital economy. We also consider the same preferences and technology parameters. We first introduce the following definition of constant capital government spending multipliers.

Definition 1 *Constant capital government spending multipliers refer to changes in aggregate variables consecutive to a unit increase in government spending expenditures g in an economy defined by intra-temporal allocations only. The constant-capital output multiplier is denoted*

$$\left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}}$$

The constant-capital investment multiplier $\left. \frac{\Delta x}{\Delta g} \right|_{\bar{k}}$ is null by assumption. The resource constraint

$$\left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}} = \left. \frac{\Delta c}{\Delta g} \right|_{\bar{k}} + \underbrace{\left. \frac{\Delta x}{\Delta g} \right|_{\bar{k}}}_{=0} + \underbrace{\left. \frac{\Delta g}{\Delta g} \right|_{\bar{k}}}_{=1}$$

ties the constant-capital consumption multiplier $\left. \frac{\Delta c}{\Delta g} \right|_{\bar{k}}$ to the constant-capital output multiplier,

$$\left. \frac{\Delta c}{\Delta g} \right|_{\bar{k}} = \left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}} - 1.$$

An increase in government spending plays a negative income effect because households have access to less final goods, *ceteris paribus*. This static economy shows how much less they choose to consume and how much more they choose to work (and produce).³ In that fixed capital economy, deep parameters move the constant capital multipliers on output and consumption in the same direction, although their signs are opposite.

³In an open economy, international trade provides another adjustment margin, which explains that fiscal multipliers are lower (in absolute values). In a small open economy where the real interest rate is exogenous, trade completely offsets any change in government spending – leaving consumption and labor input unchanged.

Proposition 1 *When capital is held constant in the economy described in section 1.1, the output multiplier equals*

$$\left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}} = \frac{1}{1 + \frac{\theta}{1-\theta} (1 - \beta\theta - \bar{g}/\bar{y})} > 0 \quad (9)$$

where \bar{g}/\bar{y} denotes the steady-state share of government spending.

The consumption multiplier equals

$$\left. \frac{\Delta c}{\Delta g} \right|_{\bar{k}} = \frac{1}{1 + \frac{\theta}{1-\theta} (1 - \beta\theta - \bar{g}/\bar{y})} - 1 < 0. \quad (10)$$

Proof: See Appendix A.

Proposition 1 shows that an increase in government spending always reduces private consumption, $\left. \frac{\Delta c}{\Delta g} \right|_{\bar{k}} < 0$, assuming a strictly positive consumption-to-output ratio.⁴ This implies that the output multiplier $\left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}}$ is smaller than one (as in Hall (2009) and Woodford (2011)), investment being held fixed in constant-capital economies. The output multiplier does tend to one when the capital share $\theta \rightarrow 0$ because in that limit case, output is linear in labor and capital is irrelevant.

Note that the constant-capital output multiplier remains below one despite an infinitely elastic labor supply, as implied by the linear disutility of labor. The value of $\left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}}$ in Proposition 1 needs therefore to be interpreted as an upper bound, as will be shown in Section 3.

Constant capital multipliers serve as a useful first step for two reasons. First, many positive models used to study fiscal stabilization policy do not model capital accumulation. Models with nominal rigidities display forward-looking inflation dynamics; capital accumulation adds a backward-looking dimension which imposes numerical solutions. Hence, fiscal multipliers in this model without capital accumulation share some features with existing multipliers in the literature. Second, these multipliers will show up as special cases of the economies with dynamic features we now study.

2 Dynamic Government Spending Multipliers

We now analyze our simple model with capital accumulation and stochastic government spending. Households face a dynamic problem on top of the static consumption-leisure tradeoff seen in Section 1.2. Increases in government spending still have negative income and wealth effects which may lead

⁴The consumption-to-output ratio is equal to $1 - \beta\theta - \bar{g}/\bar{y}$. A positive ratio at non-stochastic steady-state thus implies that $\bar{g}/\bar{y} < 1 - \beta\theta$.

households to consume less and work more. But households can now transfer resources from one period to the other through the physical asset and display richer saving behaviors.

To solve the intertemporal rational expectations equilibrium of this model, we need to specify how government spending evolves over time. The log of government spending g_t (in deviation from its deterministic steady state value \bar{g}) follows a simple stochastic process

$$\log g_t = \rho \log g_{t-1} + (1 - \rho) \log \bar{g} + \varepsilon_t$$

The previous equation simply rewrites

$$\hat{g}_t = \rho \hat{g}_{t-1} + \varepsilon_t \quad (11)$$

where $\hat{g}_t = \log g_t - \log \bar{g} \simeq (g_t - \bar{g})/\bar{g}$, $|\rho| \leq 1$ and ε is a white noise shock to government spending with zero mean and variance equal to σ_ε^2 . In what follows, ρ will characterize the persistence of expected government spending, as $E_t \hat{g}_{t+i} = \rho^i \hat{g}_t$. Despite its simplicity, this simple specification of the government spending is widely used by the DSGE literature (see e.g. Aiyagari et al. (1992), Christiano and Eichenbaum (1992), Campbell (1994), Smets and Wouters (2007), Coenen et al. (2012)).

We solve analytically the log-linear approximation of this economy around its non-stochastic steady state. The closed form solution fully characterizes the time series properties of aggregate variables, described in Proposition 2. To our knowledge, this paper is the first to compute a closed-form solution to this class of problem. It differs from McCallum (1989) which does not consider government spending.⁵ Hansen-Rogerson preferences help simplify the dynamical structure of our model economy. The assumption of complete depreciation, on the other hand, does not change the nature of the dynamical system. It only simplifies the exposition of the closed form solution.

Proposition 2 *In the economy with capital accumulation described in section 1.1, equilibrium consumption \hat{c}_t (in relative deviations from steady-state) follows a first-order autoregressive process while equilibrium investment \hat{x}_t follows an autoregressive process of order two. Denoting L the lag operator, the stochastic process of consumption and investment write*

$$(1 - \theta L) \hat{c}_t = -\frac{\theta s_g}{1 - \theta + \theta s_c} \left(\frac{1 - \beta \theta^2}{1 - \beta \theta \rho} \right) \varepsilon_t \quad (12)$$

$$[1 - (\theta + \rho)L + \theta \rho L^2] \hat{x}_t = s_g \left(\frac{\rho - \theta}{1 - \beta \theta \rho} \right) \varepsilon_t. \quad (13)$$

⁵McCallum's full depreciation approach is applicable when exogenous government expenditures are perfect substitutes to private consumption. In that case, shocks to government spending are perfectly offset and the time profile of government spending does not matter.

The stochastic process of output is a linear combination of two autoregressive processes on order one and one autoregressive process of order two:

$$\widehat{y}_t = s_c \widehat{c}_t + s_x \widehat{x}_t + s_g \widehat{g}_t,$$

where $s_c = 1 - \beta\theta - \bar{g}/\bar{y}$, $s_x = \beta\theta$ and $s_g = \bar{g}/\bar{y}$ are the consumption to output ratio, the investment to output ratio and government spending to output ratio, respectively.

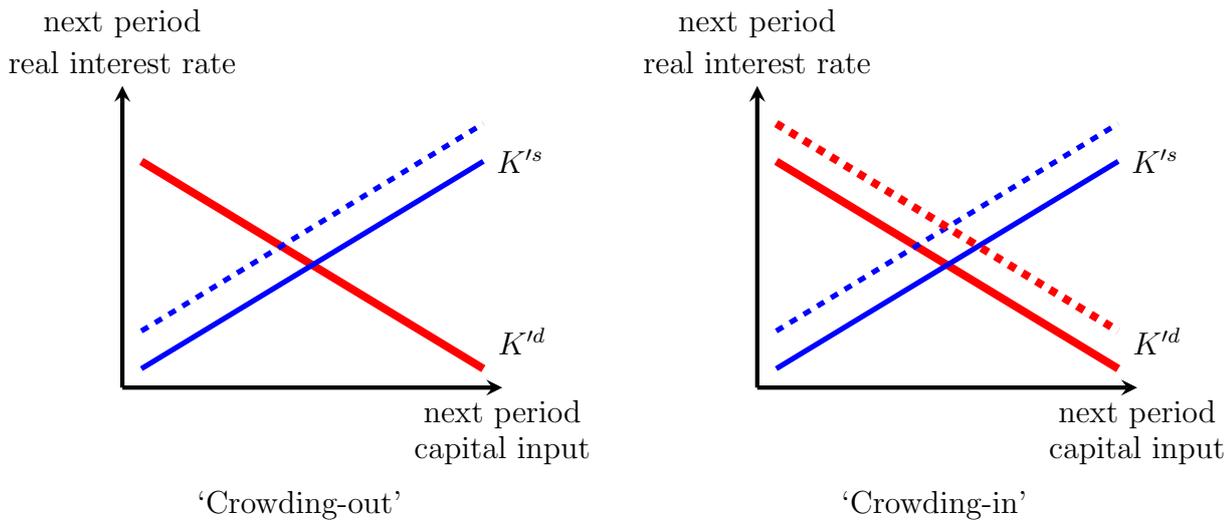
Proof: See Appendix B.

This analytic characterization delivers the impulse response function of each aggregate variable following a government spending shock $\varepsilon_0 > 0$. The main results of this paper, laid out in the next section (propositions 3 and 4), show how the investment channel shapes impact fiscal multipliers.

2.1 Impact Government Spending Multipliers

The impact response of investment to a government spending shock has an ambiguous sign. Most of the existing literature on fiscal multiplier (including Hall (2009), Christiano et al. (2011) or Woodford (2011)) mentions crowding-out type effects on investment, in which the rise in real interest rate consecutive to an increase in government spending reduces investment. Implicitly, this effect describes the response of savings (represented in blue in Figure 1), *i.e.* movements along the capital demand schedule.

Figure 1: The ambiguous response of capital to a government spending shock



But policies which stimulate employment also raise the marginal product of capital, and therefore shift the demand for capital services (in red), making the shift in equilibrium capital ambiguous. The capital demand effect is not specific to our setup (see Aiyagari et al. (1992) and Baxter and King (1993)). It is present as soon as factors are substitutable and employment increases.

The position of the capital demand schedule (displayed in red in Figure 1) depends on expectations of future employment. Hence, the persistence of government spending shock determines how much capital demand shifts up, hence the net effect on investment. On Figure 1, shifts in the demand and supply for capital services perfectly offset. Next proposition characterizes the persistence of government spending for which this result holds.

Proposition 3 *In the economy with capital accumulation described in section 1.1, the impact response of investment to an increase in government spending, $\frac{\Delta x_0}{\Delta g_0}$, can have both signs. It is determined by a cutoff rule on the persistence of government spending: there exists a threshold value $0 < \rho^* < 1$ of the persistence of government spending such that*

- when $\rho = \rho^*$, investment does not react to a change in government spending;
- the impact investment multiplier $\frac{\Delta x_0}{\Delta g_0}$ is strictly positive for any $\rho > \rho^*$;
- the impact investment multiplier $\frac{\Delta x_0}{\Delta g_0}$ is strictly negative for any $\rho < \rho^*$.

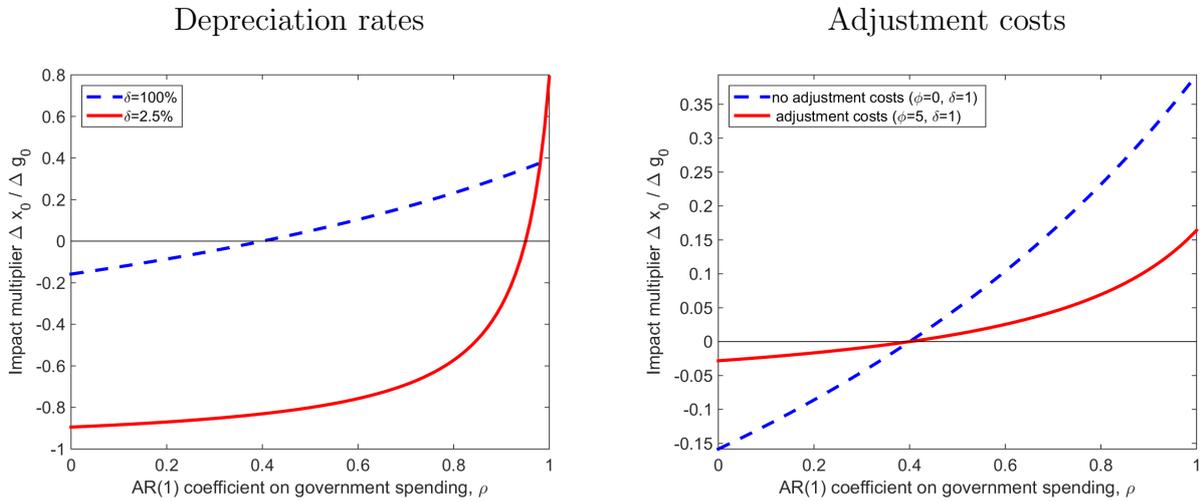
Proof: Short-run investment multipliers write $\frac{\Delta x_h}{\Delta g_0} = \frac{s_x}{s_g} \frac{\partial \hat{x}_{t+h}}{\partial \varepsilon_t}$ for $h = 0, 1, 2, \dots$. The impact investment multiplier is obtained using the stochastic process described in the previous proposition: $\frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \theta}{\frac{1}{\beta\theta} - \rho}$. Since $0 < \beta < 1$ and $0 < \theta < 1$, this multiplier is an increasing convex function of ρ and has the sign of its numerator, $\rho - \theta$. Hence, $\rho^* = \theta$. ■

The increase in government spending, which acts as a drain on resources, has two opposite effects on investment. On the one hand, households want to smooth their consumption and eat part of the existing capital (a crowding-out like effect). On the other hand, it stimulates employment and the marginal productivity of capital, increasing the demand for capital services. What matters for capital accumulation and investment is in fact the expectations of next period labor input. The more persistent the shock, the larger is that expectation. Capital accumulation is therefore desirable when government spending and employment are highly persistent, while households facing very temporary fiscal shocks exhibit negative savings. When the persistence parameter of government spending is equal to the threshold, $\rho = \theta$, the crowding-out and crowding-in effects exactly cancel out. In that

case, capital accumulation will never be affected and fiscal multiplier are identical to those of the constant–capital economy, as reported in Proposition 1. Note that the size of the shock does matter for the response of next period labor input and current investment, but not for the value of multipliers which are scaled objects.

The impact investment multiplier $\frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \theta}{\frac{1}{\beta\theta} - \rho}$ is not only an increasing function of the persistence parameter of government spending, ρ , but also a convex one (see the dashed blue line in Figure 2 for an illustration). Highly persistent government spending processes stimulate investment as well as employment, and that magnification becomes larger and larger as the persistence of government spending increases.

Figure 2: Investment multiplier and government spending persistence



Note: The parameters chosen for this illustration are $\beta = .99$, $\theta = .4$ and $\frac{\bar{y}}{y} = .2$.

As in Proposition 3, a cutoff rule will remain valid in all the extensions we consider later. While in other versions of the model, the threshold value will be a possibly complicated function of the underlying parameters, it is particularly simple in this economy with complete depreciation and linear disutility of labor: the cutoff value ρ^* is equal to θ , the capital share. For a given value of the persistence parameter ρ , the impact investment multiplier is a decreasing function of the elasticity of next period output with respect to today's investment, θ . This means that, when returns to capital are low, agents need to invest a lot today to relax future resource constraints. We can also perform comparative statics with respect to the discount factor β , and see that the impact investment multiplier increases when agents value the future more, strengthening the discounted utility benefits

of current investment.

The next proposition will show why the response of investment is crucial to understand fiscal multipliers.

Proposition 4 *The impact government spending multiplier on output in the economy with capital accumulation $\frac{\Delta y_0}{\Delta g_0}$ combines the static output multiplier in the economy $\frac{\Delta y}{\Delta g}\Big|_{\bar{k}}$ and the impact investment multiplier $\frac{\Delta x_0}{\Delta g_0}$:*

$$\frac{\Delta y_0}{\Delta g_0} = \frac{\Delta y}{\Delta g}\Big|_{\bar{k}} \times \left(1 + \frac{\Delta x_0}{\Delta g_0}\right).$$

The same decomposition holds for the impact consumption multiplier

$$\frac{\Delta c_0}{\Delta g_0} = \frac{\Delta c}{\Delta g}\Big|_{\bar{k}} \times \left(1 + \frac{\Delta x_0}{\Delta g_0}\right).$$

as well as the impact employment multiplier

$$\frac{\Delta n_0}{\Delta g_0} = \frac{\Delta n}{\Delta g}\Big|_{\bar{k}} \times \left(1 + \frac{\Delta x_0}{\Delta g_0}\right).$$

Proof: From Proposition 2 and appendix B, the impact output multiplier, $\frac{\Delta y_0}{\Delta g_0} = \frac{1}{s_g} \frac{\partial \hat{y}_t}{\partial \varepsilon_t}$, equals $\frac{\Delta y_0}{\Delta g_0} = \frac{1}{1 + \frac{\theta}{1-\theta}(1-\beta\theta-\bar{g}/\bar{y})} \left(\frac{1-\beta\theta^2}{1-\beta\theta\rho}\right) = \frac{\Delta y}{\Delta g}\Big|_{\bar{k}} \times \left(\frac{\frac{1}{\beta\theta}-\theta}{\frac{1}{\beta\theta}-\rho}\right) = \frac{\Delta y}{\Delta g}\Big|_{\bar{k}} \times \left(1 + \frac{\rho-\theta}{\frac{1}{\beta\theta}-\rho}\right)$. The impact consumption multiplier, $\frac{\Delta c_0}{\Delta g_0} = \frac{s_c}{s_g} \frac{\partial \hat{c}_t}{\partial \varepsilon_t}$ equals $\frac{\Delta c_0}{\Delta g_0} = -\frac{\frac{\theta}{1-\theta}(1-\beta\theta-\bar{g}/\bar{y})}{1 + \frac{\theta}{1-\theta}(1-\beta\theta-\bar{g}/\bar{y})} \left(\frac{1-\beta\theta^2}{1-\beta\theta\rho}\right) = \frac{\Delta c}{\Delta g}\Big|_{\bar{k}} \times \left(\frac{\frac{1}{\beta\theta}-\theta}{\frac{1}{\beta\theta}-\rho}\right)$. Finally, employment is proportional to the consumption-output ratio due to Hansen-Rogerson preferences. ■

According to that decomposition, the response of investment may amplify or dampen the multiplier on employment, output and consumption, with respect to the constant capital case. Such a decomposition is already present in Aiyagari et al. (1992), and holds as long as investment and government spending are composed of the same good.

While Proposition 4 characterizes the relative values of impact multipliers with and without capital, it does have an implication for the absolute value of the output multiplier in the general model. Remember that the constant-capital output multiplier is smaller than one. Therefore, a positive impact investment multiplier is a necessary condition for the impact output multiplier to exceed one. A large part of the literature on fiscal multipliers focuses on constant-capital effects. Taking into account the investment channel offers an alternative potential amplification mechanism.

2.2 Long-run Government Spending Multipliers

We turn to the long-run response of the economy. For that reason, we consider permanent shocks to government spending, *i.e.* when $\rho \rightarrow 1$. The long-run response of real quantities strikingly exemplifies how investment shapes fiscal multiplier, any adjustment in physical capital being completed in the long-run. The asymptotic results which follow are robust to any form of rigidity that would disappear in steady state, including nominal contracts, habit persistence, adjustment costs, and so on.

Consumption dynamics is the easiest to study. Proposition 2 has established that consumption follow a first-order autoregressive process, with an autoregressive coefficient $\theta < 1$. Consumption therefore converges back towards its initial steady-state value. The invariance of steady-state consumption is a direct consequence of the infinite elasticity of labor supply assumed so far. While a permanent increase in government spending stimulates labor supply and equilibrium employment, it does not affect the steady-state real rate interest rate which is pinned down by the psychological discount factor β , nor the steady-state real wage rate (which in turns depends on the capital share θ). With Hansen-Rogerson preferences, consumption is therefore not affected and the steady-state output multiplier increases with β and θ , which jointly determine the steady-state capitalistic intensity $\frac{\bar{k}}{\bar{y}}$.

When $\rho \rightarrow 1$, Proposition 2 implies that the growth rate of investment follows an autoregressive process of order one: $(1-\theta L)(1-L)\hat{x}_t = s_g \left(\frac{1-\theta}{1-\beta\theta} \right) \varepsilon_t$. After a permanent government spending shock, investment raises gradually and eventually converges to its new steady-state value. Proposition 5 establishes the asymptotic responses of output, consumption and investment.

Proposition 5 *In the economy with capital accumulation described in section 1.1, the asymptotic multipliers associated to permanent changes in government spending are given by*

$$\begin{aligned} \frac{\Delta c_\infty}{\Delta g_\infty} &= 0 \\ \frac{\Delta x_\infty}{\Delta g_\infty} &= \frac{\beta\theta}{1-\beta\theta} > 0 \\ \frac{\Delta y_\infty}{\Delta g_\infty} &= \frac{1}{1-\beta\theta} > 1 \end{aligned}$$

The asymptotic, constant-capital and impact multipliers on output, $\frac{\Delta y_\infty}{\Delta g_\infty}$, $\frac{\Delta y}{\Delta g} \Big|_{\bar{k}}$ and $\frac{\Delta y_0}{\Delta g_0}$, rank as follows:

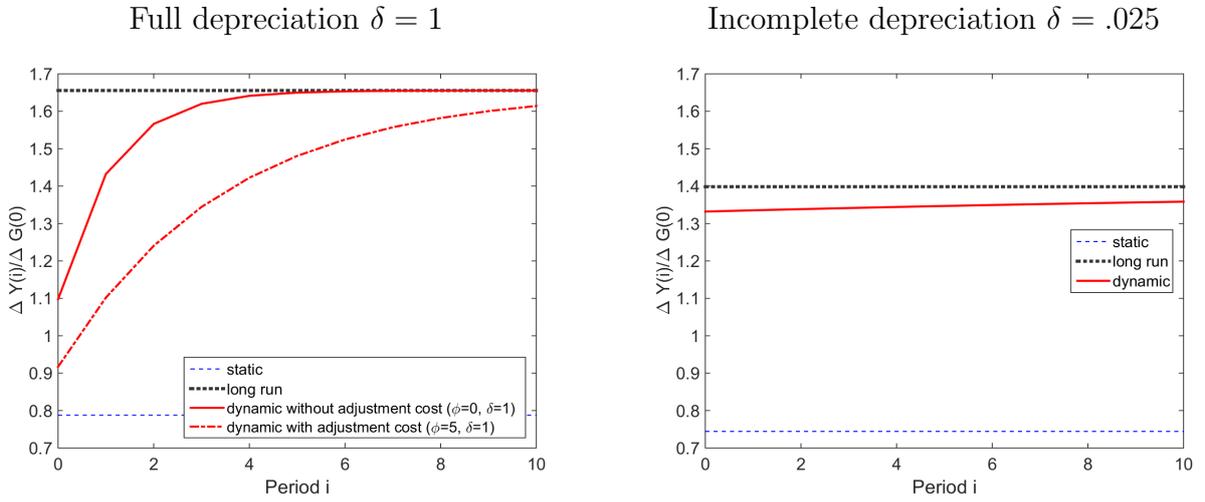
$$\frac{\Delta y_\infty}{\Delta g_\infty} > \lim_{\rho \rightarrow 1} \frac{\Delta y_0}{\Delta g_0} > \frac{\Delta y}{\Delta g} \Big|_{\bar{k}}.$$

The asymptotic, constant-capital and impact multipliers on investment, $\frac{\Delta x_\infty}{\Delta g_\infty}$, $\frac{\Delta x}{\Delta g}\Big|_{\bar{k}}$ and $\frac{\Delta x_0}{\Delta g_0}$, rank as follows:

$$\frac{\Delta x_\infty}{\Delta g_\infty} > \lim_{\rho \rightarrow 1} \frac{\Delta x_0}{\Delta g_0} > \frac{\Delta x}{\Delta g}\Big|_{\bar{k}}.$$

Proof: $\frac{\Delta x_\infty}{\Delta g_\infty} = \frac{s_x}{s_g} \lim_{h \rightarrow \infty} \frac{\partial \hat{x}_{t+h}}{\partial \varepsilon_t} = \frac{s_x}{s_g} \left(\frac{s_g}{1-\beta\theta} \right) = \frac{\beta\theta}{1-\beta\theta}$, which implies $\frac{\Delta y_\infty}{\Delta g_\infty} = \frac{\Delta c_\infty}{\Delta g_\infty} + \frac{\Delta x_\infty}{\Delta g_\infty} + \frac{\Delta y_\infty}{\Delta g_\infty} = 1 + \frac{\Delta x_\infty}{\Delta g_\infty} = \frac{1}{1-\beta\theta}$. Regarding the ranking of output multipliers, Proposition 4 implies $\lim_{\rho \rightarrow 1} \frac{\Delta y_0}{\Delta g_0} = \frac{1-\theta}{1-\theta+\theta s_c} \left(\frac{1-\beta\theta^2}{1-\beta\theta} \right)$ while Proposition 1 computes $\frac{\Delta y}{\Delta g}\Big|_{\bar{k}} = \frac{1-\theta}{1-\theta+\theta s_c} < 1$. For investment multipliers, Proposition 3 gives $\frac{\Delta x_0}{\Delta g_0} = \frac{\rho-\theta}{\frac{1}{\beta\theta}-\rho}$ and $\frac{\Delta x}{\Delta g}\Big|_{\bar{k}}$ is null by definition. ■

Figure 3: The dynamics of output multipliers: illustration



Note: The parameters chosen for these illustrations are $\beta = .99$, $\theta = .4$, $\frac{\bar{g}}{\bar{y}} = .2$ and $\rho = .99$.

When shocks to government spending are permanent, the asymptotic fiscal multiplier taking into account capital adjustments exceeds the fiscal multiplier of a similar economy where capital is held fixed. The long-run output multiplier $\frac{1}{1-\beta\theta}$ can in fact be very large when agents are very patient (β close to one) and/or returns to capital very high (θ large). The asymptotic multiplier also exceeds the impact multiplier (see the left panel of Figure 3 for an illustration), which is itself larger than the constant-capital multiplier when government spending are very persistent, as implied by the cutoff rule. The impact multiplier does incorporate a response of investment, but does not account for any adjustment of the capital stock yet. In a sense, the constant-capital multiplier and steady-state multiplier are two polar benchmarks where either consumption or investment react to the change in

government spending, but not simultaneously (as they do in the general case).

2.3 Capital Adjustment Costs and the Investment Channel

Our analysis of the aggregate multiplier has emphasized the key role of investment. To generalize our results to plausible quantitative investment dynamics, we introduce capital adjustment costs in our setup. We consider the following adjustment cost function

$$\frac{\phi}{2} \frac{(k_{t+1} - k_t)^2}{k_{ss}}$$

where $\phi \geq 0$ and k_{ss} denotes the steady-state capital stock.⁶ In the full depreciation case, $k_{t+1} = x_t$ and this specification is similar to dynamic adjustment costs on investment. The aggregate resources constraint writes

$$y_t = c_t + x_t + g_t + \frac{\phi}{2} \frac{(k_{t+1} - k_t)^2}{k_{ss}}$$

The log-linearized version of the resource constraint is left unaffected by adding capital adjustment costs. The Euler equation on consumption is modified and becomes:

$$\frac{q_t}{c_t} = \beta \theta E_t \left(\theta \frac{y_{t+1}}{k_{t+1}} + q_{t+1} - 1 \right) \frac{1}{c_{t+1}}$$

where q_t denotes the Tobin's Q and verifies

$$q_t = 1 + \phi \left(\frac{k_{t+1} - k_t}{k_{ss}} \right)$$

We solve the model backward-forward and then determine the short-run investment multiplier. The next proposition reports the impact investment multiplier. We do not report the output and consumption multiplier as the results of Proposition 4 are maintained.

Proposition 6 *The short-run investment multiplier is given by*

$$\frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \theta}{[(\beta\theta)^{-1} - \eta_{kk} + \theta - \rho] + \phi \frac{1-\theta+s_c}{\beta} [(1 + \beta(1 - \rho - \eta_{kk})] },$$

with $0 \leq \eta_{kk} \leq 1$ is the stable root on capital accumulation and $s_c = 1 - \beta\theta - \bar{g}/\bar{y}$ the consumption to output ratio.

⁶We can also consider the more usual specification $\frac{\phi}{2} \left(\frac{k_{t+1} - k_t}{k_t} \right)^2 k_t$. This yields the same findings, but analytics are less transparent. Results are available from the authors upon request. We have investigated other specifications of the adjustment costs function and financial frictions (see Carlstrom and Fuerst (1997)) and we obtain similar qualitative results (see Carrillo and Fève (2016)).

Proof: See Appendix C.

When $\phi = 0$, we retrieve the investment multiplier without capital adjustment costs analyzed in Proposition 3 (see the proof). Conversely, when $\phi \rightarrow \infty$, the government spending multiplier tends to zero, whatever the value of ρ . In this case, the capital stock remains constant after a government spending shock. Output and consumption multipliers perfectly coincide with the constant capital multipliers of section 1.2. The main finding of this exercise is that the cut-off point is not altered by the presence of capital adjustment cost. This result is not surprising: for the threshold value of government spending persistence computed absent adjustment costs, investment (next period capital) remains constant after a fiscal shock. The adjustment cost function is zero in that situation, meaning that the threshold is not affected.

However, the adjustment cost parameter ϕ does alter the slope of the curve between the investment multiplier and the value of persistence parameter ρ in government spending, as illustrated by the Figure 2 (right panel). As investment reacts less to fiscal stimulus, the impact output multiplier is much closer to the one of an economy with constant capital. However, as the horizon increases, the dynamic multiplier departs from the repeated static economy one. For $\rho \rightarrow 1$, the response of output is much more persistent and converge at a slower rate to the long-run multiplier (see the left panel of Figure 3).

2.4 Discussion

Several theoretical papers have already compared the effect of permanent and transitory government spending shocks. Aiyagari et al. (1992) emphasize the shift in capital demand due to permanent income mechanism when government spending follow a Markow process. They also establish the decomposition of output effects into a constant-capital term and the response of investment, whose sign remains undetermined. Baxter and King (1993) consider T -years increases in spending (framed as wars) to contrast temporary and permanent (when $T \rightarrow \infty$) movements in government purchases. They show numerically that the output multiplier increases with the duration of spending T . They also suggest to study ends of wars, *i.e.* one-off reductions in government spending. Campbell (1994) models government spending as a first-order autoregressive process. He emphasizes the interaction between the labor supply elasticity and the persistence of government spending shocks and computes numerically consumption, capital, employment and output elasticities for selected values. His numerical results only exhibit an increase in next period's capital in response to a positive government

purchases shock when government spending is permanent. Our paper echoes the findings of Aiyagari et al. (1992), Baxter and King (1993) and Campbell (1994), and provides deeper analytical results. First, we deliver closed form solutions for the model, including the threshold value for persistence such that investment remains constant after a government spending shock. Second, we shed new light on different multiplier concepts. We show why constant-capital multiplier may strongly differ from dynamic multipliers taking into account investment decisions. We also connect long-run and dynamic multipliers. To preserve space, we do not present results with T -years government spending plans à la Baxter and King (1993). But the positive effects on investment of expected future government expenses carries over to that process: the impact investment multiplier is an increasing function of the spending duration T and can display both signs. We can determine the mapping between T and ρ yielding identical dynamic investment multiplier for the two processes, $1 - (\beta\theta)^T = \frac{1-\beta\theta}{1-\beta\theta\rho}$. This mapping is increasing and $T \rightarrow \infty$ when $\rho \rightarrow 1$. The cutoff rule on T depends on the same set of parameters that the cutoff on ρ we provide. While estimates of the first-order autoregressive coefficient of US government spending exist and are typically large (0.97 in Smets and Wouters (2007) and Leeper et al. (2010)), we do not have knowledge of an estimate of the average duration spending.

Corsetti et al. (2012) focus, as we do, on expectations of future policies. They model spending reversals adding to an AR(1) process a feedback from current public debt. Agents from the private sector know that rule and take it as given. In a closed economy context, expectations consistent with this type of policy may yield larger multiplier if monetary policy is accomodative enough.

Can we connect our results with empirical evidence on the investment response to government spending shocks? A large VAR literature exists, whose results are inconclusive: for instance, Mountford and Uhlig (2009) obtain a negative response to unanticipated government expenditure shocks while Edelberg et al. (1999) find that nonresidential investment rises (see Ramey (2016) for a recent survey). That literature faces the concern that government spending shocks may be partly anticipated and not pure surprises. If so, VAR representations would be nonfundamental (Leeper et al. (2013)). Ramey (2011b) addresses this question with narrative techniques and finds a negative response of investment over the 1939-2008 sample. Forni and Gambetti (2011) turn to dynamic factor models and obtain a positive investment response. Our results can help interpret the large dispersion in estimated responses. We have shown in a simple neoclassical model that the response of investment varies with the persistence of government spendings, meaning that no policy-invariant prediction can be made regarding investment. None of the empirical approach mentioned is equipped

to control for the persistence of the shock process, but structural DSGE models do. Recently, Leeper et al. (2011) consider a large-scale DSGE model (a multi-sector open economy model with real and nominal frictions, non-optimizing agents and rich monetary and fiscal policy rules) to explore which parameters matter for the fiscal multipliers. Their quantitative analysis pinpoints the persistence of government spending as the parameter with the highest predictive power. They also find that investment decreases unless government spending are highly persistent. It is easy to interpret Leeper et al. (2011)'s findings regarding the response of investment as a consequence of our cutoff rule.

Finally, we can wonder how general are the results obtained in our setup. Two steps are critical for our analysis. First, that positive shocks to government spending actually raise labor input. Second, that increases in employment shift capital demand up.

In which models does the second step fail? The marginal product of capital is no longer an increasing function of labor input when the technology is Leontieff and too much capital is available (or, in a less interesting case, when capital and labor are perfect substitutes). This configuration requires investment irreversibility, plus a shock large enough that desired capital exceeds current one, despite capital depreciation. Back to the first step, the increase in employment is achieved in our model through a standard wealth effect. Other mechanisms can stimulate employment, such as productive government spending. Regardless of their cause, employment increases trigger the second step of the investment channel.

3 Extensions under Incomplete Depreciation

In this section, we extend our analysis to an incomplete depreciation setup, where the capital stock evolves according to the law of motion

$$k_{t+1} = (1 - \delta) k_t + x_t,$$

where $0 < \delta < 1$ is a constant depreciation rate.⁷

Using this law of motion on capital, we will also consider more general specifications of the utility function. We will show that two results established in the previous section are left unaffected. First, short-run fiscal multipliers combine a constant-capital effect and the response of investment. Second, the sign of the investment multiplier obeys the same cutoff rule.

⁷Capital utilization decisions which affect the depreciation rate would not change our qualitative analysis. Models with endogenous capital utilization are observationally equivalent to models without this intensive margin, but displaying a larger elasticity of output with respect to labor input. Therefore, capital utilization would reduce the threshold persistence value for which investment does not react to government spending.

From now on, we will generically denote \mathcal{P} the vector of deep parameters (the set of which will be environment-specific), with the exception of the persistence of government spending ρ . We will present several extensions of our basic setup in which the cutoff rule and impact output multiplier exhibit the following generic representation:

$$\frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \mathcal{C}(\mathcal{P})}{\mathcal{U}(\mathcal{P}) - \rho} \quad (14)$$

$$\frac{\Delta y_0}{\Delta g_0} = \mathcal{K}(\mathcal{P}) \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P})}{\mathcal{U}(\mathcal{P}) - \rho} \right] \quad (15)$$

with $\mathcal{C}(\mathcal{P})$, $\mathcal{K}(\mathcal{P})$ and $\mathcal{U}(\mathcal{P})$ three reduced-form coefficients, functions of the parameter vector \mathcal{P} . $\mathcal{K}(\mathcal{P})$ denotes the constant-capital output multiplier, defined as in the previous section (see Definition 1). $\mathcal{U}(\mathcal{P})$ is the unstable root of the dynamical system and is always larger than one. The third reduced-form coefficient, $\mathcal{C}(\mathcal{P})$, which shows up as a threshold on ρ in the numerator, drives the expected dynamics of consumption when capital is held fixed. It appears in a log-linear version of the Euler equation embedding optimal labor choices, of the form

$$E_t \widehat{c}_{t+1} = \mathcal{C}(\mathcal{P}) \widehat{c}_t - \mathcal{X}(\mathcal{P}) \widehat{k}_{t+1}. \quad (16)$$

As already pointed out, Hansen-Rogerson preferences imply that $\mathcal{X}(\mathcal{P}) = 0$ and the dynamics of consumption is autonomous.

In the environment studied in the previous section, \mathcal{P} contains the value of β , θ and \bar{g}/\bar{y} . The reduced-form coefficients respectively equal

$$\mathcal{K}(\mathcal{P}) = \frac{1}{1 + \frac{\theta}{1-\theta} (1 - \bar{x}/\bar{y} - \bar{g}/\bar{y})}, \quad \mathcal{C}(\mathcal{P}) = \theta < 1 \quad \text{and} \quad \mathcal{U}(\mathcal{P}) = \frac{1}{\beta\theta} > 1.$$

Complete depreciation was helpful to simplify these expressions: the cutoff persistence level of government spending under incomplete depreciation $\mathcal{C}(\mathcal{P})$ becomes

$$\mathcal{C}(\mathcal{P}) = \frac{\theta}{1 - \beta(1 - \theta)(1 - \delta)}. \quad (17)$$

A large depreciation rate δ means that capital can adjust quickly. The persistence of government spending above which capital accumulation becomes optimal is therefore lower. This is illustrated by the left panel of Figure 3 that reports three multiplier concepts (static, long-run and dynamic) when $\rho \rightarrow 1$. In this figure, ρ is larger than $\mathcal{C}(\mathcal{P})$, implying that the dynamic multiplier exceeds the static one for all periods. When the depreciation rate is lower (in the right panel of figure 3, we set δ equal to 1.5% per quarter), the capital stock adjusts very slowly and so does the dynamic multiplier.

Incomplete depreciation also reinforces the convexity with respect to ρ of the investment multiplier (see the solid red line in Figure 2, left panel).

In the incomplete depreciation case, the cutoff value is much larger than under complete depreciation. For the standard calibration we have used to draw our figures, $\mathcal{C}(\mathcal{P}) = 0.95$ (as compared to 0.4 when $\delta = 1$). This value is close to, but usually lower than, available estimates of first-order autoregressive coefficient of actual government spending for US data (0.97 in Smets and Wouters (2007) and Leeper et al. (2010)). Our results suggest that stimulus packages displaying less persistence than average actual government spending do would lack the serial correlation required to take advantage of the investment channel. The convexity of investment multiplier implies that small changes in ρ can trigger very large increases in multipliers in that zone of high persistence (see the left panel of Figure 2).

Throughout the section, we will emphasize whether a specific modification of the environment affects the constant-capital multiplier $\mathcal{K}(\mathcal{P})$, the impact response of investment $\frac{\Delta x_0}{\Delta g_0}$ (which can occur through a shift in the cutoff $\mathcal{C}(\mathcal{P})$ or through a change in the denominator due to $\mathcal{U}(\mathcal{P})$), or both. In what follows, \mathcal{P} also contain parameters specific to each economy studied.

3.1 Non-unit Intertemporal Elasticity of Substitution in Consumption

The desire to smooth consumption through saving is one of the factors which shape investment in our economy. To investigate the role of consumption smoothing, we allow for a more general specification of utility with respect to consumption than the log specification. The instantaneous utility rewrites

$$u(c_t, n_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \eta(1 - n_t) + v(g_t) \quad (18)$$

where $\sigma \in [0, [1, +\infty[0$ is the inverse of the intertemporal elasticity of substitution (IES) in consumption. This economy reduces to the benchmark studied in Sections 1 and 2 when $\sigma = 1$ and $\delta = 1$. We note \mathcal{P}_σ the vector of relevant deep parameters excluding the persistence of government spending, *i.e.* $\mathcal{P}_\sigma = \{\beta, \delta, \bar{g}/\bar{y}, \theta, \sigma\}$.

Utility function (18) modifies two optimality conditions: the static consumption-leisure choice (7) and the Euler equation (8). These equations rewrite

$$\eta = (1 - \theta) \frac{y_t}{n_t} c_t^{-\sigma} \quad (19)$$

$$c_t^{-\sigma} = \beta \mathbb{E}_t \left[\left(1 - \delta + \theta \frac{y_{t+1}}{k_{t+1}} \right) c_{t+1}^{-\sigma} \right] \quad (20)$$

Next proposition shows how a non-unit intertemporal elasticity of substitution affects the fiscal multipliers.

Proposition 7 *With the utility function (18) and incomplete depreciation,*

1. *The constant capital government spending multipliers on output and consumption equal*

$$\begin{cases} 0 < \left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}} = \frac{1}{1 + \frac{\theta}{(1-\theta)\sigma} s_c} = \frac{1}{1 + \frac{\theta}{(1-\theta)\sigma} \left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)} - \frac{\bar{g}}{\bar{y}} \right]} = \mathcal{K}(\mathcal{P}_\sigma) < 1 \\ \left. \frac{\Delta c}{\Delta g} \right|_{\bar{k}} = \mathcal{K}(\mathcal{P}_\sigma) - 1 < 0 \end{cases}$$

The constant capital output multiplier is positive but smaller than unity while the constant capital consumption multiplier is negative. Both multipliers increase with the curvature of the utility function σ (i.e. the multipliers are decreasing functions of the intertemporal elasticity of substitution).

2. *The impact multipliers on investment, output and consumption are given by*

$$\begin{cases} \frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \mathcal{C}(\mathcal{P}_\sigma)}{\mathcal{U}(\mathcal{P}_\sigma) - \rho} \begin{matrix} \leq 0 \\ > 0 \end{matrix} \\ \frac{\Delta y_0}{\Delta g_0} = \mathcal{K}(\mathcal{P}_\sigma) \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\sigma)}{\mathcal{U}(\mathcal{P}_\sigma) - \rho} \right] > 0 \\ \frac{\Delta c_0}{\Delta g_0} = [\mathcal{K}(\mathcal{P}_\sigma) - 1] \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\sigma)}{\mathcal{U}(\mathcal{P}_\sigma) - \rho} \right] < 0 \end{cases}$$

with $\mathcal{C}(\mathcal{P}_\sigma) = \frac{1}{1 + \frac{1-\theta}{\theta} [1-\beta(1-\delta)]} < 1$ and $\mathcal{U}(\mathcal{P}_\sigma) = \frac{1}{\beta \mathcal{C}(\mathcal{P}_\sigma)} = \frac{1 + \frac{1-\theta}{\theta} [1-\beta(1-\delta)]}{\beta} > 1$. The threshold value of persistence $\mathcal{C}(\mathcal{P}_\sigma)$ is invariant to the intertemporal elasticity of substitution.

3. *The long-run government spending multipliers (following a permanent shock) equal*

$$\frac{\Delta y_\infty}{\Delta g_\infty} = \frac{1}{1 - s_x}, \quad \frac{\Delta c_\infty}{\Delta g_\infty} = 0, \quad \frac{\Delta x_\infty}{\Delta g_\infty} = \frac{s_x}{1 - s_x}$$

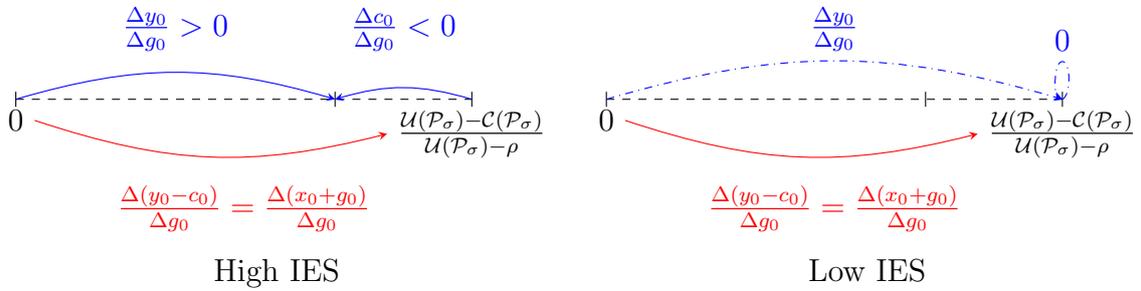
where $s_x = \frac{\beta\delta\theta}{1-\beta(1-\delta)}$ is the steady-state share of investment. They are invariant to the intertemporal elasticity of substitution. The steady-state output multiplier always exceeds the constant-capital one.

Proof: See Appendix D.

This proposition shows that the impact output multiplier $\frac{\Delta y_0}{\Delta g_0}$ is a decreasing function of the intertemporal elasticity of substitution $1/\sigma$. However, the output multiplier is only affected through the constant capital multiplier $\mathcal{K}(\mathcal{P}_\sigma)$, while the investment multiplier is invariant to the consumption smoothing parameter σ which affects neither $\mathcal{C}(\mathcal{P}_\sigma)$ nor $\mathcal{U}(\mathcal{P}_\sigma)$.

The parameter σ pins down how much the raise in aggregate savings is achieved through a decrease in consumption and how much through an increase in output (or leisure reduction). This breakdown is entirely static. In the constant capital economy, national savings increase through a relatively small increase in output and a relatively large reduction of consumption for values of σ close to 1. This case is displayed in solid blue in the left panel of Figure 4. Agents with large σ want their consumption profile to be extremely smooth and reduce their consumption less when they face higher government spending and future taxes (in fact, $\frac{\Delta c_0}{\Delta g_0} \rightarrow 0$ when $\sigma \rightarrow +\infty$ as in the right panel, dash-dotted blue line). They produce the required resources through a stronger increase in labor supply, which yields larger constant-capital output multipliers.

Figure 4: National savings and the intertemporal elasticity of substitution



The results obtained with the utility specification (18) also hold in other settings with rich intertemporal consumption decisions. For instance, the minimal consumption model

$$u(c_t, n_t) = \log(c_t - c_m) + \eta(1 - n_t) + v(g_t)$$

where $c_m \geq 0$ is a minimal level of consumption, can be interpreted as a proxy for habits in consumption decision (without adding a new state variable that complicates the derivation of the policy function). In the log-linear approximation of the model, we have $\sigma = (1 - c_m/\bar{c})$, where c_m/\bar{c} is the steady-state share of minimal consumption. When interpreted as a proxy for habit, setting $\sigma = 4$ is equivalent to an habit parameter equal to $c_m/\bar{c} = 0.75$, a value commonly obtained when it comes to estimated versions of medium-scaled DSGE models (see e.g. Smets and Wouters, 2007).

3.2 Finite Elasticity of Labor Supply

We have just investigated various levels of the intertemporal elasticity of substitution when utility is linear in leisure. We now consider finite elasticities of labor supply when utility is log in consumption.⁸

The instantaneous utility rewrites

$$u(c_t, n_t) = \log c_t - \frac{\eta}{1 + \varphi} n_t^{1+\varphi} + v(g_t) \quad (21)$$

with $\varphi \geq 0$ the inverse of the Friscean elasticity of labor supply and $\eta > 0$ a scale parameter. This representation of preferences allows to investigate the role (and consequences) of labor supply elasticities, because the parameter φ can take values between zero (infinite elasticity, as in section 1), and infinity (inelastic labor supply). In the latter case, the wealth effect of government spending disappears. We denote \mathcal{P}_φ the vector $\{\beta, \delta, \bar{g}/\bar{y}, \varphi\}$.

Additive separability between consumption and leisure implies that the Euler equation is unchanged with respect to Section 1 after accounting for incomplete depreciation. The only optimality condition affected is the static consumption-leisure choice (7), which rewrites

$$\eta n_t^\varphi = (1 - \theta) \frac{y_t}{n_t} c_t. \quad (22)$$

Solving the model gets more complicated because the dynamics of consumption is no longer autonomous (current consumption is no longer a sufficient statistic to compute the real wage and the real interest rate). This implies that $\mathcal{X}(\mathcal{P}) \neq 0$ in equation (16). The responses of aggregate variables to an unexpected shock on government spending, reported in Appendix E, point out the interplay of government spending persistence and capital adjustment in short and long-run output multipliers.

Proposition 8 *With a finite elasticity of labor supply as in (21) and incomplete depreciation,*

1. *The constant capital government spending multipliers on output and consumption equal*

$$\begin{cases} 0 < \frac{\Delta y}{\Delta g} \Big|_{\bar{k}} = \frac{1}{1 + \frac{\theta + \varphi}{1 - \theta} s_c} = \frac{1}{1 + \frac{\theta + \varphi}{1 - \theta} \left[1 - \frac{\beta \delta \theta}{1 - \beta(1 - \delta) - \frac{\bar{g}}{\bar{y}}} \right]} = \mathcal{K}(\mathcal{P}_\varphi) < 1 \\ \frac{\Delta c}{\Delta g} \Big|_{\bar{k}} = \mathcal{K}(\mathcal{P}_\varphi) - 1 < 0 \end{cases}$$

The constant capital output multiplier is positive but smaller than unity while the constant capital consumption multiplier is negative. Both multipliers decrease with the labor supply parameter φ (i.e. the multipliers are increasing functions of the labor supply elasticity).

⁸We can also combine non-unit intertemporal elasticity of substitution in consumption with finite elasticity of labor supply. To simplify the exposition, we prefer to consider each mechanism in isolation. Results are available from the authors upon request.

2. The impact multipliers on investment, output and consumption are given by

$$\begin{cases} \frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \mathcal{C}(\mathcal{P}_\varphi)}{\mathcal{U}(\mathcal{P}_\varphi) - \rho} \begin{matrix} \leq 0 \\ > 0 \end{matrix} \\ \frac{\Delta y_0}{\Delta g_0} = \mathcal{K}(\mathcal{P}_\varphi) \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\varphi)}{\mathcal{U}(\mathcal{P}_\varphi) - \rho} \right] > 0 \\ \frac{\Delta c_0}{\Delta g_0} = [\mathcal{K}(\mathcal{P}_\varphi) - 1] \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\varphi)}{\mathcal{U}(\mathcal{P}_\varphi) - \rho} \right] < 0 \end{cases}$$

with $\mathcal{C}(\mathcal{P}_\varphi) = \frac{1}{1 + \beta\theta \frac{1-\theta}{\theta+\varphi} \frac{\bar{y}}{k}} = \frac{1}{1 + \frac{1-\theta}{\theta+\varphi} [1-\beta(1-\delta)]} < 1$ and $\mathcal{U}(\mathcal{P}_\varphi) > 1$. The complete expression of the unstable root of the system is given in Appendix E. Notice that the threshold value of persistence $\mathcal{C}(\mathcal{P}_\varphi)$ decreases with the elasticity of labor supply $1/\varphi$.

3. The long-run government spending multipliers (following permanent shocks) equal

$$\frac{\Delta y_\infty}{\Delta g_\infty} = \frac{1}{1 - s_x + \varphi s_c}, \quad \frac{\Delta c_\infty}{\Delta g_\infty} = -\frac{\varphi s_c}{1 - s_x + \varphi s_c}, \quad \frac{\Delta x_\infty}{\Delta g_\infty} = \frac{s_x}{1 - s_x + \varphi s_c}$$

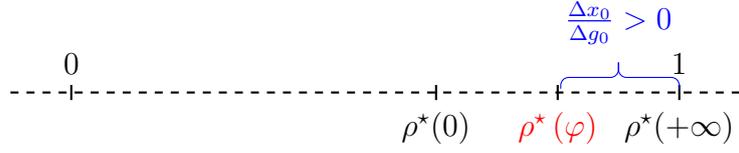
where $s_x = \frac{\beta\delta\theta}{1-\beta(1-\delta)}$ is the steady-state share of investment, s_c is the steady-state share of consumption $s_c = 1 - s_x - s_g$ and $s_g = \bar{g}/\bar{y}$. The steady-state output multiplier $\frac{\Delta y_\infty}{\Delta g_\infty}$ is a decreasing function of φ . The steady-state output multiplier always exceeds the constant-capital one and is greater than one if $\varphi < \frac{s_x}{s_c}$.

Proof: See Appendix E.

The elasticity of labor supply shifts the constant-capital multiplier, as did the intertemporal elasticity of substitution, because hours worked respond less when φ is large (in the limit case $\varphi \rightarrow +\infty$, labor supply is inelastic and $\mathcal{K}(\mathcal{P}_\varphi) \rightarrow 0$). This labor-supply parameter also affects the threshold persistence of government spending for which investment is not affected by the government spending shock: $\mathcal{C}(\mathcal{P}_\varphi)$ increases with the labor supply elasticity parameter φ .⁹ For a given persistence of government spending, higher labor supply elasticity stimulates employment, hence the marginal product of capital which itself boosts investment. Symmetrically, the less elastic is employment, the more persistence it takes for investment to increase after a government spending shock as in apparent in Figure 5. Note however that the long-run multiplier exceeds unity as long as $\varphi < \frac{s_x}{s_c}$ (a condition automatically satisfied when the utility is linear in labor supply).

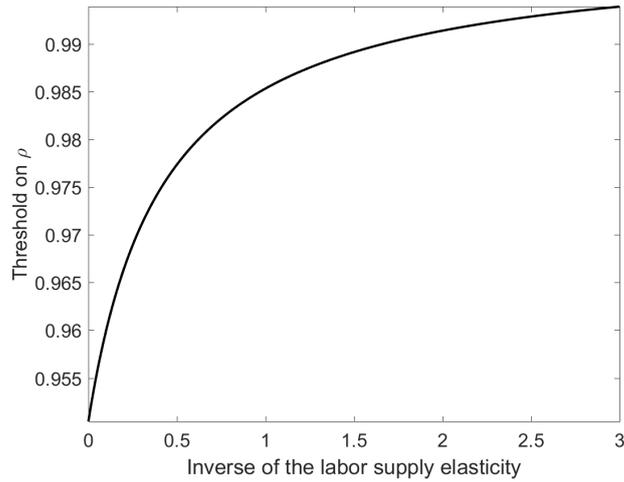
⁹The elasticity of labor supply also affects the unstable root of the system, $\mathcal{U}(\mathcal{P}_\varphi)$. The expression of the unstable root is relatively simple in the case of linear utility in leisure, but not when the elasticity of labor supply is finite. As previously mentioned, when utility is linear in labor, both the equilibrium wage rate and the equilibrium rate of interest do not depend on the capital stock, making the dynamics of consumption autonomous. This technical reason explains that the stable root of the system is in that case equal to the threshold value $\mathcal{C}(\mathcal{P}_\varphi)$. Since the product of the stable and unstable root always equal $1/\beta$ in frictionless setups, the unstable root is easy to compute.

Figure 5: Threshold persistence value and convex disutility of labor



An extreme case in our analysis of labor supply elasticity allows to isolate crowding-out effects on private spending. When $\varphi \rightarrow +\infty$, employment (and therefore output) remains constant after any change in government spending. This case exemplifies crowding-out effects: capital demand is unaffected by government spendings (meaning that the blue line in Figure 1 no longer shifts), while savings drop and the real interest increases (along the red line in Figure 1). The investment multiplier remains negative for all values of ρ : $\frac{\Delta x_0}{\Delta g_0} = \frac{\rho^{-1}}{\mathcal{U}(\mathcal{P}_\varphi)_{\varphi \rightarrow +\infty} - \rho} \leq 0$. The persistence in government spendings pins down how the reduction in private spending is split between consumption and saving. The impact consumption response, $\frac{\Delta c_0}{\Delta g_0} = -\frac{\mathcal{U}(\mathcal{P}_\varphi)_{\varphi \rightarrow +\infty} - 1}{\mathcal{U}(\mathcal{P}_\varphi)_{\varphi \rightarrow +\infty} - \rho}$, is always negative and decreasing in the persistence of public spending ρ . When shocks are quasi-permanent, consumption adjusts one for one, leaving savings unchanged as in any permanent income setup. With pure transitory shocks, the consumption is weakly affected and most of the adjustment concerns savings.

Figure 6: Elasticity of labor supply and persistence threshold



Note: The parameters chosen for these illustrations are $\beta = .99$, $\delta = .025$, $\theta = .4$ and $\frac{\bar{q}}{\bar{y}} = .2$.

Figure 6 illustrates the sensitivity of the threshold persistence level to the elasticity of labor supply when depreciation is incomplete. As shown by Proposition 8, the threshold is a decreasing function of labor supply elasticity. The persistence level in equation (17) must therefore be viewed as a lower bound. For values typically used in the DSGE literature, $1 \leq \varphi \leq 2$, the threshold values with log utility in consumption (solid line) range between .985 and .991, which exceeds most point estimates for US data (0.97 in Smets and Wouters (2007) and Leeper et al. (2010)), but remains within the estimated 90% interval.

4 Market Imperfections

We study in this section two variants of the benchmark model which embed market imperfections. We can extend our analysis of government spending multipliers to non-optimal equilibrium allocations.

4.1 Endogenous External Discounting

We consider a formulation of endogenous discounting where households do not internalize the fact that their discount factor depends on their own levels of consumption.¹⁰ We assume that the discount factor depends on the average level of consumption per capita, \tilde{c}_t , which individual households take as given:

$$\beta_{t+1} = \beta(\tilde{c}_t)\beta_t$$

with $\beta_0 = 1$. As usual, we assume $\partial\beta(\tilde{c}_t)/\partial\tilde{c}_t < 0$, *i.e.* agents are more impatient when aggregate consumption increases. The foundations of this specification relies both on “jealousy” or “catching up with the Joneses” effect, as the individual household is more impatient and wants to consume more today when the aggregate (or reference social group) does. Here we denote ω the elasticity of the discount factor with respect to consumption. We also assume $\beta(\bar{c}) = \beta$, which implies that the long-run multiplier is not affected by this modification in discounting. This economy nests our benchmark economy when $\omega = 0$ and we label $\mathcal{P}_\omega = \{\beta, \delta, \bar{g}/\bar{y}, \omega\}$ the relevant vector of structural parameters. We consider again linear utility in leisure ($\varphi = 0$).

The only condition modified is the Euler equation on consumption

$$\frac{1}{c_t} = \beta(\tilde{c}_t)\mathbf{E}_t \left(1 - \delta + \theta \frac{y_{t+1}}{k_{t+1}} \right) \frac{1}{c_{t+1}}.$$

¹⁰See Schmitt-Grohé and Uribe (2003) for an application in a small open economy setup.

In equilibrium, individual and average per capita variables are identical, $\tilde{c}_t = c_t$, and this equation rewrites

$$\frac{1}{c_t} = \beta(c_t)E_t \left(1 - \delta + \theta \frac{y_{t+1}}{k_{t+1}} \right) \frac{1}{c_{t+1}}$$

As before, the short-run multipliers are obtained by solving the log-linear approximations about the non-stochastic steady state of the FOCs and equilibrium conditions. Although endogenous discounting does not alter the multiplier with constant capital, this mechanism increases the short-run response of the economy to a government spending shock for a given persistence level.

Proposition 9 *With endogenous external discounting and incomplete depreciation,*

1. *The constant capital government spending multipliers on output and consumption are identical to the benchmark model, i.e.*

$$\begin{cases} 0 < \left. \frac{\Delta y}{\Delta g} \right|_{\bar{k}} = \frac{1}{1 + \frac{\theta}{(1-\theta)s_c}} = \frac{1}{1 + \frac{\theta}{(1-\theta)} \left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)} - \frac{\bar{g}}{\bar{y}} \right]} = \mathcal{K}(\mathcal{P}_\omega) < 1 \\ \left. \frac{\Delta c}{\Delta g} \right|_{\bar{k}} = \mathcal{K}(\mathcal{P}_\omega) - 1 < 0 \end{cases}$$

The constant capital output multiplier is positive but smaller than unity while the constant capital consumption multiplier is negative.

2. *The impact multipliers on investment, output and consumption are given by*

$$\begin{cases} \frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \mathcal{C}(\mathcal{P}_\omega)}{\mathcal{U}(\mathcal{P}_\omega) - \rho} \begin{matrix} \leq \\ > \end{matrix} 0 \\ \frac{\Delta y_0}{\Delta g_0} = \mathcal{K}(\mathcal{P}_\omega) \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\omega)}{\mathcal{U}(\mathcal{P}_\omega) - \rho} \right] > 0 \\ \frac{\Delta c_0}{\Delta g_0} = [\mathcal{K}(\mathcal{P}_\omega) - 1] \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\omega)}{\mathcal{U}(\mathcal{P}_\omega) - \rho} \right] < 0 \end{cases}$$

The threshold value of persistence $\mathcal{C}(\mathcal{P}_\omega) = \frac{1-\omega}{1 + \frac{1-\theta}{\theta}[1-\beta(1-\delta)]} < 1$ is a decreasing function of ω , the elasticity of the discount factor with respect to aggregate consumption. The unstable root $\mathcal{U}(\mathcal{P}_\omega) = \frac{1 + \frac{1-\theta}{\theta}[1-\beta(1-\delta)]}{\beta} > 1$ is invariant to ω .

3. *The long-run government spending multipliers (following permanent shocks) are identical to the benchmark model, i.e.*

$$\frac{\Delta y_\infty}{\Delta g_\infty} = \frac{1}{1 - s_x}, \quad \frac{\Delta c_\infty}{\Delta g_\infty} = 0, \quad \frac{\Delta x_\infty}{\Delta g_\infty} = \frac{s_x}{1 - s_x}$$

where $s_x = \frac{\beta\delta\theta}{1-\beta(1-\delta)}$ is the steady-state share of investment. The steady-state output multiplier always exceeds the constant-capital one.

Proof: See Appendix F.

The multiplier with constant capital is the same as in the benchmark model, $\mathcal{K}(\mathcal{P}) = \frac{1}{1 + \frac{\theta}{(1-\theta)} s_c} = \frac{1}{1 + \frac{\theta}{(1-\theta)} \left[1 - \frac{\beta \delta \theta}{1 - \beta(1-\delta)} - \frac{\bar{g}}{\bar{y}} \right]}$, because the discount factor does not modify the intra-temporal allocation.

The impact output multiplier is affected by endogenous discounting through the second term, the impact response of investment. Endogenous discounting modifies $\mathcal{C}(\mathcal{P}_\omega)$, the persistence of government spending for which investment is not affected by the government spending shock, in a simple way: $\mathcal{C}(\mathcal{P}_\omega) = (1 - \omega) \mathcal{C}(\mathcal{P})$. This change in dynamics is easy to understand. After a positive shock on public spending, individual households reduce their consumption due to the negative wealth effect. In equilibrium, all households take the same decisions, so average per capita consumption \tilde{c}_t is reduced. This makes agents more patient since the discount factor is a decreasing function of the aggregate consumption. Households have an additional incentive to save and are more willing to increase their capital stock, which amplifies the investment channel.

Since $\mathcal{C}(\mathcal{P}_\omega) < \mathcal{C}(\mathcal{P})$, investment would increase for a wider range of ρ when discounting is endogenous. The unstable root $\mathcal{U}(\mathcal{P}_\omega)$, contrarily to the stable root, is left unchanged. Therefore, the more elastic the discount factor to aggregate consumption, the larger is the impact response of investment.

4.2 Hand-to-Mouth Consumers

The last environment we consider deviates from Ricardian equivalence through a fraction of hand-to-mouth consumers. In addition, labor markets are imperfectly competitive.

The fraction of non-savers is given by $\lambda \in [0, 1]$, so aggregate consumption is defined as

$$c_t = \lambda c_t^{ns} + (1 - \lambda) c_t^s$$

where c_t^s and c_t^{ns} respectively denote consumption of savers and non-savers.

Non-savers do not have access to any store of value. They consume each period their entire disposable income, which corresponds to the labor income net of taxation, plus some government transfers (see Galí et al. (2007)). For presentation clarity, we only consider the labor income into their budget constraint: $c_t^{ns} = w_t n_t$.

Both savers and non-savers maximize their (intertemporal) utility function with respect to their own consumption and labor supply subject to their respective budget constraint. We depart from Galí et al. (2007) and assume that unions will set the real wage putting no weight on non-savers and

accounting only for the marginal rate of substitution between consumption and leisure of savers.¹¹

We acknowledge that we consider here a very special case in which non-savers work the same number of hours as savers (determined by the level of labor demand from the representative firm) and the real wage w_t is determined by savers (from unions that value only the consumption/leisure decisions of that group). If we consider the real wage specification usually adopted in the literature¹², we can not obtain in our simple setup an increase of total consumption after a positive government spending shock (see Furlanetto (2011) for a discussion on the interplay between limited asset market participation and nominal rigidities). This economy boils down to the benchmark we have studied in Section 2 when the fraction of hand-to-mouth consumers is zero¹³, *i.e.* $\lambda = 0$. As usual, the relevant parameter vector is noted \mathcal{P}_λ and contains $\beta, \delta, \theta, \bar{g}/\bar{y}$ and λ .

The first order and equilibrium conditions of this economy are given by:

$$\begin{aligned} \frac{1}{c_t^s} &= \beta \mathbb{E}_t \left[\left(1 - \delta + \theta \frac{y_{t+1}}{k_{t+1}} \right) \frac{1}{c_{t+1}^s} \right] \\ c_t^{ns} &= (1 - \theta) y_t \\ \eta &= (1 - \theta) \frac{y_t}{n_t} \frac{1}{c_t^s} \\ k_{t+1} &= (1 - \delta) k_t + x_t \\ y_t &= A k_t^\theta n_t^{1-\theta} \\ y_t &= c_t + x_t + g_t \\ c_t &= \lambda c_t^{ns} + (1 - \lambda) c_t^s \\ \log g_t &= \rho \log g_{t-1} + (1 - \rho) \log \bar{g} + \varepsilon_t \end{aligned}$$

Next proposition characterizes multipliers when a fraction λ of agents is constrained to consume their current labor income.

Proposition 10 *With a fraction λ of hand-to-mouth consumers and incomplete depreciation,*

1. *The constant capital government spending multipliers on output and consumption equal*

$$\begin{cases} 0 < \frac{\Delta y}{\Delta g} \Big|_{\bar{k}} = \frac{1-\theta}{(1-\lambda)\theta s_c + (1-\theta)(1-s_c\lambda)} = \mathcal{K}(\mathcal{P}_\lambda) \stackrel{\leq}{\geq} 1 \\ \frac{\Delta c}{\Delta g} \Big|_{\bar{k}} = \mathcal{K}(\mathcal{P}_\lambda) - 1 = \frac{\lambda-\theta}{(1-\lambda)\theta s_c + (1-\theta)(1-s_c\lambda)} s_c \stackrel{\leq}{\geq} 0 \end{cases}$$

¹¹See Appendix A in Galí et al. (2007) for a complete characterization of the wage setting rule.

¹²In Galí et al. (2007), the real wage equation is obtained under the assumption that unions weight savers and non-saver according to the fraction of these two groups in the economy. If we adopt this setup, we obtain a real wage equation $\eta c_t = (1 - \theta)(y_t/n_t)$, which differs from our equation given by $\eta c_t^s = (1 - \theta)(y_t/n_t)$, where c_t^s denotes the consumption of savers. See below the first order and equilibrium conditions of this economy.

¹³For simplicity, we keep the same notations as before, thus ignoring (constant) markups resulting from an imperfectly competitive labor market.

where $s_c = \frac{\bar{c}}{\bar{y}} = 1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)} - \frac{\bar{g}}{\bar{y}}$ is the steady-state share of consumption. The constant-capital consumption and output multipliers are increasing functions of the share λ of hand-to-mouth consumers.

2. The impact multipliers on investment, output and consumption are given by

$$\begin{cases} \frac{\Delta x_0}{\Delta g_0} = \frac{\rho - \mathcal{C}(\mathcal{P}_\lambda)}{\mathcal{U}(\mathcal{P}_\lambda) - \rho} \begin{matrix} \leq 0 \\ > 0 \end{matrix} \\ \frac{\Delta y_0}{\Delta g_0} = \mathcal{K}(\mathcal{P}_\lambda) \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\lambda)}{\mathcal{U}(\mathcal{P}_\lambda) - \rho} \right] \begin{matrix} \leq 1 \\ > 1 \end{matrix} \\ \frac{\Delta c_0}{\Delta g_0} = [\mathcal{K}(\mathcal{P}_\lambda) - 1] \times \left[1 + \frac{\rho - \mathcal{C}(\mathcal{P}_\lambda)}{\mathcal{U}(\mathcal{P}_\lambda) - \rho} \right] \begin{matrix} \leq 0 \\ > 0 \end{matrix} \end{cases}$$

The threshold value of persistence $\mathcal{C}(\mathcal{P}_\lambda) = \frac{1}{1 + \frac{1-\theta}{\theta}[1-\beta(1-\delta)]} < 1$ is invariant to the share λ of non-savers in the population. The unstable root $\mathcal{U}(\mathcal{P}_\lambda) = 1 - \delta + [1 - s_c\lambda]\bar{y}/\bar{k} = 1 - \delta + [1 - s_c\lambda]\frac{1/\beta - 1 + \delta}{\theta} > 1$ decreases with the share of hand-to-mouth consumers.

3. The long-run government spending multipliers (following permanent shocks) equal

$$\frac{\Delta y_\infty}{\Delta g_\infty} = \frac{1}{1 - s_x - \lambda s_c}, \quad \frac{\Delta c_\infty}{\Delta g_\infty} = \frac{\lambda s_c}{1 - s_x - \lambda s_c}, \quad \frac{\Delta x_\infty}{\Delta g_\infty} = \frac{s_x}{1 - s_x - \lambda s_c}$$

where $s_x = \frac{\beta\delta\theta}{1-\beta(1-\delta)}$ is the steady-state share of investment. The steady-state output multiplier always exceeds the constant-capital one.

Proof: See Appendix G.

The share of non-savers λ does not modify the threshold persistence level $\mathcal{C}(\mathcal{P}_\lambda)$, but nevertheless affects the impact response of investment. The proportion of savers shows up in the denominator $\mathcal{U}(\mathcal{P}_\lambda) - \rho$, which decreases with λ . At the limit, the unstable root is close to (but above) unity when $\lambda \rightarrow 1$.

When the persistence of government spending precisely equals the threshold $\mathcal{C}(\mathcal{P}_\lambda)$, the impact investment multiplier is zero regardless of the proportions of savers and non-savers. Remember that this threshold is determined by the adjustment speed of savings, which is not affected by hand-to-mouth agents. For other persistence levels, on the contrary, the share of hand-to-mouth consumers will magnify the response of savings to the government spending shock. The presence of agents who do not save adds inertia to the dynamics of aggregate consumption, making savers invest relatively more in the face of highly persistent government spending when the fraction of hand-to-mouth agents is larger.

The effect of hand-to-mouth consumers on impact output multiplier combines as usual the constant-capital effect and the impact response of investment. The model can yield very large output multiplier when the fraction of savers is arbitrary small.

The case of consumption multiplier is of particular interest. After a positive shock to government spending shock, rule-of-thumb consumers always consume more, contrarily to what savers do – and in fact, precisely because savers supply more labor which raises aggregate output. In the constant-capital case, the response of aggregate consumption is positive as soon as λ exceeds the capital share θ .

When the elasticity of labor supply is finite, the share of hand-to-mouth above which aggregate consumption increases in response to a government spending shock becomes $\frac{\theta+\varphi}{1+\varphi}$. This threshold decreases with the elasticity of labor supply $1/\varphi$ and tends to one in the limit case of inelastic labor supply.

This environment is helpful to disentangle the role of consumption and investment in the output multiplier. Our analysis has shown that the sign of the investment response is pinned down by the persistence parameter ρ . The sign of the aggregate consumption response depends on the share of hand-to-mouth consumers λ , whose consumption increase after an increase in government spending as opposed to savers who reduce their consumption plans. To isolate the relative contributions of consumption and investment, we consider three iso-multipliers. These iso-multipliers are the loci of (λ, ρ) for which the dynamic multipliers (on impact) for output, consumption and investment respectively satisfy

$$\frac{\Delta y_0}{\Delta g_0} = 1 \quad , \quad \frac{\Delta c_0}{\Delta g_0} = 0 \quad , \quad \frac{\Delta x_0}{\Delta g_0} = 0$$

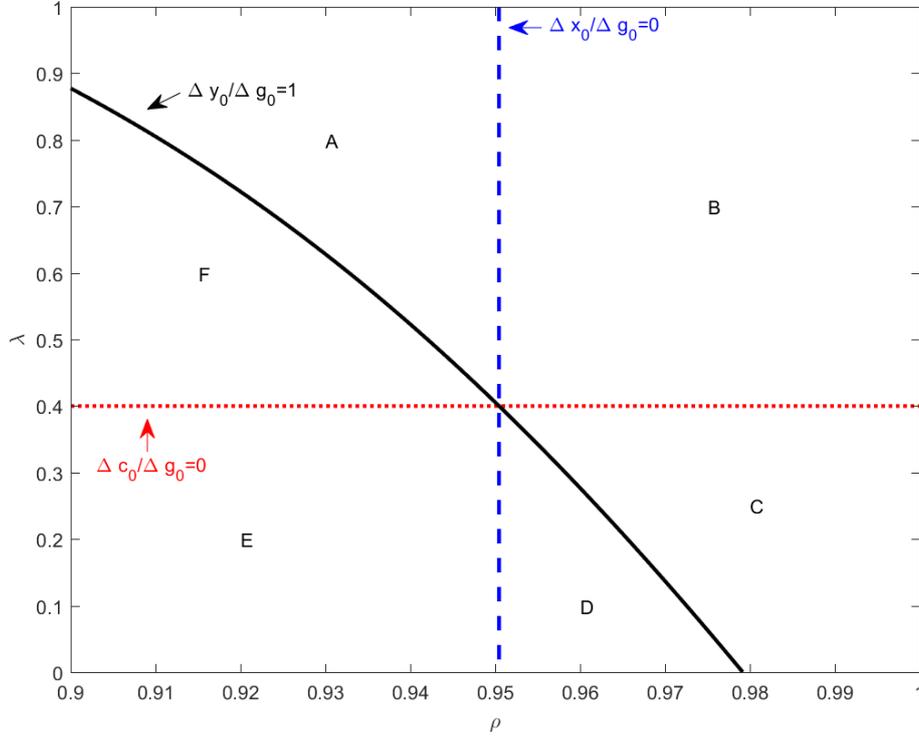
The consumption and investment loci are easy to determine given Proposition 10. $\mathcal{C}(\mathcal{P}_\lambda)$, and therefore the sign of the investment multiplier, is independent from λ . The $\frac{\Delta x_0}{\Delta g_0} = 0$ locus is $(\mathcal{C}(\mathcal{P}_\lambda), \lambda)$. On the contrary, the consumption multiplier is null iff the constant-capital multiplier $\mathcal{K}(\mathcal{P}_\lambda) = 1$. As previously stated, this condition holds for a unique value of λ , irrespective of the persistence of government spending ρ . The $\frac{\Delta c_0}{\Delta g_0} = 0$ locus is (ρ, θ) . In Figure 7, these two loci respectively show up as a vertical dashed blue line and an horizontal dotted red line.¹⁴

The black solid line displays all (λ, ρ) pairs such that the impact output multiplier equals unity. The impact multiplier depends on both parameters, because the share of non-savers λ affects the constant-capital multiplier and the persistence of government spending ρ shapes the impact response

¹⁴These results are established assuming an infinite labor supply elasticity. When labor supply becomes less elastic, both thresholds associated with consumption and investment iso-multipliers increase.

of investment. On the North-East of this locus, the short-run output multiplier exceeds unity.

Figure 7: A partition of multipliers



- A:** $\Delta c_0 / \Delta g_0 > 0$, $\Delta x_0 / \Delta g_0 < 0$ and $\Delta y_0 / \Delta g_0 > 1$.
- B:** $\Delta c_0 / \Delta g_0 > 0$, $\Delta x_0 / \Delta g_0 > 0$ and $\Delta y_0 / \Delta g_0 > 1$.
- C:** $\Delta c_0 / \Delta g_0 < 0$, $\Delta x_0 / \Delta g_0 > 0$ and $\Delta y_0 / \Delta g_0 > 1$.
- D:** $\Delta c_0 / \Delta g_0 < 0$, $\Delta x_0 / \Delta g_0 > 0$ and $\Delta y_0 / \Delta g_0 < 1$.
- E:** $\Delta c_0 / \Delta g_0 < 0$, $\Delta x_0 / \Delta g_0 > 0$ and $\Delta y_0 / \Delta g_0 < 1$.
- F:** $\Delta c_0 / \Delta g_0 > 0$, $\Delta x_0 / \Delta g_0 < 0$ and $\Delta y_0 / \Delta g_0 < 1$.

Note: The parameters chosen for these illustrations are $\beta = .99$, $\delta = .025$, $\theta = .4$ and $\frac{\bar{g}}{\bar{y}} = .2$.

Two areas are interesting. Area C, on the lower right part of this Figure, larger than unity output multipliers are obtained through increases in investment despite negative consumption multipliers. On the contrary, zone F on the upper left part features an increase in consumption. Yet, the output multiplier is below unity due to the negative response of investment triggered by a low persistence of government spending. Our analysis shows that a positive consumption multiplier is neither necessary nor sufficient to achieve an output multiplier above unity.

5 Conclusion

In this paper, we provide new analytics of government spending multipliers. We notably show that the investment channel matters a lot for the effectiveness of the output multiplier. Depending on the persistence of government spending, this channel dampens or amplifies the output multiplier obtained in a static setup. We also examine these multipliers in various dimensions: capital adjustment costs, intertemporal elasticity of consumption, Frisch elasticity of labor supply, external endogenous discounting and imperfect financial and labor markets under the form of hand-to-mouth consumers. In all these environments, we inspect the mechanisms at work and show how they can modify the multipliers.

In our framework, we deliberately abstracted from other relevant features in order to highlight, as transparently as possible, the main mechanisms at work. The existing literature insists on other modeling issues that might potentially enrich our results. We mention three of them. First, we assumed away any description of distortionary taxes. Leeper et al. (2010) considers capital and labor income taxes. Our sole emphasis is on the profile of government spending. Second, we consider that government spending enters the utility function in an additive way. The literature has already proposed models wherein government spending affects the marginal utility of consumption (see e.g. Aschauer (1985), Bailey (1971), Barro (1981), Christiano and Eichenbaum (1992)). With this specification, both intra-temporal and inter-temporal allocations are modified and thus the resulting multipliers. Third, we assume for simplicity an autoregressive process of order one for government spending. More realistic processes, say an order two process as in Uhlig (2010), may first better approximate the time profile of recent recovery plans (in US and Euro Area) and offer new perspectives for the analysis of multipliers.

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Appendix

A Static economy: Proof of Proposition 1

The equilibrium of this economy is summarized by the following static equations, where we omit the time index for simplification

$$y = c + \bar{x} + g \quad (\text{A.1})$$

$$y = A\bar{k}^\theta n^{1-\theta} \quad (\text{A.2})$$

$$\eta = \frac{1}{c}(1-\theta)\frac{y}{n} \quad (\text{A.3})$$

Equation (A.1) defines the resource constraint on the good market with constant investment. Equations (A.2) and (A.3) are the production function and the marginal rate of substitution between consumption and leisure at equilibrium. The Euler equation is excluded in that restricted setup, because agents do not have access to a store of value. We could include state contingent claims without modifying the results. Differentiating equation (A.1), (A.2) and (A.3) with respect to g yields

$$\frac{dy}{dg} = \frac{dc}{dg} + 1 \quad (\text{A.4})$$

$$\frac{dy}{dg} = (1-\theta)\frac{y}{n}\frac{dn}{dg} \quad (\text{A.5})$$

$$\frac{dc}{dg} = \frac{c}{y}\frac{dy}{dg} - \frac{c}{n}\frac{dn}{dg} \quad (\text{A.6})$$

Plugging equation (A.5) into (A.6), one deduces

$$\frac{dc}{dg} = -\frac{\theta}{1-\theta}\frac{c}{y}\frac{dy}{dg}$$

Using the above equation and (A.4), we get

$$\frac{dy}{dg} = \frac{1}{1 + \frac{\theta}{1-\theta}s_c} \text{ and } \frac{dc}{dg} = -\frac{\frac{\theta}{1-\theta}s_c}{1 + \frac{\theta}{1-\theta}s_c}$$

where $s_c = \bar{c}/\bar{y} \equiv 1 - \beta\theta - \bar{g}/\bar{y}$. This completes the proof.

B Stochastic processes of endogenous variables in the dynamic economy: Proof of Proposition 2

The log-linearization about the non-stochastic steady state yields

$$\widehat{k}_{t+1} = \frac{1}{\beta\theta}\widehat{y}_t - \frac{s_c}{\beta\theta}\widehat{c}_t - \frac{s_g}{\beta\theta}\widehat{g}_t \quad (\text{B.1})$$

$$\widehat{y}_t = \theta\widehat{k}_t + (1-\theta)\widehat{n}_t \quad (\text{B.2})$$

$$\widehat{n}_t = \widehat{y}_t - \widehat{c}_t \quad (\text{B.3})$$

$$\text{E}_t\widehat{c}_{t+1} = \widehat{c}_t + \text{E}_t(\widehat{y}_{t+1} - \widehat{k}_{t+1}) \quad (\text{B.4})$$

$$\widehat{g}_t = \rho\widehat{g}_{t-1} + \varepsilon_t \quad (\text{B.5})$$

where s_c and s_g are defined as in appendix A. After substitution of (B.3) into (B.2), one gets

$$\widehat{y}_t - \widehat{k}_t = -\frac{1-\theta}{\theta}\widehat{c}_t \quad (\text{B.6})$$

Using (B.6), (B.4) becomes

$$\text{E}_t\widehat{c}_{t+1} = \theta\widehat{c}_t, \quad (\text{B.7})$$

and (B.1) rewrites

$$\widehat{k}_{t+1} = \nu_1\widehat{k}_t - \nu_2\widehat{c}_t - \nu_3\widehat{g}_t, \quad (\text{B.8})$$

with $\nu_1 = (\beta\theta)^{-1}$, $\nu_2 = \frac{1-\theta+s_c}{\beta\theta} > 0$ and $\nu_3 = \frac{s_g}{\beta\theta} > 0$. Because $0 < \beta < 1$ and $0 < \theta < 1$, $\nu_1 > 1$ and equation (B.8) must be solved forward

$$\widehat{k}_t = \left(\frac{\nu_2}{\nu_1}\right) \lim_{T \rightarrow \infty} \text{E}_t \sum_{i=0}^T \left(\frac{1}{\nu_1}\right)^i \widehat{c}_{t+i} + \left(\frac{\nu_3}{\nu_1}\right) \lim_{T \rightarrow \infty} \text{E}_t \sum_{i=0}^T \left(\frac{1}{\nu_1}\right)^i \widehat{g}_{t+i} + \lim_{T \rightarrow \infty} \text{E}_t \left(\frac{1}{\nu_1}\right)^T \widehat{k}_{t+T}.$$

Excluding explosive paths, *i.e.* $\lim_{T \rightarrow \infty} \text{E}_t (1/\nu_1)^T \widehat{k}_{t+T} = 0$ and taking the limit, we obtain

$$\widehat{k}_t = \left(\frac{\nu_2}{\nu_1}\right) \sum_{i=0}^{\infty} \left(\frac{1}{\nu_1}\right)^i \text{E}_t \widehat{c}_{t+i} + \left(\frac{\nu_3}{\nu_1}\right) \sum_{i=0}^{\infty} \left(\frac{1}{\nu_1}\right)^i \text{E}_t \widehat{g}_{t+i}. \quad (\text{B.9})$$

Future expected values of consumption and government spending are computed according to (C.6) and (B.7), yielding

$$\widehat{k}_t = \frac{\nu_2}{\nu_1 - \theta}\widehat{c}_t + \frac{\nu_3}{\nu_1 - \rho}\widehat{g}_t,$$

from which we deduce the decision rule on consumption:

$$\widehat{c}_t = \frac{\nu_1 - \theta}{\nu_2}\widehat{k}_t - \frac{\nu_3}{\nu_2} \cdot \frac{\nu_1 - \theta}{\nu_1 - \rho}\widehat{g}_t. \quad (\text{B.10})$$

After substituting (B.10) into (B.8), the dynamics of capital is given by:

$$\widehat{k}_{t+1} = \theta \widehat{k}_t + \nu_3 \left(\frac{\rho - \theta}{\nu_1 - \rho} \right) \widehat{g}_t = \theta \widehat{k}_t + s_g \left(\frac{\rho - \theta}{1 - \beta\theta\rho} \right) \widehat{g}_t \quad (\text{B.11})$$

given the values ν_1 and ν_3 . Using $\widehat{k}_{t+1} = \widehat{x}_t$ and (C.6), (B.11) displays a second-order autoregressive process for investment:

$$\widehat{x}_t = (\theta + \rho)\widehat{x}_{t-1} - \theta\rho\widehat{x}_{t-2} + s_g \left(\frac{\rho - \theta}{1 - \beta\theta\rho} \right) \varepsilon_t. \quad (\text{B.12})$$

Combining (B.10) and (B.11), we obtain a first-order autoregressive process for consumption:

$$\widehat{c}_t = \theta \widehat{c}_{t-1} - \frac{\theta s_g}{1 - \theta + \theta s_c} \left(\frac{1 - \beta\theta^2}{1 - \beta\theta\rho} \right) \varepsilon_t \quad (\text{B.13})$$

Finally, the stochastic process of output is a linear combination of the processes respectively defined in equations (C.6), (B.13) and (B.12), according to

$$\widehat{y}_t = s_c \widehat{c}_t + s_x \widehat{x}_t + s_g \widehat{g}_t.$$

Short-run multipliers are obtained using the expressions

$$\frac{\Delta y_h}{\Delta g_0} = \frac{1}{s_g} \frac{\partial \widehat{y}_{t+h}}{\partial \varepsilon_t}, \quad \frac{\Delta c_h}{\Delta g_0} = \frac{s_c}{s_g} \frac{\partial \widehat{c}_{t+h}}{\partial \varepsilon_t}, \quad \frac{\Delta x_h}{\Delta g_0} = \frac{s_x}{s_g} \frac{\partial \widehat{x}_{t+h}}{\partial \varepsilon_t} \quad \text{for } h = 0, 1, 2, \dots$$

In particular, we determine impact multipliers for $h = 0$

$$\frac{\Delta y_0}{\Delta g_0} = \frac{1}{1 + \frac{\theta}{1-\theta} s_c} \left(\frac{1 - \beta\theta^2}{1 - \beta\theta\rho} \right) \quad (\text{B.14})$$

$$\frac{\Delta c_0}{\Delta g_0} = -\frac{\frac{\theta}{1-\theta} s_c}{1 + \frac{\theta}{1-\theta} s_c} \left(\frac{1 - \beta\theta^2}{1 - \beta\theta\rho} \right) \quad (\text{B.15})$$

$$\frac{\Delta x_0}{\Delta g_0} = \beta\theta \left(\frac{\rho - \theta}{1 - \beta\theta\rho} \right) \quad (\text{B.16})$$

This completes the proof.

C Capital Adjustment Costs: Proof of Proposition 6

The log-linearization about the non-stochastic steady state yields

$$\widehat{k}_{t+1} = \frac{1}{\beta\theta}\widehat{y}_t - \frac{s_c}{\beta\theta}\widehat{c}_t - \frac{s_g}{\beta\theta}\widehat{g}_t \quad (\text{C.1})$$

$$\widehat{y}_t = \theta\widehat{k}_t + (1-\theta)\widehat{n}_t \quad (\text{C.2})$$

$$\widehat{n}_t = \widehat{y}_t - \widehat{c}_t \quad (\text{C.3})$$

$$E_t\widehat{c}_{t+1} = \widehat{c}_t - \widehat{q}_t + E_t(\widehat{y}_{t+1} - \widehat{k}_{t+1}) + \beta E_t\widehat{q}_{t+1} \quad (\text{C.4})$$

$$\widehat{q}_t = \phi(\widehat{k}_{t+1} - \widehat{k}_t) \quad (\text{C.5})$$

$$\widehat{g}_t = \rho\widehat{g}_{t-1} + \varepsilon_t \quad (\text{C.6})$$

where s_c and s_g are defined as in Appendix A. After substitution of (C.3) into (C.2), one gets

$$\widehat{y}_t - \widehat{k}_t = -\frac{1-\theta}{\theta}\widehat{c}_t \quad (\text{C.7})$$

Using (C.7), (C.4) becomes

$$E_t\widehat{c}_{t+1} = \theta\widehat{c}_t - \theta\widehat{q}_t + \beta\theta E_t\widehat{q}_{t+1}, \quad (\text{C.8})$$

and (C.1) rewrites

$$\widehat{k}_{t+1} = \nu_1\widehat{k}_t - \nu_2\widehat{c}_t - \nu_3\widehat{g}_t, \quad (\text{C.9})$$

with $\nu_1 = (\beta\theta)^{-1}$, $\nu_2 = \frac{1-\theta+s_c}{\beta\theta} > 0$ and $\nu_3 = \frac{s_g}{\beta\theta} > 0$. Finally, using (C.9), equation (C.4) is now given by

$$E_t\widehat{c}_{t+1} = \theta\widehat{c}_t + \beta\theta\phi E_t\widehat{k}_{t+2} - \theta\phi(1+\beta)\widehat{k}_{t+1} + \theta\phi\widehat{k}_t \quad (\text{C.10})$$

Compared to Appendix B, the Euler equation is no longer autonomous in consumption. To solve the model, we use the method of undetermined coefficients (see Campbell (1994) for a similar approach). We guess and verify the two following linear equations for the (logs of) capital and private consumption

$$\widehat{k}_{t+1} = \eta_{kk}\widehat{k}_t + \eta_{kg}\widehat{g}_t \quad (\text{C.11})$$

$$\widehat{c}_t = \eta_{ck}\widehat{k}_t + \eta_{cg}\widehat{g}_t \quad (\text{C.12})$$

where the unknown coefficients $\{\eta_{kk}, \eta_{kg}, \eta_{ck}, \eta_{cg}\}$ can be identified using (C.9), (C.10) and (C.6). As usual with this method, we use the restrictions provided by the conditions (C.9)–(C.10) and the process of government spending (C.6) to identify the unknown policy rule coefficients $\{\eta_{kk}, \eta_{kg}, \eta_{ck}, \eta_{cg}\}$.

After replacement of (C.11) and (C.12) into (C.9) and (C.12) and using (C.6), it comes

$$(\eta_{ck} - \beta\theta\phi\eta_{kk})\eta_{kk} = \theta\eta_{ck} - \theta\phi(1 + \beta)\eta_{kk} + \theta\phi \quad (\text{C.13})$$

$$(\eta_{ck} - \beta\theta\phi\eta_{kk})\eta_{kg} = \rho(\eta_{cg} - \beta\theta\phi\eta_{kg}) + \theta\eta_{cg} - \theta\phi(1 + \beta)\eta_{kg} \quad (\text{C.14})$$

$$\eta_{kk} = \nu_1 - \nu_2\eta_{ck} \quad (\text{C.15})$$

$$\eta_{kg} = -\nu_2\eta_{cg} - \nu_3 \quad (\text{C.16})$$

From these four equations (C.13)–(C.16), we can now identify the four unknown coefficients $\{\eta_{kk}, \eta_{kg}, \eta_{ck}, \eta_{cg}\}$.

First, combine (C.13) and (C.15). This yields after some manipulations

$$(1 + \beta\theta\phi\nu_2)\eta_{kk}^2 - [\theta + \nu_1 + \theta\phi(1 + \beta)\nu_2]\eta_{kk} + (\theta\nu_1 + \theta\phi\nu_2) = 0$$

or equivalently

$$\eta_{kk}^2 - \frac{(\theta + \nu_1 + \theta\phi(1 + \beta)\nu_2)}{1 + \beta\theta\phi\nu_2}\eta_{kk} + \frac{\theta\nu_1 + \theta\phi\nu_2}{1 + \beta\theta\phi\nu_2} = 0$$

Factorizing this equation yields

$$P(\eta_{kk}) = (\eta_{kk} - \zeta_1)(\eta_{kk} - \zeta_2),$$

where ζ_1 and ζ_2 are the two roots associated to this polynomial. We immediately deduce

$$P(0) = \zeta_1\zeta_2 \equiv \frac{\theta\nu_1 + \theta\phi\nu_2}{1 + \beta\theta\phi\nu_2} \equiv \frac{1}{\beta} > 1$$

so, there exists at least one root for η_{kk} that exceeds unity. Moreover, we obtain

$$P(1) = (1 - \zeta_1)(1 - \zeta_2) \equiv \frac{\theta(1 + \beta) - 1 - \beta\theta^2}{\beta\theta} < 0$$

for $\beta \in]0, 1[$ and $\theta \in]0, 1[$. So there exists a root smaller than one. When $\phi = 0$, this stable root is given by $\eta_{kk} = \theta$ and when $\phi \rightarrow \infty$, the parameter $\eta_{kk} \rightarrow 1$.

We now determine η_{kg} . Using (C.15) and (C.16), we deduce

$$\eta_{cg} = -\frac{\nu_3 + \eta_{kg}}{\nu_2} \quad \text{and} \quad \eta_{ck} = \frac{\nu_1 - \eta_{kk}}{\nu_2}$$

After replacement into (C.14), we obtain

$$\left(\frac{\nu_1 - \eta_{kk}}{\nu_2} - \beta\theta\phi\eta_{kk}\right)\eta_{kg} + \rho\left(-\frac{\nu_3 + \eta_{kg}}{\nu_2} - \beta\theta\phi\eta_{kg}\right) = -\theta\frac{\nu_3 + \eta_{kg}}{\nu_2} - (1 + \beta)\theta\phi\eta_{kg}$$

Factorizing η_{kg} yields

$$\left\{\frac{\nu_1 - \eta_{kk} + \theta - \rho}{\nu_2} - \beta\theta\phi\eta_{kk} + \theta\phi[1 + \beta(1 - \rho)]\right\}\eta_{kg} = \frac{\nu_3}{\nu_2}(\rho - \theta),$$

or equivalently

$$\{\nu_1 - \eta_{kk} + \theta - \rho + \nu_2 \theta \phi [1 + \beta(1 - \rho - \eta_{kk})]\} \eta_{kg} = \nu_3 (\rho - \theta),$$

and we obtain η_{kg}

$$\eta_{kg} = \nu_3 \frac{\rho - \theta}{(\nu_1 - \eta_{kk} + \theta - \rho) + \nu_2 \theta \phi [1 + \beta(1 - \rho - \eta_{kk})]}.$$

The impact investment multiplier is obtained dividing η_{kg} by ν_3 . The denominator is always positive $\forall \beta, \theta, \rho$. When the capital adjustment cost parameter ϕ increases, the denominator increases and the size of the investment multiplier decreases. When $\phi \rightarrow \infty$, $\eta_{kg} \rightarrow 0$.

This completes the proof.

D Non-unit intertemporal elasticity of substitution in consumption: Proof of Proposition 7

In the case of the utility function (18), the log-linear approximations of first-order and equilibrium conditions rewrite:

$$\widehat{k}_{t+1} = (1 - \delta)\widehat{k}_t + \frac{\bar{y}}{\bar{k}}\widehat{y}_t - s_c \frac{\bar{y}}{\bar{k}}\widehat{c}_t - s_g \frac{\bar{y}}{\bar{k}}\widehat{g}_t \quad (\text{D.1})$$

$$\widehat{y}_t = \theta\widehat{k}_t + (1 - \theta)\widehat{n}_t \quad (\text{D.2})$$

$$\widehat{n}_t = \widehat{y}_t - \sigma\widehat{c}_t \quad (\text{D.3})$$

$$\text{E}_t \widehat{c}_{t+1} = \widehat{c}_t + \frac{\beta\theta}{\sigma} \frac{\bar{y}}{\bar{k}} \text{E}_t (\widehat{y}_{t+1} - \widehat{k}_{t+1}) \quad (\text{D.4})$$

$$\widehat{g}_t = \rho\widehat{g}_{t-1} + \varepsilon_t \quad (\text{D.5})$$

where $\bar{y}/\bar{k} = (1 - \beta(1 - \delta))/(\beta\theta)$ is the inverse of the steady state capital-output ratio, $s_g = \bar{g}/\bar{y}$ and $s_c = 1 - \delta\bar{k}/\bar{y} - s_g$ denotes the consumption to output ratio. After substitution of (D.3) into (D.2), one gets

$$\widehat{y}_t - \widehat{k}_t = -\sigma \frac{1 - \theta}{\theta} \widehat{c}_t. \quad (\text{D.6})$$

Using (D.6), (D.1) and (D.4) rewrite

$$\text{E}_t \widehat{c}_{t+1} = \mu_1 \widehat{c}_t \quad (\text{D.7})$$

$$\widehat{k}_{t+1} = \nu_1 \widehat{k}_t - \nu_2 \widehat{c}_t - \nu_3 \widehat{g}_t, \quad (\text{D.8})$$

with

$$\begin{aligned}\mu_1 &= \frac{1}{1 + \beta\theta\frac{(1-\theta)\bar{y}}{\bar{k}}} \in [0, 1] \\ \nu_1 &= 1 - \delta + \frac{\bar{y}}{\bar{k}} > 1 \\ \nu_2 &= \left(\frac{1-\theta}{\theta+\varphi} + s_c\sigma\right)\frac{\bar{y}}{\bar{k}} > 0 \\ \nu_3 &= s_g\frac{\bar{y}}{\bar{k}} > 0\end{aligned}$$

Using the same solving procedure as before (see appendix B), we can determine the short-run multipliers. This completes the proof.

E Finite labor supply elasticity: Proof of Proposition 8

In the case of the utility function (21), the log-linear approximations of first-order and equilibrium conditions rewrite:

$$\widehat{k}_{t+1} = (1 - \delta)\widehat{k}_t + \frac{\bar{y}}{\bar{k}}\widehat{y}_t - s_c\frac{\bar{y}}{\bar{k}}\widehat{c}_t - s_g\frac{\bar{y}}{\bar{k}}\widehat{g}_t \quad (\text{E.1})$$

$$\widehat{y}_t = \theta\widehat{k}_t + (1 - \theta)\widehat{n}_t \quad (\text{E.2})$$

$$\widehat{n}_t = \frac{1}{1 + \varphi}(\widehat{y}_t - \widehat{c}_t) \quad (\text{E.3})$$

$$\text{E}_t\widehat{c}_{t+1} = \widehat{c}_t + \beta\theta\frac{\bar{y}}{\bar{k}}\text{E}_t(\widehat{y}_{t+1} - \widehat{k}_{t+1}) \quad (\text{E.4})$$

$$\widehat{g}_t = \rho\widehat{g}_{t-1} + \varepsilon_t \quad (\text{E.5})$$

where $\bar{y}/\bar{k} = (1 - \beta(1 - \delta))/(\beta\theta)$ is the inverse of the steady state capital-output ratio, $s_g = \bar{g}/\bar{y}$ and $s_c = 1 - \delta\bar{k}/\bar{y} - s_g$ denotes the consumption to output ratio. After substitution of (E.3) into (E.2), one gets

$$\widehat{y}_t = \frac{\theta(1 + \varphi)}{\theta + \varphi}\widehat{k}_t - \frac{1 - \theta}{\theta + \varphi}\widehat{c}_t \quad (\text{E.6})$$

or equivalently

$$\widehat{y}_t - \widehat{k}_t = -\frac{\varphi(1 - \theta)}{\theta + \varphi}\widehat{k}_t - \frac{1 - \theta}{\theta + \varphi}\widehat{c}_t \quad (\text{E.7})$$

Using (E.7), (E.1) and (E.4) rewrite

$$\text{E}_t\widehat{c}_{t+1} = \mu_1\widehat{c}_t - \mu_2\widehat{k}_{t+1} \quad (\text{E.8})$$

$$\widehat{k}_{t+1} = \nu_1\widehat{k}_t - \nu_2\widehat{c}_t - \nu_3\widehat{g}_t, \quad (\text{E.9})$$

with

$$\begin{aligned}\mu_1 &= \frac{1}{1 + \beta\theta\frac{(1-\theta)\bar{y}}{\theta+\varphi}\bar{k}} \in [0, 1] \\ \mu_2 &= \frac{\beta\theta\varphi\frac{1-\theta}{\theta+\varphi}\frac{\bar{y}}{\bar{k}}}{1 + \beta\theta\frac{1-\theta}{\theta+\varphi}\frac{\bar{y}}{\bar{k}}} \geq 0 \\ \nu_1 &= 1 - \delta + \theta\frac{1+\varphi}{\theta+\varphi}\frac{\bar{y}}{\bar{k}} > 1 \\ \nu_2 &= \left(\frac{1-\theta}{\theta+\varphi} + s_c\right)\frac{\bar{y}}{\bar{k}} > 0 \\ \nu_3 &= s_g\frac{\bar{y}}{\bar{k}} > 0\end{aligned}$$

Compared to Appendices B and D, the dynamics of consumption is no longer autonomous when $\mu_2 > 0$ in (E.8). We use again the method of undetermined coefficients. We guess and verify the two following linear equations for the (logs of) capital and private consumption

$$\widehat{k}_{t+1} = \eta_{kk}\widehat{k}_t + \eta_{kg}\widehat{g}_t \quad (\text{E.10})$$

$$\widehat{c}_t = \eta_{ck}\widehat{k}_t + \eta_{cg}\widehat{g}_t \quad (\text{E.11})$$

where the unknown coefficients $\{\eta_{kk}, \eta_{kg}, \eta_{ck}, \eta_{cg}\}$ can be identified using (E.8), (E.9) and (E.5). After replacement of (E.10) and (E.11) into (E.8) and (E.9) and using (E.5), it comes

$$(\eta_{ck} + \mu_2)\eta_{kk} = \mu_1\eta_{ck} \quad (\text{E.12})$$

$$(\eta_{ck} + \mu_2)\eta_{kg} = (\mu_1 - \rho)\eta_{cg} \quad (\text{E.13})$$

$$\eta_{kk} = \nu_1 - \nu_2\eta_{ck} \quad (\text{E.14})$$

$$\eta_{kg} = -\nu_2\eta_{cg} - \nu_3 \quad (\text{E.15})$$

From these four equations (E.12)–(E.15), we can now identify the four unknown parameters $\{\eta_{kk}, \eta_{kg}, \eta_{ck}, \eta_{cg}\}$.

First, combine (E.12) and (E.14). This yields

$$\eta_{kk}^2 - (\mu_1 + \nu_1 + \mu_2\nu_2)\eta_{kk} + \mu_1\nu_1 = 0$$

The discriminant of the characteristic polynomial is equal to $(\mu_1 + \nu_1 + \mu_2\nu_2)^2 - 4\mu_1\nu_1 \equiv (\nu_1 - \mu_1)^2 + 2\mu_2\nu_2(\mu_1 + \nu_1) + \mu_2^2\nu_2^2$ and it is positive. So, the roots are real.

Notice that when $\varphi = 0$, then $\mu_2 = 0$ and the stable root of the characteristic polynomial is $\eta_{kk} = \mu_1 < 1$ (the unstable root is $\nu_1 > 1$ and $\mu_1\nu_1 = 1/\beta$). When $\varphi > 0$, then $\mu_2 > 0$ and the stable

root is given by

$$\eta_{kk} = \frac{\mu_1 + \nu_1 + \mu_2\nu_2 - \sqrt{(\mu_1\nu_1 + \mu_2\nu_2)^2 - 4\mu_1\nu_1}}{2} \in [0, 1]$$

and the unstable root is given by $1/(\beta\eta_{kk})$.

From (E.14), we get

$$\eta_{ck} = \frac{\nu_1 - \eta_{kk}}{\nu_2} > 0$$

because $\nu_1 > 1$, $\eta_{kk} < 1$ and $\nu_2 > 0$. From (E.13), (E.15) and the previous expression, we deduce

$$\eta_{cg} = -\frac{\nu_3}{\nu_2} \left(\frac{\nu_1 - \eta_{kk} + \mu_2\nu_2}{\mu_1 + \nu_1 + \mu_2\nu_2 - \eta_{kk} - \rho} \right)$$

From the stable root η_{kk} of the characteristic polynomial we deduce

$$\begin{aligned} \nu_1 + \mu_1 + \nu_2\mu_2 - \eta_{kk} - \rho &= \frac{\mu_1 + \nu_1 + \mu_2\nu_2 + \sqrt{(\mu_1\nu_1 + \mu_2\nu_2)^2 - 4\mu_1\nu_1}}{2} - \rho \\ &= \frac{1}{\beta\eta_{kk}} - \rho \end{aligned}$$

The coefficient η_{cg} simply rewrites

$$\eta_{cg} = -\frac{\nu_3}{\nu_2} \left(\frac{\nu_1 - \eta_{kk} + \mu_2\nu_2}{\frac{1}{\beta\eta_{kk}} - \rho} \right)$$

Since $\{\nu_1, 1/(\beta\eta_{kk})\} > 1$, $\{\mu_2, \nu_2\} > 0$ and $\rho \in [0, 1]$, it follows that η_{cg} is negative, so the private consumption decreases after a rise in government spending. Finally, we can derive η_{kg} from η_{cg} and (E.15):

$$\begin{aligned} \eta_{kg} &= -\nu_2 \left[-\frac{\nu_3}{\nu_2} \left(\frac{\nu_1 - \eta_{kk} + \mu_2\nu_2}{\frac{1}{\beta\eta_{kk}} - \rho} \right) \right] - \nu_3 \\ &= \nu_3 \left(\frac{\nu_1 - \eta_{kk} + \mu_2\nu_2}{\frac{1}{\beta\eta_{kk}} - \rho} - 1 \right) \\ &= \nu_3 \left(\frac{\nu_1 - \eta_{kk} + \mu_2\nu_2 - \frac{1}{\beta\eta_{kk}} + \rho}{\frac{1}{\beta\eta_{kk}} - \rho} \right) \end{aligned}$$

Since $\mu_1 = -(\nu_1 - \eta_{kk} + \mu_2\nu_2 - (\beta\eta_{kk})^{-1})$, it comes:

$$\eta_{kg} = \nu_3 \left(\frac{\rho - \mu_1}{\frac{1}{\beta\eta_{kk}} - \rho} \right)$$

So, the sign of η_{kg} is of the sign of $\rho - \mu_1$. In fact, μ_1 appears a particular value for ρ such that $\eta_{kg} = 0$.

Notice that when $\varphi = 0$, the expression of η_{kg} simplifies a lot since $\mu_2 = 0$ and $\nu_1 = 1/(\beta\eta_{kk})$:

$$\eta_{kg}|_{\varphi=0} = \nu_3 \left(\frac{\rho - \mu_1}{\nu_1 - \rho} \right)$$

From the above identifications, we now turn to the characterization of the multiplier (output, consumption and investment). Let us first consider, the output multiplier. From the impact response of consumption, given by η_{cg} , we can obtain the impact response of output after replacement into the production function (*i.e.* $-(1-\theta)(1+\varphi)\eta_{cg}/(\theta+\varphi)$). The impact output multiplier is given by

$$\begin{aligned}\frac{\Delta y_0}{\Delta g_0} &= -\eta_{cg} \frac{1-\theta}{\theta+\varphi} \frac{\bar{y}}{\bar{g}} \\ &= \frac{(1-\theta)}{\theta+\varphi} \frac{\bar{y}}{\bar{g}} \frac{\nu_3}{\nu_2} \left(\frac{\nu_1 - \eta_{kk} + \mu_2 \nu_2}{\frac{1}{\beta \eta_{kk}} - \rho} \right) \\ &= \frac{1}{1 + \frac{\theta+\varphi}{(1-\theta)} \frac{\bar{c}}{\bar{y}}} \left(\frac{\nu_1 - \eta_{kk} + \mu_2 \nu_2}{\frac{1}{\beta \eta_{kk}} - \rho} \right)\end{aligned}$$

So the multiplier is positive and it is an increasing function of ρ . The impact multiplier on consumption is directly deduced from η_{cg} :

$$\begin{aligned}\frac{\Delta c_0}{\Delta g_0} &= \eta_{cg} \frac{\bar{c}}{\bar{g}} \\ &= -\frac{\nu_3}{\nu_2} \left(\frac{\nu_1 - \eta_{kk} + \mu_2 \nu_2}{\frac{1}{\beta \eta_{kk}} - \rho} \right) \\ &= -\frac{1}{1 + \frac{(1-\theta)}{\theta+\varphi} \frac{\bar{y}}{\bar{c}}} \left(\frac{\nu_1 - \eta_{kk} + \mu_2 \nu_2}{\frac{1}{\beta \eta_{kk}} - \rho} \right)\end{aligned}$$

The multiplier on consumption is negative and takes larger negative values when the government spending shock is more persistent. The multiplier on investment is deduced from η_{kg}

$$\begin{aligned}\frac{\Delta x_0}{\Delta g_0} &= \frac{\eta_{kg}}{\delta} \frac{\bar{x}}{\bar{g}} \\ &= \frac{\nu_3}{\delta} \frac{\bar{x}}{\bar{g}} \left(\frac{\rho - \mu_1}{\frac{1}{\beta \eta_{kk}} - \rho} \right) \\ &= \frac{\rho - \mu_1}{\frac{1}{\beta \eta_{kk}} - \rho}\end{aligned}$$

This completes the proof.

F Endogenous external discounting: Proof of Proposition 9

The log-linearization about the non-stochastic steady state

$$\widehat{k}_{t+1} = \nu_1 \widehat{k}_t - \nu_2 \widehat{c}_t - \nu_3 \widehat{g}_t \quad (\text{F.1})$$

$$\widehat{y}_t = \theta \widehat{k}_t + (1 - \theta) \widehat{n}_t \quad (\text{F.2})$$

$$\widehat{n}_t = \widehat{y}_t - \widehat{c}_t \quad (\text{F.3})$$

$$\text{E}_t \widehat{c}_{t+1} = (1 - \omega) \widehat{c}_t + \beta \theta \frac{y}{k} \text{E}_t (\widehat{y}_{t+1} - \widehat{k}_{t+1}) \quad (\text{F.4})$$

$$\widehat{g}_t = \rho \widehat{g}_{t-1} + \varepsilon_t \quad (\text{F.5})$$

where, as before, $\nu_1 = 1 - \delta + y/k > 1$, $\nu_2 = \frac{\bar{y}}{k}(s_c + (1 - \theta)/\theta)$, $\nu_3 = \frac{\bar{y}}{k}s_g$ and $\frac{\bar{y}}{k} = (1 - \beta(1 - \delta))/(\beta\theta)$ is the steady output to capital ratio. Substituting (F.3) into (F.2), we obtain

$$\widehat{y}_t = \widehat{k}_t - \frac{1 - \theta}{\theta} \widehat{c}_t \quad (\text{F.6})$$

and substituting (F.6) into (F.4), one gets

$$\text{E}_t \widehat{c}_{t+1} = (1 - \omega) \mu \widehat{c}_t \quad (\text{F.7})$$

where $\mu < 1$ is defined as previously,

$$\mu = \frac{1}{1 + \beta \theta \frac{\bar{y}}{k} \frac{1 - \theta}{\theta}}$$

The reader can check that $(1 - \omega) \mu \nu_1 \neq 1/\beta$ unless $\omega = 0$ (no externality in discounting). Using the same solving procedure as before (see appendix B), we can determine the short-run multipliers. This completes the proof.

G Hand-to-mouth consumers: Proof of Proposition 10

The log-linearization of the non-savers consumption about the non-stochastic steady state implies that their consumption is proportional to aggregate output

$$\widehat{c}_t^{ns} = \widehat{y}_t$$

and thus the aggregate consumption is given by

$$\widehat{c}_t = \lambda \widehat{c}_t^{ns} + (1 - \lambda) \widehat{c}_t^s \equiv \lambda \widehat{y}_t + (1 - \lambda) \widehat{c}_t^s$$

This yields the following equations

$$\widehat{k}_{t+1} = \tilde{\nu}_1 \widehat{k}_t - \tilde{\nu}_2 \widehat{c}_t^s - \nu_3 \widehat{g}_t \quad (\text{G.1})$$

$$\widehat{y}_t = \theta \widehat{k}_t + (1 - \theta) \widehat{n}_t \quad (\text{G.2})$$

$$\widehat{n}_t = \widehat{y}_t - \widehat{c}_t^s \quad (\text{G.3})$$

$$\text{E}_t \widehat{c}_{t+1}^s = \widehat{c}_t^s + \beta \theta \frac{\bar{y}}{k} \text{E}_t (\widehat{y}_{t+1} - \widehat{k}_{t+1}) \quad (\text{G.4})$$

$$\widehat{g}_t = \rho \widehat{g}_{t-1} + \varepsilon_t \quad (\text{G.5})$$

where

$$\tilde{\nu}_1 = 1 - \delta + \frac{\bar{y}}{k} (1 - s_c \lambda) \equiv \nu_1 - s_c \lambda \frac{\bar{y}}{k}$$

and ν_1 is defined as before

$$\nu_1 = 1 - \delta + \frac{\bar{y}}{k} > 1$$

It is easy to verify that $\tilde{\nu}_1 > 1, \forall \lambda \in [0, 1]$ and $s_g > 0$. So the model is determinate and (G.1) can be solved forward. The parameter $\tilde{\nu}_2$ is given by

$$\tilde{\nu}_2 = \nu_2 - \frac{\bar{y}}{k} \frac{s_c \lambda}{\theta},$$

where $\nu_2 = \frac{\bar{y}}{k} (s_c + (1 - \theta)/\theta)$ is defined as before. The parameter $\nu_3 = \frac{\bar{y}}{k} s_g$ is the same as before and $\frac{\bar{y}}{k} = (1 - \beta(1 - \delta))/(\beta\theta)$ is the steady output to capital ratio.

Substituting (G.3) into (G.2), we obtain

$$\widehat{y}_t = \widehat{k}_t - \frac{1 - \theta}{\theta} \widehat{c}_t^s \quad (\text{G.6})$$

and substituting (G.6) into (G.4), one gets

$$\text{E}_t \widehat{c}_{t+1}^s = \mu \widehat{c}_t^s \quad (\text{G.7})$$

where $\mu < 1$ is defined as previously,

$$\mu = \frac{1}{1 + \beta \theta \frac{\bar{y}}{k} \frac{1 - \theta}{\theta}}$$

It comes immediately that $\mu \tilde{\nu}_1 \neq 1/\beta$ unless $\lambda = 0$ (the share of non-savers is zero). Now, using the same solving procedure as before (see appendix B), we can determine the short-run multipliers. This completes the proof.