Supplementary Material for

"**Tail expectile process and risk assessment**"

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Simulation results are discussed in Section A. The proofs of all theoretical results in the main paper and additional technical results are provided in Section B.

A Some simulation evidence

The aim of this section is to explore some features that were mentioned in Section 5.2 of the main article.

We have undertaken simulation experiments to evaluate finite-sample performance of the composite versions $\widetilde{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}$, $\overline{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}(\beta)$ and $\widehat{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}(\beta)$ studied in Theorem 9. These composite expectile-based estimators estimate the same conventional expected shortfall QES_{p_n} as the direct quantile-based estimator $\widehat{\mathrm{QES}}^{\star}_{p_n} \equiv \widehat{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}(\beta = 1)$.

In order to illustrate the behavior of the presented estimation procedures, we use the same considerations as in Section 5 of the main paper. Namely, we consider the Student t-distribution with degree of freedom $1/\gamma$, the Fréchet distribution $F(x) = e^{-x^{-1/\gamma}}$, $x > 0$, and the Pareto distribution $F(x) = 1 - x^{-1/\gamma}$, $x > 1$. The finite-sample performance of the different estimators is evaluated through their relative Mean-Squared Error (MSE) and bias, computed over 200 replications. The accuracy of the weighted estimators is investigated for various values of the weight $\beta \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. All the experiments have sample size $n = 500$ and tail index $\gamma \in \{0.05, 0.25, 0.45\}$. In our simulations we used the extreme level $p_n = 1 - \frac{1}{n}$ and the intermediate level $\tau_n = 1 - \frac{k}{n}$, where the integer k can be viewed as the effective sample size for tail extrapolation.

We first examined the accuracy of $\widetilde{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}$ and $\widehat{\mathrm{QES}}^{\star}_{p_n}$ (both independent of β) in comparison with $\overline{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}(\beta)$, in Figures 1-2, and with $\widehat{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}(\beta)$ in Figures 3-4.

Figures 1 and 2 give, respectively, the MSE (in log scale) and bias estimates of $\widehat{\mathrm{QES}}^{\star}_{p_n}/\mathrm{QES}_{p_n}$ (grey curves), $\widetilde{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}/\mathrm{QES}_{p_n}$ (black curves) and $\overline{\mathrm{XES}}^{\star}_{\hat\tau_n'(p_n)}(\beta)/\mathrm{QES}_{p_n}$ (colored curves), against k. In the case of the Student distribution (top panels), it may be seen that the black curves perform globally quite well in terms of MSE and bias. In the cases of Fréchet distribution (panels in the middle) and Pareto distribution (bottom panels), the grey and orange curves seem to be superior.

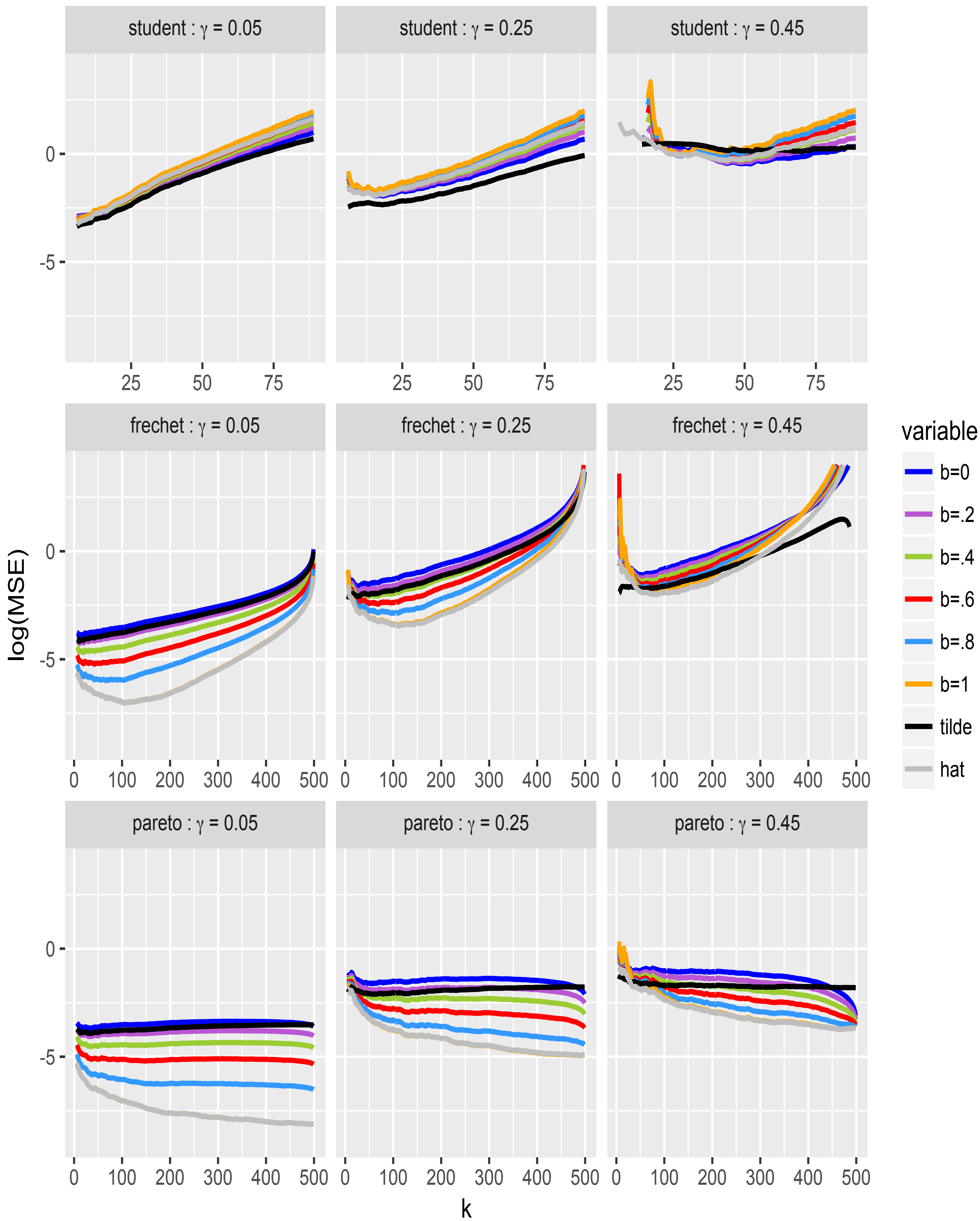

Figure 1: *MSE estimates (in log scale) of* $\widehat{QES}^{\star}_{p_n}/QES_{p_n}$ *(grey),* $\widetilde{XES}^{\star}_{\hat{\tau}'_n(p_n)}/QES_{p_n}$ *(black) and* $\overline{XES}^{\star}_{\hat{\tau}'_n(p_n)}(\beta)/QES_{p_n}$ *(colour-scheme), against* k*, for Student (top), Fréchet (middle) and Pareto (bottom) distributions, with* $\gamma = 0.05$ *(left),* $\gamma = 0.25$ *(middle) and* $\gamma = 0.45$ *(right).*

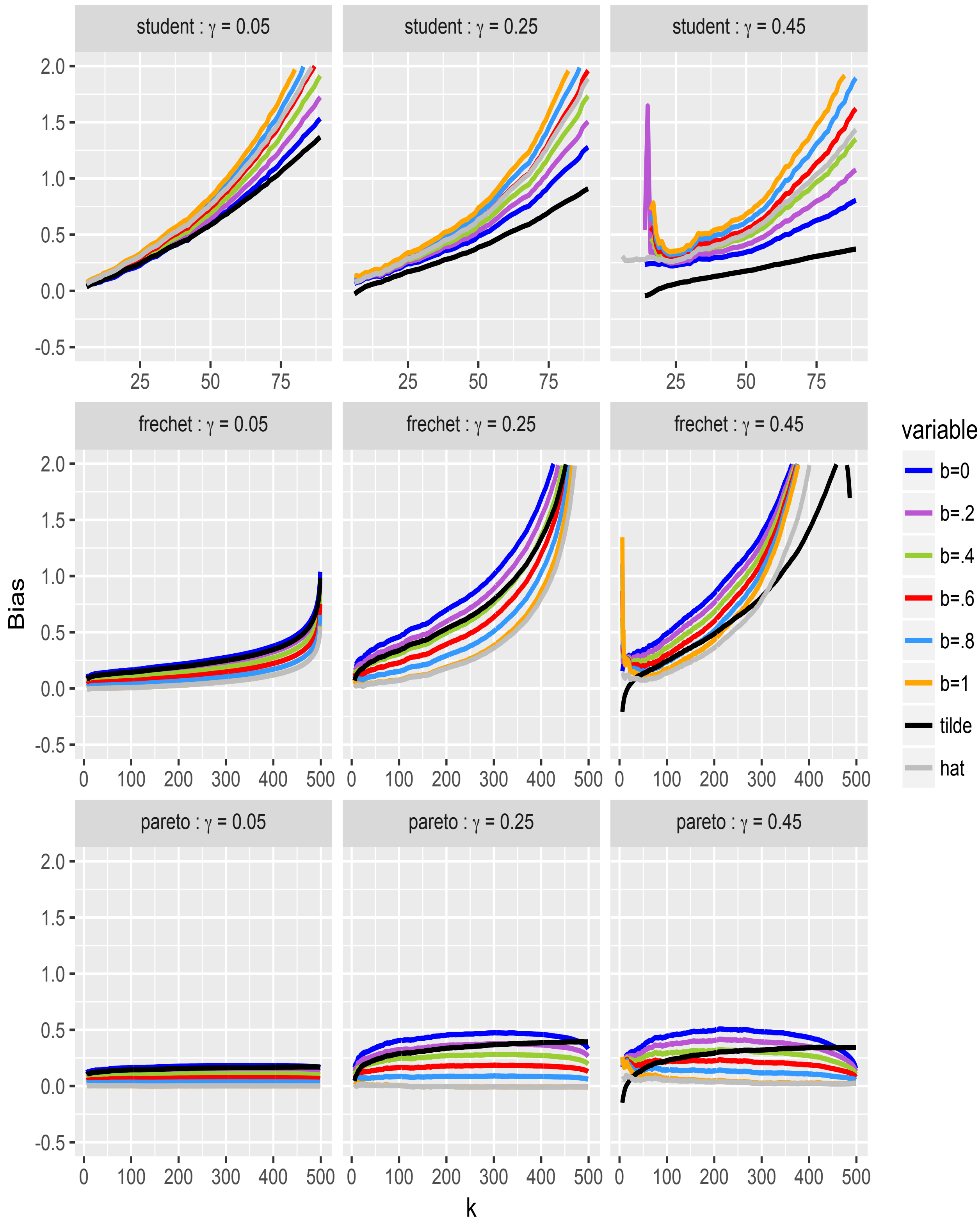

Figure 2: *Bias estimates of* $\widehat{QES}^\star_{p_n}/QES_{p_n}$ *(grey),* $\widetilde{XES}^\star_{\widehat{\tau}'_n(p_n)}/QES_{p_n}$ *(black) and* $\overline{XES}^\star_{\widehat{\tau}'_n(p_n)}(\beta)/QES_{p_n}$ *(colour-scheme).*

Figures 3 and 4 give, respectively, the MSE (in log scale) and bias estimates of $\widetilde{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}/\mathrm{QES}_{p_n}$ (black curves) and $\widehat{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta)/\mathrm{QES}_{p_n}$ (colored curves), against k. Note that the orange curves ($\beta = 1$) correspond to the Monte-Carlo estimates of $\widehat{\mathrm{QES}}^{\star}_{p_n}/\mathrm{QES}_{p_n}$, since $\widehat{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta = 1) \equiv \widehat{\mathrm{QES}}^{\star}_{p_n}$. In the case of the Student distribution (top panels), the black curves still perform quite well. By contrast, in both cases of the Fréchet distribution (panels in the middle) and Pareto distribution (bottom panels), the orange curves seem to be superior.

When comparing the four estimators $\widehat{\mathrm{QES}}^{\star}_{p_n}$, $\widetilde{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}$, $\overline{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta)$ and $\widehat{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta)$ with each other, we arrive at the following tentative conclusions:

- In the case of the real-valued profit-loss Student distribution, the best estimator seems to be $\widetilde{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}$;
- In the case of the non-negative Fréchet and Pareto loss distributions, the best estimators seem to be $\overline{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta = 1)$ and/or $\widehat{\mathrm{QES}}^{\star}_{p_n} \equiv \widehat{\mathrm{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta = 1)$.

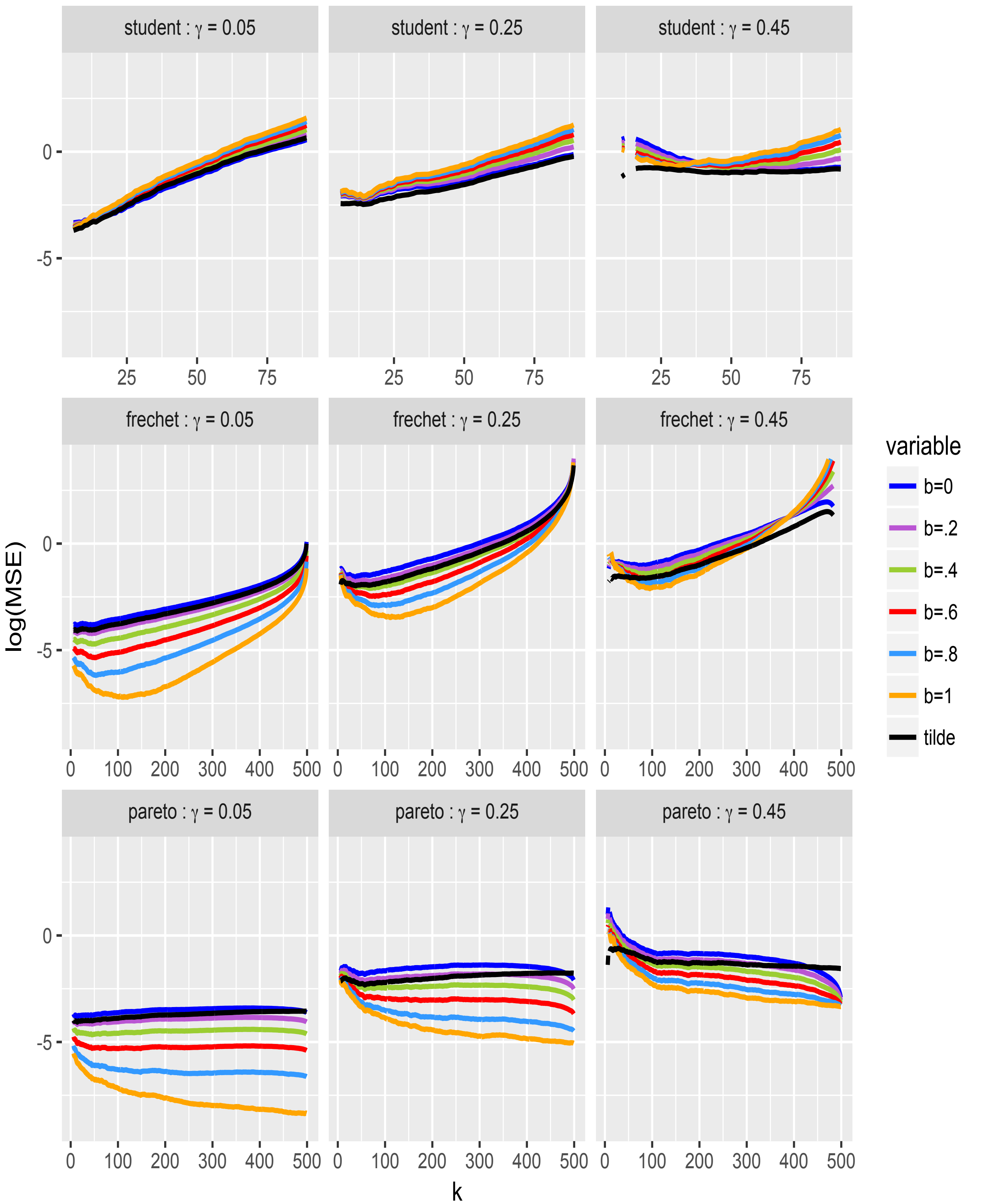

Figure 3: *MSE estimates (in log scale) of* $\widetilde{XES}^{\star}_{\widehat{\tau}'_n(p_n)}/QES_{p_n}$ *(black) and* $\widehat{XES}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta)/QES_{p_n}$ *(colour-scheme), against* k*, for Student (top), Fréchet (middle) and Pareto (bottom) distributions, with* $\gamma = 0.05$ *(left),* $\gamma = 0.25$ *(middle) and* $\gamma = 0.45$ *(right).*

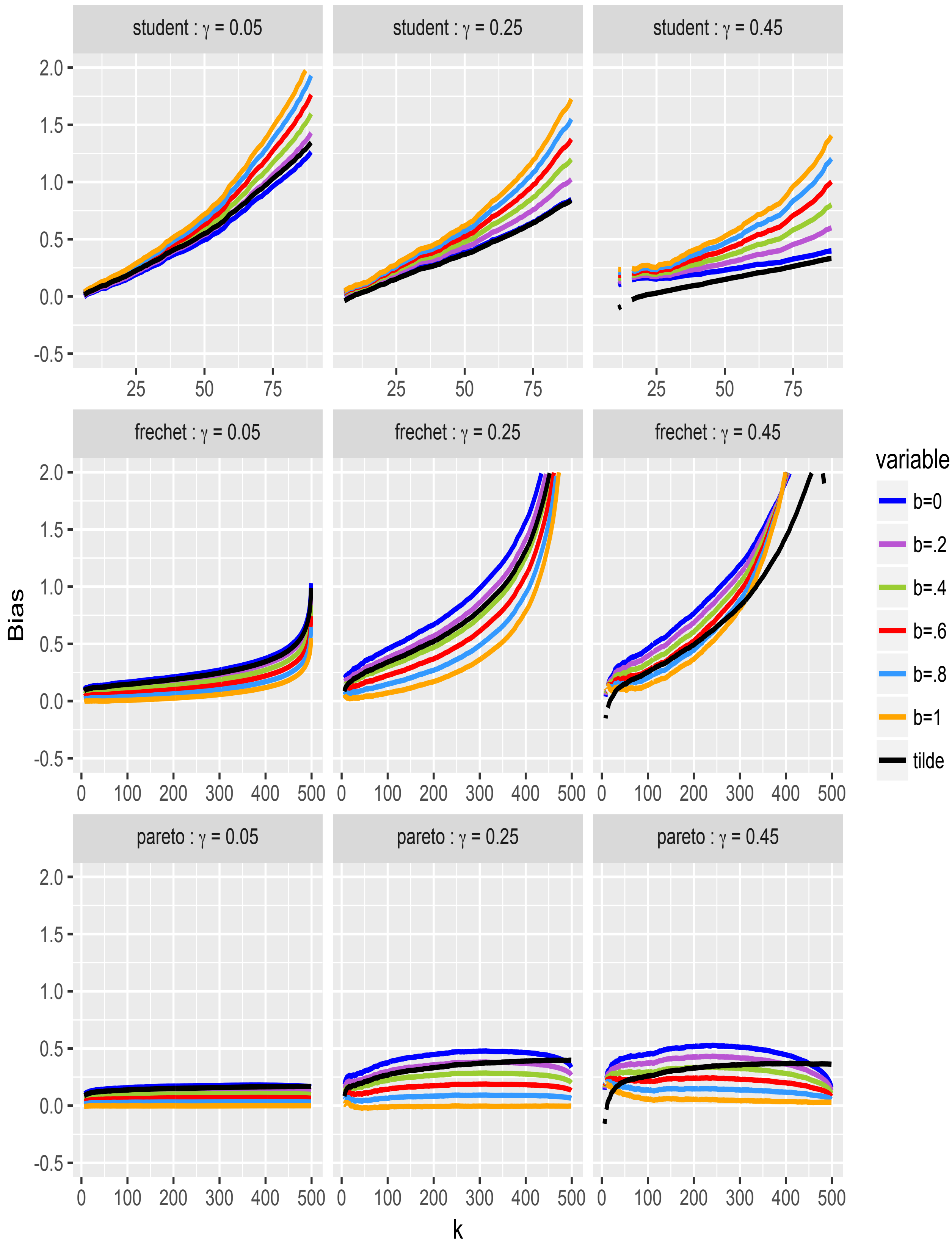

Figure 4: *Bias estimates of* $\widetilde{XES}^{\star}_{\widehat{\tau}'_n(p_n)}/QES_{p_n}$ *(black) and* $\widehat{XES}^{\star}_{\widehat{\tau}'_n(p_n)}(\beta)/QES_{p_n}$ *(colour-scheme).*

B Proofs

In all proofs, the sequence τ_n is replaced by the sequence $k = n(1-\tau_n)$.

Proof of Proposition 1. We start by showing (i). By Proposition 1 in Daouia *et al.* (2018):

$$\begin{aligned} \frac{\overline{F}(\xi_\tau)}{1-\tau} &= (\gamma^{-1}-1)(1+\varepsilon(\tau)) \\ \text{with } \varepsilon(\tau) &= -\frac{(\gamma^{-1}-1)^{\gamma}}{q_\tau}(\mathbb{E}(Y)+\mathrm{o}(1)) - \frac{(\gamma^{-1}-1)^{-\rho}}{\gamma(1-\gamma-\rho)}A((1-\tau)^{-1})(1+\mathrm{o}(1)) \text{ as } \tau \to 1. \end{aligned}$$

Using this convergence together with local uniformity of condition $\mathcal{C}_2(\gamma,\rho,A)$, we find that

$$\frac{1}{A((1-\tau)^{-1})}\left[\frac{U(1/\overline{F}(\xi_\tau))}{U((1-\tau)^{-1})} - (\gamma^{-1}-1)^{-\gamma}(1+\varepsilon(\tau))^{-\gamma}\right] \to (\gamma^{-1}-1)^{-\gamma}\frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho}$$

as $\tau \to 1$, or equivalently

$$\begin{aligned} \frac{U(1/\overline{F}(\xi_\tau))}{q_\tau} &= (\gamma^{-1}-1)^{-\gamma}\Bigg(1+\frac{\gamma(\gamma^{-1}-1)^{\gamma}}{q_\tau}(\mathbb{E}(Y)+\mathrm{o}(1)) \\ &\quad + \left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}+\frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho}+\mathrm{o}(1)\right)A((1-\tau)^{-1})\Bigg) \text{ as } \tau \to 1. \end{aligned}$$

A use of Lemma 1 at $t = \xi_\tau$ makes it possible to replace $U(1/\overline{F}(\xi_\tau))$ by ξ_τ asymptotically, thus completing the proof of (i).

To show (ii), first note that if $s = 1$, there is nothing to prove. Otherwise, write

$$\frac{\xi_{1-ks/n}}{\xi_{1-k/n}} = \frac{\xi_{1-ks/n}}{q_{1-ks/n}} \times \frac{q_{1-k/n}}{\xi_{1-k/n}} \times \frac{q_{1-ks/n}}{q_{1-k/n}}. \tag{B.1}$$

With alternatively $\tau = 1-k/n$ and $\tau = 1-ks/n$ in (i), we obtain

$$\begin{aligned} \frac{\xi_{1-k/n}}{q_{1-k/n}} &= (\gamma^{-1}-1)^{-\gamma}\Bigg(1+\frac{\gamma(\gamma^{-1}-1)^{\gamma}}{q_{1-k/n}}(\mathbb{E}(Y)+\mathrm{o}(1)) \\ &\quad + \left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}+\frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho}+\mathrm{o}(1)\right)A(n/k)\Bigg) \end{aligned}$$

and

$$\begin{aligned} \frac{\xi_{1-ks/n}}{q_{1-ks/n}} &= (\gamma^{-1}-1)^{-\gamma}\Bigg(1+s^{\gamma}\frac{\gamma(\gamma^{-1}-1)^{\gamma}}{q_{1-k/n}}(\mathbb{E}(Y)+\mathrm{o}(1)) \\ &\quad + \left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}+\frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho}+\mathrm{o}(1)\right)s^{-\rho}A(n/k)\Bigg) \end{aligned}$$

because of the regular variation property of $t \mapsto q_{1-t^{-1}}$ and $|A|$. Besides, it is a consequence of condition $\mathcal{C}_2(\gamma,\rho,A)$ that

$$\frac{q_{1-ks/n}}{q_{1-k/n}} = \frac{U(n/ks)}{U(n/k)} = s^{-\gamma}\left(1+A(n/k)\frac{s^{-\rho}-1}{\rho}+\mathrm{o}(A(n/k))\right).$$

Combining these three expansions with (B.1) yields the desired result. ∎

Proof of Theorem 1. Note that, for any $\tau \in (0,1)$,

$$\widetilde{\xi}_\tau - \frac{1}{n}\sum_{i=1}^n Y_i = \frac{2\tau-1}{1-\tau} \times \frac{1}{n}\sum_{i=1}^n (Y_i - \widetilde{\xi}_\tau)\mathbb{1}_{\{Y_i > \widetilde{\xi}_\tau\}}$$

which is a straightforward consequence of the definition of $\widetilde{\xi}_\tau$ as a minimiser (see *e.g.* Equation (2.7) in Newey and Powell, 1987, applied to the empirical distribution function). We use this with $\tau = 1 - ks/n$, $s \in (0,1]$, in order to write

$$\frac{ks/n}{1-2ks/n}(\widetilde{\xi}_{1-ks/n} - \overline{Y}_n) = \int_{\widetilde{\xi}_{1-ks/n}}^{\infty} \widehat{\overline{F}}_n(u)du \tag{B.2}$$

where $\overline{Y}_n$ denotes the empirical mean and $\widehat{\overline{F}}_n(u) = n^{-1}\sum_{i=1}^n \mathbb{1}_{\{Y_i > u\}}$ is the empirical survival function of the sample. The idea is now to obtain a uniform (in s) "asymptotic expansion" of the integral on the right-hand side.

Our main tool will be Lemma 2(ii): we may enlarge the underlying sample space and choose a suitable version of the empirical process $\widehat{F}_n$ so that there is a sequence of standard Brownian motions $\widetilde{W}_n$ such that for any $\varepsilon > 0$ small enough (which we shall fix later):

$$\frac{n}{k}\widehat{\overline{F}}_n\left(xq_{1-k/n}\right) - x^{-1/\gamma} = \frac{1}{\sqrt{k}}\left(\widetilde{W}_n(x^{-1/\gamma}) + \sqrt{k}A(n/k)x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho} + x^{(\varepsilon-1/2)/\gamma}\,\mathrm{o}_{\mathbb{P}}(1)\right)$$

uniformly in half-lines of the form $x \in [x_0, \infty)$, for $x_0 > 0$. Note then that, as a consequence of the monotonicity of expectiles together with convergence

$$\frac{\xi_\tau}{q_\tau} \to \left(\gamma^{-1}-1\right)^{-\gamma} \quad \text{as} \quad \tau \to 1 \tag{B.3}$$

(see Bellini and Di Bernardino, 2017) and Lemma 3, we have

$$\forall s \in (0,1], \ \frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}} \geqslant \frac{\widetilde{\xi}_{1-k/n}}{q_{1-k/n}} \xrightarrow{\mathbb{P}} (\gamma^{-1}-1)^{-\gamma} \quad \text{as} \quad n \to \infty.$$

Consequently

$$\mathbb{P}\left(\forall s \in (0,1], \ \frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}} > \frac{1}{2}(\gamma^{-1}-1)^{-\gamma}\right) \to 1 \quad \text{as} \quad n \to \infty. \tag{B.4}$$

It then follows from the above approximation by a sequence of Brownian motions that, with arbitrarily large probability:

$$\begin{aligned}
&\int_{\widetilde{\xi}_{1-ks/n}}^{\infty} \widehat{\overline{F}}_n(u)du \\
&= q_{1-k/n}\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} \widehat{\overline{F}}_n(xq_{1-k/n})dx \\
&= \frac{k}{n}q_{1-k/n}\left(\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{-1/\gamma}\,dx + \frac{1}{\sqrt{k}}\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} \widetilde{W}_n(x^{-1/\gamma})\,dx\right. \\
&+ \left. A(n/k)\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\,dx + \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{(\varepsilon-1/2)/\gamma}\,dx\right)\right)
\end{aligned} \tag{B.5}$$

uniformly in $s \in (0,1]$. Note that the last term is indeed well-defined, if ε is taken close enough to 0, because $\gamma \in (0,1/2)$. We choose such an ε here and in the sequel.

The next step is to use Lemma 4, primarily to remove the randomness in the lower bound of the integral of the Brownian motion $\widetilde{W}_n$ in (B.5). Lemma 4 only allows us to do so on the restricted range $[k^{-1+\delta},1]$, and we therefore focus on this case for now; we will take care of the case $s \in (0,k^{-1+\delta})$ separately afterwards. Use first (B.4) to get, for any sufficiently small $\delta > 0$ and with arbitrarily large probability irrespective of $s \in [k^{-1+\delta},1]$:

$$\begin{aligned}
& s^{\gamma}\left|\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} \widetilde{W}_n(x^{-1/\gamma})\,dx - \int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} \widetilde{W}_n(x^{-1/\gamma})\,dx\right| \\
= & \left|\int_{s^{\gamma}\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} \widetilde{W}_n(s\,u^{-1/\gamma})\,du - \int_{(\gamma^{-1}-1)^{-\gamma}}^{\infty} \widetilde{W}_n(s\,u^{-1/\gamma})\,du\right| \\
\leqslant & \ s^{\gamma}\left|\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}} - (\gamma^{-1}-1)^{-\gamma}s^{-\gamma}\right| \times \sup_{0\leqslant t\leqslant(\gamma^{-1}-1)/2^{-1/\gamma}} |\widetilde{W}_n(st)|.
\end{aligned}$$

Self-similarity of the Brownian motion $\widetilde{W}_n$ w.r.t. scaling gives

$$\sup_{0\leqslant t\leqslant(\gamma^{-1}-1)/2^{-1/\gamma}} |\widetilde{W}_n(st)| \stackrel{d}{=} \sqrt{s}\sup_{0\leqslant t\leqslant(\gamma^{-1}-1)/2^{-1/\gamma}} |\widetilde{W}_n(t)| = \mathrm{O}_{\mathbb{P}}(\sqrt{s})$$

uniformly in s, because a standard Brownian motion is almost surely bounded on any compact interval by almost sure continuity of its sample paths. A use of Lemma 4 then entails

$$\sup_{k^{-1+\delta}\leqslant s\leqslant 1} s^{\gamma-1/2}\left|\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} \widetilde{W}_n(x^{-1/\gamma})\,dx - \int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} \widetilde{W}_n(x^{-1/\gamma})\,dx\right| = \mathrm{o}_{\mathbb{P}}(1).$$

Similarly,

$$\begin{aligned}
\sup_{k^{-1+\delta}\leqslant s\leqslant 1} s^{\gamma-1}\left|\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\,dx - \int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\,dx\right| &= \mathrm{o}_{\mathbb{P}}(1) \\
\text{and} \quad \sup_{k^{-1+\delta}\leqslant s\leqslant 1} s^{\gamma-1/2+\varepsilon}\left|\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{(\varepsilon-1/2)/\gamma}\,dx - \int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} x^{(\varepsilon-1/2)/\gamma}\,dx\right| &= \mathrm{o}_{\mathbb{P}}(1).
\end{aligned}$$

Therefore, we have, uniformly in $s \in [k^{-1+\delta},1]$ and with arbitrarily large probability, that

$$\begin{aligned}
\int_{\widetilde{\xi}_{1-ks/n}}^{\infty} \widehat{\widehat{F}}_n(u)du = & \frac{k}{n}q_{1-k/n}\left(\frac{\gamma}{1-\gamma}\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}\right]^{1-1/\gamma} + \frac{1}{\sqrt{k}}\int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} \widetilde{W}_n(x^{-1/\gamma})\,dx\right. \\
& + A(n/k)\int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\,dx + \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} x^{(\varepsilon-1/2)/\gamma}\,dx\right) \\
& \left. + \ \mathrm{o}_{\mathbb{P}}\left(\frac{s^{-\gamma+1/2-\varepsilon}}{\sqrt{k}}\right)\right).
\end{aligned}$$

We now rewrite each integral as follows: firstly, a change of variables and self-similarity of the Brownian motion w.r.t. scaling yield

$$\begin{aligned}\int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} \widetilde{W}_n(x^{-1/\gamma})\,dx &= (\gamma^{-1}-1)^{-\gamma}\gamma\int_0^s \widetilde{W}_n((\gamma^{-1}-1)t)\,t^{-\gamma-1}\,dt \\ &= (\gamma^{-1}-1)^{1/2-\gamma}\gamma\int_0^s W_n(t)\,t^{-\gamma-1}\,dt \qquad \text{(B.6)}\end{aligned}$$

where $W_n(t) := (\gamma^{-1}-1)^{-1/2}\widetilde{W}_n((\gamma^{-1}-1)t)$ defines another sequence of standard Brownian motions. Secondly, a straightforward integration gives

$$\int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\,dx = \frac{(\gamma^{-1}-1)^{1-\gamma}s^{1-\gamma}}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}s^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right].$$

Thirdly and finally, another direct integration entails

$$\int_{(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}}^{\infty} x^{(\varepsilon-1/2)/\gamma}\,dx = \mathrm{O}\left(s^{-\gamma+1/2-\varepsilon}\right).$$

All in all, and combining these calculations with (B.2), we obtain, uniformly in $s \in [k^{-1+\delta}, 1]$:

$$\begin{aligned}&\frac{s}{1-2ks/n}\left(\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}-\frac{\overline{Y}_n}{q_{1-k/n}}\right)\\ =\ & \frac{\gamma}{1-\gamma}\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}\right]^{1-1/\gamma} + \frac{1}{\sqrt{k}}(\gamma^{-1}-1)^{1/2-\gamma}\gamma\int_0^s W_n(t)\,t^{-\gamma-1}\,dt \\ +\ & A(n/k)\frac{(\gamma^{-1}-1)^{1-\gamma}s^{1-\gamma}}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}s^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right] + \mathrm{o}_{\mathbb{P}}\left(\frac{s^{-\gamma+1/2-\varepsilon}}{\sqrt{k}}\right). \qquad \text{(B.7)}\end{aligned}$$

Recall now the following equivalent characterisation of population expectiles:

$$\xi_\tau - \mathbb{E}(Y) = \frac{2\tau-1}{1-\tau}\mathbb{E}((Y-\xi_\tau)\mathbb{1}_{\{Y>\xi_\tau\}}). \qquad \text{(B.8)}$$

We use this identity with $\tau = 1-k/n$ to get:

$$\begin{aligned}&\frac{1}{1-2k/n}\left(\frac{\xi_{1-k/n}}{q_{1-k/n}}-\frac{\mathbb{E}(Y)}{q_{1-k/n}}\right)\\ =\ & \frac{n}{k}\times\frac{1}{q_{1-k/n}}\int_{\xi_{1-k/n}}^{\infty}\overline{F}(u)du \\ =\ & \frac{n}{k}\overline{F}(q_{1-k/n})\left(\frac{\gamma}{1-\gamma}\left[\frac{\xi_{1-k/n}}{q_{1-k/n}}\right]^{1-1/\gamma} + A(n/k)\frac{(\gamma^{-1}-1)^{1-\gamma}}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right]\right) \\ +\ & \mathrm{o}(A(n/k))\end{aligned}$$

thanks to convergence (B.3), the asymptotic equivalence $\overline{F}(q_{1-k/n}) \sim k/n$ from Lemma 1(ii) and used inside the regularly varying function A, Lemma 5 and calculations identical to those we have carried out so far. Using the condition $\sqrt{k}A(n/k) = \mathrm{O}(1)$ and the convergence

$$\lim_{n\to\infty}\frac{1}{A(n/k)}\left(\frac{n}{k}\overline{F}(q_{1-k/n})-1\right) = 0$$

which follows from Lemma 1(ii), we obtain

$$\begin{aligned}
&\frac{1}{1-2k/n}\left(\frac{\xi_{1-k/n}}{q_{1-k/n}}-\frac{\mathbb{E}(Y)}{q_{1-k/n}}\right)\\
=\ &\frac{\gamma}{1-\gamma}\left[\frac{\xi_{1-k/n}}{q_{1-k/n}}\right]^{1-1/\gamma}+A(n/k)\frac{(\gamma^{-1}-1)^{1-\gamma}}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right]+\mathrm{o}\left(\frac{1}{\sqrt{k}}\right). \qquad \text{(B.9)}
\end{aligned}$$

Dividing (B.7) by (B.9) and using convergence (B.3) together with a Taylor expansion, we get

$$\begin{aligned}
&s\frac{1-2k/n}{1-2ks/n}\times\frac{\widetilde{\xi}_{1-ks/n}-\overline{Y}_n}{\xi_{1-k/n}-\mathbb{E}(Y)}\\
=\ &\left[\frac{\gamma}{1-\gamma}\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}\right]^{1-1/\gamma}+\frac{1}{\sqrt{k}}(\gamma^{-1}-1)^{1/2-\gamma}\gamma\int_0^s W_n(t)\,t^{-\gamma-1}\,dt\right.\\
+\ &\left.A(n/k)\frac{(\gamma^{-1}-1)^{1-\gamma}s^{1-\gamma}}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}s^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right]+\mathrm{o}_{\mathbb{P}}\left(\frac{s^{-\gamma+1/2-\varepsilon}}{\sqrt{k}}\right)\right]\\
\times\ &\frac{1-\gamma}{\gamma}\left[\frac{q_{1-k/n}}{\xi_{1-k/n}}\right]^{1-1/\gamma}\left(1-A(n/k)\frac{\gamma^{-1}-1}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right]+\mathrm{o}\left(\frac{1}{\sqrt{k}}\right)\right)\\
=\ &\left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}}\right]^{1-1/\gamma}+\frac{1}{\sqrt{k}}\gamma\sqrt{\gamma^{-1}-1}\int_0^s W_n(t)\,t^{-\gamma-1}\,dt\\
+\ &A(n/k)\frac{(\gamma^{-1}-1)s^{1-\gamma}}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}s^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right]\\
-\ &A(n/k)\left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}}\right]^{1-1/\gamma}\frac{\gamma^{-1}-1}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}-\frac{1}{1-\gamma}\right]+\mathrm{o}_{\mathbb{P}}\left(\frac{s^{-\gamma+1/2-\varepsilon}}{\sqrt{k}}\right).
\end{aligned}$$

Define now a random process $s\mapsto r_n(s)$ by the equality

$$s^{\gamma}\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}}=1+r_n(s).$$

We know, by a combination of convergence (B.3) and Lemma 4, that $r_n(s)\xrightarrow{\mathbb{P}}0$ uniformly in $s\in[k^{-1+\delta},1]$. The above expansion then simplifies as

$$\begin{aligned}
s\frac{1-2k/n}{1-2ks/n}\times\frac{\widetilde{\xi}_{1-ks/n}-\overline{Y}_n}{\xi_{1-k/n}-\mathbb{E}(Y)}=&\left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}}\right]^{1-1/\gamma}+\frac{1}{\sqrt{k}}\gamma\sqrt{\gamma^{-1}-1}\int_0^s W_n(t)\,t^{-\gamma-1}\,dt\\
&+A(n/k)\times\frac{(\gamma^{-1}-1)^{1-\rho}}{1-\gamma-\rho}\times s^{1-\gamma}\frac{s^{-\rho}-1}{\rho}+\mathrm{o}_{\mathbb{P}}\left(\frac{s^{-\gamma+1/2-\varepsilon}}{\sqrt{k}}\right). \qquad \text{(B.10)}
\end{aligned}$$

We now work on the left-hand side of the above identity. Note that we can write, uniformly in $s\in(0,1]$:

$$\frac{1-2k/n}{1-2ks/n}=1-\frac{2k}{n}\times\frac{1-s}{1-2ks/n}=1-\frac{2k}{n}(1-s)\left[1+\mathrm{O}\left(\frac{k}{n}\right)\right].$$

Moreover,

$$\begin{aligned}\frac{\widetilde{\xi}_{1-ks/n}-\overline{Y}_n}{\xi_{1-k/n}-\mathbb{E}(Y)} &= \left(\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}}-1\right)\left(1+\frac{\mathbb{E}(Y)}{\xi_{1-k/n}-\mathbb{E}(Y)}\right)+\frac{\xi_{1-k/n}-\overline{Y}_n}{\xi_{1-k/n}-\mathbb{E}(Y)} \\ &= 1+\left(\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}}-1\right)\left(1+\frac{(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)}{q_{1-k/n}}(1+\mathrm{o}(1))\right)+\mathrm{O}_{\mathbb{P}}\left(\frac{1}{q_{1-k/n}\sqrt{n}}\right)\end{aligned}$$

by asymptotic proportionality of $q_{1-k/n}$ and $\xi_{1-k/n}$, and the central limit theorem. Since $\gamma < 1/2$, we have by regular variation of $t \mapsto q_{1-t^{-1}}$ that

$$\frac{1}{q_{1-k/n}\sqrt{n}} \bigg/ \frac{1}{\sqrt{k}} = \sqrt{\frac{k}{n}}\, q_{1-k/n} = \mathrm{o}(1).$$

Consequently

$$s^{\gamma}\frac{\widetilde{\xi}_{1-ks/n}-\overline{Y}_n}{\xi_{1-k/n}-\mathbb{E}(Y)} = 1+r_n(s)\,(1+\mathrm{o}_{\mathbb{P}}(1))+(1-s^{\gamma})\frac{(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)}{q_{1-k/n}}(1+\mathrm{o}_{\mathbb{P}}(1))+\mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\right).$$

Notice finally that, by the mean value theorem:

$$1 \leqslant \sup_{0\leqslant s<1}\left\{\frac{1-s}{1-s^{\gamma}}\right\} < \infty$$

so that, using the relationship $q_{1-k/n} = \mathrm{o}(n/k)$, we get again that

$$\begin{aligned}&\frac{1-2k/n}{1-2ks/n}\times s^{\gamma}\frac{\widetilde{\xi}_{1-ks/n}-\overline{Y}_n}{\xi_{1-k/n}-\mathbb{E}(Y)} \\ &= 1+r_n(s)\,(1+\mathrm{o}_{\mathbb{P}}(1))+(1-s^{\gamma})\frac{(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)}{q_{1-k/n}}(1+\mathrm{o}_{\mathbb{P}}(1))+\mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\right).\end{aligned}$$

Because, uniformly in $s \in [k^{-1+\delta}, 1]$,

$$\left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}}\right]^{1-1/\gamma} = s^{1-\gamma}(1+r_n(s))^{1-1/\gamma} = s^{1-\gamma}\left(1+\left[1-\frac{1}{\gamma}\right]r_n(s)(1+\mathrm{o}_{\mathbb{P}}(1))\right)$$

we obtain using (B.10) that:

$$\begin{aligned}&1+r_n(s)\,(1+\mathrm{o}_{\mathbb{P}}(1))+(1-s^{\gamma})\frac{(\gamma^{-1}-1)^{\gamma}}{q_{1-k/n}}(\mathbb{E}(Y)+\mathrm{o}_{\mathbb{P}}(1))+\mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\right) \\ &= 1+\left[1-\frac{1}{\gamma}\right]r_n(s)(1+\mathrm{o}_{\mathbb{P}}(1))+\frac{1}{\sqrt{k}}\gamma\sqrt{\gamma^{-1}-1}\,s^{\gamma-1}\int_0^s W_n(t)\,t^{-\gamma-1}\,dt \\ &+ A(n/k)\times\frac{(\gamma^{-1}-1)^{1-\rho}}{1-\gamma-\rho}\times\frac{s^{-\rho}-1}{\rho}+\mathrm{o}_{\mathbb{P}}\left(\frac{s^{-1/2-\varepsilon}}{\sqrt{k}}\right).\end{aligned}$$

Rearrange and solve for $r_n(s)$ to get, uniformly in $s \in [k^{-1+\delta}, 1]$:

$$\begin{aligned}r_n(s) &= (s^{\gamma}-1)\frac{\gamma(\gamma^{-1}-1)^{\gamma}}{q_{1-k/n}}(\mathbb{E}(Y)+\mathrm{o}_{\mathbb{P}}(1)) \\ &+ \frac{1}{\sqrt{k}}\gamma^2\sqrt{\gamma^{-1}-1}\,s^{\gamma-1}\int_0^s W_n(t)\,t^{-\gamma-1}\,dt \\ &+ A(n/k)\times\frac{(1-\gamma)(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho}\times\frac{s^{-\rho}-1}{\rho}+\mathrm{o}_{\mathbb{P}}\left(\frac{s^{-1/2-\varepsilon}}{\sqrt{k}}\right).\end{aligned}$$

This is precisely what we wanted to show, but in the restricted case $s \in [k^{-1+\delta}, 1]$.

We conclude the proof by focusing on the case $s \in (0, k^{-1+\delta})$. To this end, we choose $\delta \in (0, \varepsilon/(2\varepsilon + 1 + 2\gamma))$ and we note that $\sqrt{k}\, s^{1/2+\varepsilon} \to 0$ uniformly in $s \in (0, k^{-1+\delta})$. It then follows that, by a direct calculation:

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{1/2+\varepsilon} \left(\frac{1}{q_{1-k/n}} + \frac{1}{\sqrt{k}} s^{\gamma-1} \left| \int_0^s W_n(t)\, t^{-\gamma-1}\, dt \right| + |A(n/k)| \right) \xrightarrow{\mathbb{P}} 0.$$

It is then enough to show that

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{1/2+\varepsilon} \left| s^{\gamma} \frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}} - 1 \right| = \sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{\gamma+1/2+\varepsilon} \left| \frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}} - s^{-\gamma} \right| \xrightarrow{\mathbb{P}} 0.$$

Recall that expectiles of an arbitrary distribution are monotonically increasing and exactly cover its support, and apply this to the empirical distribution to get $\widetilde{\xi}_{1-ks/n} \leqslant \widetilde{\xi}_1 = Y_{n,n}$ for any $s \in (0,1)$. Write then

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{\gamma+1/2+\varepsilon} \left| \frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}} - s^{-\gamma} \right| \leqslant k^{1/2+(-1+\delta)(\gamma+1/2+\varepsilon)} \frac{Y_{n,n}}{\xi_{1-k/n}} + \mathrm{o}(1).$$

Using Lemma 2(i) with $s = 1/(2k)$ and $\varepsilon/2$ in place of what was an arbitrary η there, gives:

$$\frac{Y_{n,n}}{q_{1-k/n}} = \mathrm{o}_{\mathbb{P}}(k^{\gamma+\varepsilon/2})$$

and therefore, by a use of (B.3) again, we get

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{\gamma+1/2+\varepsilon} \left| \frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}} - s^{-\gamma} \right| = \mathrm{o}_{\mathbb{P}} \left(k^{\delta(\gamma+1/2+\varepsilon)-\varepsilon/2} \right) + \mathrm{o}(1).$$

Recalling that $\delta < \varepsilon/(2\varepsilon + 1 + 2\gamma)$, we obtain

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{\gamma+1/2+\varepsilon} \left| \frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}} - s^{-\gamma} \right| = \mathrm{o}_{\mathbb{P}}(1).$$

This concludes the proof of the approximation result for the tail expectile process.

To complete the proof, just note that the sequence W_n has the closed form expression

$$W_n(t) = \frac{1}{\sqrt{\gamma^{-1} - 1}} \widetilde{W}_n \left((\gamma^{-1} - 1)t \right),$$

where $\widetilde{W}_n$ denotes the sequence of standard Brownian motions appearing in Lemma 3(ii), see (B.6). This sequence of Brownian motions is also the one appearing in Lemma 3(i), which is nothing but the Gaussian approximation of the tail quantile process. We omit the remaining straightforward technical details. ∎

Proof of Theorem 2. The idea is to use (B.7) in the proof of Theorem 1 together with an analogue of (B.9), with $\xi_{1-k/n}$ replaced by $\xi_{1-ks/n}$ and valid uniformly in $s \in (0,1]$. To prove such an analogue relationship, note first that

$$\frac{\xi_{1-ks/n}}{q_{1-k/n}} = \frac{\xi_{1-ks/n}}{q_{1-ks/n}} \times \frac{U(n/ks)}{U(n/k)}.$$

Recall that since $\rho < 0$, the function $t \mapsto U(t)$ is equivalent to a constant multiple of $t \mapsto t^{\gamma}$ in a neighbourhood of infinity, see p.49 of de Haan and Ferreira (2006). Using (B.3), we obtain

$$\frac{\xi_{1-ks/n}}{q_{1-k/n}} = (\gamma^{-1} - 1)^{-\gamma} s^{-\gamma}(1 + \mathrm{o}(1)) \tag{B.11}$$

uniformly in $s \in (0,1]$. Use then (B.8) with $\tau = 1 - ks/n$ to get

$$\frac{s}{1 - 2ks/n}\left(\frac{\xi_{1-ks/n}}{q_{1-k/n}} - \frac{\mathbb{E}(Y)}{q_{1-k/n}}\right) = \frac{n}{k} \times \frac{1}{q_{1-k/n}} \int_{\xi_{1-ks/n}}^{\infty} \overline{F}(u)du.$$

Use now the asymptotic equivalence $\overline{F}(q_{1-k/n}) \sim k/n$ following from Lemma 1(ii) and used inside the regularly varying function A together with Lemma 5 to obtain, for any small $\kappa > 0$,

$$\begin{aligned}
&\frac{s}{1 - 2ks/n}\left(\frac{\xi_{1-ks/n}}{q_{1-k/n}} - \frac{\mathbb{E}(Y)}{q_{1-k/n}}\right) \\
= \;& \frac{n}{k}\overline{F}(q_{1-k/n}) \left[\frac{\xi_{1-ks/n}}{q_{1-k/n}}\right]^{1-1/\gamma} \left(\frac{\gamma}{1-\gamma} + A(n/k)\frac{1}{\rho}\left[\frac{1}{1-\gamma-\rho}\left[\frac{\xi_{1-ks/n}}{q_{1-k/n}}\right]^{\rho/\gamma} - \frac{1}{1-\gamma}\right]\right) \\
+ \;& \mathrm{o}\left(A(n/k)\left[\frac{\xi_{1-ks/n}}{q_{1-k/n}}\right]^{1-(1-\rho)/\gamma+\kappa}\right)
\end{aligned}$$

uniformly in $s \in (0,1]$. According to (B.11),

$$\sup_{0<s\leqslant 1}\left[\frac{\xi_{1-ks/n}}{q_{1-k/n}}\right]^{\rho/\gamma+\kappa} \leqslant 2(\gamma^{-1}-1)^{-\rho-\kappa\gamma} \sup_{0<s\leqslant 1} s^{-\rho-\kappa\gamma} = 2(\gamma^{-1}-1)^{-\rho-\kappa\gamma} < \infty$$

for κ small enough (recall that $\rho < 0$) and n large enough. Therefore, by (B.11) again:

$$\begin{aligned}
&\frac{s}{1 - 2ks/n}\left(\frac{\xi_{1-ks/n}}{q_{1-k/n}} - \frac{\mathbb{E}(Y)}{q_{1-k/n}}\right) \\
= \;& \frac{\gamma}{1-\gamma}\left[\frac{\xi_{1-ks/n}}{q_{1-k/n}}\right]^{1-1/\gamma}\left(1 + A(n/k)\frac{\gamma^{-1}-1}{\rho}\left[\frac{(\gamma^{-1}-1)^{-\rho}s^{-\rho}}{1-\gamma-\rho} - \frac{1}{1-\gamma}\right] + \mathrm{o}\left(\frac{1}{\sqrt{k}}\right)\right)
\end{aligned}$$

uniformly in $s \in (0,1]$. Divide (B.7) by this expansion and use once again (B.11) to get:

$$\begin{aligned}
&\frac{\widetilde{\xi}_{1-ks/n} - \overline{Y}_n}{\xi_{1-ks/n} - \mathbb{E}(Y)} \\
= \ & \left[\frac{\gamma}{1-\gamma} \left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}} \right]^{1-1/\gamma} + \frac{1}{\sqrt{k}} (\gamma^{-1}-1)^{1/2-\gamma} \gamma \int_0^s W_n(t)\, t^{-\gamma-1}\, dt \right. \\
+ \ & \left. A(n/k) \frac{(\gamma^{-1}-1)^{1-\gamma} s^{1-\gamma}}{\rho} \left[\frac{(\gamma^{-1}-1)^{-\rho} s^{-\rho}}{1-\gamma-\rho} - \frac{1}{1-\gamma} \right] + \mathrm{o}_{\mathbb{P}}\left(\frac{s^{-\gamma+1/2-\varepsilon}}{\sqrt{k}} \right) \right] \\
\times \ & \frac{1-\gamma}{\gamma} \left[\frac{q_{1-k/n}}{\xi_{1-ks/n}} \right]^{1-1/\gamma} \left(1 - A(n/k) \frac{\gamma^{-1}-1}{\rho} \left[\frac{(\gamma^{-1}-1)^{-\rho} s^{-\rho}}{1-\gamma-\rho} - \frac{1}{1-\gamma} \right] + \mathrm{o}\left(\frac{1}{\sqrt{k}} \right) \right) \\
= \ & \left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} \right]^{1-1/\gamma} + \frac{1}{\sqrt{k}} \gamma \sqrt{\gamma^{-1}-1}\, s^{\gamma-1} \int_0^s W_n(t)\, t^{-\gamma-1}\, dt \\
+ \ & A(n/k) \frac{\gamma^{-1}-1}{\rho} \left[\frac{(\gamma^{-1}-1)^{-\rho} s^{-\rho}}{1-\gamma-\rho} - \frac{1}{1-\gamma} \right] \left(1 - \left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} \right]^{1-1/\gamma} \right) + \mathrm{o}_{\mathbb{P}}\left(\frac{s^{-1/2-\varepsilon}}{\sqrt{k}} \right)
\end{aligned}$$

uniformly in $s \in [k^{-1+\delta}, 1]$ and with arbitrarily large probability (here, as in the proof of Theorem 1, δ is a sufficiently small positive number to be chosen later). Define a random process $s \mapsto R_n(s)$ by the equality

$$\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} = 1 + R_n(s).$$

We know, by a combination of convergence (B.11) and Lemma 4, that $R_n(s) \xrightarrow{\mathbb{P}} 0$ uniformly in $s \in [k^{-1+\delta}, 1]$. Recalling that $\sqrt{k}A(n/k) = \mathrm{O}(1)$, the above expansion then reads

$$\begin{aligned}
&\frac{\widetilde{\xi}_{1-ks/n} - \overline{Y}_n}{\xi_{1-ks/n} - \mathbb{E}(Y)} \\
= \ & \left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} \right]^{1-1/\gamma} + \frac{1}{\sqrt{k}} \gamma \sqrt{\gamma^{-1}-1}\, s^{\gamma-1} \int_0^s W_n(t)\, t^{-\gamma-1}\, dt + \mathrm{o}_{\mathbb{P}}\left(\frac{s^{-1/2-\varepsilon}}{\sqrt{k}} \right). \qquad \text{(B.12)}
\end{aligned}$$

We now work on the left-hand side of this identity:

$$\begin{aligned}
\frac{\widetilde{\xi}_{1-ks/n} - \overline{Y}_n}{\xi_{1-ks/n} - \mathbb{E}(Y)} &= \left(\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} - 1 \right) \left(1 + \frac{\mathbb{E}(Y)}{\xi_{1-ks/n} - \mathbb{E}(Y)} \right) + \frac{\xi_{1-ks/n} - \overline{Y}_n}{\xi_{1-ks/n} - \mathbb{E}(Y)} \\
&= 1 + \left(\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-k/n}} - 1 \right) (1 + \mathrm{o}(1)) + \mathrm{O}_{\mathbb{P}}\left(\frac{1}{q_{1-k/n}\sqrt{n}} \right)
\end{aligned}$$

by asymptotic proportionality of $q_{1-k/n}$ and $\xi_{1-k/n}$ and the central limit theorem. Since moreover $\gamma < 1$, we obtain

$$\frac{\widetilde{\xi}_{1-ks/n} - \overline{Y}_n}{\xi_{1-ks/n} - \mathbb{E}(Y)} = 1 + R_n(s)\,(1 + \mathrm{o}_{\mathbb{P}}(1)) + \mathrm{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}} \right).$$

Because, uniformly in $s \in [k^{-1+\delta}, 1]$,

$$\left[\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}}\right]^{1-1/\gamma} = (1 + R_n(s))^{1-1/\gamma} = \left(1 + \left[1 - \frac{1}{\gamma}\right] R_n(s)(1 + \mathrm{o}_{\mathbb{P}}(1))\right)$$

we obtain, using (B.12) and solving for $R_n(s)$, that:

$$R_n(s) = \frac{1}{\sqrt{k}}\gamma^2\sqrt{\gamma^{-1} - 1}\, s^{\gamma-1} \int_0^s W_n(t)\, t^{-\gamma-1}\, dt + \mathrm{o}_{\mathbb{P}}\left(\frac{s^{-1/2-\varepsilon}}{\sqrt{k}}\right).$$

This is the desired result in the restricted case $s \in [k^{-1+\delta}, 1]$.

We conclude the proof by focusing on the case $s \in (0, k^{-1+\delta})$. The idea is very similar to that of the final stages of the proof of Theorem 1. Choose $\delta \in (0, \varepsilon/(2\varepsilon + 1 + 2\gamma))$: it is enough to show that

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{1/2+\varepsilon} \left|\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} - 1\right| \xrightarrow{\mathbb{P}} 0.$$

Write then

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{1/2+\varepsilon} \left|\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} - 1\right| \leqslant \sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{1/2+\varepsilon} \left\{\frac{Y_{n,n}}{\xi_{1-ks/n}}\right\} + \mathrm{o}(1).$$

Using (B.11) again, we obtain

$$\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{1/2+\varepsilon} \left|\frac{\widetilde{\xi}_{1-ks/n}}{\xi_{1-ks/n}} - 1\right| = \mathrm{O}\left(\sqrt{k} \sup_{0<s<k^{-1+\delta}} s^{\gamma+1/2+\varepsilon} \left\{\frac{Y_{n,n}}{q_{1-k/n}}\right\}\right) + \mathrm{o}(1).$$

Argue then as in the end of the proof of Theorem 1 to conclude the present proof. ■

Proof of Theorem 3. Let us start by remarking that

$$\widehat{\gamma}_k = \int_0^1 \log\left(\frac{\widehat{q}_{1-\lfloor k\rfloor s/n}}{q_{1-\lfloor k\rfloor/n}}\right) ds - \log\left(\frac{\widehat{q}_{1-\lfloor k\rfloor/n}}{q_{1-\lfloor k\rfloor/n}}\right).$$

Note that, in Theorem 1, the sequence of Brownian motions is left unchanged if k is changed into $\lfloor k\rfloor$ or $\lceil k\rceil$; this is indeed the fundamental argument behind the proof of Lemma 3(i). Set then $s_n = \lfloor k\rfloor^{-(1-\varepsilon)/(1+2\varepsilon)}$ for some sufficiently small $\varepsilon \in (0, 1/4)$, and write

$$\widehat{\gamma}_k + \log\left(\frac{\widehat{q}_{1-\lfloor k\rfloor/n}}{q_{1-\lfloor k\rfloor/n}}\right) = \int_0^{s_n} \log\left(\frac{\widehat{q}_{1-\lfloor k\rfloor s/n}}{q_{1-\lfloor k\rfloor/n}}\right) ds + \int_{s_n}^1 \log\left(\frac{\widehat{q}_{1-\lfloor k\rfloor s/n}}{q_{1-\lfloor k\rfloor/n}}\right) ds =: I_{n,1} + I_{n,2}. \qquad \text{(B.13)}$$

$I_{n,1}$ is controlled by writing

$$|I_{n,1}| \leqslant s_n \log\left(\frac{Y_{n,n}}{\widehat{q}_{1-\lfloor k\rfloor/n}}\right).$$

Using further the heavy-tailed assumption on the distribution on Y, it follows from Theorem 1.1.6, Theorem 1.2.1 and Lemma 1.2.9 in de Haan and Ferreira (2006) that

$$\frac{Y_{n,n}}{U(n)} \xrightarrow{d} 1 + \gamma G_\gamma$$

where G_γ has distribution function $x \mapsto \exp(-(1+\gamma x)^{-1/\gamma})$, for $x > -1/\gamma$. It follows that the limiting variable $1+\gamma G_\gamma$ is positive and thus $\log(Y_{n,n}/U(n)) = \mathrm{O}_{\mathbb{P}}(1)$ by the continuous mapping theorem. Besides, $\widehat{q}_{1-\lfloor k\rfloor/n}/U(n/\lfloor k\rfloor) = \widehat{q}_{1-\lfloor k\rfloor/n}/q_{1-\lfloor k\rfloor/n} \xrightarrow{\mathbb{P}} 1$, by Lemma 2(i) again. Therefore

$$\log\left(\frac{Y_{n,n}}{\widehat{q}_{1-\lfloor k\rfloor/n}}\right) = \log\left(\frac{U(n)}{U(n/\lfloor k\rfloor)}\right) + \mathrm{O}_{\mathbb{P}}(1) = \mathrm{O}_{\mathbb{P}}(\log k)$$

by Potter bounds (see *e.g.* Proposition B.1.9.5 in de Haan and Ferreira, 2006). Recalling that $s_n = k^{-(1-\varepsilon)/(1+2\varepsilon)}$ with $\varepsilon < 1/4$, it is now straightforward to get

$$\sqrt{k}|I_{n,1}| = \mathrm{O}_{\mathbb{P}}\left(s_n \times \sqrt{k}\log k\right) = \mathrm{o}_{\mathbb{P}}(1).$$

Combining then (B.13) with this convergence along with Theorem 1, a Taylor expansion of the logarithm function within $I_{n,2}$ and some straightforward calculus, we find that

$$\sqrt{k}(\widehat{\gamma}_k - \gamma) = \frac{\lambda_1}{1-\rho} + \gamma\sqrt{\gamma^{-1}-1}\left(\int_0^1 \frac{1}{s} W_n\left(\frac{s}{\gamma^{-1}-1}\right) ds - W_n\left(\frac{1}{\gamma^{-1}-1}\right)\right) + \mathrm{o}_{\mathbb{P}}(1). \quad \text{(B.14)}$$

Using Theorem 1 twice more, we can also write

$$\sqrt{k}\left(\frac{\widehat{q}_{1-k/n}}{q_{1-k/n}} - 1\right) = \gamma\sqrt{\gamma^{-1}-1}\, W_n\left(\frac{1}{\gamma^{-1}-1}\right) + \mathrm{o}_{\mathbb{P}}(1) \quad \text{(B.15)}$$

as well as

$$\sqrt{k}\left(\frac{\widetilde{\xi}_{1-k/n}}{\xi_{1-k/n}} - 1\right) = \gamma^2\sqrt{\gamma^{-1}-1}\int_0^1 W_n(t)t^{-\gamma-1}dt + \mathrm{o}_{\mathbb{P}}(1). \quad \text{(B.16)}$$

As a consequence, the random vector

$$\sqrt{k}\left(\widehat{\gamma}_k - \gamma, \frac{\widehat{q}_{1-k/n}}{q_{1-k/n}} - 1, \frac{\widetilde{\xi}_{1-k/n}}{\xi_{1-k/n}} - 1\right)$$

is asymptotically trivariate Gaussian. To complete the proof, we analyse the marginal asymptotic behaviour of each of the three components in this vector, as well as their pairwise asymptotic covariance structure.

Marginal asymptotic behaviour of $\widehat{\gamma}_k$*:* We know from Theorem 3.2.5 in de Haan and Ferreira (2006) that

$$\sqrt{k}(\widehat{\gamma}_k - \gamma) \xrightarrow{d} \mathcal{N}\left(\frac{\lambda_1}{1-\rho}, \gamma^2\right).$$

Marginal asymptotic behaviour of $\widehat{q}_{1-k/n}$*:* It is a straightforward byproduct of Equation (B.15) that

$$\sqrt{k}\left(\frac{\widehat{q}_{1-k/n}}{q_{1-k/n}} - 1\right) \xrightarrow{d} \mathcal{N}(0, \gamma^2).$$

Marginal asymptotic behaviour of $\widetilde{\xi}_{1-k/n}$*:* It is a direct consequence of Theorem 1 that

$$\sqrt{k}\left(\frac{\widetilde{\xi}_{1-k/n}}{\xi_{1-k/n}} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{2\gamma^3}{1-2\gamma}\right).$$

Asymptotic covariance structure of $(\widehat{\gamma}_k, \widehat{q}_{1-k/n})$*:* It is a consequence of the asymptotic representation of $\widehat{\gamma}_k - \gamma$ obtained in the proof of Theorem 3.2.5 in de Haan and Ferreira (2006) together with Lemma 3.2.3 therein that $\widehat{\gamma}_k - \gamma$ and $\widehat{q}_{1-k/n}/q_{1-k/n} - 1$ are asymptotically independent.

Asymptotic covariance structure of $(\widehat{\gamma}_k, \widetilde{\xi}_{1-k/n})$*:* It follows from Equations (B.14) and (B.16) that the limiting covariance of $\sqrt{k}(\widehat{\gamma}_k - \gamma, \widetilde{\xi}_{1-k/n}/\xi_{1-k/n} - 1)$ is

$$\mathcal{COV} = \gamma^2(1-\gamma)\left[\int_0^1 \int_0^1 \frac{\min((\gamma^{-1}-1)^{-1}s, t)}{s} t^{-\gamma-1} ds\, dt - \int_0^1 \min((\gamma^{-1}-1)^{-1}, t) t^{-\gamma-1} dt\right].$$

Direct computations yield

$$\mathcal{COV} = \gamma^2\left(\frac{(\gamma^{-1}-1)^{\gamma}}{(1-\gamma)^2} - 1\right) - \gamma^2\left(\frac{(\gamma^{-1}-1)^{\gamma}}{1-\gamma} - 1\right) = \frac{\gamma^3(\gamma^{-1}-1)^{\gamma}}{(1-\gamma)^2}.$$

Asymptotic covariance structure of $(\widehat{q}_{1-k/n}, \widetilde{\xi}_{1-k/n})$*:* Combining Equations (B.15) and (B.16), we find that the limiting covariance of $\sqrt{k}(\widehat{q}_{1-k/n}/q_{1-k/n} - 1, \widetilde{\xi}_{1-k/n}/\xi_{1-k/n} - 1)$ is

$$\gamma^2(1-\gamma)\int_0^1 \min(t, (\gamma^{-1}-1)^{-1}) t^{-\gamma-1} dt = \gamma^2\left(\frac{(\gamma^{-1}-1)^{\gamma}}{1-\gamma} - 1\right)$$

after some straightforward calculations.

Combining these arguments on marginal convergence and asymptotic covariance structure, we get

$$\sqrt{k}\left(\widehat{\gamma}_k - \gamma, \frac{\widehat{q}_{1-k/n}}{q_{1-k/n}} - 1, \frac{\widetilde{\xi}_{1-k/n}}{\xi_{1-k/n}} - 1\right) \xrightarrow{d} \mathcal{N}(\mathfrak{m}, \mathfrak{V})$$

with $\mathfrak{m}$ and $\mathfrak{V}$ as in the statement of Theorem 3. This concludes the proof. ∎

Proof of Theorem 4. Applying Theorem 3 and arguing as in the proof of Theorem 1 in Daouia *et al.* (2018), we get the joint convergence

$$\sqrt{k}\left(\frac{\widehat{\xi}_{1-k/n}}{\xi_{1-k/n}} - 1, \frac{\widetilde{\xi}_{1-k/n}}{\xi_{1-k/n}} - 1\right) \xrightarrow{d} \left([(1-\gamma)^{-1} - \log(\gamma^{-1}-1)]\Gamma + \Theta - \lambda,\, \Xi\right)$$

where (Γ, Θ, Ξ) is the limiting vector in Theorem 3, and

$$\lambda := \left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho} + \frac{(\gamma^{-1}-1)^{-\rho} - 1}{\rho}\right)\lambda_1 + \gamma(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)\lambda_2.$$

Then clearly

$$\sqrt{k}\left(\frac{\overline{\xi}_{1-k/n}(\beta)}{\xi_{1-k/n}} - 1\right) \xrightarrow{d} [(1-\gamma)^{-1} - \log(\gamma^{-1}-1)]\beta\Gamma + \beta\Theta + (1-\beta)\Xi - \beta\lambda.$$

Rearrange the bias component to complete the proof. ∎

Proof of Theorem 5. Define $p_n = 1 - \tau_n'$ and note that

$$\log\left(\frac{\overline{\xi}^{\star}_{1-p_n}(\beta)}{\xi_{1-p_n}}\right) = (\widehat{\gamma}_{1-k/n} - \gamma)\log\left(\frac{k}{np_n}\right) + \log\left(\frac{\overline{\xi}_{1-k/n}(\beta)}{\xi_{1-k/n}}\right) - \log\left(\left[\frac{np_n}{k}\right]^{\gamma}\frac{\xi_{1-p_n}}{\xi_{1-k/n}}\right).$$

The convergence $\log[k/(np_n)] \to \infty$ yields

$$\frac{\sqrt{k}}{\log[k/(np_n)]}\log\left(\frac{\overline{\xi}_{1-k/n}(\beta)}{\xi_{1-k/n}}\right) = \mathrm{O}_{\mathbb{P}}\left(1/\log[k/(np_n)]\right) = \mathrm{o}_{\mathbb{P}}(1) \tag{B.17}$$

and

$$\begin{aligned} &\frac{\sqrt{k}}{\log[k/(np_n)]}\log\left(\left[\frac{np_n}{k}\right]^{\gamma}\frac{\xi_{1-p_n}}{\xi_{1-k/n}}\right) \\ &= \frac{\sqrt{k}}{\log[k/(np_n)]}\left(\log\left(\frac{\xi_{1-p_n}}{q_{1-p_n}}\right) - \log\left(\frac{\xi_{1-k/n}}{q_{1-k/n}}\right) + \log\left(\left[\frac{np_n}{k}\right]^{\gamma}\frac{q_{1-p_n}}{q_{1-k/n}}\right)\right) \\ &= \mathrm{O}\left(\frac{\sqrt{k}}{\log[k/(np_n)]}\left[\frac{1}{q_{1-k/n}} + |A(n/k)|\right]\right) = \mathrm{o}(1). \end{aligned} \tag{B.18}$$

Here, convergence (B.17) is a consequence of Theorem 4. Convergence (B.18) follows from a combination of Proposition 1, Theorem 2.3.9 in de Haan and Ferreira (2006) and the regular variation of $|A|$. Combining these convergences and using the delta-method leads to the desired result. ∎

Proof of Proposition 2. Statement (i) is a clear consequence of the fact that the expectile-based ES at level τ is an increasing linear functional of the restriction of the expectile function on the interval $[\tau, 1]$, in the sense that

$$\xi_t^{(1)} \leqslant \xi_t^{(2)} \ \forall t \in [\tau, 1] \ \Rightarrow \ \mathrm{XES}_{\tau}^{(1)} := \frac{1}{1-\tau}\int_{\tau}^{1}\xi_t^{(1)}dt \leqslant \frac{1}{1-\tau}\int_{\tau}^{1}\xi_t^{(2)}dt =: \mathrm{XES}_{\tau}^{(2)}.$$

To show statement (ii), note that, for $\tau \geqslant 1/2$, XTCE_{τ} is clearly translation invariant and positive homogeneous (because so are expectiles above level $\tau \geqslant 1/2$, and conditional expectations). A simple counter-example to monotonicity and subadditivity is the following: set $\tau = 1/2$, so that

$$\mathrm{XTCE}_{1/2}(Z) = \mathbb{E}(Z \,|\, Z > \xi_{1/2}(Z)) = \mathbb{E}(Z \,|\, Z > \mathbb{E}(Z)).$$

We then actually show that $\mathrm{XTCE}_{1/2}$ is neither monotonic nor subadditive. For this, we consider a uniform random variable U on $[0, 1]$ and we set

$$X = 2\,\mathbb{1}_{\{5/6 \leqslant U < 1\}} \quad \text{and} \quad Y = \mathbb{1}_{\{1/2 \leqslant U < 5/6\}} + 2\,\mathbb{1}_{\{5/6 \leqslant U < 1\}}.$$

Then clearly $X \leqslant Y$ with probability 1, and X and Y are discrete variables taking values in the set $\{0, 1, 2\}$, with $\mathbb{E}(X) = \mathbb{E}(X\mathbb{1}_{\{X>0\}}) = 1/3$ and $\mathbb{E}(Y) = \mathbb{E}(Y\mathbb{1}_{\{Y>0\}}) = 2/3$. As such

$$\begin{aligned} &\mathbb{E}(X \,|\, X > \mathbb{E}(X)) = \mathbb{E}(X \,|\, X > 0) = 2 \\ \text{and} \quad &\mathbb{E}(Y \,|\, Y > \mathbb{E}(Y)) = \mathbb{E}(Y \,|\, Y > 0) = \frac{2/3}{1/2} = \frac{4}{3}. \end{aligned}$$

This establishes that $\mathbb{E}(Y \,|\, Y > \mathbb{E}(Y)) < \mathbb{E}(X \,|\, X > \mathbb{E}(X))$: $\text{XTCE}_{1/2}$ is not a monotonic risk measure. Besides,

$$X + Y = \mathbb{1}_{\{1/2 \leqslant U < 5/6\}} + 4\,\mathbb{1}_{\{5/6 \leqslant U < 1\}}$$

so that $\mathbb{E}(X+Y) = \mathbb{E}([X+Y]\mathbb{1}_{\{X+Y>0\}}) = 1$ and then

$$\mathbb{E}(X+Y \,|\, X+Y > \mathbb{E}(X+Y)) = \mathbb{E}(X+Y \,|\, X+Y > 1) = 4.$$

This shows that $\mathbb{E}(X+Y \,|\, X+Y > \mathbb{E}(X+Y)) > \mathbb{E}(X \,|\, X > \mathbb{E}(X)) + \mathbb{E}(Y \,|\, Y > \mathbb{E}(Y))$, proving that $\text{XTCE}_{1/2}$ is not a subadditive risk measure either. ■

Proof of Proposition 3. It follows from the asymptotic proportionality relationship $\xi_\tau/q_\tau \sim (\gamma^{-1}-1)^{-\gamma}$ as $\tau \to 1$ (see Bellini and Di Bernardino, 2017) that

$$\text{XES}_\tau = \frac{1}{1-\tau}\int_\tau^1 \xi_\alpha \, d\alpha = (\gamma^{-1}-1)^{-\gamma}\left\{\frac{1}{1-\tau}\int_\tau^1 q_\alpha(1+r(\alpha))d\alpha\right\}$$

where $r(\alpha) \to 0$ as $\alpha \to 1$. It is then clear that

$$\text{XES}_\tau \sim (\gamma^{-1}-1)^{-\gamma}\left\{\frac{1}{1-\tau}\int_\tau^1 q_\alpha \, d\alpha\right\} = (\gamma^{-1}-1)^{-\gamma}\text{QES}_\tau \quad \text{as} \quad \tau \to 1.$$

This proves that

$$\frac{\text{XES}_\tau}{\text{QES}_\tau} \sim (\gamma^{-1}-1)^{-\gamma} \sim \frac{\xi_\tau}{q_\tau} \quad \text{as} \quad \tau \to 1,$$

by asymptotic proportionality again. Besides, the equality $q_\alpha = U((1-\alpha)^{-1})$ and a change of variables entail

$$\frac{\text{QES}_\tau}{q_\tau} = \frac{1}{1-\tau}\int_\tau^1 \frac{q_\alpha}{q_\tau} d\alpha = \int_1^\infty y^{-1}\frac{U((1-\tau)^{-1}y)}{U((1-\tau)^{-1})}\frac{dy}{y}.$$

The condition $\gamma < 1$ and a uniform convergence theorem such as Proposition B.1.10 in de Haan and Ferreira (2006, p.360) entail

$$\frac{\text{QES}_\tau}{q_\tau} \to \int_1^\infty y^{\gamma-2}dy = \frac{1}{1-\gamma} \quad \text{as} \quad \tau \to 1.$$

Consequently

$$\frac{\text{XES}_\tau}{\xi_\tau} \sim \frac{\text{QES}_\tau}{q_\tau} \to \frac{1}{1-\gamma} \quad \text{as} \quad \tau \to 1.$$

Let us now turn to the terms $\text{XTCE}_\tau/\text{QTCE}_\tau$ and $\text{XTCE}_\tau/\xi_\tau$. On the one hand, we have

$$\text{XTCE}_\tau = \frac{\mathbb{E}\left[(Y-\xi_\tau)_+\right]}{\overline{F}(\xi_\tau)} + \xi_\tau \quad \text{and} \quad \text{QTCE}_\tau = \frac{\mathbb{E}\left[(Y-q_\tau)_+\right]}{\overline{F}(q_\tau)} + q_\tau,$$

where $y_+ = \max(y,0)$. On the other hand, it follows from the proof of Theorem 11 in Bellini *et al.* (2014) that

$$\frac{\mathbb{E}\left[(Y-t)_+\right]}{\overline{F}(t)} \sim \frac{t}{\gamma^{-1}-1} \quad \text{as} \quad t \to \infty.$$

Therefore

$$\frac{\text{XTCE}_\tau}{\xi_\tau} \sim \frac{1}{1-\gamma} \quad \text{and} \quad \frac{\text{QTCE}_\tau}{q_\tau} \sim \frac{1}{1-\gamma} \quad \text{as} \quad \tau \to 1.$$

Whence $\text{XTCE}_\tau/\text{QTCE}_\tau \sim \xi_\tau/q_\tau$ as $\tau \to 1$, which completes the proof. ■

Proof of Proposition 4. The starting point to show the first expansion is Proposition 1(i), which yields

$$\begin{aligned}
\mathrm{XES}_\tau &= \frac{1}{1-\tau}\int_\tau^1 \xi_\alpha\, d\alpha \\
&= (\gamma^{-1}-1)^{-\gamma}\Bigg(\mathrm{QES}_\tau + \gamma(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)(1+\mathrm{o}(1)) \\
&+ \left\{\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho} + \frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho} + \mathrm{o}(1)\right\}\frac{1}{1-\tau}\int_\tau^1 q_\alpha A((1-\alpha)^{-1})d\alpha\Bigg).
\end{aligned}$$

Use a change of variables to get

$$\frac{1}{1-\tau}\int_\tau^1 q_\alpha A((1-\alpha)^{-1})d\alpha = U((1-\tau)^{-1})A((1-\tau)^{-1})\int_1^\infty y^{-1}\frac{U((1-\tau)^{-1}y)A((1-\tau)^{-1}y)}{U((1-\tau)^{-1})A((1-\tau)^{-1})}\frac{dy}{y}.$$

This entails, using a uniform convergence theorem such as Proposition B.1.10 in de Haan and Ferreira (2006, p.360), that

$$\begin{aligned}
\frac{1}{1-\tau}\int_\tau^1 q_\alpha A((1-\alpha)^{-1})d\alpha &\sim U((1-\tau)^{-1})A((1-\tau)^{-1})\int_1^\infty y^{\gamma+\rho-2}dy \quad \text{as} \quad \tau\to 1 \\
&= \frac{q_\tau A((1-\tau)^{-1})}{1-\gamma-\rho}.
\end{aligned}$$

Since $\mathrm{QES}_\tau \sim q_\tau/(1-\gamma)$, our earlier expansion yields

$$\begin{aligned}
\frac{\mathrm{XES}_\tau}{\mathrm{QES}_\tau} &= (\gamma^{-1}-1)^{-\gamma}\Bigg(1 + \frac{\gamma(1-\gamma)(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)}{q_\tau}(1+\mathrm{o}(1)) \\
&+ \left\{\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho} + \frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho} + \mathrm{o}(1)\right\}\frac{1-\gamma}{1-\gamma-\rho}A((1-\tau)^{-1})\Bigg). \qquad \text{(B.19)}
\end{aligned}$$

Furthermore, it is a consequence of a uniform inequality such as Theorem 2.3.9 in de Haan and Ferreira (2006) applied to the function U that

$$\begin{aligned}
\frac{\mathrm{QES}_\tau}{q_\tau} &= \int_1^\infty y^{-1}\frac{U((1-\tau)^{-1}y)}{U((1-\tau)^{-1})}\frac{dy}{y} \\
&= \int_1^\infty y^{-1}\left(y^\gamma + A((1-\tau)^{-1})y^\gamma\frac{y^\rho-1}{\rho}(1+\mathrm{o}(1))\right)\frac{dy}{y} \\
&= \int_1^\infty y^{\gamma-2}dy + \frac{A((1-\tau)^{-1})}{\rho}\int_1^\infty \left(y^{\gamma+\rho-2}-y^{\gamma-2}\right)dy(1+\mathrm{o}(1)) \\
&= \frac{1}{1-\gamma}\left(1 + \frac{1}{1-\gamma-\rho}A((1-\tau)^{-1})(1+\mathrm{o}(1))\right). \qquad \text{(B.20)}
\end{aligned}$$

Finally, Proposition 1(i) reads

$$\begin{aligned}
\frac{q_\tau}{\xi_\tau} &= (\gamma^{-1}-1)^{\gamma}\Bigg(1 - \frac{\gamma(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)}{q_\tau}(1+\mathrm{o}(1)) \\
&- \left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\gamma-\rho} + \frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho} + \mathrm{o}(1)\right)A((1-\tau)^{-1})\Bigg). \qquad \text{(B.21)}
\end{aligned}$$

A use of the identity

$$\frac{\text{XES}_\tau}{\xi_\tau} = \frac{\text{XES}_\tau}{\text{QES}_\tau} \times \frac{\text{QES}_\tau}{q_\tau} \times \frac{q_\tau}{\xi_\tau}$$

and a combination of (B.19), (B.20) and (B.21) complete the proof after some straightforward computations. ■

Proof of Theorem 6. By Theorem 2:

$$\begin{aligned}\frac{\widetilde{\text{XES}}_{1-k/n}}{\text{XES}_{1-k/n}} - 1 &= \frac{1}{\sqrt{k}}\gamma^2\sqrt{\gamma^{-1}-1} \times \frac{\int_0^1 (\int_0^s W_n(t)t^{-\gamma-1}\,dt)s^{\gamma-1}\xi_{1-ks/n}\,ds}{\int_0^1 \xi_{1-ks/n}\,ds} \\ &+ \text{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}} \times \frac{\int_0^1 s^{-1/2-\varepsilon}\xi_{1-ks/n}\,ds}{\int_0^1 \xi_{1-ks/n}\,ds}\right).\end{aligned}$$

Using (B.11) and the fact that $\gamma < 1/2$, we obtain:

$$\frac{\widetilde{\text{XES}}_{1-k/n}}{\text{XES}_{1-k/n}} - 1 = \frac{1}{\sqrt{k}}(\gamma[1-\gamma])^{3/2}\int_0^1 \left(\int_0^s W_n(t)t^{-\gamma-1}\,dt\right)\frac{ds}{s} + \text{o}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\right).$$

Denoting by W a standard Brownian motion, we get, using an integration by parts, that:

$$\sqrt{k}\left(\frac{\widetilde{\text{XES}}_{1-k/n}}{\text{XES}_{1-k/n}} - 1\right) \stackrel{d}{\longrightarrow} (\gamma[1-\gamma])^{3/2}\int_0^1 W(s)s^{-\gamma-1}\log(s)\,ds.$$

Since the rhs above is a centred Gaussian random variable, it only remains to compute its variance, which is

$$v = \gamma^3(1-\gamma)^3\int_0^1\int_0^1 \min(s,t)s^{-\gamma-1}t^{-\gamma-1}\log(s)\log(t)\,ds\,dt.$$

It then follows from straightforward but lengthy computations that

$$v = \frac{2\gamma^3(1-\gamma)(3-4\gamma)}{(1-2\gamma)^3}$$

as required. ■

Proof of Theorem 7. The proof of this result is entirely similar to that of Theorem 5 (applying Theorem 6 instead of Theorem 4, and Proposition 4 instead of Proposition 1). We omit the details. ■

Proof of Theorem 8. We examine first the convergence of $\overline{\text{XES}}^\star_{1-p_n}(\beta)$. Define $p_n = 1-\tau_n'$ and write

$$\log\left(\frac{\overline{\text{XES}}^\star_{1-p_n}(\beta)}{\text{XES}_{1-p_n}}\right) = \log\left(\frac{\overline{\xi}^\star_{1-p_n}(\beta)}{\xi_{1-p_n}}\right) + \log\left(\frac{[1-\widehat{\gamma}_{1-k/n}]^{-1}}{[1-\gamma]^{-1}}\right) - \log\left(\frac{\text{XES}_{1-p_n}}{[1-\gamma]^{-1}\xi_{1-p_n}}\right).$$

By Theorem 5 and the delta-method,

$$\frac{\sqrt{k}}{\log[k/(np_n)]}\log\left(\frac{\overline{\xi}^{\star}_{1-p_n}(\beta)}{\xi_{1-p_n}}\right)\xrightarrow{d}\mathcal{N}\left(\frac{\lambda_1}{1-\rho},\gamma^2\right). \tag{B.22}$$

Using then Theorem 3.2.5 in de Haan and Ferreira (2006), the delta-method and the convergence $\log[k/(np_n)]\to\infty$, we get

$$\frac{\sqrt{k}}{\log[k/(np_n)]}\log\left(\frac{[1-\widehat{\gamma}_{1-k/n}]^{-1}}{[1-\gamma]^{-1}}\right)\xrightarrow{\mathbb{P}}0. \tag{B.23}$$

Using finally a combination of Proposition 1(i), Proposition 4 and the regular variation of $|A|$ and $t\mapsto q_{1-t^{-1}}$, we obtain

$$\frac{\sqrt{k}}{\log[k/(np_n)]}\log\left(\frac{\mathrm{XES}_{1-p_n}}{[1-\gamma]^{-1}\xi_{1-p_n}}\right)\to 0. \tag{B.24}$$

Combining convergences (B.22), (B.23) and (B.24), it follows that

$$\frac{\sqrt{k}}{\log[k/(np_n)]}\log\left(\frac{\overline{\mathrm{XES}}^{\star}_{1-p_n}(\beta)}{\mathrm{XES}_{1-p_n}}\right)\xrightarrow{d}\mathcal{N}\left(\frac{\lambda_1}{1-\rho},\gamma^2\right).$$

Another use of the delta-method completes the proof of the convergence of $\overline{\mathrm{XES}}^{\star}_{1-p_n}(\beta)$.

We now show the convergence of $\widehat{\mathrm{XES}}^{\star}_{1-p_n}(\beta)$. For this we write

$$\begin{aligned}\log\left(\frac{\widehat{\mathrm{XES}}^{\star}_{1-p_n}(\beta)}{\mathrm{XES}_{1-p_n}}\right) &= \log\left(\frac{\overline{\xi}^{\star}_{1-p_n}(\beta)}{\xi_{1-p_n}}\right)+\log\left(\frac{\widehat{\mathrm{QES}}_{1-k/n}}{\widehat{q}_{1-k/n}}\cdot\frac{q_{1-k/n}}{\mathrm{QES}_{1-k/n}}\right)\\ &\quad+\log\left(\frac{\mathrm{QES}_{1-k/n}}{q_{1-k/n}}\right)-\log\left(\frac{\mathrm{XES}_{1-p_n}}{\xi_{1-p_n}}\right)\end{aligned}$$

where we set

$$\widehat{\mathrm{QES}}_{1-k/n}:=\frac{1}{\lfloor k\rfloor}\sum_{i=1}^{\lfloor k\rfloor}Y_{n-i+1,n}=\int_0^1\widehat{q}_{1-\lfloor k\rfloor s/n}\,ds.$$

Remark now that, since $\widehat{q}_{1-\lfloor k\rfloor/n}=Y_{n-\lfloor k\rfloor,n}=\widehat{q}_{1-k/n}$, we have

$$\log\left(\frac{\widehat{\mathrm{QES}}_{1-k/n}}{\widehat{q}_{1-k/n}}\cdot\frac{q_{1-k/n}}{\mathrm{QES}_{1-k/n}}\right)=\log\left(\int_0^1\frac{\widehat{q}_{1-\lfloor k\rfloor s/n}}{\widehat{q}_{1-\lfloor k\rfloor/n}}ds\right)-\log\left(\frac{\mathrm{QES}_{1-k/n}}{q_{1-k/n}}\right).$$

Combine then Theorem 1, the delta-method, and (B.20) together with a Taylor expansion to obtain

$$\frac{\sqrt{k}}{\log[k/(np_n)]}\log\left(\frac{\widehat{\mathrm{QES}}_{1-k/n}}{\widehat{q}_{1-k/n}}\cdot\frac{q_{1-k/n}}{\mathrm{QES}_{1-k/n}}\right)=\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\log[k/(np_n)]}\right)=\mathrm{o}_{\mathbb{P}}(1). \tag{B.25}$$

Besides, a combination of Equation (B.20) and Proposition 4 with a Taylor expansion yields

$$\begin{aligned}&\frac{\sqrt{k}}{\log[k/(np_n)]}\left[\log\left(\frac{\mathrm{QES}_{1-k/n}}{q_{1-k/n}}\right)-\log\left(\frac{\mathrm{XES}_{1-p_n}}{\xi_{1-p_n}}\right)\right]\\ &=\mathrm{O}\left(\frac{\sqrt{k}}{\log[k/(np_n)]}\left[\frac{1}{q_{1-k/n}}+|A(n/k)|\right]\right)=\mathrm{o}(1).\end{aligned} \tag{B.26}$$

Finally, use together (B.22), (B.25) and (B.26) and the delta-method to complete the proof. ■

Proof of Theorem 9. We only show the result for $\widetilde{\text{XES}}^{\star}_{\widehat{\tau}'_n(p_n)}$ as the proofs of the other convergences are entirely similar. The key point is to write

$$\widetilde{\text{XES}}^{\star}_{\widehat{\tau}'_n(p_n)} = \left(\frac{1-\widehat{\tau}'_n(p_n)}{1-\tau'_n(p_n)} \right)^{-\widehat{\gamma}_{\tau_n}} \widetilde{\text{XES}}^{\star}_{\tau'_n(p_n)}. \tag{B.27}$$

It is, moreover, shown as part of the proof of Theorem 6 in Daouia *et al.* (2018) that

$$\frac{1-\widehat{\tau}'_n(p_n)}{1-\tau'_n(p_n)} = 1 + \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}} \right)$$

(combine (B.52), (B.53), (B.54) and (B.55) in the supplementary material document of [Daouia *et al.*, 2018], noting that the strict monotonicity of F_Y is not required thanks to Proposition 1(i) in the present paper; this also results in a corrected version of (B.51) in the former paper). Therefore, by the $\sqrt{n(1-\tau_n)}$−convergence of $\widehat{\gamma}_{\tau_n}$,

$$\left(\frac{1-\widehat{\tau}'_n(p_n)}{1-\tau'_n(p_n)} \right)^{-\widehat{\gamma}_{\tau_n}} = 1 + \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{n(1-\tau_n)}} \right). \tag{B.28}$$

Furthermore, using Proposition 5, we conclude that the conditions of Theorem 7 are satisfied if the parameter τ'_n there is set equal to $\tau'_n(p_n)$. By Theorem 7 then:

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n(p_n))]} \left(\frac{\widetilde{\text{XES}}^{\star}_{\tau'_n(p_n)}}{\text{XES}_{\tau'_n(p_n)}} - 1 \right) \xrightarrow{d} \mathcal{N}\left(\frac{\lambda_1}{1-\rho}, \gamma^2 \right).$$

Now

$$\log\left[\frac{1-\tau_n}{1-\tau'_n(p_n)} \right] = \log\left[\frac{1-\tau_n}{1-p_n} \right] + \log\left[\frac{1-p_n}{1-\tau'_n(p_n)} \right]$$

and in the right-hand side of this identity, the first term tends to infinity, while the second term converges to a finite constant in view of Proposition 5. As a conclusion

$$\log\left[\frac{1-\tau_n}{1-\tau'_n(p_n)} \right] \sim \log\left[\frac{1-\tau_n}{1-p_n} \right].$$

Hence the convergence

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-p_n)]} \left(\frac{\widetilde{\text{XES}}^{\star}_{\tau'_n(p_n)}}{\text{XES}_{\tau'_n(p_n)}} - 1 \right) \xrightarrow{d} \mathcal{N}\left(\frac{\lambda_1}{1-\rho}, \gamma^2 \right). \tag{B.29}$$

We conclude the proof by writing

$$\text{XES}_{\tau'_n(p_n)} = \text{QES}_{p_n} \times \left\{ (1-\gamma) \frac{\text{XES}_{\tau'_n(p_n)}}{\xi_{\tau'_n(p_n)}} \right\} \times \left\{ (1-\gamma) \frac{\text{QES}_{p_n}}{q_{p_n}} \right\}^{-1}$$

(since $\xi_{\tau'_n(p_n)} \equiv q_{p_n}$ by definition). By a combination of Propositions 4 and 5 with the regular variation of the functions $|A|$ and $t \mapsto q_{1-t^{-1}}$, one gets

$$(1-\gamma) \frac{\text{XES}_{\tau'_n(p_n)}}{\xi_{\tau'_n(p_n)}} = 1 + \mathrm{o}\left(\frac{\log[(1-\tau_n)/(1-p_n)]}{\sqrt{n(1-\tau_n)}} \right).$$

Similarly and by (B.20),

$$(1-\gamma)\frac{\mathrm{QES}_{p_n}}{q_{p_n}} = 1 + \mathrm{o}\left(\frac{\log[(1-\tau_n)/(1-p_n)]}{\sqrt{n(1-\tau_n)}}\right).$$

Therefore

$$\frac{\mathrm{XES}_{\tau_n'(p_n)}}{\mathrm{QES}_{p_n}} - 1 = \mathrm{o}\left(\frac{\log[(1-\tau_n)/(1-p_n)]}{\sqrt{n(1-\tau_n)}}\right).$$

Together with (B.29), this entails

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-p_n)]}\left(\frac{\widetilde{\mathrm{XES}}^{\star}_{\tau_n'(p_n)}}{\mathrm{QES}_{p_n}} - 1\right) \xrightarrow{d} \mathcal{N}\left(\frac{\lambda_1}{1-\rho}, \gamma^2\right). \tag{B.30}$$

Combining (B.27), (B.28) and (B.30) completes the proof. ∎

Appendix: Preliminary results and their proofs

The first preliminary lemma, which we will use to show Proposition 1, is a technical result on second-order regular variation that seems to be informally known in the literature. We prove it for the sake of completeness.

Lemma 1. *Assume that condition $\mathcal{C}_2(\gamma,\rho,A)$ holds. Then we have the following two convergences:*

(i) $\displaystyle\lim_{t\to\infty}\frac{1}{A(1/\overline{F}(t))}\left(\frac{U(1/\overline{F}(t))}{t} - 1\right) = 0;$

(ii) $\displaystyle\lim_{t\to\infty}\frac{1}{A(t)}\left(\frac{1/\overline{F}(U(t))}{t} - 1\right) = 0.$

Proof of Lemma 1. The proof of this lemma is based on that of Theorem B.3.19 in de Haan and Ferreira (2006). We only show (i), the proof of (ii) being entirely similar. Recall that

$$U(t) = \inf\{x \,|\, 1/\overline{F}(x) \geqslant t\}$$

so that $U(1/\overline{F}(t)) \leqslant t$. Furthermore, condition $\mathcal{C}_2(\gamma,\rho,A)$ is nothing but second-order extended regular variation in the sense of convergence (B.3.3) in de Haan and Ferreira (2006), which is known to be locally uniform in $x \in (0,\infty)$ (see Remark B.3.8.1 in de Haan and Ferreira, 2006). Pick $\varepsilon \in \mathbb{R}$ arbitrarily close to 0: by using condition $\mathcal{C}_2(\gamma,\rho,A)$ with t replaced by $1/\overline{F}(t)$ and $x = 1+\varepsilon A(1/\overline{F}(t))$, $t\to\infty$, we get

$$\lim_{t\to\infty}\frac{1}{A(1/\overline{F}(t))}\left[\frac{U([1+\varepsilon A(1/\overline{F}(t))]/\overline{F}(t))}{U(1/\overline{F}(t))} - (1+\varepsilon A(1/\overline{F}(t)))^{\gamma}\right] = 0$$

or equivalently

$$\lim_{t\to\infty}\frac{1}{A(1/\overline{F}(t))}\left[\frac{U([1+\varepsilon A(1/\overline{F}(t))]/\overline{F}(t))}{U(1/\overline{F}(t))} - 1\right] = \gamma\varepsilon.$$

Assume that A is positive and take $\varepsilon > 0$; the proof in the other case is similar by taking $\varepsilon < 0$ instead. Using the definition of U again, we find that $U([1+\varepsilon A(1/\overline{F}(t))]/\overline{F}(t)) \geqslant t$, and thus

$$0 \leqslant \liminf_{t\to\infty} \frac{1}{A(1/\overline{F}(t))}\left(\frac{t}{U(1/\overline{F}(t))} - 1\right) \leqslant \limsup_{t\to\infty} \frac{1}{A(1/\overline{F}(t))}\left(\frac{t}{U(1/\overline{F}(t))} - 1\right) \leqslant \gamma\varepsilon.$$

Let $\varepsilon \downarrow 0$ to complete the proof. ■

The second lemma is a generalisation of the weighted approximation of the tail empirical quantile process tailored to our purpose. Its main contribution is to give a precise representation of the Gaussian term that is of independent interest, for example when evaluating the correlation between two quantiles or expectiles at different orders.

Lemma 2. *Suppose that condition $\mathcal{C}_2(\gamma,\rho,A)$ holds. Let $k = k(n) \to \infty$ be a positive sequence such that $k/n \to 0$ and $\sqrt{k}A(n/k) = \mathrm{O}(1)$. Then, subject to a potential enlargement of the underlying probability space and to choosing a suitable version of the empirical process $\widehat{F}_n$, there exists a sequence $W_n = W_n^{(k)}$ of standard Brownian motions such that, for any $\varepsilon > 0$ sufficiently small:*

(i) We have

$$\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}} = s^{-\gamma} + \frac{1}{\sqrt{k}}\left(\gamma s^{-\gamma-1} W_n(s) + \sqrt{k}A(n/k) s^{-\gamma}\frac{s^{-\rho}-1}{\rho} + s^{-\gamma-1/2-\varepsilon}\, \mathrm{o}_{\mathbb{P}}(1)\right)$$

uniformly in $s \in (0,1]$.

(ii) If $\widehat{\overline{F}}_n(u) = n^{-1}\sum_{i=1}^n \mathbb{1}_{\{Y_i > u\}}$ is the empirical survival function of the Y_i, we have

$$\frac{n}{k}\widehat{\overline{F}}_n\left(xq_{1-k/n}\right) - x^{-1/\gamma} = \frac{1}{\sqrt{k}}\left(W_n(x^{-1/\gamma}) + \sqrt{k}A(n/k)x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho} + x^{(\varepsilon-1/2)/\gamma}\, \mathrm{o}_{\mathbb{P}}(1)\right)$$

uniformly on half-lines of the form $[x_0,\infty)$, for $x_0 > 0$.

Moreover, the sequence W_n can be chosen as $W_n(s) = W_n^{(k)}(s) = \sqrt{n/k}\,\overline{W}_n(ks/n)$, where $\overline{W}_n$ is a sequence of Brownian motions which is fixed across all possible choices of k.

Proof of Lemma 2. Note that (i) is exactly Theorem 2.4.8 in de Haan and Ferreira (2006), recalling that the function A_0 therein is asymptotically equivalent to A, in the case when k is a sequence of integers. If the sequence k is not a sequence of integers, the result can easily be proven by noting that

$$\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}} \leqslant \frac{\widehat{q}_{1-\lfloor k\rfloor s/n}}{q_{1-\lfloor k\rfloor/n}}\left(\frac{q_{1-\lfloor k\rfloor/n}}{q_{1-k/n}} - 1\right) + \frac{\widehat{q}_{1-\lfloor k\rfloor s/n}}{q_{1-\lfloor k\rfloor/n}}$$

for n sufficiently large and by using local uniformity of condition $\mathcal{C}_2(\gamma,\rho,A)$ (see *e.g.* Theorem 2.3.9 in de Haan and Ferreira, 2006) as well as the regular variation property of $|A|$. We omit the details.

Besides, if the sequence k is made of integers, there is a sequence $\widetilde{W}_n$ of (potentially different) Brownian motions such that, for a suitable version of the empirical process $\widehat{F}_n$:

$$\frac{n}{k}\widehat{\overline{F}}_n\left(xq_{1-k/n}\right) - x^{-1/\gamma} = \frac{1}{\sqrt{k}}\left(\widetilde{W}_n(x^{-1/\gamma}) + \sqrt{k}A(n/k)x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho} + x^{(\varepsilon-1/2)/\gamma}\,\mathrm{o}_{\mathbb{P}}(1)\right)$$

uniformly on half-lines of the form $[x_0, \infty)$. This follows from Theorem 5.1.4 in de Haan and Ferreira (2006). The adaptation of this expansion to an arbitrary sequence k is then also straightforward.

To prove the Lemma, it remains to show that $\widetilde{W}_n$ can be taken equal to W_n, and that the latter can be chosen as indicated in the final statement. Work throughout with the above version of $\widehat{F}_n$, and denote by $s \mapsto \widehat{q}_{1-ks/n}$ the related tail quantile process. Our goal is to show that for any $\eta, \delta > 0$, we have, for n large enough,

$$\mathbb{P}\left(\sup_{0<s\leqslant 1} s^{\gamma+1/2+\varepsilon}\left|\sqrt{k}\left(\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}} - s^{-\gamma}\right) - \gamma s^{-\gamma-1}\widetilde{W}_n(s) - \sqrt{k}A(n/k)s^{-\gamma}\frac{s^{-\rho}-1}{\rho}\right| > \eta\right) < \delta.$$

First note that, by the triangle inequality and self-similarity of the Brownian motion $\widetilde{W}_n$, one can choose $a > 0$ so small that

$$\mathbb{P}\left(\sup_{0<s\leqslant a} s^{\gamma+1/2+\varepsilon}\left|\gamma s^{-\gamma-1}\widetilde{W}_n(s) + \sqrt{k}A(n/k)s^{-\gamma}\frac{s^{-\rho}-1}{\rho}\right| > \frac{\eta}{4}\right) < \frac{\delta}{4}. \tag{B.31}$$

Using statement (i) together with the triangle inequality, and repeating exactly the same argument, we obtain that we can choose $a > 0$ so small that

$$\mathbb{P}\left(\sup_{0<s\leqslant a} s^{\gamma+1/2+\varepsilon}\left|\sqrt{k}\left(\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}} - s^{-\gamma}\right)\right| > \frac{\eta}{4}\right) < \frac{\delta}{4}. \tag{B.32}$$

Combining (B.31) and (B.32) results, for such a choice of $a > 0$, in the inequality

$$\mathbb{P}\left(\sup_{0<s\leqslant a} s^{\gamma+1/2+\varepsilon}\left|\sqrt{k}\left(\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}} - s^{-\gamma}\right) - \gamma s^{-\gamma-1}\widetilde{W}_n(s) - \sqrt{k}A(n/k)s^{-\gamma}\frac{s^{-\rho}-1}{\rho}\right| > \frac{\eta}{2}\right) < \frac{\delta}{2}.$$

Noting that $s^{\gamma+1/2+\varepsilon} \geqslant a^{\gamma+1/2+\varepsilon}$ on $(0, a]$, it is therefore sufficient to show that for any $a > 0$

$$\sup_{a\leqslant s\leqslant 1}\left|\sqrt{k}\left(\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}} - s^{-\gamma}\right) - \gamma s^{-\gamma-1}\widetilde{W}_n(s) - \sqrt{k}A(n/k)s^{-\gamma}\frac{s^{-\rho}-1}{\rho}\right| = \mathrm{o}_{\mathbb{P}}(1). \tag{B.33}$$

By statement (i), we have

$$\sqrt{k}\sup_{a\leqslant s\leqslant 1}\left|\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}} - s^{-\gamma}\right| = \mathrm{O}_{\mathbb{P}}(1).$$

Set then $x = x_n(s) = \widehat{q}_{1-ks/n}/q_{1-k/n}$ in the approximation of $\widehat{\overline{F}}_n\left(xq_{1-k/n}\right)$ to get, uniformly in $s \in [a, 1]$,

$$\begin{aligned} &\frac{\lfloor ks\rfloor}{k} - [x_n(s)]^{-1/\gamma} \\ &= \frac{1}{\sqrt{k}}\left(\widetilde{W}_n([x_n(s)]^{-1/\gamma}) + \sqrt{k}A(n/k)[x_n(s)]^{-1/\gamma}\frac{[x_n(s)]^{\rho/\gamma}-1}{\gamma\rho} + [x_n(s)]^{(\varepsilon-1/2)/\gamma}\,\mathrm{o}_{\mathbb{P}}(1)\right). \end{aligned}$$

By the uniform convergence of $x_n(s)$ to $s^{-\gamma}$ on $[a,1]$, as well as the continuity properties of Brownian motion, this entails

$$\left(\frac{\widehat{q}_{1-ks/n}}{q_{1-k/n}}\right)^{-1/\gamma} = [x_n(s)]^{-1/\gamma} = s - \frac{1}{\sqrt{k}}\left(\widetilde{W}_n(s) + \sqrt{k}A(n/k)s\frac{s^{-\rho}-1}{\gamma\rho} + \mathrm{o}_{\mathbb{P}}(1)\right)$$

uniformly in $s \in [a,1]$. A Taylor expansion then shows (B.33).

That W_n can be chosen as indicated in the final statement can be shown as follows. Proposition 2.4.9 in de Haan and Ferreira (or equivalently, Theorem 6.2.1 in Csörgő and Horváth, 1993) yields that, for a suitable choice of an independent sequence $(Z_i)_{i\geqslant 1}$ of unit Pareto random variables, there is a sequence of Brownian bridges B_n such that

$$\sup_{1/(n+1)\leqslant t\leqslant n/(n+1)} n^{\varepsilon}t^{\varepsilon-1/2}(1-t)^{\varepsilon-1/2}\left|\sqrt{n}(1-t)^{\gamma+1}\left(\frac{Z_{\lceil nt\rceil,n}^{\gamma}-1}{\gamma} - \frac{(1-t)^{-\gamma}-1}{\gamma}\right) - B_n(t)\right|$$

is stochastically bounded. Setting $t = 1 - ks/n$ and rearranging yields in particular that

$$\sup_{k^{-1}\leqslant s\leqslant 1} s^{\gamma+1/2+\varepsilon}\left|\sqrt{k}\left(\frac{(\frac{k}{n}Z_{n-\lfloor ks\rfloor,n})^{\gamma}-1}{\gamma} - \frac{s^{-\gamma}-1}{\gamma}\right) - \sqrt{\frac{n}{k}}\,s^{-\gamma-1}B_n(1-ks/n)\right| = \mathrm{o}_{\mathbb{P}}(1).$$

Set now $\overline{B}_n(t) = B_n(1-t)$, which makes $\overline{B}_n$ a sequence of Brownian bridges as well, and let $\overline{W}_n$ be any sequence of Brownian motions such that $\overline{B}_n(t) = \overline{W}_n(t) - t\overline{W}_n(1)$ (for instance, $\overline{W}_n(t) = \overline{B}_n(t) + tV_n$, where for each n, V_n is a standard Gaussian random variable independent of the process $\overline{B}_n$). Note that the sequence $\overline{W}_n$ is constructed independently of k. We have

$$\sqrt{\frac{n}{k}}\,s^{-\gamma-1}B_n(1-ks/n) = \sqrt{\frac{n}{k}}\,s^{-\gamma-1}\overline{W}_n(ks/n) - \sqrt{\frac{n}{k}}\,s^{-\gamma-1} \times \frac{ks}{n}\overline{W}_n(1)$$

and clearly

$$\sup_{k^{-1}\leqslant s\leqslant 1} s^{\gamma+1/2+\varepsilon}\left|\sqrt{\frac{n}{k}}\,s^{-\gamma-1} \times \frac{ks}{n}\overline{W}_n(1)\right| = \left|\sqrt{\frac{k}{n}}\overline{W}_n(1)\right| = \mathrm{O}_{\mathbb{P}}\left(\sqrt{\frac{k}{n}}\right) = \mathrm{o}_{\mathbb{P}}(1).$$

It follows that

$$\sup_{k^{-1}\leqslant s\leqslant 1} s^{\gamma+1/2+\varepsilon}\left|\sqrt{k}\left(\frac{(\frac{k}{n}Z_{n-\lfloor ks\rfloor,n})^{\gamma}-1}{\gamma} - \frac{s^{-\gamma}-1}{\gamma}\right) - s^{-\gamma-1}\sqrt{n/k}\,\overline{W}_n(ks/n)\right| = \mathrm{o}_{\mathbb{P}}(1).$$

The sequence of Brownian motions $W_n^{(k)}(s) = \sqrt{n/k}\,\overline{W}_n(ks/n)$ can then be shown to be the sequence W_n in the statement of our Lemma, by noting that $(Y_i)_{i\geqslant 1} \stackrel{d}{=} (U(Z_i))_{i\geqslant 1}$ and combining (2.4.23), (2.4.24) and (2.4.25) on p.59 of de Haan and Ferreira (2006). ■

The third lemma is a preliminary consistency result for intermediate sample expectiles, under a weaker moment condition than that of Theorem 2 in Daouia *et al.* (2018).

Lemma 3. *Let $k = k(n) \to \infty$ be a positive sequence such that $k/n \to 0$. Suppose further that the distribution of Y is heavy-tailed with tail index $\gamma \in (0, 1/2)$, and assume that $\mathbb{E}|Y_-|^2 < \infty$. Then*

$$\frac{\widetilde{\xi}_{1-k/n}}{\xi_{1-k/n}} \xrightarrow{\mathbb{P}} 1 \quad as \quad n \to \infty.$$

Proof of Lemma 3. We adapt the proof of Theorem 2 in Daouia *et al.* (2018), which was an asymptotic normality result formulated using the parametrisation $\tau_n = 1 - k/n$, where $\tau_n \to 1$ is such that $n(1 - \tau_n) \to \infty$. To make it easier for the reader to relate the present proof with the one of Daouia *et al.* (2018), we adopt this parametrisation here. We shall therefore show that $\widetilde{\xi}_{\tau_n}/\xi_{\tau_n} \xrightarrow{\mathbb{P}} 1$, and we will actually prove the stronger statement

$$v_n \left(\frac{\widetilde{\xi}_{\tau_n}}{\xi_{\tau_n}} - 1 \right) \xrightarrow{\mathbb{P}} 0 \text{ provided } v_n \to \infty \text{ and } v_n = \mathrm{o}\left(\sqrt{n(1 - \tau_n)} \right).$$

Note that

$$v_n \left(\frac{\widetilde{\xi}_{\tau_n}}{\xi_{\tau_n}} - 1 \right) = \arg\min_{u \in \mathbb{R}} \psi_n(u)$$

$$\text{with} \qquad \psi_n(u) := \frac{v_n^2}{n(1 - \tau_n)} \sum_{i=1}^{n} \frac{1}{2\xi_{\tau_n}^2} \left[\eta_{\tau_n} \left(Y_i - \xi_{\tau_n} - \frac{u\xi_{\tau_n}}{v_n} \right) - \eta_{\tau_n}(Y_i - \xi_{\tau_n}) \right].$$

Denoting the derivative of $y \mapsto \eta_\tau(y)/2$ by $\varphi_\tau(y) := |\tau - \mathbb{1}_{\{y \leqslant 0\}}|y$, it is straightforward to get (*e.g.* using Lemma 2 in Daouia *et al.*, 2018):

$$\begin{aligned} \psi_n(u) &= -uT_{1,n} + T_{2,n}(u) \qquad \text{(B.34)} \\ \text{with } T_{1,n} &:= \frac{v_n}{n(1 - \tau_n)} \sum_{i=1}^{n} \frac{1}{\xi_{\tau_n}} \varphi_{\tau_n}(Y_i - \xi_{\tau_n}) =: \sum_{i=1}^{n} S_{n,i} \\ \text{and } T_{2,n}(u) &:= -\frac{v_n^2}{n(1 - \tau_n)\xi_{\tau_n}^2} \sum_{i=1}^{n} \int_0^{u\xi_{\tau_n}/v_n} (\varphi_{\tau_n}(Y_i - \xi_{\tau_n} - t) - \varphi_{\tau_n}(Y_i - \xi_{\tau_n}))dt. \end{aligned}$$

The random variables $S_{n,i}$ are independent, identically distributed, and also centred, by differentiating the expectile minimisation criterion under the expectation sign. Now note that

$$\mathrm{Var}(T_{1,n}) = \mathrm{O}\left(\frac{v_n^2}{n(1 - \tau_n)} \right) \to 0$$

by Lemma 4 in Daouia *et al.* (2018). Because $\mathbb{E}(T_{1,n}) = 0$, Chebyshev's inequality then yields

$$T_{1,n} \xrightarrow{\mathbb{P}} 0. \qquad \text{(B.35)}$$

It is, meanwhile, readily shown by following the proof of Theorem 2 in Daouia *et al.* (2018) that

$$\forall u \in \mathbb{R}, \; T_{2,n}(u) \xrightarrow{\mathbb{P}} \frac{u^2}{2\gamma} \qquad \text{(B.36)}$$

(the only change to make is, with their notation, to re-define $I_n(u) = (0, |u|\xi_{\tau_n}/v_n)$ and note that $t/\xi_{\tau_n} \to 0$ uniformly in t such that $|t| \in I_n(u)$). Combining (B.34), (B.35) and (B.36) entails

$$\forall u \in \mathbb{R}, \ \psi_n(u) \xrightarrow{\mathbb{P}} \frac{u^2}{2\gamma} \ \text{ as } \ n \to \infty.$$

We conclude by noting that (ψ_n) is a random sequence of continuous convex functions and its pointwise limit defines a nonrandom continuous convex function of u which has a unique minimum at $u^\star = 0$. Applying Theorem 5 in Knight (1999) completes the proof. ∎

The fourth lemma is the key to the computation of the various terms appearing in the implicit relationship linking the tail expectile process to the tail parameters.

Lemma 4. *Suppose that $\mathbb{E}|Y_-| < \infty$. Assume further that condition $\mathcal{C}_2(\gamma, \rho, A)$ holds for some $0 < \gamma < 1/2$. Let $k = k(n) \to \infty$ be such that $k/n \to 0$ and $\sqrt{k}A(n/k) = \mathrm{O}(1)$. Then we have, for any $\delta > 0$ sufficiently small:*

$$\sup_{k^{-1+\delta} \leqslant s \leqslant 1} s^\gamma \left| \frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}} - (\gamma^{-1} - 1)^{-\gamma} s^{-\gamma} \right| \xrightarrow{\mathbb{P}} 0.$$

Proof of Lemma 4. All the $\mathrm{o}_{\mathbb{P}}$ and $\mathrm{O}_{\mathbb{P}}$ terms in the present proof should be understood as uniform in $s \in [k^{-1+\delta}, 1]$; moreover, we work throughout this proof with the version of the tail expectile process induced by the version of the empirical process $\widehat{F}_n$ leading to (B.5). Recall that any Brownian motion W satisfies, for any $\eta > 0$:

$$\forall c > 0, \ \sup_{0 < t \leqslant c} t^{-1/2+\eta} |W(t)| < \infty \ \text{ almost surely.}$$

It then comes as a consequence of (B.4) that

$$\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} W_n(x^{-1/\gamma})\, dx = \mathrm{O}_{\mathbb{P}}\left(\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{(\eta-1/2)/\gamma}\, dx \right).$$

Moreover, since $\sqrt{k}A(n/k)$ remains bounded:

$$A(n/k) \int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho}\, dx = \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}} \int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{(\eta-1/2)/\gamma}\, dx \right).$$

All in all, combining these two bounds with (B.5) gives:

$$\int_{\widetilde{\xi}_{1-ks/n}}^{\infty} \widehat{\overline{F}}_n(u) du = \frac{k}{n} q_{1-k/n} \left(\int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{-1/\gamma}\, dx + \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}} \int_{\widetilde{\xi}_{1-ks/n}/q_{1-k/n}}^{\infty} x^{(\eta-1/2)/\gamma}\, dx \right) \right)$$

or equivalently

$$\int_{\widetilde{\xi}_{1-ks/n}}^{\infty} \widehat{\overline{F}}_n(u) du = \frac{k}{n} q_{1-k/n} \times \frac{\gamma}{1-\gamma} \left(\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}} \right]^{1-1/\gamma} + \mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}} \left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}} \right]^{1-(1/2-\eta)/\gamma} \right) \right).$$

Plugging this back into (B.2) entails

$$\frac{s}{1-2ks/n}\left(\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}-\frac{\overline{Y}_n}{q_{1-k/n}}\right)=\frac{\gamma}{1-\gamma}\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}\right]^{1-1/\gamma}+\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}\right]^{1-(1/2-\eta)/\gamma}\right).$$

Note that $\overline{Y}_n \xrightarrow{\mathbb{P}} \mathbb{E}(Y) < \infty$ by the law of large numbers, and $\widetilde{\xi}_{1-ks/n} \geqslant \widetilde{\xi}_{1-k/n} \xrightarrow{\mathbb{P}} +\infty$ by Lemma 3. Therefore

$$s\,(1+\mathrm{o}_{\mathbb{P}}(1))=\frac{\gamma}{1-\gamma}\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}\right]^{-1/\gamma}+\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}\left[\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}\right]^{-(1/2-\eta)/\gamma}\right).$$

Define now a random process $s \mapsto R_n(s)$ by the equality

$$\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}=(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}(1+R_n(s)).$$

In particular, $1+R_n(s)>0$ for any $s\in(0,1]$, and

$$1+\mathrm{o}_{\mathbb{P}}(1)=(1+R_n(s))^{-1/\gamma}+\mathrm{O}_{\mathbb{P}}\left(\frac{1}{\sqrt{k}}s^{-1/2-\eta}(1+R_n(s))^{-(1/2-\eta)/\gamma}\right).$$

We infer from this equality that, uniformly in $s\in[k^{-1+\delta},1]$ for $\delta=\delta(\eta)=4\eta/(4\eta+1)>0$,

$$1+\mathrm{o}_{\mathbb{P}}(1)=(1+R_n(s))^{-1/\gamma}+\mathrm{o}_{\mathbb{P}}\left((1+R_n(s))^{-(1/2-\eta)/\gamma}\right).$$

It directly follows from this last identity, whose left-hand side should remain bounded uniformly in s, that $1+R_n(s)$ must remain bounded away from 0, uniformly in $s\in[k^{-1+\delta},1]$ with arbitrarily large probability as $n\to\infty$. The fact that the left-hand side converges in probability to 1 uniformly in s now entails that $1+R_n(s)$ should do so as well, which yields

$$\sup_{k^{-1+\delta}\leqslant s\leqslant 1}|R_n(s)|\xrightarrow{\mathbb{P}}0 \quad\text{as}\quad n\to\infty.$$

Equivalently

$$\sup_{k^{-1+\delta}\leqslant s\leqslant 1} s^{\gamma}\left|\frac{\widetilde{\xi}_{1-ks/n}}{q_{1-k/n}}-(\gamma^{-1}-1)^{-\gamma}s^{-\gamma}\right|\xrightarrow{\mathbb{P}}0. \tag{B.37}$$

And since η was arbitrarily small, $\delta=4\eta/(4\eta+1)$ was arbitrarily small as well, concluding the proof. ■

The final lemma is a technical result on second-order regular variation which will be used several times in the proofs of Theorems 1 and 2.

Lemma 5. *Assume that condition $\mathcal{C}_2(\gamma,\rho,A)$ holds with $\gamma>0$. Then one can find a function B, asymptotically equivalent to $t\mapsto A(1/\overline{F}(t))$ in a neighbourhood of infinity, satisfying the following: for any ε, $\delta>0$ there exists $t_0=t_0(\varepsilon,\delta)>0$ such that for t, $tx\geqslant t_0$,*

$$\left|\frac{1}{B(t)}\left(\frac{\overline{F}(tx)}{\overline{F}(t)}-x^{-1/\gamma}\right)-x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\right|\leqslant \varepsilon x^{-(1-\rho)/\gamma}\max\left(x^{-\delta},x^{\delta}\right).$$

Proof of Lemma 5. Note that, according to Theorem 2.3.9 in de Haan and Ferreira (2006), condition $\mathcal{C}_2(\gamma, \rho, A)$ is equivalent to

$$\forall x > 0, \ \lim_{t\to\infty} \frac{1}{A(1/\overline{F}(t))} \left(\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-1/\gamma} \right) = x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho}.$$

Define $f(x) = x^{1/\gamma}\overline{F}(x)$; it is straightforward to show that this condition entails

$$\forall x > 0, \ \lim_{t\to\infty} \frac{f(tx) - f(t)}{\gamma^{-2} f(t) A(1/\overline{F}(t))} = \frac{x^{\rho/\gamma} - 1}{\rho/\gamma}.$$

The conclusion then follows by applying Theorem B.2.18 in de Haan and Ferreira (2006). ■

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