

Supplementary Material for

EXTREMILES: A NEW PERSPECTIVE ON

ASYMMETRIC LEAST SQUARES

Supplement [A](#) contains technical lemmas and the proofs of all theoretical results in the main paper. Supplement [B](#) provides further simulation results.

A Proofs

Proof of Proposition 1 Simply note that, for $t \in (y_\ell, y_u)$,

$$\begin{aligned}
 \mathbb{P}(q_{K_\tau^{-1}(F(Y))} \leq t) &= \mathbb{P}(q_{K_\tau^{-1}(U)} \leq t \text{ for } U \stackrel{d}{=} U(0, 1) \quad (\text{because } F \text{ is continuous}) \\
 &= \int_0^1 \mathbb{I}_{\{K_\tau^{-1}(u) \leq F(t)\}} du \\
 &= \int_0^1 \mathbb{I}_{\{u \leq K_\tau(F(t))\}} du \\
 &= K_\tau(F(t)) = \mathbb{P}(Z_\tau \leq t),
 \end{aligned}$$

thus proving the desired result. \square

Proof of Proposition 2 For part **(i)** we have clearly, by (7), $|\xi_\tau| \leq s(\tau)\mathbb{E}|Y|$ if $\tau \in (0, 1/2]$, and $|\xi_\tau| \leq r(\tau)\mathbb{E}|Y|$ if $\tau \in [1/2, 1)$. So $\mathbb{E}|Y| < \infty$ implies $|\xi_\tau| < \infty$. Furthermore, it is a clear consequence of the dominated convergence theorem that the function $\tau \mapsto \xi_\tau$ is continuous on both of the intervals $(0, 1/2)$ and $(1/2, 1)$. Since $r(1/2) = s(1/2) = 1$, we conclude by the dominated convergence theorem again that the function $\tau \mapsto \xi_\tau$ is continuous at $\tau = 1/2$, and as such is continuous on the whole interval $(0, 1)$. Note also (from Equation (5)) that $\tau \mapsto r(\tau)$ is increasing on $(1/2, 1)$ and $\tau \mapsto s(\tau)$ is decreasing on $(0, 1/2)$, and this implies that the function $\tau \mapsto \xi_\tau$ is increasing on both the intervals

$(0, 1/2)$ and $(1/2, 1)$, and therefore is also increasing on the whole interval $(0, 1)$ by its continuity at $\tau = 1/2$. Finally, the fact that the extremile function maps $(0, 1)$ onto the range of F is an immediate consequence of its continuity together with the limits $\lim_{\tau \downarrow 0} \xi_\tau = y_\ell$ and $\lim_{\tau \uparrow 1} \xi_\tau = y_u$.

(ii) The necessary condition is trivial. For the sufficient condition, suppose $\xi_{Y,\tau} = \xi_{\tilde{Y},\tau}$ for all $\tau \in (0, 1)$. For every integer $s \geq 1$ and all independent copies Y^1, \dots, Y^s of Y , we have $\mathbb{E}[\min(Y^1, \dots, Y^s)] \equiv \xi_{Y,\tau(s)}$, where $\tau(s) = 1 - (1/2)^{1/s}$. Likewise, for s independent copies $\tilde{Y}^1, \dots, \tilde{Y}^s$ of \tilde{Y} , we have $\mathbb{E}[\min(\tilde{Y}^1, \dots, \tilde{Y}^s)] \equiv \xi_{\tilde{Y},\tau(s)}$. Then by assumption

$$\mathbb{E}[\min(Y^1, \dots, Y^s)] = \mathbb{E}[\min(\tilde{Y}^1, \dots, \tilde{Y}^s)] \quad \text{for } s = 1, 2, \dots$$

This implies $F_Y = F_{\tilde{Y}}$ as established by Chan (1967).

(iii) Following (7), we have $\xi_{\tilde{Y},\tau} = \int_0^1 J_\tau(t) F_{\tilde{Y}}^{-1}(t) dt$. Also, $J_\tau(1-t) = J_{1-\tau}(t)$. The assertion follows then immediately from the location and scale equivariance of quantiles

$$F_{\tilde{Y}}^{-1}(\tau) = \begin{cases} a + bF_Y^{-1}(\tau) & \text{if } b > 0 \\ a + bF_Y^{-1}(1-\tau) & \text{if } b \leq 0. \end{cases} \quad (\text{A.1})$$

(iv) Using a change of variables in the first equality in formula (7) in conjunction with the facts that $J_\tau(1-t) = J_{1-\tau}(t)$ and $q_{1-t} = 2\mu - q_t$, we easily get

$$\xi_{1-\tau} = \int_0^1 J_{1-\tau}(t) q_t dt = 2\mu - \int_0^1 J_\tau(t) q_t dt = 2\mu - \xi_\tau$$

for either cases $\tau \leq 1/2$ and $\tau \geq 1/2$.

(v) The comonotonic additivity follows immediately from (7) in conjunction with the fact that $F_{Y+\tilde{Y}}^{-1}(t) = F_Y^{-1}(t) + F_{\tilde{Y}}^{-1}(t)$ for comonotonic variables Y and \tilde{Y} . \square

Lemma A.1 *For any $s < 1$, we have*

$$(\log 2) \int_0^\infty 2^{-t} t^{-s} dt = \Gamma(1-s)(\log 2)^s =: \mathcal{G}(s).$$

Proof This is a straightforward consequence of the use of the change of variables $u = t \log 2$. \square

Lemma A.2 (i) For any $t > 0$ it holds that

$$(1 - (1 - \tau)t)^{r(\tau)-1} \rightarrow 2^{-t} \quad \text{as } \tau \uparrow 1.$$

(ii) For any τ close enough to 1, we have

$$0 \leq (1 - (1 - \tau)t)^{r(\tau)-1} \mathbb{I}_{\{0 < t < (1-\tau)^{-1}\}} \leq (\sqrt{2})^{-t}.$$

(iii) For any $t > 0$ it holds that

$$(1 - \tau)^{-1} \left[(1 - (1 - \tau)t)^{r(\tau)-1} - 2^{-t} \right] \rightarrow 2^{-t} \left(t \left[1 + \frac{\log 2}{2} \right] - t^2 \frac{\log 2}{2} \right) \quad \text{as } \tau \uparrow 1.$$

(iv) For any $c \in (0, 1)$, there is a constant $C > 0$ such that for any τ close enough to 1, we have

$$(1 - \tau)^{-1} \left| (1 - (1 - \tau)t)^{r(\tau)-1} - 2^{-t} \right| \mathbb{I}_{\{0 < t < c(1-\tau)^{-1}\}} \leq C(\sqrt{2})^{-t}.$$

Proof Convergence (i) is immediate since $r(\tau) = \log(1/2)/\log(\tau)$ is equivalent to $(1 - \tau)^{-1} \log 2$ as $\tau \uparrow 1$. Statement (ii) is shown by recalling that $\log(1 - x) \leq -x$ for all $x \in [0, 1)$, implying in particular

$$\frac{1}{-\log(\tau)} \leq (1 - \tau)^{-1} \quad \text{and} \quad \forall a > 1, \quad \forall t \in (0, (1 - \tau)^{-1}), \quad (1 - (1 - \tau)t)^{(1-\tau)^{-1} \log a} \leq a^{-t}.$$

Combining these two inequalities entails, for τ close enough to 1 and $t \in (0, (1 - \tau)^{-1})$,

$$\forall a \in (1, 2), \quad (1 - (1 - \tau)t)^{r(\tau)-1} \leq (1 - (1 - \tau)t)^{(1-\tau)^{-1} \log a} \leq a^{-t} \quad (\text{A.2})$$

as required. Convergence (iii) follows from a straightforward Taylor expansion. To prove statement (iv), we first recall that a consequence of the mean value theorem is

$$\forall x, y \in \mathbb{R}, \quad |e^y - e^x| \leq |y - x| e^{\max(x, y)}.$$

Pick then $a \in (\sqrt{2}, 2)$ and use (A.2) to obtain that eventually as $\tau \uparrow 1$ and for any $t \in (0, (1 - \tau)^{-1})$:

$$(1 - \tau)^{-1} \left| (1 - (1 - \tau)t)^{r(\tau)-1} - 2^{-t} \right| \leq (1 - \tau)^{-1} |(r(\tau) - 1) \log(1 - (1 - \tau)t) + t \log 2| a^{-t}. \quad (\text{A.3})$$

Now, the Taylor expansion $r(\tau) = (1 - \tau)^{-1}(\log 2) - (\log 2)/2 + o(1)$ entails, together with the triangle inequality, that eventually as $\tau \uparrow 1$ and for all $t \in (0, (1 - \tau)^{-1})$:

$$(1 - \tau)^{-1} |(r(\tau) - 1) \log(1 - (1 - \tau)t) + t \log 2| \leq (1 + \log 2) \frac{|\log(1 - (1 - \tau)t)|}{1 - \tau} + (\log 2) \frac{|\log(1 - (1 - \tau)t) + (1 - \tau)t|}{(1 - \tau)^2}.$$

The mean value theorem and Taylor's theorem entail, for any $c \in (0, 1)$:

$$\sup_{0 < x < c} \frac{|\log(1 - x)|}{x} \leq \frac{1}{1 - c} \quad \text{and} \quad \sup_{0 < x < c} \frac{|\log(1 - x) + x|}{x^2} \leq \frac{1}{2(1 - c)^2},$$

implying therefore that there is a constant $C_1 > 0$ with

$$(1 - \tau)^{-1} |(r(\tau) - 1) \log(1 - (1 - \tau)t) + t \log 2| \mathbb{I}_{\{0 < t < c(1 - \tau)^{-1}\}} \leq C_1 t(1 + t) \mathbb{I}_{\{t > 0\}}.$$

Recall now that since $a > \sqrt{2}$, there is a constant $C_2 > 0$ with $t(1 + t)a^{-t} \leq C_2(\sqrt{2})^{-t}$ for all $t > 0$, and report the above inequality into (A.3) to complete the proof. \square

Proof of Proposition 3 To prove (i), set $\delta = \inf\{t \in (0, 1) \mid q_t > 0\}$. Then $\delta \in [0, 1)$ since we work in the heavy-tailed case. If $\delta > 0$, then for a sufficiently large $\tau > 1/2$, and because $q_\delta \leq 0$ and q is nondecreasing, we have

$$\left| \frac{\int_0^\delta r(\tau) t^{r(\tau)-1} q_t dt}{q_\tau} \right| \leq \frac{r(\tau) \delta^{r(\tau)-1}}{q_\tau} \int_0^\delta -q_t dt = O\left(\frac{r(\tau) \delta^{r(\tau)-1}}{q_\tau}\right).$$

Recall that $r(\tau) = \log(1/2)/\log(\tau) \sim (1 - \tau)^{-1} \log 2$ as $\tau \uparrow 1$, and use Proposition 1.3.6(v) in Bingham *et al.* (1987, p.16) to get for any $\varepsilon \in (0, 1)$:

$$\frac{r(\tau) \varepsilon^{r(\tau)-1}}{q_\tau} = O\left((1 - \tau)^{-2(1-\gamma)} \exp(-(1 - \tau)^{-1}(\log(2) \times \log(1/\varepsilon))/2)\right) = o(1). \quad (\text{A.4})$$

As such

$$\left| \frac{\int_0^\delta r(\tau) t^{r(\tau)-1} q_t dt}{q_\tau} \right| = o(1). \quad (\text{A.5})$$

This is of course also trivially true if $\delta = 0$. Furthermore, the condition $F \in \text{DA}(\Phi_\gamma)$ is equivalent to

$$\lim_{s \rightarrow \infty} \frac{q_{1-(sx)^{-1}}}{q_{1-s^{-1}}} = x^\gamma \quad \text{for all } x > 0 \quad (\text{A.6})$$

(see, *e.g.*, de Haan and Ferreira (2006), Corollary 1.2.10). Therefore, it follows by Proposition B.1.10 of de Haan and Ferreira (2006) that there is $s_0 > 0$, which we may take to be larger than $(1 - \delta)^{-1}$, such that

$$\forall s > 0, \forall x > 0, s, sx \geq s_0 \Rightarrow \left| \frac{q_{1-(sx)^{-1}}}{q_{1-s^{-1}}} - x^\gamma \right| \leq \max(x^{(1+\gamma)/2}, 1). \quad (\text{A.7})$$

Write then

$$\xi_\tau = \int_0^\delta r(\tau) t^{r(\tau)-1} q_t dt + \int_\delta^{1-s_0^{-1}} r(\tau) t^{r(\tau)-1} q_t dt + \int_{1-s_0^{-1}}^1 r(\tau) t^{r(\tau)-1} q_t dt.$$

The second term above is controlled just like in (A.5), yielding

$$\frac{\xi_\tau}{q_\tau} = \int_{1-s_0^{-1}}^1 r(\tau) t^{r(\tau)-1} \frac{q_t}{q_\tau} dt + o(1). \quad (\text{A.8})$$

Use then the change of variables $t = 1 - (1 - \tau)/w$ to obtain that the integral on the right-hand side of (A.8) is equivalent to

$$(\log 2) \int_{(1-\tau)s_0}^\infty \left(1 - \frac{1-\tau}{w} \right)^{r(\tau)-1} \frac{q_{1-[(1-\tau)^{-1}w]^{-1}}}{q_{1-[(1-\tau)^{-1}]^{-1}}} \frac{dw}{w^2}.$$

A combination of Lemma A.2(i) and (ii), (A.6), (A.7) and of the dominated convergence theorem shows that

$$\int_{(1-\tau)s_0}^\infty \left(1 - \frac{1-\tau}{w} \right)^{r(\tau)-1} \frac{q_{1-[(1-\tau)^{-1}w]^{-1}}}{q_{1-[(1-\tau)^{-1}]^{-1}}} \frac{dw}{w^2} \rightarrow \int_0^\infty 2^{-1/w} w^\gamma \frac{dw}{w^2} \quad \text{as } \tau \uparrow 1.$$

Report this into (A.8) and use the change of variables $t = 1/w$ together with Lemma A.1 to get the required result.

The proof of (ii) uses the fact that, for $\gamma < 0$, $F \in \text{DA}(\Psi_\gamma)$ is equivalent to (de Haan and Ferreira (2006), Corollary 1.2.10)

$$y_u = \sup\{y : F(y) < 1\} < \infty \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{y_u - q_{1-(sx)^{-1}}}{y_u - q_{1-s^{-1}}} = x^\gamma \quad \text{for all } x > 0$$

and is entirely similar to the proof of (i).

Finally, the proof of (iii) is based on the fact that if $F \in \text{DA}(\Lambda)$, then

$$\lim_{s \rightarrow \infty} \frac{q_{1-(sx)^{-1}}}{q_{1-s^{-1}}} = 1 \quad \text{for all } x > 0$$

when $y_u = \infty$, and

$$\lim_{s \rightarrow \infty} \frac{y_u - q_{1-(sx)^{-1}}}{y_u - q_{1-s^{-1}}} = 1 \quad \text{for all } x > 0$$

when $y_u < \infty$ (see, *e.g.*, de Haan and Ferreira (2006), Lemma 1.2.9). The same arguments used to prove **(i)** yield once again the desired result, and so we omit the details. \square

Proof of Theorem 1 Write first

$$\widehat{\xi}_\tau^L = \frac{1}{n} \sum_{i=1}^n c_{i,n} Y_{i,n} \quad \text{with} \quad c_{i,n} = n \left\{ K_\tau \left(\frac{i}{n} \right) - K_\tau \left(\frac{i-1}{n} \right) \right\}.$$

Since K_τ is continuously differentiable on $(0, 1)$ with derivative J_τ , we have

$$\forall i \in \{1, \dots, n\}, \exists t_{i,n} \in \left[\frac{i-1}{n}, \frac{i}{n} \right], \quad c_{i,n} = J_\tau(t_{i,n}).$$

Define then $J_n(t) = c_{i,n}$ for $t \in [(i-1)/n, i/n]$. Note that J_n and J_τ are uniformly bounded on $[0, 1]$. Besides, if $\mathbb{E}|Y|^\kappa < \infty$ then q necessarily satisfies, for some $M > 0$,

$$|q_t| \leq M t^{-1/\kappa} (1-t)^{-1/\kappa}, \quad t \in (0, 1), \tag{A.9}$$

see Remark 1 in Shorack and Wellner (1986, p.663). Finally, note that the function $t \mapsto t(1-t)J'_\tau(t)$ is clearly bounded on $(0, 1)$.

(i) Here the constant κ in (A.9) satisfies $1/\kappa < 1$. The result then follows directly from Theorem 3 in Shorack and Wellner (1986, p.665).

(ii) Since now (A.9) holds with $1/\kappa < 2$, the result follows immediately from Theorem 1(ii) of Shorack and Wellner (1986, p.664).

(iii) If J'_τ is Lipschitz of order $\delta > \frac{1}{3}$ on $(0, 1)$, then the Berry-Esséen rate $O(n^{-1/2})$ follows from Theorem C of Serfling (1980, p.287). This is clearly true when $r(\tau) \geq 3$ or equivalently $\tau \geq (\frac{1}{2})^{1/3}$ in the right tail, and $s(\tau) \geq 3$ or equivalently $\tau \leq 1 - (\frac{1}{2})^{1/3}$ in the left tail. \square

Proof of Theorem 2 It is not hard to check the stated convergence by applying Theorem 4.2 of Shorack (2000, p.442). \square

Recall the second-order condition

$$\lim_{t \rightarrow \infty} \frac{1}{A(t)} \left\{ \frac{q_{1-(tx)^{-1}}}{q_{1-t^{-1}}} - x^\gamma \right\} = x^\gamma \frac{x^\rho - 1}{\rho} \text{ for all } x > 0. \quad (\text{A.10})$$

Proof of Proposition 4 As in the proof of Proposition 3, set $\delta = \inf\{t \in (0, 1) \mid q_t > 0\} \in (0, 1)$. Apply then Theorem 2.3.9 in de Haan and Ferreira (2006) to get that there is $s_0 > 0$, which we may take to be larger than $(1 - \delta)^{-1}$, such that

$$s, sx \geq s_0 \Rightarrow \left| \frac{1}{A_0(s)} \left(\frac{q_{1-(sx)^{-1}}}{q_{1-s^{-1}}} - x^\gamma \right) - x^\gamma \frac{x^\rho - 1}{\rho} \right| \leq \max(x^{(1+\gamma)/2}, x^{\gamma+\rho-1}). \quad (\text{A.11})$$

Here A_0 is a function that is equivalent to A in a neighborhood of infinity. By (A.4) in the proof of Proposition 3, it is then clear that there is $C > 0$ with

$$\frac{\xi_\tau}{q_\tau} = \int_{1-s_0^{-1}}^1 r(\tau) t^{r(\tau)-1} \frac{q_t}{q_\tau} dt + o(\exp(-C(1-\tau)^{-1})). \quad (\text{A.12})$$

Use then the change of variables $t = 1 - (1 - \tau)/w$ to obtain that the integral on the right-hand side above is

$$\begin{aligned} & (1 - \tau)r(\tau) \int_{(1-\tau)s_0}^\infty \left(1 - \frac{1 - \tau}{w}\right)^{r(\tau)-1} \left(\frac{q_{1-[(1-\tau)^{-1}w]^{-1}}}{q_{1-[(1-\tau)^{-1}]^{-1}}} - w^\gamma \right) \frac{dw}{w^2} \\ & + (1 - \tau)r(\tau) \int_{(1-\tau)s_0}^\infty \left(1 - \frac{1 - \tau}{w}\right)^{r(\tau)-1} w^\gamma \frac{dw}{w^2} =: I_1(\tau) + I_2(\tau). \end{aligned} \quad (\text{A.13})$$

A combination of Lemma A.2(i) and (ii), condition (A.10), (A.11) and the dominated convergence theorem entails that

$$I_1(\tau) = (\log 2) \int_0^\infty 2^{-1/w} w^\gamma \frac{w^\rho - 1}{\rho} \frac{dw}{w^2} \times A((1 - \tau)^{-1}) + o(A((1 - \tau)^{-1})) \text{ as } \tau \uparrow 1.$$

Using the change of variables $t = 1/w$ and Lemma A.1 it is easy to see that this entails

$$I_1(\tau) = A((1 - \tau)^{-1})C_1(\gamma, \rho) + o(A((1 - \tau)^{-1})). \quad (\text{A.14})$$

We now work on $I_2(\tau)$: use the change of variables $z = 1/w$ to get that

$$\frac{I_2(\tau)}{(1 - \tau)r(\tau)} = \int_0^{(1-\tau)^{-1}s_0^{-1}} (1 - (1 - \tau)z)^{r(\tau)-1} z^{-\gamma} dz.$$

A use of Lemma A.1 and of the bound $2^{-z}z^{-\gamma} = O(2^{-z})$ as $z \rightarrow \infty$ entails

$$\begin{aligned} (\log 2) \frac{I_2(\tau)}{(1-\tau)r(\tau)} &= \Gamma(1-\gamma)(\log 2)^\gamma + (\log 2) \int_0^{(1-\tau)^{-1}s_0^{-1}} \left[(1-(1-\tau)z)^{r(\tau)-1} - 2^{-z} \right] z^{-\gamma} dz \\ &\quad + o\left(\exp(-C(1-\tau)^{-1})\right). \end{aligned} \quad (\text{A.15})$$

Meanwhile, Lemma A.1, Lemma A.2(iii) and (iv) and the dominated convergence theorem yield

$$\begin{aligned} &(\log 2) \int_0^{(1-\tau)^{-1}s_0^{-1}} \left[(1-(1-\tau)z)^{r(\tau)-1} - 2^{-z} \right] z^{-\gamma} dz \\ &= (1-\tau)(\log 2) \int_0^\infty 2^{-z} \left(z \left[1 + \frac{\log 2}{2} \right] - z^2 \frac{\log 2}{2} \right) z^{-\gamma} dz + o(1-\tau) \\ &= (1-\tau) \left\{ \left[1 + \frac{\log 2}{2} \right] \mathcal{G}(\gamma-1) - \frac{\log 2}{2} \mathcal{G}(\gamma-2) \right\} + o(1-\tau). \end{aligned}$$

Using the Taylor expansion $r(\tau) = (1-\tau)^{-1}(\log 2) - (\log 2)/2 + o(1)$, it is then clear from (A.15) that

$$I_2(\tau) = \mathcal{G}(\gamma) + (1-\tau)C_2(\gamma) + o(1-\tau). \quad (\text{A.16})$$

Combine finally (A.12), (A.13), (A.14), (A.16) and the regular variation of $|A|$ (see Theorem 2.3.3 in de Haan and Ferreira, 2006) to complete the proof. \square

Proof of Theorem 3 Putting $k = n(1-\tau_n)$ and $d_n = (1-\tau_n)/(1-\tau'_n)$, we have

$$\begin{aligned} \frac{\sqrt{k}}{\log d_n} \left(\frac{\hat{\xi}_{\tau'_n}^{\mathcal{Q},*}}{\xi_{\tau'_n}} - 1 \right) &= \frac{\sqrt{k}}{\log d_n} \left(\frac{\hat{q}_{\tau'_n}^*}{q_{\tau'_n}} - 1 \right) \frac{q_{\tau'_n}}{\xi_{\tau'_n}} \mathcal{G}(\hat{\gamma}) + \frac{\sqrt{k}}{\log d_n} (\mathcal{G}(\hat{\gamma}) - \mathcal{G}(\gamma)) \frac{q_{\tau'_n}}{\xi_{\tau'_n}} \\ &\quad + \frac{\sqrt{k}}{\log d_n} \left[\mathcal{G}(\gamma) - \frac{\xi_{\tau'_n}}{q_{\tau'_n}} \right] \frac{q_{\tau'_n}}{\xi_{\tau'_n}}. \end{aligned}$$

Note that $q_\tau/\xi_\tau \rightarrow 1/\mathcal{G}(\gamma)$ by Proposition 3 and $\sqrt{k}(\mathcal{G}(\hat{\gamma}) - \mathcal{G}(\gamma)) = O_{\mathbb{P}}(1)$ by the delta-method; combining this with Theorem 4.3.8 in de Haan and Ferreira (2006), it follows that the sum of the first two terms above converges in distribution to Z . Besides, Proposition 4 entails

$$\mathcal{G}(\gamma) - \frac{\xi_{\tau'_n}}{q_{\tau'_n}} = O(A((1-\tau'_n)^{-1})) + O(1-\tau'_n) = O(A((1-\tau_n)^{-1})) + O\left(\frac{1}{n}\right)$$

due to the regular variation of $|A|$ with negative index. It only remains to use the assumptions that $\sqrt{n(1-\tau_n)}A((1-\tau_n)^{-1}) = O(1)$ and $\log d_n \rightarrow \infty$ to obtain that the third term of the above decomposition converges to 0, which completes the proof. \square

Lemma A.3 *Suppose $k = k(n) \rightarrow \infty$ is a positive sequence with $n/k \rightarrow \infty$. Then:*

(i) *We have*

$$J_{1-k/n}(1/n) = o\left(\exp\left[-\frac{\log 2}{2} \times \frac{n \log n}{k}\right]\right).$$

(ii) *For any $\delta \in (0, 1)$, we have*

$$\sup_{0 < t < \delta} \{t^{-(n \log 2)/(2k)} J_{1-k/n}(t)\} = o\left(\frac{n}{k}\right).$$

Proof To show (i), note that

$$\begin{aligned} J_{1-k/n}(1/n) &= \frac{\log 2}{-\log(1-k/n)} \exp\left(\left[\frac{\log 2}{\log(1-k/n)} + 1\right] \log n\right) \\ &= \frac{n}{k} \log 2(1+o(1)) \times \exp\left(-\frac{n \log n}{k} \log 2(1+o(1))\right) \\ &= o\left(\exp\left(-\frac{\log 2}{2} \times \frac{n \log n}{k}\right)\right) \end{aligned}$$

as required. To prove (ii), write

$$\sup_{0 < t < \delta} \{t^{-(n \log 2)/(2k)} J_{1-k/n}(t)\} = \frac{n}{k} \log 2(1+o(1)) \sup_{0 < t < \delta} \exp\left(\left[-1 - \frac{\log 2}{\log(1-k/n)} - \frac{n \log 2}{k} \frac{1}{2}\right] \log(t)\right).$$

Now $-(\log 2)/\log(1-k/n) - (n \log 2)/(2k) \rightarrow +\infty$, and on $(0, \delta)$, $\log(t) < \log(\delta) < 0$, so that eventually

$$\sup_{0 < t < \delta} \exp\left(\left[-1 - \frac{\log 2}{\log(1-k/n)} - \frac{n \log 2}{k} \frac{1}{2}\right] \log(t)\right) \leq \exp\left(\left[-1 - \frac{\log 2}{\log(1-k/n)} - \frac{n \log 2}{k} \frac{1}{2}\right] \log(\delta)\right)$$

and the upper bound in this inequality converges to 0, proving (ii). \square

Lemma A.4 *Suppose:*

- *the second-order regular variation condition (A.10) holds, with $\gamma < 1/2$;*

- $k = k(n) \rightarrow \infty$, $n/k \rightarrow \infty$ and $\sqrt{k} A(n/k) = O(1)$ as $n \rightarrow \infty$.

Then there is a positive sequence (u_n) , such that $u_n \rightarrow 0$, $nu_n/k \rightarrow \infty$, $1/u_n$ is an integer for any n , and the following all hold:

(i) We have

$$u_n^{-\gamma} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \rightarrow 1 \quad \text{and} \quad u_n^{-\rho} \frac{A(nu_n/k)}{A(n/k)} \rightarrow 1.$$

(ii) There exist $\varepsilon \in (0, 1/2 - \gamma)$, a sequence of Brownian motions (W_n) and a function A_0 which is asymptotically equivalent to A , such that:

$$\begin{aligned} & \left| \sqrt{\frac{k}{u_n}} \left(\frac{\hat{q}_{1-ks/(nu_n)} - q_{1-ks/(nu_n)}}{q_{1-k/(nu_n)}} \right) - \gamma s^{-\gamma-1} W_n(s) \right| \\ &= s^{-\gamma-\rho-\varepsilon} o \left(u_n^\varepsilon \sqrt{\frac{k}{u_n}} A \left(\frac{nu_n}{k} \right) \right) + s^{-\gamma-1/2-\varepsilon} o_{\mathbb{P}}(u_n^\varepsilon) \end{aligned}$$

uniformly in $s \in (0, 1]$.

Proof Apply Proposition B.1.10 in de Haan and Ferreira (2006, p.369) to construct by induction an increasing sequence (x_p) tending to infinity such that for any positive integer p :

$$\forall t > 0, \forall x \in (0, 1), tx \geq x_p \Rightarrow \max \left(\left| x^{-\gamma} \frac{q_{1-1/(tx)}}{q_{1-1/t}} - 1 \right|, \left| x^{-\rho} \frac{A(tx)}{A(t)} - 1 \right| \right) \leq \frac{1}{2^p} x^{-1/2}.$$

Use now Theorem 2.4.8 in de Haan and Ferreira (2006, p.52) to construct, for a suitably small fixed $\varepsilon \in (0, 1/2 - \gamma)$, and for any positive integer p , a sequence of Brownian motions $(\widetilde{W}_{n,p})$ and a positive sequence of random variables $(\widetilde{Z}_{n,p})$ such that $\widetilde{Z}_{n,p} = o_{\mathbb{P}}(1)$ as $n \rightarrow \infty$, satisfying:

$$s^{\gamma+1/2+\varepsilon} \left| \sqrt{2^p k} \left(\frac{\hat{q}_{1-2^p k s/n}}{q_{1-2^p k/n}} - s^{-\gamma} \right) - \gamma s^{-\gamma-1} \widetilde{W}_{n,p}(s) - \sqrt{2^p k} A_0 \left(\frac{n}{2^p k} \right) s^{-\gamma} \frac{s^{-\rho} - 1}{\rho} \right| \leq \widetilde{Z}_{n,p}$$

for all $s \in (0, 1]$. Here A_0 is a suitable function equivalent to A at infinity. An inspection of the proof of Theorem 2.4.8 in de Haan and Ferreira (2006) shows that the interval of possible choices of ε and the choice of A_0 only depend on the behaviour of Y in its right tail, and as such these quantities can indeed be fixed independently of p . Since for any

p , $\tilde{Z}_{n,p} = o_{\mathbb{P}}(1)$ as $n \rightarrow \infty$, we may construct an increasing sequence of integers (N_p) such that

$$\forall p \geq 1, \forall n \geq N_p, \mathbb{P} \left(\tilde{Z}_{n,p} > \left(\frac{1}{2^p} \right)^{1+\varepsilon} \right) < \frac{1}{2^p}.$$

Apply finally Theorem 2.3.9 in de Haan and Ferreira (2006, p.48) to construct by induction an increasing sequence (t_p) tending to infinity such that for any positive integer p :

$$\forall t > 0, \forall s \in (0, 1), t \geq t_p \Rightarrow s^{\gamma+\rho+\varepsilon} \left| \frac{1}{A_0(t)} \left(\frac{q_{1-s/t}}{q_{1-1/t}} - s^{-\gamma} \right) - s^{-\gamma} \frac{s^{-\rho} - 1}{\rho} \right| \leq \left(\frac{1}{2^p} \right)^{1+\varepsilon}.$$

Define now two sequences (\bar{u}_n) and (\tilde{u}_n) as follows:

$$\bar{u}_n = \frac{1}{2^p} \text{ if } 2^p \max(x_p, t_p) \leq \frac{n}{k} < 2^{p+1} \max(x_{p+1}, t_{p+1}) \text{ and } \tilde{u}_n = \frac{1}{2^p} \text{ if } N_p \leq n < N_{p+1}.$$

Note that (\bar{u}_n) is indeed well-defined since $n/k \rightarrow \infty$, that $\bar{u}_n \rightarrow 0$, $\tilde{u}_n \rightarrow 0$ and $n\bar{u}_n/k \rightarrow \infty$ by construction, and that both $1/\bar{u}_n$ and $1/\tilde{u}_n$ are sequences of integers.

Set finally $u_n = \max(\bar{u}_n, \tilde{u}_n)$. Then, as announced, (u_n) is a positive sequence such that $u_n \rightarrow 0$, $nu_n/k \rightarrow \infty$ and $1/u_n$ is an integer for any n . Furthermore, if n and p are such that $u_n = 1/2^p$, then $\bar{u}_n \leq 1/2^p$ and as such $nu_n/k \geq n\bar{u}_n/k \geq x_p$, by construction of \bar{u}_n .

We then get:

$$\max \left(\left| u_n^{-\gamma} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} - 1 \right|, \left| u_n^{-\rho} \frac{A(nu_n/k)}{A(n/k)} - 1 \right| \right) \leq \left(\frac{1}{2^p} \right)^{1/2} = \sqrt{u_n}. \quad (\text{A.17})$$

This shows that the sequence (u_n) satisfies (i). Define further $W_n = \tilde{W}_{n,p}$ and $Z_n = \tilde{Z}_{n,p}$ if and only if $u_n = 1/2^p$. Then (W_n) is a sequence of Brownian motions; besides, by construction, if $u_n = 1/2^p$ then $\tilde{u}_n \leq 1/2^p$, which entails $n \geq N_p$ and thus

$$\mathbb{P} (u_n^{-\varepsilon} Z_n > u_n) = \mathbb{P} \left(\tilde{Z}_{n,p} > \left(\frac{1}{2^p} \right)^{1+\varepsilon} \right) < \frac{1}{2^p} = u_n.$$

Since $u_n \rightarrow 0$ this shows that $u_n^{-\varepsilon} Z_n = o_{\mathbb{P}}(1)$, or equivalently that $Z_n = o_{\mathbb{P}}(u_n^{\varepsilon})$. And by construction, for any $s \in (0, 1]$,

$$\begin{aligned} & s^{\gamma+1/2+\varepsilon} \left| \sqrt{\frac{k}{u_n}} \left(\frac{\hat{q}_{1-ks/(nu_n)}}{q_{1-k/(nu_n)}} - s^{-\gamma} \right) - \gamma s^{-\gamma-1} W_n(s) - \sqrt{\frac{k}{u_n}} A_0 \left(\frac{nu_n}{k} \right) s^{-\gamma} \frac{s^{-\rho} - 1}{\rho} \right| \\ & \leq Z_n = o_{\mathbb{P}}(u_n^{\varepsilon}). \end{aligned} \quad (\text{A.18})$$

Finally, if n is such that $u_n = 1/2^p$, then $\bar{u}_n \leq 1/2^p$ and as such $nu_n/k \geq n\bar{u}_n/k \geq t_p$, by construction of \bar{u}_n . Therefore, for any $s \in (0, 1]$,

$$s^{\gamma+\rho+\varepsilon} \left| \frac{1}{A_0(nu_n/k)} \left(\frac{q_{1-ks/(nu_n)}}{q_{1-k/(nu_n)}} - s^{-\gamma} \right) - s^{-\gamma} \frac{s^{-\rho} - 1}{\rho} \right| \leq \left(\frac{1}{2^p} \right)^{1+\varepsilon} = u_n^{1+\varepsilon}$$

which we rewrite as

$$\begin{aligned} & s^{\gamma+\rho+\varepsilon} \left| \sqrt{\frac{k}{u_n}} \left(\frac{q_{1-ks/(nu_n)}}{q_{1-k/(nu_n)}} - s^{-\gamma} \right) - \sqrt{\frac{k}{u_n}} A_0 \left(\frac{nu_n}{k} \right) s^{-\gamma} \frac{s^{-\rho} - 1}{\rho} \right| \\ & \leq \left\{ \frac{A_0(nu_n/k)}{A(nu_n/k)} \right\} \times \sqrt{\frac{k}{u_n}} A \left(\frac{nu_n}{k} \right) \times u_n^{1+\varepsilon} \end{aligned}$$

for any $s \in (0, 1]$. Use the fact that A_0 is asymptotically equivalent to A to get

$$s^{\gamma+\rho+\varepsilon} \left| \sqrt{\frac{k}{u_n}} \left(\frac{q_{1-ks/(nu_n)}}{q_{1-k/(nu_n)}} - s^{-\gamma} \right) - \sqrt{\frac{k}{u_n}} A_0 \left(\frac{nu_n}{k} \right) s^{-\gamma} \frac{s^{-\rho} - 1}{\rho} \right| = o \left(u_n^\varepsilon \sqrt{\frac{k}{u_n}} A \left(\frac{nu_n}{k} \right) \right) \quad (\text{A.19})$$

uniformly in $s \in (0, 1]$. Combine (A.18) and (A.19) to obtain (ii). This ends the proof.

□

Proof of Theorem 4 Choose n so large that $1/2 < \tau_n < (n-1)/n$. Denote by k the quantity $n(1 - \tau_n)$, so that $\tau_n = 1 - k/n$. Our first main goal is to prove the desired convergence for the estimator $\hat{\xi}_{\tau_n}^L = \hat{\xi}_{1-k/n}^L$. Observe that

$$\hat{\xi}_\tau^L = \sum_{i=1}^n \left\{ K_\tau \left(\frac{i}{n} \right) - K_\tau \left(\frac{i-1}{n} \right) \right\} Y_{i,n} = Y_{n,n} + \sum_{i=1}^{n-1} K_\tau \left(\frac{i}{n} \right) [Y_{i,n} - Y_{i+1,n}].$$

Since, for any $t \in (0, 1)$, $K_\tau(t)$ is a decreasing function of $\tau \in (1/2, 1)$, we obtain that $\hat{\xi}_\tau^L$ is a sample-wise nondecreasing function of $\tau \in (1/2, 1)$, and as such

$$\hat{\xi}_{1-[k]/n}^L \leq \hat{\xi}_{\tau_n}^L \leq \hat{\xi}_{1-k/n}^L.$$

Writing then, for n large enough:

$$\begin{aligned} \sqrt{n(1 - \tau_n)} \left(\frac{\hat{\xi}_{\tau_n}^L}{\xi_{\tau_n}} - 1 \right) &= \sqrt{k} \left(\frac{\hat{\xi}_{1-k/n}^L}{\xi_{1-k/n}} - 1 \right) \\ &\leq \sqrt{k} \left(\frac{\hat{\xi}_{1-[k]/n}^L}{\xi_{1-[k]/n}} - 1 \right) \frac{\xi_{1-[k]/n}}{\xi_{1-k/n}} + \sqrt{k} \left(\frac{\xi_{1-[k]/n}}{\xi_{1-k/n}} - 1 \right), \end{aligned}$$

it comes as a straightforward consequence of Proposition 4 that

$$\sqrt{n(1-\tau_n)} \left(\frac{\hat{\xi}_{\tau_n}^L}{\xi_{\tau_n}} - 1 \right) \leq \sqrt{k} \left(\frac{\hat{\xi}_{1-[k]/n}^L}{\xi_{1-[k]/n}} - 1 \right) (1 + o(1)) + o(1).$$

A similar lower bound holds, and so it suffices to consider the convergence of $\hat{\xi}_{1-k/n}^L$ in the case when $k = n(1 - \tau_n)$ is a sequence of integers with $k \rightarrow \infty$, $n/k \rightarrow \infty$ and $\sqrt{k}A(n/k) = O(1)$.

By Proposition 3(i),

$$\sqrt{k} \left(\frac{\hat{\xi}_{1-k/n}^L}{\xi_{1-k/n}} - 1 \right) = \frac{1}{\Gamma(1-\gamma)(\log 2)^\gamma} \left(\sqrt{k} \frac{\hat{\xi}_{1-k/n}^L - \xi_{1-k/n}}{q_{1-k/n}} \right) (1 + o(1)).$$

It is then enough to show the following convergence:

$$\sqrt{k} \int_0^1 J_{1-k/n}(t) \frac{\hat{q}_t - q_t}{q_{1-k/n}} dt \xrightarrow{d} \gamma(\log 2)^{\gamma+1/2} \int_0^\infty e^{-s} s^{-\gamma-1} W(s) ds \quad (\text{A.20})$$

where W is a standard Brownian motion. The idea for this is to control the process $\hat{q}_t - q_t$ separately in the left tail of Y , in the center of the distribution of Y , and then in the right tail of the distribution of Y . More precisely, we break the integral in the left-hand side of (A.20) as follows:

$$\begin{aligned} \sqrt{k} \int_0^1 J_{1-k/n}(t) \frac{\hat{q}_t - q_t}{q_{1-k/n}} dt &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4} + I_{n,5} \\ \text{with } I_{n,1} &= \sqrt{k} \int_0^{1/n} J_{1-k/n}(t) \frac{\hat{q}_t - q_t}{q_{1-k/n}} dt, \\ I_{n,2} &= \sqrt{k} \int_{1/n}^\delta J_{1-k/n}(t) \frac{\hat{q}_t - q_t}{q_{1-k/n}} dt, \\ I_{n,3} &= \sqrt{k} \int_\delta^{1-\delta} J_{1-k/n}(t) \frac{\hat{q}_t - q_t}{q_{1-k/n}} dt, \\ I_{n,4} &= \sqrt{k} \int_{1-\delta}^{1-k/(nu_n)} J_{1-k/n}(t) \frac{\hat{q}_t - q_t}{q_{1-k/n}} dt, \\ \text{and } I_{n,5} &= \sqrt{k} \int_{1-k/(nu_n)}^1 J_{1-k/n}(t) \frac{\hat{q}_t - q_t}{q_{1-k/n}} dt, \end{aligned} \quad (\text{A.21})$$

where $\delta \in (0, 1/2)$ is chosen such that

$$\forall t \geq 1 - \delta, \quad 0 < \frac{1}{f(q_t)} \leq \frac{2}{\gamma} \frac{q_t}{1-t} \quad (\text{A.22})$$

(this is possible by the von Mises condition (14) in the main paper) and (u_n) is constructed by applying Lemma A.4. We study each term separately.

Control of $I_{n,1}$: Note that

$$\left| \int_0^{1/n} J_{1-k/n}(t) \widehat{q}_t dt \right| \leq J_{1-k/n}(1/n) \int_0^1 |\widehat{q}_t| dt = O_{\mathbb{P}}(J_{1-k/n}(1/n)),$$

by the law of large numbers. Furthermore

$$\left| \int_0^{1/n} J_{1-k/n}(t) q_t dt \right| \leq J_{1-k/n}(1/n) \int_0^1 |q_t| dt = J_{1-k/n}(1/n) \mathbb{E}|Y|.$$

Because $k/n \rightarrow 0$ and $q_{1-k/n} \rightarrow \infty$, the following crude bound then applies:

$$|I_{n,1}| = o_{\mathbb{P}}(\sqrt{n} J_{1-k/n}(1/n)).$$

Apply now Lemma A.3(i) to get

$$|I_{n,1}| = o_{\mathbb{P}}\left(\sqrt{n} \exp\left(-\frac{\log 2}{2} \times \frac{n \log n}{k}\right)\right) = o_{\mathbb{P}}\left(\exp\left(\left[\frac{1}{2} - \frac{n \log 2}{k}\right] \log n\right)\right)$$

which, since $n/k \rightarrow \infty$, translates into

$$|I_{n,1}| = o_{\mathbb{P}}(1). \tag{A.23}$$

Control of $I_{n,2}$: Use the approximation of Theorem 6.2.1 in Csörgő and Horváth (1993) (see also Proposition 2.4.9 in de Haan and Ferreira, 2006) to obtain that for any $\varepsilon \in (0, 1/2)$, there is a sequence of Brownian bridges (B_n) with

$$\widehat{q}_t - q_t = \frac{1}{\sqrt{n}} \left[\frac{B_n(t)}{f(q_t)} + O_{\mathbb{P}}\left(\frac{n^{-\varepsilon} t^{-\varepsilon+1/2} (1-t)^{-\varepsilon+1/2}}{f(q_t)}\right) \right] \tag{A.24}$$

uniformly in $t \in [1/n, (n-1)/n]$. Report this into $I_{n,2}$ to get

$$I_{n,2} = \sqrt{\frac{k}{n}} \int_{1/n}^{\delta} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{B_n(t)}{f(q_t)} dt + O_{\mathbb{P}}\left(\sqrt{\frac{k}{n}} \int_{1/n}^{\delta} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{n^{-\varepsilon} t^{-\varepsilon+1/2} (1-t)^{-\varepsilon+1/2}}{f(q_t)} dt\right).$$

Recall that any Brownian bridge B is such that $B(t) \stackrel{d}{=} W(t) - tW(1)$ with W being a standard Brownian motion. Because

$$\sup_{0 < t < 1-\delta} \frac{|W(t)|}{t^{1/2-\varepsilon}} < \infty \quad \text{almost surely} \tag{A.25}$$

(for instance as a consequence of the law of the iterated logarithm, see Theorem 1.9 and Corollary 1.10 in Chapter II of Revuz and Yor, 1999), we obtain in particular that

$$\sup_{0 < t < 1-\delta} \frac{|B_n(t)|}{t^{1/2-\varepsilon}} = O_{\mathbb{P}}(1). \quad (\text{A.26})$$

As such, and noting that $1/f(q_t) = q'_t$, the derivative of $t \mapsto q_t$, we obtain

$$|I_{n,2}| = O_{\mathbb{P}} \left(\sqrt{\frac{k}{n}} \int_{1/n}^{\delta} \frac{J_{1-k/n}(t)}{q_{1-k/n}} t^{-\varepsilon+1/2} q'_t dt \right).$$

Set now $c = (\log 2)/4 > 0$ and apply Lemma A.3(ii) to get

$$|I_{n,2}| = O_{\mathbb{P}} \left(\sqrt{\frac{n}{k}} \frac{\delta^{cn/k}}{q_{1-k/n}} \int_{1/n}^{\delta} t q'_t dt \right).$$

Notice that

$$\int_{1/n}^{\delta} t q'_t dt = \delta q_{\delta} - \frac{1}{n} q_{1/n} - \int_{1/n}^{\delta} q_t dt.$$

Since $\mathbb{E}|Y| < \infty$, the integral on the right-hand side is clearly bounded as $n \rightarrow \infty$, and it is a simple consequence of the Markov inequality that $n^{-1}q_{1/n}$ must also stay bounded as $n \rightarrow \infty$. Consequently

$$|I_{n,2}| = O_{\mathbb{P}} \left(\sqrt{\frac{n}{k}} \frac{\delta^{cn/k}}{q_{1-k/n}} \right) = O_{\mathbb{P}} \left(\sqrt{\frac{n}{k}} \delta^{cn/k} \right).$$

Since $n/k \rightarrow \infty$ and $\delta \in (0, 1)$, this gives

$$|I_{n,2}| = o_{\mathbb{P}}(1). \quad (\text{A.27})$$

Control of $I_{n,3}$: Use the previous Brownian bridge approximation together with (A.26), and note that the function $t \mapsto t^{-\varepsilon+1/2}(1-t)^{-\varepsilon+1/2}/f(q_t)$ is obviously bounded on $[\delta, 1-\delta]$ to get

$$|I_{n,3}| = O_{\mathbb{P}} \left(\sqrt{\frac{k}{n}} \int_{\delta}^{1-\delta} \frac{J_{1-k/n}(t)}{q_{1-k/n}} dt \right).$$

Now the function $J_{1-k/n}$ has unit integral, so that

$$|I_{n,3}| = O_{\mathbb{P}} \left(\sqrt{\frac{k}{n}} \times \frac{1}{q_{1-k/n}} \right).$$

Finally, because the function $t \mapsto t^{-1/2}q_{1-t^{-1}}$ is regularly varying at infinity with index $\gamma - 1/2 < 0$, it follows by Proposition B.1.9.1 in de Haan and Ferreira (2006, p.366) that

$$|I_{n,3}| = o_{\mathbb{P}}(1). \quad (\text{A.28})$$

Control of $I_{n,4}$: Use again the Brownian bridge approximation together with the boundedness of the function $t \mapsto t^{-\varepsilon+1/2}$ on a neighborhood of 1 to write

$$\begin{aligned} I_{n,4} &= \sqrt{\frac{k}{n}} \int_{1-\delta}^{1-k/(nu_n)} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{B_n(t)}{f(q_t)} dt \\ &+ O_{\mathbb{P}} \left(\sqrt{\frac{k}{n}} \int_{1-\delta}^{1-k/(nu_n)} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{n^{-\varepsilon}(1-t)^{-\varepsilon+1/2}}{f(q_t)} dt \right). \end{aligned} \quad (\text{A.29})$$

We control the two terms on the right-hand side separately. Recalling that the covariance function of a Brownian bridge at times s and t is $\min(s, t) - st$ (see p.37 of Revuz and Yor, 1999), we get that the first term has variance

$$\begin{aligned} &\text{Var} \left(\sqrt{\frac{k}{n}} \int_{1-\delta}^{1-k/(nu_n)} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{B_n(t)}{f(q_t)} dt \right) \\ &= \frac{k}{n} \frac{1}{[q_{1-k/n}]^2} \int_{1-\delta}^{1-k/(nu_n)} \int_{1-\delta}^{1-k/(nu_n)} J_{1-k/n}(s) J_{1-k/n}(t) \frac{\min(s, t) - st}{f(q_s) f(q_t)} ds dt. \end{aligned}$$

Using (A.22), the fact that $s \mapsto q_s$ and $s \mapsto (1-s)^{-1}$ are increasing functions, the inequality $\min(s, t) - st \leq s(1-t)$ and the fact that the function $J_{1-k/n}$ has unit integral, we then have:

$$\text{Var} \left(\sqrt{\frac{k}{n}} \int_{1-\delta}^{1-k/(nu_n)} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{B_n(t)}{f(q_t)} dt \right) \leq \frac{4}{\gamma^2} u_n \left(\int_{1-\delta}^{1-k/(nu_n)} J_{1-k/n}(t) dt \right)^2 \leq \frac{4}{\gamma^2} u_n \rightarrow 0.$$

As the term whose variance we bound is a centered random variable, we finally obtain

$$\sqrt{\frac{k}{n}} \int_{1-\delta}^{1-k/(nu_n)} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{B_n(t)}{f(q_t)} dt = o_{\mathbb{P}}(1). \quad (\text{A.30})$$

The control of the remainder term in $I_{n,4}$ follows the same ideas:

$$\left| \sqrt{\frac{k}{n}} \int_{1-\delta}^{1-k/(nu_n)} \frac{J_{1-k/n}(t)}{q_{1-k/n}} \frac{n^{-\varepsilon}(1-t)^{-\varepsilon+1/2}}{f(q_t)} dt \right| \leq \frac{2}{\gamma} \frac{u_n^{1/2+\varepsilon}}{k^{\varepsilon}} = o(1). \quad (\text{A.31})$$

Combining (A.29), (A.30) and (A.31), we obtain

$$|I_{n,4}| = o_{\mathbb{P}}(1). \quad (\text{A.32})$$

Control of $I_{n,5}$: Use the change of variables $t = 1 - ks/(nu_n)$ and then Lemma A.4(ii) to obtain that there exist $\varepsilon \in (0, 1/2 - \gamma)$ and a sequence of Brownian motions (W_n) such that:

$$\begin{aligned}
I_{n,5} &= \frac{k}{n\sqrt{u_n}} \int_0^1 J_{1-k/n}(1 - ks/(nu_n)) \sqrt{\frac{k}{u_n}} \left(\frac{\hat{q}_{1-ks/(nu_n)} - q_{1-ks/(nu_n)}}{q_{1-k/n}} \right) ds \\
&= \frac{1}{\sqrt{u_n}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) \gamma s^{-\gamma-1} W_n(s) ds \\
&\quad + o \left(\frac{1}{u_n^{1/2-\varepsilon}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \left\{ \sqrt{\frac{k}{u_n}} A \left(\frac{nu_n}{k} \right) \right\} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) s^{-\gamma-\rho-\varepsilon} ds \right) \\
&\quad + o_{\mathbb{P}} \left(\frac{1}{u_n^{1/2-\varepsilon}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) s^{-\gamma-1/2-\varepsilon} ds \right). \tag{A.33}
\end{aligned}$$

To control the first term above, we use first the change of variables $s = tu_n$ and the self-similarity of the standard Brownian motion w.r.t. scaling to get, if W denotes a generic Brownian motion,

$$\begin{aligned}
&\frac{1}{\sqrt{u_n}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) \gamma s^{-\gamma-1} W_n(s) ds \\
&\stackrel{d}{=} \left\{ u_n^{-\gamma} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \right\} \int_0^{1/u_n} \frac{k}{n} J_{1-k/n}(1 - kt/n) \gamma t^{-\gamma-1} W(t) dt.
\end{aligned}$$

Using Lemma A.4(i) and the definition of the function $J_{1-k/n}$, this entails

$$\begin{aligned}
&\frac{1}{\sqrt{u_n}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) \gamma s^{-\gamma-1} W_n(s) ds \\
&\stackrel{d}{=} \gamma \log 2 \int_0^{1/u_n} (1 - kt/n)^{r(1-k/n)-1} t^{-\gamma-1} W(t) dt (1 + o(1)).
\end{aligned}$$

Note that, pointwise, the integrand in the right-hand side above converges to $2^{-t} t^{-\gamma-1} W(t)$; let us then consider

$$S_n := \int_0^{1/u_n} (1 - kt/n)^{r(1-k/n)-1} t^{-\gamma-1} W(t) dt \quad \text{and} \quad T_n := \int_0^\infty 2^{-t} t^{-\gamma-1} W(t) dt.$$

Remark that T_n is indeed well-defined with probability 1, in virtue of the combination of (A.25), the inequality $\gamma < 1/2$ and the self-similarity of the Brownian motion w.r.t. time-inversion. Write then, thanks to the equality $\mathbb{E}|W(t)| = \sqrt{t} \mathbb{E}|W(1)| = \sqrt{2t/\pi}$,

$$\begin{aligned}
\mathbb{E}|S_n - T_n| &\leq \sqrt{\frac{2}{\pi}} \int_0^\infty \left| (1 - kt/n)^{r(1-k/n)-1} - 2^{-t} \right| t^{-\gamma-1/2} \mathbb{I}_{\{t \leq 1/u_n\}} dt \\
&\quad + \sqrt{\frac{2}{\pi}} \int_0^\infty 2^{-t} t^{-\gamma-1/2} \mathbb{I}_{\{t > 1/u_n\}} dt.
\end{aligned}$$

Since $\gamma < 1/2$, we may use a conjunction of Lemma A.2(i) and (ii) and the dominated convergence theorem to get $\mathbb{E}|S_n - T_n| \rightarrow 0$. In particular, $S_n = T_n + o_{\mathbb{P}}(1)$ and therefore

$$\frac{1}{\sqrt{u_n}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) \gamma s^{-\gamma-1} W_n(s) ds \xrightarrow{d} \gamma \log 2 \int_0^\infty 2^{-t} t^{-\gamma-1} W(t) dt. \quad (\text{A.34})$$

The second term in (A.33) is controlled by using the change of variables $s = tu_n$ and then by using Lemma A.4(i):

$$\begin{aligned} & \frac{1}{u_n^{1/2-\varepsilon}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \left\{ \sqrt{\frac{k}{u_n}} A\left(\frac{nu_n}{k}\right) \right\} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) s^{-\gamma-\rho-\varepsilon} ds \\ &= \left\{ u_n^{-\gamma} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \right\} \left\{ u_n^{-\rho} \frac{A(nu_n/k)}{A(n/k)} \right\} \left\{ \sqrt{k} A\left(\frac{n}{k}\right) \right\} \int_0^{1/u_n} \frac{k}{n} J_{1-k/n}(1 - kt/n) t^{-\gamma-\rho-\varepsilon} dt \\ &= O\left(\int_0^{1/u_n} (1 - kt/n)^{r(1-k/n)-1} t^{-\gamma-\rho-\varepsilon} dt \right). \end{aligned}$$

Using Lemma A.2(i) and (ii) and the dominated convergence theorem, we get

$$\frac{1}{u_n^{1/2-\varepsilon}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \left\{ \sqrt{\frac{k}{u_n}} A\left(\frac{nu_n}{k}\right) \right\} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) s^{-\gamma-\rho-\varepsilon} ds = O(1). \quad (\text{A.35})$$

The third term in (A.33) is again controlled by using the change of variables $s = tu_n$ and then by using Lemma A.4(i):

$$\begin{aligned} & \frac{1}{u_n^{1/2-\varepsilon}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) s^{-\gamma-1/2-\varepsilon} ds \\ &= \left\{ u_n^{-\gamma} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \right\} \int_0^{1/u_n} \frac{k}{n} J_{1-k/n}(1 - kt/n) t^{-\gamma-1/2-\varepsilon} dt \\ &= O\left(\int_0^{1/u_n} (1 - kt/n)^{r(1-k/n)-1} t^{-\gamma-1/2-\varepsilon} dt \right). \end{aligned}$$

Use again Lemma A.2(i) and (ii) and apply the dominated convergence theorem to obtain

$$\frac{1}{u_n^{1/2-\varepsilon}} \frac{q_{1-k/(nu_n)}}{q_{1-k/n}} \int_0^1 \frac{k}{n} J_{1-k/n}(1 - ks/(nu_n)) s^{-\gamma-1/2-\varepsilon} ds = O(1). \quad (\text{A.36})$$

Combining (A.33), (A.34), (A.35) and (A.36) yields

$$I_{n,5} \xrightarrow{d} \gamma \log 2 \int_0^\infty 2^{-t} t^{-\gamma-1} W(t) dt.$$

The change of variables $s = t \log 2$ and the self-similarity of W w.r.t. scaling now clearly entail

$$I_{n,5} \xrightarrow{d} \gamma(\log 2)^{\gamma+1/2} \int_0^\infty e^{-s} s^{-\gamma-1} W(s) ds. \quad (\text{A.37})$$

It only remains to combine (A.21), (A.23), (A.27), (A.28), (A.32) and (A.37) to prove (A.20) and therefore complete the proof of the stated convergence for $\hat{\xi}_{\tau_n}^L$.

We now prove that this convergence implies that of $\hat{\xi}_{\tau_n}^{LM}$ as well as that of $\hat{\xi}_{\tau_n}^M$. Noting that J_{τ_n} has integral 1, we have

$$\frac{1}{n} \sum_{i=1}^n J_{\tau_n} \left(\frac{i}{n} \right) - 1 = \sum_{i=1}^n \int_{(i-1)/n}^{i/n} \left[J_{\tau_n} \left(\frac{i}{n} \right) - J_{\tau_n}(t) \right] dt.$$

Since J_{τ_n} is an increasing function for n large enough, this entails

$$0 \leq \frac{1}{n} \sum_{i=1}^n J_{\tau_n} \left(\frac{i}{n} \right) - 1 \leq \frac{1}{n} \sum_{i=1}^n \left[J_{\tau_n} \left(\frac{i}{n} \right) - J_{\tau_n} \left(\frac{i-1}{n} \right) \right] = \frac{J_{\tau_n}(1)}{n}.$$

Because $(1 - \tau_n)J_{\tau_n}(1) \rightarrow \log(2)$, this implies

$$\sqrt{n(1 - \tau_n)} \left(\frac{1}{n} \sum_{i=1}^n J_{\tau_n} \left(\frac{i}{n} \right) - 1 \right) = O \left(\frac{1}{\sqrt{n(1 - \tau_n)}} \right) \rightarrow 0.$$

In other words,

$$\hat{\xi}_{\tau_n}^M = \hat{\xi}_{\tau_n}^{LM} \left(1 + o \left(\frac{1}{\sqrt{n(1 - \tau_n)}} \right) \right).$$

As a consequence,

$$\sqrt{n(1 - \tau_n)} \left(\frac{\hat{\xi}_{\tau_n}^M}{\hat{\xi}_{\tau_n}} - 1 \right) = \sqrt{n(1 - \tau_n)} \left(\frac{\hat{\xi}_{\tau_n}^{LM}}{\hat{\xi}_{\tau_n}} - 1 \right) (1 + o(1)) + o(1)$$

and it is enough to prove the convergence of $\hat{\xi}_{\tau_n}^{LM}$. Define then $J_{\tau_n}^{\text{step}}(t) = J_{\tau_n}(\lceil nt \rceil / n)$, and notice that

$$\hat{\xi}_{\tau_n}^{LM} = \frac{1}{n} \sum_{i=1}^n J_{\tau_n} \left(\frac{i}{n} \right) Y_{i,n} = \int_0^1 J_{\tau_n}^{\text{step}}(t) \hat{q}_t dt.$$

Then clearly

$$\frac{\sqrt{n(1 - \tau_n)}}{\hat{\xi}_{\tau_n}} \left| \hat{\xi}_{\tau_n}^{LM} - \hat{\xi}_{\tau_n}^L \right| \leq \frac{\sqrt{n(1 - \tau_n)}}{\hat{\xi}_{\tau_n}} \int_0^1 |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\hat{q}_t| dt.$$

We next let $t_0 \in (0, 1)$ be such that $q_{t_0} > 1$ (this is possible, since Y has a heavy right tail), and we rewrite the above bound as

$$\begin{aligned} \frac{\sqrt{n(1-\tau_n)}}{\xi_{\tau_n}} \left| \widehat{\xi}_{\tau_n}^{LM} - \widehat{\xi}_{\tau_n}^L \right| &\leq \frac{\sqrt{n(1-\tau_n)}}{\xi_{\tau_n}} \int_0^{t_0} |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\widehat{q}_t| dt \\ &+ \frac{\sqrt{n(1-\tau_n)}}{\xi_{\tau_n}} \int_{t_0}^1 |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\widehat{q}_t| dt. \end{aligned} \quad (\text{A.38})$$

To control the integrals in this upper bound, we note that for n large enough, by the mean value theorem,

$$|J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| \leq \left(\frac{\lfloor nt \rfloor}{n} - t \right) J'_{\tau_n} \left(\frac{\lfloor nt \rfloor}{n} \right) \leq \frac{1}{n} J'_{\tau_n} \left(\frac{\lfloor nt \rfloor}{n} \right) \quad \text{for all } t \in (0, 1). \quad (\text{A.39})$$

Observe also that

$$\int_0^{t_0} |\widehat{q}_t| dt \leq \int_0^1 |\widehat{q}_t| dt = O_{\mathbb{P}}(1)$$

by the law of large numbers, therefore yielding

$$\frac{\sqrt{n(1-\tau_n)}}{\xi_{\tau_n}} \int_0^{t_0} |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\widehat{q}_t| dt = o_{\mathbb{P}} \left(\sqrt{\frac{1-\tau_n}{n}} J'_{\tau_n} \left(\frac{\lfloor nt_0 \rfloor}{n} \right) \right).$$

Recalling that $J'_{\tau_n}(t) = r(\tau_n)[r(\tau_n) - 1]t^{r(\tau_n)-2}$ and $r(\tau_n) = (1 - \tau_n)^{-1} \log(2)(1 + o(1))$, we easily obtain, thanks to the convergence $\lfloor nt_0 \rfloor/n \rightarrow t_0 < 1$, that

$$\sqrt{\frac{1-\tau_n}{n}} J'_{\tau_n} \left(\frac{\lfloor nt_0 \rfloor}{n} \right) = o \left((1 - \tau_n)^{-1} t_0^{(1-\tau_n)^{-1} \log(2)/2} \right) \rightarrow 0.$$

It follows that

$$\frac{\sqrt{n(1-\tau_n)}}{\xi_{\tau_n}} \int_0^{t_0} |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\widehat{q}_t| dt = o_{\mathbb{P}}(1). \quad (\text{A.40})$$

Besides, we have, on the interval $(t_0, 1)$ and for n large enough,

$$\frac{1}{n} J'_{\tau_n} \left(\frac{\lfloor nt \rfloor}{n} \right) \leq \frac{r(\tau_n) - 1}{\lfloor nt_0 \rfloor} J_{\tau_n} \left(\frac{\lfloor nt \rfloor}{n} \right) \leq \frac{2 \log 2}{t_0} \frac{1}{n(1-\tau_n)} J_{\tau_n}(t). \quad (\text{A.41})$$

Here, the upper bound

$$\frac{J_{\tau_n}(\lfloor nt \rfloor/n)}{J_{\tau_n}(t)} \leq \left(1 + \frac{1}{nt_0} \right)^{r(\tau_n)-1}, \quad t \in (t_0, 1)$$

was used, together with the convergence

$$\left(1 + \frac{1}{nt_0} \right)^{r(\tau_n)-1} = \exp \left((1 - \tau_n)^{-1} \log(2) \log \left[1 + \frac{1}{nt_0} \right] (1 + o(1)) \right) \rightarrow 1$$

which is valid since $n(1 - \tau_n) \rightarrow \infty$. Applying (A.39) and (A.41), we get, for n large enough:

$$\begin{aligned} \frac{\sqrt{n(1 - \tau_n)}}{\xi_{\tau_n}} \int_{t_0}^1 |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\hat{q}_t| dt &\leq \frac{\sqrt{n(1 - \tau_n)}}{\xi_{\tau_n}} \int_{t_0}^1 \frac{1}{n} J'_{\tau_n} \left(\frac{[nt]}{n} \right) |\hat{q}_t| dt \\ &\leq \frac{2 \log 2}{t_0} \times \frac{1}{\xi_{\tau_n} \sqrt{n(1 - \tau_n)}} \int_{t_0}^1 J_{\tau_n}(t) |\hat{q}_t| dt. \end{aligned}$$

Observe now that $\hat{q}_{t_0} \xrightarrow{\mathbb{P}} q_{t_0}$ (this is a consequence of, for instance, Theorem 6.2.1 in Csörgő and Horváth, 1993). It follows that, with arbitrarily large probability as $n \rightarrow \infty$, one has $\hat{q}_{t_0} > 1/2$, and therefore

$$\frac{\sqrt{n(1 - \tau_n)}}{\xi_{\tau_n}} \int_{t_0}^1 |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\hat{q}_t| dt \leq \frac{2 \log 2}{t_0} \times \frac{1}{\xi_{\tau_n} \sqrt{n(1 - \tau_n)}} \int_{t_0}^1 J_{\tau_n}(t) \hat{q}_t dt.$$

Writing

$$\int_{t_0}^1 J_{\tau_n}(t) \hat{q}_t dt = \hat{\xi}_{\tau_n}^L - \int_0^{t_0} J_{\tau_n}(t) \hat{q}_t dt,$$

it follows that

$$\frac{\sqrt{n(1 - \tau_n)}}{\xi_{\tau_n}} \int_{t_0}^1 |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\hat{q}_t| dt = o_{\mathbb{P}} \left(1 + \frac{1}{\xi_{\tau_n}} \int_0^{t_0} J_{\tau_n}(t) |\hat{q}_t| dt \right).$$

Finally, using the law of large numbers again, we get

$$\int_0^{t_0} J_{\tau_n}(t) |\hat{q}_t| dt \leq r(\tau_n) t_0^{r(\tau_n)-1} \int_0^1 |\hat{q}_t| dt = O_{\mathbb{P}} \left((1 - \tau_n)^{-1} t_0^{(1-\tau_n)^{-1} \log(2)/2} \right) = o_{\mathbb{P}}(1).$$

Consequently

$$\frac{\sqrt{n(1 - \tau_n)}}{\xi_{\tau_n}} \int_{t_0}^1 |J_{\tau_n}^{\text{step}}(t) - J_{\tau_n}(t)| |\hat{q}_t| dt = o_{\mathbb{P}}(1). \quad (\text{A.42})$$

Combining (A.38), (A.40) and (A.42) results in

$$\frac{\sqrt{n(1 - \tau_n)}}{\xi_{\tau_n}} \left| \hat{\xi}_{\tau_n}^{LM} - \hat{\xi}_{\tau_n}^L \right| \xrightarrow{\mathbb{P}} 0.$$

In particular,

$$\sqrt{n(1 - \tau_n)} \left(\frac{\hat{\xi}_{\tau_n}^{LM}}{\xi_{\tau_n}} - 1 \right) = \sqrt{n(1 - \tau_n)} \left(\frac{\hat{\xi}_{\tau_n}^L}{\xi_{\tau_n}} - 1 \right) + o_{\mathbb{P}}(1)$$

which, by using the convergence of $\hat{\xi}_{\tau_n}^L$, concludes the proof. \square

Proof of Theorem 5 Write

$$\log \left(\frac{\hat{\xi}_{\tau'_n}^{M,\star}}{\xi_{\tau'_n}} \right) = (\hat{\gamma} - \gamma) \log \left(\frac{1 - \tau_n}{1 - \tau'_n} \right) + \log \left(\frac{\hat{\xi}_{\tau_n}^M}{\xi_{\tau_n}} \right) - \log \left(\left[\frac{1 - \tau'_n}{1 - \tau_n} \right]^\gamma \frac{\xi_{\tau'_n}}{\xi_{\tau_n}} \right).$$

The convergence $\log[(1 - \tau_n)/(1 - \tau'_n)] \rightarrow \infty$ yields

$$\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \log \left(\frac{\hat{\xi}_{\tau_n}^M}{\xi_{\tau_n}} \right) = O_{\mathbb{P}} \left(\frac{1}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \right) = o_{\mathbb{P}}(1), \quad (\text{A.43})$$

$$\begin{aligned} \text{and } & \frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \log \left(\left[\frac{1 - \tau'_n}{1 - \tau_n} \right]^\gamma \frac{\xi_{\tau'_n}}{\xi_{\tau_n}} \right) \\ &= \frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \left(\log \left(\frac{\xi_{\tau'_n}}{q_{\tau'_n}} \right) - \log \left(\frac{\xi_{\tau_n}}{q_{\tau_n}} \right) + \log \left(\left[\frac{1 - \tau'_n}{1 - \tau_n} \right]^\gamma \frac{q_{\tau'_n}}{q_{\tau_n}} \right) \right) \\ &= O \left(\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} [1 - \tau_n + |A((1 - \tau_n)^{-1})| + 1 - \tau'_n + |A((1 - \tau'_n)^{-1})|] \right) \\ &= O \left(\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} [1 - \tau_n + |A((1 - \tau_n)^{-1})|] \right) \\ &= o(1). \end{aligned} \quad (\text{A.44})$$

Convergence (A.43) is a consequence of our Theorem 4. Convergence (A.44) follows from a combination of Proposition 4, Theorem 2.3.9 in de Haan and Ferreira (2006) and from the regular variation of $|A|$. Combining these elements and using the Delta-method leads to the desired conclusion. \square

Proof of Proposition 5 We have for $\tau < 1/2$ that $\xi_\tau = \int_0^1 J_\tau(t) q_t dt$, where $J_\tau(\cdot)$ is a decreasing positive function satisfying the normalization condition $\int_0^1 J_\tau(t) dt = 1$. Then, according to Acerbi (2002, Theorem 2.5), $-\xi_\tau$ provides a coherent risk measure. \square

Proof of Proposition 6 Since $F \in \text{DA}(\Phi_\gamma)$ with $\gamma < 1$, it follows from Theorem 11 in Bellini *et al.* (2014) together with asymptotic inversion that

$$e_\tau \sim (\gamma^{-1} - 1)^{-\gamma} q_\tau \quad \text{as } \tau \rightarrow 1.$$

Hence $\xi_\tau/e_\tau \sim (\gamma^{-1} - 1)^\gamma \Gamma(1 - \gamma) \{\log 2\}^\gamma$ as $\tau \rightarrow 1$, in view of Proposition 3 (i). On the

other hand, it follows from Hua and Joe (2011) that

$$\frac{\mathbb{E}[Y|Y > q_\tau]}{q_\tau} \sim \frac{1}{1-\gamma} \quad \text{as } \tau \rightarrow 1.$$

Hence, in view of Proposition 3 (i), $\xi_\tau/\mathbb{E}[Y|Y > q_\tau] \sim (1-\gamma)\Gamma(1-\gamma)\{\log 2\}^\gamma = \Gamma(2-\gamma)\{\log 2\}^\gamma$ as $\tau \rightarrow 1$. This completes the proof. \square

B Additional simulations

When empirical extremiles are used to estimate the same quantity as empirical quantiles, our simulation experiments provide Monte-Carlo evidence that the extremile estimators are the most efficient in case of usual short and light-tailed distributions. This benefit in terms of efficiency comes at the price of non-robustness against heavy-tailed distributions. Yet, by considering trimmed extremiles, we recover smaller mean squared errors with respect to sample quantiles that estimate the same quantity.

B.1 Ordinary extremiles

For continuous distributions, extremiles are identical to quantiles but with different orders. Indeed, $\xi_\tau = q_\alpha$ implies $\alpha = \alpha_\tau := F(\xi_\tau)$. Therefore, an empirical extremile $\hat{\xi}_\tau$ and quantile \hat{q}_{α_τ} estimate the same quantity $\xi_\tau \equiv q_{\alpha_\tau}$. To evaluate finite-sample performance of these two estimators we have undertaken some simple Monte Carlo experiments. The simulation experiments all employ the levels $\tau = 0.1, 0.3, \dots, 0.9$. We have considered 2,000 replications for samples of size $n = 100, 200, \dots, 3000$, simulated from various scenarios: Normal(0, 1), Exponential(1), Uniform(0, 1), Beta(2, 2), Chi-square(3), Log-normal(0, 0.5), Weibull(1, π), and Gamma($a = 3, s = 1$) whose density function is $F'(y; a, s) = y^{a-1}e^{-y/s}/(s^a\Gamma(a))$, $y > 0$. The evolution of the ratio $\text{MSE}(\hat{\xi}_\tau^L)/\text{MSE}(\hat{q}_{\alpha_\tau})$ between the Mean Squared Errors with respect to the sample size n , displayed in Figure 1, provides Monte-Carlo evidence that the empirical extremile estimator $\hat{\xi}_\tau^L$ is efficient relative to the empirical quantile for all these usual short and light-tailed distributions.

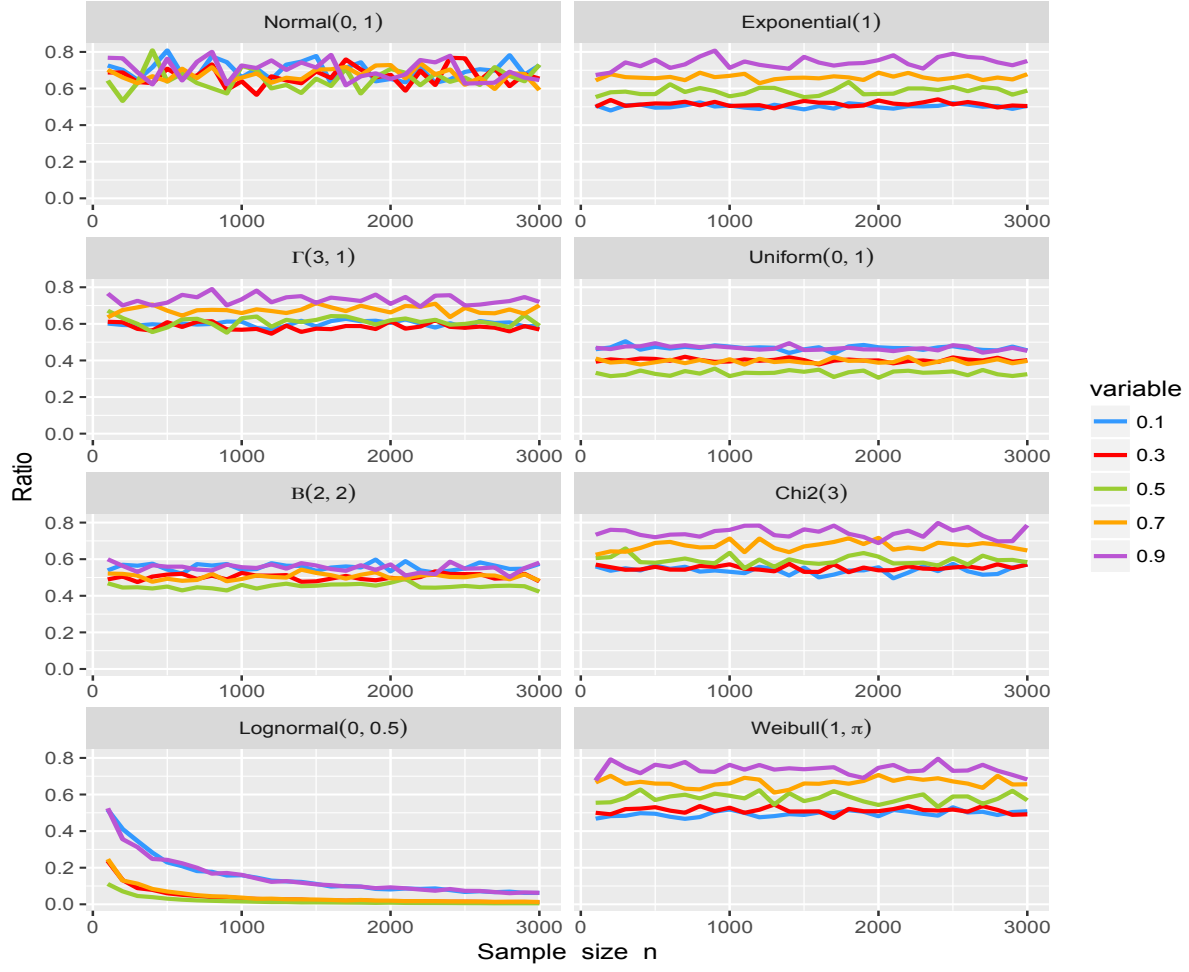


Figure 1: *Some simulation evidence – Ordinary extremes.* The evolution of the ratio $MSE(\hat{\xi}_\tau^L)/MSE(\hat{q}_{\alpha_\tau})$ against the sample size n , at the levels $\tau = 0.1, 0.3, 0.5, 0.7$ and 0.9 .

B.2 Trimmed extremes

We have undertaken some simulation experiments to evaluate the performance of the empirical trimmed extremile $\hat{\xi}_\tau(k_n, k'_n)$ in comparison with the sample quantile \hat{q}_α when they estimate the same quantity $\xi_\tau(k_n, k'_n) \equiv q_\alpha$. We have considered the Burr(4, 1), Pareto(4) and Student(3) distributions, whose heavy tails are likely to affect the ordinary sample extremile $\hat{\xi}_\tau^L$. Here, the Burr(k, c) and Pareto(a) distribution functions are, respectively, $F(y) = 1 - (1 + y^c)^{-k}$, $y > 0$, and $F(y) = 1 - y^{-a}$, $y \geq 1$. The Monte Carlo ratios $MSE(\hat{\xi}_\tau(k_n, k'_n))/MSE(\hat{q}_\alpha)$, shown in Figure 2, were computed over 2,000 replications for $n = 100, 200, \dots, 3000$. The simulation experiments all employ the extremile

levels $\tau = 0.1, 0.3, \dots, 0.9$, and the corresponding quantile levels $\alpha = F(\xi_\tau(k_n, k'_n))$ so that $\xi_\tau(k_n, k'_n) \equiv q_\alpha$. We chose $k'_n = \lceil (5/100) \cdot n / [n^{0.1}] \rceil$ in the left panels of the figure, and $k'_n = \lceil (10/100) \cdot n / [n^{0.1}] \rceil$ in the right panels. Note that k'_n corresponds to $\lceil 5\% \cdot n \rceil$ for $n < 1024$ and to $\lceil 2.5\% \cdot n \rceil$ for $n \geq 1024$. We also chose $k_n = k'_n$ only for the symmetric Student distribution, otherwise $k_n = 0$. Our tentative conclusion from this exercise is that the accuracy of the trimmed extremile estimator is quite respectable with respect to the robust sample quantile, since the MSE ratios are overall smaller than one.

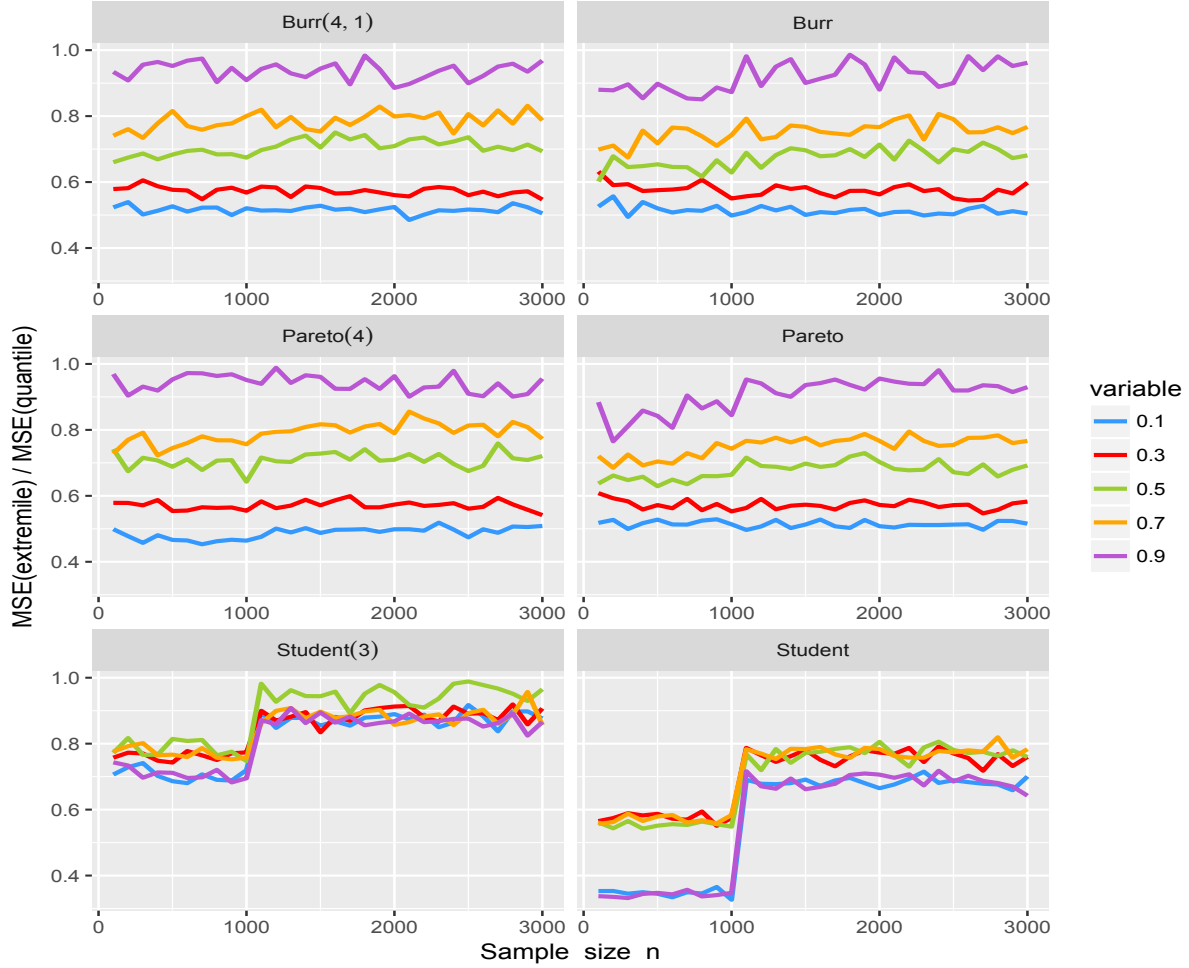


Figure 2: *Trimmed extremes* – From top to bottom, the Burr(4,1), Pareto(4) and Student(3) distributions. From left to right, $k'_n = \lceil (5\%) \cdot n / [n^{0.1}] \rceil$ and $k'_n = \lceil (10\%) \cdot n / [n^{0.1}] \rceil$.

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