# Remarks on existence and uniqueness of Cournot-Nash equilibria in the non-potential case 

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#### Abstract

Keywords: Cournot-Nash equilibria.


## 1 Introduction

Explquer le concept d eq
Faire un catalogue des dependances possibles, expliquer la congestion, les interactions etc....

Dire ce qu on fait et ce qu on a fait dans l autre article: potential games
QQS notations et hypotheses. Throughout the paper both the type space $X$ and the strategy space $Y$ will be assumed to be compact metric spaces.

Throughout the paper, the type space $X$ and the action space $Y$ will be assumed to be compact metric spaces. Given a Borel probability measures $m$ on $X$ (which we shall simply denote $m \in \mathcal{P}(X)$ ) and $T$ a Borel map: $X \rightarrow$ $Y$, the pushforward (or image measure) of $m$ through $T$, is the probability measure $T_{\#} m$ on $Y$ defined by $T_{\#} m(B)=m\left(T^{-1}(B)\right)$ for every Borel subset $B$ of $Y$. The canonical projections on $X \times Y$ will be denoted $\pi_{X}$ and $\pi_{Y}$ respectively. For $m_{1} \in \mathcal{P}(X)$ and $m_{2} \in \mathcal{P}(Y)$, we shall denote by $\Pi\left(m_{1}, m_{2}\right)$ the set of measures $\gamma \in \mathcal{P}(X \times Y)$ having $m_{1}$ and $m_{2}$ as marginals i.e. such that $\pi_{X \#} \gamma=m_{1}$ and $\pi_{Y \#} \gamma=m_{2}$.

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## 2 The regular case

In this section, we first recall the existence of Cournot-Nash in a regular setting in which one can easily apply a fixed-point argument to derive an existence result. What follows is essentially due to Mas-Colell [9], we give a short proof for the sake of completeness. Throughout this section, we suppose here for every $\nu \in \mathcal{P}(Y), C(., .,[\nu])$ is continuous on $X \times Y$ and that $\nu \mapsto C(., .,[\nu])$ is a continuous map from $(\mathcal{P}(Y), \mathrm{w}-*)$ to $\left(C(X \times Y),\|\cdot\|_{\infty}\right)$
where $\mathrm{w}-*$ stands for the weak-* topology on $\mathcal{P}(Y)$. In this setting, CournotNash equilibria are naturally defined as:

Definition 2.1. A Cournot-Nash then consists of a joint probability measure $\gamma \in \mathcal{P}(X \times Y)$ whose first marginal is the fixed measure $\mu \in \mathcal{P}(X)$ and such that, denoting by $\nu$ its second marginal we have

$$
\begin{equation*}
\gamma\left(\left\{(x, y) \in X \times Y: C(x, y,[\nu])=\min _{z \in Y} C(x, z,[\nu])\right\}\right)=1 \tag{2.2}
\end{equation*}
$$

Theorem 2.2. Assume that $X$ and $Y$ are compact metric spaces and that (2.1) holds then there exists at least one Cournot-Nash equilibrium.

Proof. Let

$$
K:=\left\{\gamma \in \mathcal{P}(X \times Y): \pi_{X \#} \gamma=\mu\right\}
$$

Obviously, $K$ is a convex and weakly-* compact subset of $\mathcal{P}(X \times Y)$. Then define for every $\gamma=\mu \otimes \gamma^{x} \in K$

$$
F(\gamma):=\left\{\mu \otimes \eta^{x}: \eta^{x} \in \mathcal{P}\left(\mathcal{Y}_{\gamma}(x)\right)\right\}
$$

where $\mathcal{Y}_{\gamma}(x)$ denotes the closed set

$$
\mathcal{Y}_{\gamma}(x):=\operatorname{argmin}_{y \in Y} C(x, y,[\nu]) \text { with } \nu:=\pi_{Y \#} \gamma .
$$

Note that, for $\gamma$ in $K$, setting $\nu:=\pi_{Y \#} \gamma$, and

$$
\varphi_{\nu}(x):=\min _{z \in Y} C(x, z,[\nu])
$$

(so that $\varphi_{\nu}$ is continuous) $F(\gamma)$ can also be expressed as

$$
F(\gamma)=\left\{\theta \in K: \int_{X \times Y}\left(C(x, y,[\nu])-\varphi_{\nu}(x)\right) d \theta(x, y)=0\right\} .
$$

Hence, $F$ is clearly a (weak-*) closed and convex valued set-valued map $K \rightrightarrows K$. Let us now prove that $F$ has a (weak-*) closed graph, since the
weak star topology is metrizable, it is enough to deal with a sequence $\left(\gamma_{n}, \theta_{n}\right)$ such that $\gamma_{n} \in K, \theta_{n} \in F\left(\gamma_{n}\right), \gamma_{n}$ weakly $*$ converges to some $\gamma$ and $\theta_{n}$ weakly * converges to some $\theta$ in $K$. Setting $\nu:=\pi_{Y \#} \gamma$ and $\nu_{n}:=\pi_{Y \#} \gamma_{n}, \nu_{n}$ weakly * converges to $\nu$. So, thanks to (2.1), C(.,.,,$\left.\left.\nu_{n}\right]\right)$ uniformly converges to $C(., ., .[\nu]), \varphi_{\nu_{n}}$ uniformly converges to $\varphi_{\nu}$, we may therefore pass to the limit in

$$
\int_{X \times Y}\left(C\left(x, y,\left[\nu_{n}\right]\right)-\varphi_{\nu_{n}}(x)\right) d \theta_{n}(x, y)=0
$$

to deduce that $\theta \in F(\gamma)$. It thus follows from Ky Fan's theorem that $F$ admits a fixed-point $\gamma$, defining $\nu:=\pi_{Y \#} \gamma$, it is then easy to see that $(\gamma, \nu)$ is an equilibrium.

The previous result is not fully satisfactory. First, the regularity assumption (2.1) is very demanding since it rules out purely local effects (congestion for instance), there are some extensions to a less regular setting (see e.g. [5]) but to the best of our knowledge all these extensions require some form of lower semicontinuity so that none of them enables one to cope with a local dependence in the cost. Another drawback of an abstract proof relying on a fixed-point theorem is that it is nonconstructive and we'd actually like to say more on uniqueness and charcaterization of equilibria....

## 3 The separable case

Our aim now is to consider costs $(x, y, \nu) \mapsto C(x, y,[\nu])$ with a possible local dependence that is a dependence in $\nu(y)$, in such a case $\nu$ has to be absolutely continuous with respect to some fixed reference measure $m_{0}$ on the action space $Y$ and $\nu(y)$ has to be understood as the value of the Radon-Nikodym derivative of $\nu$ with respect to $m_{0}$ at $y$. This is motivated by congestion i.e. the fact that more frequently played strategies may be more costly and a natural way to take the congestion effect is to consider a term of the form $f(y, \nu(y))$ where $f(y,$.$) is increasing in the total cost C$. As soon as one incorporates local congestion effects, assumption (2.1) is violated and to make the problem still reasonably tractable, we shall now restrict ourselves to the separable case where

$$
\begin{equation*}
C(x, y,[\nu])=c(x, y)+V[\nu](y) \tag{3.1}
\end{equation*}
$$

where $c \in C(X \times Y)$ is a transport (or matching) cost depending only on the agent type and her strategy, whereas the function $V[\nu]$ (which may be defined only $m_{0}$-a.e. and only for $\nu$ 's that are absolutely continuous with respect to $m_{0}$ in the case of a congestion effect) captures an additional cost
created by the whole population of players. The typical case we have in mind is

$$
\begin{equation*}
V[\nu](y):=f(y, \nu(y))+W[\nu](y) \tag{3.2}
\end{equation*}
$$

where the first term is a congestion cost (again one has to understand $\nu$ as its density), $f$ is nondecreasing in its second argument (congestion) and $W[\nu]$ is regular in the sense that that $W[\nu] \in C(Y)$ for every $\nu \in \mathcal{P}(Y)$ with

$$
\begin{equation*}
\nu \mapsto W[\nu] \text { is a continuous map from }(\mathcal{P}(Y), \mathrm{w}-*) \text { to }\left(C(Y),\|\cdot\|_{\infty}\right) \text {. } \tag{3.3}
\end{equation*}
$$

Typical regular costs are those given by averages i.e. $W[\nu](y)=\int_{Y} \phi(y, z) d \nu(z)$ where $\phi$ is continuous. Of course if the congestion cost $f$ is zero, we are left to the regular (in the sense of the previous paragraph) and separable case. Taking the strategy distribution $\nu$ as given, an agent of type $x$ therefore aims to minimize in $y$ the cost $c(x, y)+V[\nu](y)$, since the latter need not be a continuous or even lower semicontinuous, the definition of an equilibrium has to be modified as follows. First, when $V[\nu]$ is regular (in the sense of (3.3) i.e. when $f \equiv 0$ ) let us set $\mathcal{D}:=\mathcal{P}(Y)$, in the congested case where some reference measure $m_{0} \in \mathcal{P}(Y)$ is fixed according to which the congestion is measured and $V[\nu]$ is of the form (3.2), we define the domain:

$$
\begin{equation*}
\mathcal{D}:=\left\{\nu \in \mathrm{L}^{1}\left(m_{0}\right): \int_{Y} f(y, \nu(y)) d m_{0}(y)<+\infty\right\} . \tag{3.4}
\end{equation*}
$$

Note that when $f$ satisfies the power growth condition:

$$
\begin{equation*}
\frac{1}{C}\left(t^{\alpha}-1\right) \leq f(y, t) \leq C\left(t^{\alpha}+1\right) \tag{3.5}
\end{equation*}
$$

for some $\alpha>0$ and $C>0$ and every $(y, t)$ then $\mathcal{D}=L^{p}\left(m_{0}\right)$ for $p=1+\alpha$. As before a Cournot-Nash is a joint type-strategy probability measure $\gamma$ that is consistent with the cost minimizing behavior of agents, in ou setting, this leads to the defi

Definition 3.1. $\gamma \in \mathcal{P}(X \times Y)$ is a Cournot-Nash equilibria if its first marginal is $\mu$, its second marginal, $\nu$, belongs to $\mathcal{D}$ and there exists $\varphi \in \mathbf{C}(X)$ such that

$$
\begin{equation*}
c(x, y)+V[\nu](y) \geq \varphi(x) \text { for } \mu \otimes m_{0} \text {-a.e. }(x, y) \text { with equality } \gamma \text {-a.e.. } \tag{3.6}
\end{equation*}
$$

A Cournot-Nash equilibrium $(\gamma, \nu)$ is called pure if it is of the form $\gamma=$ $(\mathrm{id}, T)_{\#} \mu$ for some Borel map $T: X \rightarrow Y$ (that is agents with the same type use the same strategy).

As noted in [3], in the separable case, Cournot-Nash are very much related to optimal transport. More precisely, for $\nu \in \mathcal{P}(Y)$, let $\Pi(\mu, \nu)$ denote the set of probability measures on $X \times Y$ having $\mu$ and $\nu$ as marginals and let $\mathcal{W}_{c}(\mu, \nu)$ be the least cost of transporting $\mu$ to $\nu$ for the cost $c i . e$. the value of the Monge-Kantorovich optimal transport problem:

$$
\mathcal{W}_{c}(\mu, \nu):=\inf _{\gamma \in \Pi(\mu, \nu)} \iint_{X \times Y} c(x, y) \mathrm{d} \gamma(x, y) .
$$

It is obvious that the optimal transport problem above admits solutions since the admissible set is convex and weakly-* compact. Let us then denote by $\Pi_{o}(\mu, \nu)$ the set of optimal transport plans i.e.

$$
\Pi_{o}(\mu, \nu):=\left\{\gamma \in \Pi(\mu, \nu): \iint_{X \times Y} c(x, y) \mathrm{d} \gamma(x, y)=\mathcal{W}_{c}(\mu, \nu)\right\} .
$$

The link between Cournot-Nash equilibria and optimal transport is based on the following straightforward observation: if $\gamma$ is a Cournot-Nash equilibrium and $\nu$ denotes its second marginal then $\gamma \in \Pi_{o}(\mu, \nu)$. Indeed, if $\varphi \in C(X)$ is such that (3.6) holds and if $\eta \in \Pi(\mu, \nu)$ then we have

$$
\begin{aligned}
& \iint_{X \times Y} c(x, y) \mathrm{d} \eta(x, y) \geq \iint_{X \times Y}(\varphi(x)-V[\nu](y)) \mathrm{d} \eta(x, y) \\
= & \int_{X} \varphi(x) \mathrm{d} \mu(x)-\int_{Y} V[\nu](y) \mathrm{d} \nu(y)=\iint_{X \times Y} c(x, y) \mathrm{d} \gamma(x, y)
\end{aligned}
$$

so that $\gamma \in \Pi_{o}(\mu, \nu)$.
The previous proof also shows that $\varphi$ solves the dual of $\mathcal{W}_{c}(\mu, \nu)$ i.e. maximizes the functional

$$
\int_{X} \varphi(x) \mathrm{d} \mu(x)+\int_{Y} \varphi^{c}(y) \mathrm{d} \nu(y)
$$

where $\varphi^{c}$ denotes the $c$-transform of $\varphi$ i.e.

$$
\begin{equation*}
\varphi^{c}(y):=\min _{x \in X}\{c(x, y)-\varphi(x)\} \tag{3.7}
\end{equation*}
$$

In an euclidean setting, there are well-known conditions on $c$ (the socalled generalized Spence-Mirrlees condition, see [4]) and $\mu$ which guarantee that such an optimal $\gamma$ necessarily is pure whatever $\nu$ is:

Corollary 3.2. Assume that $X=\bar{\Omega}$ where $\Omega$ is some open connected bounded subset of $\mathbb{R}^{d}$ with negligible boundary, that $\mu$ is absolutely continuous with respect to the Lebesgue measure, that $c$ is differentiable with respect to its
first argument, that $\nabla_{x} c$ is continuous on $\mathbb{R}^{d} \times Y$ and that it satisfies the twist condition:

$$
\text { for every } x \in X \text {, the map } y \in Y \mapsto \nabla_{x} c(x, y) \text { is injective, }
$$

then for every $\nu \in \mathcal{P}(Y), \Pi_{0}(\mu, \nu)$ consists of a single element and the latter is of the form $\gamma=(\mathrm{id}, T)_{\#} \mu$ hence every Cournot-Nash equilibrium is pure (and actually fully determined by its second marginal).

In dimension 1, the condition above on $c$ roughly amounts to the usual Spence-Mirrlees singe-crossing condition i.e. the strict monotonicity in $y$ of $\partial_{x} c$ or the following condition on the mixed partial derivative

$$
\partial_{x y}^{2} c \text { has constant sign . }
$$

## 4 Uniqueness under monotonicity

In the framework of Mean-Field Games, Lions and Lasry [7] established that some simple monotonicity property of $\nu \mapsto V[\nu]$ is enough to guarantee uniqueness of the equilibrium. A simple adaptation of their argument gives the elementary uniqueness result:

Theorem 4.1. If $\nu \mapsto V[\nu]$ is strictly monotone in the sense that for every $\nu_{1}$ and $\nu_{2}$ in $\mathcal{P}(Y)$, one has

$$
\int_{Y}\left(V\left[\nu_{1}\right]-V\left[\nu_{2}\right]\right) d\left(\nu_{1}-\nu_{2}\right) \geq 0
$$

and the inequality is strict whenever $\nu_{1} \neq \nu_{2}$ (the fact that the previous integral is well-defined is actually part of the assumption: it holds for instance whenever $V[\nu]$ is a continuous function for every $\nu$ or when $Y$ is finite, or when $V[\nu]$ is local and some restrictions are imposed on $\nu \ldots$.) then all equilibria have the same second marginal.

Proof. Assume that $\left(\nu_{1}, \gamma_{1}\right)$ and $\left(\nu_{2}, \gamma_{2}\right)$ are two equilibria, and let $\varphi_{1}, \varphi_{2}$ in $C(X)$ such that

$$
V\left[\nu_{i}\right](y) \geq \varphi_{i}(x)-c(x, y), i=1,2,
$$

on $X \times Y$ (in the case of definition 2.1) or for every $x$ and $m_{0}$-a.e. $y$ (in the case of definition 3.1) with an equality $\gamma_{i}$-a.e.. Integrating with respect to $\gamma_{i}$ and using the fact that $\gamma_{i} \in \Pi\left(\mu, \nu_{i}\right)$, we get

$$
\int_{Y} V\left[\nu_{i}\right] d \nu_{i}=\int_{X} \varphi_{i} d \mu-\int_{X \times Y} c d \gamma_{i}, i=1,2
$$

whereas for $i \neq j$

$$
\int_{Y} V\left[\nu_{i}\right] d \nu_{j} \geq \int_{X} \varphi_{i} d \mu-\int_{X \times Y} c d \gamma_{j}
$$

substracting, we get $\int_{Y} V\left[\nu_{1}\right] d\left(\nu_{1}-\nu_{2}\right) \leq \int_{X \times Y} c d\left(\gamma_{2}-\gamma_{1}\right)$ and $\int_{Y} V\left[\nu_{2}\right] d\left(\nu_{2}-\right.$ $\left.\nu_{1}\right) \leq \int_{X \times Y} c d\left(\gamma_{1}-\gamma_{2}\right)$, the monotonicity assumption thus allows us to prove that $\nu_{1}=\nu_{2}$.

Typical example of strictly monotone maps are given by purely local congestion terms $V[\nu](y)=f(y, \nu(y))$ with $f$ increasing in its second argument. On the contrary, typical regular nonlocal terms are not monotone. Let us give however an example where the congestion effect dominates the nonocal interaction term, consider

$$
V[\nu](y):=\nu(y)+\int_{Y} \phi(y, z) \nu(z) d z
$$

(so that $\left.\mathcal{D}=L^{2}\left(m_{0}\right)\right)$ then if

$$
\int_{Y \times Y} \phi^{2}(y, z) m_{0}(d y) m_{0}(d z)<1
$$

as a simple application of Cauchy-Schwarz inequality, we have

$$
\int_{Y}\left(V\left[\nu_{1}\right]-V\left[\nu_{2}\right]\right) d\left(\nu_{1}-\nu_{2}\right) \geq\left\|\nu_{1}-\nu_{2}\right\|_{L^{2}\left(m_{0}\right)}^{2}\left(1-\|\phi\|_{L^{2}\left(m_{0} \otimes m_{0}\right)}^{2}\right),
$$

so that our uniqueness result applies.

## 5 Equilibria by best-reply iteration

In this section, we adopt a direct approach when $c$ is quadratic and $V[\nu]$ satisfies some suitable convexity condition which makes solving type $x$ agents program, given $\nu$, more explicit by a first-order condition. Throughout this paragraph, we will assume that $X=\bar{\Omega}, Y=\bar{U}$, where $\Omega$ and $U$ are some open bounded convex subsets of $\mathbb{R}^{d}$, that the cost is quadratic:

$$
c(x, y):=\frac{1}{2}|x-y|^{2}, \forall(x, y) \in \mathbb{R}^{d} \times \mathbb{R}^{d}
$$

that $\mu$ is absolutely continuous with respect to the Lebesgue measure on $X$ and has a bounded density, that $V[\nu]$ is a smooth and convex function for every $\nu \in \mathcal{P}(Y)$, which is the case if $V[\nu]$ has the form

$$
V[\nu](y):=\int \phi(y, z) d \nu(z)
$$

with $\phi$ smooth and convex with respect to its first argument. We also assume that for every $\nu \in \mathcal{P}(Y)$ and every $x \in X$, the solution of

$$
\begin{equation*}
\inf _{y \in Y}\left\{\frac{1}{2}|x-y|^{2}+V[\nu](y)\right\} \tag{5.1}
\end{equation*}
$$

belongs to $U$ (which is the case as soon as $V[\nu]$ fulfills some coercivity assumption and $U$ is chosen large enough). In this case the solution of (5.1) is obtained by a first-order condition which gives

$$
y=(\operatorname{Id}+\nabla V[\nu])^{-1}(x) .
$$

The resolvent operator $(\operatorname{Id}+\nabla V[\nu))^{-1}$ is a very natural operator in convex analysis where it is known as the proximal operator of $V[\nu]$. If agents have a prior $\nu$ on the other agents actions, their cost-minimizing behavior leads to another a posteriori measure on the action space $Y$, namely

$$
\begin{equation*}
T \nu:=(\operatorname{Id}+\nabla V[\nu])_{\#}^{-1} \mu . \tag{5.2}
\end{equation*}
$$

One easily checks that $(\gamma, \nu)$ is an equilibrium if and only if $\nu=T \nu$ and $\gamma=\left(\operatorname{Id},(\operatorname{Id}+\nabla V[\nu])^{-1}\right)_{\#} \mu$ is the optimal transport plan between $\mu$ and $\nu$ for the quadratic cost. Finding an equilibrium thus amounts to finding a fixed point of $T$ and we shall see some additional conditions that ensure that $T$ is a contraction of $\mathcal{P}(Y)$ endowed with the 1-Wasserstein distance $W_{1}$ (we refer to Villani's textbooks [10], [11] for more on Wasserstein distances):

$$
W_{1}\left(\nu_{1}, \nu_{2}\right):=\inf _{\eta \in \Pi\left(\nu_{1}, \nu_{2}\right)} \int_{Y \times Y}\left|y_{1}-y_{2}\right| d \eta\left(y_{1}, y_{2}\right)
$$

Since $\left(\mathcal{P}(Y), W_{1}\right)$ is a complete metric space, these conditions will therefore imply the existence and the uniqueness of an equilibrium (and more importantly, from a numerical point, this equilibrium can be approximated by the iterates of $T$ applied to any $\nu_{0} \in \mathcal{P}(Y)$ ). Our additional assumptions read as : there exists $\lambda>0, C \geq 0$ and $M>0$ such that for every $\left(\nu_{1}, \nu_{2}\right) \in \mathcal{P}(Y) \times \mathcal{P}(Y)$ the following inequalities hold

$$
\begin{gather*}
D^{2} V\left[\nu_{1}\right] \geq \lambda \operatorname{Id} \text { on } X  \tag{5.3}\\
\operatorname{det}\left(\operatorname{Id}+D^{2} V\left[\nu_{1}\right]\right) \leq M \text { on } X  \tag{5.4}\\
\int_{Y}\left|\nabla V\left[\nu_{1}\right](y)-\nabla V\left[\nu_{2}\right](y)\right| d y \leq C W_{1}\left(\nu_{1}, \nu_{2}\right) \tag{5.5}
\end{gather*}
$$

Theorem 5.1. Under the assumptions of this section, if (5.3), (5.4) and (5.5) hold and if

$$
\begin{equation*}
M C\|\mu\|_{L^{\infty}}<1+\lambda \tag{5.6}
\end{equation*}
$$

then the map $T$ defined by (5.2) is contraction of $\left(\mathcal{P}(Y), W_{1}\right)$. There exists therefore a unique equilibrium $(\gamma, \nu)=\left(\left(\operatorname{Id},(\operatorname{Id}+\nabla V[\nu])^{-1}\right)_{\#} \mu,(\operatorname{Id}+\right.$ $\left.\nabla V[\nu])^{-1}\right)_{\#} \mu$ ) and for every $\nu_{0} \in \mathcal{P}(Y)$, the sequence $T^{n} \nu_{0}$ converges to $\nu$ in the distance $W_{1}$ (that is for the weak-* topology).

Proof. Let $\nu_{1}, \nu_{2} \in \mathcal{P}(Y) \times \mathcal{P}(Y)$, since $\left(\left(\operatorname{Id}+\nabla V\left[\nu_{1}\right]\right)^{-1},\left(\operatorname{Id}+\nabla V\left[\nu_{2}\right]\right)^{-1}\right)_{\#} \mu$ belongs to $\Pi\left(T \nu_{1}, T \nu_{2}\right)$, we first have

$$
\begin{equation*}
W_{1}\left(T \nu_{1}, T \nu_{2}\right) \leq \int_{X}\left|\left(\operatorname{Id}+\nabla V\left[\nu_{1}\right]\right)^{-1}-\left(\operatorname{Id}+\nabla V\left[\nu_{2}\right]\right)^{-1}\right| d \mu \tag{5.7}
\end{equation*}
$$

Now let $x \in X$ and $y_{i}:=\left(\operatorname{Id}+\nabla V\left[\nu_{i}\right]\right)^{-1}(x)$, we then write

$$
\begin{aligned}
y_{1}-y_{2} & =\nabla V\left[\nu_{2}\right]\left(y_{2}\right)-\nabla V\left[\nu_{1}\right]\left(y_{1}\right) \\
& =\nabla V\left[\nu_{1}\right]\left(y_{2}\right)-\nabla V\left[\nu_{1}\right]\left(y_{1}\right)+\left(\nabla V\left[\nu_{2}\right]-\nabla V\left[\nu_{1}\right]\right)\left(y_{2}\right)
\end{aligned}
$$

taking the inner product with $y_{1}-y_{2}$ and using (5.3) (recalling that $D^{2} f \geq$ $\left.\lambda \operatorname{Id} \Rightarrow\left(\nabla f\left(y_{1}\right)-\nabla f\left(y_{2}\right)\right) \cdot\left(y_{1}-y_{2}\right) \geq \lambda\left|y_{1}-y_{2}\right|^{2}\right)$, we get

$$
\begin{aligned}
&\left|y_{1}-y_{2}\right|^{2}=\left(y_{1}-y_{2}\right) \cdot\left(V\left[\nu_{1}\right]\left(y_{2}\right)-\nabla V\left[\nu_{1}\right]\left(y_{1}\right)+\left(\nabla V\left[\nu_{2}\right]-\nabla V\left[\nu_{1}\right]\right)\left(y_{2}\right)\right) \\
& \leq-\lambda\left|y_{1}-y_{2}\right|^{2}+\left|y_{1}-y_{2}\right|\left|\nabla\left(V\left[\nu_{2}\right]-\nabla V\left[\nu_{1}\right]\right)\left(y_{2}\right)\right|
\end{aligned}
$$

so that

$$
\begin{array}{r}
\left|\left(\left(\operatorname{Id}+\nabla V\left[\nu_{1}\right]\right)^{-1}-\left(\operatorname{Id}+\nabla V\left[\nu_{2}\right]\right)^{-1}\right)(x)\right|=\left|y_{1}-y_{2}\right| \\
\leq \frac{1}{1+\lambda}\left|\left(\nabla V\left[\nu_{2}\right]-\nabla V\left[\nu_{1}\right]\right)\left(y_{2}\right)\right| \\
=\frac{1}{1+\lambda}\left|\left(\nabla V\left[\nu_{2}\right]-\nabla V\left[\nu_{1}\right]\right) \circ\left(\operatorname{Id}+\nabla V\left[\nu_{2}\right]\right)^{-1}(x)\right| .
\end{array}
$$

Recalling (5.7) and using the fact that $\left(\operatorname{Id}+\nabla V\left[\nu_{2}\right]\right)_{\#}^{-1} \mu=T \nu_{2}$, we then get

$$
\begin{equation*}
W_{1}\left(T \nu_{1}, T \nu_{2}\right) \leq \frac{1}{1+\lambda} \int_{Y}\left|\nabla V\left[\nu_{2}\right]-\nabla V\left[\nu_{1}\right]\right| d T \nu_{2} \tag{5.8}
\end{equation*}
$$

Now it follows from the fact that $\left(\operatorname{Id}+\nabla V\left[\nu_{2}\right]\right)_{\#} T \nu_{2}=\mu$, the injectivity of $\operatorname{Id}+\nabla V\left[\nu_{2}\right]$ and the change of variables formula that $T \nu_{2}$ has a density with respect to the Lebesgue measure (again denoted $T \nu_{2}$ ) for $y \in(\mathrm{Id}+$ $\left.\nabla V\left[\nu_{2}\right]\right)^{-1}(X)$ given by

$$
T \nu_{2}(y)=\mu\left(y+\nabla V\left[\nu_{2}\right](y)\right) \operatorname{det}\left(\operatorname{Id}+D^{2} V\left[\nu_{2}\right](y)\right) .
$$

Finally, using (5.8)-(5.4) and (5.5), we obtain

$$
\begin{array}{r}
W_{1}\left(T \nu_{1}, T \nu_{2}\right) \leq \frac{\|\mu\|_{L^{\infty} M}}{1+\lambda} \int_{Y}\left|\nabla V\left[\nu_{2}\right](y)-\nabla V\left[\nu_{1}\right](y)\right| d y \\
\leq \frac{\|\mu\|_{L^{\infty}} M C}{1+\lambda} W_{1}\left(\nu_{1}, \nu_{2}\right)
\end{array}
$$

the conclusion thus follows from assumption (5.6) and Banach's fixed point theorem.

It may seem difficult at first glance to check the assumptions of theorem 5.1 we therefore now give a class of examples. Namely, we consider the case where

$$
\begin{equation*}
V[\nu](y)=V_{0}(y)+\varepsilon \int_{Y} \phi(y, z) d \nu(z) \tag{5.9}
\end{equation*}
$$

where $\varepsilon>0$ is a scalar parameter (capturing the size of interaction, say), $V_{0}$ is a smooth and convex function such that $D^{2} V_{0} \geq \lambda_{0}$ Id on $Y$ with $\lambda_{0}>0$ and $\phi$ is $C^{2}$ on $\mathbb{R}^{d} \times \mathbb{R}^{d}$.

Corollary 5.2. Assume that $\nu \mapsto V[\nu]$ has the form (5.9) and that the previous assumptions are satisfied, then for $\varepsilon$ small enough, the map $T$ defined by (5.2) satisfies (5.3)- (5.4)- (5.5)- (5.6) and there is a unique equilibrium.
Proof. Let $\Lambda_{0} \geq \lambda_{0}$ be such that $D^{2} V_{0} \leq \Lambda_{0}$ Id on $Y$, then (5.3) and (5.4) hold respectively with $\lambda=\lambda_{0}+O(\varepsilon)$ and $M=\left(1+\Lambda_{0}+O(\varepsilon)\right)^{d}$. As far as (5.5) is concerned, we recall the Kantorovich duality formula for $W_{1}$ (see [10], [11] for details):

$$
W_{1}\left(\nu_{1}, \nu_{2}\right):=\sup \left\{\int_{Y} u d\left(\nu_{1}-\nu_{2}\right): u \text { 1-Lipschitz }\right\} .
$$

Hence, for any Lipschitz continuous function $u$ on $Y$ and any pair of probability measures $m_{1}, m_{2}$ on $Y$ one has $\left|\int_{Y} u d\left(m_{1}-m_{2}\right)\right| \leq \operatorname{Lip}(u, Y) W_{1}\left(m_{1}, m_{2}\right)$ where $\operatorname{Lip}(u, Y)$ denotes the Lipschitz constant of $u$ on $Y$. Since for $\left(\nu_{1}, \nu_{2}\right) \in$ $\mathcal{P}(Y) \times \mathcal{P}(Y)$ and $y \in Y$ we have

$$
\nabla V\left[\nu_{1}\right](y)-\nabla V\left[\nu_{2}\right](y)=\varepsilon \int_{Y} \nabla_{y} \phi(y, z) d\left(\nu_{1}-\nu_{2}\right)(z)
$$

and since $\phi$ is $C^{2}, \nabla_{y} \phi$ is locally Lipschitz, so that we obtain

$$
\int_{Y}\left|\nabla V\left[\nu_{2}\right](y)-\nabla V\left[\nu_{1}\right](y)\right| d y \leq \varepsilon\left(\int_{Y} \operatorname{Lip}\left(\nabla_{y} \phi(y, .) d y\right) W_{1}\left(\nu_{1}, \nu_{2}\right)\right.
$$

so that (5.5) holds with $C=O(\varepsilon)$ and thus (5.6) is satisfied for small enough $\varepsilon$.


Figure 1: Convergence to the equilibrium in the case $\varepsilon=10^{-2}, V_{0}(x)=0.5(x-$ $0.2)^{2}, \phi(z)=0.25 z^{4}, \mu$ consists of two bumps summetric with respect to 0.5 and the initial guess $\nu_{0}$ is a truncated parabola with support $[0,1]$. The figure on the left shows the iterates for the density $\nu_{k}$, and the figure on the right shows the coresponding cumulative functions, convergence is very fast, the equilibrium has a shape that is similar to that of $\mu$, it is slightly shifted to the left (because of $V_{0}$ ) and more concentrated (because of the interaction term).


Figure 2: Same as before but with larger interaction factor, namely $\varepsilon=1$, resulting in a more concentrated equilibrium configuration.

## 6 Combining a variational approach with a fixed point argument

### 6.1 Variational approach

In [3], we obtained Cournot-Nash equilibria by a variational approach related to optimal transport. As already recalled in paragraph 3, under the separable form (3.1), if $\gamma$ is a Cournot-Nash equilibrium and $\nu$ denotes its second marginal then $\gamma \in \Pi_{o}(\mu, \nu)$ i.e. it solves the optimal transport problem:

$$
\begin{equation*}
W_{c}(\mu, \nu):=\inf _{\gamma \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) d \gamma(x, y) . \tag{6.1}
\end{equation*}
$$

If, in addition, externalities take the typical form

$$
V[\nu](y)=f(y, \nu(y))+W[\nu](y), W[\nu](y)=\int_{Y} \phi(y, z) d \nu(z)
$$

with $f(y,$.$) increasing (congestion) satisfying the growth condition (3.5) and$ $\phi$ is continuous and symmetric i.e. $\phi(y, z)=\phi(z, y)$, then we can associate to $V[\nu]$ the functional

$$
E[\nu]=\int_{Y} F(y, \nu(y)) d m_{0}(y)+\frac{1}{2} \iint_{Y \times Y} \phi(y, z) \mathrm{d} \nu(y) \mathrm{d} \nu(z) .
$$

In this setting, $V$ is the first variation of $E, V[\nu]=\frac{\delta E}{\delta \nu}$ in the sense that for every $(\rho, \nu) \in \mathcal{D}^{2}$, one has

$$
\lim _{\varepsilon \rightarrow 0^{+}} \frac{E[(1-\varepsilon) \nu+\varepsilon \rho]-E[\nu]}{\varepsilon}=\int_{Y} V[\nu] \mathrm{d}(\rho-\nu) .
$$

It is therefore natural to consider the variational problem

$$
\begin{equation*}
\inf _{\nu \in \mathcal{D}} J_{\mu}[\nu] \quad \text { where } \quad J_{\mu}[\nu]:=W_{c}(\mu, \nu)+E[\nu] . \tag{6.2}
\end{equation*}
$$

It is not hard to check that the first-order optimality condition for (6.2) actually gives Cournot-Nash and the assumptions above easily guarantee the existence of a minimizer, we thus have (see [3]):

Theorem 6.1 (Minimizers are equilibria). Assume that $X=\bar{\Omega}$ where $\Omega$ is some open bounded connected subset of $\mathbb{R}^{d}$ with negligible boundary, that $\mu$ is equivalent to the Lebesgue measure on $X$ (that is both measures have the same negligible sets) and that for every $y \in Y, c(., y)$ is differentiable with $\nabla_{x} c$ bounded on $X \times Y$. If $\nu$ solves (6.2) and $\gamma$ solves $W_{c}(\mu, \nu)$ then $\gamma$ is a Cournot-Nash equilibrium. In particular there exist CNE.

In other words, the situation described above may be related to potential games. The main drawback of the previous result lies in the symmetry assumption for the interaction term. Symmetry is essential for $V$ to have a potential but assuming symmetry is not particularly realistic, we shall see in the next paragraph how to cope with more general nonsymmetric interactions.

### 6.2 Existence for nonsymmetric interactions

We now assume that $V[\nu]$ is the sum of a local congestion term and of a regular term:

$$
\begin{equation*}
V[\nu](y):=f(y, \nu(y))+W[\nu](y) \tag{6.3}
\end{equation*}
$$

where $f(y,$.$) is increasing, satisfies the power growth condition:$

$$
\begin{equation*}
\frac{1}{C}\left(t^{\alpha}-1\right) \leq f(y, t) \leq C\left(t^{\alpha}+1\right) \tag{6.4}
\end{equation*}
$$

for some $\alpha>0$ and $C>0$ and every $(y, t)$ and that $W[\nu] \in C(Y)$ for every $\nu \in \mathcal{P}(Y)$ with

$$
\begin{equation*}
\nu \mapsto W[\nu] \text { is a continuous map from }(\mathcal{P}(Y), \mathrm{w}-*) \text { to }\left(C(Y),\|\cdot\|_{\infty}\right) \tag{6.5}
\end{equation*}
$$

This framework covers the case of a general (pairwise) interaction term

$$
W[\nu](y):=\int_{Y} \phi(y, z) d \nu(z)
$$

or more generally

$$
W[\nu](y):=\int_{Y} \phi\left(y, z_{1}, \cdots, z_{n}\right) d \nu\left(z_{1}\right) \cdots d \nu\left(z_{n}\right)
$$

with an arbitrary continuous $\phi$ whereas the variational approach of [3] requires $\phi$ to be symmetric.

We then have
Theorem 6.2. In addition to the assumptions of theorem 6.1, assume that $V[\nu]$ is of the form (6.3) with $f$ satisfying (6.2) and $W$ satisfying (6.5) then there exists at least one Cournot-Nash equilibrium.

Proof. Let $p=\alpha+1, K$ be the set of $L^{p}$ probability densities, for $\nu \in K$ let us consider the minimization problem

$$
\inf _{\theta \in K} W_{c}(\mu, \theta)+\int F(y, \theta(y)) d y+\int W[\nu] d \theta
$$

where $F(y,$.$) is a primitive of f(y,$.$) . By standard lower semi-continuity$ arguments, this problem possesses at least a solution that is in fact unique by strict convexity of $F(y,$.$) (and convexity of the other terms). Let us$ denote by $G(\nu)$ this minimizer, it is easy to check that (6.5) implies that the map $G$ is continuous with respect to the weak topology of $L^{p}$ and the growth condition (6.2) implies that $G(K)$ is bounded in $L^{p}$ hence relatively compact for the weak topology of $L^{p}$. Thanks to Schauder's fixed point theorem, there exists $\nu \in K$ such that $\nu=G(\nu)$. Arguing as in [3], one easily sees that any $\gamma$ that solves $W_{c}(\mu, \nu)$ actually is a Cournot-Nash equilibrium.

### 6.3 An ODE for equilibria in dimension one

We now consider the one-dimensional case where $X=Y=[0,1]$ (say), $m_{0}$ is the Lebesgue measure on $X, \mu$ is equivalent to the Lebesgue measure and the cost $c$ is of class $C^{2}$ and satisfies the Spence-Mirrlees condition:

$$
\partial_{x y}^{2} c(x, y)>0 .
$$

We again consider a separable total cost of the form $c(x, y)+f(\nu(y))+$ $\int_{Y} \phi(y, z) \nu(z) d m_{0}(z)$ with $f$ increasing and $\phi$ continuous (and not necessarily symmetric). We just note that replacing the interaction term $\int_{Y} \phi(y, z) \nu(z) d z$ by a more general one like $F\left(y, \int_{Y} \phi\left(y, z_{1}, \ldots, z_{n}\right) \nu\left(z_{1}\right) d m_{0}\left(z_{1}\right) \ldots \nu\left(z_{n}\right) d m_{0}\left(z_{n}\right)\right)$ actually costs no generality for what follows. To fix ideas, we'll take two cases for the congestion cost $f$ :

$$
f(\nu)=\nu \text { or } f(\nu)=\log (\nu) .
$$

As shown in [3], in the case $f(\nu)=\log (\nu)$, the Inada condition holds which guarantees that $\nu$ is positive everywhere on $[0,1]$, this need not be the case when $f(\nu)=\nu$ (or other power functions, which can be considered as well). Because of the Spence-Mirrlees condition, we know that equilibria are pure i.e. if $(\gamma, \nu)$ is an equilibrium then $\gamma=(\mathrm{id}, T)_{\#} \mu$ for some map $T$ which is the optimal transport between $\mu$ and $\nu$. This map is well-known to be the unique nondecreasing map which transports $\mu$ to $\nu$ (and it is easy to compute, once $\nu$ is known by the formula $T=F_{\nu}^{-1} \circ F_{\mu}$ where $F_{\mu}$ is the cdf of $\mu$ and $F_{\nu}^{-1}$ is the quantile function of $\nu$ ). Finding an equilibrum $(\gamma, \nu)$ thus amounts to find the transport map $T$ which as we shall see is characterized by some nonlinear and nonlocal ODE.

The equilibrium condition can be rewritten as

$$
\begin{array}{r}
\min _{x \in[0,1]}\{c(x, y)-\varphi(x)\}+f(\nu(y))+\int_{0}^{1} \phi(y, z) \nu(z) d z \\
=\varphi^{c}(y)+f(\nu(y))+\int_{0}^{1} \phi(y, z) \nu(z) d z \geq 0 \tag{6.6}
\end{array}
$$

with an equality for $y=T(x)$ which is the point which realizes the minimum above, i.e.

$$
\varphi(x)=c(x, T(x))-\varphi^{c}(T(x))=\min _{y \in[0,1]}\{c(x, y)-\varphi(y)\}
$$

the smoothness of $c$ implies that $\varphi$ is Lipschitz hence differentiable a.e, for a point of differentiability of $\varphi$, the envelope theorem therefore gives

$$
\begin{equation*}
\varphi^{\prime}(x)=\partial_{x}(x, T(x)) \text { hence } \varphi(x)=\varphi(0)+\int_{0}^{x} \partial_{x} c(s, T(s)) d s \tag{6.7}
\end{equation*}
$$

## Log Case

In the case $f(\nu)=\log (\nu)$, as already mentioned $\nu$ is positive everywhere on $[0,1]$ so that $T$ is increasing on $[0,1], T(0)=0$ and $T(1)=1$ and using (6.6)-(6.7) and the fact that $T_{\#} \mu=\nu$, we get

$$
\begin{aligned}
\nu(T(x)) & =\exp \left(-\varphi^{c}(T(x))-\int_{0}^{1} \phi(T(x), z) \nu(z) d z\right) \\
& =\exp \left(\varphi(x)-c(x, T(x))-\int_{0}^{1} \phi(T(x), T(y)) d \mu(y)\right) \\
& =\exp \left(\varphi(0)+\int_{0}^{x} \partial_{x} c(s, T(s)) d s-c(x, T(x))-\int_{0}^{1} \phi(T(x), T(y)) d \mu(y)\right)
\end{aligned}
$$

but the fact that $T_{\#} \mu=\nu$ can be expressed as

$$
\begin{equation*}
\mu(x)=\nu(T(x)) T^{\prime}(x) \tag{6.8}
\end{equation*}
$$

Replacing and setting $C:=e^{-\varphi(0)}$ we find the following equation where only $T$ appears:
$T^{\prime}(x)=C \mu(x) \exp \left(-\int_{0}^{x} \partial_{x} c(s, T(s)) d s+c(x, T(x))+\int_{0}^{1} \phi(T(x), T(y)) d \mu(y)\right)$
supplemented with the initial condition $T(0)=0$ and since $T(1)=1$ the constant $C$ is given by
$\frac{1}{C}=\int_{0}^{1} \exp \left(-\int_{0}^{x} \partial_{x} c(s, T(s)) d s+c(x, T(x))+\int_{0}^{1} \phi(T(x), T(y)) d \mu(y)\right) \mu(x) d x$.
ITERATIVE SIMULATION ALGO (hope to find several equilibria):
Given $T_{k}$ increasing with $T_{k}(0)=0, T_{k}(1)=1$, define

$$
\begin{aligned}
\frac{1}{C_{k}} & :=\int_{0}^{1} \exp \left(-\int_{0}^{x} \partial_{x} c\left(s, T_{k}(s)\right) d s+c\left(x, T_{k}(x)\right)+\int_{0}^{1} \phi\left(T_{k}(x), T_{k}(y)\right) d \mu(y)\right) \mu(x) d x, \\
S_{k}(x) & :=C_{k} \mu(x) \exp \left(-\int_{0}^{x} \partial_{x} c\left(s, T_{k}(s)\right) d s+c\left(x, T_{k}(x)\right)+\int_{0}^{1} \phi\left(T_{k}(x), T_{k}(y)\right) d \mu(y)\right), \\
T_{k+1}(x) & :=\int_{0}^{x} S_{k}(s) d s
\end{aligned}
$$

## Linear case

Let us consider now the case where $f(\nu)=\nu$, the equilibrium condition can then be written as $\nu(y)+\varphi^{c}(y)+\int_{0}^{1} \phi(y, z) \nu(d z) \geq \lambda$ (for some constant $\lambda)$ with an equality whenever $\nu(y)>0$ which can be rewritten as

$$
\nu(y)=\left(\lambda-\varphi^{c}(y)-\int_{0}^{1} \phi(y, z) \nu(z) d z\right)_{+} .
$$

Since $\nu$ may vanish, $T$ may be discontinuous and the situation is actually more involved than in the log case (one cannot use an ODE for $T$ but just its integrated form $F_{\nu} \circ T=F_{\mu}$ ). Actually, it is better here to forget about $T$ and to look for the optimal transport between $\nu$ and $\mu$ (which may have flat zones but is continuous) and is given by

$$
S=F_{\mu}^{-1} \circ F_{\nu} .
$$

Normalizing $\varphi^{c}(0)=0$ (actually the integration constant is already in the $\lambda$ above), we then have as before

$$
\left.\varphi^{c}(y)=\int_{0}^{y} \partial_{y} c(S(s), s)\right) d s
$$

ITERATIVE SIMULATION ALGO (hope to find several equilibria):
Start with a probability density $\nu_{k}$ on $[0,1]$, then do:

- Compute the optimal transport between $\nu_{k}$ and $\mu$ :

$$
S_{k}=F_{\mu}^{-1} \circ F_{\nu_{k}},
$$

- compute the Kantorovich potential $\varphi_{k}^{c}$ by

$$
\varphi_{k}^{c}(y)=\int_{0}^{y} \partial_{y} c\left(S_{k}(s), s\right) d s
$$

- Compute the new density $\nu_{k+1}$ by

$$
\nu_{k+1}(y)=\left(\lambda_{k}-\varphi_{k}^{c}(y)-\int_{0}^{1} \phi(y, z) \nu_{k}(z) d z\right)_{+},
$$

where $\lambda_{k}$ is such that the function above has total mass 1 (this is just finding the root of a monotone function...).

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