Dynamic optimisation for economists

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Chapter 1

Introduction

Dynamic optimisation is a central pillar of modern economic analysis. It allows modeling and analyzing intertemporal decision-making in various contexts: consumption and savings, optimal control of economic systems, dynamic game theory, etc. A rigorous mathematical approach is essential for fully mastering its implications and ensuring the correct application of results. This course, by emphasizing the formal structure and rigour of the employed methods, aims to provide students with a deep and applicable understanding of dynamic optimisation tools.

The relevance of dynamic optimisation extends beyond theoretical interest; it has practical applications in fields such as financial modeling, resource management, and policy design. By understanding how agents optimize their decisions over time, we can derive insights into economic stability, growth trajectories, and optimal investment strategies. The mathematical rigour underlying these analyses ensures that the derived conclusions are logically sound and broadly applicable.

Dynamic optimisation relies on deep mathematical concepts such as differential equations, variational inequalities, and functional analysis. While economists often use these tools in an applied manner, a rigorous understanding of their foundations is essential to avoid conceptual errors and ensure correct interpretation of results.

An important aspect of this lecture is the ability to formally prove optimality conditions. Many economic problems involve constraints, whether they are budgetary, technological, or institutional. Understanding the underlying mathematical structure of these constraints enables us to apply sophisticated tools such as Lagrange multipliers and the Hamiltonian approach to derive meaningful and actionable insights.

Even though the numerical aspects will not be studied in this lecture, mathematical reasoning also enhances computational methods. With the increasing use of numerical techniques in economic research, a strong foundation in mathematics ensures the correct application of algorithms such as dynamic programming and numerical solutions to Hamilton-Jacobi-Bellman equations. Without a proper mathematical background, researchers risk misinterpreting numerical results or applying methods outside their valid range.

Each chapter of this course begins with a reminder of the mathematical tools necessary for the topics covered. This ensures that students have a strong foundation before delving into the core concepts of dynamic optimisation. Additionally, at the end of each chapter, students will find exercises designed to reinforce their understanding, featuring numerous applications in economics. These exercises will cover a wide range of topics, from optimal investment strategies to macroeconomic policy design, providing hands-on experience in applying theoretical concepts to real-world problems.

The objective is to enable students, whether economists or mathematicians, to acquire a rigorous and applied understanding of these concepts. Through exercises and practical applications, we will demonstrate how these tools are used in real economic contexts. Examples will include optimal consumption paths, environmental policy optimisation, and monetary policy adjustments.

Furthermore, this course will emphasize the connections between mathematical optimisation techniques and their real-world economic interpretations. By bridging the gap between abstract mathematical formulations and applied economic reasoning, students will develop the ability to both construct and critically evaluate dynamic models in economic research.

1.1 Reminders

Definition 1 (Partial order)**.** *A partial order, is a map on a set P that is such that for all a, b and c in P:*

- *Reflexivity: $a \leq a$, i.e. every element is related to itself.*
- *Antisymmetry: if $a \leq b$ and $b \leq a$ then $a = b$, i.e. no two distinct elements precede each other.*
- *Transitivity: if $a \leq b$ and $b \leq c$ then $a \leq c$.*

Remark 1. In consumer theory the consumption space is some set X, usually the positive orthant of some vector space so that each $x \in X$ represents a quantity of consumption specified for each existing commodity in the economy. Preferences of a consumer are usually represented by a total preorder $\preceq$ so that $x, y \in X$ and $x \preceq y$ reads: x is «at most as preferred as y». When $x \preceq y$ and $y \preceq x$ it is interpreted that the consumer is indifferent between x and y but is no reason to conclude that $x = y$. preference relations are never assumed to be antisymmetric.

Exercise 1. Check if the following are partial orders sets:

1. The real numbers ordered by the standard «less or equal than» relation $\leq$.
2. The real numbers $\mathbb{R}$, the usual «less than» relation $<$.
3. The real numbers $\mathbb{R}$, the usual «bigger or equal than» relation $\geq$.
4. The set of subsets of a given set ordered by inclusion.
5. The set of natural numbers equipped with the relation of divisibility.
6. The set $\mathbb{N} \times \mathbb{N}$, ordered with the lexicographical order: $(a, b) \leq (c, d)$ if $a < c$ or ($a = c$ and $b \leq d$).
7. The set $\mathbb{N} \times \mathbb{N}$, ordered with the product order: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.

Definition 2 (Upper bound, infimum)**.** *Let $(P, \leq)$ be a partially ordered set. Consider a subset S of P.*

An upper bound *of S is an element y of P such that*

$$\forall x \in S \qquad x \leq y \ .$$

An upper bound b in S is called a supremum *(or least upper bound, or join) of S if for all upper bounds z of S in P,*

$$b \leq z$$

(b is less than any other upper bound).

A lower bound *of S is an element y of P such that*

$$\forall x \in S \qquad y \leq x \ .$$

An lower bound b in S is called an infimum *(or greatest lower bound, or meet) of S if for all lower bounds z of S in P,*

$$z \leq b$$

(b is larger than any other lower bound).

Exercise 2. Do suprema necessarily exist?

▶ No. Existence of an supremum of a subset S of P can fail if S has no upper bound at all, or if the set of upper bounds does not contain a smallest element. (An example of this is the subset $\{x \in \mathbb{Q} : x^2 > 2\}$ of $\mathbb{Q}$. It has upper bounds, such as 1.5, but no supremum in $\mathbb{Q}$.)

Exercise 3. In $\mathbb{R}$, determine the supremum of

1. $\sup\{x \in \mathbb{R} : 0 < x < 1\}$
2. $\sup\{x \in \mathbb{R} : 0 \leq x \leq 1\}$.
3. $\sup\{1 - 1/n : n = 1, 2, 3, \ldots\}$.
4. $\sup\{(-1)^n - 1/n : n = 1, 2, 3, \ldots\}$.
5. $\sup\left\{x \in \mathbb{Q} : x^2 < 2\right\}$.

▶

1. 1
2. 1
3. If a is an upper boiund smaller than 1 then we would have
$$1 - \frac{1}{n} \leq a \Leftrightarrow \frac{1}{1-a} \leq n$$
which is a contradiction because n can be taken as large as we want.
4. 1
5. $\sqrt{2}$. The supremum of a set of rationals is irrational, which means that the rationals are incomplete.

Exercise 4. Let $(a, b) \in (\mathbb{R}_+^*)^2$. Do the following set have upper bounds? suprema?

1. $\{a + bn;\ n \in \mathbb{N}\}$,
2. $\{a + (-1)^n b;\ n \in \mathbb{N}\}$,
3. $\{a - b/n;\ n \in \mathbb{N}^*\}$

▶

1. A has no upper bound : we can always have $a + bn \leq M$ for all $n \in \mathbb{N}$, if not we would have $n \leq (M - a)/b$ and $\mathbb{N}$ would have an upper bound.
2. If $n \in 2\mathbb{N}$, $a + (-1)^n b = a + b$ and if $n \in \mathbb{N} \setminus 2\mathbb{N}$, $a + (-1)^n b = a - b$. The set is thus made of two elements $a + b$ and $a - b$. It thus has an upper bound and $\sup(A) = a + b$.
3. The elementes of A are $a - b, a - b/2, a - b/3, ...$ So that A has a upper bound a. Let us prove that a is the supremum of A. If c was an upper bound of A smaller than a, then for all $n \in \mathbb{N}^*$, we would have
$$a - \frac{b}{n} \leq c \leq a \text{ or } -\frac{b}{n} \leq c - a \leq 0.$$
But whatever is $c - a$ we can always choose n large enough so that this inequality is not satisfied. Hence a is the least upper bound, the supremum.

Definition 3 (Maximum, minimum)**.** *Let $(P, \leq)$ be a preordered set and let $S \subseteq P$. A* maximal element *of S with respect to $\leq$ is an element $m \in S$ such that*

$$\forall s \in S, \quad m \leq s \Rightarrow s \leq m.$$

Similarly, a minimal element *of S with respect to $\leq$ is an element $m \in S$ such that*

$$\forall s \in S, \quad s \leq m, \Rightarrow m \leq s.$$

Example 1.

1. In Pareto efficiency, a Pareto optimum is a maximal element with respect to the partial order of Pareto improvement, and the set of maximal elements is called the Pareto frontier.
2. In decision theory, an admissible decision rule is a maximal element with respect to the partial order of dominating decision rule.
3. In modern portfolio theory, the set of maximal elements with respect to the product order on risk and return is called the efficient frontier.

Exercise 5. Consider

$$A = \left\{ \frac{n}{mn+1};\ (m,n) \in \mathbb{N}^{*2} \right\}, \qquad B = \left\{ \frac{n}{mn+1};\ (m,n) \in \mathbb{N}^{2} \right\}.$$

Do A and B have an upper bound? A lower bound? A supremum? An infimum? A maximum? A minimum?

▶ For $n, m \in \mathbb{N}^*$, we have

$$0 \leq \frac{n}{nm+1} \leq \frac{n}{nm} \leq \frac{1}{m} \leq 1.$$

A has an upper bound 1 and a lower bound 0. Let us prove that $\inf(A) = 0$. If $c > 0$ is a lower bound of A, then for all $(m,n) \in \mathbb{N}^{*2}$, we would have

$$c \leq \frac{n}{nm+1}.$$

Taking $n = 1$, we obtain

$$c \leq \frac{1}{m+1} \iff m \leq \frac{1}{c} - 1.$$

which is a contradiction. Thus 0 is the greatest lower bound of A, and $\inf(A) = 0$.

Let us now prove that $1 = \sup(A)$. If $d < 1$ is an upper bound of A, then for all $(m,n) \in \mathbb{N}^{*2}$, we would have

$$d \geq \frac{n}{nm+1}.$$

Taking $m = 1$, we obtain

$$d \geq \frac{n}{n+1} \iff d \geq n(1-d) \iff n \leq \frac{d}{1-d}.$$

Hence $\sup(A) = 1$. Moreover 0 does not belong to A and neither does 1, as $nm + 1 > n$ for $n, m \geq 1$.

Now for B: 0 is still a lower bound of B, but this time it belongs to B. Hence $\inf(B) = \min(B) = 0$. Moreover $\mathbb{N} \subset B$ (take $m = 0$). So that, B has no upper bound.

Exercise 6. Let A be a bounded non-empty set of $\mathbb{R}$. We denote $B = \{|x - y|;\ (x,y) \in A^2\}$.

1. Justify that B has a upper bound.
2. Let $\delta(A)$ be the supremum of B. Prove that $\delta(A) = \sup(A) - \inf(A)$.

▶

1. Let $(x, y) \in A^2$ and $M \in \mathbb{R}$ be such that $x \in A \implies |x| \leq M$. We have

$$|x - y| \leq |x| + |y| \leq 2M,$$

2. Set $m = \inf(A)$ and $M = \sup(A)$. Let $(x, y) \in \mathbb{R}^2$. We have

$$m \leq x \leq M \text{ and } -M \leq -y \leq -m \implies -(M - m) \leq x - y \leq M - m$$

so that $|x - y| \leq M - m$. We deduve that $M - m$ is an upper bound of B and that $\delta(A) \leq M - m$.
To prove the reverse inequality, set $\varepsilon \geq 0$. There exists $(x, y) \in A^2$ such that

$$x \geq M - \varepsilon/2 \text{ and } y \leq m + \varepsilon/2.$$

Then $x - y \geq M - m - \varepsilon$.

Exercise 7. Let A and B be two non-empty subsets of $\mathbb{R}$. Consider a bounded map $f : A \times B \to \mathbb{R}$. Compare $\inf(\sup(f(x, y); x \in A); y \in B)$ and $\sup(\inf(f(x, y); y \in B); x \in A)$.

▶ Let $x_0 \in A$. Consider $y \in B$. We have

$$f(x_0, y) \leq \sup(f(x, y); x \in A).$$

Let us take the infimum in this inequality for $y \in B$. On a donc

$$\inf(f(x_0, y); y \in B) \leq \inf(\sup(f(x, y); x \in A); y \in B).$$

We then take the supremum on $x_0 \in A$ to obtain

$$\sup(\inf(f(x_0, y); y \in B); x_0 \in A) \leq \inf(\sup(f(x, y); x \in A); y \in B)$$

The reverse inequality is not true in general: take $A = [0, 1]$, $B = [0, 1]$ and $f(x, y) = |x - y|$. Set $x \in A$. We have $\inf(f(x, y); y \in B) = 0$ by taking $y = x$. So that, $\sup(\inf(f(x, y); y \in B); x \in A) = 0$. While setting $y \in B$, we have

$$\sup(f(x, y); y \in B) \geq 1/2$$

(choose either $x = 0$ or $x = 1$ depending if $y \geq 1/2$ or $y \leq 1/2$). We then have

$$\inf(\sup(f(x, y); y \in B); x \in A) \geq 1/2.$$

1.2 Dynamic programming problems

We are interested in optimisation problems with finite horizon of the form:

$$\sup_{(x_t)} \left\{ \sum_{t=0}^{T-1} V_t(x_t, x_{t+1}) + V_T(x_T)) \right\}$$

under the constraints $x_0 = x$ is given, and for all $t \in \{0, \cdots, T-1\}$, $x_t \in A$ and $x_{t+1} \in \Gamma(x_t)$. Here T is the finite horizon, A models the contraints on the dynamics and is called the state space, Γ_t is a correspondance from A to A, $\Gamma(x_t)$ is the set of all possible successors of x_t, the functions V_t are the instantaneous payoff and V_T is the terminal payoff.

We will also consider problems with infinite horizon with discount factor of the form

$$\sup_{(x_t)} \left\{ \sum_{t=0}^{\infty} \beta^t V_t(x_t, x_{t+1}) \right\}$$

under the constraints $x_0 = x$ is given, and for all $t \geq 0$, $x_t \in A$ and $x_{t+1} \in \Gamma(x_t)$. we interpret A, Γ and V as before and $\beta \in (0,1)$ is the discount factor.

The natural questions include: existence of solution and tractable characterisation. We will use the recursive strucure of these problems by making use of very efficient and inuitive ideas due mainly to Richard Bellman: the dynamic programming program, the value function and Bellman's equation.

These problems are frequently met in economics: growth models, stock management, natural ressources exploitation. Let us see a few examples :

1.3 Examples

1.3.1 Shortest path problem

The problem of the shortest route can be thought of as transmitting a message from one computer to another. I prefer this interpretation to that of optimising the journey of a commercial traveller, because the commercial traveller and his boss have lost sight of the fact that it is beneficial to waste time, to be open to encounters and to the unknown. In praise of slowness. "There are empty, hollow hours that carry within them destiny" Stefan Sweig.

This is a typical finite-horizon dynamic programming problem with a finite state space, which amounts to an optimisation problem on a graph. Its solution illustrates the principle of dynamic programming in a simple way.

Consider a commercial traveller who has to get from city A to city E via several intermediate cities. The possible paths are modelled by a graph with A and E as initial and final vertices (the other vertices representing the stopover cities), the edges of this graph represent the intermediate routes. From A we can go either to B (time to travel this edge: 1) or to B' (time to travel this edge: 1). From B we can go either to C (time to travel this edge: 2) or to C' (time to travel this edge: 1). From B' we can go either to C' (time to travel this edge: 2) or to C'' (time to travel this edge: 4). From C we can to D (time to travel this edge: 1). From C' we can go either to D (time to travel this edge: 2) or to D' (time to travel this edge: 1). From C'' we can go to D' (time to travel this edge: 1). From D we can go to E (time to travel this edge: 5). From D' we can go to E (time to travel this edge: 2).

Exercise 8. Draw the problem in a graph.

Exercise 9. Prove that the problem is a problem of finite horizon (precise A, the x_t, $\Gamma(x_t)$ and the $V_t(x_t, x_{t+1})$

► We have

- $A = \{A, B, B', C, C', C'', D, D', E\}$,
- $\Gamma(A) = \{B, B'\}$, $\Gamma(B) = \{C, C'\}$, $\Gamma(B') = \{C', C''\}$, $\Gamma(C) = \{D\}$, $\Gamma(C') = \{D, D'\}$, $\Gamma(C'') = \{D'\}$, $\Gamma(D) = \Gamma(D') = \{E\}$.
- We also have $V_4(D, E) = 5$, $V_4(D', E) = 2$, $V_3(C, D) = 1$, $V_3(C', D) = 2$, $V_3(C', D') = 1$, $V_3(C'', D') = 1$, $V_2(B, C) = 2$, $V_2(B, C') = 1$, $V_2(B', C') = 2$, $V_2(B', C'') = 4$, $V_1(A, B) = 1$, $V_1(A, B') = 1$.

To determine the shortest path or paths we could of course try them all, but it is much better to use the following remark (which is precisely the principle of dynamic programming in its simplest version):

If an optimal path from A to E passes through M, then it is still optimal between M and E.

Let us introduce the value function $v(M)$:= "the minimum travel time between M and E". Obviously, V is easily calculated by starting from the end:

Exercise 10. Compute $v(D)$ and $v(D')$.

▶ We have

$$v(D) = 5 \qquad v(D') = 2 \ .$$

The principle of dynamic programming suggest to then proceed by backward induction.

Exercise 11. Compute $v(C)$, $v(C')$ and $v(C'')$.

▶ We have

$$v(C) = 6 \qquad v(C') = \min\{1 + v(D), 1 + v(D')\} = 3 \qquad v(C'') = 3$$

Exercise 12. Iterating, compute $v(B)$ and $v(B')$.

▶ We obtain

$$v(B) = \min\{1 + v(C), 2 + v(C')\} = 4 \qquad v(B') = \min\{2 + v(C'), 4 + v(C'')\} = 5$$

Finally

Exercise 13. Compute $v(A)$.

▶ We have

$$v(A) = \min\{1 + v(B), 1 + v(B')\} = 5 \ .$$

Exercise 14. Conclude.

▶ The minimal time path is then equal to 5 and corresponds to the unique path $ABC'D'E$.

1.3.2 Optimal growth

I am not going to get up in arms about the fact that we are calling growth the razing of an area teeming with life to build a supermarket car park. But I would call growth more social links, a connection to nature, philosophy and poetry.

Consider an economy in which a single good is produced each period for both consumption and investment. We note respectively c_t, i_t, k_t and y_t the consomption, the investment, the capital and the production at time t. We assume that $y_t = F(k_t)$, F being the production function. We also assume that the capital is depreciated at a rate $\delta \in [0, 1]$. We then have

$$c_t + i_t = y_t = F(k_t) \quad \text{and} \quad k_{t+1} = (1 - \delta)k_t + i_t \ .$$

We impose that c_t and k_t are non-negatif. We finally assume that the economy maximises the intertemporal utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \ . \tag{1.1}$$

Exercise 15. Setting

$$f(k) := F(k) + (1 - \delta)k$$

1. Rewrite c_t as a function of (k_t).

2. Rewrite the non-negativity constraints on k_t and c_t.
3. Rewrite (1.1).

▶

1. We have
$$c_t = f(k_t) - k_{t+1} \ .$$
2. So that
$$0 \leq k_{t+1} \leq f(k_t) \ . \tag{1.2}$$
3. As a function of the capital the problem becomes
$$\sup \left\{ \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) \right\} \ .$$
under the constraint (1.2), k_0 being given.

1.3.3 A forest exploitation problem

With a title like that, you might expect me to protest against the unreasonable exploitation of Nature for the profit of a few, to the detriment of the human and non-human animal populations that inhabit or visit it, as well as all the species of plants, minerals and invisible entities. Not to mention the dramatic aspect of the forest as a financial investment or the madness of licences to destroy biodiversity. But no, I am here to simply give you the tools you need to arrive at the cold conclusion of an exploitation devoid of all affect.

Consider a forest of initial size x_0 , where x_t is its size at date t (state variable). A logger chooses at each period a cutting level v_t (control variable), and the evolution of the forest is assumed to be governed by the dynamics :
$$x_{t+1} = H(x_t) - v_t$$
where H is an exogeneous growth function of the forest. Assuming that the price of the wood is constant equal to 1 (ouf no financial market hopefully) and that the cost of felling the tree is C, the actualised payoff to exploit the forest is
$$\sum_{t=0}^{\infty} \beta^t \left(v_t - C(v_t)\right) \ .$$
We impose $v_t \geq 0$ and $x_t \geq 0$.

Exercise 16. Rewrite the problem as a function of the state variable x_t.

▶ We obtain
$$\sup \left\{ \sum_{t=0}^{\infty} \beta^t \left[H(x_t) - x_{t+1} - C\left(H(x_t) - x_{t+1}\right)\right] \right\} \ .$$
under the constraint
$$0 \leq x_{t+1} \leq H(x_t) \qquad \forall t \geq 0 \ .$$

Chapter 2

Finite horizon

2.1 Reminders

Definition 4 (Multivalued function)**.** *Let X and Y be two sets. The multivalued function $f : X \rightrightarrows Y$ can be seen as a map from X to the set $\mathcal{P}(Y)$ of all the parties of Y (=the set of all the subsets of Y).*

We denote $\mathrm{graph}(f)$ *the graph of the multivalued function f:*

$$\mathrm{graph}(f) := \{(x, y) \in X \times Y \; : \; y \in f(x)\} \, .$$

Example 2.

- If f is an ordinary function, it is a multivalued function by taking its graph

$$\mathrm{graph}(f) = \; \{(x, f(x)) \; : \; x \in X\}.$$

- The antiderivative can be considered as a multivalued function. The antiderivative of a function is the set of functions whose derivative is that function.

- Each nonzero complex number has two square roots, three cube roots, and in general n nth roots.

- The reciprocal function of a non-injective application: any point in its image is matched by the reciprocal image formed by the antecedents of that point.

2.2 Dynamic programming problems in discrete time and finite horizon

We propose to study dynamic programming problems in discrete time and finite horizon:

$$\sup_{(x_t)} \left\{ \sum_{t=0}^{T-1} V_t(x_t, x_{t+1}) + V_T(x_T)) \right\} \tag{2.1}$$

under the constraints $x_0 = x$ is given (x_0 is the *initial condition*), A is a given set (called the *state space*) and for all $t \in \{0, \cdots, T\}$, $x_t \in A$ and $x_{t+1} \in \Gamma(x_t)$ ($\Gamma(x_t)$ is the set of all possible successors of x_t ; Γ is a multivalued function which models the constraints of the dynamics). Here T is call the *finite horizon*, the functions $V_t : A \times A \in \mathbb{R}$ are the instantaneous payoff and V_T is the terminal payoff. With no loss of generality we will assume here that $V_T = 0$.

We solved a problem of type (2.1) in the previous chapter. In this chapter, which is intended to be as non-technical as possible, we will see how to generalise the strategy for solving the shortest path problem of Section 1.3.1.

We will assume that for all $x \in A$, $\Gamma_t(x) \neq \emptyset$.

2.3 Dynamic programming principle

Definition 5 (Admissible sequence). *A sequence $(x_\tau, \cdots, x_T)$ is said to be admissible for the problem $v(\tau, x)$ if for all $t \in \{\tau, \cdots, T-1\}$, $x_{t+1} \in \Gamma_t(x_t)$ and $x_\tau = x$.*

Given the recursive structure of the problem, it makes sense to introduce value functions at different dates. For $x \in A$ we define:

$$\begin{aligned}
v(0,x) &:= \sup_{\{\forall t \in \{0,\cdots,T-1\},\, x_{t+1}\in\Gamma_t(x_t), x_0 = x\}} \sum_{t=0}^{T-1} V_t(x_t, x_{t+1}) \\
v(1,x) &:= \sup_{\{\forall t \in \{1,\cdots,T-1\},\, x_{t+1}\in\Gamma_t(x_t), x_1 = x\}} \sum_{t=1}^{T-1} V_t(x_t, x_{t+1}) \\
&\vdots \\
v(\tau,x) &:= \sup_{\{\forall t \in \{\tau,\cdots,T-1\},\, x_{t+1}\in\Gamma_t(x_t), x_\tau = x\}} \sum_{t=\tau}^{T-1} V_t(x_t, x_{t+1}) \\
&\vdots \\
v(T-1,x) &:= \sup_{\{x_T \in \Gamma_{T-1}(x_{T-1}), x_{T-1} = x\}} V_{T-1}(x_{T-1}, x_T) \\
v(T,x) &:= \sup_{\{x_T = x\}} V_T(x_T) = 0
\end{aligned}$$

The dynamic programming principle states the following:

Proposition 1 (Dynamic programming principle). *Soit $x \in A$, if $(x_0, x_1, \cdots, x_T) = (x, x_1, \cdots, x_T)$ is a solution to problem $v(0,x)$ then for all $\tau \in \{1, \cdots, T-1\}$, the sequence $(x_\tau, \cdots, x_T)$ is a solution to problem $v(\tau, x_\tau)$.*

Proof. Assume that at a time τ the admissible sequence $(x_\tau, \cdots, x_T)$ is not a solution of $v(\tau, x_\tau)$.

Exercise 17.

1. Prove that there exists an admissible sequence $(z_\tau, \cdots, z_T)$ such that

$$\sum_{t=\tau}^{T-1} V_t(x_t, x_{t+1}) < \sum_{t=\tau}^{T-1} V_t(z_t, z_{t+1}) \,.$$

2. Prove that the sequence $(y_0, \cdots, y_T) = (x, x_1, \cdots, x\tau = z_\tau, z_{\tau+1}, \cdots, z_T)$ is admissible for $v(0,x)$ and

$$v(0,x) < \sum_{t=0}^{T-1} V_t(y_t, y_{t+1}) \,,$$

3. Conclude.

▶

1. By definition of the supremum.

2. By definition of admissible sequence. The equation comes easily.

3. This contredicts the definition of $v(0,x)$.

□

2.4 Backward induction

Note that the proposition assumes the existence of an optimal sequence. Without making this assumption (and allowing the value functions to take the value $+\infty$), we obtain recursive functional relationships, called *Bellman's equations*, linking the value functions to successive dates:

Proposition 2 (Bellman's equations). *Let $x \in A$. For all $t \in \{0, \cdots, T-1\}$ we have*

$$v(t,x) := \sup_{y \in \Gamma_t(x)} \{V_t(x,y) + v(t+1,y)\}$$

Proof. We prove the proposition for $t = 0$ as the proof would be very similar for any $t \in \{0, \cdots, T-1\}$.

- Let us first prove $v(0,x) \geq \sup_{y \in \Gamma_0(x)} \{V_0(x,y) + v(1,y)\}$: Consider the sequence $(x, y_1, \cdots, y_T)$ where for all $t \in \{1, \cdots, T-1\}$, $y_{t+1} \in \Gamma(y_t)$ so that $(x, y_1, \cdots, y_T)$ is admissible for $v(0,x)$.

  Exercise 18. 1. Prove

  $$v(0,x) \geq V_0(x,y_1) + \sum_{t=1}^{T-1} V_t(y_t, y_{t+1}) \; .$$

  2. Deduce that for all $y_1 \in \Gamma(x)$

  $$v(0,x) \geq V_0(x,y_1) + v(1,y_1) \; .$$

  3. Deduce that

  $$v(0,x) \geq \sup_{y \in \Gamma(x)} \{V_0(x,y) + v(1,y)\} \; .$$

  ▶

  1. We have

  $$v(0,x) \geq \sum_{t=0}^{T-1} V_t(y_t, y_{t+1}) = V_0(x,y_1) + \sum_{t=1}^{T-1} V_t(y_t, y_{t+1}) \; .$$

  2. We take the supremum in $(y_2, \cdots, y_T)$.
  3. We take the supremum in $y_1 \in \Gamma_0(x)$.

- Let $\varepsilon > 0$ and the sequence $(x_0 = x, \cdots, x_T)$ be admissible for $v(0,x)$ let us now prove that

  $$v(0,x) - \varepsilon \leq \sum_{t=0}^{T-1} V_t(x_t, x_{t+1}) \; .$$

  Exercise 19. 1. Prove

  $$\sup_{y \in \Gamma_0(x)} \{V_0(x,y) + v(1,y)\} \geq V_0(x,x_1) + v(1,x_1)$$

  2. Prove

  $$V_0(x,x_1) + v(1,x_1) \geq \sum_{t=0}^{T-1} V_t(x_t, x_{t+1})$$

  3. Deduce

  $$\sup_{y \in \Gamma_0(x)} \{V_0(x,y) + v(1,y)\} \geq v(0,x) - \varepsilon \; .$$

▶ We have

$$\sup_{y\in\Gamma_0(x)} \{V_0(x,y)+v(1,y)\} \geq V_0(x,x_1)+v(1,x_1) \geq \sum_{t=0}^{T-1} V_t(x_t,x_{t+1}) \geq v(0,x)-\varepsilon\ .$$

Exercise 20. Conclude.

▶ This completes the proof as ε can be taken arbitrarily small.

□

Using Proposition 2 and the terminal relation $v(T,x) = V_T(x)$ for all $x \in A$, it is possible, at leat in theory but also in many easy applications, to compute all the value functions starting from the final time T by *backward induction*:

$$\begin{aligned} v(T-1,x) &= \sup_{y\in\Gamma_{T-1}(x)} V_{T-1}(x,y) \\ v(T-2,x) &= \sup_{y\in\Gamma_{T-2}(x)} \{V_{T-2}(x,y)+v(T-1,y)\} \\ &\vdots \\ v(0,x) &= \sup_{y\in\Gamma_0(x)} \{V_0(x,y)+v(1,y)\} \end{aligned}$$

Knowing $v(0,x)$, $\cdots$, $v(T-1,x)$, we easily deduce

Proposition 3 (Characterise the optimal sequences (or politics))**.** *The sequence* $(x,x_1,\cdots,x_T)$ *is a solution of* $v(0,x)$ *if and only if for all* $t \in \{0,\cdots,T-1\}$*,* x_{t+1} *is a solution to*

$$\sup_{y\in\Gamma_t(x_t)} \{V_t(x_t,y)+v(t+1,y)\}$$

Note that in practice, when solving Bellman equations, we have often already calculated the solutions to the static problems appearing in Proposition 2

It is important to remember the two-step approach of this chapter:

1. value functions are determined by backward induction,
2. we then determine the optimal policies (if any) by solving the sequence of static Proposition 3 which consist in determining the optimal successors x_1 of x_0 then the optimal successors of x_1 etc...

Finally, it should be noted that the method presented here (the same as that adopted in the shortest path problem) is robust because it can also solve all the intermediate problems posed at any intermediate date with any initial condition at that date.

2.5 Exercises

Exercise 21. Consider a sequence $(x_t)_{t\in\{0,1,2\}}$ which satisfies

$$\forall t \in \{0,1\}, \qquad x_{t+1} = \frac{x_t}{2}$$

where x_0 is given.

The instantaneous utility being defined by

$$V_t(x_t,x_{t+1}) = x_t - x_{t+1}^2$$

the agent wants to maximize the total payoff:

$$\sum_{t=0}^{1} V_t(x_t,x_{t+1})\ .$$

1. Let v be the value function. Identify $v(2, x_2)$ and determine the Bellman equation for this problem.
2. Using backward induction:
   (a) Find $v(1, x_1)$,
   (b) Find $v(0, x_0)$.
3. In the case $x_0 = 4$, compute the optimal payoff

$$\sum_{t=0}^{1} V_t(x_t, x_{t+1}) .$$

▶

1.

$$v(t, x) = \max_{y=x/2} [V_t(x, y) + v(t+1, y)] .$$

As there is no terminal payoff we have for all x_2, $v(2, x_2) = 0$.

2. (a) At $t = 1$:

$$\begin{aligned} v(1, x_1) &= \max_{y=x_1/2} [V_1(x_1, y) + v(2, y)] \\ &= x_1 - y^2 \\ &= x_1 - \frac{x_1^2}{4} \end{aligned}$$

(b) At $t = 0$:

$$\begin{aligned} v(0, x_0) &= \max_{y=x_0/2} [V_0(x_0, y) + v(1, y)] \\ &= \left[x_0 - \left(\frac{x_0}{2}\right)^2 + \left(\frac{x_0}{2} - \frac{1}{4}\left(\frac{x_0}{2}\right)^2\right)\right] \\ &= \frac{3}{2}x_0 - \frac{5}{16}x_0^2 \end{aligned}$$

In this example the state variable is trivial so that we could have solved it directly. Ex: $x_0 = 4$, $x_1 = 2$ and $x_2 = 1$. While the instantaneous utilities can be computed at each time. Hence the total payoff too. We recover the computations made above.

Exercise 22. We consider the finite-horizon problem

$$\sup_{x_1, x_2, x_3} \{f(x_1, x_0) + g(x_2, x_1) + h(x_3, x_2)\}$$

subject to

$$x_1 \in \Gamma_0(x_0), \quad x_2 \in \Gamma_1(x_1), \quad x_3 \in \Gamma_2(x_2),$$

where $x_0 \geq 0$ is given and

$$\begin{aligned} \Gamma_0(x_0) &= [0, x_0^4 + 2x_0 + 3], \\ \Gamma_1(x_1) &= \left[\frac{x_1}{2}, x_1^2 + x_1\right], \\ \Gamma_2(x_2) &= \left[0, \frac{x_2^2 + 4}{x_2^2}\right]. \end{aligned}$$

The payoff functions are

$$\begin{aligned} f(x_1, x_0) &= 2x_1x_0 - x_1^2 + x_1, \\ g(x_2, x_1) &= -\frac{1}{2x_2} + x_2x_1 - \frac{x_2^2}{2}, \\ h(x_3, x_2) &= \sqrt{x_3} - \frac{x_2x_3}{2}. \end{aligned}$$

▶

Stage 3

For given x_2,

$$v(3, x_2) = \sup_{x_3 \in \Gamma_2(x_2)} \left(\sqrt{x_3} - \frac{x_2}{2} x_3 \right).$$

The first-order condition for an interior solution is

$$\frac{1}{2\sqrt{x_3}} - \frac{x_2}{2} = 0,$$

which yields

$$x_3^*(x_2) = \frac{1}{x_2^2}.$$

This choice is feasible since

$$\frac{1}{x_2^2} \leq \frac{x_2^2 + 4}{x_2^2}.$$

The value function is therefore

$$v(3, x_2) = \sqrt{\frac{1}{x_2^2}} - \frac{x_2}{2} \cdot \frac{1}{x_2^2} = \frac{1}{2x_2}.$$

Stage 2

$$\begin{aligned} v(2, x_1) &= \sup_{x_2 \in \Gamma_1(x_1)} [g(x_2, x_1) + v_3(x_2)] \\ &= \sup_{x_2 \in [\frac{x_1}{2}, x_1^2 + x_1]} \left(x_2 x_1 - \frac{x_2^2}{2} \right). \end{aligned}$$

The first-order condition is

$$x_1 - x_2 = 0,$$

so

$$x_2^*(x_1) = x_1,$$

which is feasible for all $x_1 \geq 0$.

The value function becomes

$$v(2, x_1) = x_1^2 - \frac{x_1^2}{2} = \frac{x_1^2}{2}.$$

Stage 1

$$\begin{aligned} v(1, x_0) &= \sup_{x_1 \in \Gamma_0(x_0)} [f(x_1, x_0) + v_2(x_1)] \\ &= \sup_{x_1 \in [0, x_0^4 + 2x_0 + 3]} \left(2x_0 x_1 + x_1 - \frac{x_1^2}{2} \right). \end{aligned}$$

The first-order condition is

$$2x_0 + 1 - x_1 = 0,$$

which yields

$$x_1^*(x_0) = 2x_0 + 1.$$

This choice is feasible for all $x_0 \geq 0$.

Substituting back, the value function is

$$v(1, x_0) = \frac{(2x_0 + 1)^2}{2}.$$

Optimal policies

The optimal policy functions are

$$\boxed{\begin{aligned} x_1^*(x_0) &= 2x_0+1, \\ x_2^*(x_1) &= x_1, \\ x_3^*(x_2) &= \frac{1}{x_2^2}. \end{aligned}}$$

Value functions

$$\boxed{\begin{aligned} v_3(x_2) &= \frac{1}{2x_2}, \\ v_2(x_1) &= \frac{x_1^2}{2}, \\ v_1(x_0) &= \frac{(2x_0+1)^2}{2}. \end{aligned}}$$

Exercise 23. Let $x \geq 0$ and $N \in \mathbb{N}^*$. Consider

$$V_N(x) = \sup\left\{\prod_{i=1}^N x_i \;:\; x_i \geq 0\,, \sum_{i=1}^N x_i = x\right\}.$$

1. Compute V_1.
2. Prove that
$$V_N(x) = \sup_{y\in[0,x]}\left\{y\, V_{N-1}(x-y)\right\}$$
3. Prove that
$$V_N(x) = \frac{x^N}{N^N}$$
4. Deduce the arithmetico-geometric inequality:
$$\left(\prod |x_i|\right)^{1/N} \leq \frac{1}{N}\sum_{i=1}^N |x_i|\,.$$

▶

1. We have obviously
$$V_1(x) = x$$
2. Let us consider the sequence solution to
$$V_N(x) = \sup_{y\in[0,x]}\left\{y\, V_{N-1}(x-y)\right\}$$
We can rewrite it
$$V_N(x) = \sup_{x_N\in[0,x]}\left\{x_N\, V_{N-1}(x-x_N)\right\}$$
or
$$V_N(x) = \sup_{x_N\in[0,x]}\left\{x_N \sup_{x_{N-1}\in[0,x-x_N]}\left\{x_{N-1}V_{N-2}(x-x_N-x_{N-1})\right\}\right\}$$
Iterating the procedure we obtain
$$V_N(x) = \sup_{x_N\in[0,x]}\ \sup_{x_{N-1}\in[0,x-x_N]} \cdots \sup_{x_1\in[0,x-\sum_{i=2}^N x_i]}\left\{x_N\, x_{N-1}\cdots x_1\right\}$$

The constraints require $x_i \geq 0$ for all $i \in \{1, \cdots, N\}$. For the bound from above, the most restrictive contraint is $x_1 \in [0, x - \sum_{i=2}^{N} x_i$ or $\sum_{i=1}^{N} x_i = x$, we can rewrite the problem as

$$V_N(x) = \left\{ \prod_{i=1}^{N} x_i \ : \ x_i \geq 0 \, , \sum_{i=1}^{N} x_i = x \right\} .$$

Which is the desired result.

3. Let $\mathcal{P}(n)$ be the proposition

$$V_n(x) = \frac{x^n}{n^n}$$

- By the first question $\mathcal{P}(1)$ is true.
- Assume that $\mathcal{P}(n)$ is true. We have

$$V_{n+1}(x) = \sup_{y \in [0,x]} \left\{ \frac{y^n}{n^n}(x - y) \right\} = \frac{x^{n+1}}{(n+1)^{n+1}} .$$

  Hence $\mathcal{P}(n)$ true implies that $\mathcal{P}(n+1)$ is true.
- As a consquence $\mathcal{P}(n)$ is true for any $n \geq 1$.

4. We have for all positive x_i:

$$\frac{x^N}{N^N} \geq \left\{ \prod_{i=1}^{N} x_i \ : \ x_i \geq 0 \, , \sum_{i=1}^{N} x_i = x \right\}$$

So that

$$\frac{1}{N} \sum_{i=1}^{N} x_i \geq \left(\prod_{i=1}^{N} x_i \right)^{1/N}$$

The inequality is true for $|x_i|$.

Chapter 3

Infinite horizon

3.1 Reminders

3.1.1 Banach and compact spaces

Let S be a normed vector space.

Definition 6 (Banach space). *A sequence $(x_n)_n$ in S converges to $x \in S$ if*

$$\forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}, \ \forall n \in \mathbb{N}, \quad (n \geq n_0 \Rightarrow \|x_n - x\| < \varepsilon).$$

A sequence $(x_n)_n$ in S is a Cauchy sequence if

$$\forall \varepsilon > 0, \ \exists n_0, \ \forall (p,q) \in \mathbb{N}^2, \quad (p \geq n_0, \ q \geq n_0 \ \Rightarrow \|x_p - x_q\| < \varepsilon).$$

The space S is a Banach space if the metric space $(S, \|.\|)$ is complete (that is, every Cauchy sequence in S converges to an element in S).

Exercise 24. Let $X \subset \mathbb{R}^l$, and let $\mathcal{B}(X)$ be the set of bounded functions $f : X \to \mathbb{R}$ with the sup norm,

$$\|f\| = \sup_{x \in X} |f(x)|$$

Prove that $\mathcal{B}(X)$ is a Banach space.

▶ Let (f_n) be a Cauchy sequence in $B(X)$. Set $x \in X$. For all p, q in $\mathbb{N}$, we have : $|f_p(x) - f_q(x)| \leq \|f_p - f_q\|$. The sequence $(f_n(x))$ is thus a Cauchy sequence in $\mathbb{R}$. As $\mathbb{R}$ is complete, the sequence $(f_n(x))$ converges. Let us denote $f(x)$ its limit. We can thus define a map $f : X \to \mathbb{R}$.

The sequence (f_n) is a Cauchy sequence it is bouded by M. Hence for all $x \in X$, we have : $|f_n(x)| \leq \|f_n\|_\infty \leq M$. Passing to the limit we have $|f(x)| \leq M$. This proves that $f \in B(X)$ with $\|f\| \leq M$.

Remains to prove that (f_n) converges toward f in $B(X)$. Let $\varepsilon > 0$. The exists N such that for all $p, q > N$, we have:

$$\|f_p - f_q\| \leq \varepsilon.$$

Set $x \in X$. We have:

$$|f_p(x) - f_q(x)| \leq \varepsilon.$$

Letting p going to $+\infty$. We obtain :

$$|f(x) - f_q(x)| \leq \varepsilon.$$

As this inequality is true for any $x \in X$, we have :

$$\|f - f_q\| \leq \varepsilon.$$

This is true for any $q \geq N$, hence (f_q) converges toward f.

Finally any Cauchy sequence (f_n) of $B(X)$ converges, hence $B(X)$ is complete.

Exercise 25. Let $X \subset \mathbb{R}^l$, and let $\mathcal{C}(X)$ be the set of the bounded continuous functions $f : X \to \mathbb{R}$ with the sup norm,

$$\|f\| = \sup_{x \in X} |f(x)|$$

Prove that $\mathcal{C}(X)$ is a Banach space.

Definition 7 ((Sequential) compactness)**.** *A set X is compact if any bounded sequence $(x_n)_n$ of point of X admits a subsequence which converges to $x \in X$.*

Exercise 26. Let X be a compact set. If Y is a closed subset of X then Y is compact.

▶ Consider a bounded sequence in Y. As it is a bounded sequence in X which is compact there exists a subsequence converging in X. But as the subsequence is made of points in Y which is closed, the subsequence converges in Y. Hence Y is also compact.

3.1.2 The contraction mapping theorem

Definition 8 (Contraction mapping)**.** *Let S be a normed vector space and $T : S \to S$ be a function mapping S into itself. T is a contraction mapping (with modulus β) if for some $\beta \in (0,1)$,*

$$\forall (x,y) \in S^2, \quad \|Tx - Ty\| \leq \beta \|x - y\|.$$

Theorem 1 (Contraction mapping theorem)**.** *Let S be a non-empty Banach space and $T : S \to S$ be a contraction mapping with modulus β. We have*

1. *T has exactly one fixed point v in S,*
2. *for any $v_0 \in S$, and all $n \geq 0$, $\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$.*

Corollary 1 (N-stage Contraction Theorem)**.**
Let S be non-empty Banach space and $T : S \to S$. Assume that for some integer N, $T^N : S \to S$ is a contraction mapping with modulus β. Then

1. *T has exactly one fixed point v in S,*
2. *for any $v_0 \in S$ and all $k \geq 0$, $\|T^{kN} v_0 - v\| \leq \beta^k \|v_0 - v\|$.*

Exercise 27. Are the following maps contractions on $(\mathbb{R}_+, |\cdot|)$?

1. $$\begin{array}{rccc} f: & \mathbb{R}_+ & \to & \mathbb{R}_+ \\ & x & \mapsto & \sqrt{x+1} \end{array}$$

2. $$\begin{array}{rccc} f: & \mathbb{R}_+ & \to & \mathbb{R}_+ \\ & x & \mapsto & \sqrt{x^2+1} \end{array}$$

Exercise 28. Consider the map

$$\begin{array}{rccc} T: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & x & \mapsto & Ax \end{array}$$

where

$$\begin{pmatrix} 1/2 & 5 \\ 0 & 1/2 \end{pmatrix}$$

1. Is T a contraction on $\mathbb{R}^2$ equipped with the norm $\|(x,y)\| = |x| + |y|$?
2. Is T a contraction on $\mathbb{R}^2$ equipped with the norm $\|(x,y)\| = |x| + 20|y|$?

Exercise 29 (A Cauchy-Lipschitz theorem)**.** Let $M > 0$. The aim of this exercise is to prove that in $\mathcal{C}([0, M], \mathbb{R})$ there is a unique solution to

$$\begin{cases} y'(t) = y(t) & \forall t \in [0, M] \\ y(0) = 1 \end{cases} \tag{3.1}$$

Consider the operator

$$\begin{array}{cccc} \Phi : & \mathcal{C}([0, M], \mathbb{R}) & \to & \mathcal{C}([0, M], \mathbb{R}) \\ & y & \mapsto & \left(t \mapsto 1 + \int_0^t y(s) \mathrm{d}s \right) \end{array}$$

1. If $M < 1$, prove that Φ is a contraction on $(\mathcal{C}([0, M], \mathbb{R}), \| \cdot \|_\infty)$.
2. Prove that for all $(y, z) \in \mathcal{C}([0, M], \mathbb{R})^2$,

$$\forall t \in [0, M], \qquad |\Phi^n(y)(t) - \Phi^n(z)(t)| \leq \|y - z\|_\infty \frac{t^n}{n!}$$

3. Prove that for any $M > 0$ there is a n, depending on M, such that Φ^n is a contraction on $(\mathcal{C}([0, M], \mathbb{R}), \| \cdot \|_\infty)$.
4. Deduce that for any $M > 0$ there is a unique $y \in \mathcal{C}([0, M], \mathbb{R})$ of (3.1).

▶

1. For all $t \in [0, M]$ we have

$$|\Phi(y)(t) - \Phi(z)(t)| \leq \int_0^t |y(s) - z(s)| \mathrm{d}s \leq t\|y - z\|_\infty < M\|y - z\|_\infty$$

2. Consider $\mathcal{P}$ the proposition defined by $\mathcal{P}(n)$:"

$$\forall t \in [0, M], \qquad |\Phi^n(y)(t) - \Phi^n(z)(t)| \leq \|y - z\|_\infty \frac{t^n}{n!}$$

   " We have already proven that $\mathcal{P}(1)$ is true.
   Let us assume that $\mathcal{P}(n)$ is true. We have

$$|\Phi^{n+1}(y)(t) - \Phi^{n+1}(z)(t)| \leq \int_0^t |\Phi^n(y)(s) - \Phi^n(z)(s)| \mathrm{d}s \leq \|y - z\|_\infty \int_0^t \frac{t^n}{n!} = \|y - z\|_\infty \frac{t^{n+1}}{(n+1)!}$$

   Hence $\mathcal{P}$ is true for any $n \geq 1$.

3. From the previous question we have

$$\forall t \in [0, M], \qquad |\Phi^n(y)(t) - \Phi^n(z)(t)| \leq \|y - z\|_\infty \frac{M^n}{n!}$$

   As $M^n/n!$ converges to 0 when $n \to \infty$, there exists N large enough such that for all $n > N$, $M^n/n! < 1$. For all $n > N$, Φ^n is a contraction.

4. By the fixed point theorem, there exists a unique fixed point of Φ^n. Let us call y this fixed point we have

$$y = \Phi^n(y) \ .$$

   Thus applyng Φ:

$$\Phi(y) = \Phi\left[\Phi^n(y)\right] = \Phi^n\left[\Phi(y)\right]$$

   So that $\Phi(y)$ is a fixed point of Φ^n. But as Φ^n has a unique fixed point $\Phi(y) = y$. Hence Φ also has a fixed point.

Remark 2. The same strategy can be adopted to the following: Consider $k \in \mathcal{C}([0,1]\times\mathbb{R},\mathbb{R})$ be uniformly Lipschitz with a Lipschitz constant $L > 0$ with respect to the second variable. Let $\alpha \in \mathbb{R}$. The aim of this exercise is to prove that in $\mathcal{C}([0,M],\mathbb{R})$ there is a unique solution to the Cauchy problem

$$\begin{cases} y'(t) = k(t, y(t)) & \forall t \in [0,1] \\ y(0) = \alpha \end{cases}$$

Let $E = \mathcal{C}([0,1],\mathbb{R})$ be equipped with $\|\cdot\|_\infty$. Consider the operator

$$\begin{array}{rccl} \Phi: & E & \to & E \\ & y & \mapsto & t \mapsto \alpha + \int_0^t k(s, y(s))ds \end{array}$$

1. Assume $L < 1$. Prove that Φ is a contraction on $(E, \|\cdot\|_\infty)$.
2. Deduce that there is a unique $y \in E$ such that

$$\forall t \in [0,1], \qquad y(t) = \alpha + \int_0^t k(s, y(s))ds\,.$$

3. Conclude for the case $L < 1$.
4. Let now $L \geq 1$. Define

$$\|y\|_L = \sup_{t\in[0,1]} \mathrm{e}^{-2Lt}|y(t)|\,.$$

   (a) Prove that the norms $\|\cdot\|_L$ and $\|\cdot\|_\infty$ are equivalent.

   (b) Prove that Φ is a contraction on $(E, \|\cdot\|_L)$.

   (c) Conclude for the case $L \geq 1$.

▶

1. For all $t \in [0,1]$ we have

$$|\Phi(y)(t) - \Phi(z)(t)| \leq \int_0^t |k(s,y(s)) - k(s,z(s)|)\,\mathrm{d}s \leq L\int_0^t |y(s) - z(s)|\mathrm{d}s \leq L\|y - z\|_\infty$$

2. We can thus apply the fixed point theorem.
3. There is a unique solution to the Cauchy-problem.
4. (a) For all $t \in [0,1]$ we have

$$\mathrm{e}^{-2L} \leq \mathrm{e}^{-2Lt} \leq 1 \Leftrightarrow \mathrm{e}^{-2L}|y(t)| \leq \mathrm{e}^{-2Lt} \leq |y(t)| \Rightarrow \mathrm{e}^{-2L}\|y\|_\infty \leq \|y\|_L \leq \|y\|_\infty$$

   (b) For all $t \in [0,1]$ we have

$$\begin{aligned} |\Phi(y)(t) - \Phi(z)(t)| &\leq \int_0^t |k(s,y(s)) - k(s,z(s)|)\,\mathrm{d}s \leq L\int_0^t |y(s) - z(s)|\mathrm{d}s \\ &\leq L\int_0^t \mathrm{e}^{2Ls}\mathrm{e}^{-2Ls}|y(s) - z(s)|\mathrm{d}s \leq L\frac{\mathrm{e}2Ls}{2L}\|y - z\|_L = \frac{\mathrm{e}^{2Ls}}{2}\|y - z\|_L \end{aligned}$$

   Hence Φ is a contraction as $\mathrm{e}^{2Ls}/2 < 1$.

   (c) By the fixed point theorem, there exists a unique fixed point which is the unique solution to the Cauchy problem.

3.1.3 Continuity of multi-valued functions

Definition 9 (Upper hemicontinuity). *Let A and B be non-empty compact sets. A multi-valued function $\Gamma : A \rightrightarrows B$ is said to be* upper hemicontinuous (uhc) *at a point $x \in A$ if, for every sequence $(x_n)_n$ converging to x in A, and all sequence of terms $y_n \in \Gamma(x_n)$, there exists a subsequence $(y_{n_k})_k$ converging to $\Gamma(x)$.*

Definition 10 (Lower hemicontinuity). *Let A and B be non-empty compact sets. A multi-valued function $\Gamma : A \rightrightarrows B$ is said to be* lower hemicontinuous (lhc) *at the point $x \in A$ if for every sequence $(x_n)_n$ converging to x in A, and all $y \in \Gamma(x)$, there exist subsequences $(x_{n_k})_k$ and $(y_{n_k})_k$ of points in $\Gamma(x_{n_k})$ such that $\lim y_{n_k} = y$.*

Definition 11 (Continuity). *If a multi-valued function is both upper hemicontinuous and lower hemicontinuous, it is said to be* continuous.

Remark 3. When X is compact, the closed graph theorem states that Γ is uhc is equivalent to say that its graph

$$\text{graph}(\Gamma) := \{(x, y) : x \in X, y \in \Gamma(x)\}$$

is closed.

Exercise 30. Are the following function from $[-1, 1] \to [-1, 1]$ upper hemicontinuous? lower hemicontinuous? Continuous?

1.
$$x \mapsto \begin{cases} 0 & \text{if } x < 0 \\ [0, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

2.
$$x \mapsto \begin{cases} 0 & \text{if } x \leq 0 \\ [-1, 1] & \text{if } x > 0 \end{cases}$$

3.
$$x \mapsto |x|$$

▶

1. The function is not lower hemicontinuous at 0. To see this, consider a sequence that converges to 0 from the left. The image of 0 is a vertical line that contains the point 1/2. But every sequence in the image of Γ is contained in the bottom horizontal line, so it cannot converge to 1/2. In contrast, the function is upper hemicontinuous everywhere. For example, considering any sequence $(x_n)_n$ that converges to 0 from the left or from the right, and any corresponding sequence $(y_n)_n$, the limit of $(y_n)_n$ is contained in the vertical line that is the image of the limit of $(x_n)_n$.

2. The function is not upper hemicontinuous at 0. To see this, let $(x_n)_n$ be a sequence that converges to 0 from the right. The image of $(x_n)_n$ contains vertical lines, so there exists a corresponding sequence $(y_n)_n$ in which all elements are 1/2. The image of the limit of the sequence contains a single point $\Gamma(x)$, so it does not contain the limit of $(y_n)_n$. In contrast, that function is lower hemicontinuous everywhere. For example, for any sequence a that converges to x, from the left or from the right, $\Gamma(x)$ contains a single point, and there exists a corresponding sequence $(y_n)_n$ that converges to $\Gamma(x)$.

3. Any single-value continous function is continous

$$\forall (x_n)_{n \in \mathbb{N}} \subset D : \lim_{n \to \infty} x_n = c \Rightarrow \lim_{n \to \infty} f(x_n) = f(c)\,.$$

3.2 Dynamic programming problems in discrete time and infinite horizon

We are intersted in

$$\sup_{(x_t)} \left\{ \sum_{t=0}^{\infty} \beta^t V_t(x_t, x_{t+1}) \right\} \tag{3.2}$$

under the constraints $x_0 = x$ is given, and for all $t \geq 0$, $x_t \in A$ and $x_{t+1} \in \Gamma(x_t)$. we interpret A, Γ and V as before and $\beta \in (0,1)$ is the discount factor.

Definition 12 (Admissible sequences)**.** *For $x \in A$, let us denote* Adm(x) *the set of all the* admissible sequences *starting from x:*

$$\mathrm{Adm}(x) = \{(x_t)_{t\geq 0} \in A^{\mathbb{N}} \,:\, x_0 = x \text{ and } \forall t \geq 0,\, x_{t+1} \in \Gamma(x_t)\} \,.$$

3.3 Existence of solutions

Assumption H 1. Let (A, d) be a compact metric space.

Let $A^{\mathbb{N}}$ the set of all the sequences with values in A. We equip $A^{\mathbb{N}}$ with the following distance

$$\forall\, ((u_i)_i, (v_i)_i) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \qquad d_\infty((u_i)_i, (v_i)_i) = \sum_{i=0}^{\infty} \frac{1}{2^i} d(u_i, v_i) \,.$$

Exercise 31. Prove that d_∞ is finite and defines a distance.

▶ Obvious. Check that

- The distance between an object and itself is always zero.
- The distance between distinct objects is always positive.
- Distance is symmetric: the distance from x to y is always the same as the distance from y to x.
- Distance satisfies the triangle inequality: if x, y, and z are three objects, then $d(x,z) \leq d(x,y) + d(y,z)$.

Proposition 4. *The set $(A^{\mathbb{N}}, d_\infty)$ is a compact set.*

Proof. The proof is classical (it uses the compactness of A and a diagonal extraction). □

Assumption H 2. Assume that Γ is continuous.

Lemma 1. *For any $x \in A$,* Adm(x) *is a compact set of $(A^{\mathbb{N}}, d_\infty)$.*

Proof. It is enough to prove that Adm(x) is closed. Let $((x_t^n)_{t\geq 0})_n$ a sequence of Adm(x) converging to $(x_t)_{t\geq 0}$ in $A^{\mathbb{N}}$ (for d_∞).

Exercise 32. 1. Prove that $x_0 = x$.

2. Prove that for all $t \geq 0$, $x_{t+1} \in \Gamma(x_t)$.

▶

1. By definition of the sequence, for all $t \geq 0$, $(x_t^n)_n$ converges to x_t in A.
2. For all n, $(x_t^n, x_{t+1}^n) \in \mathrm{graph}(\Gamma)$ and Γ is a closed graph hence for all $t \geq 0$, $x_{t+1} \in \Gamma(x_t)$.

□

Assumption H 3. Assume that V is continuous on $A \times A$.

Let $x \in A$. For any $(x_t)_{t\geq 0}$ in $\mathrm{Adm}(x)$ define

$$u\left((x_t)_{t\geq 0}\right) := \sum_{t=0}^{\infty} \beta^t V(x_t, x_{t+1}) \,.$$

Lemma 2. *The function u is continuous on $(A^{\mathbb{N}}, d_\infty)$.*

Proof. Let $(x_t)_{t\geq 0} \in A^{\mathbb{N}}$.

Exercise 33. 1. For t large: Let $\varepsilon > 0$. Prove that there exists $\tau > 0$ such that

$$\sum_{t=\tau}^{\infty} \beta^t V(x_t, x_{t+1}) \leq \frac{\varepsilon}{4} \,. \tag{3.3}$$

2. For t smaller: Prove that for any $\varepsilon' > 0$ there exists δ_0 such that for all $(y, z) \in A \times A$:

$$d(x_t, y) + d(x_{t+1}, z) < \delta_0 \Rightarrow |V(x_t, x_{t+1}) - V(y, z)| < \varepsilon' \tag{3.4}$$

3. Prove that, for $t < \tau - 1$,

$$d_\infty((x_t)_t, (y_t)_t) < \frac{\delta_0}{2^{\tau+1}} \Rightarrow d(x_t, y_t) + d(x_{t+1}, y_{t+1}) < \delta_0 \,.$$

4. Conclusion: Prove for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(y_t)_t$

$$d_\infty((x_t)_t, (y_t)_t) < \delta \Rightarrow |u\left((x_t)_t\right) - u\left((y_t)_t\right)| < \varepsilon \,.$$

▶

1. The function V is continuous and $A \times A$ is a compact set, so for any t_0

$$\sum_{t=\tau}^{\infty} \beta^t V(x_t, x_{t+1}) \leq \max_{(x,y)\in A\times A} |V(x, y)| \sum_{t=\tau}^{\infty} \beta^t \,.$$

As $\beta \in (0, 1)$ the serie converges which gives the desired result.

2. It comes from the continuity of V on $A \times A$.

3. We have

$$\begin{aligned} d_\infty((x_t)_t, (y_t)_t) = \sum_{t=0}^{\infty} \frac{1}{2^t} d(x_t, y_t) < \frac{\delta_0}{2^{\tau+1}} \quad &\Rightarrow \quad \sum_{t=0}^{\tau} \frac{1}{2^t} d(x_t, y_t) < \frac{\delta_0}{2^{\tau+1}} \\ &\Rightarrow \quad \sum_{t=0}^{\tau} \frac{2^{\tau+1}}{2^t} d(x_t, y_t) < \delta_0 \\ &\Rightarrow \quad 2\sum_{t=0}^{\tau} d(x_t, y_t) < \delta_0 \\ &\Rightarrow \quad \forall t \in \{0, \cdots, \tau - 1\},\ d(x_t, y_t) + d(x_{t+1}, y_{t+1}) < \delta_0 \end{aligned}$$

4. Set

$$\delta = \frac{\delta_0}{2^{\tau+1}} \text{ and } \varepsilon' = \frac{\varepsilon}{2\tau} \,.$$

Applying (3.4) to $(y, z) = (y_t, y_{t+1})$ and (3.3), we have

$$\begin{aligned} |u\left((x_t)_t\right) - u\left((y_t)_t\right)| &\leq \sum_{t=0}^{\tau-1} \beta^t \left|V(x_t, x_{t+1}) - V(y_t, y_{t+1})\right| + \sum_{t=\tau}^{\infty} \beta^t \left|V(x_t, x_{t+1}) - V(y_t, y_{t+1})\right| \\ &\leq \sum_{t=0}^{\tau-1} \frac{\varepsilon}{2\tau} + \sum_{t=\tau}^{\infty} \beta^t \left|V(x_t, x_{t+1})\right| + \sum_{t=\tau}^{\infty} \beta^t \left|V(y_t, y_{t+1})\right| \\ &\leq \frac{\varepsilon}{2} + 2\frac{\varepsilon}{4} = \varepsilon \,. \end{aligned}$$

□

Theorem 2. *For all $x \in A$, there exists $(x_t)_{t\geq 0}$ in* $\mathrm{Adm}(x)$ *which is optimal.*

Proof. This result is a direct consequence of the two previous lemmata by Weierstrass' theorem. □

For any $x \in A$, Problem (3.2) can be rephrased as

$$v(x) := \sup_{(x_t)_{t\geq 0}\in \mathrm{Adm}(x)} u\left((x_t)_{t\geq 0}\right) . \tag{3.5}$$

We will say that $(x_t)_{t\geq 0}$ is a solution to $v(x)$. Theorem 2 states that this problem has a solution and that the supremum is actually a maximum.

3.4 Optimal politics

Mimicking the proof of Propositions 1 and 2 it is easy to check the two following propositions:

Proposition 5 (Dynamic programmation principle)**.** *Let $x \in A$. If $(x_t)_t \geq 0 \in \mathrm{Adm}(x)$ is a solution to $v(x)$ then for any $\tau \geq 0$ the sequence $(x_t)_{t\geq\tau}$ is a solution to $v(x_\tau)$.*

Proposition 6 (Bellman's equation)**.** *Moreover, v is solution to Bellman's equation:*

$$v(x) = \sup_{y\in\Gamma(x)} \{V(x,y) + \beta v(y)\} .$$

However the backward induction of the previous section cannot be used here. The idea is to define

$$Tf(x) := \sup_{y\in\Gamma(x)} \{V(x,y) + \beta f(y)\}$$

and to prove that T has a fixed point, which would be a solution to Bellman's equation. In order to apply the fixed point theorem we state:

Theorem 3 (Blackwell's theorem)**.** *Let H be an operator $\mathcal{B}(A) \to \mathcal{B}(A)$ satisjying*

1. *H is monotone i.e.*

$$\forall (f,g) \in \mathcal{B}(A)\times\mathcal{B}(A), \quad (f \leq g \text{ on } A \ \Rightarrow Hf \leq Hg \text{ on } A\ )$$

2. *there exists $\eta \in (0,1)$ such that*

$$\forall a > 0, \forall f \in \mathcal{B}(A),\ H(f+a) \leq Hf + \eta a$$

then H is a η-contraction of $\mathcal{B}(A)$ for $\|\cdot\|_\infty$.

Proof.

Exercise 34. 1. Using $f \leq g + \|f-g\|_\infty$, prove

$$Hf \leq Hg + \eta\|f-g\|_\infty .$$

2. Deduce

$$\|Hf - Hg\|_\infty \leq \eta\|f-g\|_\infty .$$

▶

1. By the assumptions on H,

$$Hf \leq H(g + \|f-g\|_\infty) \leq Hg + \eta\|f-g\|_\infty .$$

2. Inverting the roles of f and g in the previous exercice, and taking the supremum we obtain the desired result.

□

Corollary 2. *Bellman's equation, Proposition 6, has a unique solution which a solution to* (3.5). *Moreover, for any $f \in \mathcal{B}(A)$, v is the uniform limit of the sequence $(T^n f)_n$.*

As in the previous chapter, $(x_t)_{t\geq 0} \in \mathrm{Adm}(x)$ is a solution to $v(x)$ if and only if for all $t \geq 0$, x_{t+1} is a solution to the following static problem

$$\max_{y\in\Gamma(x_t)} \{V(x_t, y) + \beta v(y)\} \ .$$

Hence to determine the optimal politics we first determine v by solving Bellman's equation. We then define the multi-valued function

$$M(x) := \{y \in \Gamma(x) \, : \, v(x) = V(x, y) + \beta v(y)\} \ .$$

The multi-valued function $M(x)$ can be interpreted as the set of all the optimal successors of x, and the optimal politics starting from x are the iterates of this multi-valued function.

In this chapter, we only considered the case when V is continuous and A bounded. It is, however, natural in economic applications to consider unbounded utility functions (typically a logarithm), and the essential ingredients of the chapter are generally easily adapted.

3.5 Exercises

Exercise 35. We consider the following problem:

$$\max_{\{c_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^\alpha}{\alpha}, \qquad \alpha \in (0, 1),$$

under the constraints

$$k_{t+1} = a k_t - c_t, \qquad k_0 > 0,$$

where $0 < \beta < 1$, $a > 1$, and for all t, $k_t \geq 0$ and $c_t \geq 0$.

1. Write the Bellman equation (in k_t).
2. Write the first order condition in the Bellman equation.
3. Determine the envelop equation.
4. Determine the optimal politics c_t.
5. Assuming that there exists θ such that $k_t = c_t/(a\theta)$, write the euqation satisfied by k_t and determine k_t.

▶

1. The Bellman equation reads:

$$v(x) = \sup_{0\leq y\leq ax} \left\{ \frac{(ax - y)^\alpha}{\alpha} + \beta v(y) \right\} .$$

2. The first order condition is:

$$-(ax - y)^{\alpha-1} + \beta v'(y) = 0,$$

or

$$\beta v'(y) = (ax - y)^{\alpha-1} \ .$$

3. We have:

$$v'(x) = \frac{\partial}{\partial x}\left[\frac{(ax-y^*)^\alpha}{\alpha}\right] + \frac{\partial}{\partial x}y^*(x)\cdot\frac{\partial}{\partial y}\left\{\frac{(ax-y)^\alpha}{\alpha}+\beta v(y)\right\} = a(ax-y^*)^{\alpha-1}.$$

4. Combining the first order condition in t:

$$\beta v'(k_{t+1}) = (ak_t - k_{t+1})^{\alpha-1} = c_t^{\alpha-1}\ .$$

and the envelop equation in $t+1$:

$$v'(k_{t+1}) = (ak_{t+1} - k_{t+2})^{\alpha-1} = ac_{t+1}^{\alpha-1}.$$

we obtain:

$$c_{t+1} = (\beta a)^{\frac{1}{1-\alpha}}c_t.$$

And hence

$$c_t = (\beta a)^{t/(1-\alpha)}c_0\ .$$

5. The dynamics on k_t is given by

$$k_{t+1} = ak_t - c_0 q^t,$$

We can look for a solution on the form:

$$k_t = Aa^t + Dq^t,$$

where $(A,D)\in\mathbb{R}^2$. We can look for a particular solution on the form Dq^t, we have:

$$D(q-a) = -c_0 \quad\Rightarrow\quad D = \frac{c_0}{a-q}.$$

Hence

$$k_t = Aa^t + \frac{c_0}{a-q}q^t.$$

Now, if $A\neq 0$ the solution is not bounded, hence $A=0$ and :

$$k_t = \frac{c_0}{a-q}q^t.$$

Exercise 36. Consider the optimization problem:

$$\max_{(k_t)_t}\sum_{t=0}^{+\infty}\beta^t\ln(c_t)\ ,$$

where $\beta\in(0,1)$, $k_{t+1} = ak_t - c_t$ with $a>0$, $k_0>0$ being given. We also impose that for all t, $k_t>0$ and $c_t>0$.

1. Write the Bellman equation associated to this problem (as a function of c).
2. Assuming that $v(x) = A + B\ln(x)$ write the first order condition.
3. Use the Bellman equation to determine B.
4. Deduce the optimal politics c.
5. Deduce the optimal capital.

▶

1. The problem can be rewritten

$$\max_{(k_t)_t} \sum_{t=0}^{T-1} \beta^t \ln(ak_t - k_{t+1}) \ ,$$

under the constraint $0 < k_t < ak_{t-1}$. The Bellman equation is

$$v(y) = \max_{0<y<ax} \{\ln(c) + \beta v(ax - y)\}$$

2. The first order conditions reads

$$\frac{1}{c} - \frac{\beta}{ax - c} = 0$$

so that the optimal c satisfies

$$c^* = \frac{ax}{1 + B\beta}$$

3. As

$$A + B\ln(x) = \ln\left(\frac{ax}{1 + B\beta}\right) + \beta\left\{A + B\ln\left(ax - \frac{ax}{1 + B\beta}\right)\right\}$$

we have

$$A + B\ln(x) = \ln(a) + \ln(x) - \ln(1 + B\beta) + A\beta + B\beta\left[\ln(B\beta) + \ln(a) + \ln(x) - \ln(1 + B\beta)\right] \ .$$

So that, equaling the $\ln(k)$ terms we obtain

$$B = \frac{1}{1 - \beta} \ .$$

4. Replacing we obtain

$$c_t = (1 - \beta)ak_t$$

5. As $k_{t+1} = ak_t - c_t = \beta\, ak_t$ we obtain

$$k_t = (a\beta)^t k_0 \ .$$

Exercise 37. Consider the infinite-horizon discrete-time problem:

$$\max_{\{c_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to the law of motion:

$$k_{t+1} = k_t^\alpha - c_t, \qquad k_0 > 0,$$

where $0 < \beta < 1$, $0 < \alpha < 1$, and $c_t > 0$.

1. Write the Bellman equation using next period capital as the control variable.
2. Derive the first-order condition and the envelope condition.
3. Guess the functional form of the value function and verify it.
4. Derive the optimal policy functions for k_{t+1} and c_t.
5. Describe the optimal dynamics of capital and consumption.

▶

1. The Bellman equation is:

$$v(x) = \sup_{0<y<x^\alpha} \left\{ \ln(x^\alpha - y) + \beta v(y) \right\}.$$

2. For an interior solution, the first-order condition with respect to y is:

$$-\frac{1}{x^\alpha - y} + \beta v'(y) = 0,$$

or equivalently:

$$\frac{1}{x^\alpha - y} = \beta v'(y).$$

3. By the envelope theorem:

$$v'(x) = \frac{\partial}{\partial x} \ln(x^\alpha - y^*) = \frac{\alpha x^{\alpha-1}}{x^\alpha - y^*}.$$

Thus,

$$v'(x) = \frac{\alpha x^{\alpha-1}}{c(x)},$$

where $c(x) = x^\alpha - y^*(x)$.

4. Conjecture:

$$v(x) = A + B \ln x.$$

Then:

$$v'(x) = \frac{B}{x}.$$

Using the envelope condition:

$$\frac{B}{x} = \frac{\alpha x^{\alpha-1}}{x^\alpha - y} \quad \Rightarrow \quad x^\alpha - y = \frac{\alpha}{B} x^\alpha.$$

Hence,

$$y = \left(1 - \frac{\alpha}{B}\right) x^\alpha.$$

5. From the first-order condition:

$$\frac{1}{x^\alpha - y} = \frac{\beta B}{y}.$$

Substituting:

$$\frac{1}{(\alpha/B) x^\alpha} = \frac{\beta B}{(1 - \alpha/B) x^\alpha}.$$

Simplifying yields:

$$B = \alpha(1 + \beta B).$$

Solving:

$$B(1 - \alpha\beta) = \alpha \quad \Rightarrow \quad B = \frac{\alpha}{1 - \alpha\beta}.$$

6. Substituting B back:

$$k_{t+1} = \alpha\beta\, k_t^\alpha,$$

and

$$c_t = (1 - \alpha\beta)\, k_t^\alpha.$$

7. The optimal dynamics of capital is:

$$k_{t+1} = \alpha\beta\, k_t^\alpha.$$

Consumption follows:

$$c_t = (1 - \alpha\beta)\, k_t^\alpha.$$

Chapter 4

Calculus of Variation

We are now interested in (finite horizon) problems in continuous time. Such problems consist in maximising a criterion of the type

$$J(x) = \int_0^T L(t, x(t), \dot{x}(t))\mathrm{d}t + g\left(x(T)\right) + f\left(x(0)\right)$$

on a set of functions $[0, T] \to \mathbb{R}^n$. The good functional framework is the Sobolev space but to simplify the lecture we will be working here in the set of $\mathcal{C}^1$-functions or piecewise $\mathcal{C}^1$-functions.

The map $(t, x, v) \mapsto L(t, x, v)$ is called the Lagrangian. The maps g and f are the terminal and intial gain. We will assuma that $L \in \mathcal{C}^1([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$, $g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ and $f \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$.

We can also consider a variant of the problem with prescribed limit conditions: we want to maximise

$$J(x) = \int_0^T L(t, x(t), \dot{x}(t))\mathrm{d}t$$

on the set of functions such that $x(0) = x_0$ and $x(T) = x_T$.

Historically, the calculation of variations has developed since the 17th century (the brachistochrone problem solved by Bernoulli) in conjunction with the development of physics (mechanics in particular, but also the problem of minimum resistance posed by Newton in his Principia, which is still largely unresolved today) and the geometry (of geodesics or harmonic applications, for example). Some of the great names among the mathematicians of the past three centuries have left their mark on its development: Euler, Lagrange, Hamilton, Jacobi, Legendre, Weierstrass, Noether, Carathéodory... Its use in economics is more recent, and it became truly popular in the 1960s in models of growth, investment, stock management and, more recently, in the theory of incentives and auctions. In finance, it is also common practice to use continuous-time models, since the realistic dynamics in this framework are of a random nature, stochastic control is often used instead.

About existence

Solving a calculus of variations problem means solving an optimisation problem in an infinite-dimensional functional space. There is therefore nothing obvious a priori about the existence of solutions, and I would like to warn the reader on this point. The aim here is not to develop a theory of existence - for that, consult the book by I. Ekeland and R. Temam [**?**, **?**,] for example - but to point out that most existence results require the concavity (if we maximise; convexity if we minimise) of the Lagrangian with respect to the variable v.

Exercise 38. Consider the problem

$$\inf J(x) \quad \text{where} \quad J(x) := \int_0^1 \left[\left(\dot{x}(t)^2 - 1\right)^2 + x^2(t)\right] \mathrm{d}t$$

where $x(0) = x(1) = 0$. Let for $t \in [0, 1]$,

$$t \mapsto \frac{1}{2} - \left|t - \frac{1}{2}\right| ,$$

which we extend in a map u_0 to $\mathbb{R}$ by periodicity. Consider the sequence

$$u_n(t) = \frac{u_0(nt)}{n}$$

1. Prove that u_n satisfies the limit conditions.
2. Determine the limit of $J(u_n)$.
3. Deduce the value of the infimum.
4. Conclude.

▶

1. Indeed, $u_n(0) = u_n(1) = 0$.
2. As $\dot{u_n} = 1$ alsmost everywhere and $|u_n| \leq 1/(4n)$, $J(u_n)$ converges to 0 when n goes to ∞.
3. The infimum is then 0.
4. If this infimum is a minimum: consider u solution to the problem. We would have $J(u) = 0$. Due to the structure of J is would imply $u \equiv 0$ and $\dot{u} \in \{-1, 1\}$ alsmost everywhere which is impossible.

4.1 The Euler-Lagrange equation

We denote

$$L_{x_i} := \frac{\partial L}{\partial x_i}, \quad L_{v_i} := \frac{\partial L}{\partial v_i}, \quad \nabla_x L := \begin{pmatrix} L_{x_1} \\ \vdots \\ L_{x_d} \end{pmatrix} \text{ and } \nabla_v L := \begin{pmatrix} L_{v_1} \\ \vdots \\ L_{v_d} \end{pmatrix}$$

Theorem 4 (Euler-Lagrange equation)**.** *Let x be a solution of classe $\mathcal{C}^1([0,T], \mathbb{R}^d)$ to*

$$\sup_{x \in \mathcal{C}^1([0,T],\mathbb{R}^d)} J(x) \tag{4.1}$$

We have

1. *The map x is a solution to the* Euler-Lagrange equation*:*
$$\frac{d}{dt} \left[\nabla_v L(t, x(t), \dot{x}(t))\right] = \nabla_x L(t, x(t), \dot{x}(t)) \,.$$
2. *The map x satisfies the* transversality conditions*:*
$$\nabla_v L(0, x(0), \dot{x}(0)) = \nabla f\left(x(0)\right)$$
and
$$\nabla_v L(T, x(T), \dot{x}(T)) = -\nabla g\left(x(T)\right) \,.$$

The proof of this result is based on the following:

Lemma 3 (Dubois-Reymond lemma)**.** *Let φ and ψ be in $\mathcal{C}^0([0,T], \mathbb{R})$. The two propositions are equivalent*

1. *ψ is in $\mathcal{C}^1([0,T], \mathbb{R})$ and $\psi' = \varphi$,*
2. *for all $h \in E_1 := \{h \in \mathcal{C}^1([0,T], \mathbb{R}) \ : \ h(0) = h(T) = 0\}$*
$$\int_0^T (\varphi(t)\, h(t) + \psi(t)\, \dot{h}(t)) \mathrm{d}t = 0 \,.$$

Proof.

Exercise 39. 1. Prove the implication.

2. Let us now prove the reverse implication. Let Φ be an antiderivative of φ.
   (a) Prove
   $$\int_0^T (\psi(t) - \Phi(t))\dot{h}(t)\mathrm{d}t = 0, \quad \forall h \in E_1 \ .$$
   (b) Let us define
   $$c := \frac{1}{T}\int_0^T (\psi(t) - \Phi(t))\,\mathrm{d}t$$
   and the map
   $$\alpha : t \mapsto \int_0^t (\psi(s) - \Phi(s) - c)\mathrm{d}s \ .$$
   Prove that α belongs to E_1.
   (c) Prove that
   $$\int_0^T (\psi(t) - \Phi(t) - c)\dot{h}(t)\mathrm{d}t = 0, \quad \forall h \in E_1 \ .$$
   (d) Prove that
   $$\int_0^T (\psi(t) - \Phi(t) - c)^2\mathrm{d}t = 0 \ .$$
3. Conclude.

▶

1. The implication is obvious by integration by parts.
2. (a) By intergration by parts.
   (b) As $h \in E_1$ and c is a constant.
   (c) We can take the above α as a test function.
3. This implies $\psi = \Phi + c$ and gives the result.

□

Proof of Theorem 4.

Exercise 40. To avoid heavy notations let us write $x^* = x$.

1. Let $h \in \mathcal{C}^1([0,T],\mathbb{R}^d)$. Compute
$$\lim_{\varepsilon\to 0} \frac{J(x+\varepsilon h) - J(x)}{\varepsilon}$$
2. Prove
$$\int_0^T \left[\nabla_x L(t,x(t),\dot{x}(t)) \cdot h(t) + \nabla_v L(t,x(t),\dot{x}(t)) \cdot \dot{h}(t)\right]\mathrm{d}t \\ + \nabla g\left(x(T)\right) \cdot h(T) + \nabla f\left(x(0)\right) \cdot h(0) = 0 \ .$$
3. Prove that for any $i \in \{1,\cdots,d\}$,
$$\int_0^T \left[L_{x_i}(t,x(t),\dot{x}(t)) \cdot h(t) + L_{v_i}(t,x(t),\dot{x}(t)) \cdot \dot{h}(t)\right]\mathrm{d}t = 0$$
4. Prove that x satisfies the Euler-Lagrange equation.

5. Deduce that, for any $i \in \{1, \cdots, d\}$,

$$\frac{d}{dt}\left[L_{v_i}(t, x(t), \dot{x}(t))\right] = L_{x_i}(t, x(t), \dot{x}(t)) \; .$$

6. Prove that

$$[\nabla_v L(T, x(T), \dot{x}(T)) + \nabla g\,(x(T))] \cdot h(T) + [\nabla f\,(x(0)) - \nabla_v L(0, x(0), \dot{x}(0))] \cdot h(0) = 0 \; .$$

7. Conclude

▶

1. We use the mean value and the dominated convergence theorem to prove

$$\lim_{\varepsilon \to 0} \frac{J(x + \varepsilon h) - J(x)}{\varepsilon} = \int_0^T \nabla_x L(t, x(t), \dot{x}(t)) \cdot h(t) + \nabla_v L(t, x(t), \dot{x}(t)) \cdot \dot{h}(t) \mathrm{d}t + \nabla g\,(x(T)) \cdot h(T) + \nabla f\,(x(0)) \cdot h(0)$$

2. As x is the maximum

$$\lim_{\varepsilon \to 0} \frac{J(x + \varepsilon h) - J(x)}{\varepsilon} \leq 0 \; .$$

Taking $h \mapsto -h$ we obtain the result.

3. It is just a rewriting of the Euler-Lagrange equation line by line.

4. Let

$$E_d := \{h \in \mathcal{C}^1([0, T], \mathbb{R}^d) \;:\; h(0) = h(T) = 0\} \; .$$

Taking $h \in E_d$ we obtain from the previous question

$$\int_0^T \left[\nabla_x L(t, x(t), \dot{x}(t)) \cdot h(t) + \nabla_v L(t, x(t), \dot{x}(t)) \cdot \dot{h}(t)\right] \mathrm{d}t = 0 \; .$$

The result comes by writing this equality coordinates by coordinate.

5. Let h be in $\mathcal{C}^1([0, T], \mathbb{R}^d)$. As h is arbitrary, an integration by part in the Question 1 provides the desired result.

6. By integration by parts.

7. We use the previous lemma to conclude.

□

Theorem 5. *Under the same assumptions as in Theorem 4. We moreover assume that g and f are concave functions in $\mathbb{R}^d$ and for all $t \in [0, T]$, $L(t, \cdot, \cdot)$ is concave in $\mathbb{R}^d$. The map $x \in \mathcal{C}^1([0, T], \mathbb{R}^d)$ is a solution to the Euler-Lagrange equation and the transversality condition if and only if x is a solution to Problem 4.1.*

Proof.

Exercise 41. Let x and y be two solutions to Problem 4.1.

1. Prove

$$J(y) - J(x) \leq \int [\nabla_x L(t, y(t), \dot{y}(t)) \cdot (y(t) - x(t)) + \nabla_v L(t, y(t), \dot{y}(t)) \cdot (\dot{y}(t) - \dot{x}(t))] \,\mathrm{d}t + \nabla g\,(y(T)) \cdot (y(T) - x(T)) + \nabla f\,(y(0)) \cdot (y(0) - x(0))$$

2. Conclude.

▶

1. By concavity:

$$J(x) - J(y) \geq \langle \nabla J(y), x - y \rangle$$

where here ∇ is the "full" gradient. We then perform an integration by parts.

2. By the Euler-Lagrange equation and the transversality condition the right hand side is equal to 0.

□

In the case the criterion to maximize is of the type

$$J(x) = \int_0^T L(t, x(t), \dot{x}(t)) \mathrm{d}t + g\,(x(T))$$

on a set of functions $[0, T] \to \mathbb{R}^n$ such that $x(0)$ is given. The solution satisfies the *Euler-Lagrange equation* and the *transversality condition*:

$$\nabla_v L(T, x(T), \dot{x}(T)) = -\nabla g\,(x(T))\ .$$

When maximising

$$J(x) = \int_0^T L(t, x(t), \dot{x}(t)) \mathrm{d}t + f\,(x(0))$$

$x(T)$ being given lead to the transversality condition:

$$\nabla_v L(0, x(0), \dot{x}(0)) = \nabla f\,(x(0))\ .$$

4.2 Dynamic programming principle

We define the value function:

$$v(t, x) := \sup \left\{ \int_t^T L(s, y(s), \dot{y}(s)) \mathrm{d}s + g\,(y(T)) \ :\ y \in \mathcal{C}^1([t, T], \mathbb{R}^d),\ y(t) = x \right\}.$$

It is clear that for all $x \in \mathbb{R}^d$, $v(T, x) = g(x)$ so that v satisfies the limit condition. Using a proof similar to the case of finite horizon we can prove:

Proposition 7 (Dynamic programming principle)**.** *The value function satisfies for all $x \in \mathbb{R}^d$ and $t \in [0, T]$*

$$v(0, x) := \sup \left\{ \int_0^t L(s, y(s), \dot{y}(s)) \mathrm{d}s + v\,(t, y(t)) \ :\ y \in \mathcal{C}^1([t, T], \mathbb{R}^d),\ y(0) = x \right\}.$$

This proposition states that «if a curve y starting in x in $t = 0$ is optimal between 0 and T then it is still optimal between t and T among the curve starting from $y(t)$ at time t.

4.3 Hamilton-Jacobi equation

In this section we will prove that v is the solution to a partial differential equation called *Hamilton-Jacobi* equation.

Proposition 8. *Assume that v is smooth. The value function v is solution to the* Hamilton-Jacobi *equation.*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t, x) + H\,(t, x, \nabla_x v(t, x)) = 0$$

where H is the Hamiltonian defined for all $p \in \mathbb{R}^d$ by:

$$H(t, x, p) := \sup_{q \in \mathbb{R}^d} \left\{ q \cdot p + L(t, x, q) \right\}.$$

The proof we will provide is formal as we will assume that some optimal trajectories and that v is regular. In general this is not the case. To obtain this result without this unrealistic regularity assumption we can work with viscosity solution, see [?].

Proof. Let $[t, t+\Delta t] \subset [t, T]$, $q \in \mathbb{R}^d$. Consider z an optimal solution of $v(t+\Delta t, x+q\,\Delta t)$. Set

$$y : s \mapsto \begin{cases} x + q(s-t) & \text{on } [t, t+\Delta t] \\ z & \text{on } [t+\Delta t, T] \end{cases}$$

Exercise 42.

1. Prove that $y(t) = x$ and that it is continuous on $[t, T]$.
2. Prove that
$$v(t,x) \geq v(t,x) + \Delta t\,[L(t,x,q) + \partial_t v(t,x) + q \cdot \nabla_x v(t,x)] + o(1)$$
3. Deduce in this case that
$$\partial_t v(t,x) + q \cdot \nabla_x v(t,x) + L(t,x,q) \leq 0$$
4. Conclude that v is a sub-solution of the Hamilton-Jacobi equation *i.e.*
$$\partial_t v(t,x) + H\left(t, x, \nabla_x v(t,x)\right) \leq 0$$

▶

1. Obvious.
2. By definition on the integral, we have
$$\begin{aligned} v(t,x) &\geq \int_t^{t+\Delta t} L(s, x+q(s-t), q)\mathrm{d}s + v(t+\Delta t, x+q\,\Delta t) \\ &\geq \Delta t L(t,x,q) + v(t,x) + \Delta t \partial_t v(t,x) + \Delta t\, q \cdot \nabla_x v(t,x) + o(\Delta t) \\ &\geq v(t,x) + \Delta t\,[L(t,x,q) + \partial_t v(t,x) + q \cdot \nabla_x v(t,x) + o(1)] \end{aligned}$$
3. The result comes by dividing by Δt and letting Δt going to 0.
4. As this is true for any q, we can take the supremum in q and obtain the desired result.

Exercise 43. Let now y be optimal for the problem $v(t,x)$.

1. Prove that y is also optimal for $v(t+\Delta t, y(t+\Delta t))$.
2. Prove that
$$0 = \Delta t\,[L(t,x,\dot{y}(t)) + \partial_t v(t,x) + \dot{y}(t) \cdot \nabla_x v(t,x) + o(1)]$$
3. Deduce that
$$\partial_t v(t,x) + \dot{y}(t) \cdot \nabla_x v(t,x) + L(t,x,\dot{y}(t)) = 0$$
4. Conclude that v is a super-solution to the Hamilton-Jacobi equation *i.e.*
$$\partial_t v(t,x) + H\left(t, x, \nabla_x v(t,x)\right) \geq 0$$

▶

1. It is an immediate consequence of the dynamic programming principle.
2. As y is also optimal for $v(t+\Delta t, y(t+\Delta t))$, we have
$$\begin{aligned} v(t,x) &= \int_t^{t+\Delta t} L(s, y(s), \dot{y}(s))\mathrm{d}s + v(t+\Delta t, y(t+\Delta t)) \\ &= v(t,x) + \Delta t\,[L(t,x,\dot{y}(t)) + \partial_t v(t,x) + \dot{y}(t) \cdot \nabla_x v(t,x)] + o(\Delta t) \end{aligned}$$
3. The result comes by dividing by Δt and letting Δt going to 0.
4. By definition of the supremum we obtain the desired result.

□

4.4 Exercises

Exercise 44. Solve the problem

$$\inf J(x)$$

where

$$J(x) := \int_0^1 \left[\dot{x}(t)^2 + t\,x(t)\right] \,\mathrm{d}t + x(1)^2 \,.$$

▶ Introduce

$$L(t,x,v) = tx + v^2 \,.$$

The Euler equation is

$$2x''(t) = t$$

which general form of the solutions is

$$t \mapsto \frac{t^3}{12} + \alpha t + \beta \,.$$

Using the transversality conditions

$$2x'(0) = 0 \qquad \text{and} \qquad 2x'(1) = -2x(1)$$

The problem is convex so the unique solution is

$$t \mapsto \frac{t^3}{12} - \frac{1}{3}$$

Exercise 45. Solve the problem

$$\inf J(x)$$

where

$$J(x) := \int_0^1 \left[\dot{x}(t)^2 + t\,x(t)\right] \mathrm{d}t + x(1)^2$$

with $x(0) = 1$.

▶ The Euler-Lagrange equation is the same as above and so is the transvesality condition in 1. But there is no transversality condition in 0 but we use $x(0) = 1$. This leads to determine the unique solution, the problem being convex again,

$$t \mapsto \frac{t^3}{12} - \frac{2}{3}t + 1$$

Exercise 46. Solve the problem

$$\inf J(x)$$

where

$$J(x) := \int_0^1 \left[\dot{x}(t)^2 + x(t)^2\right] \mathrm{d}t \,.$$

▶ The integrand is positive, the infimum is obviously

$$t \mapsto 0 \,.$$

Exercise 47. Solve the problem

$$\inf J(x)$$

where

$$J(x) := \int_0^1 \left[\dot{x}(t)^2 + x(t)^2\right] \mathrm{d}t \,.$$

where $x(0) = 0$ and $x(1) = 1$.

▶ The general form of the solutions is

$$t \mapsto \alpha \mathrm{e}^t + \beta \mathrm{e}^{-t}, \qquad (\alpha, \beta) \in \mathbb{R}^2 \,.$$

Using the boundary conditions we find

$$\alpha = \frac{\mathrm{e}}{\mathrm{e}^2 - 1} \qquad \text{and} \qquad \beta = -\frac{\mathrm{e}}{\mathrm{e}^2 - 1}$$

As the problem is convex the unique solution is

$$t \mapsto \frac{\mathrm{e}}{\mathrm{e}^2 - 1} \left(\mathrm{e}^t - \mathrm{e}^{-t}\right) \,.$$

Exercise 48. Consider the problem

$$\inf J(x)$$

where

$$J(x) := \int_0^1 \left[\dot{x}(t)^2 + x(t)^2\right] \mathrm{d}t \,.$$

where $x(0) = 1$.

1. Solve this problem.
2. What is the equation satisfied by the value function?

▶ The general form of the solutions is

$$t \mapsto \alpha \mathrm{e}^t + \beta \mathrm{e}^{-t}, \qquad (\alpha, \beta) \in \mathbb{R}^2 \,.$$

Using the boundary and transversality condition we find

$$\alpha = \frac{1}{1 + \mathrm{e}} \qquad \text{and} \qquad \beta = \frac{\mathrm{e}}{\mathrm{e} + 1}$$

As the problem is convex the unique solution is

$$t \mapsto \frac{1}{1 + \mathrm{e}^2} \mathrm{e}^t + \frac{\mathrm{e}}{1 + \mathrm{e}^2} \mathrm{e}^{-t} \,.$$

Exercise 49. We are interested in the problem

$$\inf_{\{x \,:\, x(0)=1, x(1)=0\}} \int_0^1 t \dot{x}(t)^2 \mathrm{d}t$$

1. What is the value of the integral for

$$x_N : t \mapsto \begin{cases} 1 & \text{on } \left[0, \frac{1}{N}\right] \\ -\frac{\log(t)}{\log(N)} & \text{on } \left(\frac{1}{N}, 1\right] \end{cases}$$

2. What is the infimum?

3. Is the infimum attained?

Exercise 50. Using a clever change of function, solve

$$\inf \left\{ \int_0^1 \left[\frac{x'(t)^2}{x(t)^2} - \log\left(x(t)^2\right) \right] \mathrm{d}t \right\}$$

on the set such that $x(0) = 1$ and $x(1) = \mathrm{e}$.

▶ We perform the change of variable $y(t) = \log(x(t))$. For y the problem is

$$\inf \left\{ \int_0^1 \left[y'(t)^2 - 2y(t) \right] \mathrm{d}t \right\}$$

on the set such that $y(0) = 0$ and $y(1) = 1$. The Euler-Lagrange equation is

$$y''(t) = -1 \ .$$

As a consequence the candidat to be a solution is

$$t \mapsto -\frac{t^2}{2} + \frac{3}{2}t \ .$$

Coming back to x we obtain

$$t \mapsto \mathrm{e}^{-t^2/2+3t/2}$$

The value is $-1/12$.

Exercise 51. Let $f : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^2$ function which is strictly convex.

$$\inf \int_0^1 f(\dot{x}(t))\mathrm{d}t$$

such that $x(0) = x_0$ and $x(1) = 1$.

▶ The Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} f'(x(t)) = 0 \ .$$

Hence, there exists C such that

$$f'(x(t)) = 0 \ .$$

As f is strictly convex, f' is a one-to-one correspondance and there exists v such that we have

$$\dot{x}(t) = v \ .$$

As a consequence the solution is of the forme $t \mapsto vt + \alpha$ where $(v, \alpha) \in \mathbb{R}^2$. Using the initial and final conditions we find that, as it is a convex problem, the unique solution is the straight line

$$t \mapsto (1 - x_0)t + x_0.$$

Exercise 52. We are interested in the optimal way to eat a cake between the dates 0 and T. Initially the cake is of size 1. The objective of the consumer is to consume the cake in the most satisfactory way. The satisfaction being measured by

$$V(c) := \int_0^T U(c(t))\mathrm{e}^{-\delta t}\mathrm{d}t \ ,$$

where $c(t)$ designs the instantaneous consumption at time t. The map U is a $\mathcal{C}^1$ map which is strictly concave and non-decreasing.

1. Give a relation between the size of the cake and the consumption.
2. Write the problem as a calculus of variation problem.
3. Prove that V is strictcy concave.
4. What is the relation that should satisfy the optimal solution?
5. Solve the problem in the case $U(c) = \log(c)$.

▶

1. Let $x(t)$ be the size of the cake at time t. We have
$$x(t) = 1 - \int_0^t c(s)\mathrm{d}s \;.$$
2. The problem is then
$$\sup_x \int_0^T U(\dot{x}(t))\mathrm{e}^{-\delta t}\mathrm{d}t \;,$$
with $x(0) = 1$ and $x(T) = 0$.
3. As U is strictly concave so is V.
4. We set the Lagrangian $L(t, x, v) = U(v)\mathrm{e}^{-\delta t}$. The Euler-Lagrange equation is
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[U'\left(\dot{x}(t)\right)\mathrm{e}^{-\delta t}\right] = 0 \;.$$
As a consequence there exists a constant K such that
$$U'\left(\dot{x}(t)\right)\mathrm{e}^{-\delta t} = K \;.$$
5. In this case the Euler-Lagrange equation becomes:
$$\frac{1}{\dot{x}(t)} = K\mathrm{e}^{\delta t} \;.$$
Using the initial conditions we find that the unique solution (remind that the problem is concave) is
$$x \mapsto 1 - \frac{1 - \mathrm{e}^{-\delta t}}{1 - \mathrm{e}^{-\delta T}} \;.$$
Coming back to c we obtain the optimal consumption:
$$t \mapsto \frac{\delta \mathrm{e}^{-\delta t}}{1 - \mathrm{e}^{-\delta T}} \;.$$

Exercise 53 (A saving-consumption economics model)**.** Let $T > 0$ and $\delta > 0$. We consider a household whose instantaneous wealth, wage, consumption and saving are denoted respectively $x(t)$, $w(t)$, $c(t)$ and $s(t)$. The wage $w(t)$ is assumed to be a constant w exogeneously given. Let V be a given continuous map. The household tries to maximise its utility given by
$$\int_0^T \mathrm{e}^{-\delta t}\log\left(c(t)\right)\mathrm{d}t + \mathrm{e}^{-\delta T}V\left(x(T)\right) \;.$$
We have the relations
$$w(t) = c(t) + s(t) \qquad \text{and} \qquad \dot{x}(t) = s(t) + rx(t) \;.$$
where r is a exogeneous and constant interest rate.

1. The initial wealth $x(0)$ being given and equal to x_0. Write the variational problem in terms of x.
2. Characterise the solution in terms of c.
3. Characterise the solution in terms of x.
4. Conclude.

▶

1. The variational problem is
$$\max_{x\,:\,x(0)=x_0} J(x)$$
where
$$J(x) = \int_0^T \mathrm{e}^{-\delta t} \log\left(w(t) + rx(t) - \dot{x}(t)\right) \mathrm{d}t + \mathrm{e}^{-\delta T} V\left(x(T)\right)$$
2. It is a concave problem. Setting
$$L(t,x,v) := \mathrm{e}^{-\delta t} \log\left[w(t) + rx - v\right] \ .$$
The Euler-Lagrange equation is
$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{e}^{-\delta t}}{c(t)} = \frac{r\mathrm{e}^{-\delta t}}{c(t)} \ .$$
Setting
$$y(t) := \frac{\mathrm{e}^{-\delta t}}{c(t)}$$
we obtain
$$y(t) = \frac{\mathrm{e}^{-rt}}{c(0)}$$
and hence
$$c(t) = \mathrm{e}^{(r-\delta)t} c(0) \ .$$
3. Let us come back to the state variable x. As
$$\dot{x}(t) - rx(t) = w(t) - c(t)$$
we have to solve
$$\dot{x}(t) - rx(t) = w(t) - \mathrm{e}^{(r-\delta)t} c(0)$$
A solution to the homogeneous equation is given by
$$t \mapsto \lambda \mathrm{e}^{rt} \ .$$
Using varying the constant method we consider a solution of the form
$$t \mapsto \lambda(t)\mathrm{e}^{rt} \ .$$
Plugging into the equation we obtain
$$\dot{\lambda}(t) = w(t)\mathrm{e}^{-rt} + \mathrm{e}^{-\delta t} c(0)$$
and hence
$$\lambda(t) = \int_0^t \left[w(s)\mathrm{e}^{-rs} + \mathrm{e}^{-\delta s} c(0)\right]$$
As w is assumed to be constant we obtain
$$\lambda(t) = w\frac{1-\mathrm{e}^{-rt}}{r} + \frac{1-\mathrm{e}^{-\delta t}}{\delta} c(0) \ .$$

Finally we obtain that the solution is of the form

$$t \mapsto \left[w\frac{1-\mathrm{e}^{-rt}}{r} + \frac{1-\mathrm{e}^{-\delta t}}{\delta}c(0)\right]\mathrm{e}^{rt}\ .$$

Using the transversality condition

$$V'\left(x(T)\right) = \frac{1}{c(T)}$$

where we compute from the expression of x

$$c(T) = 2w + \mathrm{e}^{(r-\delta)T}c(0)\ .$$

We determine $c(0)$ as the unique solution to

$$V'\left\{\left[w\frac{1-\mathrm{e}^{-rT}}{r} + \frac{1-\mathrm{e}^{-\delta T}}{\delta}c(0)\right]\mathrm{e}^{rT}\right\} = \frac{1}{2w + \mathrm{e}^{(r-\delta)T}c(0)}\ .$$

Exercise 54. Let $T > 0$ and consider the variational problem

$$\sup_{x\in\mathcal{C}^1(0,T)\,;\,x(0)=x_0}\left\{\int_0^T -\frac{1}{2}\dot{x}(t)^2\mathrm{d}t - \frac{1}{2}x(T)^2\right\}\ .$$

1. Derive the equations satisfied by an optimal trajectory.
2. Compute explicitly the Hamiltonian and write the corresponding Hamilton–Jacobi equation.
3. Assuming that a solution is in separable form, *i.e.* can be written $v(t,x) = f(t)g(x)$, where f and g are scalar functions to determine, solve the Hamilton–Jacobi equation.

▶

1. The Lagrangian is

$$L(t,x,v) = -\frac{1}{2}v^2.$$

The Euler–Lagrange equation is

$$\ddot{x}(t) = 0.$$

The transversality condition is

$$\dot{x}(T) = x(T).$$

2. We have

$$v(t,x) = \inf_x\left\{-\int_t^T \frac{1}{2}\dot{x}(s)^2\mathrm{d}s - \frac{1}{2}x(T)^2\right\},$$

where $x(t) = x$.

The Hamilton–Jacobi equation satisfied by v is

$$\partial_t v(t,x) + H\big(x, \partial_x v(t,x)\big) = 0$$

with terminal condition

$$v(T,x) = \frac{1}{2}x^2.$$

where the Hamiltonian is defined by:

$$H(x,p) = \sup_{q\in\mathbb{R}}\left\{pq + L(x,q)\right\}.$$

3. Since
$$L(x,q) = -\frac{1}{2}q^2,$$
the first-order condition gives $q = p$.
Substituting,
$$H(x,p) = \frac{1}{2}p^2.$$
Therefore the Hamilton–Jacobi equation becomes
$$\partial_t v + \frac{1}{2}(\partial_x v)^2 = 0$$
with
$$v(T,x) = \frac{1}{2}x^2.$$

4. We compute
$$\partial_x v(t,x) = P(t)x, \quad \text{and} \quad \partial_t v(t,x) = \frac{1}{2}P'(t)x^2.$$
Substituting into the Hamilton–Jacobi equation gives
$$\frac{1}{2}P'(t)x^2 + \frac{1}{2}P(t)^2x^2 = 0.$$
Since this identity must hold for all x, we obtain the ordinary differential equation
$$P'(t) + P(t)^2 = 0.$$
Which solutions are of the form
$$\frac{1}{P(t)} = t - C.$$
From
$$v(T,x) = \frac{1}{2}x^2,$$
we deduce
$$P(T) = 1.$$
So that
$$C = T - 1.$$
Therefore,
$$P(t) = \frac{1}{1+T-t}.$$
The solution of the Hamilton–Jacobi equation is
$$v(t,x) = \frac{x^2}{2(1+T-t)}.$$

Exercise 55. Let $T > 0$. We want to maximise
$$J[x(\cdot)] = -\int_0^T \left(\frac{1}{2}\dot{x}(t)^2 + \frac{1}{2}\omega^2 x(t)^2\right) \mathrm{d}t - \frac{1}{2}x(T)^2,$$
where $\omega > 0$ is a constant and $x(0) = x_0$.

1. Derive the Euler–Lagrange equation satisfied by an optimal trajectory.
2. Define the Hamiltonian associated with the Lagrangian.

3. Write the value function.

4. Write the Hamilton–Jacobi equation satisfied by the value function.

5. Assuming that a solution is in separable form, *i.e.* can be written $v(t,x) = f(t)g(x)$, where f and g are scalar functions to determine, solve the Hamilton–Jacobi equation.

6. Determine the solutions.

▶

1. The Lagrangian is
$$L(x,v) = -\frac{1}{2}v^2 - \frac{1}{2}\omega^2 x^2.$$
The Euler–Lagrange equation is
$$-\ddot{x}(t) - \omega^2 x(t) = 0.$$

2. The momentum is
$$p = -\frac{\partial L}{\partial q} = q.$$
The Hamiltonian is obtained by:
$$H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2.$$

3. The value function is
$$v(t,x) = \inf_x \left\{ -\int_t^T \left(\frac{1}{2}\dot{x}(s)^2 + \frac{1}{2}\omega^2 x(s)^2 \right) \mathrm{d}s - \frac{1}{2}x(T)^2 \right\}.$$

4. The value function satisfies
$$\partial_t v(t,x) + \frac{1}{2}(\partial_x v(t,x))^2 - \frac{1}{2}\omega^2 x^2 = 0$$
with terminal condition
$$v(T,x) = \frac{1}{2}x^2.$$

5. From the terminal condition we identify $g(x) = x^2/2$. We have
$$\partial_x v = P(t)x, \qquad \partial_t v = \frac{1}{2}P'(t)x^2.$$
Substituting into the Hamilton–Jacobi equation:
$$\frac{1}{2}P'(t)x^2 + \frac{1}{2}P(t)^2 x^2 - \frac{1}{2}\omega^2 x^2 = 0.$$
Since this holds for all x, we obtain
$$P'(t) + P(t)^2 - \omega^2 = 0,$$
with terminal condition
$$P(T) = 1.$$

6. We have
$$\frac{1}{2\omega}\ln\left|\frac{\omega+P(t)}{\omega-P(t)}\right| = t + C.$$
Exponentiating:
$$\frac{\omega+P(t)}{\omega-P(t)} = Ke^{2\omega t},$$
where $K = e^{2\omega C}$. Or equivalently
$$P(t) = \omega\frac{Ke^{2\omega t}-1}{Ke^{2\omega t}+1}.$$
Using the terminal condition $P(T) = 1$ we obtain
$$Ke^{2\omega T} = \frac{\omega+1}{\omega-1}.$$
Hence
$$P(t) = \omega\frac{\frac{\omega+1}{\omega-1}e^{2\omega(t-T)}-1}{\frac{\omega+1}{\omega-1}e^{2\omega(t-T)}+1}.$$
The solution of the Hamilton–Jacobi equation is therefore
$$V(t,x) = \frac{1}{2}P(t)x^2,$$
where $P(t)$ is given above.

Chapter 5

Introduction to optimal control

This section is a short introduction to optimal control. Control theory is concerned with dynamic systems that depend on a parameter (called control or command) that can be acted upon to, for example, move the position of the system from one point to another. In optimal control, the aim is to act on the control of the dynamic system in such a way as to optimise a given criterion. Dynamic systems can be of different types (discrete or continuous time, with or without noise, etc.) and have different origins (mechanical, electrical, chemical, economic, etc.). For example, in mathematical finance, the evolution of a portfolio is frequently modelled as a stochastic dynamic system that is influenced (in discrete or continuous time) by the sale or purchase of financial assets. Another example of an application in economics is the theory of rational expectations, which makes extensive use of optimal control (see the monograph by Lucas and Stokey, which contains a large number of economic examples and also deals with Markov programming, which is outside the scope of this course).

We will see that it is also possible to write necessary conditions for optimality in the context of optimal control: the Pontryagin's maximum principle.

5.1 Introduction

Let V be a metric space, $T > 0$, and $x \in \mathbb{R}^d$. Consider $u : [0, T] \to V$ a continuous map and $f : [0, T] \times \mathbb{R}^d \times V \to \mathbb{R}^d$ a continuous map such that

$$|f(t, x, u) - f(t, y, u)| \leq L|x - y| \qquad \text{and} \qquad |f(t, x, u)| \leq C(1 + |x|) \ .$$

Consider the Cauchy problem

$$\begin{cases} \dot{y}(t) = f\left(t, y(t), u(t)\right) \\ y(0) = x \end{cases} \tag{5.1}$$

This differential equation is *controlled.* The variable u, called the *control variable*, influences the dynamics of the state variable y.

Under the above assumption the Cauchy-Lipschitz theorem applies: for any $x \in \mathbb{R}^d$ and $u \in \mathcal{C}^0([0, T], V)$ there exists a unique solution which we denote $y_{x,u}$ or y_u when no confusion is possible.

A Lagrangian $L \in \mathcal{C}^0([0, T] \times \mathbb{R}^d \times V, \mathbb{R}^d)$ and a terminal gain function $g \in \mathcal{C}^0(\mathbb{R}^d, \mathbb{R})$ being given, we are interested in the following *optimal control* problem:

$$\sup_u J(u) \tag{5.2}$$

where

$$J(u) := \int_0^T L\left(t, y(t), u(t)\right) \mathrm{d}t + g\left(y(T)\right) \ .$$

Note that J also depends on x and y as the state variable y is linked to u by the dynamics (5.1) with x given.

Remark 4. In the case the dynamics is given by $\dot{y} = u$, this problem is a problem of calculus of variation.

The adapted functional framework with the concept of viscosity solutions is out of the scope of this lecture. We will rather assume here that the control u is piecewise continuous.

5.2 Pontriaguine principle

Let us add a new variable $p \in \mathbb{R}^d$, called the *adjoint variable* and can be interpreted as a multiliper associated to (5.1). We define the pre-Hamiltonian (or present value Hamiltonian):

$$\begin{array}{rrcl} \tilde{H}: & [0,T] \times \mathbb{R}^d \times V \times \mathbb{R}^d & \to & \mathbb{R} \\ & (t,y,u,p) & \mapsto & L(t,y,u) + p \cdot f(t,y,u) \end{array}$$

and the Hamiltonian

$$\begin{array}{rrcl} H: & [0,T] \times \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{R} \\ & (t,y,p) & \mapsto & \sup_{u\in V} \left\{ \tilde{H}(t,y,u,p) \right\} . \end{array}$$

In this lecture, we will assume that H is continuous, differentiable with respect to x and p. We shall not give a proof of Pontriaguine principle, it would take at least 15 pages. However, let us notice that similarly to KKT's theorem, Pontriaguine principle requires a qualification condition. The interested reader could refer to the book of the Soviet school: [**?**].

Theorem 6. *Under suitable regularity and qualification assumptions; if u is an optimal control which is piecewise continuous solution of* (5.2) *and y is the associated trajectory, then there exists $p \in \mathcal{C}^0([0,t], \mathbb{R}^d) \cap \mathcal{C}^1_{\mathrm{p}}([0,t], \mathbb{R}^d)$ such that*

$$H(t, y(t), p(t)) = \sup_{u \in V} \left\{ \tilde{H}(t, y(t), u(t), p(t)) \right\} .$$

Moreover, (y,p) is a solution to the Hamiltonian system:

$$\begin{cases} \dot{p}(t) & = -\nabla_x H(t, y(t), p(t)) \quad \textit{(co-state equation)} \\ \dot{y}(t) & = \nabla_p H(t, y(t), p(t)) \quad \textit{(state equation)} \end{cases}$$

with the limit conditions

$$y(0) = x \quad \textit{and, the transversality condition,} \quad p(T) = \nabla g(y(T)) .$$

If «everything goes well» (*i.e.* if a global existence-uniqueness result in Cauchy-Lipschitz type applies), the Hamiltonian system, which is a first-order system admits a unique solution satisfying the boundary conditions given by the theorem. If we're «lucky», we can also calculate this solution (y,p). A necessary condition on the command u is then provided by the maximisation condition. However, we must be aware that Pontriaguine's principle only provides necessary conditions for optimality. In other words, even in the «nice» cases where the maximum principle makes it possible to determine a control u, the corresponding state y and the adjoint variable p, there is no guarantee that u is actually an optimal control. We shall see below, using ideas from dynamic programming and Bellman's approach, that we can obtain sufficient conditions for optimality. These are often used in practice to check whether a solution provided by the by the maximum principle is indeed an optimal control.

5.3 Dynamic programming principle

We define the value function of the control problem:

$$v(t,x) := \sup_{u\,:\,y(t)=x} \left\{ \int_t^T L(s,y(s),u(s))\mathrm{d}s + g(y(T)) \right\} .$$

It is obvious that v satisfies the limit condition:

$$\forall x \in \mathbb{R}^d, \qquad v(T,x) = g(x)$$

The dynamic programming principle states that «if a control u is optimal between 0 and T for the initial condition x then u is also optimal between t and T with the intial condition $y(t)$ at the date t». Mathematically we state it:

Proposition 9. *For all $x \in \mathbb{R}^d$ and $t \in [0,T]$,*

$$v(0,x) = \sup_{u\,:\,y(0)=x} \left\{ \int_0^t L(s,y(s),u(s))\mathrm{d}s + v(t,y(t)) \right\}$$

5.4 Hamilton-Jacobi-Bellman equation

Following a proof very similar to the previous chapter we can prove:

Proposition 10. *Assume that v is regular. The map v is solution to the Hamilton-Jacobi-Bellman (HJB) equation:*

$$\frac{\partial}{\partial t} v(t,x) + H(t,x,\nabla_x v(t,x)) = 0 \ .$$

and

$$v(T,x) = g(x) \ .$$

5.5 Pontryagin vs Hamilton–Jacobi–Bellman: A Unification

5.5.1 Verification Theorem

Theorem 7 (Verification theorem)**.** *Let v satisfy the Hamilton–Jacobi–Bellman equation and u^* be a control which attains the maximum in the Hamiltonian. Assume that v is of class C^1. Then u^* is optimal for Problem* (5.2) *and*

$$v(t,x) = J(u^*(t)).$$

Remember that J depends on x, through the dynamics of y, eventhough it is not made explicit in the notation.

Proof.

Exercise 56. For any admissible control u:

1. Prove
$$\frac{d}{ds} v(s,y(s)) \leq -L(s,y(s),u(s)).$$
2. Prove
$$v(t,x) \geq \int_t^T L(s,y(s),u(s))\mathrm{d}s + g(x(T)).$$
3. Conclude.

▶

1. By differentiating, for any u,

$$\frac{d}{ds}v(s,y(s)) = \partial_t v(s,y(s)) + \dot{y}(s)\cdot\nabla_x v(s,y(s)) = \partial_t v(s,y(s)) + \nabla_x v(s,y(s))\cdot f(s,y(s),u(s)).$$

As the Hamilton–Jacobi–Bellman equation is

$$\partial_t v + \sup_u\{L(s,y(s),u(s)) + \nabla_x v\cdot f(s,y(s),u(s))\} = 0\ ,$$

we obtain

$$\frac{d}{ds}v(s,y(s)) \leq -L(s,y(s),u(s))$$

2. Integrating from t to T:

$$v(T,y(T)) - v(t,y(t)) \leq -\int_t^T L(s,y(s),u(s))\mathrm{d}s.$$

Since $v(T,x) = g(x)$, we obtain

$$v(t,x) \geq \int_t^T L(s,y(s),u(s))\mathrm{d}s + g(x(T)).$$

3. In the above computations equality holds for u^*.

□

5.5.2 Relation between p and v

Theorem 8 (Equivalence: $p(t) = \nabla_x v(t,y^*(t))$)**.** *Let $v \in C^2$ be a solution of the Hamilton–Jacobi–Bellman (HJB) equation, u^* an optimal control that attains the maximum in the Hamiltonian, and $y^* : [0,T] \to \mathbb{R}^n$ the associated optimal state trajectory.*

Define the costate along the trajectory by

$$p(t) := \nabla_x v(t,y^*(t)).$$

Then p satisfies the adjoint equation of Pontryagin's Maximum Principle:

$$\dot{p}(t) = -\nabla_x H(t,y^*(t),p(t)), \qquad p(T) = \nabla_x g(y^*(T)).$$

Proof.

Exercise 57. 1. Prove

$$\frac{\mathrm{d}}{\mathrm{d}t}v_x(t,y^*(t)) = v_{xt}(t,y^*(t)) + v_{xx}(t,y^*(t))\cdot\dot{y}^*(t)$$

2. Prove

$$v_{xt}(t,y^*(t)) + v_{xx}(t,y^*(t))\cdot\dot{y}^*(t) = -\nabla_x H(t,y^*(t),p(t)).$$

3. Deduce that

$$\frac{d}{dt}v_x(t,y^*(t)) = -\nabla_x H(t,y^*(t),v_x(t,y^*(t))).$$

4. Conclude.

▶

1. By the chain rule along $y^*(t)$,

$$\frac{d}{dt}\nabla_x v(t,y^*(t)) = \partial_t\nabla_x v(t,y^*(t)) + D^2_{xx}v(t,y^*(t))\dot{y}^*(t),$$

where $D^2 v_{xx}$ is the Hessian with respect to x.

2. The HJB equation reads
$$\partial_t v(t,x) + H(t,x,v_x(t,x)) = 0.$$
Differentiating HJB with respect to x gives
$$\partial_t \nabla_x v(t,x) + \nabla_x H(t,x,v_x(t,x)) + \nabla_p H(t,x,v_x(t,x)) D^2_{xx} v(t,x) = 0.$$
Applying to $x = y^*(t)$, since we have by the state equation $\nabla_p H(t, y^*(t), p) = \dot{y}^*(t)$ we obtain
$$\partial_t \nabla_x v(t, y^*(t)) + D^2_{xx} v(t, y^*(t)) \cdot \dot{y}^*(t) = -\nabla_x H(t, y^*(t), v_x(t, y^*(t))).$$

3. Combining with the first question, we obtain
$$\frac{d}{dt} \nabla_x v(t, y^*(t)) = -\nabla_x H(t, y^*(t), \nabla_x(t, y^*(t))).$$

4. By Pontryagin's Maximum Principle, the costate p satisfies
$$\dot{p}(t) = -\nabla_x H(t, y^*(t), p(t)), \qquad p(T) = \nabla_x g(y^*(T)).$$
We have the same ODE for p and $\nabla_x v(\cdot, y^*)$, with identical terminal conditions. By uniqueness of the solution to the ODE,
$$p(t) = v_x(t, y^*(t)), \quad \forall t \in [0,T].$$

□

5.6 Application to Ramsey's optimal growth model

Exercise 58. We are interested here in Ramsey's optimal growth model for a single production sector with a finite horizon $T > 0$. We will denote $c(t)$ the instantaneous consumption at time t of a representative household whose satisfaction is assumed to be measured by the quantity
$$\int_0^T U\left(c(t)\right) \mathrm{e}^{-\delta t} \mathrm{d}t + \gamma \ln(k(T)) \,,$$
The instantaneous consumption $c(t)$ has to be non-negative for all time t. The rate $\delta > 0$ is given and U is strictly concave, non-decreasing, and differentiable. We will denote $y(t)$, $k(t)$ and $i(t)$ the production, capital and investment in the economy at time t. We have $y(t) = f(k(t))$ where the production function is supposed to be strictly concave, non-decreasing and differentiable. We also have
$$y(t) = c(t) + i(t)$$
and
$$i(t) = \dot{k}(t)$$
and $k(0) = k_0$ is given.

1. Let us solve the problem using the Pontriaguine principle:
   (a) Write the problem in the form of an optimal control problem.
   (b) Write the necessary condition given by the Pontriaguine principle.
   (c) When $U = \ln$ and $f(k) = Ak^\alpha$, $\alpha \in (0,1)$. Deduce from the previous question the equations satisfied by k and c.
   (d) Solve it explicitly when $\alpha = 1$.
2. Still in the case $U = \ln$ and $f(k) = Ak$. Let us now solve this problem using the Hamilton-Jacobi-Bellman equation.

(a) Write the Hamilton-Jacobi-Bellman equation.

(b) Solve it in the case $U = \ln$ and $f(k) = Ak$. Hint: try

$$v(k,t) = \alpha(t) + \beta(t)\ln k.$$

(c) Conclude.

▶

1. (a) The state variable is k (capital), and the control variable is c (consumption). The planner chooses a consumption path c in order to

$$\max_{c(\cdot)} \int_0^T U(c(t))e^{-\delta t}\mathrm{d}t + g(k(T))$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t),$$

where $k(0) = k_0$ and with $c(t) \geq 0$.

(b) Define the present-value Hamiltonian:

$$\tilde{H}(k,c,\lambda,t) = U(c)e^{-\delta t} + \lambda(f(k) - c).$$

- The first-order condition with respect to c gives:

$$\frac{\partial \tilde{H}}{\partial c} = U'(c)e^{-\delta t} - \lambda = 0, \quad \text{or} \quad \lambda(t) = U'(c(t))e^{-\delta t}.$$

- State equation:

$$\dot{k}(t) = f(k(t)) - c(t).$$

- Costate equation:

$$\dot{\lambda}(t) = -\frac{\partial \tilde{H}}{\partial k} = -\lambda(t)f'(k(t)).$$

- Transversality condition:

$$\lambda(T) = g'(k(T)) = \frac{\gamma}{k(T)}.$$

- Initial condition:

$$k(0) = k_0.$$

(c) Differentiating in time $\lambda(t) = U'(c(t))e^{-\delta t}$ gives

$$\dot{\lambda}(t) = e^{-\delta t}\big(U''(c(t))\dot{c}(t) - \delta U'(c(t))\big).$$

Using the costate equation we obtain:

$$U''(c(t))\dot{c}(t) = (\delta - f'(k(t)))U'(c(t)).$$

In the case $U = \ln$ and $f(k) = Ak^\alpha$ it simplifies to

$$\begin{cases} \dot{c} = (\alpha A k^{\alpha-1} - \delta)c, \\ \dot{k} = Ak^\alpha - c \\ \dot{\lambda} = -\lambda \alpha A k^{\alpha - 1} \end{cases}$$

(d) For $\alpha = 1$, the dynamic system becomes:

$$\begin{cases} \dot{k} = Ak - c, \\ \dot{c} = (A - \delta)c \\ \dot{\lambda} = -A\lambda \end{cases}$$

So that

$$\lambda(t) = \frac{\gamma}{k(T)} e^{-A(T-t)}$$

Solving the ODE, we find for c:

$$c(t) = \alpha e^{(A-\delta)t}$$

where α has to be determined.

The equation for k is a first order linear ODE with constant coefficients so that:

$$k(t) = e^{At}\left[k_0 - \frac{k(T)e^{AT}}{\gamma(2A+\delta)}\left(1 - e^{-(2A+\delta)t}\right)\right].$$

We can determine $k(T)$ as being

$$k(T) = \frac{k_0 e^{AT}}{1 + \frac{e^{2AT}}{\gamma(2A+\delta)}(1 - e^{-(2A+\delta)t})}$$

2. (a) The value function is

$$v(k,t) = \max_{c(\cdot)} \int_t^T U(c(s))e^{-\delta s}\, ds + g(k(T)).$$

The HJB equation reads:

$$\partial_t v(k,t) + \max_{c \geq 0}\{e^{-\delta t} \ln c + v_k(k,t)(Ak - c)\} = 0,$$

with terminal condition

$$v(k,T) = g(k) = \gamma \ln k.$$

(b) The first-order condition is:

$$U'(c)e^{-\delta t} = v_k(k,t).$$

And gives

$$c^*(k,t) = \frac{e^{-\delta t}}{v_k(k,t)}$$

With the ansatz: $v(k,t) = \alpha(t) + \beta(t) \ln k$, we obtain

$$v_k(k,t) = \frac{\beta(t)}{k}, \quad v_t(k,t) = \alpha'(t) + \beta'(t) \ln k.$$

So that

$$c^*(k,t) = \frac{ke^{-\delta t}}{\beta(t)}$$

Using this control in the state equation we obtain

$$\dot{k}(t) = Ak(t) - \frac{k(t)e^{-\delta t}}{\beta(t)} = k(t)\left(A - \frac{e^{-\delta t}}{\beta(t)}\right)$$

To determine β we substitute into the HJB equation to obtain:

$$\alpha'(t) + \beta'(t) \ln k + e^{-\delta t} \ln \frac{ke^{-\delta t}}{\beta(t)} + \frac{\beta(t)}{k}\left(Ak - \frac{ke^{-\delta t}}{\beta(t)}\right) = 0$$

So that

$$\begin{cases} \beta'(t) + e^{-\delta t} = 0, \\ \alpha'(t) + e^{-\delta t}(-\delta t - \ln \beta(t)) + A\beta(t) - e^{-\delta t} = 0, \end{cases}$$

with terminal conditions:

$$\beta(T) = \gamma, \quad \alpha(T) = 0.$$

As a consequence $\beta(t)$ is:

$$\beta(t) = \gamma + \int_t^T e^{-\delta s} ds = \gamma + \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \quad (\delta \neq 0).$$

Coming back to k we obtain

$$k(t) = k_0 \exp \left[At - \int_0^t \delta e^{-\delta s} \delta\gamma + e^{-\delta s} - e'^{\delta T} \right]$$

Hence the optimal consumption is given by:

$$c^*(k,t) = \frac{k(t)e^{-\delta t}}{\beta(t)}.$$

5.7 Exercises

Exercise 59. For $x > 0$ and $T > 0$, we consider the real optimal control problem

$$\sup_u \{-x(T)\}$$

where x is a solution to

$$\dot{x}(t) = -u(t)\,x(t) - \frac{1}{2}u(t)^2$$

with $x(0) = x_0$.

1. (a) Compute the Hamitonian.
   (b) Write the necessary conditions given by the Pontriaguine principle.
   (c) Find the solutions x^*, u^* and p^* of the Hamiltonian system.
2. (a) Give that HJB equation associated to this problem.
   (b) Unsing the solution x^* of the previous, solve the HJB equation associated to the initial condition x at the initial time t.

▶

1. (a) Since the cost is purely terminal, the present-value Hamiltonian is

$$\tilde{H}(x,u,p) = p\left(-ux - \frac{1}{2}u^2\right).$$

The fisrt order condition gives

$$u = -x\ .$$

Hence the Hamiltonian is

$$H(x,p) = p\frac{x^2}{2}\ .$$

   (b) The Hamiltonian system is
   - State equation:

$$\dot{x} = -ux - \frac{1}{2}u^2.$$

- Costate equation:
$$\dot{p} = -\frac{\partial H}{\partial x} = pu.$$
- Transversality condition (since $g(x(T)) = -x(T)$):
$$p(T) = -1.$$
- Initial condition
$$x(0) = x_0$$

(c)
- We can start by computing x^*: as $u = -x$, the state equation becomes an automomeous equation in x:
$$\dot{x} = \frac{x^2}{2}.$$
Therefore using the intial condition $x(0) = x_0$ we obtain
$$x^*(t) = \frac{2x_0}{2 - xt}.$$
- Let us compute u^*: As $u^* = -x$ we have
$$u^*(t) = -\frac{2x_0}{2 - x_0 t}.$$
- Let us compute p^*: Since $u^* = -x^*$, the costate equation becomes
$$\dot{p} = px.$$
So that
$$p(t) = \exp\left(- \int_t^T x^*(s) \mathrm{d}s \right).$$
We compute
$$\int x^*(s) \mathrm{d}s = \int \frac{2x_0}{2 - x_0 s} \mathrm{d}s = -2\ln(2 - x_0 s).$$
Hence
$$p^*(t) = \left(\frac{2' x_0 T}{2 - x_0 t} \right)^2.$$

2. (a) Consider the value function
$$v(t, x) = \inf_u x(T).$$
The HJB equation is
$$v_t + \min_u \left\{ v_x \left(ux + \frac{1}{2}u^2 \right) \right\} = 0,$$
with terminal condition
$$v(T, x) = x.$$
The minimization gives
$$\frac{\partial}{\partial u} \left[v_x (ux + \frac{1}{2}u^2) \right] = v_x(x + u) = 0.$$
Thus
$$u^* = -x.$$
Substituting back we obtain the following HJB equation
$$v_t - \frac{1}{2}x^2 v_x = 0, \qquad v(T, x) = x.$$

(b) Using Pontriaguine's principle, the optimal trajectory starting from x in t is

$$x^*(s) = \frac{x}{1 + \frac{x}{2}(s-t)}.$$

As here $v(t,x) = x^*(T)$ we obtain

$$v(t,x) = \frac{x}{1 + \frac{x}{2}(T-t)}.$$

It is indeed a solution to the HJB equation. We finally obtain the solution:

$$v(0,x) = J(u^*) = \frac{x}{1 + \frac{x}{2}T},$$

and the control is a minimiser of the present value Hamiltonian:

$$u^*(t) = -x^*(t).$$

Solving the state equation with this optimal control u^* we would recover

$$x^*(t) = \frac{x}{1 + \frac{x}{2}t}.$$

Exercise 60. Consider

$$\max_{u(\cdot)} \int_0^T \left(-\frac{1}{2}u(t)^2\right) \mathrm{d}t - \frac{1}{2}x(T)^2$$

subject to

$$\dot{x}(t) = u(t), \qquad x(0) = x_0.$$

1. Write the Hamiltonian.
2. Derive the Pontryagin necessary conditions.
3. Solve the Hamiltonian system.
4. Write the HJB equation.
5. Consider the ansatz $v(t,x) = -a(t)x^2/2$, solve the HJB.

▶

1. The present-value Hamiltonian is

$$\tilde{H}(x,u,p) = -\frac{1}{2}u^2 + pu.$$

The first order condition is $u + p = 0$ so that the Hamiltonian is

$$H(x,p) = \frac{1}{2}p^2\ .$$

2. The Hamiltonian system gives:
The State equation:

$$\dot{x} = u.$$

The costate equation:

$$\dot{p} = -\frac{\partial H}{\partial x} = 0.$$

3. By the costate equation p is constant. The terminal condition gives

$$p(t) = p = p(T) = g'(x(T)) = -x(T).$$

As the optimality condition gave: $u^* = p$. The control is constant and

$$u^*(t) = -x(T)\ .$$

By the state equation,

$$\dot{x} = p.$$

Hence

$$x(T) = x_0 + pT.$$

But $p = -x(T)$ thus

$$p = -(x_0 + pT).$$

Solving:

$$p(1 + T) = -x_0.$$

gives

$$p = -\frac{x_0}{1 + T}.$$

Hence

$$u^*(t) = -\frac{x_0}{1 + T}.$$

We have

$$x^*(t) = x_0 - \frac{x_0}{1 + T}t.$$

4. The HJB is:

$$v_t + \max_u \left\{ -\frac{1}{2}u^2 + v_x u \right\} = 0,$$

with

$$v(T, x) = -\frac{1}{2}x^2.$$

The first-order condition being:

$$-u + v_x = 0 \quad \Rightarrow \quad u^* = v_x,$$

we obtain the the HJB equation is

$$v_t + \frac{1}{2}v_x^2 = 0.$$

5. With the ansazt we have

$$v_x = -a(t)x, \quad \text{and} \quad v_t = -\frac{1}{2}a'(t)x^2.$$

Plugging into the HJB equation gives:

$$-\frac{1}{2}a'(t)x^2 + \frac{1}{2}a(t)^2x^2 = 0.$$

Thus

$$a'(t) = a(t)^2.$$

As the terminal condition is

$$a(T) = 1.$$

We obtain

$$a(t) = \frac{1}{1 + T - t}.$$

Hence

$$v(t, x) = -\frac{1}{2}\frac{x^2}{1 + T - t}.$$

Exercise 61. Consider the optimal control problem:

$$\max_{u(\cdot)} \int_0^T x(t)\mathrm{d}t$$

subject to the dynamics

$$\dot{x}(t) = u(t), \quad x(0) = x_0,$$

and the control constraint

$$u(t) \in [0, 1].$$

1. Write the Hamiltonian.
2. Derive the Pontryagin necessary conditions.
3. Determine the structure of the optimal control.
4. Solve for the optimal trajectory $x^*(t)$.

▶

1. The present-value Hamiltonian is

$$H(x, p, u) = x + pu.$$

Its maximum in u is linear in u and depends on the sign of p:

$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > 0, \\ 0 & \text{if } p(t) < 0. \end{cases}$$

But the adjoint equation is

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = -1,$$

with terminal condition $p(T) = 0$. As a consequence

$$p(t) = T - t.$$

Since $p(t) = T - t > 0$ for all $t \in [0, T)$, the optimal control is

$$u^*(t) = 1 \quad \forall t \in [0, T].$$

The state equation is

$$\dot{x} = u^*(t) = 1, \quad x(0) = x_0,$$

so

$$x^*(t) = x_0 + t.$$