

# Memento of algebra

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August 22, 2025

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# 1 Vector spaces

## 1.1 Vector spaces

**Definition 1.1 (Vector spaces)** Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . A  $\mathbb{K}$ - vector space is a set  $V$  together with two binary operations,

- vector addition:  $V \times V \rightarrow V$  denoted  $v + w$ , where  $v, w$  are in  $V$ , and
- scalar multiplication:  $\mathbb{K} \times V \rightarrow V$  denoted  $av$ , where  $a \in \mathbb{K}$  and  $v \in V$ ,

satisfying the axioms below.

1. Vector addition is associative:

$$\forall(u, v, w) \in V^3, \quad u + (v + w) = (u + v) + w.$$

2. Vector addition is commutative:

$$\forall(u, v, w) \in V^3, \quad v + w = w + v.$$

3. Vector addition has an identity element: There exists an element  $0 \in V$  such that

$$\forall v \in V, \quad v + 0 = v.$$

4. Vector addition has inverse elements:

$$\forall v \in V, \exists w \in V : \quad v + w = 0.$$

5. Distributivity holds for scalar multiplication over vector addition:

$$\forall a \in \mathbb{K}, \forall(v, w) \in V^2, \quad a(v + w) = av + aw.$$

6. Distributivity holds for scalar multiplication over field addition:

$$\forall(a, b) \in \mathbb{K}^2, \forall v \in V, \quad (a + b)v = av + bv.$$

7. Scalar multiplication is compatible with multiplication in the field of scalars:

$$\forall(a, b) \in \mathbb{K}^2, \forall v \in V, \quad a(bv) = (ab)v.$$

8. Scalar multiplication has an identity element:

$$\forall v \in V, \quad 1v = v.$$

**Example 1** • Let  $n \in \mathbb{N}$ . The set  $\mathbb{R}^n$  with the usual addition of vectors and the scalar multiplication is a vector space.

- The set of matrices,
- Let  $(a, b) \in \mathbb{R} \cup \{-\infty, +\infty\}$ . The set of maps from  $a, b$  to  $\mathbb{R}$  is a vector space.

**Definition 1.2 (Vector sub-space)** Let  $(V, +, \cdot)$  be a vector space. A non-empty sub-space  $F$  of  $V$  is a vector sub-space equipped with the operation "+" and "." if it satisfies

$$\forall(u, v) \in F^2, \forall \lambda \in \mathbb{R} \quad u + \lambda \cdot v \in F.$$

**Example 2** • The set of polynomial functions of degree less than  $n$ ,

- The set of polynomial functions of any degree.

**Exercise 1** In  $\mathbb{R}^3$ , the plane defined by  $\{(x, y, z) \in \mathbb{R}^3 : y = 2x\}$  is a vector sub-space of  $\mathbb{R}^3$  (see Section 1.3 for other examples).

**Exercise 2** In  $\mathbb{R}^2$ , let  $F_1 = \{(x, y) \in \mathbb{R}^2 : y = x\}$  and  $F_2 = \{(x, y) \in \mathbb{R}^2 : y = -x\}$ . Is  $F_1 \cup F_2$  is a vector sub-space of  $\mathbb{R}^2$ .

**Proposition 1.3** Let  $E$  be a vector space and  $F_1$  and  $F_2$  be two vector sub-spaces of  $E$ . The set  $F_1 \cap F_2$  is a vector sub-space of  $E$ .

**Exercise 3** Proof left as an exercise.

## 1.2 Linear combination of vectors

**Definition 1.4 (Linear combination of vectors)** Let  $E$  be a vector space and  $F$  be a vector sub-space of  $E$ . Consider  $(u_i)_{i \in [[1, p]]} = (u_1, \dots, u_p) \in F^p$ . We call linear combination of  $(u_i)_{i \in [[1, p]]}$  any vector  $\sum_{i=1}^p \lambda_i u_i$ .

The set of all the linear combination of  $(u_i)_{i \in [[1, p]]}$  is denoted  $\text{Span}\{u_1, u_2, \dots, u_p\}$ .

The family  $(u_i)_{i \in [[1, p]]}$  spans  $F$  if  $F \in \text{Span}\{u_1, u_2, \dots, u_p\}$ .

**Exercise 4** In  $\mathbb{R}^3$ , under which condition a vector  $(x, y, z) \in \mathbb{R}^3$  is a linear combination of  $(1, 0, -1)$  and  $(0, 1, -1)$ .

**Exercise 5** Determine  $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \in \text{Span}\{(1, 0, -1), (0, 1, -1)\}\}$ .

**Proposition 1.5** Let  $V$  be a vector space and  $(u_i)_{i \in [[1, p]]} \in V^p$ . The set  $\text{Span}\{u_1, u_2, \dots, u_p\}$  is a vector sub-space of  $V$ .

**Exercise 6** Proof left as an exercise.

**Definition 1.6 (Linear independence)** Let  $V$  be a vector space and  $(u_i)_{i \in [[1, p]]} \in V^p$ . The family  $(u_i)_{i \in [[1, p]]}$  is linearly independent if

$$\sum_{i=1}^p \lambda_i u_i = 0_V \Rightarrow \lambda_i = 0, \quad \forall i \in [[1, p]].$$

The family  $(u_i)_{i \in [[1, p]]}$  is linearly dependent if it is not linearly independent.

**Example 3** The family  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  is linearly independent.

**Exercise 7** 1. In  $\mathbb{R}^3$ , prove that  $\{(0, 1, 2), (1, 0, 2), (1, 2, 0)\}$  is a linearly independent family.

2. In  $\mathbb{R}^3$ , prove that  $\{(0, 1, 2), (1, 0, 2), (3, 2, 10)\}$  is a linearly dependent family.

3. Let  $u = (2, 1, 0)$ ,  $v = (1, -1, 2)$  and  $w = (0, 3, -4)$ . Determine an equation of  $F = \text{Span}\{u, v, w\}$ .

4. Let  $P = X + 1$ ,  $Q = X^2$  and  $R = X - 1$ . Determine  $\text{Span}\{P, Q, R\}$ .

5. Let  $P = X - 2$  and  $Q = 3X - 6$ . Determine  $\text{Span}\{P, Q\}$ .

6. Are the following families linearly independent?

- (a)  $v_1 = (1, 0, 1)$ ,  $v_2 = (0, 2, 2)$  and  $v_3 = (3, 7, 1)$  in  $\mathbb{R}^3$ .
- (b)  $v_1 = (1, 0, 0)$ ,  $v_2 = (0, 1, 1)$  and  $v_3 = (1, 1, 1)$  in  $\mathbb{R}^3$ .
- (c)  $v_1 = (1, 2, 1, 2)$ ,  $v_2 = (2, 1, 2, 1)$ ,  $v_3 = (1, 0, 1, 1)$  and  $v_4 = (0, 1, 0, 0)$  in  $\mathbb{R}^4$ .
- (d)  $u = 2 - 5X + 6X^2 - X^3$ ,  $v = 3 + 2X - 4X^2 + 5X^3$  in  $\mathbb{R}[X]$ .
- (e) (\*)  $\cos$  and  $\sin$
- (f) (\*)  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto (x + y)$  and  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto (x - y)$
- (g) (\*\*)  $u : x \mapsto e^{i\pi x}$ ,  $v : x \mapsto e^{-i\pi x}$  and  $\cos$ .

**Example 4** The family  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  is a spanning family of  $\mathbb{R}^3$ .

**Exercise 8** (i) In  $\mathbb{R}^2$ ,  $\{(1, 1), (-1, 1)\}$  is a spanning family of  $\mathbb{R}^2$ .

(ii) In  $\mathbb{R}^3$ ,  $\{(1, 0, 0), (0, 0, 1)\}$  is not a spanning family of  $\mathbb{R}^3$ .

(iii) In  $\mathbb{R}^2$ ,  $\{(1, 1), (-1, 1), (0, 1)\}$  is a spanning family of  $\mathbb{R}^2$ .

**Remark 1** Any spanning family of  $\mathbb{R}^n$  contains at least  $n$  vectors.

**Definition 1.7 (Basis)** Let  $E$  be a vector sub-space. If a family  $(u_i)_{i \in [[1, p]]}$  is linearly independent and spanning of  $E$  we say that  $(u_i)_{i \in [[1, p]]}$  is a basis of  $E$ . We say that  $E$  is of dimension  $p$ .

**Exercise 9** What is the dimension of  $\mathbb{R}^n$ ?

**Example 5** The family  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  is a basis of  $\mathbb{R}^3$ .

**Exercise 10** In  $\mathbb{R}^2$ ,  $\{(1, 1), (-1, 1), (0, 1)\}$  is not a basis of  $\mathbb{R}^2$ .

**Exercise 11**

Find a basis of  $\mathcal{G} = \{(x, y, z) \in \mathbb{R}^4, x + y = 0 \text{ and } x + 3y - 2z = 0\}$ .

**Example 6 (Canonic base of  $\mathbb{R}^n$ )** Let  $(e_i)_{i \in [[1, n]]}$  be the vector whose component are 0 except the  $i$ -th component which is 1. The family  $(e_i)_{i \in [[1, n]]}$  is called the canonic base of  $\mathbb{R}^n$ .

**Theorem 1.8** Any base of  $\mathbb{R}^n$  contains exactly  $n$  vectors.

**Exercise 12** 1. Prove that  $u = (1, 2, 3)$ ,  $v = (0, 1, 2)$  and  $w = (0, 0, 1)$  is a basis of  $\mathbb{R}^3$ .

2. Determine a basis of  $\{(x, y, z, t) \in \mathbb{R}^4 : x + y - z + t = 0 \text{ and } y - 2z + t = 0\}$ .

3. Let  $F := \text{Span}\{(1, 1, 0, 1), (2, 2, 0, 1), (0, 2, 2, 0), (1, 0, 0, 0)\}$ . Complete  $\{(1, 3, 2, 1), (2, 6, 4, 1)\}$  to obtain a basis of  $F$ .

4. Let  $G := \{(x, y, z, t) \in \mathbb{R}^4 : x - y + z + t = 0\}$ .

(a) Prove that  $\mathcal{F} := \{(1, 1, 1, -1), (0, 1, 2, -1)\}$  is a linearly independent family of  $\mathbb{R}^4$ .

(b) Complete  $\mathcal{F}$  to obtain a basis of  $G$ .

**Theorem 1.9** Let  $E$  be a vector space of dimension  $N$

### 1.3 Geometry

**Definition 1.10 (Geometric vector space)** Let  $E$  be a vector space and  $F$  be a vector subspace of  $E$ .

- If  $\dim(F) = 1$  then  $F$  is called a (vectorial) straight line,
- If  $\dim(F) = 2$  then  $F$  is called a (vectorial) plan.

**Proposition 1.11** Let  $F$  and  $G$  be vector spaces.  $F = G$  is and only if  $F \subset G$  and  $\dim(F) = \dim(G)$ .

**Exercise 13** 1. Determine  $\dim(E \cap F)$  where

$$E = \{(x, y, z, t) \in \mathbb{R}^4 : x + y + z + t = 0\} \quad \text{and} \quad F = \{(x, y, z, t) \in \mathbb{R}^4 : x + y = z + t\} .$$

2. Determine the geometric nature of  $F \cap G$  where

$$F = \text{Span}\{(1, 0, 1, -1), (2, 1, 2, 1), (3, 1, 2, 0)\} \quad \text{and} \quad G = \text{Span}\{(1, 1, 3, -1), (2, -1, -4, 4)\}$$

3. Let  $F := \text{Span}\{(-1, 1, 1), (1, 0, 1)\}$  and  $G := \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0\}$ . Prove that  $F = G$ .

4. Prove that  $\text{Span}\{(1, -1, -2), (2, 3, -1)\} = \text{Span}\{(3, 7, 0), (5, 0, -7)\}$ .

5. Prove that  $\text{Span}\{X, X^2\} = \text{Span}\{X(X - 1), X^2\}$ .

## 2 Matrix

### 2.1 Definition

**Definition 2.1 (Matrix)** Let  $n$  and  $p$  in  $\mathbb{N}$ . We call (real coefficient) matrix, the collection of  $n \times p$  reals denoted  $(a_{ij})_{i \in [[1, n]], j \in [[1, p]]}$ . We say that  $a_{ij}$  is the general term of the matrix  $A$ . We represente the matrix in a table:

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1p} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{ip} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{np} \end{bmatrix}$$

The first index is the index of the line, the second is the index of the column.

We denoted by  $\mathcal{M}_{n,p}(\mathbb{R}^n)$  the set of all the matrix with  $n$  lines and  $p$  column.

**Example 7** In  $\mathbb{R}^n$ , a vector "is" a matrix in  $\mathcal{M}_{n,1}(\mathbb{R}^n)$ .

**Remark 2 (Representation of a family of vectors)** Let  $S := (u_i)_{i \in [[1, p]]}$  a family of  $p$  vectors of  $\mathbb{R}^n$ . For any  $j \in [[1, p]]$ , there exists  $n$  reals  $(a_{ij})_{i \in [[1, n]]}$  such that  $u_j = \sum_{i=1}^n a_{ij} e_i$  (where  $(u_i)_{i \in [[1, p]]}$  is the canonical base of  $\mathbb{R}^n$ ). The matrix  $A := (a_{ij})_{i \in [[1, n]], j \in [[1, p]]}$  is a representation of the family  $S$  in the canonic base of  $\mathbb{R}^n$ .

The  $j$ -th column of  $A$  is made of the coordinates of  $u_j$  in the canonic base.

**Exercise 14** (i) In  $\mathbb{R}^3$ , determine the matrix of the canonic base?

(ii) Determine is the matricial representation of  $\{(1, 0, 0, 0), (0, 1, 2, 0), (1, 0, 0, 3)\}$ .

**Definition 2.2** Let  $A = (a_{ij})_{i \in [[1, n]], j \in [[1, p]]}$  and  $B = (b_{ij})_{i \in [[1, n]], j \in [[1, p]]}$  be two matrix of  $\mathcal{M}_{n,p}(\mathbb{R}^n)$ .

- $A$  is equal to  $B$  if for all  $i \in [[1, n]]$  and  $j \in [[1, p]]$ ,  $a_{ij} = b_{ij}$ ,
- The sum  $A + B$  is a matrix of  $\mathcal{M}_{n,p}(\mathbb{R}^n)$  of general term  $a_{ij} + b_{ij}$ ,
- Let  $\lambda \in \mathbb{R}$ . The scalar product of the matrix  $A$  by  $\lambda$  denoted,  $\lambda A$  is the matrix of general term  $\lambda a_{ij}$ .

**Exercise 15** Define

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 3 & 4 \\ 2 & 1 & 6 \end{bmatrix}$$

Compute  $3A$ ,  $A + B$  and  $3A - B$ .

**Proposition 2.3** The space  $\mathcal{M}_{n,p}(\mathbb{R}^n)$  with the addition and the scalar product as defined above is a vector space.

**Definition 2.4 (Matricial product)** Let  $A = (a_{ij})_{i \in [[1, n]], j \in [[1, p]]}$  be a matrix in  $\mathcal{M}_{n,p}(\mathbb{R}^n)$  and  $B = (b_{ij})_{i \in [[1, p]], j \in [[1, q]]}$  be a matrix of  $\mathcal{M}_{p,q}(\mathbb{R}^n)$ . The product of  $A$   $B$  is the matrix in  $\mathcal{M}_{n,q}(\mathbb{R}^n)$  of general term  $c_{ij} := \sum_{k=1}^n a_{ik} b_{kj}$  for any  $i \in [[1, n]]$  and  $j \in [[1, q]]$ .

**Exercise 16** (i) Consider

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 2 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

Compute if possible  $AB$  and  $BA$ .

(ii) Consider

$$C := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Compute if possible  $CD$  and  $DC$ .

**Remark 3 (Block product)**

$$A \begin{bmatrix} C_1 & \dots & C_q \end{bmatrix} = \begin{bmatrix} AC_1 & \dots & AC_q \end{bmatrix}$$

and

$$\begin{bmatrix} L_1 \\ \vdots \\ L_p \end{bmatrix} B = \begin{bmatrix} L_1 B \\ \vdots \\ L_p B \end{bmatrix}$$

**Definition 2.5 (Transpose)** Let  $A$  of general term  $a_{ij}$  be a matrix in  $\mathcal{M}_{n,p}$ . The transpose of  $A$ , denoted  $A^t$ , is the matrix in  $\mathcal{M}_{p,n}$  of general term  $a_{ji}$ .

**Exercise 17** Consider  $A$  defined in Exercise 15. Compute  $A^t$ .

## 2.2 Application to a vector family

**Remark 4 (Linear combination of vectors)** Let  $u_i = \sum_{j=1}^n a_{ij}e_j$ . The relation  $(y_1, \dots, y_n) = \sum_{i=1}^p \lambda_i u_i$  can be written as

$$\begin{cases} y_1 = a_{11}\lambda_1 + \dots + a_{1j}\lambda_j + \dots + a_{1p}\lambda_p \\ \vdots \\ y_i = a_{i1}\lambda_1 + \dots + a_{ij}\lambda_j + \dots + a_{ip}\lambda_p \\ \vdots \\ y_n = a_{n1}\lambda_1 + \dots + a_{nj}\lambda_j + \dots + a_{np}\lambda_p \end{cases}$$

or thanks to matrix by  $Y = AX$  where

$$Y := \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad X := \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

and  $A$  is the matrix representation of the family of vectors  $(u_i)_{i \in \llbracket 1, p \rrbracket}$  (see Remark 2).

**Exercise 18** What is the matricial representation of the family  $\{u_1, u_2, u_3\}$  where  $u_1 := (1, 0, 1, -1)$ ,  $u_2 := (0, 1, 2, 1)$  and  $u_3 := (1, 2, 2, 1)$ ? What are the coordinates of  $u_1 - 2u_2 + u_3$ ?

## 2.3 Square matrix

**Definition 2.6** A matrix in  $\mathcal{M}_{n,p}$  is said to be a square matrix if  $n = p$ . We denote this space  $\mathcal{M}_n$ .

**Notation (Kronecker symbol)** We denote  $\delta_{ij}$  the real defined by  $\delta_{ij} = 0$  if  $i \neq j$  and  $\delta_{ii} = 1$ .

**Definition 2.7 (Identity matrix)** In  $\mathbb{R}^n$ , the square matrix in  $\mathcal{M}_n$  of general term  $\delta_{ij}$ . We denote it  $I_n$ .

**Example 8** In  $\mathbb{R}^2$ , the matrix

$$I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the identity matrix of  $\mathbb{R}^2$ .

**Proposition 2.8** For any matrix  $A \in \mathcal{M}_n$ , we have  $AI_n = I_n A$ .

**Definition 2.9** In  $\mathbb{R}^n$ , let  $I$  be the identity matrix. We say that a square matrix  $A \in \mathcal{M}_n$  is invertible if there exists a matrix  $B \in \mathcal{M}_n$  such that  $AB = BA = I$ .

If the matrix is not invertible we say that it is singular.

**Proposition 2.10** Let  $A$  and  $B$  be two invertible (square) matrix.  $(AB)^{-1} = B^{-1}A^{-1}$ .

**Theorem 2.11 (Criterion for invertibility (Part 1))**  $A \in \mathcal{M}_n$  invertible  $\Leftrightarrow$  the family of the column vectors of  $A$  is linearly independent  $\Leftrightarrow$  the family of the lines of  $A$  is linearly independent  $\Leftrightarrow$  the family of the column vectors of  $A$  is a basis of  $\mathbb{R}^n \Leftrightarrow$  the family of the lines of  $A$  is a basis of  $\mathbb{R}^n$ .

### 3 Linear systems

#### 3.1 Definitions

**Definition 3.1 (Elementary operation)** We call elementary operation on the lines one of the following transformation:

- Multiply the  $i$ -th line by a **non-nul** real  $\alpha$  (we write  $L_i \leftarrow \alpha L_i$ ),
- Exchange the lines  $i$ -th and  $j$ -th (we write  $L_i \leftrightarrow L_j$ ),
- Add to the  $i$ -th line the  $j$ -line multiplied by  $\alpha$  (we write  $L_i \leftarrow L_i + \alpha L_j$ ),
- Add to  $\alpha$  times the  $i$ -th line the  $j$ -line multiplied by  $\beta$  where  $\alpha \neq 0$  and  $i \neq j$  (we write  $L_i \leftarrow \alpha L_i + \beta L_j$ ),.

**Proposition 3.2** Let  $\mathcal{S}$  be a system. We obtain a system equivalent to  $\mathcal{S}$  by doing on one line one of the above elementary operation.

**Definition 3.3 (Echelon form)** A matrix is in echelon form if

- All non-zero rows are above any rows of all zeroes, and
- The leading coefficient of a row is always strictly to the right of the leading coefficient of the row above it.

The first non-zero entry in each row is called a pivot.

#### 3.2 Resolution

Let  $\mathcal{S}$  be a system. The aim of the Gauss method is to use successively the elementary operations on the lines of  $\mathcal{S}$  to obtain a equivalent system in echelon form.

**Gauss algorithm:**

For  $k = 1$  to  $n$

If there exists a line  $i \geq k$  such that  $a_{ik}^{k-1} \neq 0$  do  $L_i \leftrightarrow L_k$

$$L_k^k \leftarrow \frac{1}{a_{kk}^{k-1}} L_k^{k-1}$$

for  $i = 1$  to  $n$  and  $i \neq k$

$$L_i^k \leftarrow L_i^{k-1} - a_{ik}^{k-1} \times L_k^k$$

$k = k + 1$ .

The echelon system obtained after the Gauss algorithm:

$$R \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$$

is equivalent to the initial system  $\mathcal{S}$  ( $R$  is in echelon form).

**Resolution algorithm:**

Let  $q$  be the number of pivots.

- If ( $q < n$  and  $\exists i \in [[q + 1, n]]$  such that  $\beta_i \neq 0$ ) then  $\mathcal{S}$  is **impossible**
- If ( $q = n$ ) or ( $q < n$  and  $\forall i \in [[q + 1, n]]$ ,  $\beta_i = 0$ )

we solve the echelon system (by transferring if  $q < p$  the auxiliary unknown in the right hand side)

if ( $p = q$ ),  $\mathcal{S}$  admits a unique solution. We say that  $\mathcal{S}$  is a **Cramer system**,

if ( $p < q$ ),  $\mathcal{S}$  is **undetermined**.

**Exercise 19** (i) Solve

$$\begin{cases} 2x + y + z = 7 \\ x + 2y + z = 8 \\ x + y + 2z = 9 \end{cases}$$

(ii) Solve

$$\begin{cases} x + y + z = 3 \\ 2x + y + 3z = 1 \\ x - y + 3z = 5 \end{cases}$$

(iii) Solve

$$\begin{cases} x + 2y + 3z = 0 \\ 3x + 4y + 5z = 0 \\ 5x + 8y + 11z = 0 \\ x + y + z = 0 \end{cases}$$

**Remark 5 (Application to linearly independent family)** Let  $(u_i)_{i \in [1,p]}$  be a family of vectors of  $\mathbb{R}^n$  and  $A$  its matrix representation in the canonic base. The family is linearly independent if and only if the Gauss reduction of  $A$  has  $p$  pivots.

**Exercise 20** Is the family  $\{(1, 3, 5, 1), (2, 4, 8, 1), (3, 5, 11, 1)\}$  linearly independent?

**Consequence**

- A family of  $p$  vectors of  $\mathbb{R}^n$  cannot be linearly independent if  $p > n$ ,
- If  $F$  is a family of  $n$  vectors of  $\mathbb{R}^n$ .  $F$  linearly independent  $\Leftrightarrow F$  base  $\Leftrightarrow F$  is a spanning family of  $\mathbb{R}^n$ .

### 3.3 Application to the inverse of a matrix

**Theorem 3.4 (Criterion for invertibility (Part 2))** Let  $A \in \mathcal{M}_n$ . The matrix  $A$  is invertible if and only if it has  $n$  pivots.

**Gauss-Jordan method:** See TD.

**Exercise 21** What is the inverse of

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

## 4 Linear map

**Definition 4.1 (Linear map)** Let  $E$  and  $F$  be vector spaces. A map  $f : E \rightarrow F$  is linear if

- $\forall (x, y) \in E^2 \quad f(x + y) = f(x) + f(y)$ ,
- $\forall x \in E, \forall \lambda \in \mathbb{R} \quad f(\lambda x) = \lambda f(x)$

We denote  $\mathcal{L}(E, F)$  the set of linear maps from  $E$  to  $F$ .

**Matrix representation:** Reciproquely, let  $A$  be a matrix of  $\mathcal{M}_{n,p}$ . The relation  $X \mapsto AX$  is isomorph to a linear map  $f : \mathbb{R}^p \rightarrow \mathbb{R}^n$ .

Let  $E$  and  $F$  be vector spaces of finite dimnsion and  $f \in \mathcal{L}(E, F)$ . Let  $\mathcal{E} := \{e_i\}_{\{1, \dots, n\}}$  be basis of  $E$  and  $\mathcal{F}$  be a basis of  $F$ . The matricial representation of  $f$  in the basis  $\mathcal{E}$ ,  $FF$  denoted  $\text{Mat}(f, \mathcal{E}, \mathcal{F})$ , is the matrix which  $i$ -th column is made of the coordinates of  $f(e_i)$  in the basis  $\mathcal{F}$ .

**Exercise 22** (i) What is the matricial representation of the application  $f$  defined by  $f(x, y, z) = (x, z)$  in the canonical basis?

(ii) What is the application corresponding to

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}?$$

**Exercise 23** Let  $f$  which matricial representation in the canonical basis is

$$\begin{pmatrix} 3 & 1 & -3 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Set  $u_1 := (1, 1, 1)_C$ ,  $u_2 := (1, -1, 0)_C$  and  $u_3 := (1, 0, 1)_C$ . Let  $\mathcal{B}' := \{u_1, u_2, u_3\}$ .

1. Prove that  $\mathcal{B}'$  is a basis of  $\mathbb{R}^3$ .
2. Determine the matricial representation of  $f$  in the basis  $\mathcal{B}'$ .

**Definition 4.2 (Image, Kernel)** Let  $E$  and  $F$  be two sets and  $f$  a map from  $E$  to  $F$ .

- We call kernel of  $f$  the sub-set of  $E$  defined by

$$\text{Ker } f := \{x \in E : f(x) = 0\}.$$

- We call image of  $f$  the sub-set of  $F$  defined by

$$\text{Im } f := f(E) = \{y \in F : \exists x \in E \text{ such that } y = f(x)\}.$$

**Proposition 4.3** Let  $E$  and  $F$  be two vector spaces. Consider  $f \in \mathcal{L}(E, F)$ . The kernel of  $f$  is a vector sub-space of  $E$  and the image of  $f$  is a vector sub-space of  $F$ .

Proof Left to the reader.

**Proposition 4.4** Let  $E$  and  $F$  be two vector spaces. Consider  $f \in \mathcal{L}(E, F)$ . The image of  $f$  is spanned by the columns of any matricial representation of  $f$ .

**Application to the notions of injectivity, surjectivity and bijections**

**Definition 4.5 (Injectivity)** Let  $f : E \rightarrow F$ . If

$$\forall (x, x') \in E^2, f(x) = f(x') \Rightarrow x = x'$$

then  $f$  is injective.

**Proposition 4.6** Let  $f \in \mathcal{L}(E, F)$ .  $f$  is injective if and only if  $\text{ker}(f) = \{0\}$ .

proof Left to the reader.

**Definition 4.7 (Surjectivity)** Let  $f : E \rightarrow F$ . If

$$\forall y \in F, \exists x \in E \text{ such that } f(x) = y$$

then  $f$  is surjective on  $F$ .

By definition of the image we have

**Proposition 4.8** Let  $f : E \rightarrow F$ .  $f$  is surjective if and only if  $\text{Im}(f) = F$ .

**Definition 4.9 (Bijectivity)** Let  $f : E \rightarrow F$ . If then  $f$  is injective and surjective on  $F$  then  $f$  is bijective from  $E$  to  $F$ .

**Exercise 24** Let  $f$  which matricial representation in the canonical basis is

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

1. Determine a basis of  $B$  of  $\text{Im}(f)$  and a basis  $B'$  of  $\text{Ker}(f)$ .
2. Determine the matricial representation of  $f$  in the basis which first vectors are the vectors of  $B$  and the last vectors the vectors of  $B'$ .
3. Describe  $f$  in this basis.
4. Describe  $f$  in the canonical basis.

**Exercise 25** Let  $f$  which matricial representation in the canonical basis is

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

1. Determine a basis of  $B$  of  $\text{Im}(f)$  and a basis  $B'$  of  $\text{Ker}(f)$ .
2. Determine the matricial representation of  $f$  in the basis which first vectors are the vectors of  $B$  and the last vectors the vectors of  $B'$ .
3. Describe  $f$  in this basis.
4. Describe  $f$  in the canonical basis.

### Application to linear systems

Let  $A \in \mathcal{M}_{n,p}$  and  $B \in \mathcal{M}_{n,1}$ . The system  $AX = B$  of unknowns  $X \in \mathcal{M}_{p,1}$  has a solution if and only if  $B \in \text{Im}(A)$ . Moreover if  $X$  and  $X'$  are two solutions to this system  $X - X' \in \text{ker}(A)$ . Indeed, subtracting the equalities  $AX = B$  and  $AX' = B$  we obtain  $A(X - X') = 0$ . As a consequence any solution to  $AX = B$  is of the form  $X_P + \text{ker}(A)$  where  $X_P$  is any particular solution.

**Exercise 26** Consider

$$f : \begin{array}{ccc} \mathbb{R}^3 & \rightarrow & \mathbb{R}^3 \\ (x, y, z) & \mapsto & (y + z, x + y, x + 2y + z) . \end{array}$$

1. Prove that  $f$  is a linear map from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .

2. Determine the matricial representation of  $f$  in the canonical basis.
3. Determine a basis of  $\text{Ker}(f)$ .
4. Determine a basis of  $\text{Im}(f)$ .
5. Determine  $\text{Im}(f) \cap \text{Ker}(f)$ .
6. Discuss the number of solutions to  $f(x, y, z) = (1, 1, 1)$ .
7. Discuss the number of solutions to  $f(x, y, z) = (1, 1, 2)$ .
8. Discuss, depending on  $(a, b) \in \mathbb{R}^2$ , the number of solutions to  $f(x, y, z) = (1, a, b)$ .

**Definition 4.10 (Rank)** The rank of  $f$  is the dimension of the image of  $f$ . It is denoted  $\text{rk} f$ .

**Exercise 27** Consider

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Determine the rank of  $A$ . Find a basis of  $\text{Im} A$  and  $\text{Ker} A$ . Prove that  $\text{Im} A \cap \text{Ker} A = 0$ .

**Theorem 4.11 (Rank-nullity theorem)** Let  $E$  and  $F$  be two vector space and  $f$  a linear mapping from  $E$  to  $F$ . We have

$$\dim E = \text{rk} f + \dim \text{Ker} f .$$

**Remark 6** These quantity are also defined at the matrix level thanks to the matricial representation of a linear mapping.

As a consequence of the rank-nullity theorem we have

**Proposition 4.12** Let  $E$  and  $F$  be vector spaces of same dimension  $n$ . Consider  $f \in \mathcal{L}(E, F)$ . The following statements are equivalent

- $f$  is a one-to-one correspondance,
- $\ker(f) = \{0\}$ ,
- $\dim \ker(f) = 0$ ,
- $\text{Im}(f) = F$ ,
- $\dim \text{Im}(f) = n$ .

All these statements are equivalent to  $\text{Det}(f) \neq 0$ . In term of matrix this is equivalent to say that the matricial representation of  $f$  is inversible.

## 5 Diagonalisation

### 5.1 Change of basis

**Definition 5.1 (► Passage matrix)** Let  $E$  be a vector space and  $\mathcal{E}$  and  $\mathcal{E}'$  be two basis of  $E$ . We call passage matrix from  $\mathcal{E}$  to  $\mathcal{E}'$ , and denote,  $P_{\mathcal{E}}^{\mathcal{E}'}$ , the matrix

$$P_{\mathcal{E}}^{\mathcal{E}'} := \text{Mat}(\text{Id}, \mathcal{E}', \mathcal{E}) .$$

**Example 9** (►) Determine  $P_{\mathcal{C}_3}^{\mathcal{E}}$  where  $\mathcal{E} := \{(1, 1, 0), (0, 1, 0), (1, 1, 1)\}$ .

▷

$$P_{\mathcal{C}_3}^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

**Example 10** (►) In  $\mathbb{R}_2[X]$ , determine  $P_{\mathcal{C}_3}^{\mathcal{E}}$  where  $\mathcal{E} := \{1, X - 1, 2X^2 - X + 1\}$ .

▷

$$P_{\mathcal{C}_3}^{\mathcal{E}} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

**Example 11** (►) Determine  $P_{\mathcal{E}^3}^{\mathcal{C}_3}$  where  $\mathcal{E} := \{(1, 1, 0), (0, 1, 0), (1, 1, 1)\}$ .

▷ We have:

$$\begin{aligned} (1, 0, 0) &= 1 * (1, 1, 0) + (-1) * (0, 1, 0) + 0 * (1, 1, 1) = (1, -1, 0)_{\mathcal{E}} \\ (0, 1, 0) &= (0, 1, 0)_{\mathcal{E}} \\ (0, 0, 1) &= (-1, 0, 1)_{\mathcal{E}} \end{aligned}$$

So that

$$P_{\mathcal{E}^3}^{\mathcal{C}_3} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Proposition 5.2** (► *Inverse of a passage matrix*) Let  $E$  be a vector space and  $\mathcal{E}$  and  $\mathcal{E}'$  be two basis of  $E$ . We have  $P_{\mathcal{E}'}^{\mathcal{E}} = [P_{\mathcal{E}}^{\mathcal{E}'}]^{-1}$ .

**Proof 1** We have

$$P_{\mathcal{E}'}^{\mathcal{E}} \cdot P_{\mathcal{E}}^{\mathcal{E}'} = \text{Mat}(\text{Id}, \mathcal{E}', \mathcal{E}) \cdot \text{Mat}(\text{Id}, \mathcal{E}, \mathcal{E}') = \text{Mat}(\text{Id}, \mathcal{E}', \mathcal{E}') = I_n .$$

### 5.1.1 Change of basis : vector

**Proposition 5.3** (► *Change of basis : vector*) Let  $E$  be a vector space and  $\mathcal{E}$  and  $\mathcal{E}'$  be two basis of  $E$ . Consider  $x \in E$ . Let  $X_{\mathcal{E}}$  be the coordinates of  $x$  in the basis  $\mathcal{E}$  and  $X_{\mathcal{E}'}$  be the coordinates of  $x$  in the basis  $\mathcal{E}'$ . We have

$$X_{\mathcal{E}} = P_{\mathcal{E}'}^{\mathcal{E}} \cdot X_{\mathcal{E}'} .$$

**Proof 2** We have

$$P_{\mathcal{E}'}^{\mathcal{E}} \cdot X_{\mathcal{E}'} = \text{Mat}(\text{Id}, \mathcal{E}', \mathcal{E}) \cdot X_{\mathcal{E}'} = X_{\mathcal{E}} .$$

### 5.1.2 Change of basis : linear map

**Proposition 5.4** (► *Change of basis : linear map*) Let  $E$  and  $F$  be a vector space and  $\mathcal{E}$  and  $\mathcal{E}'$  be two basis of  $E$  and  $\mathcal{F}$  and  $\mathcal{F}'$  be two basis of  $F$ . Consider  $f \in \mathcal{L}(E, F)$ . We have

$$\text{Mat}(f, \mathcal{E}, \mathcal{F}) = P_{\mathcal{F}'}^{\mathcal{F}} \text{Mat}(f, \mathcal{E}', \mathcal{F}') P_{\mathcal{E}'}^{\mathcal{E}} .$$

► In the case of an endomorphisme we have

$$\text{Mat}(f, \mathcal{E}) = P \text{Mat}(f, \mathcal{E}') P^{-1} \quad \text{où} \quad P := P_{\mathcal{E}'}^{\mathcal{E}} . \quad (5.1)$$

## 5.2 Diagonalisation and eigen elements

**Definition 5.5 (► Similar matrix)** Let  $(A, B) \in \mathcal{M}_N(\mathbb{K})^2$ .  $A$  and  $B$  are similar if there exists an invertible  $P \in \mathcal{M}_N(\mathbb{K})$  such that

$$A = PBP^{-1}.$$

**Definition 5.6 (► Diagonalisable matrix)** Let  $A \in \mathcal{M}_N$ .  $A$  is diagonalisable in  $\mathbb{K}$  if it is similar in  $\mathbb{K}$  to a diagonal matrix.

elle est diagonalisable.

► We look for a basis  $\mathcal{E} := \{\varepsilon_i\}_{i \in \{1, \dots, N\}}$  such that:  $\text{Mat}(f, \mathcal{E}) = \text{diag}(\lambda_i)_{i \in \{1, \dots, N\}}$ . It means

$$\text{Mat}(f, \mathcal{E}) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

Hence

$$\forall i \in \{1, \dots, N\}, \quad f(\varepsilon_i) = \lambda_i \varepsilon_i.$$

**Definition 5.7 (► Eigenvalue, eigenvector)** Let  $A \in \mathcal{M}_N(\mathbb{K})$ . A real number  $\lambda$  is an eigenvalue of  $A$  in  $\mathbb{K}$  if there exists  $X_\lambda \neq 0$  in  $\mathbb{K}^N$  such that  $AX_\lambda = \lambda X_\lambda$ . Such a  $X_\lambda$  is called an eigenvector associated to the eigenvalue  $\lambda$ . The set of all the eigenvalues of  $A$  is called the spectrum of  $A$  and is denoted  $\text{Sp}(A)$ .

**Example 12** Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

1. Prove that  $(1, 1)$  is an eigenvector of  $A$ .
2. Prove that  $(-1, 1)$  is an eigenvector of  $A$ .

► We compute

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

hence  $(1, 1)$  is an eigenvector associated to 3. Similarly

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

hence  $(-1, 1)$  is an eigenvector associated to  $-1$ .

► To determine such a basis we are looking for  $(\lambda_i, \varepsilon_i) \in \mathbb{R} \times E \setminus \{0_E\}$  such that

$$f(\varepsilon_i) = \lambda_i \varepsilon_i.$$

or

$$\text{we are looking for } \varepsilon_i \neq 0_E \text{ such that } f(\varepsilon_i) - \lambda_i \varepsilon_i = 0_E$$

or equivalently

$$\text{we are looking for } \varepsilon_i \neq 0_E \text{ such that } (f - \lambda_i \text{Id})\varepsilon_i = 0_E$$

or

$$\text{we are looking for } \varepsilon_i \neq 0_E \text{ in } \ker(f - \lambda_i \text{Id})$$

This is possible if  $\dim(\ker(f - \lambda_i \text{Id})) > 0$ . Or  $\text{Det}(f - \lambda_i \text{Id}) = 0$ . We then have to find the zeros of  $\text{Det}(f - \lambda_i \text{Id})$  or of  $\text{Det}(A - \lambda_i I)$ .

**Definition 5.8 (► Characteristic polynomial function)** Let  $A \in \mathcal{M}_N$ . The polynomial function

$$\chi_A : \begin{array}{l} \mathbb{R} \rightarrow \mathbb{R} \\ \lambda \mapsto \text{Det}(A - \lambda I) . \end{array}$$

is the characteristic polynomial function of  $A$ .

**Proposition 5.9 (► Characterisation of an eigenvalue)** Let  $A \in \mathcal{M}_N$ . A scalar  $\lambda \in \mathbb{K}$  is an eigenvalue if and only if  $\chi_A(\lambda) = 0$ .

**Example 13** Determine the spectrum of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

**Definition 5.10 (► Eigenspace)** Let  $A \in \mathcal{M}_N$  and  $\lambda \in \mathbb{K}$ . The eigenspace is  $E_\lambda := \ker(A - \lambda I)$ .

**Proposition 5.11 (► Eigenvector)** Let  $A \in \mathcal{M}_N$  and  $\lambda \in \text{Sp}(A)$ . Any vector of  $E_\lambda$  is an eigenvector associated to  $\lambda$ .

**Example 14** Determine the eigenvector associated to  $-1$  and  $3$  of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

**Remark 1 (► )** We have

$$\chi_A = (-1)^n X^n + (-1)^{n-1} \text{Tr}(A) X^{n-1} + \dots + \text{Det}(A) .$$

so that

- $\text{Tr}(A)$  has to be equal to the sum of the eigenvalues counted as many times as their multiplicity,
- $\text{Det}(A)$  has to be equal to the product of the eigenvalues counted as many times as their multiplicity.

**Example 15** Let

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix}$$

1. Determine  $\chi_A$ .
2. Determine  $\text{Sp}(A)$ .
3. Determine a basis of eigenvectors of  $A$ .

▷

1. We compute

$$\begin{aligned} \chi_A(\lambda) &= \text{Det} \left( \begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \text{Det} \left( \begin{pmatrix} 1-\lambda & -4 \\ -4 & 7-\lambda \end{pmatrix} \right) \\ &= \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1) . \end{aligned}$$

2.  $\text{Sp}(A) = \{-1, 9\}$ .

3. We compute

$$\begin{pmatrix} x \\ y \end{pmatrix} \in E_{-1} = \ker(A + I) \Leftrightarrow \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x - 2y = 0$$

so that  $E_{-1} = \text{Span}\{(2, 1)\}$ . We can check

$$\begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = (-1) * \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

hence  $(2, 1)$  is an eigenvector associated to  $-1$ .

We compute

$$\begin{pmatrix} x \\ y \end{pmatrix} \in E_9 \Leftrightarrow \begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 2x + y = 0$$

so  $E_9 = \text{Span}\{(1, -2)\}$ .

4.  $\{(2, 1), (1, -2)\}$  is a basis of  $\mathbb{R}^2$  made of eigenvectors.

**Example 16** Let

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{pmatrix}$$

1. Determine  $\text{Sp}(A)$ .

2. Determine a basis of eigenvectors of  $A$ .

▷

1. We compute  $\chi_A(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 2)(\lambda - 1)^2$ . Hence  $\text{Sp}(A) = \{1, 2\}$ .

2. We compute  $E_2 = \text{Span}\{(1, 2, 1)\}$  and  $E_1 = \text{Span}\{(-1, 1, 0), (2, 0, 1)\}$ .

So that  $\{(1, 2, 1), (-1, 1, 0), (2, 0, 1)\}$  is a basis of  $\mathbb{R}^3$ .

**Example 17** Let

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 6 \\ 3 & 0 & 1 \end{pmatrix}$$

1. Determine  $\text{Sp}(A)$ .

2. can we determine a basis of eigenvectors of  $A$ .

▷

1. We compute  $\chi_A(\lambda) = -\lambda^3 + 6\lambda^2 - 32 = -(\lambda + 2)(\lambda - 4)^2$ .

2. We compute  $E_4 = \text{Span}\{(0, 1, 0)\}$  and  $E_{-2} = \{(-3, -2, 3)\}$ . There are only two eigenvectors so they cannot form a basis of  $\mathbb{R}^3$ .

### 5.3 Dimension criterion

**Theorem 5.12** (► *Dimension criterion*) Let  $A \in \mathcal{M}_N$  and  $\text{Sp}(A) = \{\lambda_i\}_{i \in \{1, \dots, p\}}$ .  $A$  is diagonalisable if and only if

$$\sum_{i=1}^p \dim(E_i) = N.$$

## 5.4 Multiplicity criterion

**Theorem 5.13** (► *Multiplicity criterion*) Let  $A \in \mathcal{M}_N(\mathbb{K})$ .

$A$  is diagonalisable in  $\mathbb{K} \Leftrightarrow \begin{cases} \chi_A \text{ is split on } \mathbb{K} \\ \text{for all } \lambda \in \text{Sp}(A) \text{ we have } \dim E_\lambda \text{ is equal to the multiplicity of } \lambda \text{ in } \chi_A \end{cases}$

**Corollary 5.14** (► *Characteristic polynomial is split and has simple eigenvalues*) Let  $A \in \mathcal{M}_N(\mathbb{K})$ . If  $A$  has  $N$  distinct eigenvalues in  $\mathbb{K}$  then  $A$  is diagonalisable in  $\mathbb{K}$ .

## 5.5 Cayley-Hamilton theorem

**Theorem 5.15**

$$\chi_A(A) = 0.$$

## 5.6 Applications

### 5.6.1 Application to iteration of linear maps

**Definition 5.16** (*iteration of linear maps*) Let  $f \in \mathcal{L}(E)$ . We define the iteration of  $f$  for all  $k \in \mathbb{N}^*$

$$\begin{cases} f^{(1)} = f \\ f^{(k+1)} = f \circ f^{(k)} \end{cases}.$$

**Proposition 5.17** (► *Power of a diagonalisable matrix*) Let  $M \in \mathcal{M}_n(\mathbb{R})$  similar to a diagonal matrix  $D$ . For all  $k \in \mathbb{N}^*$ ,

$$M^k = PD^kP^{-1}.$$

This equation is equivalent to  $X_{n+1} = AX_n$  with

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

By Proposition ?? we have  $X_n = A^n X_0$ . By Proposition 5.17,  $A^n = PD^nP^{-1}$ .

### 5.6.2 Application to the geometry of linear maps

**Example 18** describe geometrically

$$f : \begin{array}{ccc} \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ (x, y) & \mapsto & (2x, -2y) \end{array}$$

► The matricial representation of  $f$  is

$$A = \begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}$$

We have  $\text{Sp}(A) = \{0, 2\}$  and  $E_2 = \text{Span}(1, -1)$ ,  $E_0 = \text{Span}\{(0, 1)\}$ . Hence in the basis  $\{(1, -1), (0, 1)\}$  the matricial representation of  $f$  is

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

in this basis the linear map is the compose of a projection on the second eigenvector and an homothetia of power 2.

In the original basis  $f$  is the compose of a projection on the axis  $\text{Span}\{(1, -1)\}$  and an homothetia of power 2.

## 6 Bilinear and quadratic forms

**Definition 6.1** (► *Bilinear form*) Let  $E$  be a vector space. We consider the map  $b : E \times E \rightarrow \mathbb{K}$ .  $b$  is a bilinear map if

- For all  $y \in E$

$$\begin{aligned} b(\cdot, y) : E &\rightarrow \mathbb{K} \\ x &\mapsto b(x, y) \end{aligned}$$

is a linear map on  $E$ .

- For all  $x \in E$

$$\begin{aligned} b(x, \cdot) : E &\rightarrow \mathbb{K} \\ y &\mapsto b(x, y) \end{aligned}$$

is a linear map on  $E$ .

We denote  $\mathcal{B}(E)$  the set of bilinear forms on  $E$ .

**Example 19** Prove that

$$b : \begin{aligned} &\mathbb{R}^3 \times \mathbb{R}^3 &&\rightarrow \mathbb{K} \\ &\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &&\mapsto 2x_1y_1 + x_2y_2 + 3x_1y_2 - x_1y_3 + 5x_2y_1 + 2x_2y_3 + x_3y_1 - 6x_3y_2 - 3x_3y_3 \end{aligned}$$

is a bilinear form.

▷  $y$  being fixed

$$\begin{aligned} &\mathbb{R}^3 &&\rightarrow \mathbb{K} \\ &\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &&\mapsto x_1(2y_1 + 3y_2 - y_3) + x_2(5y_1 + y_2 + 2y_3) + x_3(y_1 - 6y_2 - 3y_3) \end{aligned}$$

is a linear form.

Similarly

$$\begin{aligned} &\mathbb{R}^3 &&\rightarrow \mathbb{K} \\ &\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &&\mapsto y_1(2x_1 + 5x_2 + x_3) + y_2(3x_1 + x_2 - 6x_3) + y_3(-x_1 + 2x_2 - 3y_3) \end{aligned}$$

is a linear form.

Hence  $b$  is a bilinear form on  $\mathbb{R}^3$ .

**Definition 6.2** (► *Symmetric bilinear form*) Let  $E$  be a vector space and  $b \in \mathcal{B}(E)$ .  $b$  is symmetric if

$$\forall (x, y) \in E^2, \quad b(x, y) = b(y, x).$$

**Example 20** (► ) Consider

$$\begin{aligned} \phi : \mathcal{C}([0, 1])^2 &\rightarrow \mathbb{R} \\ (f, g) &\mapsto \int_0^1 f(x)g(x)dx \end{aligned}$$

1. Prove that  $\phi$  is well defined.

2. Prove that  $\phi$  is a bilinear form.
3. Prove that  $\phi$  is symmetric.

▷

1. Any continuous function defined on  $[0, 1]$  is integrable on  $[0, 1]$ .
2. Let us first prove that  $\phi$  is symmetric

$$\phi(f, g) = \int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx = \phi(g, f) .$$

Remains to prove that  $f \mapsto \int_0^1 f(x)g(x)dx$  is linear which is trivial.

► Let  $\{e_i\}_{i \in \{1, \dots, N\}}$  be a basis of  $E$ . Any  $x \in E$  can be written  $x = \sum_{i=1}^N x_i e_i$  and any  $y \in E$  can be written  $y = \sum_{j=1}^N y_j e_j$ . Let  $b \in \mathcal{B}(E)$ . By bilinearity of  $b$  we have

$$b(x, y) = b \left( \sum_{i=1}^N x_i e_i, \sum_{j=1}^N y_j e_j \right) = \sum_{i=1}^N \sum_{j=1}^N x_i y_j b(e_i, e_j) .$$

So that any bilinear form can be written

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} x_i y_j .$$

where  $a_{ij}$  are real numbers. Moreover  $b$  is characterised by  $a_{ij} = b(e_i, e_j)$  for all  $(i, j) \in \{1, \dots, N\}$ .

**Definition 6.3 (► Matricial representation)** Let  $E$  be a vector space and  $\mathcal{E} := \{e_i\}_{i \in \{1, \dots, N\}}$  be a basis of  $E$ . Consider  $b \in \mathcal{B}(E)$ . The matricial representation of  $b$  in the basis  $\mathcal{E}$  is the matrix of general term  $b(e_i, e_j)$ .

**Example 21** determine the matricial representation of  $b$  defined in exercise 19.

▷ We have

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & 1 & 2 \\ 1 & -6 & -3 \end{pmatrix}$$

► Reciproquely to any  $A \in \mathcal{M}_N(\mathbb{K})$  we can associate the bilinear form

$$\begin{aligned} \mathcal{M}_N(\mathbb{K})^2 &\rightarrow \mathbb{R} \\ (X, Y) &\mapsto X^T A Y \end{aligned}$$

**Example 22** Determine the bilinear map associated to

$$\begin{pmatrix} 2 & 3 & -1 \\ 5 & 1 & 2 \\ 1 & -6 & -3 \end{pmatrix}$$

▷ We compute

$$\begin{aligned} (x_1 \quad x_2 \quad x_3) \begin{pmatrix} 2 & 3 & -1 \\ 5 & 1 & 2 \\ 1 & -6 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ = 2x_1y_1 + x_2y_2 + 3x_1y_2 - x_1y_3 + 5x_2y_1 + 2x_2y_3 + x_3y_1 - 6x_3y_2 - 3x_3y_3 . \end{aligned}$$

**Proposition 6.4 (► Change of basis in a bilinear form)** Let  $E$  be a vector space and  $\mathcal{E}$  and  $\mathcal{E}'$  be two basis of  $E$ . Consider  $b \in \mathcal{B}(E)$ . We have

$$\text{Mat}(b, \mathcal{E}) = (P_{\mathcal{E}'}^{\mathcal{E}})^T \text{Mat}(b, \mathcal{E}') P_{\mathcal{E}'}^{\mathcal{E}} .$$

## 6.1 Quadratic forms

**Definition 6.5** (► **Quadratic forms**) Let  $E$  be a vector space. A quadratic form on  $\mathbb{K}$  is a map  $q : E \rightarrow \mathbb{K}$  such that

- for all  $\lambda \in \mathbb{K}$  and all  $v \in E$ ,  $q(\lambda v) = \lambda^2 q(v)$ ,
- the map  $\phi : (u, v) \mapsto [q(u+v) - q(u) - q(v)]/2$  (called the polar form of  $q$ ) is a bilinear form

We denote  $\mathcal{Q}(E)$  the set of quadratic forms on  $E$ .

**Example 23** (► ) Consider

$$q : \begin{matrix} \mathbb{R}^3 & \rightarrow & \mathbb{K} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \mapsto & 2x_1^2 + x_2^2 + 8x_1x_2 - 4x_3x_2 - 3x_3^2 \end{matrix}$$

Prove that  $q$  is a quadratic form.

► Let us determine the polar form associated to  $q$ : for all  $u = (x_1, x_2, x_3)$  and  $v = (y_1, y_2, y_3)$  in  $\mathbb{R}^3$  we have

$$\begin{aligned} q(u+v) - q(u) - q(v) &= 2(x_1+y_1)^2 + (x_2+y_2)^2 + 8(x_1+y_1)(x_2+y_2) - 4(x_3+y_3)(x_2+y_2) - 3(x_3+y_3)^2 \\ &\quad - (2x_1^2 + x_2^2 + 8x_1x_2 - 4x_3x_2 - 3x_3^2) - (2y_1^2 + y_2^2 + 8y_1y_2 - 4y_3y_2 - 3y_3^2) \end{aligned}$$

We obtain

$$b(u, v) := \frac{1}{2} [q(u+v) - q(u) - q(v)] = 2x_1y_1 + x_2y_2 + 4x_1y_2 + 4x_2y_1 - 2x_3y_2 - 2x_2y_3 - 3x_3y_3 .$$

$b$  is a bilinear form because it is of the form  $\sum a_{ij}x_i y_j$ . Moreover  $b(u, u) = q(u)$  so that for all  $\lambda \in \mathbb{K}$  and all  $v \in E$ ,  $q(\lambda v) = b(\lambda v, \lambda v) = \lambda^2 b(v, v) = \lambda^2 q(v)$ .

**Example 24** (► ) Consider

$$\begin{matrix} \psi : & \mathcal{C}([0, 1]) & \rightarrow & \mathbb{R} \\ & f & \mapsto & \int_0^1 f(x)^2 dx \end{matrix}$$

Prove that  $\psi$  is a quadratic form.

► For all  $f \in \mathcal{C}([0, 1])$  and  $\lambda \in \mathbb{R}$ , we have

$$\psi(\lambda f) = \int_0^1 [\lambda f(x)]^2 dx = \lambda^2 \int_0^1 f(x)^2 dx = \lambda^2 \psi(f) .$$

Let us determine the polar form associated to  $\psi$ :

$$\psi(f+g) - \psi(f) - \psi(g) = \int_0^1 [f(x) + g(x)]^2 dx - \int_0^1 f(x)^2 dx - \int_0^1 g(x)^2 dx = 2 \int_0^1 f(x)g(x) dx .$$

Hence

$$\begin{matrix} b : & \mathcal{C}([0, 1])^2 & \rightarrow & \mathbb{R} \\ & (f, g) & \mapsto & \int_0^1 f(x)g(x) dx \end{matrix}$$

For any given  $g$

$$\begin{matrix} \mathcal{C}([0, 1]) & \rightarrow & \mathbb{R} \\ f & \mapsto & \int_0^1 f(x)g(x) dx \end{matrix}$$

is linear. Moreover  $b$  is symmetric hence  $b$  is a bilinear form and  $\psi$  is a quadratic form.

**Definition 6.6 (► Quadratic form associated to a bilinear form)** Let  $E$  be a vector space. Consider  $b \in \mathcal{B}(E)$ . We call quadratic form associated to a bilinear form  $b$  the form

$$\begin{aligned} E &\rightarrow \mathbb{R} \\ x &\mapsto b(x, x) \end{aligned}$$

Trick to find the polar form associated to a quadratic form in  $\mathbb{R}^n$ :  $x_i^2$  becomes  $x_i y_i$  and  $x_i x_j$  becomes  $(x_i y_j + x_j y_i)/2$ .

**Definition 6.7 (► Matricial representation of a quadratic form)** The matricial representation of a quadratic form in a basis is the matricial representation of the polar form associated in the same basis.

## 6.2 Sign of a quadratic form

**Definition 6.8 (► Positive and negative quadratic forms)** Let  $E$  be a vector space. Consider  $b \in \mathcal{B}(E)$ .

- $b$  is positive semi-definite if for all  $x \in E$ ,  $b(x, x) \geq 0$ .
- $b$  is negative semi-definite if for all  $x \in E$ ,  $b(x, x) \leq 0$ .
- $b$  is positive definite if for all  $x \in E$ ,

$$\begin{cases} b(x, x) \geq 0 \\ b(x, x) = 0 \Leftrightarrow x = 0 \end{cases}$$

- $b$  is negative definite if for all  $x \in E$ ,

$$\begin{cases} b(x, x) \leq 0 \\ b(x, x) = 0 \Leftrightarrow x = 0 \end{cases}$$

**Example 25** Consider

1.

$$q_1 : \begin{aligned} \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x_1, x_2) &\mapsto x_1^2 + x_2^2 \end{aligned}$$

2.

$$q_2 : \begin{aligned} \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x_1, x_2) &\mapsto x_1^2 + x_2^2 - 2x_1 x_2 \end{aligned}$$

3.

$$q_3 : \begin{aligned} \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x_1, x_2) &\mapsto x_1^2 + 5x_2^2 - 4x_1 x_2 \end{aligned}$$

Determine if the quadratic forms are positive definite.

▷

1. For all  $(x_1, x_2) \in \mathbb{R}^2$  we have

$$q_1(x_1, x_2) \geq 0 \quad \text{and} \quad q_1(x_1, x_2) = 0 \Leftrightarrow x_1^2 + x_2^2 = 0 \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

Hence  $q_1$  is positive definite.

2. Pour  $q_2$  on pourrait s'attendre naïvement à ce que le terme en  $-x_1x_2$  compense les termes positifs lorsque  $x_1$  et  $x_2$  sont grands. Ça n'est évidemment pas le cas puisque  $q_2(x_1, x_2) = (x_1 + x_2)^2 \geq 0$ . Par contre

$$q_2(x_1, x_2) = 0 \Leftrightarrow (x_1 + x_2)^2 = 0 \Leftrightarrow x_1 + x_2 = 0$$

mais ceci n'implique pas que  $x_1 = x_2 = 0$ . En effet,  $q_2(1, -1) = 0$  ce qui prouve que  $q_2$  n'est pas définie positive.

3. Pour  $q_3$ , on ne peut pas appliquer directement une identité remarquable. Par contre on peut faire apparaître des carrés. Cette méthode de réduction en carrés est due à Gauss et sera présentée en détail dans la section qui va suivre.

$$q_3(x_1, x_2) = x_1^2 + x_2^2 - 4x_1x_2 = (x_1 - 2x_2)^2 - (-2x_2)^2 + 5x_2^2 = (x_1 - 2x_2)^2 + x_2^2.$$

Comme  $q_3$  est une somme de carrés elle est au moins semi-définie positive. Pour ce qui est de la positivité on a

$$q_3(x_1, x_2) = 0 \Leftrightarrow (x_1 - 2x_2)^2 + x_2^2 = 0 \Leftrightarrow \begin{cases} x_1 - 2x_2 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

Donc  $q_3$  est définie positive.

### 6.3 Méthode de réduction en carrés de Gauss

Cette méthode permet toujours de déterminer la signature d'une forme quadratique. Cette méthode ouvre aussi le champs à de multiples développements. Nous en présenterons une dans la section à venir. Cette méthode est donc à maîtriser sans négociation possible.

D'après Remarque ??, toute forme quadratique sur  $\mathbb{R}^N$  s'écrit

$$q((x_i)_{\{1, \dots, N\}}) = \sum_{i=1}^N c_{ii}x_i^2 + \sum_{(i,j)=1, i < j}^N c_{ij}x_ix_j.$$

On va considérer deux cas :

- **Cas où il y a au moins un terme  $x_i^2$  :** Sans perte de généralité, quitte à renommer les variables, on peut supposer que  $c_{11} \neq 0$ . L'idée est de choisir une variable qui a un terme carré, ici on va prendre  $x_1$  et de faire apparaître le début d'un carré afin de faire disparaître cette variable du reste de la forme quadratique :

$$\begin{aligned} \sum_{i=1}^N c_{ii}x_i^2 + \sum_{(i,j)=1, i < j}^N c_{ij}x_ix_j &= c_{11} \left[ \sum_{i=1}^N \frac{c_{ii}}{c_{11}}x_i^2 + \sum_{(i,j)=1, i < j}^N \frac{c_{ij}}{c_{11}}x_ix_j \right] \\ &= c_{11} \left[ x_1^2 + x_1 \sum_{j=2}^N \frac{c_{1j}}{c_{11}}x_j + \sum_{i=2}^N \frac{c_{ii}}{c_{11}}x_i^2 + \sum_{(i,j)=2, i < j}^N \frac{c_{ij}}{c_{11}}x_ix_j \right] \\ &= c_{11} \left[ x_1 + \frac{1}{2} \sum_{j=2}^N \frac{c_{1j}}{c_{11}}x_j \right]^2 - \frac{c_{11}}{4} \left[ \sum_{j=2}^N \frac{c_{1j}}{c_{11}}x_j \right]^2 \\ &\quad + c_{11} \left[ \sum_{i=2}^N \frac{c_{ii}}{c_{11}}x_i^2 + \sum_{(i,j)=2, i < j}^N \frac{c_{ij}}{c_{11}}x_ix_j \right] \end{aligned}$$

La première ligne de la dernière égalité correspond exactement à tous les termes qui contiennent  $x_1$ . On voit que la partie

$$c_{11} \left\{ - \left[ \frac{1}{2} \sum_{j=2}^N \frac{c_{ij}}{c_{11}} x_j \right]^2 + \left[ \sum_{i=2}^N \frac{c_{ii}}{c_{11}} x_i^2 + \sum_{(i,j)=2, i < j}^N \frac{c_{ij}}{c_{11}} x_i x_j \right] \right\}$$

est une forme quadratique en  $(x_i)_{\{2, \dots, N\}}$  (donc qui ne contient plus de terme en  $x_1$ ). Donc à chaque étape on élimine une variable ce qui assure qu'en au plus  $N$  étapes la forme quadratique sera décomposée sous forme de somme de carrés.

► **Cas où il n'y a pas de terme  $x_i^2$  :** Sans perte de généralité, quitte à renommer les variables, on peut supposer que  $c_{12} \neq 0$ . L'idée est de choisir deux variables, ici on va prendre  $x_1$  et  $x_2$ , et de faire apparaître deux carrés tout en faisant disparaître ces deux variables du reste de la forme quadratique :

$$\begin{aligned} \sum_{(i,j)=1, i < j}^N c_{ij} x_i x_j &= c_{12} \left\{ x_1 x_2 + \sum_{(i,j)=2, i < j}^N \frac{c_{ij}}{c_{12}} x_i x_j \right\} \\ &= c_{12} \left\{ x_1 x_2 + x_1 \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j + x_2 \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j + \sum_{(i,j)=3, i < j}^N \frac{c_{ij}}{c_{12}} x_i x_j \right\} \\ &= c_{12} \left\{ \underbrace{\left( x_1 + \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j \right)}_{=:A} \underbrace{\left( x_2 + \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j \right)}_B + \sum_{(i,j)=3, i < j}^N \frac{c_{ij}}{c_{12}} x_i x_j \right\} \end{aligned}$$

En utilisant l'identité

$$AB = \frac{(A+B)^2}{4} - \frac{(A-B)^2}{4}$$

On obtient

$$\begin{aligned} q((x_i)_{\{1, \dots, N\}}) &= c_{12} \left\{ \frac{1}{4} \left( x_1 + \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j + x_2 + \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j \right)^2 \right. \\ &\quad \left. - \frac{1}{4} \left( x_1 + \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j - x_2 - \sum_{j=3}^N \frac{c_{ij}}{c_{12}} x_j \right)^2 + \sum_{(i,j)=3, i < j}^N \frac{c_{ij}}{c_{12}} x_i x_j \right\}. \end{aligned}$$

Pour la suite de l'algorithme, ou bien on a un terme carré et on utilise la technique vue dans le cas 1 ou bien il n'y a pas de terme carré et on utilise de nouveau cette technique vue dans ce cas 2.

En au plus  $N$  étapes l'algorithme s'arrête puisque la cas 1 fait apparaître un carré en faisant disparaître une variable du reste de la forme quadratique, alors que le cas 2 fait disparaître deux variables en faisant apparaître deux carrés. On a donc au plus  $N$  terme carrés.

Finalement cette technique permet d'écrire toute forme quadratique sous la forme

$$q(x) = \sum_{i=1}^p \alpha_i l_i^2(x)$$

où les  $\alpha_i$  sont des scalaires  $l_i$  sont des formes linéaires. De plus, comme on obtient des formes linéaires échelonnées ou de la forme  $\{(A, B) \mapsto A + B, (A, B) \mapsto A - B\}$ , la famille  $\{l_i\}_{i \in \{1, \dots, N\}}$  est libre. Il est noter qu'il est possible d'utiliser une autre méthode pour faire apparaître des carrés mais cette méthode assure la liberté de la famille des formes linéaires. Et la liberté ça compte. Le libéralisme compte aussi mais sans cœur..

**Exemple 26** Considérons

$$\begin{aligned} f : \mathbb{R}^3 &\rightarrow \mathbb{R} \\ (x, y, z) &\rightarrow x^2 + 2y^2 + 3z^2 + 4xy - 6xz - 4yz \end{aligned}$$

Est-ce que  $f$  est définie positive ?

▷ On peut commencer par choisir  $x$  car il est précédé du coefficient 1 :

$$f(x, y, z) = \underbrace{(x^2 + 2x(2y - 3z))}_{\text{tous les termes en } x} + \underbrace{2y^2 + 3z^2 - 4yz}_{\text{pas de terme en } x} \quad (6.2)$$

On reconnaît le début d'un carré dans le terme comportant tous les termes en  $x$  :

$$x^2 + 2x(2y - 3z) = [x + (2y - 3z)]^2 - \underbrace{(2y - 3z)^2}_{\text{terme carré à enlever pour corriger}} = (x + 2y - 3z)^2 - 4y^2 + 12yz - 9z^2.$$

Si on réintroduit dans l'expression 6.2 on obtient

$$f(x, y, z) = (x + 2y - 3z)^2 - 4y^2 + 12yz - 9z^2 + 2y^2 + 3z^2 - 4yz = (x + 2y - 3z)^2 - \underbrace{2y^2 - 6z^2 + 8yz}_{\text{forme quadratique en } y \text{ et } z}.$$

Puis on recommence

$$\begin{aligned} f(x, y, z) &= (x + 2y - 3z)^2 - 2(y^2 + 4z^2 - 4yz) \\ &= (x + 2y - 3z)^2 - 2[(y^2 - 4yz) + 3z^2] \\ &= (x + 2y - 3z)^2 - 2[(y - 2z)^2 - 4z^2 + 3z^2] \\ &= (x + 2y - 3z)^2 - 2(y - 2z)^2 - z^2. \end{aligned}$$

Il y a au moins un terme qui est précédé d'un coefficient négatif, nous verrons Section ?? que cela assure que  $f$  n'est pas même semi-définie positive, pour le moment il suffit d'exhiber un vecteur en lequel  $f$  n'est pas positive. Par exemple en résolvant le système échelonné :

$$\begin{cases} x + 2y - 3z = 0 \\ y - 2z = 0 \\ z = 1 \end{cases}$$

qui a pour solution  $(-1, 2, 1)$ . Donc  $f(-1, 2, 1) = -1$  et  $f$  n'est pas positive. On peut remarquer qu'elle n'est pas non plus semi-définie négative, parce qu'il y a un carré précédé d'un coefficient positif, en choisissant par exemple un vecteur tel que

$$\begin{cases} x + 2y - 3z = 1 \\ y - 2z = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 0 \\ z = 0 \end{cases}$$

en lequel  $f$  vaut  $+1$ .

**Example 27** Considérons la forme quadratique

$$f : \mathbb{R}^4 \rightarrow \mathbb{R}$$

$$(x, y, z, t) \rightarrow xy + xz + 2xt + yz - 4yt$$

Quelle est la signature de  $f$  ?

▷ On a

$$f(x, y, z, t) = \underbrace{(x + z - 4t)}_{\text{les termes multipliés par } y} \underbrace{(y + z + 2t)}_{\text{les termes multipliés par } x} - \underbrace{(z - 4t)(z + 2t)}_{\text{termes de correction, sans } x \text{ ni } y}$$

On utilise  $AB = (A + B)^2/4 - (A - B)^2/4$  et on obtient

$$f(x, y, z, t) = \frac{(x + z - 4t + y + z + 2t)^2}{4} - \frac{(x + z - 4t - y - z - 2t)^2}{4} - z^2 + 8t^2 + 2zt$$

$$= \frac{(x + y + 2z - 2t)^2}{4} - \frac{(x - y - 6t)^2}{4} - z^2 + 8t^2 + 2zt .$$

Le terme de reste, sans  $x$  ni  $y$ , comporte des carrés donc on fait apparaître, comme dans le premier cas, le début d'un carré :

$$f(x, y, z, t) = \frac{(x + y + 2z - 2t)^2}{4} - \frac{(x - y - 6t)^2}{4} - [z^2 - 8t^2 - 2zt]$$

$$= \frac{(x + y + 2z - 2t)^2}{4} - \frac{(x - y - 6t)^2}{4} - [(z - t)^2 - t^2 - 8t^2]$$

$$= \frac{(x + y + 2z - 2t)^2}{4} - \frac{(x - y - 6t)^2}{4} - (z - t)^2 + 9t^2 .$$

Ne pas oublier que le signe - dans la dernière égalité porte sur tout le terme entre parenthèses. C'est la raison pour laquelle cela facilite les calculs de commencer par les termes précédés d'un coefficient le plus simple possible.

**Theorem 6.9 (Sylvester's law)** Let  $E$  be a vector space of dimension  $N$  and  $q : E \rightarrow \mathbb{R}$  After reduction in squares with Gauss's method we obtain

$$q(X) = \sum_{i=1}^N \alpha_i [l_i(X)]^2 .$$

with  $\{l_i\}_i$  a family of linearly independent linear forms.

- If for all  $i \in \{1, \dots, N\}$ ,  $\alpha_i \geq 0$  then  $q$  is positive semi-definite.
- If for all  $i \in \{1, \dots, N\}$ ,  $\alpha_i \leq 0$  then  $q$  is negative semi-definite.
- If for all  $i \in \{1, \dots, N\}$ ,  $\alpha_i > 0$  then  $q$  is positive definite.
- If for all  $i \in \{1, \dots, N\}$ ,  $\alpha_i < 0$  then  $q$  is negative definite.

**Remark 7** The number of positive and negative coefficients in Gauss' reduction in squares does not depend on the decomposition chosen.

## 7 Scalar product

### 7.1 Definition

**Definition 7.1 (Scalar product)** Let  $E$  be a vector space. Consider  $b \in \mathcal{B}(E)$ . The map  $b$  is a scalar product if

- $b$  is symmetric
- $b$  is positive definite.

We call norm of  $u$ , and denote  $\|\cdot\|_b$  the quantity  $\|u\|_b := \sqrt{b(u, u)}$ .

**Definition 7.2 (Orthogonality)** Let  $E$  be a vector space and  $u$  and  $v$  be two vectors of  $E$ .  $u$  and  $v$  are orthogonal if  $b(u, v) = 0$ .

We will denote  $b = \langle \cdot, \cdot \rangle$ .

**Definition 7.3 (Orthogonal basis)** Let  $E$  be a vector sub-space,  $b$  be a scalar product on  $E$  and  $(u_i)_{i \in [[1, p]]}$  a basis of  $E$ . If  $b(u_i, u_j) = 0$ , for all  $i \neq j$  in  $[[1, p]]$  then  $(u_i)_{i \in [[1, p]]}$  is a  $b$ -orthogonal basis.

If moreover  $\|u_i\|_b = 1$ , for all  $i$  in  $[[1, p]]$  then the basis is said to be  $b$ -orthonormal.

### 7.2 Geometric interpretation

The norm denotes the length of a vector. We can define the angle between two vectors  $u$  and  $v$  as

$$\theta := \arccos \left( \frac{b(u, v)}{\|u\|_b \|v\|_b} \right).$$

With this definition we recover the usual identity

$$b(u, v) = \|u\|_b \|v\|_b \cos(\theta).$$

Two vectors are orthogonal if the angle between them is  $\frac{\pi}{2}$ .

### 7.3 Properties

**Proposition 7.4 (Pythagore theorem)** Let  $E$  be a vector space and  $u$  and  $v$  be two vectors of  $E$ .  $u$  and  $v$  are orthogonal if and only if  $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ .

*Proof*

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= \|v\|^2 + \|w\|^2 \end{aligned}$$

*Q.E.D.*

**Proposition 7.5 (Cauchy-Schwarz' inequality)** Let  $E$  be a vector space and  $u$  and  $v$  be two vectors of  $E$ . Then  $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$ .

*Proof*

**Step 1:** if  $v = 0$ , we have

$$\langle 0, w \rangle = \langle 0 + 0, w \rangle = \langle 0, w \rangle + \langle 0, w \rangle = 2 \langle 0, w \rangle.$$

Whence  $\langle 0, w \rangle = 0$ .

**Step 2:** We write  $w = \alpha v + u$  with , avec  $\alpha \in \mathbb{R}$  and  $u$  orthogonal to  $v$ . Observe that que  $w = \alpha v + u \Leftrightarrow u = w - \alpha v$ . Whence,

$$0 = \langle v, u \rangle = \langle v, w - \alpha v \rangle = \langle v, w \rangle - \alpha \langle v, v \rangle$$

Choose  $\alpha = \frac{\langle v, w \rangle}{\|v\|^2}$ . We can apply Pythagore theorem (Proposition 7.4):

$$\|w\|^2 = \left\| \frac{\langle v, w \rangle}{\|v\|^2} v \right\|^2 + \|u\|^2 \geq \frac{\|\langle v, w \rangle\|^2}{\|v\|^4} \|v\|^2 = \frac{\|\langle v, w \rangle\|^2}{\|v\|^2}.$$

We obtain

$$\|v\|^2 \|w\|^2 \geq \|\langle v, w \rangle\|^2$$

and so  $\|v\| \|w\| \geq \|\langle v, w \rangle\|$ .

*Q.E.D.*

**Remark 8** There is equality in the Cauchy-Schwartz inequality if and only if there exists  $\beta$  such that  $v = \beta w$ .

**Proposition 7.6 (Minkowski inequality, also known as triangular inequality)** Let  $E$  be a vector space and  $u$  and  $v$  be two vectors of  $E$ . Then  $\|x + y\| \leq \|x\| + \|y\|$ .

*Proof* Let us compute

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle \\ &\leq \|v\|^2 + 2\langle v, w \rangle + \|w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2. \end{aligned}$$

Whence  $\|v + w\| \leq \|v\| + \|w\|$ .

*Q.E.D.*

**Proposition 7.7 (Parallelogramme rule)** Let  $E$  be a vector space and  $u$  and  $v$  be two vectors of  $E$ . Then  $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ .

*Proof* Left to the reader.

**Proposition 7.8 (Polar identity)** Let  $E$  be a vector space and  $u$  and  $v$  be two vectors of  $E$ . Then  $\|x + y\|^2 - \|x - y\|^2 = 2\langle x, y \rangle + 2\langle y, x \rangle$ .

*Proof* Left to the reader.

**Exercise 28** Let

$$\begin{aligned} f : \mathbb{R}^3 \times \mathbb{R}^3 &\rightarrow \mathbb{R} \\ (x_1, x_2, x_3), (y_1, y_2, y_3) &\mapsto x_1 y_1 + 5x_2 y_2 + 6x_3 y_3 - x_1 y_2 - x_2 y_1 + x_3 y_1 + x_1 y_3 - 3x_3 y_2 - 3x_2 y_3 \end{aligned}$$

1. Prove that  $f$  is a bilinear form.
2. Determine the matricial representation of  $f$  in the canonical basis.
3. Determine the matricial representation of  $f$  in the basis  $\{(1, 1, 0), (1, 0, 1), (0, 0, 1)\}$ .
4. Determine the quadratic form  $Q$  associated to  $f$ .
5. Determine the matricial representation of  $f$  in the canonical basis.
6. Determine the matricial representation of  $f$  in the basis  $\{(1, 1, 0), (1, 0, 1), (0, 0, 1)\}$ .

**Exercise 29** Let

$$\begin{aligned} f : \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x_1, x_2), (y_1, y_2) &\mapsto x_1y_1 + 5x_2y_2 - x_1y_2 + x_2y_1 \end{aligned}$$

1. Prove that  $f$  is a bilinear form.
2. Determine the matricial representation of  $f$  in the canonical basis.
3. Determine the matricial representation of  $f$  in the basis  $\{(1, 1), (1, 0)\}$ .
4. Determine the quadratic form  $Q$  associated to  $f$ .
5. Determine the matricial representation of  $f$  in the canonical basis.
6. Determine the matricial representation of  $f$  in the basis  $\{(1, 1), (1, 0)\}$ .

**Exercise 30** Let

$$g : x_1y_1 + x_2y_2 + 4x_4y_4 + x_1y_2 + x_2y_1 - 2x_1y_4 - 2x_4y_1 + x_2y_3 + x_3y_2 - 2x_3y_4 - 2x_4y_3 .$$

1. Prove that  $g$  is a bilinear form.
2. Determine the matricial representation of  $g$  in the canonical basis.
3. What is the signature of  $g$ ?

**Exercise 31** Are the following scalar products (positive definite symmetric bilinear form)?

1.

$$(x_1, x_2, x_3), (y_1, y_2, y_3) \mapsto 2x_1y_1 + 5x_2y_2 + x_3y_3 - 3x_1y_2 - 3x_2y_1$$

2.

$$(x_1, x_2, x_3), (y_1, y_2, y_3) \mapsto x_1y_1 + x_2y_2 + x_3y_3 - 2x_1y_2 - 2x_2y_1 - 2x_1y_3 - 2x_3y_1 - 2x_2y_3 - 2x_3y_2$$

3.

$$(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \mapsto x_1y_2 + x_2y_1 - 2x_1y_3 - 2x_3y_1 + x_2y_3 + x_3y_2 - 2x_3y_4 - 2x_4y_3$$

## 7.4 Orthogonal projection

**Definition 7.9 (Projection)** Let  $E$  be a vector space. A projection is a linear mapping  $p : E \rightarrow E$  such that  $p^2 = p$ .

**Definition 7.10 (Orthogonal projection)** Let  $E$  be a vector space,  $p$  be a projection and  $P$  its matricial representation. The projection  $p$  is an orthogonal projection if  $P = P^T$ .

**Projection on a line:** If  $u$  is a vector, then the projection on the line  $\text{Span}(u)$  is given by

$$P_u = \frac{1}{\|u\|^2} u u^T.$$

This operator leaves  $u$  invariant, and it annihilates all vectors orthogonal to  $u$ .

**Projection on a vector sub-space:** If  $(u_i)_{i \in [1, p]}$  is a basis, and  $A$  is the matrix representing these vectors, then the projection on the space  $\text{Span}(u_1, \dots, u_p)$  is given by  $P_A = (A^T A)^{-1} A A^T$ .

**Distance to a straight line:** Let  $E$  be a vector space,  $F$  a vector sub-space,  $p$  be a projection on  $F$  and  $u$  be a vector of  $E$ . The orthonormal projection  $p(x)$  of  $x$  satisfies the following minimising property:

$$\forall y \in F, \|x - p(x)\| \leq \|x - y\|$$

and there is equality if and only if  $x = y$ .  $p(x)$  is thus the closest point of  $x$  which belongs to  $F$ . The distance  $\|x - p(x)\|$  is called the distance from  $x$  to  $F$ .